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Positive radial solutions for a class of quasilinear Schrödinger equations in $\mathbb{R}^3$

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Abstract. This paper is concerned with the following quasilinear Schrödinger equations of the form:

$$-\Delta u - u\Delta(u^2) + u = |u|^{p-2}u, \qquad x \in \mathbb{R}^3,$$

where $p \in (2, 12)$. By making use of the constrained minimization method on a special manifold, we prove that the existence of positive radial solutions of the above problem for any $p \in (2, 12)$.

Keywords: quasilinear Schrödinger equations, constrained minimization method.

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1 Introduction

In this paper, we are devoted to studying the following quasilinear Schrödinger equations:

$$-\Delta u - u\Delta(u^2) + u = |u|^{p-2}u, \qquad x \in \mathbb{R}^3, \tag{1.1}$$

where $p \in (2, 12)$.

Set

$$E := \left\{ u \in H_r^1\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} u^2 \left|\nabla u\right|^2 dx < \infty \right\},$$

where

$$H_r^1\left(\mathbb{R}^3\right) := \left\{ u \in H^1\left(\mathbb{R}^3\right) : u\left(|x|\right) = u\left(x\right) \right\}$$

with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 \right) dx \right)^{1/2}.$$

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A function $u \in E$ is called a weak solution of equation (1.1), if for all $\phi \in C_0^\infty(\mathbb{R}^3)$ it holds

$$\int_{\mathbb{R}^3} \nabla u \nabla \phi dx + \int_{\mathbb{R}^3} u\phi dx + 2\int_{\mathbb{R}^3} u^2 \nabla u \nabla \phi dx + 2\int_{\mathbb{R}^3} |\nabla u|^2 u\phi dx = \int_{\mathbb{R}^3} |u|^{p-2} u\phi dx.$$

Define the functional I on E by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx + \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx.$$

It is easy to check that I is continuous on E. Furthermore, given $u \in E$ and $\phi \in C_0^\infty(\mathbb{R}^3)$, we can compute the Gateaux derivative of I in the direction ϕ at u:

$$\langle I'(u), \phi \rangle = \int_{\mathbb{R}^3} \nabla u \nabla \phi dx + \int_{\mathbb{R}^3} u\phi dx + 2\int_{\mathbb{R}^3} \left(|\nabla u|^2 u\phi + u^2 \nabla u \nabla \phi\right) dx - \int_{\mathbb{R}^3} |u|^{p-2} u\phi dx.$$

Hence u is a weak solution of equation (1.1) if and only if this derivative is zero in every direction $\phi \in C_0^\infty(\mathbb{R}^3)$.

When $V(x) = 1$, $\alpha(s) = s$ and $f(x,z) = |z|^{p-2}z$, solutions of equation (1.1) are standing waves of the following quasilinear Schrödinger equations of the form:

$$iz_t + \Delta z - V(x)z + \Delta\alpha\left(|z|^2\right)\alpha'\left(|z|^2\right)z + f(x,z) = 0, \qquad x \in \mathbb{R}^3, \tag{1.2}$$

where $V(x)$ is a given potential, α and f are real functions. Equation (1.2) has been derived as models of several physical phenomena, such as [1,4–6]. It began with [11] for the studies on mathematics. Several methods can be used to deal with problem (1.2), such as, the existence of a positive ground state solution was studied by making use of the constrained minimization method in [8,12]; Liu et al. in [9] and Colin et al. in [3] obtained the existence results for equation (1.2) through making a change of variable and reducing the quasilinear problem (1.2) to a semilinear one; Nehari method was used to obtain the existence results of ground state solutions for equation (1.2) in [10]. Moreover, in [7], the existence results for the general form of quasilinear elliptic equations were studied by means of a perturbation method. Especially, in [13], Ruiz et al. proved the existence of positive radial solutions for the Schrödinger–Poisson equation by using the constrained minimization argument on the Nehari–Pohožaev manifold.

In the present paper, inspired by [13], our goal is to prove the existence of positive radial solutions for equation (1.1) via the constrained minimization method on the Nehari–Pohožaev manifold. Our main result reads as follows.

