



Triangulations of polygons and stacked simplicial complexes: separating their Stanley–Reisner ideals

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Abstract

A triangulation of a polygon has an associated Stanley–Reisner ideal. We obtain a full algebraic and combinatorial understanding of these ideals and describe their separated models. More generally, we do this for stacked simplicial complexes, in particular for stacked polytopes.

Keywords Triangulation of polygon · Stacked simplicial complex · Separation of ideal · Regular sequence · Independent vertices

Mathematics Subject Classification Primary: 13F55; Secondary: 05C69 · 05C70

1 Introduction

Triangulations of polygons constitute a basic yet rich topic going into many directions. The most classical fact about these is perhaps that they are counted by the Catalan numbers, [12, Chap. 23]. Their Stanley–Reisner ideals seem hitherto not to have been systematically studied. Here, we get a full understanding of their algebraic and combinatorial nature. Considerably more generally, we do this for the Stanley–Reisner ideals of *stacked simplicial complexes*.

Example 1.1 Consider the triangulation of the heptagon in Fig. 1. This may be built up step by step from triangles, by successively attaching the triangles

127, 257, 567, 245, 234.

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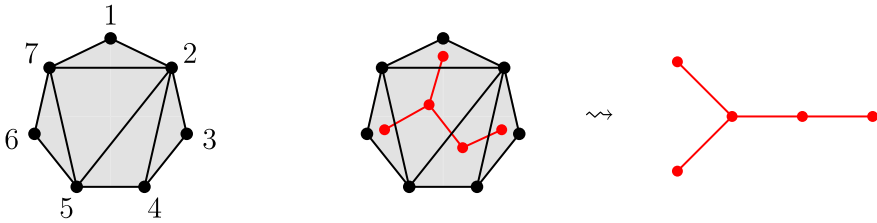


Fig. 1 The triangulation of the heptagon in Example 1.1, and its associated tree in red (Color figure online)

Each triangle after the first is attached to a single edge of some earlier triangle. This is a type of shelling called a *stacking*, and every triangulation of a polygon is a stacking. Moreover, to a triangulation of the polygon we may associate a tree (drawn in red in Fig. 1), showing how the triangles are attached to each other.

This gives our two fundamental notions: That of stacking and the associated (hyper)tree.

Let X be a simplicial complex on a set A , i.e., a family of subsets of A such that if $F \in X$ and $G \subseteq F$, then $G \in X$. Let F_1, F_2, \dots, F_k be an ordering of the facets (the maximal faces) of X . We assume that the F_i 's all have the same cardinality. Let X_p be the simplicial complex generated by F_1, \dots, F_p .

The sequence F_1, \dots, F_k is a *stacking* of X if each F_p is attached to X_{p-1} along a single codimension-one face of X_{p-1} . So we may write $F_p = G_p \cup \{v_p\}$ where G_p is a face of X_{p-1} and v_p is not a vertex of X_{p-1} . This is a shelling, but a particularly simple kind of shelling, since each F_p is attached to a single codimension-one face, in contrast to a union of one or more such faces. A simplicial complex that has a stacking as above is called a *stacked simplicial complex*. Such simplicial complexes have appeared in the literature also as “facet constructible complexes” (see [6, 11]). Although they have been previously studied from the point of view of commutative algebra, to the best of our knowledge it was with a different perspective than in this paper. For instance, in [6] the focus is more on the homological invariants and Cohen–Macaulayness of such complexes.

To a stacked simplicial complex X , we associate a (hyper)tree as in the example above. Let V be an index set for the facets of X . For a codimension-one face G of X which is on at least two facets, let $e_G = \{v \in V \mid F_v \supseteq G\}$. This gives a hypergraph on V whose edges are the sets e_G . In fact, this hypergraph is a hypertree T : it is connected, each pair of edges intersects in at most one vertex, and there are no cycles. The hypertree T is an ordinary tree, like in Fig. 1, when each codimension-one face is on at most two facets. Then, X is a triangulated ball. In fact, X may then be realized as a stacked polytope (see [13]), and every stacked polytope is of this kind. For the relationship between stacked simplicial complexes and stacked polytopes, we refer to Sect. 4.5 of [11].

Given an (ordinary) tree T , let V be the vertices of T , and E the edge set of T . Let $C \subseteq E \times V$ be the incidence relation consisting of pairs (e, v) such that v is a vertex on the edge e . Let $k[x_C]$ be the polynomial ring in the variables $x_{e,v}$ for $(e, v) \in C$. (Indexing the variables this way was suggested by Lars Hällström.) We associate a squarefree monomial ideal $I(T)$ in the polynomial ring $k[x_C]$ as follows. Given a pair

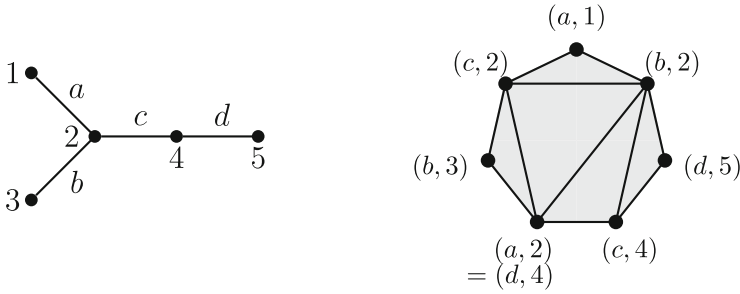
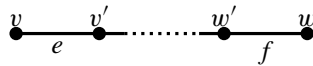


Fig. 2 The tree in Example 1.2 on the left, and on the right the triangulated heptagon that is the complex associated to the ideal obtained after taking the quotient in Example 1.2

of vertices v, w of T , there is a unique path between v and w in the tree T :



Associate with the pair of vertices $\{v, w\}$ the monomial $m_{v,w} = x_{e,v} \cdot x_{f,w}$.

The ideal $I(T)$ is the monomial ideal generated by the $m_{v,w}$ as v and w run through all distinct pairs of vertices of V .

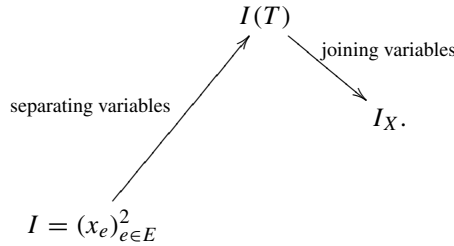
We show that the Stanley–Reisner ring of any triangulation of a polygon is obtained from $k[x_C]/I(T)$ by dividing out by a suitable regular sequence of variable differences $x_{e,v'} - x_{f,w'}$. More generally, any Stanley–Reisner ring of a stacked simplicial complex is obtained this way. The rings $k[x_C]/I(T)$ for trees T are thus the “initial objects” or “free objects” for Stanley–Reisner rings of stacked simplicial complexes. Formulated otherwise, let I be the Stanley–Reisner ring of a stacked simplicial complex. The *separated models* of I are one or more of the $I(T)$.

Example 1.2 Consider the directed tree in Fig. 2. The ideal $I(T)$ is generated by the ten monomials

$$\begin{aligned}
 m_{12} &= x_{a,1}x_{a,2} & m_{13} &= x_{a,1}x_{b,3} & m_{14} &= x_{a,1}x_{c,4} & m_{15} &= x_{a,1}x_{d,5} \\
 m_{23} &= x_{b,2}x_{b,3} & m_{24} &= x_{c,2}x_{c,4} & m_{25} &= x_{c,2}x_{d,5} & m_{34} &= x_{b,3}x_{c,4} \\
 m_{35} &= x_{b,3}x_{d,5} & m_{45} &= x_{d,4}x_{d,5}.
 \end{aligned}$$

Then, $I(T)$ is the Stanley–Reisner ideal of a stacked simplicial complex of dimension 3 (with facets of cardinality 4) with eight vertices and five facets. Dividing out by the variable difference $x_{a,2} - x_{d,4}$, we get the Stanley–Reisner ring of the triangulation of the heptagon $k[x_C]/(I(T) + (x_{a,2} - x_{d,4}))$. Figure 2, on the right, shows the triangulation with our new labelings of the vertices.

The ideals $I(T)$ are introduced in [2] where they are shown to be all possible polarizations of the square of the graded maximal ideal $(x_e)_{e \in E}^2$ in $k[x_e]_{e \in E}$. If I_X is the Stanley–Reisner ring of a stacked simplicial complex, we therefore have processes:



Each of the arrows above preserves the graded Betti numbers. Hence, every I_X has the same graded Betti numbers as a second power of a graded maximal ideal $(x_e)_{e \in E}$.

Let $k[x_C]_1 = \langle x_{e,v} \rangle_{(e,v) \in C}$ be the linear subspace of one-dimensional forms in the polynomial ring $k[x_C]$. A subspace L of this linear space is a *regular linear space* if it has a basis consisting of a regular sequence of variable differences for $k[x_C]/I(T)$. The quotient ring by the space L of linear forms will still be a polynomial ring divided by a monomial ideal. We show the following.

Theorem 5.16. *There is a one-to-one correspondence between regular linear spaces for $k[x_C]/I(T)$ and partitions of the vertex set V .*

In particular for the partition with one part, the whole of V , the regular sequence consists of all variable differences $x_{e,v} - x_{e,w}$ for $e = \{v, w\} \in E$, and the quotient ring is $k[x_e]_{e \in E}/m^2$, where $m = (x_e)_{e \in E}$ is the irrelevant maximal ideal.

Theorem 6.2. *There is a one-to-one correspondence between regular linear spaces for $k[x_C]/I(T)$ giving squarefree quotient rings, and partitions of the vertex set V into sets of independent vertices.*

These quotient rings give the Stanley–Reisner rings of stacked simplicial complexes.

