ABSTRACT

Title of dissertation: PRESENTATIONS OF SEMIGROUP ALGEBRAS OF WEIGHTED TREES

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We study presentations for subalgebras of invariants of the coordinate algebras of binary symmetric models of phylogenetic trees studied by Buczynska and Wisniewski. These algebras arise as toric degenerations of projective coordinate rings of weight varieties of the Grassmannian of two planes associated to the Plücker embedding, and as toric degenerations of rings of invariants of Cox-Nagata rings.

PRESENTATIONS OF SEMIGROUP ALGEBRAS OF WEIGHTED TREES

by

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Dedication

To my parents.

Acknowledgments

First and foremost I thank my wife, for puting up with me. I thank John Millson for introducing me to the topic of this thesis, and Ben Howard for introducing me to combinatorial algebraic geometry.

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Chapter 1

Introduction

1.1 Weightings of trivalent trees

Let $\mathcal{T}$ denote an abstract trivalent tree with leaves $V(\mathcal{T})$, edges $E(\mathcal{T})$, and non-leaf (internal) vertices $I(\mathcal{T})$, by trivalent we mean that the valence of v is three for any $v \in I(\mathcal{T})$. Let e_i be the unique edge incident to the leaf $i \in V(\mathcal{T})$. Let Y be the unique trivalent tree with three leaves. For each $v \in I(\mathcal{T})$ we pick an injective map $i_v : Y \to \mathcal{T}$, sending the unique member of $I(Y)$ to v. We denote the members of $E(Y)$ by E, F, and G. We call a leaf in $V(\mathcal{T})$ lone if it is attached to an edge which is the unique leaf-edge incident to an internal vertex. Leaves which are not lone are called paired leaves. We will be concerned with properties of weightings of trivalent trees, defined as a functions

$$\omega : E(\mathcal{T}) \to \mathbb{Z}_{\geq 0}.$$

The maps i_v define pull-back operations on weightings by the formulas

$$i_v^*(\omega)(E) = \omega(i_v(E)),$$

$$i_v^*(\omega)(F) = \omega(i_v(F)),$$

$$i_v^*(\omega)(G) = \omega(i_v(G)).$$

Definition 1.1.1 *Let $S_{\mathcal{T}}$ be the graded semigroup where $S_{\mathcal{T}}[k]$ is the set of weightings which satisfy the following conditions.*

1. *For all $v \in I(\mathcal{T})$ the numbers $i_v^*(\omega)(E)$, $i_v^*(\omega)(F)$ and $i_v^*(\omega)(G)$ satisfy*

$$|i_v^*(\omega)(E) - i_v^*(\omega)(F)| \leq i_v^*(\omega)(G) \leq |i_v^*(\omega)(E) + i_v^*(\omega)(F)|$$

   These are referred to as the triangle inequalities.

2. *$i_v^*(\omega)(E) + i_v^*(\omega)(F) + i_v^*(\omega)(G)$ is even.*

3. *$\frac{1}{2}\sum_{i \in V(\mathcal{T})} \omega(e_i) = k$*

Note that because the triangle inequalities hold for the integers $i_v^*(\omega)(E)$, $i_v^*(\omega)(F)$, and $i_v^*(\omega)(G)$ if and only if a triangle exists with these side lengths, the condition is symmetric in E, F, and G. This semigroup is also multigraded, with the grading given by the weights $\omega(e_i)$ on the leaf edges of the tree.

Definition 1.1.2 *Let $\mathbf{r} : V(\mathcal{T}) \to \mathbb{Z}_{\geq 0}$ be a vector of nonnegative integers. Let $S_{\mathcal{T}}(\mathbf{r})$ be the multigraded subsemigroup of $S_{\mathcal{T}}$ formed by the pieces $S_{\mathcal{T}}[k\mathbf{r}]$.*

Proposition 1.1.3 *If $\mathbf{r}$ has an odd total sum, then $S_{\mathcal{T}}(\mathbf{r})[1] = \emptyset$.*

Proof 1.1.4 *This follows from the parity condition. Note that it is true by definition for $\mathcal{T} = Y$. Suppose now that the result holds for every trivalent tree with $n-1$ leaves, and consider $\mathcal{T}$ with n leaves. Pick a pair of paired leaves e, f in $V(\mathcal{T})$, and let $\mathcal{T}'$ be the trivalent tree obtained by forgetting e and f, and the edges connected to them. Let g be the internal edge of $\mathcal{T}$ which shares a vertex with f and g. Note that* *we may consider g a leaf of $\mathcal{T}'$. Any weighting $\omega \in S_{\mathcal{T}}$ defines a weighting of $\mathcal{T}'$ by restriction. By the induction hypothesis, $\omega|_{\mathcal{T}'}$ weights an even number of $V(\mathcal{T}')$ with odd numbers. There are two cases, if g is weighted odd then by parity only one of e or f may be weighted odd. If g is weighted even, then either both e and f is weighted odd, or neither is weighted odd.*

Because of the previous proposition, we focus on $\mathbf{r}$ with even total sum. Forgetting the grading for a moment, geometrically $S_{\mathcal{T}}$ is the semigroup of lattice points in a cone $P_{\mathcal{T}}$ in $\mathbb{R}^{|E(\mathcal{T})|}$. The inequalities defining $P_{\mathcal{T}}$ are given by the triangle inequalities, and the parity condition defines a certain sublattice of $\mathbb{Z}^{|E(\mathcal{T})|}$. We will see now that $P_{\mathcal{T}}$ has the structure of a fibered product of cones. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be trivalent trees with N_1 and N_2 leaves, respectively. Identify the leaf 1 from $\mathcal{T}_2$ with the leaf N_1 from $\mathcal{T}_1$, relabeling the leaves of $\mathcal{T}_2$ as follows, $1 \to N_1, \ldots, N_2 \to N_1 + N_2 - 1$. This creates a tree with a unique vertex of valence 2, replace this vertex and both of its incident edges with a single edge, the resulting tree $\mathcal{T}_1 * \mathcal{T}_2$ is trivalent. We call this operation *merging*. Let $i \in V(\mathcal{T})$, and denote the projection onto the $e_i - th$ component of $\mathbb{R}^{|E(\mathcal{T})|}$ by π_i. It is simple to check that

$$P_{\mathcal{T}_1 * \mathcal{T}_2} = P(\mathcal{T}_1)_{\pi_{N_1}} \times_{\pi_1} P(\mathcal{T}_2).$$

In particular this implies that all $P(\mathcal{T})$ are fibered products of copies of $P(Y)$, so in some sense all of the algebraic information in any $P(\mathcal{T})$ can be extracted from $P(Y)$, we will call this the fibered-product principle. It is reminiscient of the theory of moduli of orientable surfaces, where structures on a surface of high genus can be glued together from structures on three-punctured spheres over a pair-of-pants decomposition. The reason for this resemblance is not entirely accidental, see [HMM] for a moduli-of-surfaces interpretation of spaces associated to the semigroup $S_{\mathcal{T}}$. Bucynski and Wisniewski defined merging in [BW], where they show that a similar formula holds for a class of semigroups of weightings which we will now introduce.

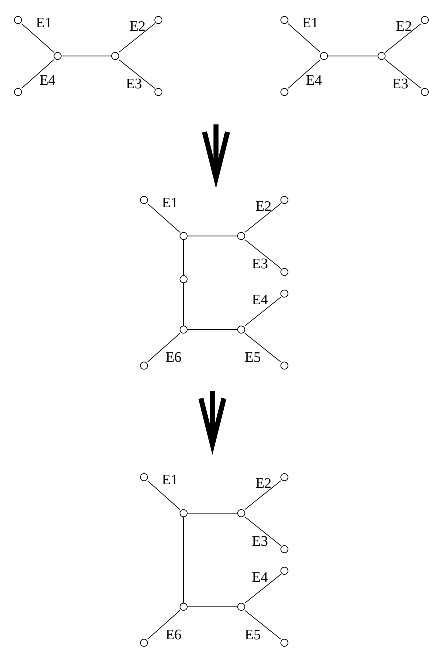

Figure 1.1: Merging two trees

Definition 1.1.5 *For a trivalent tree $\mathcal{T}$ let $\Delta(\mathcal{T})$ be the polytope in $\mathbb{R}^{|E(\mathcal{T})|}$ formed by the convex hull of weightings ω such that $\omega(e) \in \{0, 1\}$ for all $e \in E(\mathcal{T})$, and $i_v^*(\omega)(E) + i_v^*(\omega)(F) + i_v^*(\omega)(G) \in 2\mathbb{Z}$ for all $v \in I(\mathcal{T})$.*

It is shown in [BW] (Proposition 1.13) that $\Delta(\mathcal{T})$ is a fiber product of $|I(\mathcal{T})|$ copies of $\Delta(Y)$, using the same merging operation. The lattice point semigroup of $L\Delta(\mathcal{T})$ is isomorphic to the following semigroup.

Definition 1.1.6 *Let L be a positive integer. Let $S_{\mathcal{T}}^L$ be the graded semigroup where $S_{\mathcal{T}}^L[k]$ is the set of weightings ω of $\mathcal{T}$ which satisfy*

1. *For all $v \in I(\mathcal{T})$ the numbers $i_v^*(\omega)(E)$, $i_v^*(\omega)(F)$ and $i_v^*(\omega)(G)$ satisfy the triangle inequalities.*

2. *$i_v^*(\omega)(E) + i_v^*(\omega)(F) + i_v^*(\omega)(G)$ is even.*

3. *$i_v^*(\omega)(E) + i_v^*(\omega)(F) + i_v^*(\omega)(G) \leq 2kL$*

This is last item is referred to as the level condition.

Note that $S_{\mathcal{T}}^1$ has a fibered product decomposition into copies S_Y^1 in a way completely analagous to $S_{\mathcal{T}}$. To see that the lattice points of $\Delta(\mathcal{T})$ correspond with the first graded piece of $S_{\mathcal{T}}^1$, one need only use the fibered product decomposition of both objects. We observe that the lattice points of $\Delta(Y)$ are given by the degree 1 members of S_Y^1. Fixing a level L and an edge multigrade $\mathbf{r}$ picks out the multigrade $(L, \mathbf{r})$ subsemigroup $S_{\mathcal{T}}^L(\mathbf{r})$.

Each of the semigroups defined here has a corresponding semigroup algebra. The semigroup algebras $\mathbb{C}[S_{\mathcal{T}}]$ and $\mathbb{C}[S_{\mathcal{T}}^1]$ were studied in [SpSt] and [BW], respectively. In [SpSt] the authors show that the projective coordinate ring of $Gr_2(\mathbb{C}^n)$ with respect to the Plücker embedding flatly degenerates to each of the $\mathbb{C}[S_{\mathcal{T}}]$, in particular this establishes that the Hilbert function of $\mathbb{C}[S_{\mathcal{T}}]$ does not depend on $\mathcal{T}$.

