Direct Construction of Optimal Z-Complementary Code Sets with Even Lengths by Using Generalized Boolean Functions

Gobinda Ghosh, Sudhan Majhi, Palash Sarkar and Ashish Kumar Upadhaya

***Abstract*—The Z-complementary code set (ZCCS) is well-known for being used in multicarrier code-division multiple access (MC-CDMA) systems to provide interference-free communication in a quasi-synchronous environment. Based on the existing literature, the direct constructions of optimal ZCCSs are limited to their lengths. This letter proposes a direct construction of optimal ZCCSs for all possible even lengths using generalized Boolean functions. The maximum column sequence peak-to-mean envelope power ratio (PMEPR) of the proposed ZCCSs is upper-bounded by two which can benefit in managing PMEPR over a ZCCS-based MC-CDMA system.**

***Index Terms*—MC-CDMA, GBF, ZCCS, ZCZ, PMEPR.**

I. Introduction

MULTICARRIER code-division multiple access (MC-CDMA) is a multiple access scheme used in orthogonal frequency division multiplexing (OFDM) based telecommunication systems, allowing the system to support multiple users at the same time and over the same frequency band. The traditional orthogonal code such as Walsh-Hadamard, gold-codes, and m-sequences suffer from high peak-to-mean envelope power ratio (PMEPR) problem as well as multipath interference (MPI) and multiple-access interference over asynchronous environment for MC-CDMA [1], [2]. The complete complementary code (CCC) [3] has perfect cross and auto-correlation characteristics, which allows for simultaneous interference-free transmission in MC-CDMA over the asynchronous system. A major disadvantage of CCC is that the number of supported users for MC-CDMA is limited by the number of row sequences in each complementary matrix, i.e., number of subcarriers. The set size of the Z-complementary code set (ZCCS) system is much bigger than that of CCC [4]. It enables more number of users to be supported by a ZCCS-based MC-CDMA system in a quasi-synchronous environment with low computational complexity, unlike a CCC-based MC-CDMA system.

This work was supported in part by the CSIR, India, under Grant 09/1023(0033)/2019-EMR-I. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Yeong-Luh Ueng. (Corresponding author: Sudhan Majhi.)

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In [4]–[9] generalized Boolean function (GBF) based construction of complementary set has been discussed. The GBFs based construction of CCCs [10] were extended to optimal ZCCSs in [4],[5]. However, GBFs based construction of optimal ZCCS has a limitation on the sequence lengths which is in the form of power-of-two [4], [5], [8], [11]. Recently, [12]–[14] proposed a direct construction of optimal ZCCSs, which can provide non-power-of-two length sequences but are limited to a few numbers. Besides direct constructions, many indirect constructions of ZCCS can be found in [15]–[18] which are dependent on some kernel at its initial stages. The limitation on the lengths of optimal ZCCS through direct constructions in the existing literature motivates us to search for new GBFs to provide all possible even lengths.

In search of new ZCCS, in this article, we propose a direct construction of optimal ZCCS for all possible even lengths, using GBFs. It has been shown that, the proposed construction is able to maintain a minimum column sequence PMEPR of 2. The ZCCS reported in [12] appears as a special case of proposed construction.

II. Preliminary

This section provides a few fundamental concepts and lemmas that will be used throughout the proposed construction. Let $\mathbf{x}_1 = [x_{1,0}, x_{1,1}, \ldots, x_{1,N-1}]$ and $\mathbf{x}_2 = [x_{2,0}, x_{2,1}, \ldots, x_{2,N-1}]$ be two sequences whose components are complex numbers. A function is defined for integer value of τ as

$$\Theta(\mathbf{x}_1, \mathbf{x}_2)(\tau) = \begin{cases} \sum_{i=0}^{N-1-\tau} x_{1,i+\tau} x_{2,i}^*, & 0 \leq \tau < N, \\ \sum_{i=0}^{N+\tau-1} x_{1,i} x_{2,i-\tau}^*, & -N < \tau < 0, \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

