A Modal Logic for Supervised Learning

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Abstract Formal learning theory formalizes the process of inferring a general result from examples, as in the case of inferring grammars from sentences when learning a language. In this work, we develop a general framework—the *supervised learning game*—to investigate the interaction between *Teacher* and *Learner*. In particular, our proposal highlights several interesting features of the agents: on the one hand, Learner may make mistakes in the learning process, and she may also ignore the potential relation between different hypotheses; on the other hand, Teacher is able to correct Learner's mistakes, eliminate potential mistakes and point out the facts ignored by Learner. To reason about strategies in this game, we develop a *modal logic of supervised learning* and study its properties. Broadly, this work takes a small step towards studying the interaction between graph games, logics and formal learning theory.

Keywords Formal Learning Theory · Modal Logic · Dynamic Logic · Undecidability · Graph Games

1 Introduction

Formal learning theory formalizes the process of inferring a general result from examples, as in the case of inferring grammars from sentences when learning a language. A good way of understanding this general process is by treating it as

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a game played by *Learner* and *Teacher* [18]. It starts with a class of possible worlds, where one of them is the actual one chosen by Teacher. Learner's aim is to get to know which one it is. Teacher inductively provides information about the world, and whenever receiving a piece of information Learner picks a conjecture from the class, indicating which one she thinks is the case. Different success conditions for Learner can be defined. In this article we focus on the case that at some finite stage of the procedure Learner decides on a correct hypothesis. This kind of learnability is known as *finite identification* [27,24].

Although empirical evidence suggests that children can learn a language without responding to the correction of linguistic mistakes [19], the importance of teachers in many other paradigms is significant. For instance, in the paradigm of *learning from queries and counterexamples* [1], Teacher has a strong influence on whether the process is successful. Moreover, results in [18] suggest that a helpful Teacher may make learning easier. In this work, instead of focusing only on Learner, we highlight the interactive nature of learning.

As noted in [18], a concise model for characterizing the interaction between Learner and Teacher is the *sabotage game (SG)* [30]. A SG is played on a graph with a starting node and a goal node, and it goes in rounds: Teacher first cuts an edge, then Learner makes a step along one of the edges still available. Both of them win if, and only if, Learner arrives at the goal node. Roughly, the game depicts a guided learning situation. Say, a natural interpretation is the case of theorem proving. Intuitively, the starting node is given by axioms, the goal node stands for the theorem to be proved, other nodes represent lemmas conjectured by Learner, and edges capture Learner's possible inferences between them. Inferring is represented by moving along edges. The information provided by Teacher can be treated as his feedback, i.e., removing edges to eliminate wrong inferences. The success condition is given by the winning condition: the learning process has been successful if Learner reaches the goal node, i.e., by proving the theorem. For the general correspondence between SG and learning models, we refer to [18].

However, we would argue that this application of SG gives a highly restricted model of learning. For instance,

- Intuitively, all links in the graph are inferences conjectured by Learner, which may include mistakes. From the perspective of Learner, the wrong inferences cannot be distinguished from the correct ones. Although it is reasonable to assume that Teacher is able to do so, SG does not highlight that Learner lacks perfect information.
- In sabotage games, Teacher has to remove a link in each round, which looks overly restrictive and may lead to undesired outcomes. Also, Teacher can only *delete* links to decide what Learner will not learn, and thus he only teaches what Learner has already conjectured. However, during the process of learning, 'possibilities may also be ignored due to the more questionable practice if assuming that one of the theories under consideration must be true. And complexity can come to be ignored through convention or habit'

([22], pp. 260). Hence, it is natural to assume that Learner may ignore the correct relation between different hypotheses.

- Links removed represent wrong inferences between lemmas. So, whether or not a link deleted occurs in Learner's current proof (i.e., the current process) is important. If the proof includes a mistake, any inference after the mistake should not make sense. However, if a potential transition having not occurred in the proof is wrong, Learner can continue with her current proof. Clearly, SG cannot distinguish between these two cases.
- The game does not distinguish between all the various ways Learner can reach the goal. That is, as long as Learner has come to the right conclusion, the game cannot tell us whether Learner has come to this conclusion in a coherent way. Reaching the correct hypothesis by wrong transitions is not reliable. The well-known Gettier cases [14] where one has justified true belief, but not knowledge are also examples of situations in which one wrongly reaches the right conclusion. Thus, the theory developed in [18] is subject to the Gettier problems.

In this paper, we therefore propose a new framework, called *supervised learning games*. It differs from SG on several accounts, motivated by the mentioned restrictions. Unlike in SG, Teacher in SLG now has four actions to choose from: (a) do nothing, (b) add an edge, (c) delete an edge that is not currently in Learner's path or (d) delete an edge in Learner's path and thereby move Learner back to the node before that edge. Before its definition, let us define some auxiliary notions.

Let $S = \langle w_0, w_1, \cdots, w_n\rangle$ be a non-empty, finite sequence. We use $e(S)$ to denote its last element, and $S; v$ the sequence extending S with v. Define $Set(S) := \{\langle w_0, w_1\rangle, \langle w_1, w_2\rangle, \cdots, \langle w_{n-1}, w_n\rangle\}$. When S is a singleton, $Set(S) := \emptyset$. Also, for any $\langle w_i, w_{i+1}\rangle \in Set(S)$, $S|_{\langle w_i,w_{i+1}\rangle} := \langle w_0, w_1, \cdots, w_u\rangle$, where $\langle w_u, w_{u+1}\rangle = \langle w_i, w_{i+1}\rangle$ and $\langle w_u, w_{u+1}\rangle \neq \langle w_j, w_{j+1}\rangle$ for any $j < i$. Intuitively, $S|_{\langle w_i,w_{i+1}\rangle}$ is obtained by deleting all elements occurring after w_u from S, where $\langle w_u, w_{u+1}\rangle$ is the first occurrence of $\langle w_i, w_{i+1}\rangle$ in S. Say, when $S = \langle a, b, c, a, b\rangle$, we have $S|_{\langle a,b\rangle} = \langle a\rangle$. Now let us introduce SLG.

Definition 1 A **supervised learning game (SLG)** $\langle W, R_L, R_T, \langle s\rangle, g\rangle$ is given by a graph $\langle W, R_L, R_T\rangle$, the starting node s and the goal node g. A position of the game is a tuple $\langle R^i_L, S^i\rangle$. The initial position $\langle R^0_L, S^0\rangle$ is given by $\langle R_L, \langle s\rangle\rangle$. Round $n + 1$ from position $\langle R^n_L, S^n\rangle$ is as follows: first, Learner moves from $e(S^n)$ to any of its R_L-successors s'; then Teacher does nothing or acts out one of the following three choices:

(a). Extend R^n_L with some $\langle v, v'\rangle \in R_T$;
(b). Transfer $S^n; s'$ to $(S^n; s')|_{\langle v,v'\rangle}$ by deleting $\langle v, v'\rangle$ from $Set(S^n; s') \setminus R_T$;
(c). Delete some $\langle v, v'\rangle \in (R^n_L \setminus R_T) \setminus Set(S^n; s')$ from R^n_L.

The new position, denoted $\langle R^{n+1}_L, S^{n+1}\rangle$, is $\langle R^n_L, S^n\rangle$ (when Teacher does nothing), $\langle R^n_L \cup \{\langle v, v'\rangle\}, S^n; s'\rangle$ (when he chooses (a)), $\langle R^n_L \setminus \{\langle v, v'\rangle\}, (S^n; s')|_{\langle v,v'\rangle}\rangle$ (if he acts as in (b)), or $\langle R^n_L \setminus \{\langle v, v'\rangle\}, S^n; s'\rangle$ (if he chooses (c)). It ends if Learner arrives at g through an R_T-path $\langle s, \cdots, g\rangle$ or cannot make a move, with both players winning in the former case and losing in the latter.

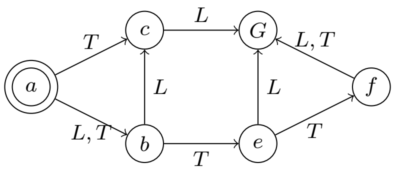

Fig. 1 A SLG game. In this graph, R_L is labelled with 'L' and R_T with 'T'. The starting node is a and the goal node is G. We show that players have a winning strategy by depicting the game to play out as follows. Learner begins by moving along the only available edge to node b. Teacher in his turn can make $\langle e, f\rangle$ 'visible' to Learner by adding it to R_L. Then, Learner proceeds to move along $\langle b, c\rangle$, and Teacher extends $\langle b, e\rangle$ to R_L. Afterwards, Learner continues on the only option $\langle c, G\rangle$. Although she now has already arrived at the goal node, her path $\langle a, b, c, G\rangle$ is not an R_T-sequence. So, Teacher can remove $\langle b, c\rangle$ moving Learner back to node b. Next, Learner has to move to e, and Teacher can delete $\langle e, G\rangle$ from R_L. Finally, Learner can arrive at G in 2 steps with Teacher doing nothing. Now we have $Set(\langle a, b, e, f, G\rangle) \subseteq R_T$. So, they win.

Intuitively, the clause for Learner illustrates that she cannot distinguish the links starting from the current position. The sequence S^i is her current learning process, which may include mistakes; R_L represents Learner's possible inferences; and R_T is the correct inferences. For any position $\langle R_L^n, S^n\rangle$, $Set(S^n) \subseteq R_L^n$. Besides, (b) and (c) focus on the case where Teacher eliminates wrong transitions, but there is an important difference. Action (b) concerns the case where Teacher gives Learner a counterexample to show that she has gone wrong somewhere in her current process, so Learner should move back to the conjecture right before the wrong transition. In contrast, (c) illustrates that Teacher eliminates a wrong transition conjectured having not occurred in Learner's process yet, therefore it does not modify Learner's current process.

From the winning condition, we know that both the players cooperate with each other. It is important to recognize that Learner's action does not conflict with her cooperative nature: to achieve the goal, she tries to move in each round. For an example of SLG, see Figure 1. The correlation between the situation of theorem proving and SLG is shown in Table 1.

Remark 1 The interpretation of SLG in Table 1 can be easily adapted to characterize other paradigms in formal learning theory, such as language learning and scientific inquiry. More generally, any single-agent games, such as solitaire and computer games, can be converted into SLG. Say, the player (Learner) does not know the correct moves well, but she knows the starting position and the goal position, and has some conjectures about the moves of the game. Besides, she can be taught by Teacher: she just attempts to play it, while Teacher instructs her positively (by revealing more correct moves) or negatively (by pointing out incorrect moves, in which case Learner may have to be moved back to the moment previous to the first incorrect move, if she made any).

Finally, we end this part by a preliminary comparison of SLG and SG.

First, note that the players in a SG can win only if the graph contains a sequence of edges from the starting node to the goal. Similarly, in a SLG,

Table 1 Correspondence between theorem proving and supervised learning games.