In [BW] Bucynska and Wiesniewski studied the algebras $\mathbb{C}[S_{\mathcal{T}}^1]$, proving that they are all deformation equivalent, and generated in degree 1 with relations generated in degree 2. Notably Bucynska and Wiesniewski constructed the algebras

$\mathbb{C}[S^1_{\mathcal{T}}]$ as a step in the study of phylogenetic algebraic geometry. They study the spaces $Proj(\mathbb{C}[S^1_{\mathcal{T}}])$ because the toric ideal of $\mathbb{C}[S^1_{\mathcal{T}}]$ vanishes on a phylogenetic statistical model used to relate taxa in phylogenetic biology. See [BW] and [BW2] for details.

Notably Bucynksa and Wiesniewski did not construct an analogue of the projective coordinate ring of the Grassmannian of two planes in proving that the $\mathbb{C}[S^1_{\mathcal{T}}]$ are deformation equivalent. This was accomplished by Sturmfels and Xu in [BW]. In this paper the authors also establish a common algebro-geometric framework for both $\mathbb{C}[S_{\mathcal{T}}]$ and $\mathbb{C}[S^1_{\mathcal{T}}]$. These results are discussed in more detail in the following sections.

1.2 Grobner degenerations of $\mathbb{C}[Gr_2(\mathbb{C}^n)]$

In this subsection we will review the construction of one Grobner deformation of $A = \mathbb{C}[Gr_2(\mathbb{C}^n)]$ (from now on assumed to be the projective coordinate ring associated to the Plücker embedding) for each trivalent tree $\mathcal{T}$ with n ordered leaves, following the work of Speyer and Sturmfels in [SpSt]. We will also review results of [HMSV] on presentations of the rings $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$.

Our presentation of the Grobner theory of A follows that in [HMM] and [SpSt]. There is a well-known presentation of the ring A as the polynomial ring over the invariants Z_{ij} with $1 \leq i < j \leq n$ modulo the ideal generated by the *Plücker equations*,

$$Z_{ij}Z_{k\ell} - Z_{ik}Z_{j\ell} + Z_{i\ell}Z_{jk} = 0,$$

for $i < j < k < \ell$. We let $w_{ij}^{\mathcal{T}}$ denote the number of edges in the unique path in $\mathcal{T}$ connecting the i-th and the j-th leaf. Now we may define the $\mathcal{T}$-weight of any monomial term in A by

$$w(\prod_{k=1}^{m} Z_{i_k j_k}) = \sum_{k=1}^{m} w_{i_k j_k}^{\mathcal{T}}.$$

This weighting induces an increasing filtration on the ring A, let $F_m^{\mathcal{T}}$ denote the m-th part of this filtration. Note that we must have $F_m^{\mathcal{T}} F_k^{\mathcal{T}} \subset F_{m+k}^{\mathcal{T}}$. From this information we can define the Reese Algebra $A^{\mathcal{T}} = \bigoplus t^m F_m^{\mathcal{T}}$ with multiplication defined by $t^m x \circ t^k y = t^{m+k} xy$. We also may define the associated graded ring $A_0^{\mathcal{T}} = \bigoplus F_m^{\mathcal{T}} / F_{m-1}^{\mathcal{T}}$. It is a standard fact of the theory of Grobner degenerations [AB] that $A^{\mathcal{T}}$ is flat over $\mathbb{C}[t]$, $A^{\mathcal{T}} \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong A[t, t^{-1}]$, and $A^{\mathcal{T}} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t) \cong A_0^{\mathcal{T}}$. To each monomial $M = \prod_{k=1}^{m} Z_{i_k j_k}$ we may assigne a multigrading $\mathbf{r}(M) = (r_1(M), \ldots, r_n(M))$ where $r_i(M)$ is the number of Z_{ij} in the product M with i as an index. This multigrading naturally extends to the Reese algebra and the associated graded algebra. The grading g corresponding to the projective embedding of $Gr_2(\mathbb{C}^n)$ is obtained from the multigrading by the formula $g(M) = \frac{1}{2} \sum r_i(M)$ (compare with the definition of the grading on $S_{\mathcal{T}}$).

Proposition 1.2.1 *With respect to the grading g, the scheme $Proj(A^{\mathcal{T}})$ is fibered in projective schemes over the affine line $\mathbb{A}_1 = Spec(\mathbb{C}[t])$, with a generic fiber homeomorphic to $Gr_2(\mathbb{C}^n)$, and the special fiber over 0 equal to $Gr_2(\mathbb{C}^n)_0^{\mathcal{T}} = Proj(A_0^{\mathcal{T}})$*

The connection to semigroup algebras of weighted trees comes from the following proposition, which can be found in [HMM].

Proposition 1.2.2 *As graded rings,* $A_0^{\mathcal{T}} \cong \mathbb{C}[S^{\mathcal{T}}]$.

Let $A(\mathbf{r})$, $A^{\mathcal{T}}(\mathbf{r})$ and $A_0^{\mathcal{T}}(\mathbf{r})$ denote the sum of the $k\mathbf{r}$ graded components of A, $A^{\mathcal{T}}$, and $A_0^{\mathcal{T}}$ respectively, over nonegative integers k. We obtain the following propositions, which can be found in [HMM].

Proposition 1.2.3 *With respect to the multigrading* $\mathbf{r}$, *the scheme* $Proj(A^{\mathcal{T}}(\mathbf{r}))$ *is fibered in projective schemes over the affine line* $\mathbb{A}_1 = Spec(\mathbb{C}[t])$, *with a generic fiber equal to* $Gr_2(\mathbb{C}^n)/_{\mathbf{r}}T$, *and the special fiber over* 0 *equal to* $Proj(A_0^{\mathcal{T}}(\mathbf{r}))$

Proposition 1.2.4 *As graded rings,* $A_0^{\mathcal{T}}(\mathbf{r}) \cong \mathbb{C}[S^{\mathcal{T}}(\mathbf{r})]$.

Here $Gr_2(\mathbb{C}^n)/_{\mathbf{r}}T$ is the $\mathbf{r}$-weight variety of the Grassmannian of 2-planes. This variety is equal to the GIT quotient of $(\mathbb{P}^1)^n$ by $SL(2,\mathbb{C})$ with respect to the character corresponding to the weight $\mathbf{r}$, see [HMM] and [HMSV]. Hence, the $\mathcal{T}$-weight construction of Speyer and Sturmfels defines one Grobner degeneration of $(\mathbb{P}^1)^n//_{\mathbf{r}}SL(2,\mathbb{C})$ for each $\mathcal{T}$ to the toric varieties $Proj(\mathbb{C}[S_{\mathcal{T}}(\mathbf{r}))$. The results of this subsection so far show that varieties $Proj(\mathbb{C}[S_{\mathcal{T}}])$ for each $\mathcal{T}$, and $Gr_2(\mathbb{C}^n)$ all lie in the same component of a multigraded Hilbert Scheme. In particular, this implies that the Hilbert functions of $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ and $\mathbb{C}[S_{\mathcal{T}}]$ (with the grading g) do not depend on $\mathcal{T}$. Of main interest is the following proposition, which is a standard result from the theory of Grobner degenerations.

Proposition 1.2.5 *A presentation of the ring A (resp. $A(\mathbf{r})$) can be lifted from a presentation of $\mathbb{C}[S_{\mathcal{T}}]$ (resp. $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$).*

This is a key step in [HMSV], where presentations of the projective coordinate ring of $(\mathbb{P}^1)^n//_{\mathbf{r}}SL(2,\mathbb{C})$ are constructed for $\mathbf{r} = (1,\ldots,1)$ (from now on this weight is denoted 1^n), and a *good* trees $\mathcal{T}$, defined below.

Definition 1.2.6 *A trivalent tree $\mathcal{T}$ is a good tree if all leaves in $V(\mathcal{T})$ are paired.*

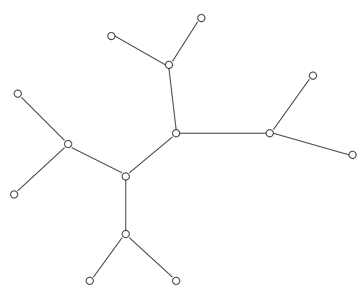

Figure 1.2: A Good Tree

In [HMSV], the authors prove that the ideal of relations of $\mathbb{C}[(\mathbb{P}^1)^n//\mathbf{r}SL(2,\mathbb{C})]$ is generated in degree at most 4. The workhorse of their proof is the following theorem, which can be found in [HMSV]. We let 1^n be the vector $(1,\ldots,1)$.

Theorem 1.2.7 *The algebra $\mathbb{C}[S_{\mathcal{T}}(1^n)]$ is generated in degree 1 if and only if $\mathcal{T}$ is a good tree. In this case relations are generated in degree less than or equal to 3.*

This result is obtained via an impressive graphical calculus. We will obtain a more general result by different techniques.

1.3 SAGBI degenerations of the Cox-Nagata rings of Sturmfels and Xu

In this subsection we review material from [StX], specifically their construction of the Cox-Nagata ring of a blow-up of $\mathbb{P}^n$ at d points, and multigraded SAGBI degenerations of these rings. First we must construct the Cox-Nagata ring R^G.

We follow [StX]. Let K be a field, and let $\ell_1, \ldots, \ell_n$ be linear forms over K which together span K^d, with $\ell_i = a_{1i}e^1 + \ldots + a_{di}e^d$. Let G be the nullspace of the linear transformation $K^n \to K^d$ defined by $e_i \to \ell_i$, so G is the space of linear relations

$$\lambda_1\ell_1 + \ldots + \lambda_n\ell_n,$$

among the linear forms ℓ_i. We define an action on the polynomial ring $R = K[\vec{x}, \vec{y}]$, $\vec{x} = \{x_1, \ldots, x_n\}$, $\vec{y} = \{y_1, \ldots, y_n\}$ defined by letting $\lambda \in G$ take $x_j \to x_j$ and $y_j \to y_j + \lambda_j x_j$. The invariant subring defined by this action is called the Cox-Nagata ring R^G. When G is generic, R^G only depends on the numbers n and d (see [StX]). The Cox-Nagata ring R^G is naturally an algebra over K and comes with a multigrading by $\mathbb{Z}^{n+1}$ given by $deg(x_i) = f_i$ and $deg(y_i) = deg(x_i) + f_0$ where $\mathbb{Z}^{n+1} = \mathbb{Z}[f_0, \ldots, f_n]$. The action of G on the polynomial ring R was used by Nagata in a special case to resolve Hilbert's 14th problem, showing that R^G is not finitely generated when G is a generic subspace of K^{16} of codimension 3. Mukai took this further in [M], proving the following theorem.

