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Ground states solutions for some non-autonomous Schrödinger–Bopp–Podolsky system

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Abstract. In this paper we study the existence of ground states solutions for non-autonomous Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + u + \lambda K(x)\phi u = b(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2\Delta^2\phi = 4\pi K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0, 2 < p \leq 4$ and both $K(x)$ and $b(x)$ are nonnegative functions in $\mathbb{R}^3$. Assuming that $\lim_{|x|\to+\infty} K(x) = K_\infty > 0$ and $\lim_{|x|\to+\infty} b(x) = b_\infty > 0$ and satisfying suitable assumptions, but not requiring any symmetry property on them. We show that the existence of a positive solution depends on the parameters λ and p. We also establish the existence of ground state solutions for the case $3.18 \approx \frac{1+\sqrt{73}}{3} < p \leq 4$.

Keywords: non-autonomous Schrödinger–Bopp–Podolsky system, variational methods, Pohožaev identity, Nehari manifold.

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1 Introduction and main results

In this paper we are concerned with the existence of ground states for Schrödinger–Bopp–Podolsky (SBP) system

$$\begin{cases} -\Delta u + u + \lambda K(x)\phi u = b(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2\Delta^2\phi = 4\pi K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $a > 0$ is the Bopp-Podolsky (BP) parameter, u represents the modulus of the wave function and ϕ the electrostatic situation. The Schrödinger–Bopp–Podolsky system has been studied in [13] for the first time in the mathematical literature. The system appears when one

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looks for stationary solutions $u(x)e^{i\omega t}$ of the Schrödinger equation coupled with the Bopp–Podolsky Lagrangian of the electromagnetic field in the purely electrostatic situation.

The Bopp–Podolsky theory is a second order for the electromagnetic field, and was proposed to deal with the so called infinity problem that appears in the classical Maxwell theory which is similar to the Mie theory [21] and its generalizations given by Born and Infeld [3–6]. In fact, by the well-known Gauss law (or Poisson's equation), the electrostatic potential ϕ for a given charge distribution whose density is ρ satisfies the equation

$$-\Delta\phi = \rho \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

If $\rho = 4\pi\delta_{x_0}$, with $x_0 \in \mathbb{R}^3$, the fundamental solution of (1.2) is $\mathcal{G}(x - x_0)$, where

$$\mathcal{G}(x) = \frac{1}{|x|},$$

and the electrostatic energy is

$$\mathcal{E}_M(\mathcal{G}) = \frac{1}{2}\int_{R^3} |\nabla\mathcal{G}|^2 = +\infty.$$

Thus, equation (1.2) is replaced by

$$-\text{div}\left(\frac{\nabla\phi}{\sqrt{1 - |\nabla\phi|^2}}\right) = \rho \quad \text{in } \mathbb{R}^3$$

in the Bopp–Infeld theory and by

$$-\Delta\phi + a^2\Delta^2\phi = \rho \quad \text{in } \mathbb{R}^3,$$

in the Bopp–Podolsky theory. In both cases, if $\rho = 4\pi\delta_{x_0}$, their solutions can be written explicitly, and the corresponding energy is finite. In this paper, we focus on the Bopp–Podolsky theory $-\Delta + a^2\Delta^2$, the fundamental solution of the equation

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_{x_0}$$

is $\mathcal{L}(x - x_0)$, where

$$\mathcal{L}(x) := \frac{1 - e^{-\frac{|x|}{a}}}{|x|},$$

which presents no singularities at x_0, since

$$\lim_{x\to x_0} \mathcal{L}(x - x_0) = \frac{1}{a}.$$

Furthermore, its energy is

$$\mathcal{E}_{\text{BP}}(\mathcal{L}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\mathcal{L}|^2\, \mathrm{d}x + \frac{a^2}{2}\int_{\mathbb{R}^3} |\Delta\mathcal{L}|^2\, \mathrm{d}x < \infty.$$

We refer to [13] for more details.

In recent years, there has been increasing attention to problems like (1.1) on the existence of positive solutions, ground state solutions, multiple solutions and normalized solutions, see e.g. [1, 10, 16, 18–20, 27] and the references therein. According to [25], we know that there are two parameters $K(x)$ and $b(x)$ have an effect on the nonlocal term and nonlinear term. Hence, we take advantage of the idea of [25]. And we know that a typical way to deal with (1.1) is to use Nehari manifold and variational methods. In this paper, we mainly solve the Pohožaev identity of (1.1), because the non-local terms and nonlinear terms are affected by $K(x)$ and $b(x)$. It has not been studied before.

Then we are concerned with existence of ground states for following generalized nonlinear system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + u + \lambda K(x)\phi u = b(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi K(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

It is known that system (1.3) can be transformed into a nonlinear Schrödinger equation with a non-local term, for example, see [2,11,24]. Then we can use the same method as in [13] to find the solution of the second equation of the system (1.3). For all $u \in H^1(\mathbb{R}^3)$, the unique $\phi_{K,u} \in \mathcal{D}$ (where $\mathcal{D}$ is a function space that will be introduced in Section 2) is given by

$$\phi_{K,u}(x) = \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|} K(y)u^2(y)dy,$$

such that $-\Delta\phi + a^2\Delta^2\phi = 4\pi K(x)u^2$ and that, substituting it into the first equation of system (1.3), gives

$$-\Delta u + u + \lambda K(x)\phi_{K,u}u = b(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

Equation (1.4) has solutions are the critical points of functional $\mathcal{J}(u)$ defined in $H^1(\mathbb{R}^3)$ as

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx. \tag{1.5}$$

Furthermore, one can see that $\mathcal{J}$ is a C^1 functional with the derivative given by

$$\langle \mathcal{J}'(u), \varphi \rangle = \int_{\mathbb{R}^3} \left(\nabla u \nabla\varphi + u\varphi + \lambda K(x)\phi_{K,u}u\varphi - b(x)|u|^{p-2}u\varphi\right) dx$$

for all $\varphi \in H^1(\mathbb{R}^3)$, where $\mathcal{J}'$ denotes the Fréchet derivative of $\mathcal{J}$. We say that a pair $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}$ is a solution of system (1.3) if and only if u is a critical point of $\mathcal{J}$. Furthermore, for system(1.3), we find that the corresponding Pohožaev identity (see section 6 for more details) is

$$\begin{aligned} 0 = &\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} u^2 dx + \frac{5\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx \\ &+ \frac{\lambda}{2}\int_{\mathbb{R}^3} \langle \nabla K(x), x\rangle \phi_{K,u}u^2 dx + \frac{\lambda}{4a}\int_{\mathbb{R}^3} K(x)\psi_{K,u}u^2 dx \\ &- \frac{3}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx - \frac{1}{p}\int_{\mathbb{R}^3} \langle \nabla b(x), x\rangle |u|^p dx, \end{aligned}$$

where $K(x), b(x) \in C^1(\mathbb{R}^3)$ and $\psi_{K,u} := e^{-\frac{|x|}{a}} * Ku^2 = \int_{\mathbb{R}^3} e^{-\frac{|x-y|}{a}} K(y)u^2(y)dy$. It appears that the Pohožaev identity of the non-autonomous case looks more complicated than that of the autonomous case [24].

Therefore, we introduce a new set that can be seen as the filtration of the Nehari manifold. That is

$$\mathcal{N}(c_\tau) = \{u \in \mathcal{N} : \mathcal{J}(u) < c_\tau\},$$

where $\mathcal{N} = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle \mathcal{J}'(u), u\rangle = 0\}$ (we can see in [22]) is the Nehari manifold and c_τ is the energy level of the functional $\mathcal{J}$. Apparently, $\mathcal{N}(c_\tau)$ is a subset of the Nehari manifold. We will show that $\mathcal{N}(c_\tau)$ can be divided into two parts

$$\mathcal{N}^{(1)}(c_\tau) = \{u \in \mathcal{N}(c_\tau) : \|u\| < C_1\} \quad \text{and} \quad \mathcal{N}^{(2)}(c_\tau) = \{u \in \mathcal{N}(c_\tau) : \|u\| > C_2\},$$

where each local minimizer of the functional $\mathcal{J}$ is a critical point of $\mathcal{J}$ in $H^1(\mathbb{R}^3)$. The approach we take is to minimize the energy functional $\mathcal{J}$ on the $\mathcal{N}^{(1)}(c_\tau)$, where the $\mathcal{J}$ is bounded below and the minimization sequence is bounded.

This paper gives the following assumptions about $b(x)$ and $K(x)$:

(G_1) $b(x)$ is a positive continuous function on $\mathbb{R}^3$ such that

$$\lim_{|x|\to\infty} b(x) = b_\infty > 0 \quad \text{uniformly on } \mathbb{R}^3,$$

and

$$b_{\max} := \sup_{x\in\mathbb{R}^3} b(x) < \frac{b_\infty}{A(p)^{\frac{p-2}{2}}},$$

where

$$A(p) = \begin{cases} \left(\frac{4-p}{2}\right)^{\frac{1}{p-2}} & \text{if } 2 < p \le 3, \\ \frac{1}{2} & \text{if } 3 < p \le 4. \end{cases}$$

(G_2) $K(x) \in L^\infty(\mathbb{R}^3)\backslash\{0\}$ is a non-negative function on $\mathbb{R}^3$ such that

$$\lim_{|x|\to\infty} K(x) = K_\infty \ge 0 \quad \text{uniformly on } \mathbb{R}^3.$$

Remark 1.1. A direct calculation shows that for $2 < p \le 4$, there holds

$$A(p) < \frac{1}{\sqrt{e}} < 1 \quad \text{and} \quad A(p)\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}} > 1.$$

Let w_0 be tha unique positive solution of the following Schrödinger equation

$$-\triangle u + u = b_\infty |u|^{p-2}u \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

Available from [17]

$$w_0(0) = \max_{x\in\mathbb{R}^3} w_0(x),$$

$$\|w_0\|^2 = \int_{\mathbb{R}^3} b_\infty |w_0|^p dx = \left(\frac{S_p^p}{b_\infty}\right)^{\frac{2}{p-2}}, \tag{1.7}$$

and

$$\alpha_\infty^0 := \inf_{u\in\mathcal{M}_\infty^0} \mathcal{J}_0^\infty(u) = \frac{p-2}{2p}\left(\frac{S_p^p}{b_\infty}\right)^{\frac{2}{p-2}},$$

where $\mathcal{J}_0^\infty$ is the energy functional of equation (1.6) in $H^1(\mathbb{R}^3)$ in the form

$$\mathcal{J}_0^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx - \frac{1}{p}\int_{\mathbb{R}^3} b_\infty |u|^p dx, \tag{1.8}$$

with

$$\mathcal{M}_\infty^0 = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} \mid \langle (\mathcal{J}_0^\infty)'(u), u\rangle = 0\}.$$

Definition 1.2. (u, ϕ) is called a ground state solution of system (1.3) , if (u, ϕ) is a solution of system (1.3) which has the least energy among all nontrivial solutions of system (1.3) .

Now, we give our main results as follows.

Theorem 1.3. *Suppose that* $2 < p \leq 4, K(x) \equiv K_\infty > 0$ *and* $b(x) \equiv b_\infty > 0$. *Then for each* $0 < \lambda < \Lambda$, *system* (1.3) *has a positive solution* $(w, \phi_{K_\infty,w}) \in H^1(\mathbb{R}^3) \times \mathcal{D}$, *and when* $2 < p < 4$ *it satisfies*

$$0 < \|w\| < \left(\frac{2S_p^p}{b_\infty(4-p)} \right)^{\frac{1}{p-2}},$$

and

$$\alpha_\infty^0 < \alpha_\infty^- := \mathcal{J}(w) < \frac{A(p)(p-2)}{2p} \left(\frac{2S_p^p}{b_\infty(4-p)} \right)^{\frac{2}{p-2}}.$$

In particular, when $p = 4$ *we have*

$$\alpha_\infty^- = \mathcal{J}(w) > \alpha_\infty^0,$$

and $(w, \phi_{K_\infty,w})$ *is a ground state solution of system* (1.3).