Theorem 1.1. *For $2 < p < 12$, problem (1.1) possesses one positive radial solution.*

2 Preliminaries and proof of main result

Lemma 2.1. *For $p \in (2,12)$, I is unbounded from below.*

Proof. Let $u \in E$ be radial and positive, and $u_t = t^{1/2}u\left(t^{-1}x\right)$ for $t > 0$. To facilitate the estimation of $I(u_t)$, we firstly compute:

$$\int_{\mathbb{R}^3} |\nabla u_t|^2 dx = t^2 \int_{\mathbb{R}^3} |\nabla u|^2 dx, \qquad \int_{\mathbb{R}^3} u_t^2 dx = t^4 \int_{\mathbb{R}^3} u^2 dx,$$

$$\int_{\mathbb{R}^3} u_t^2 |\nabla u_t|^2 dx = t^3 \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx, \qquad \int_{\mathbb{R}^3} |u_t|^p dx = t^{\frac{p+6}{2}} \int_{\mathbb{R}^3} |u|^p dx.$$

Then one has

$$\begin{aligned} I(u_t) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_t|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^3} u_t^2 dx + \int_{\mathbb{R}^3} u_t^2 |\nabla u_t|^2\, dx - \frac{1}{p}\int_{\mathbb{R}^3} |u_t|^p\, dx \\ &= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{t^4}{2}\int_{\mathbb{R}^3} u^2 dx + t^3\int_{\mathbb{R}^3} u^2 |\nabla u|^2\, dx - \frac{1}{p} t^{\frac{p+6}{2}} \int_{\mathbb{R}^3} |u|^p dx. \end{aligned}$$

Since $(p+6)/2 > 4$ for $p \in (2,12)$, we easily infer that $I(u_t) \to -\infty$ as $t \to +\infty$. □

Lemma 2.2. *Let c_1, c_2, c_3, c_4 be positive constants and $p > 2$. Then for $t > 0$, the function*

$$\eta(t) = c_1 t^2 + c_2 t^3 + c_3 t^4 - c_4 t^{\frac{p+6}{2}}$$

has a unique positive critical point which corresponds to its maximum.

Proof. The conclusion is easily obtained by elementary calculation. □

Now, in order to define the Nehari–Pohožaev manifold, we firstly need to introduce the following Pohožaev identity (see, e.g., [13, p. 1224]).

Lemma 2.3. *If $u \in E$ is a weak solution to equation (1.1), then the following Pohožaev identity holds:*

$$P(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{3}{p}\int_{\mathbb{R}^3} |u|^p dx = 0.$$

Proof. The proof is standard, so we omit it. □

As mentioned in the introduction, we will use the constrained minimization argument on a special manifold to prove the existence result of equation (1.1).

Let us justify the choice of the manifold. Assume that $u \in E$ is a critical point of I. Define, as above, $u_t(x) = t^{1/2} u(t^{-1}x)$, and consider

$$\eta(t) = I(u_t) = \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{t^4}{2}\int_{\mathbb{R}^3} u^2 dx + t^3 \int_{\mathbb{R}^3} u^2 |\nabla u|^2\, dx - \frac{1}{p} t^{\frac{p+6}{2}} \int_{\mathbb{R}^3} |u|^p dx.$$

Obviously, $\eta(t) > 0$ for small t and $\eta(t) \to -\infty$ as $t \to +\infty$. Moreover, it follows from Lemma 2.2 that $\eta(t)$ has a unique critical point which corresponds to its maximum. But since u is a critical point of I, the maximum of $\eta(t)$ should be achieved at $t = 1$ and thus $\eta'(1) = 0$. Thus we can define the manifold $\mathcal{T}$ as

$$\mathcal{T} := \left\{ u \in E \setminus \{0\} : J(u) = 0 \right\},$$

where

$$J(u) := \eta'(1) = \int_{\mathbb{R}^3} |\nabla u|^2 dx + 2\int_{\mathbb{R}^3} u^2 dx + 3\int_{\mathbb{R}^3} u^2 |\nabla u|^2\, dx - \frac{p+6}{2p}\int_{\mathbb{R}^3} |u|^p dx.$$

Clearly, $J(u) = \frac{1}{2}\langle I'(u), u\rangle + P(u)$. If u is a nontrivial solution of problem (1.1), then $u \in \mathcal{T}$. The manifold $\mathcal{T}$ can be viewed as the combination of the commonly used Nehari manifold and Pohožaev manifold. Such manifold was first introduced in [13], in which the Schrödinger–Poisson system was studied.