Theorem 1.3.1 *Let G be a generic subspace of K^n of codimension d. Then R^G is* *finitely generated if and only if*

$$\frac{1}{2} + \frac{1}{d} + \frac{1}{n-d} > 1$$

Now let $d \geq 3$ and let X_G denote the blow-up of $\mathbb{P}^{d-1}$ at the points defined by $\ell_1, \ldots, \ell_n$. Let L be divisor on X_G defined by the pullback of the hyperplane class on $\mathbb{P}^{d-1}$, and let $E_1, \ldots, E_n$ be the exceptional divisors of the blow-up. We identify $\mathbb{Z}^{n+1}$ with the Picard group of this blow-up. The $\mathbb{Z}^{n+1}$-graded ring

$$Cox(X_G) = \bigoplus \Gamma(X_G, rL + (u_1 - r)E_1 + \ldots + (u_n - r)E_n)$$

is called the Cox-ring of the blow-up X_G. The following is essentially a result of Ensalem and Iarrobino [EI].

Theorem 1.3.2 *If $d \geq 3$ then $Cox(X_G) \cong R^G$ as $\mathbb{Z}^{n+1}$ graded K-algebras.*

This explains the name Cox-Nagata ring. The multigrading on R^G defines a natural $(K^*)^{n+1}$ action on R^G, we have identified $\mathbb{Z}^{n+1}$ with the $Pic(X_G)$, so from now on we call this the action of the Picard Torus. The Cox Ring $Cox(X_G)$ is the affine coordinate ring of the $Universal\ Torsor$ over X_G. This space captures properties of all projective embeddings of X_G, in particular the statement that $Spec(Cox(X_G))$ degenerates to a toric variety implies that the same holds for *each* projective embedding of X_G.

Now we will review the SAGBI theory of R^G. Fix K to be $\mathbb{Q}(t)$, and let R be as above. Recall that the $initial\ form$ $in(f) \in \mathbb{Q}[\vec{x}, \vec{y}]$ of an element $f \in R$ is the coefficient of the lowest power of t appearing in f. For any subset $F \subset R$ we may define $in(F) \subset \mathbb{Q}[\vec{x}, \vec{y}]$. A subset F is called *moneric* if every element of $in(F)$ is a monomial. For any subalgebra $U \subset R$ we may define the algebra of initial forms $in(U) \subset \mathbb{Q}[\vec{x}, \vec{y}]$. A subset $F \subset U$ is called a SAGBI basis of U if F is moneric and the subalgebra generated by $in(F)$ equals $in(U)$. The acronym SAGBI stands for Subalgebra Analogue to Grobner Basis for Ideals, and was introduced by Robbiano and Sweedler in [RS]. The following is a standard property of SAGBI bases.

Proposition 1.3.3 *Let the algebra U have a finite SAGBI basis. Then U defines a flat deformation of $\mathbb{Q}$ algebras from $U(a)$ to $in(U)$, where $U(a)$ is a specialization of U with a generic.*

We will now see how to define SAGBI degenerations of the Cox-Nagata ring R^G for G generic, recall that these are all isomorphic. Let $G \subset K^n$ be the rowspan of a generic $2 \times n$ matrix B with entries in K. This means that the Plücker coordinates are non-zero on B and the same is true for generic specializations of B. The following follows from results in [StX].

Theorem 1.3.4 *Let R be as above and let $n = d + 2$, then for each trivalent tree $\mathcal{T}$, there is a matrix $B(\mathcal{T})$ with rowspan G such that $in(R^G) \cong \mathbb{Q}[S^1_{\mathcal{T}}]$, and all Plücker coordinates of $B(\mathcal{T})$ nonzero. The associated flat degenerations preserves the action of the Picard Torus.*

From this it follows that for $K = \mathbb{Q}$, and G generic, the Cox-Nagata ring R^G flatly degenerates to $\mathbb{Q}[S^1_{\mathcal{T}}]$ for each trivalent tree $\mathcal{T}$.

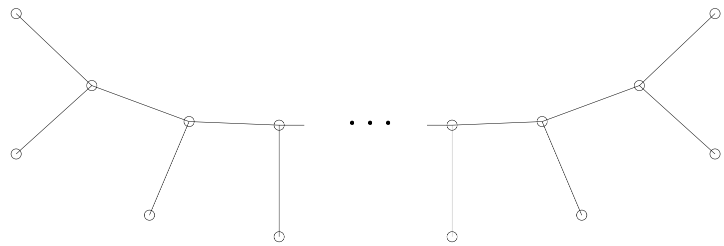

Figure 1.3: The Caterpillar Tree

Example 1.3.5 *The tree $\mathcal{T}_0$ pictured below is called the Caterpillar tree.*

The matrix

$$B = \begin{pmatrix} 1 & t & \dots & t^n \\ t^n & t^{n-1} & \dots & 1 \end{pmatrix}$$

defines the SAGBI degeneration to $\mathbb{Q}[S^1_{\mathcal{T}_0}]$.

In fact, much in the spirit of the fibered product principle, Sturmfels and Xu construct $B(\mathcal{T})$ out of the $B(\mathcal{T}_0)$ for various n. The SAGBI degenerations defined by Sturmfels and Xu preserve the multigrade by $\mathbb{Z}^{n+1}$, however there is a change of grade by an invertible matrix of determinant 1 to get the multigrade on $S^1_{\mathcal{T}}$ described above. We relabel the $(r, \vec{u})$ component of R^G with the multigrade $(L, \mathbf{r})$ with $L = (\sum_{i=1}^n u_i) - r$, $r_1 = u_1, \dots, r_{n-1} = u_{n-1}$, and $r_n = (\sum_{i=1}^n u_i) - r - u_n$. Let $R^G(L, \mathbf{r})$ be the subalgebra of components which are multiplies of $(L, \mathbf{r})$. The next theorem follows from the previous theorem.

Theorem 1.3.6 *There is a SAGBI degeneration of $R^G(L, \mathbf{r})$ to $\mathbb{Q}[S^L_{\mathcal{T}}(\mathbf{r})]$.*

Properties of presentations of $\mathbb{C}[S^L_{\mathcal{T}}(\mathbf{r})]$ lift to those of projective coordinate rings of the blow-up of $\mathbb{P}^{n-1}$ at $n+2$ points for $n = |V(\mathcal{T})|$, as well as the invariant subring $R^G(L, \mathbf{r})$ for the appropriate linearization of the action of the Picard Torus. It is also prudent to mention that the blow-up X_G for G of codimension 2 is related to the moduli space of parabolically semistable rank 2 bundles on $\mathbb{P}^1$, $N_{(0,n)}(\vec{\alpha})$ by a sequence of flops, see [B]. This implies that their Cox-rings are the same. As a consequence we get that the multigraded Hilbert function of the ring $\mathbb{C}[S^1_{\mathcal{T}}]$ is given by the Verlinde formula from mathematical physics. This establishes a combinatorial link between mathematical physics and phylogenetic algebraic geometry, and hints at a deeper representation-theoretic structure in the rings R^G. In [StX] Sturmfels and Xu construct all SAGBI degenerations of $Cox(X_G)$ when G is of codimension 1. The ring $Cox(X_G)$ in this case is isomorphic to the ring A from the last section, and Sturmfels and Xu are able to construct the Grobner degenerations discussed there. This is an attractive result as it suggests that $\mathbb{C}[S^L_{\mathcal{T}}(\mathbf{r})]$, and $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ are related by more than their combinatorial presentation, as both objects are useful in the study of blow-ups of projective spaces.

1.4 Statement of results

We now state our main results concerning the rings $\mathbb{C}[S^L_{\mathcal{T}}(\mathbf{r})]$. We begin with the definition of an admissible triple $(\mathcal{T}, \mathbf{r}, L)$.

Definition 1.4.1 *We call the triple $(\mathcal{T}, \mathbf{r}, L)$ admissible if L is even, $\mathbf{r}(i)$ is even for every lone leaf i, and $\mathbf{r}(j) + \mathbf{r}(k)$ is even for all paired leaves j, k.*

Remark 1.4.2 The assumption that $\mathbf{r}$ has an even total sum implies that an even number, $2M$ of the entries of $\mathbf{r}$ are odd. Choosing $\mathcal{T}$ with $2M$ paired leaves then

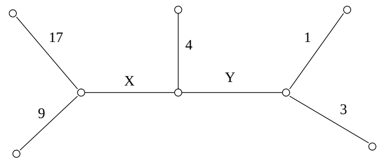

Figure 1.4: An admissible weighting

guarantees that $(\mathcal{T}, \mathbf{r}, L)$ *is admissible, provided that L is even. This is important for constructing presentations of* $R^G(\mathbf{r}, L)$*, since this ring always has a flat deformation to* $\mathbb{C}[S^L_{\mathcal{T}}(\mathbf{r})]$ *for some admissible* $(\mathcal{T}, \mathbf{r}, L)$ *when L is even. Also note that the second Veronese subring of* $\mathbb{C}[S^L_{\mathcal{T}}(\mathbf{r})]$ *is the semigroup algebra associated to* $(\mathcal{T}, 2\mathbf{r}, 2L)$*, which is always admissible.*

Theorem 1.4.3 *For* $(\mathcal{T}, \mathbf{r}, L)$ *admissible with* $L > 2$, $\mathbb{C}[S^L_{\mathcal{T}}(\mathbf{r})]$ *is generated in degree* 1*.*

Theorem 1.4.4 *For* $(\mathcal{T}, \mathbf{r}, L)$ *admissible with* $L > 2$, $\mathbb{C}[S^L_{\mathcal{T}}(\mathbf{r})]$ *has relations generated in degree at most* 3*.*

As a corollary we get the same results for $S_{\mathcal{T}}(\mathbf{r})$ when $(\mathcal{T}, \mathbf{r})$ satisfy admissibility conditions. These theorems will be proved in sections 2, 3, and 4. In section 5 we will look at some special cases, and investigate what can go wrong when $(\mathcal{T}, \mathbf{r}, L)$ is not an admissible triple. The following proposition is easy to prove from the triangle inequalities, and shows that as L becomes large the rings $\mathbb{C}[S^L_{\mathcal{T}}(\mathbf{r})]$ stabilize to $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$, and hence Theorems 1.4.4 and 1.4.3 apply without level condition as well.

Proposition 1.4.5 *There is a number* $N(\mathcal{T}, \mathbf{r})$ *such that* $\omega(e) \leq N(\mathcal{T}, \mathbf{r})$ *for every* $e \in E(\mathcal{T})$*.*

Corollary 1.4.6 *The results of Theorems 1.4.3 and 1.4.4 hold for the algebras $R^G(2L, \mathbf{r})$ and $A(\mathbf{r})$ for $L > 1$.*

We will refine this result further in section 5.