where $*$ denotes the complex conjugate. When $\mathbf{x}_1 = \mathbf{x}_2$, $\Theta(\mathbf{x}_1, \mathbf{x}_2)(\tau) = \mathcal{A}_{\mathbf{x}_1}(\tau)$. The functions Θ and $\mathcal{A}$ are known as aperiodic cross-correlation function (ACCF) of $\mathbf{x}_1$ and $\mathbf{x}_2$ and aperiodic auto-correlation function (AACF) of $\mathbf{x}_1$ respectively. Let $\mathbf{B} = \{\mathbf{B}^0, \mathbf{B}^1, \ldots, \mathbf{B}^{M-1}\}$ be a collection of M matrices each of dimensions $K \times N$, i.e., $\mathbf{B}^\delta = \left[\mathbf{b}_0^\delta, \mathbf{b}_1^\delta, \ldots, \mathbf{b}_{K-1}^\delta\right]_{K \times N}^T$ where T denotes a matrix's transpose and each $\mathbf{b}_i^\delta$ is a complex-valued sequence of length N, i.e., $\mathbf{b}_i^\delta = (b_{i,0}^\delta, b_{i,1}^\delta, \ldots, b_{i,N-1}^\delta)$. Suppose $\mathbf{B}^{\delta_1}, \mathbf{B}^{\delta_2} \in \mathbf{B}$, where $0 \leq \delta_1, \delta_2 \leq M-1$, we define ACCF between $\mathbf{B}^{\delta_1}$

and $\mathbf{B}^{\delta_2}$ as, $\Theta(\mathbf{B}^{\delta_1},\mathbf{B}^{\delta_2})(\tau) = \sum_{i=0}^{K-1}\Theta(\mathbf{b}_i^{\delta_1},\mathbf{b}_i^{\delta_2})(\tau)$. When $\delta_1=\delta_2$, we denote ACCF by AACF.

Definition 1: Code set $\mathbf{B}$ is called a ZCCS ([4]) if

$$\Theta(\mathbf{B}^{\delta_1},\mathbf{B}^{\delta_2})(\tau)=\begin{cases}KN, & \tau=0,\delta_1=\delta_2,\\ 0, & 0<|\tau|<Z,\delta_1=\delta_2,\\ 0, & |\tau|<Z,\delta_1\neq\delta_2,\end{cases} \tag{2}$$

where Z denotes zero correlation zone (ZCZ) width. With the parameter K,N,M, and Z, we denote the set of matrices $\mathbf{B}$ as $(M,Z)-ZCCS_K^N$ which is called optimal for $M=K\lfloor\frac{N}{Z}\rfloor$ and non-optimal for $M<K\lfloor\frac{N}{Z}\rfloor$ [19]. When $K=M$ and $Z=N$, we denote $\mathbf{B}$ by (K,K,N)-CCC.

A. Generalized Boolean Functions (GBFs)

A degree i monomial is a product of i distinct variables from the set $\{y_0,\ldots,y_{m-1}\}$. GBFs are functions $g{:}\{0,1\}^m\to\mathbb{Z}_q$ that are represented as a linear combination of monomials formed by the variables $\{y_0,\ldots,y_{m-1}\}$ where each $y_i's$ is a Boolean variable and coefficient of each monomial is drawn from $\mathbb{Z}_q$ where q denotes a positive integer. The order of g determines by the greatest degree monomial with a non-zero coefficient contained in the expression of g. As an example $4y_2y_1+y_0$ is a GBF of three variables y_0,y_1, and y_2 of order 2. A graph of second-order GBF g is denoted by $G(g)$ [7]. Let $\psi(g)$ denote a complex-valued sequence corresponding to a GBF g and it is defined as $\psi(g)=(\omega_q^{g_0},\ldots,\omega_q^{g_{2^m-1}})$, where ω_q denotes $\exp\left(2\pi\sqrt{-1}/q\right)$, $g_r=g(r_0,\ldots,r_{m-1})$, and $(r_0,\ldots,r_{m-1})$ is the binary vector representation of integer r ($r=\sum_{\alpha=0}^{m-1}r_\alpha 2^\alpha$). Let $\mathcal{C}=(f_1,f_2,\ldots,f_M)$ be an ordered set of M GBFs. The code $\psi(\mathcal{C})$ corresponding to $\mathcal{C}$ can be expressed as $\psi(\mathcal{C})=[\psi(f_1),\psi(f_2),\ldots,\psi(f_M)]^T$.