Theorem 1.4. *Suppose that* $2 < p \leq 4, K_\infty > 0$ *and conditions* (G_1)–(G_2) *hold. Furthermore, we assume that*

(G_3) $\int_{\mathbb{R}^3} [b(x) - b_\infty] w^p dx \geq 0$ *and* $\int_{\mathbb{R}^3} K(x)\phi_{K,w} w^2 dx \leq \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx$, *but the equality signs can not hold at the same time, where* w *is the positive solution as described in Theorem* 1.3.

Then for each $0 < \lambda < \Lambda$, *system* (1.3) *has a positive solution* $(v, \phi_{K,v}) \in H^1(\mathbb{R}^3) \times \mathcal{D}$, *and when* $2 < p < 4$ *it satisfies*

$$0 < \|v\| < \left(\frac{2S_p^p}{b_{\max}(4-p)} \right)^{\frac{1}{p-2}},$$

and

$$\frac{p-2}{4p} \left(\frac{S_p^p}{b_{\max}} \right)^{2/(p-2)} \leq \mathcal{J}(v) < \alpha_\infty^- \quad \text{for } 2 < p < 4.$$

In particular, when $p = 4$ *we have*

$$\frac{1}{4} \left(\frac{S_p^p}{b_{\max}} \right)^{2/(p-2)} \leq \mathcal{J}(v) < \alpha_\infty^-,$$

and $(v, \phi_{K,v})$ *is a ground state solution of system* (1.3).

Theorem 1.5. *Suppose that* $\frac{1+\sqrt{73}}{3} < p < 4$ *and conditions* (G_1)–(G_2) *hold. Furthermore, we assume that*

(G_4) *the functions* $b(x), K(x) \in C^1(\mathbb{R}^3)$ *satisfy* $\langle \nabla b(x), x \rangle \leq 0$ *and*

$$\frac{3p^2 - 2p - 24}{2(6-p)} K(x) + \frac{p(p-2)}{6-p} \langle \nabla K(x), x \rangle \geq 0.$$

If $(v, \phi_{K,v})$ *is the positive solution as described in Theorem* 1.4, *then* $(v, \phi_{K,v})$ *is a ground state solution of system* (1.3).

The paper is organized as follows. First, we present some notations and the lemma for the later proof in section 2. In Section 3, we give the proof Theorem 1.3. In Section 4, is devoted to proof Theorem 1.4. Section 5 is dedicated to the proof of Theorem 1.5.

2 Notations and preliminaries

We use the following notation:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$\langle u,v\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx; \qquad \|u\|^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx.$$

- H^{-1} denotes the dual space of $H^1(\mathbb{R}^3)$.
- $L^p(\Omega), 1 \le p \le +\infty, \Omega \subseteq \mathbb{R}^3$, demotes a Lebesgue space, the norm in L^p is denoted by $|u|_{p,\Omega}$ when Ω is a proper subset of $\mathbb{R}^3$, by $|\cdot|_p$ when $\Omega = \mathbb{R}^3$.
- C, C', C_i are various positive constants.
- For any $\theta > 0$ and for any $\zeta \in \mathbb{R}^3, B_\theta(\zeta)$ denotes the ball of radius θ centered at ζ.
- $\hat{S}$ is the best constant for the embedding of $H^1(\mathbb{R}^3)$ in $L^{\frac{12}{5}}\left(\mathbb{R}^3\right)$.
- $\bar{S}$ is the best Sobolev constant for the embedding of $D^{1,2}\left(\mathbb{R}^3\right)$ in $L^6\left(\mathbb{R}^3\right)$, that is

$$\bar{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_{D^{1,2}}}{|u|_6}.$$

- S_p is the best Sobolev constant for the embedding of $H^1(\mathbb{R}^3)$ is continuously embedded into $L^p(\mathbb{R}^3)$ $(2 \le p \le 6)$, that is

$$S_p = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|}{|u|_p}$$

  where

$$D^{1,2}\left(\mathbb{R}^3\right) := \left\{u \in L^6\left(\mathbb{R}^3\right) : \nabla u \in L^2\left(\mathbb{R}^3\right)\right\}.$$

Then we let

$$\Lambda = \begin{cases} \frac{(p-2)K_\infty^{-2}\bar{S}^2\hat{S}^4}{2(4-p)} \left(\frac{b_\infty(4-p)^2}{2pS_p^p}\right)^{2/(p-2)} & \text{if } 2<p<4, \\ b_\infty K_\infty^{-2}\bar{S}^2\hat{S}^4 S_4^{-4} & \text{if } p=4, \end{cases}$$

and

$$\Lambda_0 = \left[1 - A(p)\left(\frac{b_{\max}}{b_\infty}\right)^{2/(p-2)}\right]\left(\frac{b_\infty}{S_p^p}\right)^{2/(p-2)} \frac{\bar{S}^2\hat{S}^4}{K_{\max}^2}, \tag{2.1}$$

where $K_{\max} = \sup_{x\in\mathbb{R}^3} K(x)$. When $p = 12/5$, we may take $S_{12/5} = \hat{S}$. In particular, if $K(x) \equiv K_\infty$ and $b(x) \equiv b_\infty$, then equality (2.1) becomes

$$\Lambda_0 = (1 - A(p))\left(\frac{b_\infty}{S_p^p}\right)^{2/(p-2)} \frac{\bar{S}^2\hat{S}^4}{K_\infty^2}.$$

- $\mathcal{D}(\mathbb{R}^3)$ is the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$\langle \eta, \zeta\rangle_{\mathcal{D}} = \int_{\mathbb{R}^3} (\nabla\eta\nabla\zeta + a^2\Delta\eta\Delta\zeta)dx.$$

Then $\mathcal{D}$ is a Hilbert space continuously embedded into $D^{1,2}(\mathbb{R}^3)$ and consequently in $L^6(\mathbb{R}^3)$. It is interesting to point out the following properties.

Lemma 2.1 ([13]). *The space $\mathcal{D}$ is continuously embedded in $L^\infty(\mathbb{R}^3)$.*

The next lemma gives a useful characterization of the space $\mathcal{D}$.

Lemma 2.2 ([13]). *The space C_c^∞ is dense in*

$$\mathcal{A} := \{\phi \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3)\}$$

named by $\sqrt{\langle \phi, \phi \rangle_{\mathcal{D}}}$ and, therefore, $\mathcal{D} = \mathcal{A}$.

Now, by combining Lemma 3.4 in [13] with Proposition 2.1 in [27], the following lemma can be obtained.

Lemma 2.3. *For every $u \in H^1(\mathbb{R}^3)$ we have:*

(i) for every $y \in \mathbb{R}^3$, $\phi_{K,u(\cdot+y)} = \phi_{K,u}(\cdot + y)$;

(ii) $\phi_{K,u} \geq 0$ in $\mathbb{R}^3$;

(iii) $\phi_{K,u} \in \mathcal{D}$;

(iv) $\|\phi_{K,u}\|_6 \leq C\|u\|^2$;

(v) $\phi_{K,u}$ is the unique minimizer in $\mathcal{D}$ of the functional

$$E(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{a^2}{2}\|\Delta\phi\|_2^2 - \int_{\mathbb{R}^3} \phi u^2 \,\mathrm{d}x, \quad \phi \in \mathcal{D}.$$

Moreover,

(vi) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{K(x),u_n} \rightharpoonup \phi_{K(x),u}$ in $\mathcal{D}$,

$$\int_{\mathbb{R}^3} K(x)\phi_{K,u_n} u_n^2 dx \to \int_{\mathbb{R}^3} K(x)\phi_{K,u} u^2 dx,$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{K,u_n} u_n \zeta dx \to \int_{\mathbb{R}^3} K(x)\phi_{K,u} u\zeta dx, \qquad \forall \zeta \in H^1(\mathbb{R}^3).$$

Next, we define the Nehari manifold

$$\mathcal{M} := \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : \langle \mathcal{J}'(u), u \rangle = 0 \right\}.$$

Then, $u \in \mathcal{M}$ if and only if $\|u\|^2 + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u} u^2 dx - \int_{\mathbb{R}^3} b(x)|u|^p dx = 0$. It follows the Sobolev inequality that

$$\begin{aligned} \|u\|^2 &\leq \|u\|^2 + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u} u^2 dx \\ &= \int_{\mathbb{R}^3} b(x)|u|^p dx \\ &\leq S_p^{-p} b_{\max} \|u\|^p, \end{aligned}$$

for all $u \in \mathcal{M}$. Then we can get

$$\int_{\mathbb{R}^3} b(x)|u|^p dx \geq \|u\|^2 \geq \left(\frac{S_p^p}{b_{\max}}\right)^{\frac{2}{p-2}} \quad \text{for all } u \in \mathcal{M}. \tag{2.2}$$

The Nehari manifold $\mathcal{M}$ is closely linked to the behavior of the function of the form $h_u : t \to \mathcal{J}(tu)$ for $t > 0$. Such maps are known as fibering maps and were introduced by Drábek–Pohožaev [14], and were further discussed by Brown–Zhang [9] and Brown–Wu [7,8] etc. For $u \in H^1(\mathbb{R}^3)$, we find

$$\begin{aligned}
h_u(t) &= \frac{t^2}{2}\|u\|^2 + \frac{\lambda t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - \frac{t^p}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx,\\
h_u'(t) &= t\|u\|^2 + \lambda t^3 \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - t^{p-1}\int_{\mathbb{R}^3} b(x)|u|^p dx,\\
h_u''(t) &= \|u\|^2 + 3\lambda t^2 \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - (p-1)t^{p-2}\int_{\mathbb{R}^3} b(x)|u|^p dx.
\end{aligned}$$

As a direct consequence, we have

$$th_u'(t) = \|tu\|^2 + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,tu}(tu)^2 dx - \int_{\mathbb{R}^3} b(x)|tu|^p dx,$$

and so, for $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ and $t > 0, h_u'(t) = 0$ holds if and only if $tu \in \mathcal{M}$. In particular, $h_u'(1) = 0$ holds if and only if $u \in \mathcal{M}$. It is convenient to divide $\mathcal{M}$ into three parts, corresponding to local minima, local maxima, and inflection points. Following [26], we define

$$\begin{aligned}
\mathcal{M}^+ &= \left\{u \in \mathcal{M} : h_u''(1) > 0\right\},\\
\mathcal{M}^0 &= \left\{u \in \mathcal{M} : h_u''(1) = 0\right\},\\
\mathcal{M}^- &= \left\{u \in \mathcal{M} : h_u''(1) < 0\right\}.
\end{aligned}$$

Lemma 2.4. *Suppose that u_0 is a local minimizer for $\mathcal{J}$ on $\mathcal{M}$ and $u_0 \notin \mathcal{M}^0$. Then $\mathcal{J}'(u_0) = 0$ in $H^{-1}(\mathbb{R}^3)$.*

The proof of Lemma 2.4 is essentially the same as in Brown–Zhang [9], so we omitted it here.

For each $u \in \mathcal{M}$, we find that

$$\begin{aligned}
h_u''(1) &= \|u\|^2 + 3\lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - (p-1)\int_{\mathbb{R}^3} b(x)|u|^p dx\\
&= -(p-2)\|u\|^2 + (4-p)\lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx\\
&= -2\|u\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx.
\end{aligned} \tag{2.3}$$

For each $u \in \mathcal{M}^-$and $2 < p < 4$, using (2.2) and (2.3) gives

$$\begin{aligned}
\mathcal{J}(u) &= \frac{1}{4}\|u\|^2 - \frac{(4-p)}{4p}\int_{\mathbb{R}^3} b(x)|u|^p dx\\
&> \frac{p-2}{2p}\|u\|^2\\
&\geq \frac{p-2}{2p}\left(\frac{S_p^p}{b_{\max}}\right)^{2/(p-2)}.
\end{aligned}$$

Moreover, for each $u \in \mathcal{M}^-$ and $p = 4$, by virtue of (2.2) we have

$$\mathcal{J}(u) = \frac{1}{4}\|u\|^2 \geq \frac{1}{4}\left(\frac{S_p^p}{b_{\max}}\right)^{2/(p-2)}.$$

From this, the following lemma are obtained.