1.4.1 Outline of Techniques and Organization of Thesis

To prove Theorems 1.4.3 and 1.4.4 we use two main ideas. First, we employ the following trivial but useful observation.

Proposition 1.4.7 *Let $(\mathcal{T}, \mathbf{r}, L)$ be admissible, then for any weighting $\omega \in S^L_{\mathcal{T}}(\mathbf{r})$, $\omega(e)$ is an even number when e is not an edge connected to a paired leaf.*

This allows us to drop the parity condition that $i_v^*(\omega)(E) + i_v^*(\omega)(F) + i_v^*(\omega)(G)$ is even by forgetting the paired leaves and halving all remaining weights.

Definition 1.4.8 *Let $c(\mathcal{T})$ be the subtree of $\mathcal{T}$ given by forgetting all edges incident to paired leaves.*

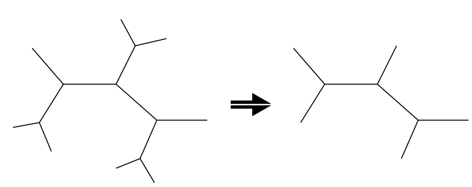

Figure 1.5: Clipping the paired leaves

Definition 1.4.9 *Let $U^L_{c(\mathcal{T})}(\mathbf{r})$ be the graded semigroup of weightings on $c(\mathcal{T})$ such that the members of $U^L_{c(\mathcal{T})}(\mathbf{r})[k]$ satisfy the triangle inequalities, the new level condition $i_v^*(\omega)(E) + i_v^*(\omega)(F) + i_v^*(\omega)(G) \leq kL$, and the following conditions.*

1. $\omega(e_i) = k\frac{\mathbf{r}(i)}{2}$ *for* i *a lone leaf of* $\mathcal{T}$.

2. $\frac{k|\mathbf{r}(i)-\mathbf{r}(j)|}{2} \leq \omega(e) \leq \frac{k|\mathbf{r}(i)+\mathbf{r}(j)|}{2}$ *for* e *the unique edge of* $\mathcal{T}$ *connected to the vertex which is connected to the paired leaves* i *and* j.

3. $\omega(e) + \frac{k\mathbf{r}(i)+k\mathbf{r}(j)}{2} \leq kL$

Let $U^L_{c(\mathcal{T})}$ be the graded semigroup of weightings which satisfy the triangle inequalities and the new level condition for L. The following is a consequence of these definitions.

Proposition 1.4.10 *For* $(\mathcal{T}, \mathbf{r}, L)$ *admissible,*

$$U^L_{c(\mathcal{T})}(\mathbf{r}) \cong S^L_{\mathcal{T}}(\mathbf{r})$$

as graded semigroups.

The next main idea is to undertake the analysis of $U^L_{c(\mathcal{T})}(\mathbf{r})$ by first considering the weightings $i_v^*(\omega) \in U^L_Y$. After constructing an object in U^L_Y, like a factorization or relation, we "glue" these objects back together along edges shared by the various $i_v(Y)$ with what amounts to a fibered product of graded semigroups. This is once again the fibered product principle. We obtain information about U^L_Y by studying the following polytope. Let $P_3(L)$ be the convex hull of $(0,0,0)$, $(\frac{L}{2}, \frac{L}{2}, 0)$, $(\frac{L}{2}, 0, \frac{L}{2})$, and $(0, \frac{L}{2}, \frac{L}{2})$.

The graded semigroups of lattice points for $P_3(L)$ is U^L_Y. By a lattice equivalence of polytopes P, $Q \subset \mathbb{R}^n$ with respect to a lattice $\Lambda \subset \mathbb{R}^n$ we mean a composition of translations by members of Λ and members of $GL(\Lambda) \subset GL_n(\mathbb{R})$ which

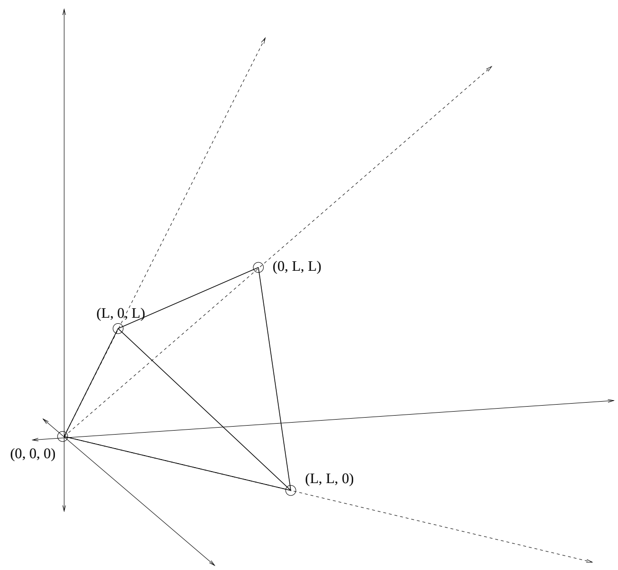

Figure 1.6: $P_3(2L)$

takes P to Q. If P and Q are lattice equivalent it is easy to show that they have isomorphic graded semigroups of lattice points. When L is an even integer (admissibility condition) the intersection of this polytope with any translate of the unit cube in $\mathbb{R}^3$, is, up to lattice equivalence, one of the polytopes shown in figure 1.7.

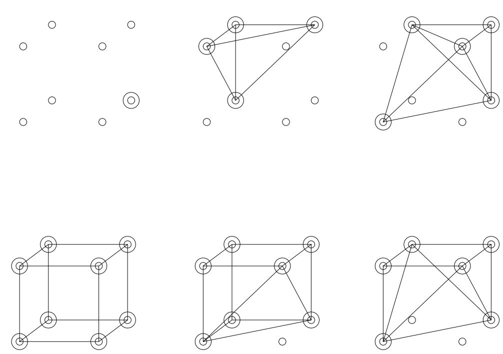

Figure 1.7: Cube Polytopes

Each of these polytopes is normal, meaning that their associated semigroup of lattice points are generated in degree 1. Also, the relations of the associated semigroups of each of these polytopes are generated in degree at most 3. Proving these two facts is the focus of Chapter 2. In Chapter 3 we will lift these properties to $U^L_{c(\mathcal{T})}(\mathbf{r})$, and therefore $S^L_{\mathcal{T}}(\mathbf{r})$ for $(\mathcal{T}, \mathbf{r}, L)$ admissible. Facts about the six polytopes above also allow us to carry out a more detailed investigation into the properties of the semigroups $S^L_{\mathcal{T}}(\mathbf{r})$ in Chapter 4, for example they allow us to show the redundancy of the cubic relations for certain $(\mathcal{T}, \mathbf{r}, L)$.

Chapter 2

The Cube Semigroups

In this chapter we will prove that the intersection of any translate of the unit cube of $\mathbb{R}^3$ with the polytope $P_3(L)$ produces a normal polytope whose semigroup of lattice points has relations generated in degree at most 3 when L is even. First we will recall some facts about Graver bases.

2.1 The Graver bases of the semigroup algebra of a polytope

We follow [St] for all information concerning Graver bases. For anything to do with the toric variety associated to a polytope we suggest Fulton's book, [Fu]. Let P be a lattice polytope in $\mathbb{R}^n$, and fix a lattice $L \subset \mathbb{R}^n$ such that $L \otimes \mathbb{R} = \mathbb{R}^n$. To P we associated a graded semigroup S_P where $S_P[k]$ is the collection of lattice points from kP, the kth Minkowski sum of P. The polytope P is called *normal* if S_P is generated in degree 1, in general this is not the case, the simplest counterexample known to author being the convex hull of $\{(0,1,1),(1,0,1),(1,1,0),(0,0,0\} \subset \mathbb{R}^3$ with regards to the standard lattice. We may form the semigroup algebra $\mathbb{C}[S_P]$. For a collection of vectors with integer entries $\{a_1,\ldots,a_n\}$, in our case the generators of S_P, we can consider the matrix $A = [a_1,\ldots,a_n]$ and its kernel as a linear transformation, $Ker(A)$. The semigroup algebra $\mathbb{C}[S_P]$ then has a presentation

$$0 \to I_A \to \mathbb{C}[x_{a_1}, \ldots, x_{a_n}] \to \mathbb{C}[S_P] \to 0,$$

with I_A generated by binomials. Let $x_A^{\vec{u}}$ be the monomial $\prod x_{a_i}^{u_i}$. For any vector with integer entries $\vec{u}$ we may rewrite $\vec{u}$ as $\vec{u} = \vec{u}^+ - \vec{u}^-$ for unique vectors with nonnegative entries. The ideal I_A is generated by the elements of the form $x_A^{u^+} - x_A^{u^-}$ for $\vec{u} \in Ker(A)$.

We write $\vec{v} \leq \vec{u}$ if the same is true for each index. A nonzero integer vector $\vec{u} \in Ker(A)$ is called primitive if there does not exist a vector $\vec{v} \in Ker(A) \setminus \{0, \vec{u}\}$ such that $\vec{v}^+ \leq \vec{u}^+$ and $\vec{v}^- \leq vecu^-$.

Definition 2.1.1 *Let $Gr(A)$ be the collection of primitive vectors for the matrix A, this is called the Graver Basis of A.*

We will use the following two properties of Graver bases.

Proposition 2.1.2 *The Graver basis $Gr(A)$ gives a generating set for the ideal I_A.*

Proposition 2.1.3 *Let $\{b_1, \ldots, b_k\} \subset \{a_1, \ldots, a_n\}$ with associated matrix B. Then $Gr(B) \subset Gr(A)$.*

This means that for any polytope P contained in a (perhaps more computationally tractable) polytope Q, generators of I_P are among the members of the Graver basis of I_Q. In particular the maximal degree of relations in $Gr(Q)$ bounds the degree of relation generation for the semigroup algebra generated by P.

2.2 The cone P_3 and the cube rooted at a lattice point

Recall from Chapter 1 that P_Y is the cone of triples of nonnegative integers which satisfy the triangle inequalities. From now on let $C(m_1, m_2, m_3)$ denote the unit cube *rooted* at $(m_1, m_2, m_3) \in \mathbb{R}^3$,

$$C(m_1, m_2, m_3) = conv\{(m_1 + \epsilon_1, m_2 + \epsilon_2, m_3 + \epsilon_3) | \epsilon_i \in \{0, 1\}\}.$$

We wish to classify the polytopes which have the presentation $C(m_1, m_2, m_3) \cap P_3$, since P_3 is symmetric we may assume that (m_1, m_2, m_3) is ordered by magnitude with m_3 the largest. In this analysis we keep track of the triangle inequalities with the quantities $n_i = m_j + m_k - m_i$. For a point (m_1, m_2, m_3) to be in P_3 is equivalent to $n_i \geq 0$ for each i. Immediately we have the following inequalities.

$$n_1 \geq n_2 \geq n_3, n_2 \geq 0$$

If $n_3 < -2$ then no member of $C(m_1, m_2, m_3)$ can belong to P_3. If $n_3 \geq -2$ then there are six distinct possibilities, we list each case along with the standard lattice members of $C(m_1, m_2, m_3) \cap P_3 - (m_1, m_2, m_3)$.