Lemma 1: (Construction of CCC [6])
Let $q\geq 2$ be an even positive integer and $g:\mathbb{Z}_2^m\to\mathbb{Z}_q$ be a second-order GBF and $\tilde{g}$ be the reversal of g, i.e, $\tilde{g}(y_0,\ldots,y_{m-1})=g(1-y_0,\ldots,1-y_{m-1})$. Let $\{\beta_0,\ldots,\beta_{n-1}\}\subset\{0,1,\ldots,m-1\}$ and the graph $G(g)$ contain vertices denoted as $y_{\beta_0},\ldots,y_{\beta_{n-1}}$ such that, after executing a deletion operation on those vertices, the resultant graph reduces to a path. Also let the edges in the path have identical weight of $\frac{q}{2}$. Also let the binary representation of the integer r be $\mathbf{r}=(r_0,\ldots,r_{n-1})$. Then the codes are defined as

$$\begin{aligned}G_r&=\left\{g+\frac{q}{2}\Big((\mathbf{v}+\mathbf{r})\cdot\mathbf{y}+v_ny_\gamma\Big):\mathbf{v}\in\{0,1\}^n,v_n\in\{0,1\}\right\},\\ \bar{G}_r&=\left\{\tilde{g}+\frac{q}{2}\Big((\mathbf{v}+\mathbf{r})\cdot\bar{\mathbf{y}}+\bar{v}_ny_\gamma\Big):\mathbf{v}\in\{0,1\}^n,v_n\in\{0,1\}\right\},\end{aligned} \tag{3}$$

where γ specifies the label for either of the end vertices in the path, $\mathbf{y}=(y_{\beta_0},\ldots,y_{\beta_{n-1}})$, $\bar{\mathbf{y}}=(1-y_{\beta_0},\ldots,1-y_{\beta_{n-1}})$, $\mathbf{v}=(v_0,\ldots,v_{n-1})$. Then $\{\Psi(G_r),\Psi^*(\bar{G}_r):0\leq r<2^n\}$ forms $(2^{n+1},2^{n+1},2^m)$-CCC, over $\mathbb{Z}_q$, where $\Psi^*(\cdot)$ denotes the complex conjugate of $\Psi(\cdot)$.

Lemma 2: ([20]) Let t and t' be two non-negative integers, where $0\leq t\neq t'<p_i$, p_i is a prime number. Then $\sum_{j=0}^{p_i-1}\omega_{p_i}^{(t-t')j}=0$.

Let l be a positive integer and $p_1,p_2,\ldots,p_l$ be prime numbers and $\mathbf{c}=(c_1,c_2,\ldots,c_l)$ where $0\leq c_i\leq p_i-1$ and $1\leq i\leq l$. Let $g{:}\{0,1\}^m\to\mathbb{Z}_q$ be a second-order GBF of m variables and let $\mathbf{Y}=(y_0,\ldots,y_{b-1})\in\mathbb{Z}_2^b$ where $b=m+\sum_{i=1}^l s_i$. The following GBFs $M^{\mathbf{c}}:\mathbb{Z}_2^b\to\mathbb{Z}_a$ and $N^{\mathbf{c}}:\mathbb{Z}_2^b\to\mathbb{Z}_a$ are defined with the help of g as

$$\begin{aligned}M^{\mathbf{c}}(\mathbf{Y})&=\frac{a}{q}g(y_0,\ldots,y_{m-1})+\sum_{i=1}^{l}c_i\frac{a}{p_i}\sum_{k=0}^{s_i-1}2^k y_{m+\sum_{j=0}^{i-1}s_j+k},\\ N^{\mathbf{c}}(\mathbf{Y})&=\frac{a}{q}\tilde{g}(y_0,\ldots,y_{m-1})+\sum_{i=1}^{l}c_i\frac{a}{p_i}\sum_{k=0}^{s_i-1}2^k y_{m+\sum_{j=0}^{i-1}s_j+k},\end{aligned} \tag{4}$$