Theorem Proving	**Supervised Learning Games**
Axioms	Starting node
Theorem	Goal node
Lemmas conjectured by Learner	Other states except the starting state and the goal state
Learner's possible inference from a to b	R_L-edge from a to b
Correct inference from a to b	R_T-edge from a to b
Inferring b from a	Transition from a to b
Proof for a	R_L-sequence from the starting node to a
Correct proof for a	R_L-sequence S from the starting node to a and $Set(S) \subseteq R_T$
Giving a counterexample to the inference from a to b in the proof S	Modifying S to $S\|_{\langle a,b\rangle}$ ($\langle a,b\rangle \in Set(S)$)
Giving a counterexample to the conjectured inference from a to b not in the proof S	Deleting $\langle a,b\rangle$ from R_L ($\langle a,b\rangle \notin Set(S)$)
Pointing out a potential inference from a to b not conjectured by Learner before	Extending R_L with $\langle a,b\rangle$

players cannot win when there exists no R_T-path from the starting node to the goal node. From the perspective of learning, both these two conditions are reasonable: the interaction between Learner and Teacher makes sense only when the goal is learnable. However, it is important to recognize that in both SG and SLG, the existence of such a path cannot guarantee their winning.

Also, there are several notable differences between SG and SLG. In a SG, Learner knows the underlying graph well, and is always on one of the paths with which she can finally arrive at the goal (if they exist). Therefore, she has the ability to move to a suitable node in the next round. In contrast, the player in a SLG does not have this ability: all R_L-links starting from her current position look 'the same' from her perspective, and she is not able to guarantee that her movements are always the good ones (even though sometimes she may move to some 'good' nodes by chance). As to Teacher, compared with that of SG, the player in SLG is more powerful: he now is not only able to remove links, but also able to add new edges to the graph. However, from another aspect, the ability of Teacher in SLG is more restrictive as well: he can only delete the wrong translations from the graph.

An interesting issue worth studying is the precise relationship between SG and SLG. One observation involving this is as follows. Given a SG including a path with which the players can win, we can build a SLG by labelling the links of the path with both 'L' and 'T' and all others in the initial graph with 'T' only. In the SLG constructed, with the same path as that of the initial SG, the players can also win: Learner just moves (in each round there exists only one R_L-successor of her current position), and Teacher does not need to do anything. The observation is restrictive, but it seems there does not exist an obvious way to encode SG into SLG generally. We leave this for future work.

In the rest of the article, we will study SLG from a modal perspective, to reason about players' strategies in the game. *Sabotage modal logic (SML)* [6, 2–4] is known to be a suitable tool to characterize SG, which extends the basic modal logic with a sabotage modality $\langle - \rangle\varphi$ stating that there is an edge such that, φ is true at the evaluation node after deleting the edge from the model. However, given the differences between SG and SLG, we will develop a novel modal *Logic of Supervised Learning (LSL)* to capture SLG.

Outline. Section 2 introduces LSL along with its application to SLG and some preliminary observations. Section 3 studies the expressivity of LSL. Section 4 investigates the model checking problem and satisfiability problem for LSL. We end this paper by Section 5 on conclusion and future work.

2 The Logic of Supervised Learning (LSL)

In this section, we introduce the language and semantics of LSL, and analyze its applications to SLG. Also, we make various observations, including some logical validities and relations between LSL and other existing logics.

2.1 Language and Semantics

We begin by considering the actions of Learner. In SML, the standard modality $\Diamond$ characterizes the transition from a node to its successors and corresponds well to Learner's actions in SG. However, operator $\Diamond$ is not any longer sufficient in our case. Note that after Teacher cuts a link $\langle w, v\rangle$ from Learner's current process S, Learner should start from w with the new path $S|_{\langle w,v\rangle}$ in the next round. Therefore, the desired operator should remember the history of Learner's movements, similar to the case of *memory logics* [5].

To capture Teacher's actions, a natural place to start is by defining operators corresponding to link addition and deletion. There is already a body of literature on logics of these modalities, such as the sabotage operator $\langle - \rangle$ and the bridge operator $\langle + \rangle$ [2–4]. As mentioned, each occurrence of $\langle - \rangle$ in a formula deletes exactly one link, whereas the bridge operator *adds* links stepwise to models. Yet, including these two modalities is still not enough: we need to take into account if or not a link deleted by Teacher occurs in the path of Learner's movements. We now introduce the language $\mathcal{L}$ of LSL.

Definition 2 Let $\mathbf{P}$ be a countable set of propositional atoms. The **language** $\mathcal{L}$ is recursively defined in the following way:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \blacklozenge\varphi \mid \langle - \rangle_{on}\varphi \mid \langle - \rangle_{off}\varphi \mid \langle + \rangle\varphi$$

where $p \in \mathbf{P}$. Notions $\top$, $\bot$, $\vee$ and $\to$ are as usual. Also, we use $\blacksquare$, $[-]_{on}$, $[-]_{off}$ and $[+]$ to denote the dual operators of $\blacklozenge$, $\langle - \rangle_{on}$, $\langle - \rangle_{off}$ and $\langle + \rangle$ respectively.

Intuitively, $\blacklozenge\varphi$ states that φ holds after extending the current path with one of its successors. $\langle - \rangle_{on}\varphi$ reads φ is the case after deleting a link *on* the current path, while $\langle - \rangle_{off}\varphi$ states that after removing a link that is not on the path, φ holds. We use different subscripts '*on*' and '*off*' to indicate the two situations. Instead of link deletion, $\langle + \rangle\varphi$ shows that after extending the model with a particular link, φ holds. Roughly, operator $\blacklozenge$ is used to capture the actions of Learner in SLG, and operators $\langle + \rangle$, $\langle - \rangle_{on}$ and $\langle - \rangle_{off}$ characterize those of Teacher. This will become clear after we introduce the semantics.

Several fragments of $\mathcal{L}$ will be studied in the article. For brevity, we use a notational convention listing in subscript all modalities of the corresponding language. For instance, $\mathcal{L}_{\blacklozenge}$ is the fragment of $\mathcal{L}$ that has only the operator $\blacklozenge$ (besides Boolean connectives $\neg$ and $\wedge$); $\mathcal{L}_{\langle - \rangle_{off}}$ has only the modality $\langle - \rangle_{off}$; $\mathcal{L}_{\blacklozenge\langle - \rangle_{on}}$ has only $\blacklozenge$ and $\langle - \rangle_{on}$, etc. We now proceed to define the models.

Definition 3 A **model** of LSL is a tuple $\mathcal{M} = \langle W, R_L, R_T, V \rangle$, where W is a non-empty set of possible worlds, $R_{i \in \{L,T\}} \subseteq W^2$ are two binary relations and $V : \mathbf{P} \to 2^W$ is a valuation function. $\mathcal{F} = \langle W, R_L, R_T \rangle$ is a **frame**. Let S be an R_L-sequence, i.e., $Set(S) \subseteq R_L$. We name $\langle \mathcal{M}, S \rangle$ a **pointed model**, and S an **evaluation sequence**.

For brevity, usually we write $\mathcal{M}, S$ instead of $\langle \mathcal{M}, S \rangle$. Also, we use $\mathfrak{M}$ to denote the class of pointed models and $\mathfrak{M}^\bullet$ the class of pointed models whose sequence S is a singleton. Let $\mathcal{M} = \langle W, R_L, R_T, V \rangle$ be a model, $w \in W$ and $i \in \{L, T\}$. We use $R_i(w) := \{v \in W \mid R_i wv\}$ to denote the set of R_i-successors of w in $\mathcal{M}$. For any sequence S, define $R_i(S) := R_i(e(S))$, i.e., the R_i-successors of a sequence S are exactly the R_i-successors of its last element. Moreover, $\mathcal{M} \ominus \langle u, v \rangle := \langle W, R_L \setminus \{\langle u, v \rangle\}, R_T, V \rangle$ is the model obtained by removing $\langle u, v \rangle$ from R_L, and $\mathcal{M} \oplus \langle u, v \rangle := \langle W, R_L \cup \{\langle u, v \rangle\}, R_T, V \rangle$ is obtained by extending R_L in $\mathcal{M}$ with $\langle u, v \rangle$. Now let us introduce the semantics of LSL.

Definition 4 Let $\langle \mathcal{M}, S \rangle$ be a pointed model and $\varphi \in \mathcal{L}$. The **semantics** of LSL is defined as follows:

$$
\begin{array}{rll}
\mathcal{M}, S \vDash p & \text{iff} & e(S) \in V(p) \\
\mathcal{M}, S \vDash \neg\varphi & \text{iff} & \mathcal{M}, S \nvDash \varphi \\
\mathcal{M}, S \vDash \varphi \wedge \psi & \text{iff} & \mathcal{M}, S \vDash \varphi \text{ and } \mathcal{M}, S \vDash \psi \\
\mathcal{M}, S \vDash \blacklozenge\varphi & \text{iff} & \exists v \in R_L(S) \text{ s.t. } \mathcal{M}, S; v \vDash \varphi \\
\mathcal{M}, S \vDash \langle - \rangle_{on}\varphi & \text{iff} & \exists \langle v, v' \rangle \in Set(S) \setminus R_T \text{ s.t. } \mathcal{M} \ominus \langle v, v' \rangle, S|_{\langle v, v' \rangle} \vDash \varphi \\
\mathcal{M}, S \vDash \langle - \rangle_{off}\varphi & \text{iff} & \exists \langle v, v' \rangle \in (R_L \setminus R_T) \setminus Set(S) \text{ s.t. } \mathcal{M} \ominus \langle v, v' \rangle, S \vDash \varphi \\
\mathcal{M}, S \vDash \langle + \rangle\varphi & \text{iff} & \exists \langle v, v' \rangle \in R_T \setminus R_L \text{ s.t. } \mathcal{M} \oplus \langle v, v' \rangle, S \vDash \varphi
\end{array}
$$

Both the truth conditions for $\langle - \rangle_{on}$ and $\langle - \rangle_{off}$ show that links deleted cannot be R_T-edges. Intuitively, whereas $\langle - \rangle_{on}$ depicts the case when Teacher deletes a link from Learner's path S, $\langle - \rangle_{off}$ captures the situation where the link deleted does not occur in S. Finally, $\langle + \rangle\varphi$ means that after extending R_L with a link of R_T, φ holds at the current sequence.

A formula φ is **satisfiable** if there exists $\langle \mathcal{M}, S \rangle \in \mathfrak{M}$ with $\mathcal{M}, S \vDash \varphi$. Also, **validity** in a model and in a frame is defined as usual. Note that the relevant class of pointed models to specify LSL is $\mathfrak{M}^\bullet$. Hence LSL is the set of $\mathcal{L}$-formulas that are valid w.r.t. $\mathfrak{M}^\bullet$.