$Condition$	$C(m_1, m_2, m_3) \cap P_3 - (m_1, m_2, m_3)$
$n_3 = -2$	$(1,1,0)$
$n_3 = -1$	$(1,1,0),(0,1,0),(1,0,0),(1,1,1)$
$n_1 = n_2 = n_3 = 0$	$(1,1,0),(0,1,1),(1,0,1),(1,1,1),(0,0,0)$
$n_1 > 0, n_2 = n_3 = 0$	$(1,1,0),(0,1,1),(1,0,1),(1,1,1),(0,0,0),(0,0,1)$
$n_1, n_2 > 0, n_3 = 0$	$(1,1,0),(0,1,1),(1,0,1),(1,1,1),(0,0,0),(0,0,1),(0,1,0)$
$n_i > 0$	all points

The figure below illustrates these arrangements.

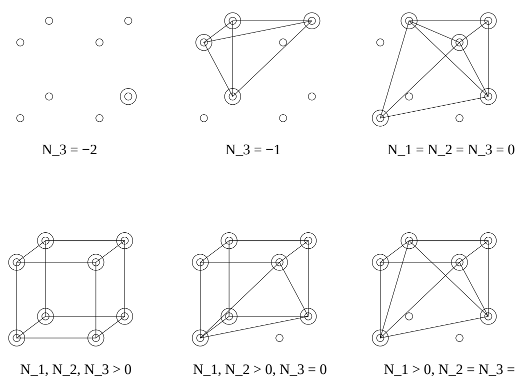

Figure 2.1: Primitive cube semigroups

Now we will see what happens when we intersect P_3 with the half space defined by the inequality $v_1 + v_2 + v_3 \leq 2L$ to get $P_3(2L)$. The reader may want to refer to figure 2.2 for this part. The convex set $C(m_1, m_2, m_3) \cap P_3(2L)$ can be one of the above polytopes (up to lattice equivalence), or one of them intersected with the half plane $v_1 + v_2 + v_3 \leq 2L$. Note that a vertex v in $C(m_1, m_2, m_3) \cap P_3(2L)$ lying on a facet of P_3 necessarily satisfies $v_1 + v_2 + v_3 = 0 \ mod\ 2$. In Figure 2.2 these points are colored black.

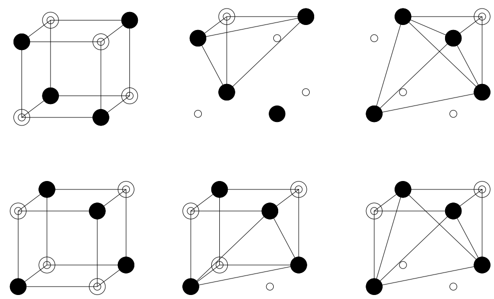

Figure 2.2: Cube semigroups with the lattice $v_1 + v_2 + v_3 = 0 \ mod \ 2$

The hyperplane defined by the equation $v_1 + v_2 + v_3 = 2L$ must intersect these polytopes at collections of three black points. If we assume that the lower left corner is $(0, 0, 0)$, these points have coordinates $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$, or $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Figure 2.2 represents the new possibilities for $C(m_1, m_1, m_3) \cap P_3(2L) - (m_1, m_2, m_3)$. The polytope pictured lower center in Figure 2.3 is the only case which is not lattice equivalent to one pictured in Figure 2.1. It is rooted at $(0, 0, 0)$ and occurs only when $L = 1$ (level condition is 2). The point $(1, 1, 1)$ in its second Minkowski sum cannot be expressed as the sum of two lattice points of degree one, so this is not a normal polytope. This is the reason we stipulate that $L > 2$ in Theorem 1.4.3. Now we analyze each $C(m_1, m_2, m_3) \cap P_3(2L)$. Since lattice equivalent polytopes have isomorphic semigroups of lattice points, it suffices to investigate the polytopes listed in Figure 2.1.

Caution 2.2.1 *In [BW], Buczynska and Wisniewski study a normal polytope with the same vertices as the non-normal polytope mentioned above. This is possible because they are using the the lattice* $v_1 + v_2 + v_3 = 0 \ mod \ 2$*, not the standard lattice.*

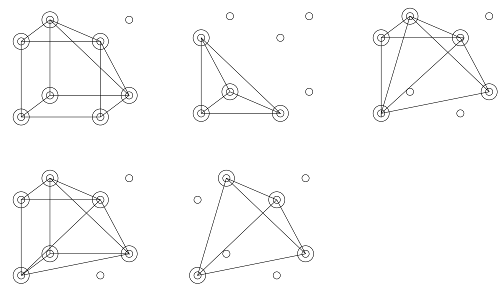

Figure 2.3: New Possibilities for $C(m_1, m_2, m_3) \cap P_3(2L)$

2.3 Graver bases of the unit cube

We make use of the computational algebra package 4ti2, [4ti2] to compute the *Graver basis* of the toric ideal of the unit 3-cube.

$$(1,0,0) + (1,1,1) = (1,0,1) + (1,1,0) \quad (0,1,0) + (1,1,1) = (0,1,1) + (1,1,0)$$

$$(0,0,0) + (1,1,1) = (0,0,1) + (1,1,0) \quad (0,0,1) + (1,1,1) = (0,0,1) + (1,0,1)$$

$$(0,0,0) + (1,1,1) = (0,1,0) + (1,0,1) \quad (0,0,1) + (1,1,0) = (0,1,0) + (1,0,1)$$

$$(0,0,0) + (1,1,1) = (0,1,1) + (1,0,0) \quad (0,0,1) + (1,1,0) = (0,1,1) + (1,0,0)$$

$$(0,1,0) + (1,0,1) = (0,1,1) + (1,0,0) \quad (0,0,0) + (1,1,0) = (0,1,0) + (1,0,0)$$

$$(0,0,0) + (1,0,1) = (0,0,1) + (1,0,0) \quad (0,0,0) + (0,1,1) = (0,0,1) + (0,1,0)$$

$$(0,1,0) + (1,0,0) + (1,1,1) = (0,0,1) + (1,1,0) + (1,1,0)$$

$$(0,0,0) + (1,1,1) + (1,1,1) = (0,1,1) + (1,0,1) + (1,1,0)$$

$$(0,0,1) + (1,0,0) + (1,1,1) = (0,1,0) + (1,0,1) + (1,0,1)$$

$$(0,0,1) + (0,0,1) + (1,1,0) = (0,0,0) + (0,1,1) + (1,0,1)$$

$$(0,0,0) + (0,1,1) + (1,1,0) = (0,1,0) + (0,1,0) + (1,0,1)$$

$$(0,0,0) + (1,0,1) + (1,0,1) = (0,1,1) + (1,0,0) + (1,0,0)$$

$$(0,0,1) + (0,1,0) + (1,1,1) = (0,1,1) + (0,1,1) + (1,0,0)$$

$$(0,0,0) + (0,0,0) + (1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$$

Operating on this set of monomials, one can show that the toric ideal of every sub-polytope of the unit 3-cube which is not a simplex has a square-free Gröbner basis. This, combined with the fact that the sub-polytopes with $n_3 = -2$ and -1 are unimodular simplices shows the following theorem, see Proposition 13.15 of [St].

Theorem 2.3.1 *Suppose $L \neq 1$, then for all (m_1, m_2, m_3), if $C(m_1, m_2, m_3) \cap P_3(2L)$ is non-empty, then it is a normal lattice polytope.*

Remark 2.3.2 This theorem implies, among other things, that if $\omega \in U_Y^{2L}[k]$, then

$$\omega = \sum_{i=1}^{k} W_i$$

for $W_i \in P_3(2L)$ with the property that each

$$W_i = X + (\epsilon_1, \epsilon_2, \epsilon_3)$$

with $\epsilon_j \in \{0, 1\}$ for all i for a fixed $X \in \mathbb{R}^3$. It is easy to show that

$$X = (\lfloor \frac{\omega(E)}{k} \rfloor, \lfloor \frac{\omega(F)}{k} \rfloor, \lfloor \frac{\omega(G)}{k} \rfloor)$$

Therefore each W_i *is* $(\frac{\omega(E)}{k}, \frac{\omega(F)}{k}, \frac{\omega(G)}{k})$ *with either floor or ceiling applied to each entry.*

Now we move on to relations, Let $S(m_1, m_2, m_3)$ be the semigroup of lattice points for $C(m_1, m_2, m_3) \cap P_3(2L) - (m_1, m_2, m_3)$, once again it suffices to treat the cases represented in Figure 2.1.

Theorem 2.3.3 *All relations for the semigroup* $S(m_1, m_2, m_3)$ *are reducible to quadrics and cubics.*

Proof 2.3.4 *This follows from the fact that the Graver basis of the unit* 3*-cube is composed of members of degree at most* 3.

Up to equivalence and after accounting for redundancy, all relations considered here are of the form

$$(1,0,0) + (0,1,0) = (1,1,0) + (0,0,0)$$

$$(1,0,1) + (0,1,0) = (1,1,1) + (0,0,0)$$

$$(1,0,1) + (1,1,0) = (1,1,1) + (1,0,0)$$

$$(1,1,1) + (1,1,1) + (0,0,0) = (1,1,0) + (1,0,1) + (0,1,1),$$

with the last one the only degree 3 relation, we refer to it as the “degenerated Segre Cubic” (see [HMSV]).

Chapter 3

Proof of main theorems

In this Chapter we will lift theorems on the cube semigroups to semigroups of weighted trees.

3.1 Proof of theorem 1.4.3

In this section we use Theorem 2.3.1 to prove that $U^L_{c(\mathcal{T})}(\mathbf{r})$ is generated in degree 1, which then proves Theorem 1.4.3. For each $v \in I(\mathcal{T})$ we have the morphism of graded semigroups

$$i_v^* : U^L_{c(\mathcal{T})}(\mathbf{r}) \to U^L_Y.$$

Given a weight $\omega \in U^L_{c(\mathcal{T})}(\mathbf{r})$ we factor $i_v^*(\omega)$ for each $v \in I(c(\mathcal{T}))$ using Theorem 2.3.1. Then, special properties of the weightings obtained by this procedure will allow us to glue the factors of the $i_v^*(\omega)$ back together along common edges to obtain a factorization of ω. First we must make sure that the factorization procedure does not disrupt the conditions at the edges of $c(\mathcal{T})$.