where $a=l.c.m(q,p_1,p_2,\ldots,p_l)$ and $s_i\in\mathbb{Z}^+$ which refers to the collection of all positive integers and $s_0=0$. From (4), it is clear that both $M^{\mathbf{c}}(\mathbf{Y})$ and $N^{\mathbf{c}}(\mathbf{Y})$ are GBFs of b variables over $\mathbb{Z}_a$. We chose s_i in such a way that $p_i\leq 2^{s_i}$ $\forall i\in\{1,2,\ldots,l\}$. Let $h:\{0,1\}^{n+1}\to\mathbb{Z}_q$ be a function. For simplicity, we denote $g^{\mathbf{V},\mathbf{r},h}(y_0,\ldots,y_{m-1})$ by $g^{\mathbf{V},\mathbf{r},h}$ and $s^{\mathbf{V},\mathbf{r},h}(y_0,\ldots,y_{m-1})$ by $s^{\mathbf{V},\mathbf{r},h}$ and define the functions as

$$\begin{aligned}g^{\mathbf{V},\mathbf{r},h}&=g(y_0,\ldots,y_{m-1})+h(\mathbf{V})+\frac{q}{2}((\mathbf{v}+\mathbf{r})\cdot\mathbf{y}+v_ny_\gamma),\\ s^{\mathbf{V},\mathbf{r},h}&=\tilde{g}(y_0,\ldots,y_{m-1})+h(\mathbf{V})+\frac{q}{2}((\mathbf{v}+\mathbf{r})\cdot\bar{\mathbf{y}}+\bar{v}_ny_\gamma),\end{aligned} \tag{5}$$

where $\mathbf{V}=(\mathbf{v},v_n)\in\{0,1\}^{n+1}$. We also denote $M^{\mathbf{V},\mathbf{r},\mathbf{c},h}(\mathbf{Y})$ by $M^{\mathbf{V},\mathbf{r},\mathbf{c},h}$ and $N^{\mathbf{V},\mathbf{r},\mathbf{c},h}(\mathbf{Y})$ by $N^{\mathbf{V},\mathbf{r},\mathbf{c},h}$ and define the functions as

$$\begin{aligned}M^{\mathbf{V},\mathbf{r},\mathbf{c},h}&=M^{\mathbf{c}}(\mathbf{Y})+\frac{a}{q}\left(h(\mathbf{V})+\frac{q}{2}\left((\mathbf{v}+\mathbf{r})\cdot\mathbf{y}+v_ny_\gamma\right)\right),\\ N^{\mathbf{V},\mathbf{r},\mathbf{c},h}&=N^{\mathbf{c}}(\mathbf{Y})+\frac{a}{q}\left(h(\mathbf{V})+\frac{q}{2}((\mathbf{v}+\mathbf{r})\cdot\bar{\mathbf{y}}+\bar{v}_ny_\gamma)\right).\end{aligned} \tag{6}$$

As per our assumption, for any choice of $\mathbf{V}$, h, and $\mathbf{r}$, the functions $g^{\mathbf{V},\mathbf{r},h}$ and $s^{\mathbf{V},\mathbf{r},h}$ are $\mathbb{Z}_q$-valued GBFs of m variables and $M^{\mathbf{V},\mathbf{r},\mathbf{c},h}$ and $N^{\mathbf{V},\mathbf{r},\mathbf{c},h}$ are $\mathbb{Z}_a$-valued GBFs of b variables. We define the complex-valued sequence as

$$\Psi(M^{\mathbf{V},\mathbf{r},\mathbf{c},h})=(\omega_a^{M_0^{\mathbf{V},\mathbf{r},\mathbf{c},h}},\ldots,\omega_a^{M_{2^b-1}^{\mathbf{V},\mathbf{r},\mathbf{c},h}}) \tag{7}$$

where $M_k^{\mathbf{V},\mathbf{r},\mathbf{c},h}=M^{\mathbf{V},\mathbf{r},\mathbf{c},h}(k_0,\ldots,k_{b-1})$, $0\leq k<2^b$ and the binary representation of the integer k is $(k_0,\ldots,k_{b-1})$. The kth element of $\Psi(M^{\mathbf{V},\mathbf{r},\mathbf{c},h})$ is given by