Lemma 2.4. *If $p \in (2,12)$, then $\mathcal{T}$ is a C^1-manifold and every critical point of $I|_{\mathcal{T}}$ is a critical point of I.*

Proof. Step 1. $0 \notin \partial\mathcal{T}$. By Sobolev's inequality, one has

$$J(u) \geq \|u\|^2 - C_1 \frac{p+6}{2p}\|u\|^p,$$

where C_1 is a positive constant. Choosing R small enough, then there exists $\rho > 0$ such that $J(u) > \rho$ for $\|u\| < R$, that is, $0 \notin \partial\mathcal{T}$.

Step 2. $\inf I|_{\mathcal{T}} > 0$. For any $u \in \mathcal{T}$, for convenience, we set

$$\alpha = \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad \beta = \int_{\mathbb{R}^3} u^2 dx, \quad \gamma = \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx, \quad \theta = \int_{\mathbb{R}^3} |u|^p dx, \quad s = I(u). \tag{2.1}$$

Then $\alpha, \beta, \gamma, \theta$ are positive, and we get

$$\begin{cases} I(u) = \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma - \frac{1}{p}\theta = s, \\ J(u) = \alpha + 2\beta + 3\gamma - \frac{p+6}{2p}\theta = 0. \end{cases} \tag{2.2}$$

By solving the system (2.2), we obtain

$$\gamma = \frac{2(p+6)s - (p+2)\alpha - (p-2)\beta}{2p} \tag{2.3}$$

and

$$\frac{p+2}{4}\alpha + \frac{p-2}{4}\beta + \frac{p}{2}\gamma = \frac{p+6}{2}s. \tag{2.4}$$

Since $\alpha, \beta, \gamma > 0$ and $p > 2$, we follow from (2.3) and (2.4) that

$$(p-2)(\alpha+\beta) < (p+2)\alpha + (p-2)\beta < 2(p+6)s \tag{2.5}$$

and

$$\gamma < \frac{p+6}{p}s. \tag{2.6}$$

Moreover, it follows from Step 1 that there exists $\varepsilon > 0$ such that $\alpha + \beta > \varepsilon$. Therefore, by (2.5) we get

$$I(u) = s > \frac{p-2}{2(p+6)}(\alpha+\beta) > 0, \tag{2.7}$$

which means $I|_{\mathcal{T}} > 0$.

Step 3. $\mathcal{T}$ is a C^1-manifold. It suffices to show that $J'(u) \neq 0$ for any $u \in \mathcal{T}$ by the implicit function theorem. Suppose that $J'(u) = 0$ for some $u \in \mathcal{T}$. In a weak sense, the equation $J'(u) = 0$ can be written as

$$-2\Delta u - 3u\Delta(u^2) + 4u = \frac{p+6}{2}|u|^{p-2}u. \tag{2.8}$$

Multiplying (2.8) by u and integrating, one has

$$\langle J'(u), u\rangle = 2\int_{\mathbb{R}^3} |\nabla u|^2 dx + 4\int_{\mathbb{R}^3} u^2 dx + 12\int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{p+6}{2}\int_{\mathbb{R}^3} |u|^p dx = 0. \tag{2.9}$$

The Pohožaev identity corresponding to (2.9) is

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx + 6\int_{\mathbb{R}^3} u^2 dx + 3\int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{3(p+6)}{2p}\int_{\mathbb{R}^3} |u|^p dx = 0. \tag{2.10}$$

Thus, using the same notations defined in (2.1), we follow from (2.9) and (2.10) that

$$\begin{cases} I(u) = \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma - \frac{1}{p}\theta = s, \\ J(u) = \alpha + 2\beta + 3\gamma - \frac{p+6}{2p}\theta = 0, \\ 2\alpha + 4\beta + 12\gamma - \frac{p+6}{2}\theta = 0, \\ \alpha + 6\beta + 3\gamma - \frac{3(p+6)}{2p}\theta = 0. \end{cases}$$

It can be checked out that for $p \in (2, 12)$, the above system of equations admits one unique solution on θ, given by

$$\theta = \frac{-24ps}{(p-2)(p+3)}.$$

Since $s > 0$, we infer $\theta < 0$, which is impossible. So $J'(u) \neq 0$ for any $u \in \mathcal{T}$, and then we conclude that $\mathcal{T}$ is a C^1-manifold.

Step 4. $I'(u) = 0$. Assume that u is a critical point of $I|_{\mathcal{T}}$. Depending on the Lagrange multiplier argument, there exists $\mu \in \mathbb{R}$ such that $I'(u) = \mu J'(u)$. We claim that $\mu = 0$.