In [10], the first author gives a one-to-one correspondence between partitions of the vertex set of a tree T into $(r + 1)$ independent sets, and partitions of the edge set of T into r sets. We recall this in the appendix. The above may then be reformulated as:

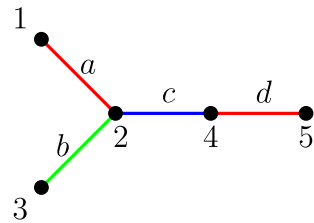
Theorem 6.3. *There is a one-to-one correspondence between regular linear spaces for $k[x_C]/I(T)$ giving squarefree quotient rings, and partitions of the edge set $E(T)$. Moreover, the dimension of the simplicial complex associated with this quotient Stanley–Reisner ring is one less than the number of parts in the partition.*

Example 1.3 Consider Example 1.2 above. The regular linear space L is the space $L = \langle x_{a,2} - x_{d,4} \rangle$. It corresponds to the partitions of vertices and partitions of edges of the tree in Fig. 3. These partitions are, respectively,

$$V = \{1, 5\} \cup \{2\} \cup \{3\} \cup \{4\}, \quad E = \{a, d\} \cup \{b\} \cup \{c\}.$$

There are three parts in the edge partition, and so the dimension of the associated simplicial complex is one less, the dimension of the triangulated polygon.

Fig. 3 The tree in Example 1.3. The partition of the edge set is illustrated by the different colors (Color figure online)



Finally, we show the following.

Theorem 8.7. *There is a one-to-one correspondence between regular linear spaces for $k[x_C]/I(T)$ giving squarefree quotient rings whose associated simplicial complex is a triangulated ball, and partitions of the edge set $E(T)$ into sets of independent edges.*

In particular, the last two theorems above give that triangulations of *simplicial polygons* correspond to partitions of the edges of trees T into three parts, each part being a set of independent edges. In particular, only trees T whose maximal vertex degree is 3 arise in this context (which is easy to see directly like in Example 1.1).

The organization of this article is as follows. In Sect. 2, we recall the notions of separating and joining variables in monomial ideals. We develop basic auxiliary results for doing this. We also recall the notion of separated model. In Sects. 3 and 4, we recall basic notions for simplicial complexes. We define stacked simplicial complexes and hypertrees. We show that the separated models of stacked simplicial complexes are the ideals $I(T)$.

Section 5 is the main technical part and gives the combinatorial description of which linear spaces of variable differences are regular linear spaces for $k[x_C]/I(T)$. Section 6 describes the regular linear spaces that give squarefree quotient rings. Section 7 describes the ordering relation between partitions of vertices that corresponds to inclusion of regular linear spaces. Lastly, in Sect. 8 we describe those regular linear spaces where the quotient ring is associated with a triangulation of a ball, or equivalently of a stacked polytope. We also describe the Stanley–Reisner ring of the boundary of these polytopes, which are simplicial spheres.

The appendix recalls the correspondence between partitions of vertices of $V(T)$ into independent sets and partitions of edges of $E(T)$.

2 Separations and joins for Stanley–Reisner ideals

We recall the notion of separation for monomial ideals I . It is a converse to the notion of dividing a quotient ring S/J out by a variable difference which is a nonzero divisor. When $I = I_X$ or $J = J_X$ is a Stanley–Reisner ideal, we describe how the simplicial complex X transforms under these processes.

For a set V denote by $k[x_V]$, the polynomial ring in the variables x_v for $v \in V$. With some abuse of notation, for $R \subseteq V$ let x_R denote the monomial $\prod_{r \in R} x_r$. (This should not cause confusion since in the polynomial ring we always surround x_V with square brackets).

2.1 Separating a variable

The following definition is from [9, Sect. 2].

Definition 2.1 Let $V' \xrightarrow{p} V$ be a surjection of finite sets with the cardinality of V' one more than that of V . Let v_1 and v_2 be the two distinct elements of V' which map to a single element v in V . Let I be a monomial ideal in the polynomial ring $k[x_V]$ and J a monomial ideal in $k[x_{V'}]$. We say J is a *simple separation* of I if the following hold:

1. The monomial ideal I is the image of J by the map $k[x_{V'}] \rightarrow k[x_V]$.
2. Both the variables x_{v_1} and x_{v_2} occur in some minimal generators of J (usually in distinct generators).
3. The variable difference $x_{v_1} - x_{v_2}$ is a nonzero divisor in the quotient ring $k[x_{V'}/J$.

More generally, if $V' \xrightarrow{p} V$ is a surjection of finite sets and $I \subseteq k[x_V]$ and $J \subseteq k[x_{V'}]$ are monomial ideals such that J is obtained by a succession of simple separations of I , J is a *separation* of I . If J has no further separation, we call J a *separated model* (of I).

Let X be a simplicial complex on the set V . This is a family of subsets of V such that $F \in X$ and $G \subseteq F$ implies $G \in X$. The set of $v \in V$ with $\{v\} \in X$ is the *support* of X . For $R \subseteq V$, the *restriction* X_R is the simplicial complex on R consisting of all $F \in X$ such that $F \subseteq R$. Denote by X_{-R} the restriction X_{R^c} where R^c is the complement of R in V . The *link* $\text{lk}_X R$ is the simplicial complex on R^c consisting of all $F \subseteq R^c$ such that $F \cup R \in X$. If $Y \subseteq X$ are simplicial complexes, denote by $X \setminus Y$ the relative simplicial complex, consisting of those $F \in X$ which are not in Y .

Let I_X be the Stanley–Reisner ideal of X , the monomial ideal in $k[x_V]$ whose generators are the monomials x_R for $R \notin X$. Suppose we use v to separate I_X to an ideal $I_{X'}$ in the polynomial ring $k[x_{V'}]$. Write the minimal set of monomial generators of $I_{X'}$ as $\mathcal{M}_0 \cup \mathcal{M}_v$, where \mathcal{M}_0 consists of those that do not contain x_v and \mathcal{M}_v of those of the form $x_v \cdot x_R$. The separated ideal $I_{X'}$ will then have minimal generators $\mathcal{M}_0 \cup \mathcal{M}_{v,1} \cup \mathcal{M}_{v,2}$ (sets of monomials in $k[x_{V'}]$). Here, $\mathcal{M}_{v,i}$ consists of those minimal generators that contain x_{v_i} . There is a bijection between $\mathcal{M}_{v,1} \cup \mathcal{M}_{v,2}$ and \mathcal{M}_v by sending $x_{v_i} \cdot x_R$ to $x_v \cdot x_R$.

2.2 Criteria for separating a variable

Here is a general description of how I_X can be separated using the variable x_v .

Proposition 2.2 *We may separate I_X using the variable x_v iff the following holds: there is a partition of the faces into two non-empty parts*

$$X_{-\{v\}} \setminus \text{lk}_X v = \mathcal{F}_1 \sqcup \mathcal{F}_2$$

where each \mathcal{F}_i is closed under taking smaller sets in the sense that if $G \subseteq F$ and $F \in \mathcal{F}_i$, then either $G \in \mathcal{F}_i$ or $G \in \text{lk}_X v$. The facets of the simplicial complex X' in the separated ideal $I_{X'}$ are then obtained from the facets F of X as follows:

- If $F = G \cup \{v\}$, then $G \cup \{v_1, v_2\}$ is a facet of X' .
- If F is in \mathcal{F}_1 , then $F \cup \{v_1\}$ is a facet of X' .
- If F is in \mathcal{F}_2 , then $F \cup \{v_2\}$ is a facet of X' .

Proof Assume first that $I_{X'}$ is a separation of I_X . If $F \in X_{-\{v\}} \setminus \text{lk}_X v$, then $x_v x_F \in I_X$ (otherwise F would be in $\text{lk}_X v$). Let \mathcal{F}_1 be the set of those F such that $x_{v_1} x_F$ is in $I_{X'}$ and x_F is not in $I_{X'}$, or equivalently in I_X . Similarly define \mathcal{F}_2 as the set of F such that $x_{v_2} x_F$ is in $I_{X'}$ and x_F is not in $I_{X'}$. Let us show that \mathcal{F}_1 and \mathcal{F}_2 are disjoint. If both $x_{v_1} x_F$ and $x_{v_2} x_F$ are in $I_{X'}$, then since $x_{v_1} - x_{v_2}$ is a nonzero divisor for $k[x_{V'}]/I_{X'}$, we have x_F in $I_{X'}$ and so in I_X . But then F could not have been in X .

Suppose conversely we have the partition $\mathcal{F}_1 \sqcup \mathcal{F}_2$. Let the ideal $I_{X'}$ be constructed as just before this Sect. 2.2, so it is generated by $\mathcal{M}_0 \cup \mathcal{M}_{v,1} \cup \mathcal{M}_{v,2}$. Let us show that $x_{v_1} - x_{v_2}$ is a nonzero divisor of $k[x_{V'}]/I_{X'}$. Suppose $(x_{v_1} - x_{v_2})x_F$ is in $I_{X'}$. Then, $x_{v_1} x_F$ and $x_{v_2} x_F$ are both in $I_{X'}$. By construction of $I_{X'}$ we must have $x_v x_F$ in I_X (and note it is not a minimal generator). So F is not in $\text{lk}_X v$. If x_F is not in I_X (the opposite of what we want), F is in $X_{-\{v\}}$. So F is in \mathcal{F}_1 or \mathcal{F}_2 . In the first case $F \cup \{v_1\}$ is a face of X' , and so $x_{v_1} x_F$ is not in $I_{X'}$, a contradiction. Similarly F is not in $\{\mathcal{F}\}_2$. Hence x_F is in I_X and so $x_{v_1} - x_{v_2}$ is not a zero divisor. \square

For $v \in V$ and X a simplicial complex, let the neighborhood of v be

$$N(v) = \{w \mid \{v, w\} \in X\} \subseteq V.$$

Note that $N(v)$ is non-empty iff v is in the support of X , in which case $v \in N(v)$.