Lemma 2.5. *Suppose that $2 < p \leq 4$. Then the energy functional $\mathcal{J}(u)$ is coercive and bounded below on $\mathcal{M}^-$. Furthermore, for all $u \in \mathcal{M}^-$, when $2 < p < 4$, there holds*

$$\mathcal{J}(u) > \frac{p-2}{4p}\left(\frac{S_p^p}{b_{\max}}\right)^{\frac{2}{p-2}},$$

if $p = 4$, there holds

$$\mathcal{J}(u) \geq \frac{1}{4}\left(\frac{S_p^p}{b_{\max}}\right)^{\frac{2}{p-2}}.$$

From the Lemma 2.3 and [24], the following properties can be obtained

Lemma 2.6. *For each $u \in H^1(\mathbb{R})^3$, the following two inequalities are true.*

(i) $\phi_{K,u} \geq 0$;

(ii) $\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \leq \bar{S}^{-2}\hat{S}^{-4}K_{\max}^2\|u\|^4$.

Citing the lemma in [25], the same inequality can be obtained here, because

$$\phi_{K,u} = \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|}K(y)u^2(y)dy \leq \int_{\mathbb{R}^3} \frac{1}{|x-y|}K(y)u^2(y)dy.$$

For any $u \in \mathcal{M}$ and $2 < p < 4$ with $\mathcal{J}(u) < A(p)\frac{(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}$, we inference that

$$\begin{aligned}
A(p)&\frac{(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} \\
&> \mathcal{J}(u) \\
&= \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx - \frac{1}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx \\
&= \frac{p-2}{2p}\|u\|^2 - \frac{\lambda(4-p)}{4p}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \\
&\geq \frac{p-2}{2p}\|u\|^2 - \lambda\left(\frac{4-p}{4p}\right)\bar{S}^{-2}\hat{S}^{-4}K_{\max}^2\|u\|^4.
\end{aligned} \tag{2.4}$$

In addition, consider the quadratic equation as follows

$$\frac{1}{4}\left(1 - A(p)\left(\frac{b_{\max}}{b_\infty}\right)^{\frac{2}{p-2}}\right)\left(\frac{b_\infty(4-p)}{pS_p^p}\right)^{\frac{2}{p-2}} x^2 - x + A(p)\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} = 0.$$

It is easy to get one of the solutions expressed as

$$
\begin{aligned}
x_1 &= \frac{2\left(1+\sqrt{1-A(p)\Big(1-A(p)\big(\frac{b_{\max}}{b_\infty}\big)^{\frac{2}{p-2}}\Big)\big(\frac{2}{p}\big)^{\frac{2}{p-2}}}\right)}{\Big(1-A(p)\big(\frac{b_{\max}}{b_\infty}\big)\Big)\big(\frac{2}{p}\big)^{\frac{2}{p-2}}}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} \\
&> 2\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}},
\end{aligned}
\tag{2.5}
$$

then we have used condition (G_1), Remark 1.1 and the fact of $\big(\frac{2}{p}\big)^{\frac{2}{p-2}} < 1$ in the last inequality.

From (2.4) and (2.5), if $2 < p < 4$ and $0 < \lambda < \frac{p-2}{2(4-p)}\big(\frac{4-p}{p}\big)^{\frac{2}{p-2}}\Lambda_0$, then there exist two positive number $D^{(1)}$ and $D^{(2)}$ satisfying

$$
\sqrt{A(p)}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{1}{p-2}} < D^{(1)} < \left(\frac{2S_p^p}{b_{\max}(4-p)}\right)^{\frac{1}{p-2}} < \sqrt{2}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{1}{p-2}} < D^{(2)}
$$

such that

$$
\|u\| < D^{(1)} \quad \text{or} \quad \|u\| > D^{(2)}.
$$

Obviously, it can be seen that when $p \to 4^-$, then $D^{(1)} \to \infty$.

So, we have

$$
\begin{aligned}
&\mathcal{M}\left[\frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}\right] \\
&\quad = \left\{u \in \mathcal{M} : \mathcal{J}(u) < \frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}\right\} \\
&\quad = \mathcal{M}^{(1)} \bigcup \mathcal{M}^{(2)},
\end{aligned}
\tag{2.6}
$$

where

$$
\mathcal{M}^{(1)} := \left\{u \in \mathcal{M}\left[\frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}\right] : \|u\| < D^{(1)}\right\},
$$

and

$$
\mathcal{M}^{(2)} := \left\{u \in \mathcal{M}\left[\frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}\right] : \|u\| > D^{(2)}\right\}.
$$

Because of $2 < p < 4$ and $0 < \lambda < \frac{p-2}{2(4-p)}\big(\frac{4-p}{p}\big)^{\frac{2}{p-2}}\Lambda_0$, we have

$$
\|u\| < D^{(1)} < \left(\frac{2S_p^p}{b_{\max}(4-p)}\right)^{\frac{1}{p-2}} \quad \text{for all } u \in \mathcal{M}^{(1)}, \tag{2.7}
$$

and

$$
\|u\| > D^{(2)} > \sqrt{2}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{1}{p-2}} \quad \text{for all } u \in \mathcal{M}^{(2)}. \tag{2.8}
$$

From the Sobolev inequality, (2.3) and (2.7)

$$h_u''(1) \le -2\|u\|^2 + (4-p)S_p^{-p} b_{\max}\|u\|^p < 0 \quad \text{for all } u \in \mathcal{M}^{(1)}.$$

Using (2.8) we deduce that

$$\begin{aligned} &\frac{1}{4}\|u\|^2 - \frac{(4-p)}{4p}\int_{\mathbb{R}^3} b(x)|u|^p dx \\ &= \mathcal{J}(u) < \frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} \\ &< \frac{p-2}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} \\ &< \frac{p-2}{4p}\|u\|^2 \quad \text{for all } u \in \mathcal{M}^{(2)}, \end{aligned}$$

this implies

$$2\|u\|^2 < (4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx \quad \text{for all } u \in \mathcal{M}^{(2)}. \tag{2.9}$$

Combining (2.3) and (2.9) results in

$$h_u''(1) = -2\|u\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx > 0 \quad \text{for all } u \in \mathcal{M}^{(1)}.$$

Therefore, we get the following result.

Lemma 2.7.

(i) If $2 < p < 4$ and $0 < \lambda < \frac{p-2}{2(4-p)}\left(\frac{4-p}{p}\right)^{\frac{2}{p-2}}\Lambda_0$, then $\mathcal{M}^{(1)} \subset \mathcal{M}^-$ and $\mathcal{M}^{(2)} \subset \mathcal{M}^+$ are C^1 sub-manifolds. Furthermore, each local minimizer of the functional $\mathcal{J}$ in the sub-manifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ is a critical point of $\mathcal{J}$ in $H^1(\mathbb{R}^3)$.

(ii) If $p = 4$ and $\lambda > 0$, then $\mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$ is a C^1 manifold and so the Nehari manifold $\mathcal{M}^{(1)}$ is a natural constraint for the functional $\mathcal{J}$.

There we define

$$Q_b(u) = \left(\frac{\|u\|^2}{\int_{\mathbb{R}^3} b(x)|u|^p dx}\right)^{\frac{1}{p-2}} \quad \text{for } u \in H^1(\mathbb{R}^3)\backslash\{0\}.$$

Lemma 2.8. *Suppose that $2 < p < 4$. then for each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying*

$$\int_{\mathbb{R}^3} b(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\lambda(4-p)K_{\max}^2}{(p-2)\bar{S}^2\hat{S}^4}\right)^{\frac{p-2}{2}}\|u\|^p,$$

there exists a constant $\bar{q}^{(1)} > \left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)$ such that

$$\inf_{t\ge 0}\mathcal{J}(tu) = \inf_{\left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < t < \bar{q}^{(1)}} \mathcal{J}(tu) < 0. \tag{2.10}$$

Proof. For any $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ and $t > 0$, it has

$$\begin{aligned}\mathcal{J}(tu) &= \frac{t^2}{2}\|u\|^2 + \frac{\lambda t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx - \frac{t^p}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx \\ &= t^4\left[b(t) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx\right] \\ &= h_u(t),\end{aligned}$$

where $b(t) = \frac{t^{-2}}{2}\|u\|^2 - \frac{t^{p-4}}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx$.

Apparently, $\mathcal{J}(tu) = 0$ if and only if $b(t) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx = 0$. It is not difficult to observe that $b(\hat{t}) = 0, \lim_{t\to 0^+} b(t) = \infty$ and $\lim_{t\to\infty} b(t) = 0$, where $\hat{t} = (\frac{p}{2})^{\frac{1}{p-2}}Q_b(u)$. Considering the derivative of $b(t)$, we get

$$\begin{aligned}b'(t) &= -t^{-3}\|u\|^2 + \frac{(4-p)}{p}t^{p-5}\int_{\mathbb{R}^3} b(x)|u|^p dx \\ &= t^{-3}\left[\frac{(4-p)t^{p-2}}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx - \|u\|^2\right],\end{aligned}$$

it means that $b(t)$ is decreasing when $0 < t < (\frac{p}{4-p})^{\frac{1}{p-2}}Q_b(u)$ and is increasing when $t > (\frac{p}{4-p})^{\frac{1}{p-2}}Q_b(u)$, and so

$$\begin{aligned}\inf_{t>0} b(t) &= b\left[\left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right] \\ &= -\frac{p-2}{2(4-p)}\left(\frac{p\|u\|^2}{(4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx}\right)^{\frac{-2}{p-2}}\|u\|^2.\end{aligned}$$

From Lemma 2.6 (ii) and the Sobolev inequality that for each $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}^3} b(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\lambda(4-p)K_{\max}^2}{(p-2)\bar{S}^2\hat{S}^4}\right)^{\frac{p-2}{2}}\|u\|^p,$$

we have

$$\begin{aligned}\inf_{t>0} b(t) &= -\frac{p-2}{2(4-p)}\left(\frac{p\|u\|^2}{(4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx}\right)^{\frac{-2}{p-2}}\|u\|^2 \\ &< -\lambda K_{\max}^2\bar{S}^{-2}\hat{S}^{-4}\|u\|^2 \\ &< -\frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx.\end{aligned}$$

Then, there exist $\bar{q}^{(1)}$ and $\bar{q}^{(2)}$ satisfying

$$0 < \bar{q}^{(2)} < \left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < \bar{q}^{(1)} \tag{2.11}$$

such that

$$b(\bar{q}^{(j)}) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx = 0 \quad \text{for } j = 1, 2.$$

That is $\mathcal{J}(\bar{q}^{(j)}u) = 0$ for $j = 1, 2$. □

So, for each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}^3} b(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\lambda(4-p)K_{\max}^2}{(p-2)\bar{S}^2\hat{S}^4}\right)^{\frac{p-2}{2}} \|u\|^p,$$

we have

$$\begin{aligned}
&\mathcal{J}\left[\left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right] \\
&\quad = \left[\left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right]\left[b\left(\left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx\right] \\
&\quad < 0,
\end{aligned}$$

and so $\inf_{t\geq 0} \mathcal{J}(tu) < 0$.

Then, we know that $h_u'(t) = 4t^3[b(t) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx] + t^4 b'(t)$, which leads to $h_u'(t) < 0$ for all $t \in (\bar{q}^{(2)}, (\frac{p}{4-p})^{\frac{1}{p-2}} Q_b(u)]$ and $h_u'(\bar{q}^{(1)}) > 0$. Finally, we get the inequality (2.10).