For any $\langle \mathcal{M}, S \rangle$ and $\langle \mathcal{M}', S' \rangle$, we say that they are **learning modal equivalent** (notation: $\langle \mathcal{M}, S \rangle \leftrightsquigarrow_l \langle \mathcal{M}', S' \rangle$) iff $\mathcal{M}, S \vDash \varphi \Leftrightarrow \mathcal{M}', S' \vDash \varphi$ for any formulas $\varphi \in \mathcal{L}$. Besides, define $\mathbb{T}^l(\mathcal{M}, S) := \{\varphi \in \mathcal{L} \mid \mathcal{M}, S \vDash \varphi\}$, denoting the **LSL theory** of S in $\mathcal{M}$. It is easy to see that two pointed models are learning modal equivalent if, and only if, they have the same LSL theory. In addition, we define a relation $\mathbf{U} \subseteq \mathfrak{M} \times \mathfrak{M}$ with $\langle\langle\mathcal{M}, S\rangle, \langle\mathcal{M}', S'\rangle\rangle \in \mathbf{U}$ iff $\langle\mathcal{M}', S'\rangle$ is $\langle\mathcal{M}, S; w\rangle$ for some state w with $\langle e(S), w\rangle \in R_L$, $\langle\mathcal{M}\ominus\langle v, v'\rangle, S|_{\langle v,v'\rangle}\rangle$ for some $\langle v, v'\rangle \in Set(S) \setminus R_T$, $\langle\mathcal{M} \ominus \langle v, v'\rangle, S\rangle$ for some $\langle v, v'\rangle \in (R_L \setminus R_T) \setminus Set(S)$, or $\langle\mathcal{M} \oplus \langle v, v'\rangle, S\rangle$ for some $\langle v, v'\rangle \in R_T \setminus R_L$. Furthermore, we can also iterate this order, to talk about models reachable in finitely many $\mathbf{U}$-steps, obtaining a new relation $\mathbf{U}^*$.

2.2 Application: Winning Strategies in SLG

By Definition 4, language $\mathcal{L}$ is able to capture the actions of both players in SLG. Also, our logic is expressive enough to describe the winning strategy (if there is one) for players in finite graphs.[1]

Given a finite SLG, let p be a distinguished atom holding only at the goal node. Generally, the winning strategy of Learner and Teacher can be described by formulas of the following form:

$$\blacksquare \bigcirc_0 \blacksquare \bigcirc_1 \blacksquare \cdots \bigcirc_n \blacksquare(p \wedge [-]_{on}\bot) \tag{1}$$

where $\bigcirc_i$ is blank or one of $\langle-\rangle_{on}$, $\langle-\rangle_{\mathit{off}}$ and $\langle+\rangle$, for each $i \leq n$. In (1), the recurring $\blacksquare$ operator depicts Learner's actions and $\bigcirc_i$ Teacher's response. The proposition p signalizes Learner's arrival at the goal, and $[-]_{on}\bot$ states that there are no edges in Learner's path that Teacher can cut. Hence, we conclude that Learner has reached the goal in a coherent way. Recall the example of SLG in Figure 1. Formula $\blacksquare\langle+\rangle\blacksquare\langle+\rangle\blacksquare\langle-\rangle_{on}\blacksquare\langle-\rangle_{\mathit{off}}\blacksquare\blacksquare(p \wedge [-]_{on}\bot)$ holds at the starting node a. Therefore, there exists a winning strategy in this specific SLG.

It is worthwhile to emphasize that in formula (1) we use $\blacksquare$, other than $\blacklozenge$, to characterize the actions of Learner, which may be different from some other cases.[2] However, the modality $\blacksquare$ used in formula (1) does not indicate that Learner is unwilling to learn. Essentially, it illustrates that she has no idea where to move in the next step, and we would claim that the form is in line with the spirit of SLG where Learner may move in wrong directions: Learner cannot distinguish different ways to the goal. In effect, all Learner can do in a SLG is to move as much as possible. Meanwhile, Teacher has to make some correct inferences 'visible' to Learner, and put Learner on track no matter what happens. Therefore, the form of formula (1) does not violate the cooperative nature of Learner.[3]

[1] Generally speaking, to define the existence of winning strategies for players, we need to extend SLG with some fixpoint operators. We leave this for future inquiry.

[2] For instance, in sabotage games, we use $\Diamond$ to capture actions of Learner in formulas describing winning strategies (if they exist). See [18].

[3] In contrast, one extreme case of non-cooperative variants of SLG might be that Learner is allowed to stay at her current position in each round: she makes no efforts to reach the goal node.

Remark 2 In SG we know that links cut by Teacher represent wrong inferences. However, SG does not tell us anything about the links that remain in the graph. Therefore, winning strategies of the players in SG cannot guarantee against situations like Gettier cases. In contrast, the formula $[-]_{on}\bot$ in (1) ensures that Teacher is not allowed to remove any more links from Learner's path. In SLG, a Gettier-style case is that Learner arrives at the goal node with some $\langle u, v\rangle \in R_L \setminus R_T$ occurring in her path, so Teacher now would be able to cut those links. Therefore Gettier cases cannot be winning strategies in SLG.

2.3 Preliminary Observations

In this section, we make some preliminary observations on LSL. In particular, we discuss the relations between LSL and other related logics, present some logical validities, and study some basic features of LSL. Let us begin with the relation between $\mathcal{L}_{\blacklozenge}$ and the standard modal logic.

Proposition 1 *Let $\mathcal{M} = \langle W, R_L, R_T, V\rangle$ be a model. For any $\langle \mathcal{M}, S\rangle \in \mathfrak{M}$ and $\varphi \in \mathcal{L}_{\blacklozenge}$, $\mathcal{M}, S \vDash \varphi \Leftrightarrow \langle W, R_L, V\rangle, e(S) \vDash \varphi^*$, where φ^* is a standard modal formula obtained by replacing every occurrence of $\blacklozenge$ in φ with $\Diamond$.*

Therefore, essentially the fragment $\mathcal{L}_{\blacklozenge}$ of $\mathcal{L}$ is the standard modal logic. Moreover, the operator $\langle -\rangle_{\mathit{off}}$ is much similar to the sabotage operator $\langle -\rangle$:

Proposition 2 *Let $\mathcal{M} = \langle W, R_L, R_T, V\rangle$ be a model, and $R = R_L \setminus R_T$. For any $\langle \mathcal{M}, w\rangle \in \mathfrak{M}^\bullet$ and $\varphi \in \mathcal{L}_{\langle -\rangle_{\mathit{off}}}$, $\mathcal{M}, w \vDash \varphi \Leftrightarrow \langle W, R, V\rangle, w \vDash \varphi'$, where φ' is a SML formula obtained by replacing each occurrence of $\langle -\rangle_{\mathit{off}}$ in φ with $\langle -\rangle$.*

Next, the following result captures the relation between $\mathcal{L}_{\blacklozenge\langle +\rangle}$ and the '*bridge modal logic (BML)*' (i.e., the modal logic extending the standard modal logic with the bridge operator):

Proposition 3 *Let $\mathcal{M} = \langle W, R_L, W^2, V\rangle$ be a model. For any $\langle \mathcal{M}, S\rangle \in \mathfrak{M}$ and $\varphi \in \mathcal{L}_{\blacklozenge\langle +\rangle}$, $\mathcal{M}, S \vDash \varphi \Leftrightarrow \langle W, R_L, V\rangle, e(S) \vDash \varphi^\star$, where $\varphi^\star$ is a BML formula obtained by replacing every occurrence of $\blacklozenge$ in φ with $\Diamond$.*[4]

Proposition 1-3 can be proved by a standard induction on formulas. From these results, we know that several fragments of LSL are similar to some existing logics. Yet, as a whole, different operators of LSL interact with each other. For instance, for any $\langle \mathcal{M}, w\rangle \in \mathfrak{M}^\bullet$, formula $[-]_{on}\varphi$ is valid, as $Set(w) = \emptyset$. However, $\blacklozenge\neg[-]_{on}\varphi$ is satisfiable. This presents a drastic difference between LSL and other logics mentioned so far: in those logics, it is impossible that the evaluation point has access to a node satisfying a contradiction. To understand how operators in LSL work, we present some other validities.

[4] By abuse of notation, for any $\varphi \in \mathcal{L}_{\blacklozenge\langle +\rangle}$, $\varphi^\star$ is a formula of the bridge modal logic.

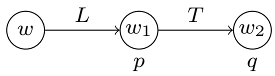

Fig. 2 A case showing that validities of $\mathcal{L}_{\blacklozenge\langle -\rangle_{on}}$ are not closed under substitution. Consider the general schema $\varphi \wedge \blacklozenge\psi \to \blacksquare[-]_{on}\varphi$ of formula (2). Let $\varphi := \blacklozenge p$ and $\psi := \blacksquare q$. It holds that $\mathcal{M}, w \vDash \blacklozenge p \wedge \blacklozenge\blacksquare q$. But, since w has exactly one R_L-successor w_1 and $\langle w, w_1\rangle \notin R_T$, we have $\mathcal{M}, w \nvDash \blacksquare[-]_{on}\blacklozenge p$.

Proposition 4 *Let $p \in \mathbf{P}$ and $\varphi, \psi \in \mathcal{L}$. The following formulas are validities of LSL (w.r.t. $\mathfrak{M}^\bullet$):*

$$p \wedge \blacklozenge\top \to \blacksquare[-]_{on}p \tag{2}$$

$$\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi) \qquad \bigcirc \in \{[-]_{off}, [+]\} \tag{3}$$

$$\blacksquare^n[-]_{on}(\varphi \to \psi) \to (\blacksquare^n[-]_{on}\varphi \to \blacksquare^n[-]_{on}\psi) \qquad n \in \mathbb{N} \tag{4}$$

$$\blacklozenge^n\langle -\rangle_{on}\varphi \to \bigvee_{m<n} \blacklozenge^m\langle -\rangle_{off}\varphi \qquad 1 \leq n \in \mathbb{N} \tag{5}$$

Their validity holds immediately by the semantics. Formula (2) states that, for any singleton w, if it is p and has some R_L-successors, then any of its extensions $\langle w, v\rangle$ with $v \in R_L(w)$ is $[-]_{on}p$, no matter whether $\langle w, v\rangle \in R_T$ or not. Principles (3) and (4) show that all operators $[-]_{off}$, $[+]$ and $[-]_{on}$ are normal operators. Formula (5) illustrates that in some situations, a formula containing $\langle -\rangle_{on}$ can be reduced to another formula containing $\langle -\rangle_{off}$.