Corollary 2.3 *Let X be a flag simplicial complex on V , i.e., I_X is generated by quadratic monomials. Let v be in the support of X . Suppose $X_{-N(v)} = X_1 \cup X_2$ where X_1 and X_2 are simplicial complexes supported on disjoint vertex sets V_1 and V_2 . Then, using x_v the ideal $I_X \subseteq k[x_V]$ may be separated to an ideal $I_{X'} \subseteq k[x_{V'}]$. The facets of X' correspond to the facets F of X as follows:*

- If $F = G \cup \{v\}$ contains v , then $G \cup \{v_1, v_2\}$ is a facet of X' .
- If F is supported on V_1 , then $F \cup \{v_1\}$ is a facet of X' .
- If F is supported on V_2 , then $F \cup \{v_2\}$ is a facet of X' .

Proof Let $U = N(v) \setminus \{v\}$. The link $\text{lk}_X v$ is supported on $U \subseteq V$. Let $F \in X_{-\{v\}} \setminus \text{lk}_X v$. Write F as a disjoint union $F_0 \cup F_1$ where $F_1 = F \cap U$. Since X is flag, if $F_0 = \emptyset$, we would have F in $\text{lk}_X v$. So F_0 is non-empty and we show it is a subset of either V_1 or V_2 . Let $a, b \in F_0$. Then, a is not in the link $\text{lk}_X v$, so a is either in V_1 or V_2 . Similarly with b . So $\{a, b\}$ is in $X_{-N(v)}$ and so in either X_1 or X_2 . Hence, a, b are in the same set V_i . The upshot is that F_0 is a subset of either V_1 or V_2 .

We then let \mathcal{F}_1 be the set of those F such that F_0 is a subset of V_1 and similarly for \mathcal{F}_2 . These will then be disjoint and closed under taking smaller sets. \square

Example 2.4 In Fig. 2, in the introduction one may apply the above corollary to the vertex $v = 5$. This gives the separated ideal $I(T)$ of Example 1.2. The vertex $v = 5$ is the only vertex we may use to get a separated ideal. These things may also be seen by Proposition 2.2.

2.3 Criteria for joining variables

We present here basic results on dividing out a Stanley–Reisner ring by variable differences.

Let X be a simplicial complex on a set V , and F a facet of X . Then, for the algebraic set $A(X)$ in the affine space \mathbb{A}_k^V defined by the Stanley–Reisner ideal $I_X \subseteq k[x_V]$, the facet F corresponds to the linear space $A(F)$ in \mathbb{A}_k^V where all coordinates $x_v = 0$ for $v \notin F$, while the x_v for $v \in F$ may take arbitrary values.

For $v_1, v_2 \in V$, let $V_1 = (V \setminus \{v_1, v_2\}) \cup \{v\}$. The natural map $V \rightarrow V_1$ sending $v_1, v_2 \mapsto v$ gives a surjection of polynomial rings $k[x_V] \rightarrow k[x_{V_1}]$. Let the ideal I be the image of I_X . Then, I may or may not be squarefree. If I is squarefree, we say that $x_{v_1} - x_{v_2}$ cuts squarefree. Then, let $I = I_{X_1}$ where X_1 is the associated simplicial complex. We then have a commutative diagram of algebraic sets:

$$\begin{array}{ccc}
 A(X_1) & \xrightarrow{\phi} & A(X) \\
 \downarrow & & \downarrow \\
 \mathbb{A}_k^{V_1} & \xrightarrow{\quad} & \mathbb{A}_k^V.
 \end{array}$$

Let e_v be the point in affine space \mathbb{A}_k^V where x_v takes value 1 and the other variables value 0. The map ϕ above sends

$$\sum_{i \in V_1} a_i e_i \mapsto a_{v_1} e_{v_1} + a_{v_2} e_{v_2} + \sum_{i \neq v} a_i e_i. \tag{1}$$

Lemma 2.5 *Let X be a simplicial complex on a set V .*

- a. *A variable difference $x_{v_1} - x_{v_2}$ where $v_1, v_2 \in V$ is a nonzero divisor for S/I_X iff for each facet F of X , at least one of the variables v_1 or v_2 is in F .*
- b. *The ideal I is squarefree iff whenever $F \cup \{v_1\}$ and $F \cup \{v_2\}$ are faces of X , then $F \cup \{v_1, v_2\}$ is a face of X .*
- c. *Let F_1, \dots, F_r be the facets of X . If the difference $x_{v_1} - x_{v_2}$ is a nonzero divisor and cuts squarefree, the facets of X_1 are G_1, \dots, G_r where:*
 - *If F_i contains exactly one of v_1 and v_2 , then $G_i = F_i \setminus \{v_1, v_2\}$.*
 - *If v_1, v_2 are both in F_i , then $G_i = (F_i \setminus \{v_1, v_2\}) \cup \{v\}$.*

Proof

- a. The associated primes of I_X are the ideals generated by variables $(x_v)_{v \notin F}$, one such ideal for each facet F . The variable difference is a nonzero divisor iff it is not in any of these ideals. This means that never both v_1 and v_2 are in such an ideal, or equivalently never both v_1 and v_2 are outside of a facet F .
- b. The ideal I is squarefree iff there is no minimal generator $x_{v_1} x_{v_2} x_F$ of I_X . But having such a generator means having faces $F \cup \{v_1\}$ and $F \cup \{v_2\}$ but not a face $F \cup \{v_1, v_2\}$.
- c. This follows by (1) above.

□

When S/I_X is a Cohen-Macaulay ring, a sequence of linear forms ℓ_1, \dots, ℓ_r is a regular sequence for S/I_X iff for every facet F , it cuts down $A(F)$ successively by one dimension for every ℓ_k [5].

Corollary 2.6 *Let X be a Cohen–Macaulay simplicial complex on a set V . Let B be a forest on V , and denote B_1, \dots, B_m the trees in B and V_i the support of B_i for each i . Then, $\{x_v - x_w \mid \{v, w\} \text{ edge of } B\}$ is a regular sequence for S/I_X iff for each facet F and each V_i , at most one of the vertices of V_i is not in F .*

Proof A facet F of X gives the irreducible component $A(F)$ of the algebraic set associated with X . When cutting down $A(F)$ by the sequence of variable differences associated with the edges of B_i we have:

- If some $u \in V_i$ is not in F , the coordinate $x_u = 0$ on points in F . If $\{u, v\}$ is an edge in B_i , then $v \in F$, and when dividing out by this variable difference, we will have $x_v = 0$ on the resulting facet. Continuing dividing out by edges in B_i , this reduces dimension by $|V_i| - 1$.
- If $V_i \subseteq F$, all coordinates x_v for $v \in V_i$ become equal. This again reduces dimension by $|V_i| - 1$.

Hence, using the edges of the forest B the linear space $A(F)$ is cut down to a linear space whose dimension is $\sum_i (|V_i| - 1)$ less than $A(F)$ for each facet F of X . But then the set of variable differences is a regular sequence.

Conversely assume the sequence is regular. If there are V_i and F with $\{v', v''\} \subseteq V_i \setminus F$, then the variable differences $x_v - x_w$ associated with edges $\{v, w\}$ in B_i would only give the restrictions $x_v = 0$ for $v \in V_i \cap F$. This only cuts down dimension at most by $|V_i| - 2$, contrary to the sequence being regular. \square

3 Stacked simplicial complexes

Let X be a simplicial complex on a set A . In the previous section, we used V for the vertex set of X , but in the sequel we reserve V for the vertex set of the hypertree T associated with the stacked simplicial complex X . In other words, V is an index set for the facets of X .

We show that I_X may be successively separated to an ideal $I_{X'}$, where X' is a stacked simplicial complex of dimension two less than the number of facets $|V|$. (See Fig. 4 for two examples of such X').

3.1 Stacked simplicial complexes and associated hypertree

A facet F of a simplicial complex X is a *leaf* if there is a vertex v of F such that F is the only facet containing v . Such a vertex is a *free vertex* of X . If v is the only free vertex of F , we say F is *stacked* on $X_{-\{v\}}$. (In general, a *free face* of a simplicial complex is a face that is not a facet and that lies on exactly one facet, see for instance [16]. The term “leaf” is quite standard in graph theory, and less common in the setting of simplicial complexes, but see for instance [8]).

Definition 3.1 A pure simplicial complex (i.e., where all the facets have the same dimension) is *stacked* if there is an ordering of its facets F_0, F_1, \dots, F_k such that if X_{p-1} is the simplicial complex generated by F_0, \dots, F_{p-1} , then F_p is stacked on X_{p-1} .

Remark 3.2 This is a special case of shellable simplicial complexes, see [15, Sect. 8.2]. It is not the same as the notion of simplicial complex being a tree as in [8], even if the tree is pure. Rather the notion of stacked simplicial complex is more general. For instance, the triangulation of the heptagon given in Example 1.1, is not a tree in the sense of [8], since removing the triangles 234 and 257 one has no facet which is a leaf.

Remark 3.3 Stacked simplicial complexes are flag complexes. Every minimal nonface is an edge. Equivalently, the Stanley–Reisner ideal is generated by quadratic monomials.

A *hypergraph* is an ordered pair $H = (V, E)$ where V is a set and E is a collection of subsets of V such that no $e \in E$ is contained in another $e' \in E$. The elements of V are called the vertices of H , and the elements of E are called the edges of H . A hypergraph H is a *hypertree* if

- (1) Any two edges intersect in either one or zero elements,
- (2) H is connected, i.e., for any two vertices v and w in V there is a sequence e_1, \dots, e_m of edges of H with $v \in e_1$ and $w \in e_m$ and such that for every $i \in \{1, \dots, m-1\}$ one has $e_i \cap e_{i+1} \neq \emptyset$, and
- (3) H has no cycle, i.e., no sequence of distinct vertices v_0, v_1, \dots, v_n save $v_n = v_0$, with $n \geq 3$ such that each pair $\{v_{i-1}, v_i\}$ is contained in an edge, but no triple $\{v_{i-1}, v_i, v_{i+1}\}$ is contained in an edge.

If T' and T are hypertrees on the same vertex set, T' is a *refinement* of T if

- (1) Every edge of T' is contained in an edge of T , and
- (2) Every edge of T is a union of edges of T' .