Lemma 2.9. *For each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying*

$$\int_{\mathbb{R}^3} b(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\lambda(4-p)K_{\max}^2}{(p-2)\bar{S}^2\hat{S}^4}\right)^{\frac{p-2}{2}} \|u\|^p \quad \text{if } 2 < p < 4,$$

or

$$\int_{\mathbb{R}^3} b(x)|u|^4 dx > \lambda K_{\max}^2 \bar{S}^{-2}\hat{S}^{-4}\|u\|^4 \quad \text{if } p = 4,$$

the following two statements are true.

(i) if $2 < p < 4$, then there exist two constants t^+ and t^- which satisfy

$$Q_b(u) < t^- < \sqrt{A(p)}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < t^+ \tag{2.12}$$

such that

$$t^{\pm}u \in \mathcal{M}^{\pm}, \qquad \mathcal{J}(t^- u) = \sup_{0\leq t\leq t^+} \mathcal{J}(tu),$$

and

$$\mathcal{J}(t^+ u) = \inf_{t\geq t^-} \mathcal{J}(tu) = \inf_{t\geq 0} \mathcal{J}(tu) < 0.$$

(ii) if $p = 4$, then there is a unique constant

$$\bar{t} = \left(\frac{\|u\|^2}{\int_{\mathbb{R}^3} b(x)u^4 dx - \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx}\right)^{\frac{1}{2}} > Q_b(u)$$

such that

$$\bar{t}u \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M},$$

and

$$\mathcal{J}(\bar{t}u) = \sup_{t\geq 0} \mathcal{J}(tu) = \sup_{t\geq Q_b(u)} \mathcal{J}(tu).$$

Proof. (i) Define $f(t) = t^{-2}\|u\|^2 - t^{p-4}\int_{\mathbb{R}^3} b(x)|u|^p dx$ for $t > 0$. Obviously, $tu \in \mathcal{M}$ if and only if $f(t) + \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx = 0$. A straightforward evaluation gives $f(Q_b(u)) = 0, \lim_{t\to 0^+} f(t) = \infty$ and $\lim_{t\to\infty} f(t) = 0$.

Since $2 < p < 4$ and $f'(t) = t^{-3}(-2\|u\|^2 + (4-p)t^{p-2}\int_{\mathbb{R}^3} b(x)|u|^p dx)$, we know that $f(t)$ is decreasing when $0 < t < (\frac{2}{4-p})^{\frac{1}{p-2}}Q_b(u)$ and is increasing when $t > (\frac{2}{4-p})^{\frac{1}{p-2}}Q_b(u)$. This gives

$$\inf_{t>0} f(t) = f\left[\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right]. \tag{2.13}$$

For each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}^3} b(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\lambda(4-p)K_{\max}^2}{(p-2)\bar{S}^2\hat{S}^4}\right)^{\frac{p-2}{2}} \|u\|^p \quad \text{if } 2 < p < 4,$$

from Lemma 2.6 (ii) and Sobolev's inequality we get

$$\begin{aligned} f\left(\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right) &= -\left(\frac{p-2}{4-p}\right)\left(\frac{2\|u\|^2}{(4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx}\right)^{\frac{-2}{p-2}} \|u\|^2 \\ &< -2\left(\frac{p}{2}\right)^{\frac{2}{p-2}} \lambda K_{\max}^2 \bar{S}^{-2}\hat{S}^{-4}\|u\|^4 \\ &< -\lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx, \end{aligned}$$

where we also used the fact that $(\frac{2}{p})^{\frac{2}{p-2}} > 1$. However, for each $2 < p < 4$, by Remark 1.1 we have

$$Q_b(u) < \sqrt{A(p)}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u), \tag{2.14}$$

and directly calculated

$$\frac{\left(\frac{2}{4-p}\right) A(p)^{\frac{p-2}{2}} - 1}{A(p)\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}} > \frac{p-2}{2(4-p)}\left(\frac{4-p}{p}\right)^{\frac{2}{p-2}}. \tag{2.15}$$

Then, from (2.13)–(2.15) that

$$\begin{aligned} f\left(\sqrt{A(p)}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right) &= -\frac{\left(\frac{2}{4-p}\right) A(p)^{\frac{p-2}{2}} - 1}{A(p)\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}}\left(\frac{\int_{\mathbb{R}^3} b(x)|u|^p dx}{\|u\|^2}\right)^{\frac{2}{p-2}} \|u\|^2 \\ &< -\lambda K_{\max}^2 \bar{S}^{-2}\hat{S}^{-4}\|u\|^4 \\ &\le -\lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx. \end{aligned}$$

Therefore, there exist two constants t^+ and $t^- > 0$ which satisfy

$$Q_b(u) < t^- < \sqrt{A(p)}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < t^+ \tag{2.16}$$

such that $f(t^\pm) + \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx = 0$. That is $t^\pm u \in \mathcal{M}$.

By calculating the second derivative, we find that

$$\begin{aligned} h''_{t^-u}(1) &= -2\|t^-u\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|t^-u|^p dx \\ &= (t^-)^5 f'(t^-) < 0, \end{aligned}$$

and

$$\begin{aligned} h''_{t^+u}(1) &= -2\|t^+u\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|t^+u|^p dx \\ &= (t^+)^5 f'(t^+) > 0. \end{aligned}$$

This means that $t^{\pm}u \in \mathcal{M}^{\pm}$ and $h'_u(t) = t^3(f(t) + \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx)$. It is known that $h'_u(t) > 0$ holds for all $t \in (0, t^-) \cup (t^+, \infty)$ and $h'_u(t) < 0$ holds for all $t \in (t^-, t^+)$. It leads to $\mathcal{J}(t^-u) = \sup_{0\leq t\leq t^+} \mathcal{J}(tu)$ and $\mathcal{J}(t^+u) = \inf_{t\geq t^-} \mathcal{J}(tu)$, and so $\mathcal{J}(t^+u) < \mathcal{J}(t^-u)$. From Lemma 2.8 that $\mathcal{J}(t^+u) = \inf_{t\geq 0}\mathcal{J}(tu) < 0$.
(ii) Let

$$\bar{f}(t) = t^{-2}\|u\|^2 \quad \text{for} \quad t > 0. \tag{2.17}$$

Apparently, $tu \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$ if and only if $\bar{f}(t) - \int_{\mathbb{R}^3} b(x)u^4dx + \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx = 0$. By (2.17), we know that $\bar{f} > 0 (t > 0)$ is decreasing, and $\lim_{t\to 0^+} \bar{f}(t) = \infty$ and $\lim_{t\to\infty} \bar{f}(t) = 0$.

For each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying $\int_{\mathbb{R}^3} b(x)|u|^4dx > \lambda K_{\max}^2 \bar{S}^{-2}\hat{S}^{-4}\|u\|^4$, by using Lemma 2.6 (ii) and (2.15), we obtain $\int_{\mathbb{R}^3} b(x)|u|^4dx > \lambda K_{\max}^2 \bar{S}^{-2}\hat{S}^{-4}\|u\|^4 \geq \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx$. Then we can get that equation $\bar{f}(t) - \int_{\mathbb{R}^3} b(x)u^4dx + \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx = 0$ has a unique positive solution $\bar{t} = \left(\frac{\|u\|^2}{\int_{\mathbb{R}^3} b(x)u^4dx - \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx}\right)^{\frac{1}{2}} > Q_b(u)$. This means that $\bar{t}u \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$. Similar to the discussion of Case (i), we get that $\mathcal{J}(\bar{t}u) = \sup_{t\geq 0}\mathcal{J}(tu) = \sup_{t\geq Q_b(u)} \mathcal{J}(tu)$. This completes the proof. □

3 Proofs of main results

3.1 Proof of Theorem 1.3

In this section, we first consider that $K(x) \equiv K_\infty > 0$ and $b(x) \equiv b_\infty > 0$. The existence of the positive ground state solutions of system (1.3) at infinity, namely,

$$\begin{cases} -\Delta u + u + \lambda K_\infty \phi u = b_\infty |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi K_\infty u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{3.1}$$

Then we consider the following equation at infinity

$$-\Delta u + u + \lambda K_\infty \phi_{K_\infty,u} u = b_\infty |u|^{p-2}u. \tag{3.2}$$

We define the associated energy functional in $H^1(\mathbb{R}^3)$ by

$$\mathcal{J}^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,u} u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} b_\infty |u|^p dx,$$

we know that solutions of equation (3.2) are critical points of the functional $\mathcal{J}^\infty(u)$.

Define

$$\mathcal{M}_\infty := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle(\mathcal{J}^\infty)'(u), u\rangle = 0\},$$

where $(\mathcal{J}^\infty)'$ denotes the Fréchet derivative of $\mathcal{J}^\infty$. Then, $u \in \mathcal{M}_\infty$ if and only if

$$\|u\|^2 + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,u} u^2 dx - \int_{\mathbb{R}^3} b_\infty |u|^p dx = 0.$$

Notice that $\mathcal{M}_\infty = \mathcal{M}$ with $K(x) \equiv K_\infty$ and $b(x) \equiv b_\infty$. We denote by $\mathcal{M}_\infty^{(j)} = \mathcal{M}^{(j)}$ with $K(x) \equiv K_\infty$ and $b(x) = b_\infty$ for $j = 1, 2$.

Since w_0 is the unique positive solution of equation (1.6), for $2 < p < 4$ and $0 < \lambda < \frac{p-2}{2(4-p)}\left(\frac{4-p}{p}\right)^{\frac{2}{p-2}} \Lambda_0$, from (1.7), we get

$$\begin{aligned}\int_{\mathbb{R}^3} b_\infty |w_0|^p dx &= b_\infty S_p^{-p} \|w_0\|^p \\ &> \frac{p}{4-p} \left(\frac{2\lambda(4-p)K_\infty^2}{\bar{S}^2 \hat{S}^4 (p-2)} \right)^{\frac{p-2}{2}} \|w_0\|^p.\end{aligned}$$

From Lemma 2.9 (i) there exist two constants t_∞^- and t_∞^+ satisfy

$$1 < t_\infty^- < \sqrt{A(p)} \left(\frac{2}{4-p} \right)^{\frac{1}{p-2}} < t_\infty^+,$$

such that $t_\infty^\pm w_0 \in \mathcal{M}_\infty^\pm$, where $\mathcal{M}_\infty^\pm = \mathcal{M}^\pm$ with $K(x) \equiv K_\infty$ and $b(x) \equiv b_\infty$. However, we obtain $\mathcal{J}^\infty(t_\infty^- w_0) = \sup_{0 \le t \le t_\infty^+} \mathcal{J}^\infty(tw_0)$ and $\mathcal{J}^\infty(t_\infty^+ w_0) = \inf_{t \ge t_\infty^-} \mathcal{J}^\infty(tw_0) = \inf_{t \ge 0} \mathcal{J}^\infty(tw_0) < 0$. Then we can get

$$\begin{aligned}\mathcal{J}^\infty(t_\infty^- w_0) &= \frac{1}{2}\|t_\infty^- w_0\|^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty, t_\infty^- w_0} (t_\infty^- w_0)^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} b_\infty |t_\infty^- w_0|^p dx \\ &= \frac{(t_\infty^-)^2}{4} \left[1 - \frac{4-p}{p} (t_\infty^-)^{p-2} \right] \|w_0\|^2 \\ &< A(p) \left(\frac{2}{4-p} \right)^{\frac{2}{p-2}} \frac{p-2}{2p} \|w_0\|^2 \\ &= A(p) \frac{p-2}{2p} \left(\frac{2S_p^p}{b_\infty(4-p)} \right)^{\frac{2}{p-2}}.\end{aligned} \tag{3.3}$$

This indicates that $t_\infty^- w_0 \in \mathcal{M}_\infty^{(1)}$. Namely, $\mathcal{M}_\infty^{(1)}$ is nonempty.