Note that principle (2) is not a schema. Although it will still be valid if we replace propositional atoms occurring in it with any other Boolean formulas, substitution fails generally. See Figure 2 for an example, which essentially illustrates the following result:

Proposition 5 *$\mathcal{L}_{\blacklozenge\langle -\rangle_{on}}$ and LSL are not closed under substitution.*

Moreover, LSL and $\mathcal{L}_{\blacklozenge\langle -\rangle_{on}}$ also have other features very different from the standard modal logic. For instance,

Proposition 6 *Both $\mathcal{L}_{\blacklozenge\langle -\rangle_{on}}$ and LSL lack the tree model property.*

Proof Let $\varphi_{\mathbb{T}}$ be the conjunction of $p \wedge \blacklozenge p \wedge \blacklozenge\neg p$ ($\mathbb{T}_1$), $\blacksquare(p \to \blacklozenge p \wedge \blacklozenge\neg p)$ ($\mathbb{T}_2$) and $\blacksquare(\neg p \to \langle -\rangle_{on}(\blacksquare p \wedge \blacksquare\blacksquare p))$ ($\mathbb{T}_3$). Clearly, $\varphi_{\mathbb{T}} \in \mathcal{L}_{\blacklozenge\langle -\rangle_{on}}$ is satisfiable w.r.t. $\mathfrak{M}^\bullet$ (see Figure 3). We now show that, for any $\mathcal{M} = \langle W, R_L, R_T, V\rangle$ and $w \in W$, $\mathcal{M}, w \vDash \varphi_{\mathbb{T}}$ entails $R_L ww$. By ($\mathbb{T}_1$), it follows that $w \in V(p)$, and it can reach some $w_1 \in V(p)$ and some $w_2 \notin V(p)$ via R_L. Besides, ($\mathbb{T}_2$) states that, via R_L, each such w_1 can also reach some p-node w_3 and $\neg p$-node w_4. Finally, from ($\mathbb{T}_3$) we know that w can only reach one $\neg p$-point by R_L and that w_1 does not have $\neg p$-successors via R_L any longer after cutting $\langle w, w_2\rangle$. So, $\langle w, w_2\rangle = \langle w_1, w_4\rangle$. Therefore, $R_L ww$. □

As observed, many instances of validities in our logic are not straightforward, and LSL has some distinguishing features. In the sections to come we will make a deeper investigation into our logic.

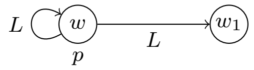

Fig. 3 A model of $\varphi_{\mathbb{T}}$. It is not hard to see that $\varphi_{\mathbb{T}}$ is true at w.

3 Expressive Power of LSL

In this section, we study the expressivity of LSL. First, we will show that LSL is still a fragment of FOL even though it looks complicated. After this, a suitable notion of bisimulation for LSL is introduced. Finally, we provide a van Benthem style characterization theorem for the logic.

3.1 First-order Translation

Given the complicated semantics, is LSL still a fragment of FOL? In this part we will provide a positive answer to this question, by describing a translation from LSL to FOL. However, compared with that for the standard modal logic [8], we now need some new devices.

Let $\mathcal{L}_1$ be the first-order language consisting of countable unary predicates $P_{i\in\mathbb{N}}$, two binary relations $R_{i\in\{L,T\}}$, and equality $\equiv$. Take any finite, non-empty sequence E of variables. Let y, y' be two fresh variables not appearing in E. When there exists $\langle x, x'\rangle \in Set(E)$ with $x \equiv y$ and $x' \equiv y'$, we define $E|_{\langle y,y'\rangle} := E|_{\langle x,x'\rangle}$. Now let us define the first-order translation.

Definition 5 Let $E = \langle x_0, x_1, \cdots, x_n\rangle$ be a finite sequence (non-empty) of variables without any variable appearing more than once, and E^- and E^+ two finite sets (maybe empty) of ordered pairs of variables. The **first-order translation** $\mathfrak{T}(\varphi, E, E^+, E^-)$ from $\varphi \in \mathcal{L}$ to first-order formulas is as follows:

$$
\begin{aligned}
\mathfrak{T}(p, E, E^+, E^-) &= Pe(E) \qquad \mathfrak{T}(\neg\varphi, E, E^+, E^-) = \neg\mathfrak{T}(\varphi, E, E^+, E^-)\\
\mathfrak{T}(\varphi\wedge\psi, E, E^+, E^-) &= \mathfrak{T}(\varphi, E, E^+, E^-) \wedge \mathfrak{T}(\psi, E, E^+, E^-)\\
\mathfrak{T}(\blacklozenge\varphi, E, E^+, E^-) &= \exists y\Big(\big(\bigvee_{\langle x,x'\rangle\in E^+}(e(E)\equiv x\wedge y\equiv x')\vee(R_L e(E)y\wedge\\
&\qquad \neg\bigvee_{\langle v,v'\rangle\in E^-}(e(E)\equiv v\wedge y\equiv v')\big)\big)\wedge\mathfrak{T}(\varphi, E; y, E^+, E^-)\Big)\\
\mathfrak{T}(\langle -\rangle_{on}\varphi, E, E^+, E^-) &= \exists y\exists y'\Big(\bigvee_{\langle x,x'\rangle\in Set(E)\setminus(E^-\cup E^+)}(y\equiv x\wedge y'\equiv x')\wedge\\
&\qquad R_L yy'\wedge\neg R_T yy'\wedge\mathfrak{T}(\varphi, E|_{\langle y,y'\rangle}, E^+, E^-\cup\{\langle y,y'\rangle\})\Big)\\
\mathfrak{T}(\langle -\rangle_{off}\varphi, E, E^+, E^-) &= \exists y\exists y'\Big(\neg\bigvee_{\langle x,x'\rangle\in Set(E)\cup E^-\cup E^+}(y\equiv x\wedge y'\equiv x')\wedge\\
&\qquad R_L yy'\wedge\neg R_T yy'\wedge\mathfrak{T}(\varphi, E, E^+, E^-\cup\{\langle y,y'\rangle\})\Big)\\
\mathfrak{T}(\langle +\rangle\varphi, E, E^+, E^-) &= \exists y\exists y'\Big(\neg\bigvee_{\langle x,x'\rangle\in E^-\cup E^+}(y\equiv x\wedge y'\equiv x')\wedge\neg R_L yy'\wedge\\
&\qquad R_T yy'\wedge\mathfrak{T}(\varphi, E, E^+\cup\{\langle y,y'\rangle\}, E^-)\Big)
\end{aligned}
$$

where y, y' are variables having not been used yet in the translation. In addition, given a set Φ of $\mathcal{L}$-formulas, we denote by $\mathfrak{T}(\Phi, E, E^+, E^-)$ the set $\{\mathfrak{T}(\varphi, E, E^+, E^-) \mid \varphi \in \Phi\}$ of first-order translations of formulas in Φ.

From the perspective of SLG, E denotes Learner's process, and E^+, E^- represent links having already been added and deleted respectively. In any translation, E^+ and E^- may be extended. For any extensions $E^+ \cup X$ and $E^- \cup Y$, $X \cap Y = \emptyset$. This is in line with our semantics: links deleted are different from those added. Furthermore, unlike the standard modal logic, generally the translation does not yield a formula with only one free variable. But, it does so when setting E, E^+ and E^- to be a singleton, $\emptyset$ and $\emptyset$ respectively.

Note that in Definition 5, the sequence E includes no variable appearing more than once, and it is not hard to see that any modification of E in a translation still has this property. Specifically, this assumption is used to guarantee that assignments are well-defined. Let σ be an assignment, S a sequence of points in a model, and E a sequence of variables with the same size as S. In what follows, when writing $\sigma_{E:=S}$, we mean a new assignment that is the same as σ except assigning variables in E to the corresponding points in S. Since all variables in E appear only once, no variable in the sequence can be assigned to different points in S. With Definition 5, we have the following result:

Lemma 1 *Let $\mathfrak{T}(\varphi, E, E^+, E^-)$ be a translation with $E^+ \cap E^- = \emptyset$, and y, y' two fresh variables. For any assignment σ and model $\mathcal{M}$, we have $\mathcal{M} \ominus \langle v, v' \rangle \vDash \mathfrak{T}(\varphi, E, E^+, E^-)[\sigma]$ iff $\mathcal{M} \vDash \mathfrak{T}(\varphi, E, E^+, E^- \cup \{\langle y, y' \rangle\})[\sigma_{y^{(\prime)}:=v^{(\prime)}}]$, for any $\langle v, v' \rangle \in R_L \setminus R_T$; and $\mathcal{M} \oplus \langle v, v' \rangle \vDash \mathfrak{T}(\varphi, E, E^+, E^-)[\sigma]$ iff $\mathcal{M} \vDash \mathfrak{T}(\varphi, E, E^+ \cup \{\langle y, y' \rangle\}, E^-)[\sigma_{y^{(\prime)}:=v^{(\prime)}}]$, for any $\langle v, v' \rangle \in R_T \setminus R_L$.*

Proof The proofs for these two cases are similar, and both of them can be shown by induction on the syntax of formulas. We focus on the first one, and only prove the cases for propositional atoms and $\langle - \rangle_{on}$. Assume that $\langle v, v' \rangle \in R_L \setminus R_T$, and $R_L^- := R_L \setminus \{\langle v, v' \rangle\}$.

(1). Formula φ is $p \in \mathbf{P}$. By Definition 5, $\mathcal{M} \ominus \langle v, v' \rangle \vDash \mathfrak{T}(\varphi, E, E^+, E^-)[\sigma]$ iff $\mathcal{M} \ominus \langle v, v' \rangle \vDash Pe(E)[\sigma]$. From the definition of $\mathcal{M} \ominus \langle v, v' \rangle$, it follows that $\mathcal{M} \ominus \langle v, v' \rangle \vDash Pe(E)[\sigma]$ iff $\mathcal{M} \vDash Pe(E)[\sigma]$. Again, by Definition 5, it holds that $\mathcal{M} \vDash Pe(E)[\sigma]$ iff $\mathcal{M} \vDash \mathfrak{T}(\varphi, E, E^+, E^- \cup \{\langle y, y' \rangle\})[\sigma_{y^{(\prime)}:=v^{(\prime)}}]$.