Definition 3.4 Let X be a stacked simplicial complex with facets F_v indexed by a set V . We associate a hypertree to X on the vertex set V . For each codimension-one face G , let $e_G = \{v \in V \mid F_v \supseteq G\}$. The edge set of the hypertree is $E = \{e_G \mid |e_G| \geq 2\}$, the set of those e_G containing at least two facets.

The simplicial complex X is a triangulated ball iff its associated hypertree T is an ordinary tree, [4, Theorem 11.4]. It can then be realized as a stacked polytope. Such polytopes are extremal in the following sense: they have the minimal number of faces, given the number of vertices (see [3]).

Observation 3.5 Let X be a stacked simplicial complex which is a cone with p vertices in the cone apex. Thus, X is a join $X_1 * \Delta_p$ where Δ_p is a simplex on p elements and X_1 is not a cone. Then, X_1 is also stacked and both X and X_1 have the same associated hypertree.

3.2 Separating stacked simplicial complexes

Lemma 3.6 *Let X be a stacked simplicial complex of dimension d with hypertree T . If T has $\geq d + 3$ vertices, then using the procedure of Corollary 2.3, X may be separated to a simplicial complex X' which is also stacked, and whose hypertree T' is a refinement of T .*

Proof Note that if $d = 0$, then X is a collection of ≥ 3 vertices and the hypertree T has one edge, the set of all facet indices V . By Corollary 2.3, X may be separated.

Let $d \geq 1$ and F_0, \dots, F_k be a stacking order for X . The facet F_k is stacked on some previous facet F_p , with $p < k$. Let v be the vertex of $F_p \setminus F_k$, w the vertex of $F_k \setminus F_p$, and $G = F_p \cap F_k$. Then, $X_{-N(v)}$ contains $\{w\}$ as a component. If there are other components, we may apply Corollary 2.3.

Suppose then $\{w\}$ is the only component. Then, $X_{-N(v)}$ must be a cone over v . Let Y be the link $\text{lk}_X v$. It is stacked, of dimension $d - 1$ and has $\geq d + 2$ facets. By induction, we may use Corollary 2.3 and separate Y , using an element v' , to Y' whose tree is a refinement of that of Y . Let

$$V_1 \cup V_2 = (V \setminus \{v, w\}) \setminus N_Y(v')$$

be the partition given in Corollary 2.3.

1. Suppose $N(v')$ (the neighborhood considered in X) contains w . Then, $X_{-N(v')}$ has two components, supported on, respectively, V_1 and V_2 , and we may apply Corollary 2.3.
2. Suppose $N(v')$ contains G and not w . Then, $X_{-N(v')}$ has components $\{w\}$ together with at least one other component and we may again apply Corollary 2.3.
3. Suppose $N(v')$ does not contain w nor G . Then, there is $u \in G$ not in $N(v')$. So u is in, say V_1 . Then, $X_{-N(v')}$ may be written as a disjoint union $X_1 \cup X_2$, with X_2 supported on V_2 and X_1 supported on $V_1 \cup \{w\}$. Again, we may apply Corollary 2.3.

Let us now show that T' is a refinement of T . Let G be a codimension-one face of X contained in two or more facets F_i for $i \in D$, so D is an edge in T . The variable x_v is the variable used in the separation.

- If G contains v , write $G = G^0 \cup \{v'\}$, and then F_i is $F_i^0 \cup \{v'\}$. Then, $G^0 \cup \{v'_1, v'_2\}$ is a codimension-one face in facets $F_i^0 \cup \{v'_1, v'_2\}$ for $i \in D$. So D is still an edge in T' .
- Suppose G does not contain v . Let $D_1 \subseteq D$ index all F_i in \mathcal{F}_1 containing G and similarly define D_2 . There might also be a facet $F = G \cup \{v'\}$ containing G , in which case we extend both D_1 and D_2 with the index of this facet. Then, D_1 and D_2 are edges of T' and they have at most one vertex in common.

□

Proposition 3.7 *Let X be a stacked simplicial complex of dimension d which is not a cone, and let T be the associated hypertree.*

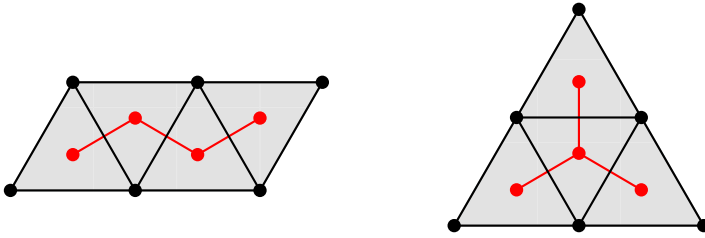


Fig. 4 The two 2-dimensional complexes with four facets, as in Example 3.8, with their corresponding trees in red (Color figure online)

- a. T has $\geq d + 2$ vertices,
- b. If T has an edge of cardinality ≥ 3 , then T has $\geq d + 3$ vertices
- c. If T has $\geq d + 3$ vertices, X may be separated to a simplicial complex X' whose tree T' is a refinement of T .
- d. If T is an (ordinary) tree with $d + 2$ vertices, then X is inseparable and the isomorphism class of X is uniquely determined by T .

Example 3.8 For $d = 2$, Fig. 4 shows the two stacked simplicial complexes of dimension 2 with four facets. The corresponding trees are also drawn in red.

Proof of Proposition 3.7 a,b. Let F_0, \dots, F_k be a stacking order of facets. Let X_p be the complex generated by F_0, \dots, F_p . Let $C_p = \bigcap_{i=0}^p F_i$ and G_p be the codimension-one face of F_p which attaches it to X_{p-1} . Then, for $p \geq 1$, $C_p = \bigcap_{i=1}^p G_i$ and $C_p = C_{p-1} \cap G_p$. Note G_p has codimension one in F_{p-1} and $C_{p-1} \subseteq F_{p-1}$. But then C_p has cardinality

$$|C_p| = |C_{p-1} \cap G_p| \geq |C_{p-1}| - 1.$$

Since $|C_0| = d + 1$, we get $|C_p| \geq d + 1 - p$ and so if X is not a cone, $k \geq d + 1$. If T has an edge of cardinality ≥ 3 , some G_p equals some G_r for $r < p$. Then, $C_{p-1} \subseteq G_r \subseteq G_p$ and we get $C_{p-1} = C_p$. Thus, $|C_q| \geq d + 2 - q$ for $q \geq p$, and so if X is not a cone, $k \geq d + 2$.

c. This is shown in Lemma 3.6.

d. Let X have associated tree T . Label the vertices of T with $\{0, 1, \dots, d + 1\}$.

We assume the labeling is such that the induced subgraph on $[0, p]$ is always a tree for $p = 0, \dots, d + 1$. Then, the corresponding ordering F_0, F_1, \dots, F_{d+1} of the facets of X is a stacking order.

Let Y be another stacked simplicial complex with tree S isomorphic to T . Transferring the labeling from T , we get a stacking order G_0, G_1, \dots, G_{d+1} of the facets of Y . Let

$$F_{d+1} \setminus F_\ell = \{v\}, \quad G_{d+1} \setminus G_\ell = \{w\},$$

where $\ell < d + 1$ is such that F_{d+1} is stacked on F_ℓ . The following restrictions are cones by part a, since they have $\leq d + 1$ vertices

$$X_{-\{v\}} = X' * \{v'\}, \quad Y_{-\{w\}} = Y' * \{w'\}$$

and X' and Y' are not cones (since X and Y are not cones). Their trees are obtained from T and S by removing the vertices labeled $d + 1$. The $F'_i = \text{lk}_{F_i} v'$ for $i = 0, 1, \dots, d$ form a stacking order for X' and similarly the $G'_i = \text{lk}_{G_i} w'$ form a stacking order for Y' .

By induction, there is a bijection between $V \setminus \{v, v'\}$ and $W \setminus \{w, w'\}$ sending the facet F'_i of X' to the facet G'_i of Y' . Extend this to a bijection between V and W by $v \mapsto w, v' \mapsto w'$. Then, the facet F_i is sent to the facet G_i for $i = 0, \dots, d$.

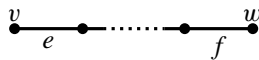
So consider the facets F_{d+1} and G_{d+1} . Let vertex $(d + 1)$ of T be attached to vertex $p \leq d$. So F_{d+1} is attached by the codimension-one face $F_{d+1} \cap F_p$. But this is $F_{d+1} \setminus \{v\}$ and does not contain v' (F_{d+1} does not contain v' since X is not a cone). So this codimension-one face is F'_p . Similarly, G_{d+1} is attached to $G'_p = G_{d+1} \setminus \{w\}$. Since F'_p is sent to G'_p , the facet F_{d+1} is sent to G_{d+1} . \square

4 Trees and the associated separated model

Given a tree T , we define the ideal $I(T)$. These ideals are the separated models of stacked simplicial complexes.

Let T be a tree whose set of vertices is V . Let $E = E(T)$ be its set of edges. The incidence relation $C \subseteq E \times V$ is the set of pairs (e, v) such that $v \in e$. It comes with a natural involution $\tau : C \rightarrow C$ sending $(e, v) \mapsto (e, w)$ where $e = \{v, w\}$.

For $v, w \in V$, denote by vTw the unique path from v to w



and let e, f be the edges incident to, respectively, v, w on this path. For a set A denote by $(A)_2$, the set of subsets $\{a_1, a_2\}$ of cardinality 2. From the directed tree T on V , we get a map

$$\Psi : (V)_2 \rightarrow (C)_2$$

$$\{v, w\} \mapsto \{(e, v), (f, w)\}.$$

For a graph G on V , those vertices that are incident to an edge of G are called the vertices of G . The edges $\Psi(E(G))$ give a graph ΨG whose vertices are in C .

- If G_1 and G_2 have disjoint vertex sets, the same holds for ΨG_1 and ΨG_2 .
- If G is a forest, then ΨG is a forest, since a cycle in ΨG must come from a cycle in G .