$$w_a^{M_k^{\mathbf{V},\mathbf{r},\mathbf{c},h}}=\omega_q^{g_{j_0}^{\mathbf{V},\mathbf{r},h}}\omega_{p_1}^{c_1(j_1)}\omega_{p_2}^{c_2(j_2)}\ldots\omega_{p_l}^{c_l(j_l)}, \tag{8}$$

where $0\leq j_0\leq 2^m-1$, $\omega_q^{g_{j_0}^{\mathbf{V},\mathbf{r},h}}$ represents the j_0th element of the sequence corresponding to the GBF $g^{\mathbf{V},\mathbf{r},h}$, $0\leq j_i\leq 2^{s_i}-1$ $\forall i\in\{1,2,\ldots,l\}$ and

$$k=j_0+j_12^m+j_22^{m+s_1}+\ldots+j_l2^{m+s_1+\ldots+s_{l-1}}.$$

Now, we truncate the sequence $\Psi(M^{\mathbf{V},\mathbf{r},\mathbf{c},h})$ by removing all the kth elements of the form $w_q^{g_{j_0}^{\mathbf{V},\mathbf{r},h}}w_{p_1}^{c_1(j_1)}w_{p_2}^{c_2(j_2)}\ldots w_{p_l}^{c_l(j_l)}$ if at least one of $j_i\geq p_i$ where $1\leq i\leq l$. Therefore, after the truncation, we left with a sequence $\Psi_{Trun}(M^{\mathbf{V},\mathbf{r},\mathbf{c},h})$ where, the k'th element of $\Psi_{Trun}(M^{\mathbf{V},\mathbf{r},\mathbf{c},h})$ is of the form

$$w_q^{g_{j_0'}^{\mathbf{V},\mathbf{r},h}}w_{p_1}^{c_1(j_1')}w_{p_2}^{c_2(j_2')}\ldots w_{p_l}^{c_l(j_l')}$$

where $0 \leq k' < 2^m \prod_{i=1}^{l} p_i$, $0 \leq j'_0 \leq 2^m - 1$, $0 \leq j'_i \leq p_i - 1$, $\forall i \in \{1, 2, \dots, l\}$ and

$$k' = j'_0 + j'_1 2^m + j'_2 p_1 2^m + \dots + j'_l p_1 p_2 \dots p_{l-1} 2^m.$$

We partition the length of $\Psi_{Trun}(M^{\mathbf{V},\mathbf{r},\mathbf{c},h})$ by $\prod_{i=1}^{l} p_i$ subsequences where each subsequences has length 2^m. The jth subsequence of $\Psi_{Trun}(M^{\mathbf{V},\mathbf{r},\mathbf{c},h})$ and $\Psi_{Trun}(N^{\mathbf{V},\mathbf{r},\mathbf{c},h})$ are given by

$$\begin{aligned} \Psi^j_{Trun}(M^{\mathbf{V},\mathbf{r},\mathbf{c},h}) &= \left(\prod_{i=1}^{l} w_{p_i}^{c_i(j'_i)}\right)\left(w_q^{g_0^{\mathbf{V},\mathbf{r},d,h}}, \dots, w_q^{g_{2^m-1}^{\mathbf{V},\mathbf{r},d,h}}\right), \\ \Psi^j_{Trun}(N^{\mathbf{V},\mathbf{r},\mathbf{c},h}) &= \left(\prod_{i=1}^{l} w_{p_i}^{c_i(j'_i)}\right)\left(w_q^{s_0^{\mathbf{V},\mathbf{r},d,h}}, \dots, w_q^{s_{2^m-1}^{\mathbf{V},\mathbf{r},d,h}}\right), \end{aligned} \tag{9}$$

where $j = j'_1 + j'_2 p_1 + \dots + j'_l p_1 \dots p_{l-1}$. Let $S = \{M^{\mathbf{V_1},\mathbf{r},\mathbf{c},h}, \dots, M^{\mathbf{V_{2^{n+1}}},\mathbf{r},\mathbf{c},h}\}$ be an ordered set of 2^{n+1} GBFs where $\mathbf{V}_i \in \{0,1\}^{n+1}$. We define $\Psi_{Trun}(S) = [\Psi_{Trun}(M^{\mathbf{V_1},\mathbf{r},\mathbf{c},h}), \dots, \Psi_{Trun}(M^{\mathbf{V_{2^{n+1}}},\mathbf{r},\mathbf{c},h})]^T$.