Lemma 3.1.1 *Let $\omega \in U^L_{c(\mathcal{T})}(\mathbf{r})[k]$, and let $v \in I(\mathcal{T})$ be connected to a leaf of $c(\mathcal{T})$, at E. Then if $i_v^*(\omega) = \eta_1 + \ldots + \eta_k$ is any factorization of $i_v^*(\omega)$ with $\eta_i \in C(\lfloor \frac{i_v^*(\omega)(E)}{k} \rfloor, \lfloor \frac{i_v^*(\omega)(F)}{k} \rfloor, \lfloor \frac{i_v^*(\omega)(G)}{k} \rfloor)$ Then $\eta_i(E)$ satisfies the appropriate edge condition for elements in $U^L_{c(\mathcal{T})}(\mathbf{r})[1]$.*

Proof 3.1.2 *If E is attached to a lone leaf of $\mathcal{T}$ then $i_v^*(\omega)(E) = k\mathbf{r}(e)$ for $i_v(E) = e$, $e \in V(\mathcal{T})$. By Remark 2.3.2*

$$\eta_i(E) = \lfloor \mathbf{r}(e) \rfloor = \mathbf{r}(e)$$

or

$$\eta_i(E) = \lfloor \mathbf{r}(e) \rfloor + 1 = \mathbf{r}(e) + 1$$

Since $\sum_{i=1}^k \eta_i(E) = k\mathbf{r}(e)$ we must have $\eta_i(E) = \mathbf{r}(e)$ for all i. If E is a stalk of paired leaves i and j in $\mathcal{T}$ then we must have

$$k\frac{|\mathbf{r}(i) - \mathbf{r}(j)|}{2} \leq \omega_Y(E) \leq k\frac{|\mathbf{r}(i) + \mathbf{r}(j)|}{2}$$

Note that both bounds are divisible by k. Since floor preserves lower bounds we have

$$\frac{|\mathbf{r}(i) - \mathbf{r}(j)|}{2} \leq \lfloor \frac{i_v^*(\omega)(E)}{k} \rfloor,$$

and since ceiling preserves upper bounds we have

$$\lceil \frac{i_v^*(\omega)(E)}{k} \rceil \leq \frac{|\mathbf{r}(i) + \mathbf{r}(j)|}{2}.$$

Therefore each η_i satisfies

$$\frac{|\mathbf{r}(i) - \mathbf{r}(j)|}{2} \leq \eta_i(E) \leq \frac{|\mathbf{r}(i) + \mathbf{r}(j)|}{2}$$

Now that we can safely use Theorem 2.3.1 with each $i_v^* : U^L_{c(\mathcal{T})}(\mathbf{r}) \to U^L_Y$, we will establish tools to extend factorization properties of U^L_Y to $U^L_{c(\mathcal{T})}(\mathbf{r})$ by exploiting the fibered product structure of the ambient semigroup $U^L_{c(\mathcal{T})}$. The following concept allows us to control conditions on the edges of two trees we wish to merge.

Definition 3.1.3 *We say a list of nonnegative integers* $\{X_1, \ldots, X_n\}$ *is balanced if* $|X_i - X_j| = 1$ *or* 0 *for all* i, j.

Lemma 3.1.4 *If two lists* $\{X_1, \ldots, X_n\}$ *and* $\{Z_1, \ldots, Z_m\}$ *are balanced, have the same total sum, and* $n = m$*, then they are the same list up to permutation.*

Proof 3.1.5 *Let* C_1 *be the smallest member of* $\{X_1, \ldots, X_n\}$*, and* C_2 *be the smallest member of* $\{Z_1, \ldots, Z_n\}$*. Let* S *be the total sum of either list. Both lists are balanced, so we must have* $S = nC_1 + k_1 = nC_2 + k_2$*, where* k_1 *and* k_2 *are nonnegative integers less than or equal to* n*. Suppose without loss of generality that* $k_2 - k_1$ *is nonnegative, then it must be divisible by* n*. By assumption, this can only happen if* $k_2 = k_1$*, in which case* $C_1 = C_2$*, and the lists have the same members.*

Proposition 3.1.6 *The semigroup* $U^L_{c(\mathcal{T})}(\mathbf{r})$ *is generated in degree* 1*.*

Proof 3.1.7 *Recall that by Remark 2.3.2, for any edge* $E \in Y$ *the edge weights of a factorization* $i^*_v(\omega) = \eta_1 + \ldots \eta_k$ *satisfy* $\eta_i(E) = \lfloor \frac{i^*_v(\omega)(E)}{k} \rfloor$ *or* $\lceil \frac{i^*_v(\omega)(E)}{k} \rceil$*. Take any two* v_1*,* v_2 *which share a common edge* E *in* $c(\mathcal{T})$*. Let* $\omega \in U^L_{c(\mathcal{T})}(\mathbf{r})[k]$ *and let* $\{\eta^1_1, \ldots, \eta^1_k\}$ *and* $\{\eta^2_1, \ldots, \eta^2_k\}$ *be factorizations of* $i^*_{v_1}(\omega)$ *and* $i^*_{v_2}(\omega)$ *respectively. Then the lists* $\{\eta^1_1(E), \ldots, \eta^1_k(E)\}$ *and* $\{\eta^2_1(E), \ldots, \eta^2_k(E)\}$ *are balanced and have the same sum, so by Lemma 3.1.4 they are the same list up to some permutation. We may glue factors* η^1_i *and* η^2_j *when* $\eta^1_i(E) = \eta^2_j(E)$*, the above observation gurantees that any* η^1_i *has an available partner* η^2_j*. The proposition now follows by induction on the number of* $v \in I(c(\mathcal{T}))$*. This implies Theorem 1.4.3.*

3.2 Proof of theorem 1.4.4

In this section we show how to get all relations in $U^L_{c(\mathcal{T})}(\mathbf{r})$ from those lifted from U^L_Y. The procedure follows the same pattern as the proof of Theorem 1.4.3. We consider the image of a relation $\omega_1 + \ldots + \omega_n = \eta_1 + \ldots + \eta_n$ under a map $i^*_v : U^L_{c(\mathcal{T})}(\mathbf{r}) \to U^L_Y$, using Theorem 2.3.3 we convert this to a trivial relation using relations of degree at most 3. We then give a recipe for lifting each of these relations back to $U^L_{c(\mathcal{T})}(\mathbf{r})$. The result is a way to convert $\omega_1 + \ldots + \omega_n = \eta_1 + \ldots + \eta_n$ to a relation which is trivial over the trinode v using quadrics and cubics. In this way we take a general relation to a trivial relation one $v \in I(c(\mathcal{T}))$ at a time.

Definition 3.2.1 *A set of degree* 1 *elements* $\{\omega_1, \ldots, \omega_k\}$ *in* $U^L_{c(\mathcal{T})}(\mathbf{r})$ *is called Balanced when the list* $\{\omega_1(E), \ldots, \omega_k(E)\}$ *is balanced for all* $E \in c(\mathcal{T})$*. A relation* $\omega_1 + \ldots + \omega_k = \eta_1 + \ldots + \eta_k$ *in* $U^L_{c(\mathcal{T})}(\mathbf{r})$ *is called Balanced when* $\{\omega_1, \ldots, \omega_k\}$ *and* $\{\eta_1, \ldots, \eta_k\}$ *are balanced.*

The following lemmas show that we need only consider balanced relations.

Lemma 3.2.2 *Any list of nonegative integers* $S = \{X_1, \ldots, X_n\}$ *can be converted to a balanced list* $T = \{Y_1, \ldots, Y_n\}$ *with* $\sum_{i=1}^n Y_i = \sum_{i=1}^n X_i$ *by replacing a pair* X_i *and* X_j *with* $\lfloor \frac{X_i+X_j}{2} \rfloor$ *and* $\lceil \frac{X_i+X_j}{2} \rceil$ *a finite number of times.*

Proof 3.2.3 *Let* $d(S)$ *be the difference between the maximum and minimum elements of* S*. It is clear that with a finite number of exchanges*

$$\{X_i, X_j\} \to \{\lfloor \frac{X_i + X_j}{2} \rfloor, \lceil \frac{X_i + X_j}{2} \rceil\}$$

We get a new set S' with $d(S) > d(S')$, unless $d(S) = 1$ or 0. Since this happens if and only of S is balanced, the lemma follows by induction.

Lemma 3.2.4 *Let*

$$\omega_1 + \ldots + \omega_k = \eta_1 + \ldots + \eta_k$$

be a relation in $U^L_{c(\mathcal{T})}(\mathbf{r})$ then it can be converted to a balanced relation

$$\omega_1' + \ldots + \omega_k' = \eta_1' + \ldots + \eta_k'$$

using only degree 2 relations.

Proof 3.2.5 *First we note that using the proof of Theorem 1.4.3 we can factor the weighting $\omega_1 + \omega_2$ into $\omega_1' + \omega_2'$ so that $\{\omega_1', \omega_2'\}$ is balanced. Using this and Lemma 3.2.2 we can find*

$$\omega_1' + \ldots + \omega_k' = \omega_1 + \ldots + \omega_k$$

such that the set $\{\omega_1'(E), \ldots, \omega_k'(E)\}$ is balanced for some specific E, using only degree 2 relations. Observe that if $\{\omega_1(F), \ldots, \omega_k(F)\}$ is balanced for some F, the same is true for $\{\omega_1'(F), \ldots, \omega_k'(F)\}$, after a series of degree 2 applications of 1.4.3. This shows that we may inductively convert $\{\omega_1, \ldots, \omega_k\}$ to $\{\omega_1', \ldots, \omega_k'\}$ with the property that $\{\omega_1'(E), \ldots, \omega_k'(E)\}$ is a balanced list for all edges E, using only degree 2 relations. Applying the same procedure to the η_i then proves the lemma.

The next lemma shows how we lift a balanced relation in U^L_Y to one in $U^L_{c(\mathcal{T})}(\mathbf{r})$.

Lemma 3.2.6 *Let $\{\omega_1 \ldots \omega_k\}$ be a balanced set of elements in $U^L_{c(\mathcal{T})}(\mathbf{r})$. Let $i_v^*(\omega_1) + \ldots + i_v^*(\omega_k) = \eta_1 + \ldots + \eta_k$ be a degree k relation in the appropriate $S(m_1, m_2, m_3) \subset$*

U^L_Y. Then the η_i may be lifted to weightings of $c(\mathcal{T})$ giving a relation of degree k in $U^L_{c(\mathcal{T})}(\mathbf{r})$ which agrees with the relation above when i^*_v is applied, and is a permutation of $i^*_{v'}(\omega_1)\ldots i^*_{v'}(\omega_k)$ for $v' \neq v$.