The following is a basic object in this article.

Definition 4.1 Let $k[x_C]$ be the polynomial ring whose variables are indexed by the incidence relation C . The *tree ideal* $I(T)$ in $k[x_C]$ associated with the tree T is the edge ideal of ΨT . It is generated by the monomials $m_{v,w} = x_{e,v}x_{f,w}$, one monomial for each pair of distinct vertices v, w in V . The edges e and f are incident to v and w , respectively, on the path vTw .

These tree ideals are introduced in [2, Sect. 5] (but in a slightly less conceptual setting by indexing the variables by $E \times \{0, 1\}$). They are shown to be all the possible separated models for the second power $(x_e \mid e \in E(T))^2$ of the irrelevant maximal ideal in the polynomial ring $k[x_e]_{e \in E(T)}$ whose variables are indexed by the edges of T . In particular, the ideals $I(T)$ are Cohen–Macaulay and their graded Betti numbers are precisely those of the graded free resolution of the second power $(x_e \mid e \in E(T))^2$ of the graded maximal ideal of $k[x_e]_{e \in E(T)}$.

The following is given in [2, Sect. 5].

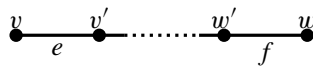
Lemma 4.2 *The facets of the simplicial complex associated with the Stanley–Reisner ideal $I(T)$ are*

$$F_v = \{(e, w) \in C \mid w \text{ vertex on } e \text{ closest to } v\},$$

one facet for each vertex $v \in V$. The cardinality of these facets is then the number of edges of T .

Corollary 4.3 *The ideal $I(T)$ defines the unique non-cone stacked simplicial complex with tree T of dimension $|E| - 1$ with $|E| + 1$ vertices, given in Proposition 3.7d.*

A variation of the map Ψ above is $\overline{\Psi} = (\tau)_2 \circ \Psi$ where $(\tau)_2 : (C)_2 \rightarrow (C)_2$ is derived from the involution τ of C . Considering the path between v and w



this variation is defined as

$$\begin{aligned} \overline{\Psi} : (V)_2 &\rightarrow (C)_2 \\ \{v, w\} &\mapsto \{(e, v'), (f, w')\}. \end{aligned}$$

We will divide the ring $k[x_C]/I(T)$ by the following variable differences:

Definition 4.4 For each pair $\{v, w\}$ in $(V)_2$ let $h_{v,w}$ be the variable difference associated with the edge $\overline{\Psi}\{v, w\}$. So

$$h_{v,w} = x_{e,v'} - x_{f,w'}.$$

Note that $h_{w,v} = -h_{v,w}$. Sometimes we write this as h_e where $e = \{v, w\}$ when this sign plays no role.

5 Regular quotients of tree ideals

We describe precisely what sequences of variable differences are regular for $k[x_C]/I(T)$. The combinatorial description is in terms of partitions of the vertex set of T , Theorem 5.16.

Definition 5.1 Let T be a tree with vertex set V .

- The sequence of vertices v, u, w is T -aligned if u is on the path in T linking v and w .
- The set $\{v, u, w\}$ is non-aligned for T , if no ordering of them makes a T -aligned sequence.

Example 5.2 Consider the second tree in Fig. 7 in the appendix. The sequence of vertices $1, 4, 8$ is T -aligned, and the set $\{1, 5, 8\}$ is non-aligned for T .

Recall the variable difference $h_{v,w}$ from Definition 4.4. The variables of the polynomial ring $k[x_C]$ (see Definition 4.1) are indexed by the incidence relation C .

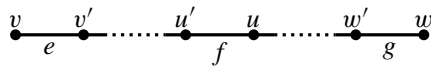
Lemma 5.3 The variable differences in $k[x_C]$ which are nonzero divisors for $k[x_C]/I(T)$ are those coming from the edges of $\overline{\Psi}T$, i.e., the differences $h_{v,w}$.

Proof This is by Lemma 2.5 and the description in Lemma 4.2 of the facets of the simplicial complex associated with $I(T)$. Given any edge outside of $\text{im } \overline{\Psi}$, one may find a facet F_v disjoint from this edge. □

The following is the basic obstruction for a sequence of $h_{v,w}$'s to be regular.

Lemma 5.4 Let v, u, w be T -aligned. Then, $h_{v,u}$ and $h_{v,w}$ do not form a regular sequence.

Proof Let the path vTw be:



We show that $h_{v,w}$ is not $k[x_C]/(I(T) + (h_{v,u}))$ -regular, by showing that $x_{g,w}$ is in the colon ideal $(I(T) + (h_{v,u})) : h_{v,w}$. Indeed

$$\begin{aligned}
 x_{g,w}h_{v,w} &= x_{g,w}(x_{e,v'} - x_{g,w'}) \\
 &= -x_{g,w}x_{g,w'} + x_{g,w}x_{e,v'} \\
 &= -x_{g,w}x_{g,w'} + x_{g,w}x_{f,u'} + x_{g,w}h_{v,u}
 \end{aligned}$$

is an element of $I(T) + (h_{v,u})$. □

The following is straight-forward.

Lemma 5.5 Let $\{u, v, w\}$ be non-aligned. Then, $h_{v,u} + h_{u,w} = h_{v,w}$.

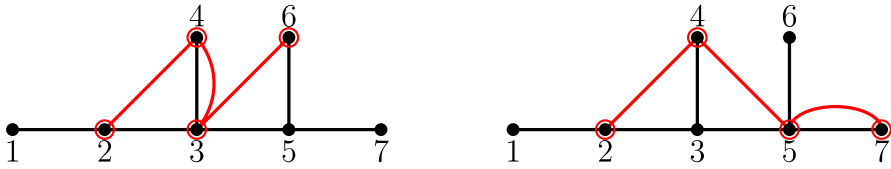


Fig. 5 The trees in Example 5.7

Definition 5.6 Let T be a tree with vertex set V . Let $U \subseteq V$ and let S be a tree on U (S is a priori unrelated to T). The tree S flows with T if whenever v, u, w are T -aligned vertices with $v, u, w \in U$, then v, u, w are S -aligned.

Example 5.7 The tree T in Fig. 5 has black edges and seven vertices. The trees S are drawn in red. In the first case, $U = \{2, 3, 4, 6\}$. The sequence of vertices 2, 3, 4 is T -aligned but not S -aligned, so S does not flow with T . In the second case, $U = \{2, 4, 5, 7\}$ and 4, 5, 7 is a T -aligned and S -aligned sequence. This tree S flows with T .

Lemma 5.8 Let $U \subseteq V$ and let S and T be trees with vertex sets U and V , respectively. Then, S flows with T iff whenever $\{v, w\}$ is an edge in S , there is no $u \in U \setminus \{v, w\}$ such that v, u, w are T -aligned.

Proof Let S flow with T , and let $\{v, w\}$ be an S -edge. If there is u such that v, u, w are T -aligned, then v, u, w would be S -aligned, which is not the case since $\{v, w\}$ is an edge in S .

Conversely suppose the condition holds for edges in S . Let v, u, w be vertices in U which are T -aligned, so $\{v, w\}$ is not an edge of S . Suppose the path vSw does not contain u . We argue by induction on the length $\ell_S(v, w)$ of vSw that this is not possible. Since $\ell_S(v, w) \geq 2$ let $r \in U$ on vSw be distinct from v, w (note that $r \neq u$). Then, $\ell_S(v, w) > \ell_S(v, r)$ and $\ell_S(v, w) > \ell_S(r, w)$.

Consider in T a path p from r to a vertex on the path vTw . We may assume only the end vertex of p is on vTw . If p first hits vTw in the path segment vTu , then r, u, w are T -aligned and with the path rSw being such that $\ell_S(r, w) < \ell_S(v, w)$. By induction, this situation is not possible. The case when p first hits vTw in uTw is similar. \square

Corollary 5.9 For any $U \subseteq V$, there is a tree S with vertices U flowing with T .

Proof Let $v \in V$. Consider v as a center from which the tree T branches out. Let U_0 be the subset of U consisting of $w \in U$ such that the path vTw contains no other vertex in U than w (in particular, if $v \in U$ then $U_0 = \{v\}$).

Now, define S to be the tree whose edges are:

- Pairs $\{u, w\} \subseteq U$ where (1) v, u, w are T -aligned (we allow $v = u$ if $v \in U$) and (2) the path uTw intersects U only in $\{u, w\}$.
- Give the vertices in U_0 a total order. If $u, w \in U_0$ are successive, let $\{u, w\}$ be an edge in S .

The tree S fulfills the criterion of the lemma above, and hence flows with T . □

Definition 5.10 If S is a tree on the vertex set $U \subseteq V$, let $L(S)$ be the linear space with basis the $h_e = h_{v,w}$ where $e = \{v, w\}$ are the edges of S . If $L(S)$ has a basis that is a regular sequence of variable differences for $k[x_C]/I(T)$, we say that $L(S)$ is a *regular linear space*. (Equivalently some basis or any basis of $L(S)$ is a regular sequence).

Lemma 5.11 *Let S be a tree with vertex set U , and assume that only the end vertices v and w of the path vTw are contained in U .*

- If S flows with T , then $h_{v,w} \in L(S)$.
- If $L(S)$ is a regular linear space, then $h_{v,w} \in L(S)$.

Proof Let $v = v_0, v_1, \dots, v_n = w$ be a path in S of length $n \geq 2$. Then, by assumption v_1, \dots, v_{n-1} are not in vTw . If the v_i -incident edges on $v_{i-1}Tv_i$ and v_iTv_{i+1} are always distinct, the paths would splice to give the unique path from v to w . This cannot be the case since this path vTw only has the end vertices in U . Hence, for at least one $v_p, 1 \leq p \leq n - 1$, these two v_p -incident edges are equal. We have three possibilities:

- (1) v_p, v_{p-1}, v_{p+1} are T -aligned,
- (2) v_p, v_{p+1}, v_{p-1} are T -aligned,
- (3) $\{v_{p-1}, v_p, v_{p+1}\}$ is non-aligned for T .