For $p = 4$ and $0 < \lambda < b_\infty K_\infty^{-2} \bar{S}^2 \hat{S}^4 S_4^{-4}$, there holds

$$\int_{\mathbb{R}^3} b_\infty |w_0|^4 dx = b_\infty S_4^{-4} \|w_0\|^4 > \lambda K_\infty^2 \bar{S}^{-2} \hat{S}^{-4} \|w_0\|^4.$$

Then, from Lemma 2.9 (ii), there exists a unique constant

$$\bar{t}^\infty = \frac{\|w_0\|^2}{\int_{\mathbb{R}^3} b_\infty |w_0|^4 dx - \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty, w_0} w_0^2 dx} > 1$$

such that $\bar{t}^\infty w_0 \in \mathcal{M}_\infty^{(1)} = \mathcal{M}_\infty^- = \mathcal{M}_\infty$ and $\mathcal{J}^\infty(\bar{t}^\infty w_0) = \sup_{t \ge 0} \mathcal{J}^\infty(tw_0) = \sup_{t > 1} \mathcal{J}^\infty(tw_0)$.

Then we define

$$\alpha_\infty^- = \inf_{u \in \mathcal{M}_\infty^{(1)}} \mathcal{J}^\infty(u) = \inf_{u \in \mathcal{M}_\infty^-} \mathcal{J}^\infty(u) \quad \text{for } 2 < p < 4,$$

$$\alpha_\infty^+ = \inf_{u\in\mathcal{M}_\infty^{(2)}} \mathcal{J}^\infty(u) = \inf_{u\in\mathcal{M}_\infty^+} \mathcal{J}^\infty(u) \quad \text{for } 2<p<4,$$

and

$$\alpha_\infty^- = \inf_{u\in\mathcal{M}_\infty^{(1)}} \mathcal{J}^\infty(u) = \inf_{u\in\mathcal{M}_\infty} \mathcal{J}^\infty(u) \quad \text{for } p=4.$$

It follows from Lemma 2.5 and (3.3), we have

$$\frac{p-2}{4p}\left(\frac{S_p^p}{b_\infty}\right)^{\frac{2}{p-2}} \le \alpha_\infty^- < \frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} \quad \text{for } 2<p<4, \tag{3.4}$$

and $\alpha_\infty^+ = -\infty$. For $p=4$, it follows from Lemma 2.5 that $\alpha_\infty^- \ge \frac{1}{4}(\frac{S_p^p}{b_\infty})^{\frac{2}{p-2}}$.

Then we are ready to prove Theorem 1.3.

Let $u_n \in \mathcal{M}_\infty^{(1)}$ be a sequence, for $2<p<4$, we have

$$\mathcal{J}^\infty(u_n) = \alpha_\infty^- + o(1) \quad \text{and} \quad (\mathcal{J}^\infty)'(u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^3). \tag{3.5}$$

According to Theorem 7.2 in [25], we obtain that for

$$0<\lambda<\frac{(p-2)\bar{S}^2\hat{S}^4}{2(4-p)K_\infty^2}\left(\frac{b_\infty(4-p)^2}{2pS_p^p}\right)^{\frac{2}{p-2}} \quad \text{if } 2<p<4,$$

or $\lambda>0$ if $p=4$, the compactness of the sequence $\{u_n\}$ holds. Then there exist a positive constant $\xi=\xi(\theta)(\theta>0)$ and a sequence $\{y_n\}\subset\mathbb{R}^3$ such that

$$\int_{[B(y_n;\xi)]^c} (|\nabla u_n(x)|^2 + u_n^2(x))dx < \theta \quad \text{uniformly for } n\ge 1. \tag{3.6}$$

Now, we define a new sequence of functions

$$v_n := u_n(\cdot + y_n) \in H^1(\mathbb{R}^3).$$

We find that $\{v_n\}\subset\mathcal{M}_\infty^{(1)}$, and

$$\phi_{K_\infty,v_n} = \phi_{K\infty,u_n}(\cdot+y_n) \quad \text{and} \quad \mathcal{J}^\infty(v_n) = \alpha_\infty^- + o(1).$$

By inequality (3.6), there exists a positive constant $\xi=\xi(\theta)(\theta>0)$ such that

$$\int_{[B(0;\xi)]^c} (|\nabla v_n(x)|^2 + v_n^2(x))dx < \theta \quad \text{uniformly for } n\ge 1. \tag{3.7}$$

For $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$, we can assume that there exist a subsequence $\{v_n\}$ and $w\in H^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup w \quad \text{in } H^1(\mathbb{R}^3), \tag{3.8}$$

$$v_n \to w \quad \text{in } L^r_{\text{loc}},\ \forall\, 2\le r<6, \tag{3.9}$$

$$v_n \to w \quad \text{a.e. in } \mathbb{R}^3.$$

For any $\theta > 0$ and sufficiently large $n(\geq 1)$, by Fatou's Lemma and (3.7)–(3.9), there exists a constant $\xi > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3} |v_n - w|^p dx &\leq \int_{B(0;\xi)} |v_n - w|^p dx + \int_{[B(0;\xi)]^c} |v_n - w|^p dx \\ &\leq \theta + S_p^{-p} \left[\int_{[B(0;\xi)]^c} (|\nabla v_n|^2 + v_n^2) dx + \int_{[B(0;\xi)]^c} (|\nabla w|^2 + w^2) dx \right]^{\frac{p}{2}} \\ &\leq \theta + S_p^{-p} (2\theta)^{\frac{p}{2}},\end{aligned}$$

then we obtain

$$v_n \to w \quad \text{in } L^r(\mathbb{R}^3),\ \forall r \in (2,6). \tag{3.10}$$

We know that $\phi : L^{\frac{12}{5}}(\mathbb{R}^3) \to \mathcal{D}$ is a continuous function. It follows from (3.10) that

$$\phi_{K_\infty, v_n} \to \phi_{K_\infty, w} \quad \text{in } \mathcal{D},$$

and

$$\int_{\mathbb{R}^3} \phi_{K_\infty, v_n} v_n^2 dx \to \int_{\mathbb{R}^3} \phi_{K_\infty, w} w^2 dx. \tag{3.11}$$

Since $\{v_n\} \subset \mathcal{M}_\infty^{(1)}$, using (2.2) and (3.10) gives

$$\int_{\mathbb{R}^3} b_\infty |w|^p dx \geq \left(\frac{S_p^p}{b_\infty} \right)^{\frac{2}{p-2}} > 0.$$

This implies that $w \neq 0$ and

$$\int_{\mathbb{R}^3} b_\infty |w|^p dx - \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty, w} w^2 dx \geq \|w\|^2 > 0.$$

Next, we proof that

$$v_n \to w \quad \text{in } H^1(\mathbb{R}^3).$$

For this, we assume the opposite. Then we have

$$\|w\| < \liminf_{n\to\infty} \|v_n\|. \tag{3.12}$$

An argument similar to Lemma 2.9, there exists a unique $t^- > 0$ such that

$$t^- w \in \mathcal{M}_\infty^- \quad \text{and} \quad (h_w^\infty)'(t^-) = 0. \tag{3.13}$$

For $v_n \in \mathcal{M}_\infty^{(1)}$, from (3.12) we get

$$(h_w^\infty)'(1) < 0. \tag{3.14}$$

Using (3.13), (3.14) and the contour of $h_w^\infty(t)$ results in $t^- < 1$. By (3.10)–(3.12), we know $(h_{v_n}^\infty)'(t^-) > 0$ for sufficiently large n. Obviously, there holds

$$(h_{v_n}^\infty)'(1) = 0 \tag{3.15}$$

due to $v_n \in \mathcal{M}_\infty^{(1)}$. Similar to the proof of Lemma 2.9, for $2 < p < 4$, we have

$$(h_{v_n}^\infty)'(t) = t^3 (f^\infty(t) + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty, v_n} v_n^2 dx,$$

where $f^\infty(t) := t^{-2}\|v_n\|^2 - t^{p-4}\int_{\mathbb{R}^3} b_\infty |v_n|^p dx$ is decreasing for $0 < t < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}}\left(\frac{\|v_n\|^2}{\int_{\mathbb{R}^3} b_\infty |v_n|^p dx}\right)^{\frac{1}{p-2}}$, and $\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}}\left(\frac{\|v_n\|^2}{\int_{\mathbb{R}^3} b_\infty |v_n|^p dx}\right)^{\frac{1}{p-2}} > 1$ by using (2.12) and (3.15). This implies that $(h^\infty_{v_n})'(t) > 0$ $(0 < t < 1)$, which indicates that $h^\infty_{v_n}$ is increasing on $(t^-, 1)$ for sufficiently large n. When $p = 4$, we have

$$(h^\infty_{v_n})'(t) = t^3\left(\bar{f}^\infty(t) - \int_{\mathbb{R}^3} b_\infty |v_n|^4 dx + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty, v_n} v_n^2 dx\right) \quad \text{for } t > 0,$$

where $\bar{f}^\infty(t) := t^{-2}\|v_n\|^2$ is decreasing for $t > 0$. This means that $(h^\infty_{v_n})'(t) > 0$ $(0 < t < 1)$, and $h^\infty_{v_n}$ is increasing on $(t^-, 1)$ for sufficiently large n. So, for,$2 < p \leq 4$, $h^\infty_{v_n}(t^-) < h^\infty_{v_n}(1)$ holds for sufficiently large n. This means that $\mathcal{J}^\infty(t^- v_n) < \mathcal{J}^\infty(v_n)$ for sufficiently large n.

Using (3.10)–(3.12) we again obtain

$$\mathcal{J}^\infty(t^- w) < \liminf_{n\to\infty} \mathcal{J}^\infty(t^- v_n) \leq \liminf_{n\to\infty} \mathcal{J}^\infty(v_n) = \alpha^-_\infty,$$

which is a contradiction. However, we get that $v_n \to w$ in $H^1(\mathbb{R}^3)$ and $\mathcal{J}^\infty(v_n) \to \mathcal{J}^\infty(w) = \alpha^-_\infty$ as $n \to \infty$.

In addition, we find that for $2 < p < 4$,

$$\frac{(p-2)\bar{S}^2\hat{S}^4}{2(4-p)K^2_\infty}\left(\frac{b_\infty(4-p)^2}{2pS^p_p}\right)^{\frac{2}{p-2}} < \frac{p-2}{2(4-p)}\left(\frac{4-p}{p}\right)^{\frac{2}{p-2}}\Lambda_0.$$

So, w is a minimizer for $\mathcal{J}^\infty$ on $\mathcal{M}^-_\infty$ for each $0 < \lambda < \Lambda$. For $2 < p < 4$, it follows from (3.2) that

$$\mathcal{J}^\infty(w) = \alpha^-_\infty \leq \mathcal{J}^\infty(t^-_\infty w_0) < \frac{A(p)(p-2)}{2p}\left(\frac{2S^p_p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}},$$

which indicates that $w \in \mathcal{M}^{(1)}_\infty$. Since $|w| \in \mathcal{M}^-_\infty$ and $\mathcal{J}^\infty(|w|) = \mathcal{J}^\infty(w) = \alpha^-_\infty$, we can see that w is a positive solution of equation (3.2) according to Lemma 2.4. It also implies that $(w, \phi_{K_\infty, w})$ is a positive solution of system (3.1).