(2). Formula φ is $\langle - \rangle_{on}\psi$. We have the following equivalences:

$$\begin{aligned}
& \mathcal{M} \ominus \langle v, v' \rangle \vDash \mathfrak{T}(\varphi, E, E^+, E^-)[\sigma] \\
\Leftrightarrow\ & \mathcal{M} \ominus \langle v, v' \rangle \vDash \exists u \exists u' \Big(\bigvee_{\langle z, z' \rangle \in Set(E) \setminus (E^- \cup E^+)} (u \equiv z \wedge u' \equiv z') \wedge R_L^- uu' \wedge \\
& \qquad \neg R_T uu' \wedge \mathfrak{T}(\psi, E|_{\langle u, u' \rangle}, E^+, E^- \cup \{\langle u, u' \rangle\}) \Big)[\sigma] \\
\Leftrightarrow\ & \mathcal{M} \vDash \exists u \exists u' \Big(\bigvee_{\langle z, z' \rangle \in Set(E) \setminus (E^+ \cup E^- \cup \{\langle y, y' \rangle\})} (u \equiv z \wedge u' \equiv z') \wedge R_L uu' \wedge \\
& \qquad \neg R_T uu' \wedge \mathfrak{T}(\psi, E|_{\langle u, u' \rangle}, E^+, E^- \cup \{\langle u, u' \rangle, \langle y, y' \rangle\}) \Big)[\sigma_{y^{(\prime)}:=v^{(\prime)}}] \\
\Leftrightarrow\ & \mathcal{M} \vDash \mathfrak{T}(\varphi, E, E^+, E^- \cup \{\langle y, y' \rangle\})[\sigma_{y^{(\prime)}:=v^{(\prime)}}]
\end{aligned}$$

The first equivalence holds directly by Definition 5. By the inductive hypothesis and the definition of R_L^-, the second one holds. The last equivalence follows from the definition of first-order translation. The proof is completed. □

With Lemma 1, we now can show the correctness of the translation:

Theorem 1 *Let $\langle \mathcal{M}, S \rangle \in \mathfrak{M}$ and E an R_L-sequence of variables with the same size as S. For any $\varphi \in \mathcal{L}$, $\mathcal{M}, S \vDash \varphi$ iff $\mathcal{M} \vDash \mathfrak{T}(\varphi, E, \emptyset, \emptyset)[\sigma_{E:=S}]$.*

Proof The proof is by induction on the structure of φ. Also, we only consider the cases for propositional atoms and $\langle - \rangle_{on}$.

(1). Formula φ is $p \in \mathbf{P}$. By the semantics, $\mathcal{M}, S \vDash \varphi$ iff $e(S) \in V(p)$. On the other hand, by Definition 5, $\mathfrak{T}(\varphi, E, \emptyset, \emptyset)$ is $Pe(E)$. So we have $\mathcal{M}, S \vDash \varphi$ iff $\mathcal{M} \vDash \mathfrak{T}(\varphi, E, \emptyset, \emptyset)[\sigma_{E:=S}]$.

(2). When φ is $\langle - \rangle_{on}\psi$, the following equivalences hold:

$$
\begin{aligned}
& \mathcal{M}, S \vDash \varphi \\
\Leftrightarrow\ & \text{there exists } \langle v, v' \rangle \in (Set(S) \setminus R_T) \text{ s.t. } \mathcal{M} \ominus \langle v, v' \rangle, S|_{\langle v, v' \rangle} \vDash \psi \\
\Leftrightarrow\ & \text{there exists } \langle v, v' \rangle \in (Set(S) \setminus R_T) \text{ s.t.} \\
& \mathcal{M} \ominus \langle v, v' \rangle \vDash \mathfrak{T}(\psi, E|_{\langle y, y' \rangle}, \emptyset, \emptyset)[\sigma_{E:=S, y^{(\prime)}:=v^{(\prime)}}] \\
\Leftrightarrow\ & \mathcal{M} \vDash \exists y \exists y' (\bigvee_{\langle v, v' \rangle \in Set(E)} (y \equiv v \wedge y' \equiv v') \wedge R_L yy' \wedge \neg R_T yy' \wedge \\
& \quad \mathfrak{T}(\psi, E|_{\langle y, y' \rangle}, \emptyset, \{\langle y, y' \rangle\}))[\sigma_{E:=S}] \\
\Leftrightarrow\ & \mathcal{M} \vDash \mathfrak{T}(\varphi, E, \emptyset, \emptyset)[\sigma_{E:=S}]
\end{aligned}
$$

The first equivalence follows from our semantics immediately. By the inductive hypothesis, the second one follows. With Lemma 1, we have the third one. The last one follows directly from Definition 5. This completes the proof. □

In the result above, we have an extra requirement on the sequence E used in the translation, i.e., $Set(E) \subseteq R_L$. Intuitively, this restriction corresponds to the definition of pointed models. When S is a singleton, E is also a singleton, and each extension of E fulfils the requirement automatically by Definition 5.

So far, by the translation, we have shown that LSL is a fragment of FOL. Also, Definition 5 gives us other information about our logic. For example, it includes immediate transfer of the compactness property of FOL to LSL. Moreover, since the complexity of the model checking problem for FOL is PSPACE-complete [11] and the translation has only a polynomial size increase, we can obtain an upper bound for that of LSL. We will return to this below.

3.2 Bisimulation and Characterization for LSL

The notion of bisimulation serves as an important tool for measuring the expressive power of modal logics. However, LSL is not invariant under the standard bisimulation [8]. So, we introduce a novel notion of 'learning bisimulation (l-bisimulation)' tailored to our logic, which finally leads to a van Benthem style characterization theorem for LSL.

Definition 6 For any $\mathcal{M} = \langle W, R_L, R_T, V \rangle$ and $\mathcal{M}' = \langle W', R'_L, R'_T, V' \rangle$, a non-empty relation $Z_l \subseteq \mathbf{U}^*(\langle \mathcal{M}, S \rangle) \times \mathbf{U}^*(\langle \mathcal{M}', S' \rangle)$ is an **l-bisimulation** between $\langle \mathcal{M}, S \rangle$ and $\langle \mathcal{M}', S' \rangle$ (notation: $\langle \mathcal{M}, S \rangle Z_l \langle \mathcal{M}', S' \rangle$) if:

Atom: $\mathcal{M}, S \vDash p$ iff $\mathcal{M}', S' \vDash p$, for each $p \in \mathbf{P}$.

Zig$_{\blacklozenge}$: If there exists $v \in R_L(e(S))$, then there exists $v' \in R'_L(e(S'))$ such that $\langle \mathcal{M}, S; v\rangle Z_l \langle \mathcal{M}', S'; v'\rangle$.

Zig$_{\langle - \rangle_{on}}$: If there is $\langle u, v\rangle \in Set(S) \setminus R_T$, then there is $\langle u', v'\rangle \in Set(S') \setminus R'_T$ with $\langle \mathcal{M} \ominus \langle u, v\rangle, S|_{\langle u,v\rangle}\rangle Z_l \langle \mathcal{M}' \ominus \langle u', v'\rangle, S'|_{\langle u',v'\rangle}\rangle$.

Zig$_{\langle - \rangle_{off}}$: If there exists $\langle u, v\rangle \in (R_L \setminus R_T) \setminus Set(S)$, then there exists $\langle u', v'\rangle \in (R'_L \setminus R'_T) \setminus Set(S')$ with $\langle \mathcal{M} \ominus \langle u, v\rangle, S\rangle Z_l \langle \mathcal{M}' \ominus \langle u', v'\rangle, S'\rangle$.

Zig$_{\langle + \rangle}$: If there exists $\langle u, v\rangle \in R_T \setminus R_L$, then there exists $\langle u', v'\rangle \in R'_T \setminus R'_L$ with $\langle \mathcal{M} \oplus \langle u, v\rangle, S\rangle Z_l \langle \mathcal{M}' \oplus \langle u', v'\rangle, S'\rangle$.

Zag$_{\blacklozenge}$, **Zag$_{\langle - \rangle_{on}}$**, **Zag$_{\langle - \rangle_{off}}$** and **Zag$_{\langle + \rangle}$**: the analogous clauses in the converse direction of **Zig$_{\blacklozenge}$**, **Zig$_{\langle - \rangle_{on}}$**, **Zig$_{\langle - \rangle_{off}}$** and **Zig$_{\langle + \rangle}$** respectively.

For brevity, we write $\langle \mathcal{M}, S\rangle \underline{\leftrightarrow}_l \langle \mathcal{M}', S'\rangle$ if there is an l-bisimulation Z_l such that $\langle \mathcal{M}, S\rangle Z_l \langle \mathcal{M}', S'\rangle$.

The clauses for $\blacklozenge$ is similar to those for $\Diamond$ in the standard bisimulation: they keep the model fixed and extend the evaluation sequence with one of its R_L-successors. However, all conditions for $\langle - \rangle_{on}$, $\langle - \rangle_{off}$ and $\langle + \rangle$ change the model. In particular, clauses for $\langle - \rangle_{off}$ and $\langle + \rangle$ do not modify the evaluation sequence, while those for $\langle - \rangle_{on}$ change both the model and the current sequence. By a straightforward induction on $\varphi \in \mathcal{L}$, we have the following result:

Theorem 2 **($\underline{\leftrightarrow}_l \subseteq \leftrightsquigarrow_l$)** *For any pointed models $\langle \mathcal{M}, S\rangle$ and $\langle \mathcal{M}', S'\rangle$, it holds that: $\langle \mathcal{M}, S\rangle \underline{\leftrightarrow}_l \langle \mathcal{M}', S'\rangle \Rightarrow \langle \mathcal{M}, S\rangle \leftrightsquigarrow_l \langle \mathcal{M}', S'\rangle$.*

Moreover, the converse direction of Theorem 2 holds for the models that are ω-saturated. For each finite set Y, we denote the expansion of $\mathcal{L}_1$ with a set Y of constants with $\mathcal{L}_1^Y$, and denote the expansion of $\mathcal{M}$ to $\mathcal{L}_1^Y$ with $\mathcal{M}^Y$. Let $\mathbf{x}$ be a finite tuple of variables. A model $\mathcal{M} = \langle W, R_L, R_T, V\rangle$ is **ω-saturated** if, for every finite subset Y of W, the expansion $\mathcal{M}^Y$ realizes every set $\Gamma(\mathbf{x})$ of $\mathcal{L}_1^Y$-formulas whose finite subsets $\Gamma'(\mathbf{x})$ are all realized in $\mathcal{M}^Y$.

Theorem 3 **($\leftrightsquigarrow_l \subseteq \underline{\leftrightarrow}_l$)** *For any ω-saturated $\langle \mathcal{M}, S\rangle$ and $\langle \mathcal{M}', S'\rangle$, it holds that: $\langle \mathcal{M}, S\rangle \leftrightsquigarrow_l \langle \mathcal{M}', S'\rangle \Rightarrow \langle \mathcal{M}, S\rangle \underline{\leftrightarrow}_l \langle \mathcal{M}', S'\rangle$.*

Proof We show that $\leftrightsquigarrow_l$ itself is an l-bisimulation. Here we only prove the cases involving clauses **Zig$_{\blacklozenge}$** and **Zig$_{\langle - \rangle_{on}}$**. Let E' be a sequence of variables over R'_L with the same size as S'.