For case (1), if S flows with T , this would give that v_p, v_{p-1}, v_{p+1} are S -aligned, which is not the case since v_{p-1}, v_p, v_{p+1} are S -aligned. Similarly, the second case (2) is excluded. If $L(S)$ is regular the first and second cases are also excluded by Lemma 5.3. Hence, only the last possibility (3) is left.

If S flows with T , then if $v_{p-1}Tv_{p+1}$ contains an element of U , distinct from v_{p-1} and v_{p+1} , such an element would be either on $v_{p-1}Tv_p$ or on v_pTv_{p+1} , and distinct from v_p . But this is not the case by Lemma 5.8 since v_{p-1}, v_p and v_p, v_{p+1} are edges in S . Then, we take out the edge $\{v_{p-1}, v_p\}$ from S and take in the edge $\{v_{p-1}, v_{p+1}\}$ to get a new tree S' which still flows with T by Lemma 5.8. By induction on the length $\ell_{S'}(v, w)$, we have $h_{v,w} \in L(S)$.

If $L(S)$ is a regular linear space, then we again replace S with S' . Due to Lemma 5.5, we have $L(S) = L(S')$ and again we get $h_{v,w} \in L(S)$. □

Proposition 5.12 *Let S be a tree on $U \subseteq V$. If $L(S)$ is a regular linear space, then S flows with T .*

Proof Let $\{v, w\}$ be an edge in S , so $h_{v,w} \in S$. Suppose v, u, w are T -aligned vertices in U . If we show this is not possible, then S flows with T by Lemma 5.8. Choose u as close as possible to v , so vTu only contains v and u from U . By Lemma 5.11, $h_{v,u}$ is in $L(S)$. So both $h_{v,w}$ and $h_{v,u}$ are in $L(S)$. By Lemma 5.4, these two elements do not form a regular sequence, contradicting the fact that $L(S)$ is regular. Hence, there can be no u such that v, u, w are T -aligned. So S flows with T . □

Lemma 5.13 *For any two trees R and S on U flowing with T , one has $L(R) = L(S)$. Thus, U determines a unique regular linear space, denoted $L(U)$.*

Proof Let $\{v, w\}$ be an R -edge. We show $h_{v,w} \in L(S)$. Since R flows with T , the path vTw does not contain any elements of U save the end vertices. Since S flows with T , Lemma 5.11 gives $h_{v,w} \in L(S)$. \square

Lemma 5.14 *Let G be a graph on vertex set V (with G a priori unrelated to T).*

- a. *If $\{h_e\}_{e \in G}$ is a regular sequence for $k[x_C]/I(T)$, then G is a forest.*
- b. *If G is a forest consisting of the trees S_1, \dots, S_r , then $\{h_e\}_{e \in G}$ is a regular sequence iff each $\{h_e\}_{e \in S_i}$ is a regular sequence.*

Proof a. It is enough to show that if G is a cycle O , then $\{h_e\}_{e \in O}$ is not a regular sequence. Denote by $L(O)$ their linear span, and let the cycle be $v_0, v_1, \dots, v_{n-1}, v_n = v_0$ of length n .

We now use induction on the length n of the cycle to show that $L(O)$ cannot be regular linear space. Not every sequence $v_{i-1}v_i v_{i+1}$ is T -aligned for $i = 1, \dots, n - 1$ since $v_0 = v_n$ and T is a tree. Suppose $v_{p-1}v_p v_{p+1}$ is not T -aligned. If say $v_p v_{p-1} v_{p+1}$ are T -aligned, then $h_{v_p, v_{p-1}}$ and $h_{v_p, v_{p+1}}$ do not form a regular sequence, against the assumption. By the same reason, $v_p v_{p+1} v_{p-1}$ are not T -aligned. Hence, $\{v_{p-1}, v_p, v_{p+1}\}$ is non-aligned. Then, $h_{v_{p-1}, v_{p+1}}$ is $h_{v_{p-1}, v_p} + h_{v_p, v_{p+1}}$. Take the edges $\{v_{p-1}, v_p\}$ and $\{v_p, v_{p+1}\}$ out from the cycle O and take in the edge v_{p-1}, v_{p+1} to make a new cycle $L(O') \subseteq L(O)$. By induction, $L(O')$ is not a regular linear space and so neither is $L(O)$.

b. Suppose each S_i gives a regular sequence. This sequence is determined by the edges of $\overline{\Psi}S_i$, and this is a forest. By Corollary 2.6, this is equivalent to each tree in $\overline{\Psi}S_i$ giving a regular sequence. But the disjoint union of the trees in the $\overline{\Psi}S_i$ are precisely the trees in $\overline{\Psi}S$. Hence, Corollary 2.6 gives the result. \square

The following is the converse of Proposition 5.12:

Proposition 5.15 *Let S be a tree on $U \subseteq V$. If S flows with T , then $L(S)$ is a regular linear space.*

Proof By Lemma 5.13 above, if U is the vertex set of S , we may choose S to be any tree on U that flows with T .

Let $v \in V$. Consider the face

$$F_v = \{(e, w') \mid w' \text{ vertex on } e \text{ closest to } v\}.$$

Let $U \subseteq V$ and define the tree S flowing with T with vertices in U as in Corollary 5.9. This tree comes with two types of edges:

- Edges $\{u, w\}$ where u and w are two successive elements in the ordering of U_0 . Then, $\overline{\Psi}\{u, w\} = \{(f, u'), (g, w')\}$ where $f = \{u, u'\}$ is the edge on uTv going out from u and similarly $g = \{w, w'\}$ the edge on vTw going out from w . These $\{(f, u'), (g, w')\}$ give a tree T_0 with vertices from the incidence relation C (actually a line graph).
- Edges $\{u', w\}$ where u' is the element in U on the path vTw closest to w . If $f = \{w', w\}$ and $g = \{u', u\}$ are the edges on $u'Tw$, one has $\overline{\Psi}\{u', w\} = \{(g, u), (f, w')\}$. Each such w gives a unique u' , but one u' may correspond to

several g 's and w 's. For each pair u', g , these edges form a tree $T_{u',g}$, a star, with vertices from C .

The trees T_0 and $T_{u',g}$ (with vertices from C) are all disjoint. Together the edges of these trees give all variable differences $h_{v,w}$ for $\{v, w\}$ an S -edge. The vertices of T_0 are contained in F_v . Each $T_{u',g}$ has all its vertices save (g, u) contained in F_v . By Corollary 2.6, the linear space $L(S)$ is regular. \square

Theorem 5.16 *There is a one-to-one correspondence between regular linear spaces for $k[x_C]/I(T)$ and partitions Q of the vertex set V . If the partition of V is $Q : U_0 \sqcup U_1 \sqcup \dots \sqcup U_r$, then this regular linear space is*

$$L(Q) = L(U_0) \oplus \dots \oplus L(U_r).$$

Proof By Lemma 5.13, each U_i determines a unique linear space $L(U_i)$. If S_i is a tree on U_i flowing with T , then $L(S_i) = L(U_i)$. Let $S = \cup_{i=0}^r S_i$. By Lemma 5.14, the edges of $\overline{\Psi}S$ give a regular sequence. This regular sequence is a basis for $L(Q)$.

Conversely if L is a regular linear space generated by the regular elements $\{h_e\}_{e \in G}$ for some graph G on V , by Lemma 5.14 the graph G decomposes into a forest and we get a partition of V where each U_i is the vertex set of each tree in the forest. (The vertices v of V not incident to any edge of G give singletons $\{v\}$ in the partition). \square

Corollary 5.17 *The length of the longest regular sequence of variable differences for $k[x_C]/I(T)$ is $|E|$. Such a sequence corresponds to the trivial partition of V with only one part, the set V itself. The corresponding tree that flows with T is just T itself. Hence, this regular sequence is given by $\{h_e\}_{e \in E}$ and the quotient ring is $k[x_E]/(x_e \mid e \in E)^2$.*

6 Squarefree quotients

We determine what regular linear spaces give quotient rings of $k[x_C]/I(T)$ whose associated ideals are squarefree. These are the Stanley–Reisner rings of stacked simplicial complexes. Let T be a tree with vertices V .

Lemma 6.1 *Let $U \subseteq V$ and let S be a tree on U . If S flows with T , then the regular quotient of $k[x_C]/I(T)$ by $\{h_e\}_{e \in S}$ is a squarefree monomial ideal iff the vertex set U is an independent vertex set in V for the tree T .*

Proof The following is essential to note: The variables in the quotient ring modulo the sequence $\{h_e\}_{e \in S}$ correspond precisely to the connected components of the graph $\overline{\Psi}S$ with vertex set the incidence relation C .

If the vertex set U is dependent, say contains end vertices of an edge $e = \{v, w\}$, then we divide out by $x_{e,v} - x_{e,w}$ and the ideal of the quotient ring will contain x_e^2 as a generator and so is not squarefree.

Suppose then that U is independent. Let $\{v, w\}$ be a pair of vertices in U . Suppose the associated monomial $x_{e,v}x_{f,w}$ becomes a square after dividing out by the regular sequence. This means that (e, v) and (f, w) are in the same connected component of $\overline{\Psi}S$. Let the edge e have vertices v, v' and the edge f vertices w', w . So v' and w' are

on the path vTw . Removing the edge e from T , we get a component T_v containing v , and similarly removing f from T we get a component T_w containing w .

Any edge in $\bar{\Psi}S$ containing (e, v) is the image of an edge $\{\bar{v}, v'\}$ in S where $\bar{v} \in T_v$. Similarly, we have an edge $\{w', \bar{w}\}$ in S where $\bar{w} \in T_w$. But since (e, v) and (f, w) are in the same connected component of $\bar{\Psi}S$, there must in S be an edge $\{\tilde{v}, \tilde{w}\}$ where \tilde{v} is in T_v and \tilde{w} is in T_w .