III. Proposed Construction of ZCCS

Theorem 1: Let $g : Z_2^m \rightarrow Z_q$ be a GBF as defined in *Lemma 1*, such that after deleting the vertices $y_{\beta_0}, \dots, y_{\beta_{n-1}}$ from the graph of g the resultant graph reduces to a path where $n \leq m$. Let y_γ be either of the end vertices in the path. Let $s_i \in \mathbb{Z}^+$ be such that $2 \leq p_i \leq 2^{s_i}, \forall i$ where, p_i's are primes and $1 \leq i \leq l$. Let $\mathbf{c} = (c_1, c_2, \dots, c_l)$ where, $0 \leq c_i < p_i$ and $a = l.c.m(q, p_1, \dots, p_l)$. We define

$$\begin{aligned} M^{\mathbf{c}}(\mathbf{Y}) &= \frac{a}{q} g(y_0, \dots, y_{m-1}) + \sum_{i=1}^{l} c_i \frac{a}{p_i} \sum_{k=0}^{s_i - 1} 2^k y_{m + \sum_{j=0}^{i-1} s_j + k}, \\ N^{\mathbf{c}}(\mathbf{Y}) &= \frac{a}{q} \tilde{g}(y_0, \dots, y_{m-1}) + \sum_{i=1}^{l} c_i \frac{a}{p_i} \sum_{k=0}^{s_i - 1} 2^k y_{m + \sum_{j=0}^{i-1} s_j + k}. \end{aligned} \tag{10}$$

Let $h : \{0,1\}^{n+1} \rightarrow \mathbb{Z}_q$ be a function such that $h \in \{\lambda, \frac{q}{2} + \lambda\}$ where $\lambda \in Z_q$. Let $\mathbf{v} \in \{0,1\}^n$, $v_n \in \{0,1\}$ and $\mathbf{V} = (\mathbf{v}, v_n)$. We define,

$$\begin{aligned} \Omega_r^{\mathbf{c}} &= \Big\{ M^{\mathbf{c}}(\mathbf{Y}) + \frac{a}{q}\left(h(\mathbf{V}) + \frac{q}{2}\Big((\mathbf{v} + \mathbf{r}) \cdot \mathbf{y} + v_n y_\gamma\Big)\right) \\ &\qquad : \mathbf{v} \in \{0,1\}^n, v_n \in \{0,1\} \Big\}, \\ \Lambda_r^{\mathbf{c}} &= \Big\{ N^{\mathbf{c}}(\mathbf{Y}) + \frac{a}{q}\left(h(\mathbf{V}) + \frac{q}{2}\Big((\mathbf{v} + \mathbf{r}) \cdot \bar{\mathbf{y}} + \bar{v}_n y_\gamma\Big)\right) \\ &\qquad : \mathbf{v} \in \{0,1\}^n, v_n \in \{0,1\} \Big\}. \end{aligned} \tag{11}$$

Then the code set $\mathcal{S} = \left\{ \psi_{Trun}(\Omega_r^{\mathbf{c}}), \psi^*_{Trun}(\Lambda_r^{\mathbf{c}}) : 0 \leq r < 2^n, 0 \leq c_i \leq p_i - 1 \right\}$, forms $(\prod_{i=1}^{l} p_i 2^{n+1}, 2^m) - ZCCS_{2^{n+1}}^{2^m \prod_{i=1}^{l} p_i}$ over $\mathbb{Z}_a$.