Proof 3.2.7 *Let $c(\mathcal{T})(E)$ be the unique connected subtrivalent tree of $c(\mathcal{T})$ which includes v and has the property that any path $\gamma \subset c(\mathcal{T})(E)$ with endpoints at a vertex $v' \neq v$ in $c(\mathcal{T})(E)$ and v includes the edge E (see Figure 3.1), define $c(\mathcal{T})(F)$ and $c(\mathcal{T})(G)$ in the same way. To make $\eta'_1 \ldots \eta'_k$ over $c(\mathcal{T})$, note that the list $\{i^*_{c(\mathcal{T})(E)}(\omega_i)(E)\}$ is the same as the list $\{\eta_i(E)\}$ up to permutation, because they are both balanced lists with the same sum and the same number of elements, so we may glue these weightings together to make a tuple over $c(\mathcal{T})$.*

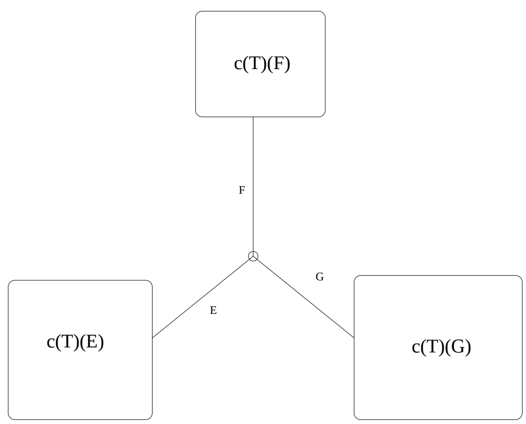

Figure 3.1: Component subtrees about a vertex

Suppose we are given a balanced relation

$$\omega_1 + \ldots + \omega_k = \eta_1 + \ldots + \eta_k.$$

We can pick any $v \in I(c(\mathcal{T}))$, and consider the relation

$$i_v^*(\omega_1) + \ldots + i_v^*(\omega_k) = i_v^*(\eta_1) + \ldots + i_v^*(\eta_k).$$

We convert this to a trivial relation using a series of relations in the appropriate $S(m_1, m_2, m_3)$, then lift the result back to $U^L_{c(\mathcal{T})}(\mathbf{r})$. For any $v' \neq v$ in $I(c(\mathcal{T}))$, this process only permutes the members of $\{i_{v'}^*(\omega_1), \ldots, i_{v'}^*(\omega_k)\}$ and $\{i_{v'}^*(\eta_1), \ldots, i_{v'}^*(\eta_k)\}$, which does not change whether or not this was a balanced relation. In this way, we may convert any balanced relation in $U^L_{c(\mathcal{T})}(\mathbf{r})$ to a trivial relation one $v \in I(c(\mathcal{T}))$ at a time.

Proposition 3.2.8 *Let N be the maximum degree of relations needed to generate all relations in the semigroups $S(m_1, m_2, m_3)$. Then the semigroup $U^L_{c(\mathcal{T})}(\mathbf{r})$ has relations generated in degree bounded above by N.*

This proposition, coupled with Theorem 2.3.3 proves Theorem 1.4.4. We recap the content of the last two sections with the following theorem.

Theorem 3.2.9 *Let $(\mathcal{T}, \mathbf{r}, L)$ be admissible. Then the ring $\mathbb{C}[U^L_{c(\mathcal{T})}(\mathbf{r})]$ has a presentation*

$$0 \longrightarrow I \longrightarrow \mathbb{C}[X] \longrightarrow \mathbb{C}[U^L_{c(\mathcal{T})}(\mathbf{r})] \longrightarrow 0$$

where X is the set of degree 1 elements of $U^L_{c(\mathcal{T})}(\mathbf{r})$, and I is the ideal generated by two types of binomials,

$$[\omega_1] \circ \ldots \circ [\omega_n] - [\eta_1] \circ \ldots \circ [\eta_n].$$

1. Binomials where $n \leq 3$, $i_v^*(\omega_1) + \ldots + i_v^*(\omega_n) = i_v^*(\eta_1) + \ldots + i_v^*(\eta_n)$ is *a balanced relation in U_Y^L for some specific v, and $\{i_{v'}^*(\omega_1), \ldots, i_{v'}^*(\omega_n)\} = \{i_{v'}^*(\eta_1), \ldots, i_{v'}^*(\eta_n)\}$ for $v \neq v'$.*

2. *Binomials where $n = 2$ and $i_v^*(\omega_1)+i_v^*(\omega_2) = i_v^*(\eta_1)+i_v^*(\eta_2)$ such that $\{i_v^*(\omega_1), i_v^*(\omega_2)\}$ is balanced for all $v \in I(c(\mathcal{T}))$.*

This induces a presentation for $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$ by isomorphism.

Chapter 4

Special cases and observations

In this chapter we collect results on some special cases of $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$. In particular we study some instances when cubic relations are unnecessary, we give some examples where the semigroup is not generated in degree 1, we analyze the case when L is allowed to be odd, and we give instances where cubic relations are necessary.

4.1 The caterpillar tree

One consequence of the proof of Theorem 2.3.3 is that a semigroup $U_{c(\mathcal{T})}^{2L}(\mathbf{r})$ which omits or only partially admits the semigroup $S(0,0,0)$ or $S(L-1,L-1,0)$ as an image of one of the morphisms i_v^* manages to avoid degree 3 relations entirely. This happens for $(\mathcal{T}_0, 2\mathbf{r}, 2L)$ where $\mathcal{T}_0$ is the Caterpillar tree.

Proposition 4.1.1 *Let $\mathcal{T}_0$ be the caterpillar tree, then $S_{\mathcal{T}_0}^{2L}(2\mathbf{r})$ is generated in degree 1, with relations generated by quadrics.*

Proof 4.1.2 *We catalogue the weights $i_v^*(\omega)$ which can appear in degree 1. For the sake of simplicity we divide all weights by 2. Suppose $i_v(G)$ is an external edge, then $i_v^*(\omega)(E)$ and $i_v^*(\omega)(F)$ satisfy the following inequalities*

$$i_v^*(\omega)(E) \leq i_v^*(\omega)(F) + \mathbf{r}(i)$$

$$i_v^*(\omega)(F) \leq i_v^*(\omega)(E) + \mathbf{r}(i)$$

$$i_v^*(\omega)(E) + i_v^*(\omega)(F) + \mathbf{r}(i) \leq 2L$$

where $i_v^*(\omega)(G) = \mathbf{r}(i)$. *These conditions define a polytope in* $\mathbb{R}^2$ *with vertices* $(L, L-\mathbf{r}(i))$, $(L-\mathbf{r}(i), L)$, $(\mathbf{r}(i), 0)$ *and* $(0, \mathbf{r}(i))$. *Pictured below is the case* $L = 9$, $\mathbf{r}(i) = 3$.

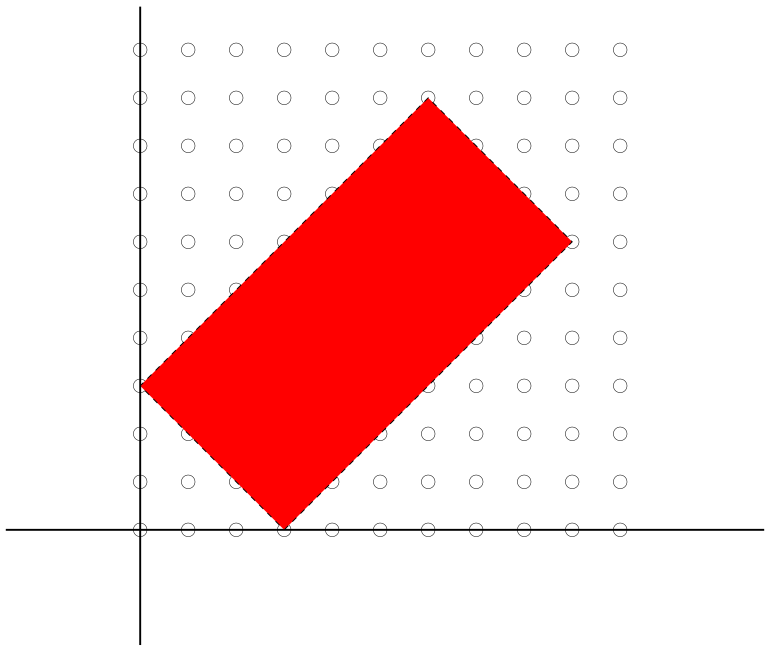

Figure 4.1: The case $L = 9$, $2\mathbf{r}(i) = 6$

When two edges are external, the polytope is an integral line segment. Note that the intersection of any lattice cube in $\mathbb{R}^2$ *with the above polytope is a simplex or a unit square. Both of these polytopes have at most quadrics for relations in their semigroup of lattice points. Hence the argument used to prove Theorem 1.4.4 shows that* $U^{2L}_{c(\mathcal{T}_0)}(\mathbf{r})$ *needs only quadric relations.*

Corollary 4.1.3 *If* $L > 1$, *and* $\mathbf{r}$ *is a vector of nonnegative integers, the ring* $R^G(2L, 2\mathbf{r})$ *has a presentation with defining ideal generated by quadrics. In particular, the second Veronese subring of any* $R^G(\mathbf{r}, L)$ *has such a presentation if* $L > 1$.

4.2 Counterexamples to degree 1 generation

Now we'll see examples of $(\mathbf{r}, \mathcal{T}, L)$ such that $S_{\mathcal{T}}^{L}(\mathbf{r})$ is not generated in degree 1. We will begin by defining a certain class of paths in the tree $\mathcal{T}$. Let $\mathcal{T}$ have an even number of leaves. We claim that there is a set of edges $A(\mathcal{T}) \subset E(\mathcal{T})$ the members of which are assigned odd numbers by any weighting ω which assigns an odd number to each leaf of $\mathcal{T}$. It suffices to establish the stronger result that the parity of members of $V(\mathcal{T})$ determines the parity of every edge in $\mathcal{T}$. To see this, first note that the parity of two edges of a trinode determines the parity of the third edge, an induction argument on the number of edges in $\mathcal{T}$ does the rest.

Proposition 4.2.1 *Let $\mathcal{T}$ be as above. The set $A(\mathcal{T})$ is a union of edges from disjoint paths in $\mathcal{T}$.*

Proof 4.2.2 *Exactly two out of three edges in each trinode can be assigned an odd number, by the parity condition. This establishes the proposition.*

From now on we let $O(\mathcal{T})$ denote the set of paths established by the previous proposition.