Note that for $2 < p < 4$, there holds

$$(4-p)\int_{\mathbb{R}^3} b_\infty |w|^p dx < 2\|w\|^2 \quad \text{and} \quad t_{b_\infty}(w)w \in \mathcal{M}^0_\infty,$$

where

$$\left(\frac{4-p}{2}\right)^{\frac{1}{p-2}} < t_{b_\infty}(w) := \left(\frac{|w|^2}{\int_{\mathbb{R}^3} b_\infty |w|^p dx}\right)^{\frac{1}{p-2}} < 1. \tag{3.16}$$

According to Lemma 2.9, for $2 < p < 4$, we have $\mathcal{J}^\infty(w) = \sup_{0\leq t\leq t^+} \mathcal{J}^\infty(tw)$, where $t^+ > \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} t_{b_\infty}(w) > 1$ by (3.16). Using this, together with (3.37), we get $\mathcal{J}^\infty(w) > \mathcal{J}^\infty(t_{b_\infty}(w)w)$. Similarly, for $p = 4$, we can also get the above inequality. So, we have

$$\begin{aligned} \alpha^-_\infty &= \mathcal{J}^\infty(w) > \mathcal{J}^\infty(t_{b_\infty}(w)w) \\ &\geq \alpha^\infty_0 + \frac{\lambda[t_{b_\infty}(w)]^4}{4}\int_{\mathbb{R}^3} K_\infty \phi_{K_\infty, w} w^2 dx \\ &> \alpha^0_\infty. \end{aligned}$$

Consequently, we complete the proof.

3.2 Proof of Theorem 1.4

Definition 3.1.

(1) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $\mathcal{J}$ if $\mathcal{J}(u_n) = \beta + o(1)$ and $\mathcal{J}(u_n)'(u_n) = o(1)$ strongly in $H^{-1}(\mathbb{R}^3)$ as $n \to \infty$.

(2) If every $(PS)_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $\mathcal{J}$ contains a convergent subsequence, we can say that $\mathcal{J}$ satisfies the $(PS)_\beta$-sequence in $H^1(\mathbb{R}^3)$.

Lemma 3.2. *Let $\{u_n\}$ be a bounded $(PS)_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $\mathcal{J}$. There exists a subsequence $\{u_n\}$, a number $l \in \mathbb{N}$, a sequence $\{x_n^{(k)}\} \subset \mathbb{R}^3$ for $1 \le k \le l$, a function $v_0 \in H^1(\mathbb{R}^3)$, and $0 \neq w^i \in H^1(\mathbb{R}^3)$ when $1 \le i \le l$ such that*

(i) $|x_n^k| \to \infty$ *and* $|x_n^k - x_n^h| \to \infty$, *as* $n \to \infty, 1 \le k \neq h \le l$;

(ii) $-\Delta v_0 + v_0 + \lambda K(x)\phi_{K,v_0}v_0 = b(x)|v_0|^{p-2}v_0$ *in* $\mathbb{R}^3$;

(iii) $-\Delta w^i + w^i + \lambda K_\infty \phi_{K_\infty,w^i} w^i = b(x)|w^i|^{p-2}w^i$ *in* $\mathbb{R}^3$;

(iv) $u_n = v_0 + \sum_{i=1}^{l}(\cdot - x_n^i) + o(1)$ *strongly in* $H^1(\mathbb{R}^3)$;

(v) $\mathcal{J}(u_n) = \mathcal{J}(v_0) + \sum_{i=1}^{l} \mathcal{J}^\infty(w^i) + o(1)$.

The proof is similar to the argument of [13] Lemma 4.5, so we omit it here.

Corollary 3.3. *Suppose that $\{u_n\} \subset \mathcal{M}^-$ is a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $\mathcal{J}$ with $0 < \beta < \alpha_\infty^-$. Then there exist a subsequence $\{u_n\}$ and a nonzero u_0 in $H^1(\mathbb{R}^3)$ such that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$ and $\mathcal{J}(u_0) = \beta$. However, (u_0, ϕ_{u_0}) is a nonzero solution of equation (1.4).*

By Theorem 1.3, we know that equation (3.2) have a positive solution $w(x) \in \mathcal{M}_\infty^-$ (up to translation) such that for $2 < p \le 4$, there holds

$$\mathcal{J}^\infty(w) = \alpha_\infty^- \quad \text{and} \quad \frac{4-p}{2}\int_{\mathbb{R}^3} b_\infty |w|^p dx < \|w\|^2.$$

Define $Q_b(w)$ as

$$\left(\frac{(4-p)b_\infty}{2b_{\max}}\right)^{\frac{1}{p-2}} < Q_b(w) := \left(\frac{\|w\|^2}{\int_{\mathbb{R}^3} b(x)|w|^p dx}\right)^{\frac{1}{p-2}}.$$

Lemma 3.4. *Suppose that $0 < \lambda < \Lambda$. Then the following two statements are true.*

(i) If $2 < p < 4$, then there exists $t^\infty > \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} t_{b_\infty}(w) > 1$ such that

$$\mathcal{J}^\infty(w) = \sup_{0 \le t \le t^\infty} \mathcal{J}^\infty(tw) = \alpha_\infty^-, \tag{3.17}$$

where $t_{b_\infty}(w)$ is defined as (3.16).

(ii) If $p = 4$, then it has

$$\mathcal{J}^\infty(w) = \sup_{t \ge 0} \mathcal{J}^\infty(tw) = \sup_{t \ge 1} \mathcal{J}^\infty(tw) = \alpha_\infty^-. \tag{3.18}$$

Proof. (i) Let

$$g^{\infty}(t) = t^{-2}\|w\|^2 - t^{p-4}\int_{\mathbb{R}^3} b_\infty |w|^p dx \quad \text{for } t > 0. \tag{3.19}$$

Obviously, it satisfies

$$g^{\infty}(1) + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx = 0 \quad \text{for all } 0 < \lambda < \Lambda. \tag{3.20}$$

Then we get $g^{\infty}(t_{b_\infty}(w)) = 0$, $\lim_{t\to 0^+} g^{\infty}(t) = 0$ and $\lim_{t\to\infty} g^{\infty}(t) = 0$.

For $2 < p < 4$ and the equality $(g^{\infty})'(t) = t^{-3}(-2\|w\|^2 + (4-p)t^{p-2}\int_{\mathbb{R}^3} b_\infty |w|^p dx)$, we find that g^{∞} is decreasing when $0 < t < (\frac{2}{4-p})^{\frac{1}{p-2}} t_{b_\infty}(w)$ and is increasing when $t > (\frac{2}{4-p})^{\frac{1}{p-2}} t_{b_\infty}(w)$. This means that

$$\inf_{t>0} g^{\infty}(t) = g^{\infty}\left(\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} t_{b_\infty}(w)\right). \tag{3.21}$$

Moreover, from (3.16) we know that

$$\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} t_{b_\infty}(w) > 1. \tag{3.22}$$

So from (3.20)–(3.22) that

$$\inf_{t>0} g^{\infty}(t) < g^{\infty}(1) = -\lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx. \tag{3.23}$$

This means that there exists $t^{\infty} > (\frac{2}{4-p})^{\frac{1}{p-2}} t_{b_\infty}(w) > 1$ such that $g^{\infty}(t^{\infty}) + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx = 0$. Using a similar argument as the proof of Lemma (2.9) (i), we get (3.17).

(ii) Let $\bar{g}^{\infty}(t) = t^{-2}\|w\|^2$ for $t > 0$. Then we get $\bar{g}^{\infty}(1) - \int_{\mathbb{R}^3} b_\infty |w|^4 dx + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx = 0$ for all $0 < \lambda < \Lambda$. we can observe that $\bar{g}^{\infty}(t)$ is decreasing when $t > 0$ and $\lim_{t\to 0^+} \bar{g}^{\infty}(t) = \infty$ and $\lim_{t\to\infty} \bar{g}^{\infty}(t) = 0$. Since w is the positive solution of equation (3.2), we have $\int_{\mathbb{R}^3} b_\infty |w|^4 dx - \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx = \|w\|^2 > 0$, which shows that $t = 1$ is a unique positive solution of the equation $\bar{g}^{\infty}(t) - \int_{\mathbb{R}^3} b_\infty |w|^4 dx + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx = 0$. By the proof of Lemma 2.9 (ii) , we get (3.18). □

Lemma 3.5. *Suppose that $0 < \lambda < \Lambda$ and conditions (G_1)–(G_3) hold. Then the following two statements are true.*

(i) If $2 < p < 4$, then there exist two constants $t^{(1)}$ and $t^{(2)}$ satisfying

$$Q_b(w) < t^{(1)} < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w) < t^{(2)},$$

such that $t^{(i)}w \in \mathcal{M}^{(i)} (i = 1, 2)$, $\mathcal{J}(t^{(1)}w) = \sup_{0\le t\le t^{(2)}} \mathcal{J}(tw) < \alpha_\infty^-$, and $\mathcal{J}(t^{(2)}w) = \inf_{t\ge t^{(1)}} \mathcal{J}(tw)$.

(ii) If $p = 4$, then there exists a unique constant

$$\tilde{t} = \left(\frac{\|w\|^2}{\int_{\mathbb{R}^3} b(x)|w|^4 dx - \lambda \int_{\mathbb{R}^3} K(x)\phi_{K(x),w} w^2 dx}\right)^{\frac{1}{2}} > Q_b(w)$$

such that $\tilde{t}w \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$ and $\mathcal{J}(\tilde{t}w) = \sup_{t\ge 0} \mathcal{J}(tw) = \sup_{t\ge Q_b(w)} \mathcal{J}(tw) < \alpha_\infty^-$.

Proof. (i) Let $g(t) = t^{-2}\|w\|^2 - t^{p-4}\int_{\mathbb{R}^3} b(x)|w|^p dx$ for $t > 0$. Clearly, $tw \in \mathcal{M}$ if and only if

$$g(t) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K(x),w} w^2 dx = 0. \tag{3.24}$$

From (3.24) gives $g(Q_b(w)) = 0$, $\lim_{t\to 0^+} g(t) = \infty$ and $\lim_{t\to\infty} g(t) = 0$.

In view of $2 < p < 4$ and $g'(t) = t^{-3}(-2\|w\|^2 + (4-p)t^{p-2}\int_{\mathbb{R}^3} b(x)|w|^p dx)$, we see that $g(t)$ is decreasing on $0 < t < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w)$ and is increasing on $t > \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w)$. Then from condition (G_3) that $Q_b(w) \le Q_{b_\infty}(w) < 1$ and $g(t) \le g^\infty(t)$, where $g^\infty(t)$ is given in (3.19). Using condition (G_3) and (3.23) again, we deduce that

$$\begin{aligned}
\inf_{t>0} g(t) &= g\left(\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w)\right) \\
&\le -\frac{p-2}{4-p}\left(\frac{4-p}{2}\right)^{\frac{2}{p-2}} \|w\|^2 \left(\frac{\|w\|^2}{\int_{\mathbb{R}^3} b(x)|w|^p dx}\right)^{\frac{-2}{p-2}} \\
&= \inf_{t>0} g^\infty(t) < -\lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx \\
&\le -\lambda \int_{\mathbb{R}^3} K(x)\phi_{K(x),w} w^2 dx.
\end{aligned}$$

Then, it can be concluded that there are two constants $t^{(1)}$ and $t^{(2)}$ satisfying $Q_b(w) < t^{(1)} < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w) < t^{(2)}$ such that $g(t^{(i)}) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w} w^2 dx = 0$ for $i = 1, 2$. That is, $t^{(i)}w \in \mathcal{M}$ $(i = 1, 2)$.

Direct calculation of the second derivative gives

$$h''_{t^{(1)}w}(1) = -2\|t^{(1)}w\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|t^{(1)}w|^p dx = (t^{(1)})^5 g'(t^{(1)}) < 0,$$

and

$$h''_{t^{(2)}w}(1) = -2\|t^{(2)}w\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|t^{(2)}w|^p dx = (t^{(2)})^5 g'(t^{(2)}) > 0.$$

Then we get $t^{(1)}w \in \mathcal{M}^-$ and $t^{(2)}w \in \mathcal{M}$.