Proof: From (9) it can be observed that the jth subsequence of $\Psi_{Trun}(M^{\mathbf{V},\mathbf{r},\mathbf{c},h})$ can be expressed as $\omega_{p_1}^{c_1(j'_1)} \omega_{p_2}^{c_2(j'_2)} \dots \omega_{p_l}^{c_l(j'_l)} \Psi(g^{\mathbf{V},\mathbf{r},h})$. From (9), (11), *Lemma 1*, and *Lemma 2*, the ACCF between $\Psi_{Trun}(\Omega_r^{\mathbf{c}})$ and $\Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'})$ for $\tau = 0$ and can be derived as

$$\begin{aligned} &\Theta(\Psi_{Trun}(\Omega_r^{\mathbf{c}}), \Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'}))(0) \\ &= \sum_{\mathbf{V}} \Theta(\Psi_{Trun}(M^{\mathbf{V},\mathbf{r},\mathbf{c},h}), \Psi_{Trun}(M^{\mathbf{V},\mathbf{r}',\mathbf{c}',h}))(0) \\ &= \sum_{\mathbf{V}} \Theta(\Psi(g^{\mathbf{V},\mathbf{r},h}), \Psi(g^{\mathbf{V},\mathbf{r}',h}))(0) \prod_{i=1}^{l} \left(\sum_{\alpha=0}^{p_i - 1} \omega_{p_i}^{(c_i - c'_i)(\alpha)} \right) \\ &= \Theta(\Psi(G_r), \Psi(G_{r'}))(0) \prod_{i=1}^{l} \left(\sum_{\alpha=0}^{p_i - 1} \omega_{p_i}^{(\mathbf{c}_i - c'_i)(\alpha)} \right) \\ &= \begin{cases} p_1 p_2 \dots p_l 2^{m+n+1}, & r = r', \mathbf{c} = \mathbf{c}', \\ 0, & r = r', \mathbf{c} \neq \mathbf{c}', \\ 0, & r \neq r', \mathbf{c} = \mathbf{c}', \\ 0, & r \neq r', \mathbf{c} \neq \mathbf{c}'. \end{cases} \end{aligned} \tag{12}$$

Now, using (9), (11), and *Lemma 1*, the ACCF between $\Psi_{Trun}(\Omega_r^{\mathbf{c}})$ and $\Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'})$ for $0 < |\tau| < 2^m$ can be derived as,

$$\begin{aligned} &\Theta(\Psi_{Trun}(\Omega_r^{\mathbf{c}}), \Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'}))(\tau) \\ &= \Theta(\Psi(G_r), \Psi(G_{r'}))(\tau) \left[\prod_{i=1}^{l} \left(\sum_{\alpha=0}^{p_i - 1} \omega_{p_i}^{(c_i - c'_i)(\alpha)} \right) \right] \\ &+ \Theta(\Psi(G_r), \Psi(G_{r'}))(\tau - 2^m) \Bigg[\sum_{\alpha=0}^{p_1 - 2} \omega_{p_1}^{c_1(\alpha+1) - c'_1(\alpha)} \\ &\left\{ \prod_{i=2}^{l} \left(\sum_{\alpha=0}^{p_i - 1} \omega_{p_i}^{c_i(\alpha) - c'_i(\alpha)} \right) \right\} \Bigg] + \Theta(\Psi(G_r), \Psi(G_{r'}))(\tau - 2^m) \\ &\Bigg[\sum_{f=1}^{l-2} \Bigg\{ \left(\prod_{d=1}^{f} \omega_{p_d}^{c_d(0) - c'_d(p_d - 1)} \right) \left(\sum_{\alpha=0}^{p_{f+1} - 2} \omega_{p_{f+1}}^{c_{f+1}(\alpha+1) - c'_{f+1}(\alpha)} \right) \\ &\left(\prod_{k=f+2}^{l} \left(\sum_{\alpha=0}^{p_k - 1} \omega_{p_k}^{c_k(\alpha) - c'_k(\alpha)} \right) \right) \Bigg\} \Bigg] + \sum_{\alpha=0}^{p_l - 2} \omega_{p_l}^{c_l(\alpha+1) - c'_l(\alpha)} \\ &\left(\prod_{d=1}^{l-1} w_{p_d}^{c_d(0) - c'_d(p_d - 1)} \right) \Theta(\Psi(G_r), \Psi(G_{r'}))(\tau - 2^m). \end{aligned} \tag{13}$$

From *Lemma 1*, we have, $\Theta(\Psi(G_r), \Psi(G_{r'}))(\tau) = 0$, $\forall \tau$, $0 < |\tau| < 2^m$. Therefore, from the above, we obtain,

$$\Theta(\Psi_{Trun}(\Omega_r^{\mathbf{c}}), \Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'}))(\tau) = 0, 0 < |\tau| < 2^m. \tag{14}$$