Then, either v' or w' from U is in the interior of the path $\tilde{v}T\tilde{w}$. Since S flows with T , this cannot be the case by Lemma 5.8. \square

Theorem 6.2 *There is a one-to-one correspondence between regular linear spaces for $k[x_C]/I(T)$ giving squarefree quotient rings, and partitions of V into sets of independent vertices.*

Proof Suppose we have a squarefree quotient ring. Each part U_i of the partition gives a regular linear space $L(U_i)$. By Lemma 6.1, U_i is independent. Conversely, if we have a partition of V into independent sets U_i , let S_i be a tree on U_i flowing with T . The images $\bar{\Psi}S_i$ have disjoint vertex sets as i varies. Lemma 6.1 above shows that the quotient is squarefree. \square

Using Theorem A.1, the above may equivalently be formulated as follows:

Theorem 6.3 *There is a one-to-one correspondence between regular linear spaces for $k[x_C]/I(T)$ giving squarefree quotient rings, and partitions of the edge set E .*

If P is a partition of the edge set corresponding to the partition Q into independent vertex sets, write $L(P) = L(Q)$.

Corollary 6.4 *The length of the longest regular sequence of variable differences giving a squarefree quotient of $k[x_C]/I(T)$ is $|E| - 1$. It corresponds to the unique partition of V into two independent sets of V for the tree T . Thus, the associated regular linear space is also unique.*

7 Partial order on partitions

If Q and Q' are partitions of the vertex set V of a tree T , we get the linear spaces $L(Q)$ and $L(Q')$. What does the inclusion relation on linear spaces correspond to on partitions? Since the linear spaces depend on additional structure coming from the tree T , this is not simply refinement of partitions.

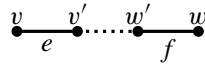
7.1 Partitions of the vertex set

Definition 7.1 Let $U' \subseteq U \subseteq V$. Then, U' is *convex* in U if for every $v, w \in U'$, all vertices on the path vTw that are contained in U are in U' .

Note that such a U' may be convex in some U while not being convex in V .

Lemma 7.2 *Let U' and U be subsets of V . If $L(U')$ is a subspace of $L(U)$, then U' is a convex subset of U , or U' is a singleton (then $L(U') = 0$). Conversely if $U' \subseteq U$ is a convex subset, then $L(U')$ is a subspace of $L(U)$.*

Proof Suppose $L(U')$ is a nonzero subspace of $L(U)$ and there exists $v \in U' \setminus U$. There is another $w \in U'$ such that $h_{v,w} \in L(U')$. Consider the path vTw in T :



Then, $h_{v,w} = x_{e,v'} - x_{f,w'}$ is in $L(U')$. If the edge e occurred in some $h_{a,b}$ generating $L(U)$, since $v \notin U$, one of a or b would have to be v' , and $v' \in U$. But then this $h_{a,b}$ would contain $x_{e,v}$ instead of $x_{e,v'}$. Hence, $U' \subseteq U$.

Let us show that U' is convex in U . Let $v, w \in U'$ be such that vTw contains some $u \in U \setminus U'$. By possibly moving v and w closer to u , and u closer to v , we may assume on vTw that v and w are the only vertices in U' , and on vTu that v and u are the only vertices in U . But then $h_{v,w} \in L(U')$ and $h_{v,u} \in L(U)$ by Lemma 5.11. If $L(U') \subseteq L(U)$, this could not be the case by Lemma 5.4, since $L(U)$ is regular. Hence, if we have inclusion, U' must be convex in U .

Conversely if U' is convex in U , then letting S' be a tree on U' flowing with T , by Lemma 5.11, for each edge e in S' we have $h_e \in L(U)$. □

The following is immediate from the above.

Theorem 7.3 *Let Q' and Q be partitions of V . Then, $L(Q') \subseteq L(Q)$ iff each part U_i of Q is a union of parts of Q' which are convex for U_i . Write then $Q' \preceq Q$.*

In a partition Q of V , if U_i and U_j are parts such that either U_i or U_j is not convex in $U_i \cup U_j$, we say that U_i and U_j are *intertwined*.

Corollary 7.4 *The maximal partitions for the partial order \preceq are the partitions Q such that any two parts U_i and U_j in the partition are intertwined.*

Example 7.5 In the introduction, looking at Fig. 3, the partition of vertices in Example 1.3 is not maximal. We may join

$$\{1, 5\} \cup \{2\} \cup \{3\} \cup \{4\} \preceq \{1, 5\} \cup \{2, 3, 4\}.$$

The latter vertex partition is maximal since it is intertwined. Also note that the first partition is not $\preceq \{1, 2, 3, 4, 5\}$, since $\{1, 5\}$ and $\{2\}$ (as well as $\{4\}$) are intertwined.

The partition $\{1, 2, 3, 4, 5\}$ corresponds to the quotient ring $k[x_E]/(x_e)_{e \in E}^2$, which is $k[x_E]$ divided by the square of the maximal graded ideal. Hence, this ring is not a quotient ring of $k[x_C]/(I(T) + (x_{a,2} - x_{d,4}))$ of Example 1.2, by a regular linear space. (But it is of course a quotient taking a suitable general linear space).

Corollary 7.6 *Let \bar{V} be the partition of V into singletons. Then, for any partition Q of V the interval $[\bar{V}, Q]$ with respect to the partial order \preceq is a Boolean lattice.*

Proof Given a subset U of V , we must show that the lattice of partitions of U into convex parts is a Boolean lattice. Let v be extremal in U in the sense that every other vertex of U is on the same side of v , i.e., there is an edge $e = \{v, w\}$ from v such that

the path from v to any other vertex of U starts with the edge e . Let $U' = U \setminus \{v\}$. By induction, the lattice of partitions of U' into convex subsets is a Boolean lattice B . The partitions Q of U into convex subsets are now of two types: either $\{v\}$ is a singleton class, or v and w are in the same class. This gives that the lattice of partitions of U identifies as $B \times \{0, 1\}$ and so is Boolean. \square

7.2 Partitions of the edge set

If $D' \subseteq D \subseteq E$ are sets of edges of T , we may as above define the notion of D' being convex in D . As above, we may show:

Proposition 7.7 *Let P' and P be partitions of E . Then, $L(P') \subseteq L(P)$ iff each part E_i of P is a union of parts of P' which are convex for E_i . Write then $P' \preceq P$.*

Corollary 7.8 *The maximal partitions for the partial order \preceq are the partitions P such that any two parts E_i and E_j in the partition are intertwined.*

Example 7.9 In the introduction, looking at Fig. 3, the partition of edges in Example 1.3 is not maximal. We may join

$$\{a, d\} \cup \{b\} \cup \{c\} \preceq \{a, d\} \cup \{b, c\}.$$

In the latter partition, the parts are intertwined and so it is maximal. It corresponds to the vertex partition $\{1, 5\} \cup \{3, 4\} \cup \{2\}$. This vertex partition is also maximal (but that does not necessarily follow from the edge partition being maximal).

Corollary 7.10 *Let \bar{E} be the partition of E into singletons. Then, for any partition P of E the interval $[\bar{E}, P]$ with respect to the partial order \preceq is a Boolean lattice.*

8 Hypertree of quotients and triangulated balls

We describe the squarefree quotients of $k[x_C]/I(T)$ by regular linear spaces whose associated simplicial complex is a *triangulated ball*. In particular, we describe when we get triangulations of polygons.

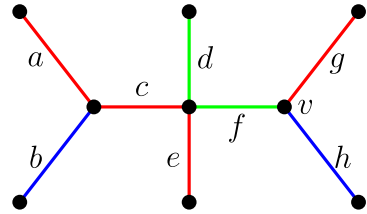
Let

$$P : E_1 \sqcup E_2 \sqcup \cdots \sqcup E_r$$

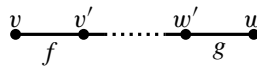
be a partition of the edge set E of the tree T . We may think of the edges of E_i as a color class. The partition corresponds by Theorem A.1 to a partition $V = U_0 \sqcup U_1 \sqcup \cdots \sqcup U_r$ of the vertex set into independent sets of vertices. Let $S = S_0 \cup S_1 \cup \cdots \cup S_r$ where the S_i are trees on U_i flowing with T . The image $\bar{\Psi}S$ is a forest, and each $\bar{\Psi}S_j$ is a collection of connected components (trees) of $\bar{\Psi}S$. Moreover, $L(P) = \bigoplus_{i=1}^r L(S_i)$. In the sequel, we also write $\bar{\Psi}P$ for $\bar{\Psi}S$.

Let us describe the variables in the quotient ring $k[x_C]/L(P)$ (this is a polynomial ring). These variables identify as subsets of of the incidence relation C . Those subsets

Fig. 6 The tree in Example 8.1. The partition of the edge set is illustrated by the different colors (Color figure online)



which contain more than one element arise as follows. For each class E_i , consider maximal sets of edges $E_{ij} \subseteq E_i$ such that for every pair of edges f, g in E_{ij} , the only edges in E_i on the unique path from f to g are f and g themselves. For given i , two such maximal E_{ij} and $E_{i'j'}$ have at most one edge in common. (In fact, the E_{ij} 's form the set of edges in a hypertree on E_i .) If f, g are edges in a E_{ij} with path



then $x_{f,v'} - x_{g,w'}$ is a variable difference in $L(P)$. It gives a class $[x_{f,v'}]$, a variable in $k[x_C]/L(P)$. This gives one variable in $k[x_C]/L(P)$ for each set E_{ij} .

Example 8.1 In Fig. 6, we have a partition of the edges into three color classes. The four red edges give eight red variables in $k[x_C]$. The red edges give two maximal sets $\{a, c\}$ and $\{c, e, g\}$, each of which combines into one variable, giving five red variables in the quotient ring $k[x_C]/L(P)$.