(1). Let $v \in R_L(S)$. We prove that there is some $v' \in R'_L(S')$ with $\langle \mathcal{M}, S; v\rangle \leftrightsquigarrow_l \langle \mathcal{M}', S'; v'\rangle$. For any finite $\Gamma \subseteq \mathbb{T}^l(\mathcal{M}, S; v)$, we have:

$$\begin{aligned} \mathcal{M}, S \vDash \blacklozenge \bigwedge \Gamma \; &\Leftrightarrow \; \mathcal{M}', S' \vDash \blacklozenge \bigwedge \Gamma \\ &\Leftrightarrow \; \mathcal{M}' \vDash \mathfrak{T}(\blacklozenge \bigwedge \Gamma, E', \emptyset, \emptyset)[\sigma_{E':=S'}] \\ &\Leftrightarrow \; \mathcal{M}' \vDash \exists y (R'_L e(E') y \wedge \mathfrak{T}(\bigwedge \Gamma, E'; y, \emptyset, \emptyset))[\sigma_{E':=S'}] \end{aligned}$$

As the pointed model $\langle \mathcal{M}', S'\rangle$ is ω-saturated, there exists $y \in R'_L(E')$ with $\mathcal{M}' \vDash \mathfrak{T}(\mathbb{T}^l(\mathcal{M}, S; v), E'; y, \emptyset, \emptyset)[\sigma_{E':=S'}]$. By Theorem 1, there is $v' \in R'_L(S')$ s.t. $\langle \mathcal{M}, S; v\rangle \leftrightsquigarrow_l \langle \mathcal{M}', S'; v'\rangle$. The proof of the **Zig$_{\blacklozenge}$** clause is completed.

(2). Let $\langle u, v\rangle \in Set(S) \setminus R_T$. We show that there exists $\langle u', v'\rangle \in Set(S') \setminus R'_T$ s.t. $\langle \mathcal{M} \ominus \langle u, v\rangle, S|_{\langle u,v\rangle}\rangle \leftrightsquigarrow_l \langle \mathcal{M}' \ominus \langle u', v'\rangle, S'|_{\langle u',v'\rangle}\rangle$. Let Γ be a finite subset of $\mathbb{T}^l(\mathcal{M} \ominus \langle u, v\rangle, S|_{\langle u,v\rangle})$, then the following equivalences hold:

$$\begin{aligned}\mathcal{M}, S \vDash \langle - \rangle_{on} \bigwedge \Gamma &\Leftrightarrow \mathcal{M}', S' \vDash \langle - \rangle_{on} \bigwedge \Gamma \\ &\Leftrightarrow \mathcal{M}' \vDash \mathfrak{T}(\langle - \rangle_{on} \bigwedge \Gamma, E', \emptyset, \emptyset)[\sigma_{E':=S'}] \\ &\Leftrightarrow \mathcal{M}' \vDash \exists y \exists z (\bigvee_{\langle x,x' \rangle \in Set(E')} (y \equiv x \wedge z \equiv x') \wedge \neg R'_T yz \wedge \\ &\qquad \mathfrak{T}(\bigwedge \Gamma, E'|_{\langle y,z \rangle}, \emptyset, \{\langle y,z \rangle\}))[\sigma_{E':=S'}]\end{aligned}$$

Since $\langle \mathcal{M}', S' \rangle$ is ω-saturated, there are y, z s.t. $\langle y, z \rangle \in Set(E') \setminus R'_T$ and $\mathcal{M}' \vDash \mathfrak{T}(\mathbb{T}^l(\mathcal{M} \ominus \langle u, v \rangle, S|_{\langle u,v \rangle}), E'|_{\langle y,z \rangle}, \emptyset, \{\langle y, z \rangle\}))[\sigma_{E':=S'}]$. W.l.o.g., assume $\sigma(y) = u'$ and $\sigma(z) = v'$. As $\langle u', v' \rangle \in Set(S') \setminus R'_T$, $\mathcal{M}' \ominus \langle u', v' \rangle \vDash \mathfrak{T}(\mathbb{T}^l(\mathcal{M} \ominus \langle u, v \rangle, S|_{\langle u,v \rangle}), E'|_{\langle y,z \rangle}, \emptyset, \emptyset))[\sigma_{E':=S'}]$ follows from Lemma 1. By Theorem 1, $\mathcal{M}' \ominus \langle u', v' \rangle, S'|_{\langle u',v' \rangle} \vDash \mathbb{T}^l(\mathcal{M} \ominus \langle u, v \rangle, S|_{\langle u,v \rangle})$. So, $\langle \mathcal{M} \ominus \langle u, v \rangle, S|_{\langle u,v \rangle} \rangle \leftrightsquigarrow_l \langle \mathcal{M}' \ominus \langle u', v' \rangle, S'|_{\langle u',v' \rangle} \rangle$. The proof of $\mathbf{Zig}_{\langle - \rangle_{on}}$ is completed. □

Thus we have established a match between learning modal equivalence and learning bisimulation for the ω-saturated models. Now, by a simple adaptation of standard arguments [8, 6, 25], we can show the following result:

Theorem 4 *For any $\alpha(x) \in \mathcal{L}_1$ with only one free variable, $\alpha(x)$ is equivalent to the translation of some $\varphi \in \mathcal{L}$ iff $\alpha(x)$ is invariant under l-bisimulation.*

Proof The direction from left to right holds by Theorem 2 directly. We now consider the other direction. Let $\alpha \in \mathcal{L}_1$ with only one free variable. Suppose that α is invariant under l-bisimulation. Define $\mathbb{C}_l(\alpha) := \{\mathfrak{T}(\varphi, x, \emptyset, \emptyset) \mid \varphi \in \mathcal{L} \text{ and } \alpha \vDash \mathfrak{T}(\varphi, x, \emptyset, \emptyset)\}$. Each formula of $\mathbb{C}_l(\alpha)$ has only one free variable x. We now show $\mathbb{C}_l(\alpha) \vDash \alpha$. Let $\langle \mathcal{M}, w \rangle \in \mathfrak{M}^\bullet$ such that $\mathcal{M} \vDash \mathbb{C}_l(\alpha)[\sigma_{x:=w}]$. First, we prove that $\Sigma = \mathfrak{T}((\mathbb{T}^l(\mathcal{M}, w), x, \emptyset, \emptyset)) \cup \{\alpha\}$ is consistent.

Suppose that Σ is not consistent. By the compactness of FOL, it holds that $\vDash \alpha \to \neg \bigwedge \Gamma$ for some finite $\Gamma \subseteq \mathfrak{T}(\mathbb{T}^l(\mathcal{M}, w), x, \emptyset, \emptyset)$. Then from the definition of $\mathbb{C}_l(\alpha)$, we know $\neg \bigwedge \Gamma \in \mathbb{C}_l(\alpha)$, which is followed by $\neg \bigwedge \Gamma \in \mathfrak{T}(\mathbb{T}^l(\mathcal{M}, w), x, \emptyset, \emptyset)$. However, it contradicts to $\Gamma \subseteq \mathfrak{T}(\mathbb{T}^l(\mathcal{M}, w), x, \emptyset, \emptyset)$.

Now we show $\mathcal{M} \vDash \alpha[\sigma_{x:=w}]$. Since Σ is consistent, there exists some $\langle \mathcal{M}', w' \rangle \in \mathfrak{M}^\bullet$ s.t. $\mathcal{M}' \vDash \Sigma[\sigma_{x:=w'}]$. Consequently, $\langle \mathcal{M}, w \rangle \leftrightsquigarrow_l \langle \mathcal{M}', w' \rangle$. Now take two ω-saturated elementary extensions $\langle \mathcal{M}_\omega, w \rangle$ and $\langle \mathcal{M}'_\omega, w' \rangle$ of $\langle \mathcal{M}, w \rangle$ and $\langle \mathcal{M}', w' \rangle$ respectively (such extensions always exist [12]). By the invariance of FOL under elementary extensions, $\mathcal{M}' \vDash \alpha[\sigma_{x:=w'}]$ entails $\mathcal{M}'_\omega \vDash \alpha[\sigma_{x:=w'}]$. Moreover, by Theorem 3 and the assumption that α is invariant for l-bisimulation, we have $\mathcal{M}_\omega \vDash \alpha[\sigma_{x:=w}]$. By the elementary extension, we obtain $\mathcal{M} \vDash \alpha[\sigma_{x:=w}]$. Therefore, $\mathbb{C}_l(\alpha) \vDash \alpha$.

Finally, we show that α is equivalent to the translation of an $\mathcal{L}$-formula. Since $\mathbb{C}_l(\alpha) \vDash \alpha$, by the compactness and deduction theorems of FOL it holds that $\vDash \bigwedge \Gamma \to \alpha$ for some finite $\Gamma \subseteq \mathbb{C}_l(\alpha)$. Besides, by the definition of $\mathbb{C}_l(\alpha)$, we have $\vDash \alpha \to \bigwedge \Gamma$. Thus, $\vDash \alpha \leftrightarrow \bigwedge \Gamma$. Now the proof is completed. □

Therefore, in terms of the expressivity, LSL is as powerful as the one free variable fragment of FOL that is invariant for l-bisimulation.

4 Model Checking and Satisfiability for LSL

In this section, we consider the model checking problem and satisfiability problem for LSL. In particular, we show that the model checking problems for both LSL and $\mathcal{L}_{\blacklozenge\langle+\rangle}$ are PSPACE-complete. Also, both LSL and $\mathcal{L}_{\blacklozenge\langle-\rangle_{on}}$ lack the finite model property, and their satisfiability problems are undecidable.

Theorem 5 *Model checking for LSL is PSPACE-complete.*

Proof As mentioned, an upper bound can be established by Definition 5, which suggests that model checking for LSL is in PSPACE. On the other hand, a lower bound can be provided by a reduction f from BML into $\mathcal{L}_{\blacklozenge\langle+\rangle}$. Precisely, f is the reverse of the translation used in Proposition 3. Clearly, f has a polynomial size increase. Let $\langle W, R_L, V\rangle$ be a standard relational model and $w \in W$. It holds that $\langle W, R_L, V\rangle, w \vDash \varphi$ iff $\langle W, R_L, W^2, V\rangle, w \vDash f(\varphi)$. Since the model checking problem for BML is also PSPACE-complete [3], the model checking for LSL is PSPACE-hard. The proof is completed. □

By the same reasoning as in the proof of Theorem 5, but now focusing on $\mathcal{L}_{\blacklozenge\langle+\rangle}$ instead of LSL, we can obtain the following result:

Theorem 6 *Model checking for $\mathcal{L}_{\blacklozenge\langle+\rangle}$ is PSPACE-complete.*

Given the form of formula (1) describing winning strategies in SLG, it is also an interesting problem concerning the game to study the complexity of the model checking for the fragment of $\mathcal{L}$ consisting only of operators $\wedge, \blacksquare, \langle-\rangle_{on}, \langle-\rangle_{on}$ and $\langle+\rangle$ (without $\neg$). We leave this as an open problem.