Note that

$$t^{(1)} < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w) \le \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} t_{b_\infty}(w) < \min\left\{\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}}, t^\infty\right\},$$

where t^∞ is the same as described in Lemma 3.4. For each $0 < \lambda < \Lambda$ and from Lemma 3.4 and condition (G_3), there holds

$$\begin{aligned}
\mathcal{J}(t^{(1)}w) &= \mathcal{J}^\infty(t^{(1)}w) - \frac{[t^{(1)}]^p}{p}\int_{\mathbb{R}^3} [b(x) - b_\infty]|w|^p dx \\
&\quad + \frac{\lambda [t^{(1)}]^4}{4}\left(\int_{\mathbb{R}^3} K(x)\phi_{K,w} w^2 dx - \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx\right) \\
&\le \sup_{0\le t\le t^\infty} \mathcal{J}^\infty(tw) - \frac{[t^{(1)}]^p}{p}\int_{\mathbb{R}^3} [b(x) - b_\infty]|w|^p dx \\
&\quad + \frac{\lambda [t^{(1)}]^4}{4}\left(\int_{\mathbb{R}^3} K(x)\phi_{K,w} w^2 dx - \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx\right) \\
&< \alpha_\infty^- < A(p)\frac{p-2}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}.
\end{aligned}$$

In other words, $t^{(1)}w \in \mathcal{M}^{(1)}$ and $\mathcal{J}(t^{(1)}w) < \alpha_\infty^-$. From the equation $h'_w(t) = t^3(g(t) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2dx)$, we notice that $h'_w(t) > 0$ for all $t \in (0, t^{(1)}) \cup (t^{(2)}, \infty)$ and $h'_w(t) < 0$ for all $t \in (t^{(1)}, t^{(2)})$. Finally, we get $\mathcal{J}(t^{(1)}w) = \sup_{0\le t\le t^{(2)}} \mathcal{J}(tw)$ and $\mathcal{J}(t^{(2)}w) = \inf_{t\ge t^{(1)}} \mathcal{J}(tw)$. That is, $\mathcal{J}(t^{(2)}w) \le \mathcal{J}(t^{(1)}w) < \alpha_\infty^-$, and so $t^{(2)}w \in \mathcal{M}^{(2)}$.

(ii) Let $\hat{g}(t) = t^{-2}\|w\|^2$ for $t > 0$. Clearly, $tw \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$ if and only if $\hat{g}(t) - \int_{\mathbb{R}^3} b(x)w^4dx + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2dx = 0$. After analysis $\hat{g}(t)$, we know that $\hat{g}(t) > 0$ is decreasing for $t > 0$, and $\lim_{t\to 0^+} \hat{g}(t) = \infty$ and $\lim_{t\to\infty} \hat{g}(t) = 0$. For $0 < \lambda < \Lambda$ and from condition (G_3) we have

$$\begin{aligned}\int_{\mathbb{R}^3} b(x)w^4dx - \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2dx &> \int_{\mathbb{R}^3} b_\infty w^4dx - \lambda \int_{\mathbb{R}^3} K_\infty\phi_{K_\infty,w}w^2dx \\ &= \|w\|^2 > 0.\end{aligned}$$

This implies that the equation $\hat{g}(t) - \int_{\mathbb{R}^3} b(x)w^4dx + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2dx = 0$ has a unique positive solution $\hat{t} = \left(\frac{\|w\|^2}{\int_{\mathbb{R}^3} b(x)w^4dx - \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2dx}\right)^{\frac{1}{2}} > Q_b(w)$. Then we get $\hat{t}w \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$. Similar to the discussion of case (i), we get that $\mathcal{J}(\hat{t}w) = \sup_{t\ge 0} \mathcal{J}(tw) = \sup_{t\ge Q_b(w)} \mathcal{J}(tw) < \alpha_\infty^-$. This completes the proof. □

Learning [23,26] we get the following result.

Lemma 3.6 ([25])**.** *Suppose that $4 < p \le 4$ and $0 < \lambda < \Lambda$. Then for each $u \in \mathcal{M}^{(1)}$, there exist $v > 0$ and a differentiable function: $t_* : B(0; v) \subset H^1(\mathbb{R}^3) \to \mathbb{R}^+$ such that $t_*(0) = 1$ and $t_*(v)(u - v) \in \mathcal{M}^{(1)}$ for all $v \in B(0; v)$, and there holds*

$$\langle (t_*)'(0), \varphi\rangle = \frac{2\int_{\mathbb{R}^3}(\nabla u \nabla \varphi + u\varphi)dx + 4\lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u\varphi dx - p\int_{\mathbb{R}^3} b(x)|u|^{p-2}u\varphi dx}{\|u\|^2 - (p-1)\int_{\mathbb{R}^3} b(x)|u|^p dx}$$

for all $\varphi \in H^1(\mathbb{R}^3)$.

By (2.6) and Lemma 2.7, for $2 < p < 4$, we define $\alpha^- = \inf_{u\in\mathcal{M}^{(1)}} \mathcal{J}(u) = \inf_{u\in\mathcal{M}^-} \mathcal{J}(u)$ and $\alpha^+ = \inf_{u\in\mathcal{M}^{(2)}} \mathcal{J}(u) = \inf_{u\in\mathcal{M}^+} \mathcal{J}(u)$. When $p = 4$, we define $\alpha^- = \inf_{u\in\mathcal{M}^{(1)}} \mathcal{J}(u) = \inf_{u\in\mathcal{M}} \mathcal{J}(u)$.

Proposition 3.7. *Suppose that $2 < p \le 4$ and $0 < \lambda < \Lambda$. Then there exists a sequence $\{u_n\} \subset \mathcal{M}^{(1)}$ such that*

$$\mathcal{J}(u_n) = \alpha^- + o(1) \quad \textit{and} \quad \mathcal{J}'(u_n) = o(1) \quad \textit{in } H^{-1}(\mathbb{R}^3). \tag{3.25}$$

Proof. According to the Ekeland variational principle [15], it follows from Lemma 2.5 that there exists a minimization sequence $\{u_n\} \subset \mathcal{M}^{(1)}$ such that $\mathcal{J}(u_n) < \alpha^- + \frac{1}{n}$ and

$$\mathcal{J}(u_n) \le \mathcal{J}(w) + \frac{1}{n}\|w - u_n\| \;\text{ for all }\; w \in \mathcal{M}^{(1)}. \tag{3.26}$$

By Lemma 3.6 with $u = u_n$, there exists a function $\bar{t}_* : B(0;\epsilon) \to \mathbb{R}$ for some $\epsilon > 0$ such that $\bar{t}_*(w)(u_n - w) \in \mathcal{M}^{(1)}$. Let $0 < \rho < \epsilon$ and $u \in H^1(\mathbb{R}^3)$ with $u \ne 0$. we set $w_\rho = \frac{\rho u}{\|u\|}$ and $z_\rho = \bar{t}_*(w_\rho)(u_n - w_\rho)$. Since $z_\rho \in \mathcal{M}^{(1)}$, from (3.26) we can get that $\mathcal{J}(z_\rho - \mathcal{J}(u_n) \ge -\frac{1}{n}\|z_\rho - u_n\|$. Generated using the median theorem

$$\langle \mathcal{J}'(u_n), z_\rho - u_n\rangle + o(\|z_\rho - u_n\|) \ge -\frac{1}{n}\|z_\rho - u_n\|,$$

and

$$\langle \mathcal{J}'(u_n), -w_\rho\rangle + (\bar{t}_*(w_\rho) - 1)\langle \mathcal{J}'(u_n), u_n - w_\rho\rangle \geq -\frac{1}{n}\|z_\rho - u_n\| + o(\|z_\rho - u_n\|). \tag{3.27}$$

Observed $\bar{t}_*(w_\rho)(u_n - w) \in \mathcal{M}^{(1)}$. From (3.27) it gives

$$\begin{aligned} &-\rho\left\langle \mathcal{J}'(u_n), \frac{u}{\|u\|}\right\rangle + \frac{(\bar{t}_*(w_\rho)-1)}{\bar{t}_*(w_\rho)}\langle \mathcal{J}'(z_\rho), \bar{t}_*(w_\rho)(u_n - w)\rangle \\ &\quad + (\bar{t}_*(w_\rho) - 1)\langle \mathcal{J}'(u_n) - \mathcal{J}'(z_\rho), u_n - w_\rho\rangle \\ &\geq -\frac{1}{n}\|z_\rho - u_n\| + o(\|z_\rho - u_n\|). \end{aligned}$$

Rewrite the above inequality as

$$\begin{aligned} \left\langle \mathcal{J}'(u_n), \frac{u}{\|u\|}\right\rangle \leq &\frac{\|z_\rho - u_n\|}{\rho n} + \frac{o(\|z_\rho - u_n\|)}{\rho} \\ &+ \frac{(\bar{t}_*(w_\rho) - 1)}{\rho}\langle \mathcal{J}'(u_n) - \mathcal{J}'(z_\rho), u_n - w_\rho\rangle. \end{aligned} \tag{3.28}$$

Then, there exist a constant $C > 0$ independent of ρ such that $\|z_\rho - u_n\| \leq \rho + C(|\bar{t}_*(w_\rho) - 1|)$ and $\lim_{\rho\to 0}\frac{|\bar{t}_*(w_\rho)-1|}{\rho} \leq \|(\bar{t}_*)'(0)\| \leq C$. Letting $\rho \to 0$ in (3.28) and using the fact that $\lim_{\rho\to 0}\|z_\rho - u_n\| = 0$, we get $\langle \mathcal{J}'(u_n), \frac{u}{\|u\|}\rangle \leq \frac{C}{n}$, this allows us to get (3.25). □

Therefore, we begin to prove the proof of Theorem 1.4.

By Proposition 3.7, for $2 < p \leq 4$, there exists a sequence $\{u_n\} \subset \mathcal{M}^{(1)}$ satisfying

$$\mathcal{J}(u_n) = \alpha^- + o(1) \quad \text{and} \quad \mathcal{J}'(u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^3).$$

From Corollary 3.3 and Lemma 3.4, 3.5, we know that equation (1.4) has a non-trivial solution $v \in \mathcal{M}^-$ such that $\mathcal{J}(v) = \alpha^-$. So, v is a minimizer for $\mathcal{J}$ on $\mathcal{M}^-$. In particular, for $2 < p < 4$, using $\alpha^- < \alpha_\infty^- < A(p)\frac{p-2}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}$, we obtain $v \in \mathcal{M}^-$. Through similar discussions, we get $|v| \in \mathcal{M}^-$ and $\mathcal{J}(|v|) = \mathcal{J}(v) = \alpha^-$. According to Lemma 2.4, v is a positive solution to equation (1.4). Therefore, $(v, \phi_{K,v})$ is a positive solution to the system (1.3).

3.3 Proof of Theorem 1.5

Lemma 3.8. *Suppose that* $\frac{1+\sqrt{73}}{3} < p < 4$ *and condition* (G_4) *holds. Let* u_0 *be a nontrivial solution of equation* (1.4) *. Then* $u_0 \in \mathcal{M}^-$.