We now describe the facets of the simplicial complex corresponding to the quotient $k[x_C]/(I(T) + L(P))$. For each $v \in V$ and color class E_i , let E_i^v be the set of edges f in E_i such that on the path from f to v the only edge in E_i is f itself. Then, E_i^v is a maximal set E_{ij} as above and hence gives a variable $x_{E_i^v}$ in $k[x_C]/L(P)$. We have $x_{E_i^v} = x_{E_j^w}$ iff $i = j$ and there is no edge from E_i on the path from v to w . Let

$$F_v = \{E_i^v \mid i = 1, \dots, r\}.$$

Example 8.2 Consider Fig. 6. The facet F_v of $k[x_C]/(I(T) + L(P))$ is of cardinality 3. Its elements are the three maximal sets

$$E_{\text{red}}^v = \{c, e, g\}, \quad E_{\text{blue}}^v = \{h, b\}, \quad E_{\text{green}}^v = \{f\}.$$

Lemma 8.3 *The facets of the simplicial complex associated with $k[x_C]/(I(T) + L(P))$ are the F_v 's, for $v \in V$. In particular, the cardinality of each facet is the number of classes in the partition P .*

Proof This follows by repeated use of Lemma 2.5. □

Lemma 8.4 *Let $e = \{v, w\}$ be an edge in T , in the class E_k . Then, $E_i^v = E_i^w$ for $i \neq k$. The facets F_v and F_w have a codimension-one face in common. It is the set*

$$F_e = \{E_i^v (= E_i^w) \mid i \neq k\}.$$

Proof This is clear. \square

Lemma 8.5 *The facets F_v and F_w have a codimension-one face G in common if and only if the path from v to w has all edges of the same color. Then, for all edges e on this path, the F_e are equal, and this is G . In particular, G is common to all facets F_u for u on this path.*

Proof Suppose the edges on the path are all of the same color red. Let the path be $v = u_0, u_1, \dots, u_m = w$ with e_i the edge $\{u_{i-1}, u_i\}$. Then, for each edge e_i

$$F_{u_{i-1}} = F_{e_i} \cup \{(e_i, u_{i-1})\}, \quad F_{u_i} = F_{e_i} \cup \{(e_i, u_i)\}$$

for suitable F_i . Since e_i and e_{i+1} are successive red edges, we divide out by the variable difference $x_{e_i, u_i} - x_{e_{i+1}, u_i}$ and so (e_i, u_i) identifies with (e_{i+1}, u_i) . We also have

$$F_{u_i} = F_{e_{i+1}} \cup \{(e_{i+1}, u_i)\}, \quad F_{u_{i+1}} = F_{e_{i+1}} \cup \{(e_{i+1}, u_{i+1})\}.$$

We must then have $F_{e_i} = F_{e_{i+1}}$. Hence, all these F_{e_i} are equal.

Suppose the edges on the path are not of the same color. Suppose going from v to w , there is first a sequence of red edges, the first one being $e = \{v = u_0, u_1\}$ and then eventually a blue edge $f = \{u_i, u_{i+1}\}$.

- The facet F_v contains (e, u_0) of color red. The facet F_w also contains a (class) of a red edge. If this red edge was e , it would have to be (e, u_1) . Hence, (e, u_0) is in F_v but not in F_w .
- Similarly, the blue (f, u_i) is in F_v , and by a similar argument as above, (f, u_i) is not in F_w .
- The upshot is that $F_v \setminus F_w$ contains at least two elements, and so F_v and F_w do not intersect in codimension one. \square

Recall that a set of edges in the tree T is *independent* if no two edges in the set are adjacent. The quotient of $k[x_C]/I(T)$ by $L(P)$ is a stacked simplicial complex. It is again a quotient of the polynomial ring $k[x_C]/L(P)$. Each part E_i of E is a subforest of T . Let T_{ij} be the trees of this subforest and $V_{ij} \subseteq V$ the support of T_{ij} . Let T' be the hypertree whose edges are the sets V_{ij} . In particular, note that if P is a partition whose parts E_i consist of independent edges, then each T_{ij} is simply an edge, and so $T' = T$.

Proposition 8.6 *Let P be a partition of the edge set of T . The quotient of $k[x_C]/I(T)$ by $L(P)$ corresponds to a stacked simplicial complex X whose associated hypertree is T' .*

Proof Consider then the tree T_{ij} . Let v, u, w be three vertices in V_{ij} . If they are T -aligned for some ordering, the facets F_v, F_u, F_w of X have a codimension-one face in common by Lemma 8.5. Suppose $\{u, v, w\}$ are non-aligned. Consider the path from v to w and let e be its last edge. Then, e is also the last edge on the path from u to

w . Write $F_w = F \cup \{(e, w)\}$. By the argument of Lemma 8.5, all x on these paths have F_x containing F . We readily get that F is a codimension-one face of every F_x for $x \in V_{ij}$. Thus, each V_{ij} form an edge in the hypertree T' associated with the simplicial complex X . \square

Theorem 8.7 *There is a one-to-one correspondence between:*

- Regular linear spaces giving squarefree quotients of $k[x_C]/I(T)$ corresponding to triangulated balls, and
- Partitions P of the edge set E of T into sets of independent edges.

The codimension-one faces of this triangulation which are on two facets are precisely the faces F_e of Lemma 8.4. Let $B(T, P)$ be the ideal generated by the $x_{F_e} = \prod_{a \in F_e} x_a$ for $e \in E(T)$. Then, $I(T) + B(T, P)$ is the Stanley–Reisner ideal in $k[x_C]/L(P)$ defining the boundary of this triangulated ball, a triangulated sphere.

Proof When the edges are partitioned into independent sets, the hypertree T' is an ordinary tree T . And when a stacked simplicial complex gives an ordinary tree T , it is a triangulated ball and may be realized as a stacked polytope.

The only faces on a stacked simplicial complex not on the boundary are the codimension one faces which are on at least two faces. This gives the statement about the Stanley–Reisner ideal of the boundary. \square

Remark 8.8 In [7], the first author et al. give the construction of large classes of triangulated balls, defined by letterplace ideals of posets. The ideal defining the boundary of triangulated balls is given in a similar way there.

In particular, triangulations of simplicial polygons correspond to partitions of trees T into *three parts*, each part being a set of independent edges. Thus, only trees T whose maximal vertex degree is 3 arise in this context.

Corollary 8.9 *The length of the longest regular sequence of variable differences giving a squarefree quotient of $k[x_C]/I(T)$ that corresponds to a triangulated ball is $|E| - \Delta$, where Δ is the maximal degree of a vertex of T .*

Proof This is because the minimal number of parts in a partition of $E(T)$ into independent edges, the edge chromatic number of the tree T , is the maximal degree of a vertex in T , [1]. \square

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Data availability This manuscript has no associated data.

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Appendix A. Partitions of the vertices and edges of a tree

We recall the basic result on trees from [10] on the correspondence between partitions of edges and partitions of vertices into independent sets. Let T be a tree with vertex set V and edge set E . We consider partitions of the vertices

$$V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_r \quad (2)$$

into disjoint sets such that each V_i is an independent set of vertices. (This is almost the same as a coloring of vertices, but not quite: The symmetric group S_r acts on colorings by permuting the color labels of the V_i . So such a partition is an orbit for the actions of S_r . The class of such orbits, or equivalently of partitions (2) are also called *non-equivalent vertex colorings*, see [14]).

We also consider partitions of the edges

$$E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_s.$$

Here, we have no independence requirements. Any partition is good.

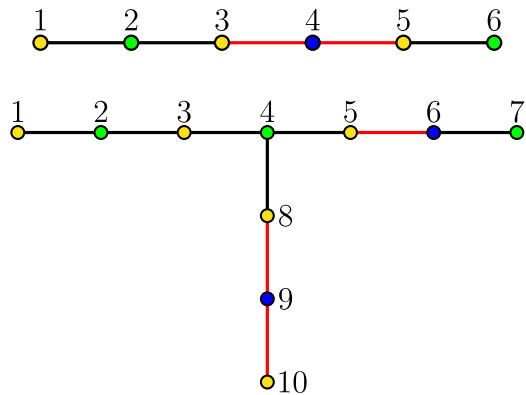
Now, we make a correspondence as follows. Given such a partition of V , make a partition of E as follows: If v and w are vertices consider the unique path in T linking v and w . Let f , respectively, g , be the edge incident to v , respectively, w , on this path. If (i) v and w are in the *same part* V_i of V and (ii) *no other* vertex on this path is in the part V_i , then put f and g into the same part of E , and write $f \sim_E g$. The partition of edges is the equivalence relation \sim_E generated by \sim_E , i.e., the smallest equivalence relation on E containing \sim_E . Note that in general \sim_E alone would not be reflexive nor transitive.

Conversely, given a partition of the edge set E , make a partition of V as follows: Let v and w be distinct vertices, and consider again the path from v to w . If (i) the edges f and g are distinct, (ii) f and g are in the *same part* E_j , and (iii) *no other* edge on this path is in the part E_j , then put v and w in the same part of V , and write $v \sim_V w$. The partition of vertices is the equivalence relation \sim_V generated by \sim_V .

Theorem A.1 ([10]) *Let T be a tree with vertex set V and edge set E . The above gives a one-to-one correspondence between partitions of the vertices V into $r + 1$ independent sets, and partitions of the edges E into r sets.*

Example A.2 Any tree has a unique partition of the vertices into two independent sets (two colors modulo S_2). This corresponds to the partition of the edges into one part (one color).

Fig. 7 The trees in Example A.3. The partitions of the vertex set and the edge set are illustrated by the different colors (Color figure online)



Example A.3 In Fig. 7, we partition the edges into red and black color classes. The vertices are then partitioned into three sets, each consisting of independent vertices. The partition of the vertex set of the first tree is

$$\{1, 3, 5\} \cup \{2, 6\} \cup \{4\},$$

and that of the second tree is

$$\{1, 3, 5, 8, 10\} \cup \{2, 4, 7\} \cup \{6, 9\}.$$

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