Now we move to considering the satisfiability problem. In particular, it will be shown that LSL is undecidable. To achieve this, in what follows we will study $\mathcal{L}_{\blacklozenge\langle-\rangle_{on}}$ instead of LSL. We first show that the fragment does not enjoy the finite model property. To prove this, we will construct a 'spy point' [9], which can see any reachable point in one step.

Theorem 7 *$\mathcal{L}_{\blacklozenge\langle-\rangle_{on}}$ does not enjoy the finite model property.*

Proof To prove this, we construct an $\mathcal{L}_{\blacklozenge\langle-\rangle_{on}}$-formula that can only be satisfied by some infinite models. Let φ_∞ be the conjunction of the following formulas:

(F_1)	$p \wedge q \wedge \blacklozenge p \wedge \blacklozenge\neg p \wedge \blacksquare\neg q$
(F_2)	$\blacksquare(p \to \blacklozenge q \wedge \blacklozenge\neg q \wedge \blacksquare p)$
(F_3)	$\blacksquare(p \to \blacksquare(q \to \blacksquare\neg q \wedge \blacklozenge\neg p))$
(F_4)	$\blacklozenge(\neg p \wedge \langle-\rangle_{on}\blacksquare(p \wedge \blacksquare(q \to \blacksquare p)))$
(F_5)	$\blacksquare(p \to \blacksquare(\neg q \to \blacklozenge q \wedge \blacklozenge\neg q \wedge \blacksquare p))$
(F_6)	$\blacksquare(p \to \blacksquare(\neg q \to \blacksquare(q \to \blacksquare\neg q \wedge \blacklozenge\neg p)))$
(F_7)	$\blacklozenge(\neg p \wedge \langle-\rangle_{on}\blacksquare\blacksquare(\neg q \to \blacksquare(q \to \blacksquare p)))$
(Spy)	$\blacksquare(p \to \blacksquare(\neg q \to \blacksquare(q \to \langle-\rangle_{on}(\neg q \wedge \blacksquare\neg q \wedge \langle-\rangle_{on}(q \wedge \blacklozenge(p \wedge \blacksquare\neg q))))))$
(Irr)	$\blacksquare(p \to \blacksquare(q \to \langle-\rangle_{on}(\neg q \wedge \blacksquare\neg q \wedge \blacksquare\blacklozenge q)))$
$(No\text{-}3cyc)$	$\neg\blacklozenge(p \wedge \blacksquare(q \to \langle-\rangle_{on}(\neg q \wedge \blacksquare(\neg q \wedge \blacksquare(q \to \langle-\rangle_{on}(\neg q \wedge \blacksquare\neg q \wedge$ $\langle-\rangle_{on}(q \wedge \blacklozenge(p \wedge \blacksquare\neg q)) \wedge \blacklozenge\blacklozenge(p \wedge \blacksquare\neg q)))))))$
$(Trans)$	$\blacksquare(p \to \blacksquare(q \to \langle-\rangle_{on}(\neg q \wedge \blacksquare\neg q \wedge \blacksquare\blacksquare(\neg q \to \blacksquare(q \to \langle-\rangle_{on}(\neg q \wedge$ $\blacksquare\neg q \wedge \langle-\rangle_{on}(p \wedge \neg\blacklozenge q \wedge \blacklozenge\blacksquare\neg q)))))))$

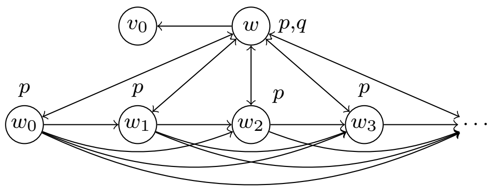

Fig. 4 A model of formula φ_∞ (every link in the model belongs to R_L, and $R_T = \emptyset$). It can be shown that the formula is true at w.

Formula φ_∞ is satisfiable (see Figure 4). Now we show that for any $\langle \mathcal{M}, w\rangle$, if $\mathcal{M}, w \vDash \varphi_\infty$, then $\mathcal{M}$ is infinite. Let $B := \{v \in W \mid v \in R_L(w) \cap V(p)\}$. In what follows, we assume that all previous conjuncts hold.

By (F_1), node w is $(p \wedge q)$, and $R_L(w) \cap V(q) = \emptyset$. Consequently, $\neg R_L ww$. Besides, $B \neq \emptyset$ and $R_L(w) \setminus B \neq \emptyset$. From (F_2), it follows that each element of B can see some $(q \wedge p)$-point(s) and $(\neg q \wedge p)$-point(s) via R_L, but cannot see any $\neg p$-points through R_L. Hence each point in B has at least one R_L-successor distinct from itself. By (F_3), for any $w_1 \in B$, each of its R_L-successors that is q can see some $\neg p$-point(s) via R_L, but cannot see any q-points by R_L. By (F_4), $R_L(w) \setminus B \neq \emptyset$ is a singleton. Moreover, each $w_1 \in B$ can see point w via R_L, and for each $w_2 \in V(q)$, $R_L w_1 w_2$ entails $w_2 = w$.

Formulas (F_2)-(F_4) show the properties of the $(\neg q \wedge p)$-points accessible from w in one step by R_L. Similarly, formulas (F_5)-(F_7) play the same roles as (F_2)-(F_4) respectively, but focusing on showing the properties of the $(\neg q \wedge p)$-points accessible from w in 2 steps via R_L. In particular, (F_7) guarantees that every $(\neg q \wedge p)$-point w_1 accessible from w in 2 steps by R_L can also see w via R_L, and that for each q-point w_2, $R_L w_1 w_2$ entails $w_2 = w$.

Formula (*Spy*) shows that, for any $(\neg q \wedge p)$-points w_1, w_2 s.t. $R_L ww_1$ and $R_L w_1 w_2$, after removing some $\langle v, v'\rangle \in \{\langle w, w_1\rangle, \langle w_1, w_2\rangle, \langle w_2, w\rangle\}$, v is $\neg q$ and does not have any q-successors. As $w \in V(q)$, $v \neq w$. Besides, if $\langle v, v'\rangle = \langle w_1, w_2\rangle$, after we cut $\langle v, v'\rangle$, v still can see $w \in V(q)$, so $\langle v, v'\rangle = \langle w_2, w\rangle$. Also, after deleting $\langle w, w_1\rangle$, w can reach a p-point w_3 via R_L s.t. $R_L(w_3) \cap V(q) = \emptyset$. Therefore, $w_3 = w_2$. Thus, (*Spy*) ensures that each $(\neg q \wedge p)$-point w_1 accessible from w in 2 steps via R_L is also accessible from w in one step via R_L.

By (*Irr*), for each $w_1 \in B$, $\neg R_L w_1 w_1$. (*No-3cyc*) shows R_L-links cannot be cycles of length 2 or 3 in B, and (*Trans*) forces R_L to transitively order B.

Hence $\langle B, R_L\rangle$ is an unbounded strict partial order, thus B is infinite and so is W. This completes the proof. □

We now proceed to show the undecidability of $\mathcal{L}_{\blacklozenge\langle - \rangle_{on}}$, by encoding the $\mathbb{N} \times \mathbb{N}$ tiling problem [21]. Inspired by [9], we will use three modalities $\blacklozenge_s$, $\blacklozenge_u$ and $\blacklozenge_r$ to stand for $\blacklozenge$. Correspondingly, a model $\mathcal{M} = \{W, R_s, R_u, R_r, R_T, V\}$ now includes four relations. We are going to construct a spy point over R_s, and relations R_u, R_r represent moving up and to the right, respectively, from one tile to the other. Intuitively, the union of these three relations can be treated as R_L in the model. Moreover, as illustrated by the following proof, they are disjoint with each other. So they are a partition of R_L. Thanks to this, we do not need any extra modalities to represent $\langle - \rangle_{on}$.

Theorem 8 *The satisfiability problem for $\mathcal{L}_{\blacklozenge\langle - \rangle_{on}}$ is undecidable.*[5]

Proof Let $T = \{T_1, \cdots, T_n\}$ be a finite set of tile types. For each T_i, $u(T_i)$, $d(T_i)$, $l(T_i)$ and $r(T_i)$ are the colors of its up, down, left and right edges respectively. Also, each T_i is coded with a fixed proposition t_i. Now we show that φ_T, the conjunction of the following formulas, is true iff T tiles $\mathbb{N} \times \mathbb{N}$.