Proof. Since u_0 is a nontrivial solution of equation (1.4), there holds

$$\int_{\mathbb{R}^3}|\nabla u_0|^2\,dx + \int_{\mathbb{R}^3}u_0^2dx + \lambda\int_{\mathbb{R}^3}K(x)\phi_{K,u_0}u_0^2dx - \int_{\mathbb{R}^3}b(x)\,|u_0|^p\,dx = 0. \tag{3.29}$$

Following the argument of [12] it is not difficult to verify that equation (1.4) satisfies the following Pohožaev type identity:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_0|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}u_0^2dx + \frac{5\lambda}{4}\int_{\mathbb{R}^3}K(x)\phi_{K,u_0}u_0^2dx \\ &\quad + \frac{\lambda}{2}\int_{\mathbb{R}^3}\langle\nabla K(x), x\rangle\phi_{K,u_0}u_0^2dx + \frac{\lambda}{4a}\int_{\mathbb{R}^3}K(x)\psi_{K,u_0}u_0^2dx \\ &= \frac{3}{p}\int_{\mathbb{R}^3}b(x)|u_0|^pdx + \frac{1}{p}\int_{\mathbb{R}^3}\langle\nabla b(x), x\rangle|u_0|^pdx. \end{aligned} \tag{3.30}$$

Combining (3.29) and (3.30) we get

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla u_0|^2 dx = \frac{3(p-2)}{6-p} \int_{\mathbb{R}^3} u_0^2 dx + \frac{5p-12}{2(6-p)} \lambda \int_{\mathbb{R}^3} K(x) \phi_{K,u_0} u_0^2 dx \\ + \frac{p\lambda}{6-p} \int_{\mathbb{R}^3} \langle \nabla K(x), x \rangle \phi_{K,u_0} u_0^2 dx + \frac{p\lambda}{2a(6-p)} \int_{\mathbb{R}^3} K(x) \psi_{K,u_0} u_0^2 dx \\ - \frac{2}{6-p} \int_{\mathbb{R}^3} \langle \nabla b(x), x \rangle |u_0|^p dx. \end{aligned} \tag{3.31}$$

From (2.3), (3.31) and condition (G_4) we obtain that

$$\begin{aligned} h''_{u_0}(1) =& -(p-2)\|u_0\|^2 + (4-p) \int_{\mathbb{R}^3} K(x) \phi_{K,u_0} u_0^2 dx \\ =& -(p-2) \int_{\mathbb{R}^3} |\nabla u_0|^2 dx - (p-2) \int_{\mathbb{R}^3} u_0^2 dx + (4-p) \int_{\mathbb{R}^3} K(x) \phi_{K,u_0} u_0^2 dx \\ =& -\frac{2p(p-2)}{6-p} \int_{\mathbb{R}^3} u_0^2 dx - \frac{p(p-2)\lambda}{2a(6-p)} \int_{\mathbb{R}^3} K(x) \psi_{K,u_0} u_0^2 dx \\ & -\lambda \int_{\mathbb{R}^3} \left(\frac{3p^2-2p-24}{2(6-p)} K(x) + \frac{p(p-2)}{6-p} \langle \nabla K(x), x \rangle\right) \phi_{K,u_0} u_0^2 dx \\ & + \frac{2(p-2)}{6-p} \int_{\mathbb{R}^3} \langle \nabla b(x), x \rangle |u_0|^p dx \\ <& \ 0. \end{aligned}$$

So, we get $u_0 \in \mathcal{M}^-$. □

We now come to the proof of Theorem 1.5.

Proof. Let v be a positive solution of equation (1.4), then we get $v \in \mathcal{M}^-$ and $\mathcal{J}(v) = \inf_{u \in \mathcal{M}^-} \mathcal{J}(u) = \alpha^-$. Next by Lemma 3.8, we know that v is a ground state solution of equation (1.4). Therefore, $(v, \phi_{K,v})$ is a positive solution of system (1.3). □

4 Appendix

In this section, we give the calculation procedure of Pohožaev identity.

Let $(u, \phi) \in H^1_\phi(\mathbb{R}^3) \times \mathcal{D}$ be a nontrivial solution of (1.1). Recall that $\phi = \phi_{K,u}$. we have

$$\|\nabla u\|_2^2 + \|u\|_2^2 + \lambda \int K(x)\phi u^2 - b(x)\|u\|_p^p = 0 \tag{4.1}$$

and

$$\|\nabla \phi\|_2^2 + a^2 \|\Delta \phi\|_2^2 = 4\pi \int K(x) \phi u^2, \tag{4.2}$$

that are usually called Nehari identities.

In fact, if (u, ϕ) solve (1.1), recalling the regularity proved in Appredix A.1. [13], for every $R > 0$, we have

$$\int_{B_R} -\Delta u \langle x \cdot \nabla u \rangle = -\frac{1}{2} \int_{B_R} |\nabla u|^2 - \frac{1}{R} \int_{\partial B_R} |x \cdot \nabla u|^2 + \frac{R}{2} \int_{\partial B_R} |\nabla u|^2, \tag{4.3}$$

$$\int_{B_R} K(x)\phi u\langle x\cdot\nabla u\rangle = -\frac{1}{2}\int_{B_R} K(x)u^2\langle x\cdot\nabla\phi\rangle - \frac{1}{2}\int_{B_R}\phi u^2\langle x\cdot\nabla K(x)\rangle \tag{4.4}$$

$$-\frac{3}{2}\int_{B_R} K(x)\phi u^2 + \frac{R}{2}\int_{\partial B_R} K(x)\phi u^2, \tag{4.5}$$

$$\int_{B_R} u\langle x\cdot\nabla u\rangle = -\frac{3}{2}\int_{B_R} u^2 + \frac{R}{2}\int_{\partial B_R} u^2, \tag{4.6}$$

$$\int_{B_R} b(x)|u|^{p-2}u\langle x\cdot\nabla u\rangle = -\frac{1}{p}\int_{B_R}\langle x\cdot\nabla b(x)\rangle|u|^p - \frac{3}{p}\int_{B_R} b(x)|u|^p dx + \frac{R}{p}\int_{\partial B_R}|u|^p, \tag{4.7}$$

where B_R is the ball of $\mathbb{R}^3$ centered in the origin and with radius R (see also [12]), and, since

$$\Delta^2\phi\langle x\cdot\nabla\phi\rangle = \operatorname{div}\left(\nabla\Delta\phi\langle x\cdot\nabla\phi\rangle - \Delta\phi\nabla\phi - \mathbb{F} + x\frac{(\Delta\phi)^2}{2}\right) + \frac{(\Delta\phi)^2}{2},$$

where $\mathbb{F}_i = \Delta\phi\langle x\cdot\nabla(\partial_i\phi)\rangle, i = 1,2,3$, then

$$\int_{B_R}\Delta^2\phi\langle x\cdot\nabla\phi\rangle = \frac{1}{2}\int_{B_R}(\Delta\phi)^2 + \int_{\partial B_R}\left(\nabla\Delta\phi\langle x\cdot\nabla\phi\rangle - \Delta\phi\nabla\phi - \mathbb{F} + x\frac{(\Delta\phi)^2}{2}\right)\cdot v. \tag{4.8}$$

Multiplying the first equation of (1.1) by $x\cdot\nabla u$ and the second equation by $x\cdot\nabla\phi$ and integrating on B_R, by (4.3), (4.4), (4.6), (4.7), and (4.8) we obtain

$$\begin{aligned}
&-\frac{1}{2}\int_{B_R}|\nabla u|^2 - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2 + \frac{R}{2}\int_{\partial B_R}|\nabla u|^2 - \frac{3}{2}\int_{B_R}u^2 + \frac{R}{2}\int_{\partial B_R}u^2\\
&\qquad -\frac{\lambda}{2}\int_{B_R}K(x)u^2\langle x\cdot\nabla\phi\rangle - \frac{\lambda}{2}\int_{B_R}\phi u^2\langle x\cdot\nabla K(x)\rangle - \frac{3\lambda}{2}\int_{B_R}K(x)\phi u^2 + \frac{\lambda R}{2}\int_{\partial B_R}K(x)\phi u^2\\
&= -\frac{1}{p}\int_{B_R}\langle x\cdot\nabla b(x)\rangle|u|^p - \frac{3}{p}\int_{B_R}b(x)|u|^p dx + \frac{R}{p}\int_{\partial B_R}|u|^p
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&4\pi\int_{B_R}K(x)u^2\langle x\cdot\nabla\phi\rangle\\
&\quad= -\frac{1}{2}\int_{B_R}|\nabla\phi|^2 - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla\phi|^2 + \frac{R}{2}\int_{\partial B_R}|\nabla\phi|^2\\
&\qquad+\frac{a^2}{2}\int_{B_R}(\Delta\phi)^2 + a^2\int_{\partial B_R}\left(\nabla\Delta\phi\langle x\cdot\nabla\phi\rangle - \Delta\phi\nabla\phi - \mathbb{F} + x\frac{(\Delta\phi)^2}{2}\right)\cdot v.
\end{aligned} \tag{4.10}$$

Substituting (4.10) into (4.9) we get

$$\begin{aligned}
&-\frac{1}{2}\int_{B_R}|\nabla u|^2 - \frac{3}{2}\int_{B_R}u^2 + \frac{\lambda}{16\pi}\int_{B_R}|\nabla\phi|^2 - \frac{\lambda a^2}{16\pi}\int_{B_R}(\Delta\phi)^2 - \frac{3\lambda}{2}\int_{B_R}K(x)\phi u^2\\
&\quad-\frac{\lambda}{2}\int_{B_R}\phi u^2\langle x\cdot\nabla K(x)\rangle + \frac{1}{p}\int_{B_R}\langle x\cdot\nabla b(x)\rangle|u|^p + \frac{3}{p}\int_{B_R}b(x)|u|^p dx\\
&=\frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2 - \frac{R}{2}\int_{\partial B_R}|\nabla u|^2 - \frac{R}{2}\int_{\partial B_R}u^2 - \frac{\lambda R}{2}\int_{\partial B_R}K(x)\phi u^2\\
&\quad+\frac{R}{p}\int_{\partial B_R}|u|^p - \frac{\lambda}{8\pi R} - \int_{\partial B_R}|x\cdot\nabla\phi|^2 + \frac{\lambda R}{16\pi}\int_{\partial B_R}|\nabla\phi|^2\\
&\quad+\frac{\lambda a^2}{8\pi}\int_{\partial B_R}\left(\nabla\Delta\phi\langle x\cdot\nabla\phi\rangle - \Delta\phi\nabla\phi - \mathbb{F} + x\frac{(\Delta\phi)^2}{2}\right)\cdot v.
\end{aligned}$$

Using the same arguments as in [12, Proof of Theorem 1.2] we have that right-hand side tends to zero as $R \to +\infty$, since

$$\int_{\partial B_R} \nabla \Delta \phi \langle x \cdot \nabla \phi \rangle \cdot v = R \int_{\partial B_R} \frac{\partial \Delta \phi}{\partial v} \frac{\partial \phi}{\partial v} \to 0,$$
$$\int_{\partial B_R} \Delta \phi \nabla \phi \cdot v = \int_{\partial B_R} \Delta \phi \frac{\partial \phi}{\partial v} \to 0,$$
$$\int_{\partial B_R} \mathbb{F} \cdot v = R \int_{\partial B_R} \frac{\partial^2 \phi}{\partial v^2} \to 0,$$
$$\frac{1}{2} \int_{\partial B_R} (\Delta \phi)^2 x \cdot v = \frac{R}{2} \int_{\partial B_R} (\Delta \phi)^2 \to 0.$$

Finally, using formula (A.3) in [13] , the Pohožaev identity can be written as

$$\begin{aligned} 0 = &\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} u^2 dx + \frac{5\lambda}{4} \int_{\mathbb{R}^3} K(x) \phi_{K,u} u^2 dx \\ &+ \frac{\lambda}{2} \int_{\mathbb{R}^3} \langle \nabla K(x), x \rangle \phi_{K,u} u^2 dx + \frac{\lambda}{4a} \int_{\mathbb{R}^3} K(x) \psi_{K,u} u^2 dx \\ &- \frac{3}{p} \int_{\mathbb{R}^3} b(x) |u|^p dx - \frac{1}{p} \int_{\mathbb{R}^3} \langle \nabla b(x), x \rangle |u|^p dx, \end{aligned}$$

where $\psi_{K,u} := e^{-\frac{|x|}{a}} * Ku^2 = \int_{\mathbb{R}^3} e^{-\frac{|x-y|}{a}} K(y) u^2(y) dy.$

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