As above, $I'(u) = \mu J'(u)$ can be written, in a weak sense, as

$$-\Delta u - u\Delta(u^2) + u - u^{p-2}u = \mu \left[-2\Delta u - 3u\Delta(u^2) + 4u - \frac{p+6}{2}u^{p-2}u \right],$$

which means

$$-(1-2\mu)\,\Delta u - (1-3\mu)\,u\Delta(u^2) + (1-4\mu)u = \left(1 - \frac{p+6}{2}\mu\right) u^{p-2}u. \tag{2.11}$$

Combining (2.2) and (2.11), we get

$$\begin{cases} I(u) = \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma - \frac{1}{p}\theta = s, \\ J(u) = \alpha + 2\beta + 3\gamma - \frac{p+6}{2p}\theta = 0, \\ \alpha + \beta + 4\gamma - \theta = 0, \\ (1-2\mu)\alpha + (1-4\mu)\beta + (4-12\mu)\gamma - \left[1 - \frac{p+6}{2}\mu\right]\theta = 0. \end{cases} \tag{2.12}$$

The third equation corresponds to $\langle I'(u), u \rangle = 0$ for $u \in \mathcal{T}$. The fourth one follows by multiplying (2.11) by u and integrating. Now we deal with this system. Considering $\alpha, \beta, \gamma, \theta$ as unknowns and denoting by D the coefficient matrix, we can get

$$\det D = \frac{(p-2)\mu}{2}.$$

Therefore, for $p \in (2, 12)$ we infer

$$\det D = 0 \Leftrightarrow \mu = 0.$$

Now we prove that $\mu = 0$ by contradiction. If $\mu \neq 0$, then $\det D \neq 0$, which means system (2.12) has a unique solution. So we can obtain

$$\theta = -\frac{12s}{p-2}.$$

This is impossible since θ must be positive. Hence $\mu = 0$, and then $I'(u) = 0$. □

Lemma 2.5. *If $p \in (2,12)$, then $c_{\mathcal{T}}$ is achieved, where $c_{\mathcal{T}} := \inf\{I(u) : u \in \mathcal{T}\}$.*

Proof. Let $\{u_n\} \subset \mathcal{T}$ be a minimizing sequence of $I|_{\mathcal{T}}$, namely that $I(u_n) \to c_{\mathcal{T}}$. Referring to (2.5) and (2.6), in a similar way we can deduce that

$$\|u_n\|^2 < \frac{2(p+6)}{p-2} I(u_n)$$

and

$$\int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 dx < \frac{p+6}{p} I(u_n).$$

Then $\{u_n\}$ is bounded in E and $\{\nabla(u_n^2)\}$ is bounded in $L^2(\mathbb{R}^3)$. Moreover, by the continuous Sobolev embedding $E \hookrightarrow L^6(\mathbb{R}^3)$ and Hölder's inequality, we conclude that there exists a positive constant C such that

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n^2|^2 dx &\le \left(\int_{\mathbb{R}^3} |u_n|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u_n|^6 dx\right)^{\frac{1}{2}} \\ &\le C\|u_n\|^4,\end{aligned}$$

which together with the boundedness of $\{\nabla(u_n^2)\}$ in $L^2(\mathbb{R}^3)$ means that $\{u_n^2\}$ is bounded in E. Therefore, by using the compact embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ for any $s \in (2,6)$ and interpolation inequality, we get

$$\begin{cases} u_n^2 \rightharpoonup u^2 & \text{in } E, \\ u_n \rightharpoonup u & \text{in } E, \\ u_n \to u & \text{in } L^q(\mathbb{R}^3), \text{ for } q \in (2,12), \\ u_n \to u & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{2.13}$$

We claim that $u \in \mathcal{T}$ and $u_n \to u$ strongly in E.