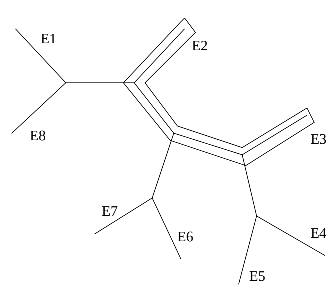

Figure 4.2: E2 and E3 are lone leaves connected by an element of $O(\mathcal{T})$

Proposition 4.2.3 *Let* $(\mathbf{r}, \mathcal{T}, L)$ *be such that the edges connected to the endpoints of each member of* $O(\mathcal{T})$ *are given the same parity by* $\mathbf{r}$*. Assume further that there is a* $\gamma \in O(\mathcal{T})$ *such that end points of* γ *are connected to edges* e *and* f *with* $\mathbf{r}(e)$ *and* $\mathbf{r}(f)$ *odd. If there is a degree* 2 *weighting which assigns* 0 *to any edge in* γ*, then* $S_{\mathcal{T}}^{2L}(\mathbf{r})$ *is not generated in degree* 1*.*

Proof 4.2.4 *All degree* 1 *elements must assign odd numbers to the edges in* γ*. No two odd numbers add to* 0*.*

Corollary 4.2.5 *For simplicity, let* $L > 1$*. The semigroup* $S_{\mathcal{T}}^{2L}(\vec{1})$ *is generated in degree* 1 *if and only if* $\mathcal{T}$ *is good.*

Proof 4.2.6 *First note that the if portion of this statement is taken care of by Theorem 1.4.3. Suppose now that* $\mathcal{T}$ *has lone leaves. Then two of these leaves are connected by a member* γ *of* $O(\mathcal{T})$*. Pick any non-leaf edge* e *in* γ*, and consider the weighting* ω *which assigns* 0 *to* e *and* 2 *to every other edge in* $\mathcal{T}$*. We have* $\omega \in S_{\mathcal{T}}^{2L}(\vec{1})[2]$ *for any* L*, and by proposition 4.2.3* ω *cannot be factored.*

4.3 The case when L is odd

When the level L is odd, the polytope $P_3(L)$ is no longer integral, however its Minkowski square $P_3(2L)$ is integral, so clearly there are elements of $P_3(2L)$ which cannot be integrally factored, specifically the corners. This observation has a generalization.

Definition 4.3.1 *Let* $IP_3(L)$ *be the convex hull of the integral points of* $P_3(L)$*. Let*

Ω be the set of elements in the graded semigroup of lattice points of $P_3(L)$ such that $\frac{1}{deg(Q)}Q \in P_3(L) \setminus IP_3(L)$.

Let $(E, F, G) = Q \in P_3(L)$ be integral with L odd, and suppose E, F, or $G \geq \frac{L-1}{2} + 1$. Then, by the triangle inequalities we must have $F + G \geq \frac{L-1}{2} + 1$, so $E + F + G \geq L + 1$, a contradiction. This shows that $IP_3(L)$ is contained in the intersection of $P_3(L)$ with the halfspaces $E, F, G \leq \frac{L-1}{2}$, this identifies $IP_3(L)$ as the convex hull of the set

$$\{(0,0,0), (\frac{L-1}{2}, \frac{L-1}{2}, 0), (\frac{L-1}{2}, 0, \frac{L-1}{2}), (0, \frac{L-1}{2}, \frac{L-1}{2}),$$
$$(\frac{L-1}{2}, \frac{L-1}{2}, 1), (\frac{L-1}{2}, 1, \frac{L-1}{2}), (1, \frac{L-1}{2}, \frac{L-1}{2})\}.$$

The case $IP_3(5)$ is pictured below.

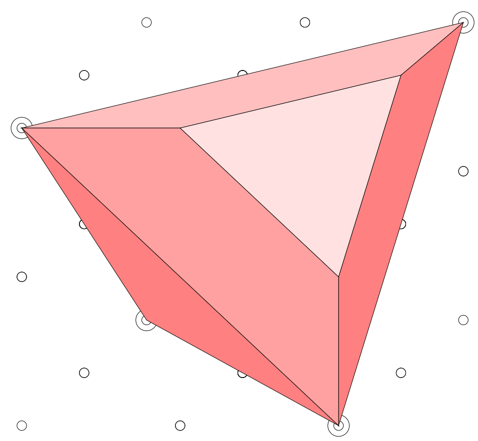

Figure 4.3: The Polytope $IP_3(5)$

Proposition 4.3.2 *Any $Q \in \Omega$ cannot be integrally factored.*

Proof 4.3.3 *This follows from the observation that if $Q = E_1 + \ldots + E_n$ then $\frac{1}{n}Q$ is in the convex hull of $\{E_1, \ldots, E_n\}$.*

A factorization of any element ω such that $i_v^*(\omega) = Q$ gives a factorization of Q. So any $\omega \in U^L_{c(\mathcal{T})}(\mathbf{r})$ with a $i_v^*(\omega) \in \Omega$ is necessarily an obstruction to generation in degree 1, this also turns out to be a sufficient obstruction criteria.

Theorem 4.3.4 *Let $\mathcal{T}$ and $\mathbf{r}$ satisfy the same conditions as admissibility, and let $L \neq 2$. Then $U^L_{c(\mathcal{T})}(\mathbf{r})$ is generated in degree 1 if and only if*

$$i_v^*(\omega) \in U_Y^L \setminus \Omega$$

for all $v \in I(c(\mathcal{T}))$, $\omega \in U^L_{c(\mathcal{T})}(\mathbf{r})$. In this case all relations are generated by those of degree at most 3.

Proof 4.3.5 *We analyze $IP_3(L)$ in the same way we did $P_3(2L)$. The reader can verify that the integral points of $C(m_1, m_2, m_3) \cap P_3(L)$ are the same as the integral points of $C(m_1, m_2, m_3) \cap IP_3(L)$. The possibilities are represented by slicing the cubes in Figure 2.2 along the plane formed by the upper right or lower left collection of three non-filled dots, depending on the cube, and then restricting to the convex hull of the remaining integral points. All cases are lattice equivalent to one of the polytopes listed in Figure 2.1, after considering two and one dimensional cases as facets of neighboring three dimensional polytopes. Since any element of U_Y^L not in Ω is necessarily a lattice point of a Minkowski sum of $IP_3(L)$, the theorem follows by the same arguments used to prove Theorems 1.4.3 and 1.4.4*

4.4 Necessity of degree 3 relations

Now we show that there are large classes of admissible $(\mathcal{T}, \mathbf{r}, L)$ which require degree 3 relations. We will exhibit a degree 3 weighting which has only two factorizations. The tree $\mathcal{T}$ with weight $\omega_{\mathcal{T}}$ is pictured below, it is an element of $S_{\mathcal{T}}(\vec{2})$. In all that follows all weightings are considered to have been halved.

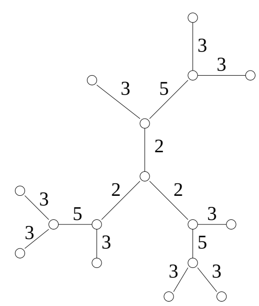

Figure 4.4: $\omega_{\mathcal{T}}$

Notice that $\omega_{\mathcal{T}}$ has 3-way symmetry about the central trinode, we will exploit this by considering the tree $\mathcal{T}'$ with restricted weighting $\omega_{\mathcal{T}'}$ pictured in Figure 4.5. We find the weightings that serve as a degree 1 factors of $\omega_{\mathcal{T}'}$. First of all, any degree 1 weighting which divides $\omega_{\mathcal{T}'}$ must be as in Figure 4.6.

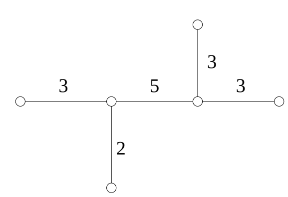

Figure 4.5: $\omega_{\mathcal{T}'}$

It suffices to find the possible values of X and Z. Both must be ≤ 2, which shows that Z can be either 2 or 1. This implies that two factors have $Z = 2$ and one factor has $Z = 1$. For X, we note that $X = 0$ cannot be paired with $Z = 2$

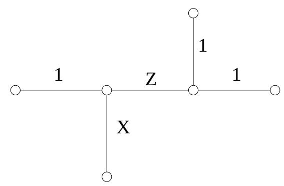

Figure 4.6:

because of the triangle inequalities. This implies that X cannot equal 2, and that both factors with $Z = 2$ have $X = 1$, and $Z = 1$ is paired with $X = 0$. This shows that there are exactly two possibilities determined by the value of X, both are shown in Figure 4.7. Any factorization of $\omega_{\mathcal{T}}$ is determined by its values on the central trinode, and these values must be weights composed entirely of 0 and 1. There are exactly two such variations, making the Degenerated Segre Cubic.

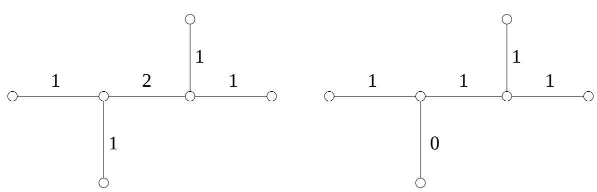

Figure 4.7:

We have not specified a level L for this weighting, but the same argument applies for any level large enough to admit $\omega_{\mathcal{T}}$ as a weighting in degree 3. For any tree $\mathcal{T}^*$, edge $e^* \in tree^*$, and weight $\omega_{\mathcal{T}^*}$ we can create a new weight on a larger tree by adding a vertex in the middle of e^*, attaching a new leaf edge at that vertex, and weighting the both sides of the split e^* with $\omega_{\mathcal{T}^*}(e^*)$, and the new edge with 0. Using this procedure on any $(\mathcal{T}^*, e^*, \omega_{\mathcal{T}^*})$, and $(\mathcal{T}, e, \omega_{\mathcal{T}})$ for any edge $e \in \mathcal{T}$, can create a new weighted tree by identifying the new 0-weighted edges. This construction is called the pointed graft of two pointed trees, and was introduced in Definition 2.25

of [BW]. An example is pictured below. In this way many examples of unremoveable degree 3 relations can be made.

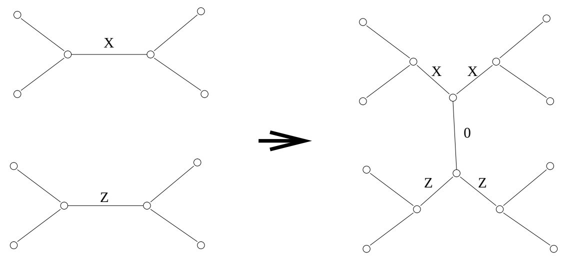

Figure 4.8: Grafting two tree weightings.

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