$$
\begin{array}{rl}
(M_1) & p \wedge q \wedge \blacklozenge_s p \wedge \blacklozenge_s \neg p \wedge \blacksquare_s \neg q \wedge \blacklozenge_s \langle - \rangle_{on} \blacksquare_s p \\
(M_2) & \blacksquare_s (p \to \blacklozenge_s \top \wedge \blacksquare_s (q \wedge \blacklozenge_s \neg p)) \\
(M_3) & \blacklozenge_s (\neg p \wedge \langle - \rangle_{on} \blacksquare_s \blacksquare_s (q \wedge \neg \blacklozenge_s \neg p)) \\
(M_4) & \blacksquare_s (p \to \blacklozenge_u \top \wedge \blacksquare_u (p \wedge \neg q \wedge \blacklozenge_s \top \wedge \blacksquare_s (q \wedge \blacklozenge_s \neg p))) \\
 & \blacksquare_s (p \to \blacklozenge_r \top \wedge \blacksquare_r (p \wedge \neg q \wedge \blacklozenge_s \top \wedge \blacksquare_s (q \wedge \blacklozenge_s \neg p))) \\
(M_5) & \blacklozenge_s (\neg p \wedge \langle - \rangle_{on} \blacksquare_s \blacksquare_u \blacksquare_s \neg \blacklozenge_s \neg p) \\
 & \blacklozenge_s (\neg p \wedge \langle - \rangle_{on} \blacksquare_s \blacksquare_r \blacksquare_s \neg \blacklozenge_s \neg p) \\
(M_6) & \blacksquare_s (p \to \blacksquare_u (\blacklozenge_u \top \wedge \blacklozenge_r \top \wedge \blacksquare_u (p \wedge \neg q) \wedge \blacksquare_r (p \wedge \neg q))) \\
 & \blacksquare_s (p \to \blacksquare_r (\blacklozenge_u \top \wedge \blacklozenge_r \top \wedge \blacksquare_u (p \wedge \neg q) \wedge \blacksquare_r (p \wedge \neg q))) \\
(M_7) & \blacksquare_s (p \to \blacksquare_s (q \wedge \langle - \rangle_{on} (\neg q \wedge \blacksquare_u (\blacklozenge_s q \wedge \neg \blacklozenge_u \neg \blacklozenge_s q)))) \\
 & \blacksquare_s (p \to \blacksquare_s (q \wedge \langle - \rangle_{on} (\neg q \wedge \blacksquare_r (\blacklozenge_s q \wedge \neg \blacklozenge_r \neg \blacklozenge_s q)))) \\
(\mathit{Spy}) & \blacksquare_s (p \to \blacksquare_u \blacksquare_s \langle - \rangle_{on} (\blacksquare_s \bot \wedge \langle - \rangle_{on} (p \wedge q \wedge \blacklozenge_s (p \wedge \blacksquare_s \bot)))) \\
 & \blacksquare_s (p \to \blacksquare_r \blacksquare_s \langle - \rangle_{on} (\blacksquare_s \bot \wedge \langle - \rangle_{on} (p \wedge q \wedge \blacklozenge_s (p \wedge \blacksquare_s \bot)))) \\
(\mathit{Func}) & \blacksquare_s (p \to \blacksquare_s \langle - \rangle_{on} (\blacksquare_s \bot \wedge \blacksquare_u \langle - \rangle_{on} (\blacksquare_s \bot \wedge \blacksquare_u \bot)) \\
 & \blacksquare_s (p \to \blacksquare_s \langle - \rangle_{on} (\blacksquare_s \bot \wedge \blacksquare_r \langle - \rangle_{on} (\blacksquare_s \bot \wedge \blacksquare_r \bot)) \\
(\mathit{No\text{-}UR}) & \blacksquare_s (p \to \blacksquare_s \langle - \rangle_{on} (\blacksquare_s \bot \wedge \blacksquare_u \blacksquare_r \blacklozenge_s q \wedge \blacksquare_r \blacksquare_u \blacklozenge_s q)) \\
(\mathit{No\text{-}URU}) & \blacksquare_s (p \to \blacksquare_s \langle - \rangle_{on} (\blacksquare_s \bot \wedge \blacksquare_u \blacksquare_r \blacksquare_u \blacklozenge_s q)) \\
(\mathit{Conv}) & \blacksquare_s (p \to \blacksquare_s \langle - \rangle_{on} (\blacksquare_s \bot \wedge \blacklozenge_u \blacksquare_s \langle - \rangle_{on} (\blacksquare_s \bot \wedge \blacklozenge_u \top \wedge \\
 & \blacklozenge_r \blacksquare_u \langle - \rangle_{on} (\blacksquare_u \bot \wedge \blacklozenge_s \blacklozenge_s (p \wedge \blacksquare_s \bot \wedge \blacklozenge_r \blacklozenge_u (p \wedge \blacksquare_u \bot)))))) \\
(\mathit{Unique}) & \blacksquare_s (p \to \bigvee_{1 \le i \le n} t_i \wedge \bigwedge_{1 \le i < j \le n} (t_i \to \neg t_j)) \\
(\mathit{Vert}) & \blacksquare_s (p \to \bigwedge_{1 \le i \le n} (t_i \to \blacklozenge_u \bigvee_{1 \le j \le n,\ u(T_i) = d(T_j)} t_j)) \\
(\mathit{Horiz}) & \blacksquare_s (p \to \bigwedge_{1 \le i \le n} (t_i \to \blacklozenge_r \bigvee_{1 \le j \le n,\ r(T_i) = l(T_j)} t_j))
\end{array}
$$

Let $\mathcal{M} = \{W, R_s, R_u, R_r, R_T, V\}$ be a model and $w \in W$ s.t. $\mathcal{M}, w \vDash \varphi_T$. We show that $\mathcal{M}$ tiles $\mathbb{N} \times \mathbb{N}$. Define $G := \{v \in W \mid v \in R_s(w) \cap V(p)\}$ where $R_s(w) := \{v \in W \mid R_s wv\}$. We will use the elements of G to represent tiles.

By (M_1), node w is $(p \wedge q)$, and $R_s(w) \cap V(q) = \emptyset$. So, $\neg R_s ww$. Besides, $R_s(w) \setminus G$ is a singleton (e.g., $\{v\}$) and $G \neq \emptyset$. By (M_2), each tile w_1 has some successor(s) via R_s, and each $w_2 \in R_s(w_1)$ is q and has some $\neg p$-successor(s) via R_s. Formulas (M_1) and (M_2) illustrate that R_s is irreflexive. Formula (M_3) ensures that each tile w_1 can see w via R_s, and that for each $w_2 \in$

[5] The four modalities used in its proof can be reduced to two by a standard argument [23], but we will omit the details because of the syntactic cost involved in writing the formulas.

$V(q)$, $R_s w_1 w_2$ entails $w_2 = w$. From (M_4), we know that each tile has some successor(s) via R_u and some successor(s) via R_r. Besides, each point accessible from a tile via R_u or R_r is $(\neg q \wedge p)$, and it has some q-successor(s) w_1 via R_s where each w_1 can see some $\neg p$-point(s) via R_s. By formula (M_5), each $w_1 \in W$ accessible from a tile via R_u or R_r can see w by R_s. Also, for each $(q \wedge p)$-point w_2, if $w_2 \in R_s(w_1)$, then $w_2 = w$. Formula (M_6) ensures that each $w_1 \in W$ accessible from some tile via R_u or R_r also has some successor(s) via R_u and some successor(s) via R_r. Besides, each such successor via R_u or R_r is $(\neg q \wedge p)$. Formula (M_7) shows that both the restrictions of R_u and R_r to $G \times G$ are irreflexive and asymmetric. By (*Spy*), w is a spy point via R_s. Note that formula (M_4) says that each tile has some tile(s) above it and some tile(s) to its right. Now, with (*Func*), we have that each tile has exactly one tile above it and exactly one tile to its right. By (*No-UR*), any tile cannot be above/below as well as to the left/right of another tile. Formula (*No-URU*) disallows cycles following successive steps of the R_u, R_r, and R_u relations, in this order. Moreover, (*Conv*) ensures that the tiles are arranged as a grid. Formula (*Unique*) guarantees that each tile has a unique type. Finally, (*Vert*) and (*Horiz*) force the colors of tiles to match properly. Thus, $\mathcal{M}$ tiles $\mathbb{N} \times \mathbb{N}$.

On the other hand, it is easy to see that any tiling of $\mathbb{N} \times \mathbb{N}$ induces a model for φ_T. Now the proof is completed. □

From Theorem 7 and Theorem 8, it follows immediately that LSL lacks the finite model property, and its satisfiability problem is undecidable.

Finally, it is worth noting that, besides $\mathcal{L}_{\blacklozenge\langle - \rangle_{on}}$, other fragments also deserve to be studied, say, $\mathcal{L}_{\blacklozenge\langle - \rangle_{off}}$. It is already known that the satisfiable problem for SML is undecidable [6] and its model checking problem is PSPACE-complete [3]. Given the similarity between $\langle - \rangle_{off}$ and $\langle - \rangle$ (recall Proposition 2), is the model checking for $\mathcal{L}_{\blacklozenge\langle - \rangle_{off}}$ PSPACE-complete? And is its satisfiability problem undecidable?

5 Conclusion and Future Work

Summary Motivated by restrictions on learning in SG, we have extended the game to SLG by naming right and wrong paths of learning, and let Teacher not only delete but also add links. Afterwards, logic LSL was presented, which enables us to reason about players' strategies in SLG. Besides, to understand the new device, we provided some interesting observations and logical validities. Next, we studied basics of its expressivity, including its first-order translation, a novel notion of bisimulation and a characterization theorem for LSL as a fragment of FOL that is invariant under the bisimulation introduced. Finally, it was proved that model checking for LSL is PSPACE-complete, and via the research on $\mathcal{L}_{\blacklozenge\langle - \rangle_{on}}$ we showed that LSL does not enjoy the finite model property and its satisfiability problem is undecidable.

Relevant Research Broadly, this work takes a small step towards studying the interaction between graph games, logics and formal learning theory. As mentioned, the success condition of learning studied in the article is finite identification. Both [27] and [24] concern this kind of learnability in the context of indexed families of recursive languages. More generally, a relaxed notion of finite identification and its relation to logics and information updates is proposed and studied in [13,15–17].

We are inspired by the work on SG [30], SML [6] and their application to formal learning theory [18]. This article is also relevant to other work studying graph games with modal logics, such as [10,20,25,28,31,26]. Technically, the logic LSL has resemblances to several recent logics with model modifiers, such as [2–4]. Besides, instead of updating links, [32] considers a logic of stepwise point deletion, which sheds light on the long-standing open problem of how to axiomatize the sabotage-style modal logics. Moreover, [32] is also helpful to understand the complexity jumps between dynamic epistemic logics of model transformations and logics of freely chosen graph changes recorded in current memory. Another relevant line of research for this paper is epistemic logics. As mentioned already, one goal of our work is to avoid the Gettier problem. Similarly, [7] uses the topological semantics to study the *full belief*.

Future Work Except what have been studied in this article, there are still various open problems deserving to be studied in the future, from the perspectives of logic, games and learning theory.

From the logic point of view, Section 2.2 shows that logic LSL is able to express the winning positions for players in finite games, but to capture those for infinite games, can LSL be expanded with some least-fixpoint operators? From the translation described in Definition 5, we know that LSL is effectively axiomatizable [29]. However, is it possible to axiomatize the logic via a Hilbert-style calculus? Also, Proposition 5 shows that validities of LSL are not closed under substitution. But, are the schematic validities of LSL decidable? Moreover, are they axiomatizable?

In terms of games, we do not know the complexity of SLG, although we have a basic observation on the necessary condition of winning. Besides, SLG includes exactly two players, and it is meaningful to study the cases that are more general. In addition, SLG studied in the article is a cooperative game, but there are also other cases corresponding to different levels of players' ability and attitude. Say, Learner may be unwilling to learn, and Teacher can also be unhelpful or not omniscient. Are there some natural variants of SLG capturing these situations? Finally, another interesting direction to investigate is the significance of cycles in SLG. Consider the case that Learner reaches the goal through a path $\langle a_0, a_1, ..., a_i, ..., a_m, ..., a_n\rangle$ from the starting node a_0 to the goal node a_n, where $Set(\langle a_0, a_1, ..., a_i\rangle) \cup Set(\langle a_m ..., a_n\rangle) \subseteq R_T$, $Set(\langle a_i ..., a_m\rangle) \nsubseteq R_T$ and $a_i = a_m$. According to our theory, Learner has not reached the goal through a correct path, so she has not learned properly. However, it can be argued that even if one in a learning situation learns some unimportant circular argument in addition to a proper argument, one has still learned the proper argument. But our game now cannot capture these scenarios. A possible solution is to define learning such that it could include 'meaningless' cycles. Nonetheless, it is an issue worth looking into.

Finally, although Section 1 discusses some applications of SLG to scenarios of learning, the relations between our framework and existing proposals of formal learning theory deserve to be studied more systematically.[6]

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[6] For more discussions on the applications of SG-style frameworks to paradigms of learning theory, we refer to [18], whose arguments also apply to SLG after minor modifications.

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