Similar to (2.1), we define

$$\alpha_n = \int_{\mathbb{R}^3} |\nabla u_n|^2 dx, \qquad \beta_n = \int_{\mathbb{R}^3} u_n^2 dx, \qquad \gamma_n = \int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 dx, \qquad \theta_n = \int_{\mathbb{R}^3} |u_n|^p dx$$

and

$$\tilde{\alpha} = \lim_{n\to\infty} \alpha_n, \qquad \tilde{\beta} = \lim_{n\to\infty} \beta_n, \qquad \tilde{\gamma} = \lim_{n\to\infty} \gamma_n, \qquad \tilde{\theta} = \lim_{n\to\infty} \theta_n.$$

In order to show $u_n \to u$ in E, we just need to prove $\|u_n\| \to \|u\|$ by the Brezis–Lieb Lemma in [2], that is, $\alpha + \beta = \tilde{\alpha} + \tilde{\beta}$. From (2.13), we infer that $\alpha \le \tilde{\alpha}, \beta \le \tilde{\beta}$ and $\gamma \le \tilde{\gamma}$. Suppose by contradiction that $\alpha + \beta < \tilde{\alpha} + \tilde{\beta}$.

Noting that $\lim_{n\to\infty} I(u_n) = c_{\mathcal{T}}$ and $J(u_n) = 0$, we infer

$$\begin{cases} \frac{1}{2}\tilde{\alpha} + \frac{1}{2}\tilde{\beta} + \tilde{\gamma} - \frac{1}{p}\tilde{\theta} = c_{\mathcal{T}}, \\ \tilde{\alpha} + 2\tilde{\beta} + 3\tilde{\gamma} - \frac{p+6}{2p}\tilde{\theta} = 0. \end{cases} \tag{2.14}$$

We first show $u \neq 0$. By (2.13), we easily infer that $\theta = \tilde{\theta}$. Thanks to Step 2 in the proof of Lemma 2.4, we get $\tilde{\alpha} + \tilde{\beta} > \varepsilon > 0$, which together with (2.14) yields to $\tilde{\theta} > 0$. Thus we infer

$$\theta = \int_{\mathbb{R}^3} |u|^p dx > 0,$$

which means $u \neq 0$.

Set

$$g(t) = \frac{1}{2}t^2\alpha + \frac{1}{2}t^4\beta + t^3\gamma - \frac{1}{p}t^{\frac{p+6}{2}}\theta, \qquad \tilde{g}(t) = \frac{1}{2}t^2\tilde{\alpha} + \frac{1}{2}t^4\tilde{\beta} + t^3\tilde{\gamma} - \frac{1}{p}t^{\frac{p+6}{2}}\tilde{\theta}.$$

Depending on Lemma 2.2, we know that both g and $\tilde{g}$ have a unique critical point, corresponding to their maxima. From (2.14), we get that $\tilde{g}'(1) = 0$, namely that $\tilde{g}(1) = c_{\mathcal{T}}$. Moreover, since $\alpha + \beta < \tilde{\alpha} + \tilde{\beta}, \gamma \leq \tilde{\gamma}$ and $\theta = \tilde{\theta}$, then $g(t) < \tilde{g}(t)$ for all $t > 0$. Let $t_0 > 0$ be the maximum of g. Then $g'(t_0) = 0$ and $g(t_0) < c_{\mathcal{T}}$.

Define $v_0(x) = t_0^{1/2}u(t_0^{-1}x)$. Then one has

$$I(v_0) = \frac{1}{2}t_0^2\alpha + \frac{1}{2}t_0^4\beta + t_0^3\gamma - \frac{1}{p}t_0^{\frac{p+6}{2}}\theta = g(t_0) < c_{\mathcal{T}}$$

and

$$J(v_0) = t_0^2\alpha + 2t_0^4\beta + 3t_0^3\gamma - \frac{p+6}{2p}t_0^{\frac{p+6}{2}}\theta = g'(t_0)\,t_0 = 0.$$

Then $v_0 \in \mathcal{T}$ and $I(v_0) < c_{\mathcal{T}}$, which is a contradiction. Therefore $\alpha + \beta = \tilde{\alpha} + \tilde{\beta}$, and then $u_n \to u$ in E. □

Proof of Theorem 1.1. By Lemma 2.5, we know that $I|_{\mathcal{T}}$ attains its minimum at u and $u \neq 0$, namely that u is a nontrivial critical point of $I|_{\mathcal{T}}$. And then from Lemma 2.4, we get that u is a nontrivial solution of equation (1.1). Since the functional I and the manifold $\mathcal{T}$ are symmetric, we easily deduce that $|u|$ is also a nontrivial solution of equation (1.1). Hence we may assume that such a solution does not change sign, i.e., $u \geq 0$. Depending on the strong maximum principle, u must be strictly positive, and then u is a positive solution of equation (1.1). □

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