From (12) and (14), we have,

$$\begin{aligned} &\theta(\Psi_{Trun}(\Omega_r^{\mathbf{c}}), \Psi_{Trun}(\Omega_{r'}^{\mathbf{c}'}))(\tau) \\ &= \begin{cases} p_1 p_2 \dots p_l 2^{m+n+1}, & r = r', \mathbf{c} = \mathbf{c}', \tau = 0, \\ 0, & r = r', \mathbf{c} \neq \mathbf{c}', 0 < |\tau| < 2^m, \\ 0, & r \neq r', \mathbf{c} = \mathbf{c}', 0 < |\tau| < 2^m, \\ 0, & r \neq r', \mathbf{c} \neq \mathbf{c}', 0 < |\tau| < 2^m. \end{cases} \end{aligned} \tag{15}$$

Similarly, it can be shown that

$$\theta(\Psi^{*}_{Trun}(\Lambda_r^{\mathbf{c}}),\Psi^{*}_{Trun}(\Lambda_{r'}^{\mathbf{c}'})(\tau)$$
$$=\begin{cases} p_1p_2\dots p_l2^{m+n+1}, & r=r',\mathbf{c}=\mathbf{c}',\tau=0,\\ 0, & r=r',\mathbf{c}\neq\mathbf{c}',0<|\tau|<2^m,\\ 0, & r\neq r',\mathbf{c}=\mathbf{c}',0<|\tau|<2^m,\\ 0, & r\neq r',\mathbf{c}\neq\mathbf{c}',0<|\tau|<2^m. \end{cases} \tag{16}$$

From *Lemma 1*, we have $\Theta(\Psi(G_r),\Psi^{*}(\bar{G}_r))(\tau)=0,\ |\tau|<2^m$. From (9), (11) and *Lemma 1*, the ACCF between $\Psi_{Trun}(\Omega_r^{\mathbf{c}})$ and $\Psi^{*}_{Trun}(\Lambda_{r'}^{\mathbf{c}'})$ for $\tau=0$ can be derived as,

$$\begin{aligned}
&\Theta(\Psi_{Trun}(\Omega_r^{\mathbf{c}}),\Psi^{*}_{Trun}(\Lambda_{r'}^{\mathbf{c}'})(0)\\
&=\sum_{\mathbf{V}}\Theta(\Psi_{Trun}(M^{\mathbf{V},\mathbf{r},\mathbf{c},h}),\Psi^{*}_{Trun}(N^{\mathbf{V},\mathbf{r}',\mathbf{c}',h}))(0)\\
&=\sum_{\mathbf{V}}\Theta(\Psi(g^{\mathbf{V},\mathbf{r},h}),\Psi^{*}(s^{\mathbf{V},\mathbf{r}',h}))(0)\prod_{i=1}^{l}\sum_{\alpha=0}^{p_i-1}\omega_{p_i}^{(c_i+c_i')(\alpha)}\\
&=\omega_q^{2\lambda}\Theta(\Psi(G_r),\Psi^{*}(\bar{G}_{r'}))(0)\prod_{i=1}^{l}\sum_{\alpha=0}^{p_i-1}\omega_{p_i}^{(c_i+c_i')(\alpha)}\\
&=0.
\end{aligned} \tag{17}$$

By the similar calculation as in (13), we have

$$\Theta(\Psi_{Trun}(\Omega_r^{\mathbf{c}}),\Psi^{*}_{Trun}(\Lambda_{r'}^{\mathbf{c}'})(\tau)=0,\forall\ 0<|\tau|<2^m. \tag{18}$$

Hence by (15), (16), (17), and (18), it is concluded that the set $\mathcal{S}$ forms a $(\prod_{i=1}^{l}p_i2^{n+1},2^m)-ZCCS_{2^{n+1}}^{2^m\prod_{i=1}^{l}p_i}$. ■

Remark 1: The proposed construction gives optimal ZCCS of length of the form $\prod_{i=1}^{l}p_i2^m$where p_i's are primes. When $m=1$, all the even lengths ZCCS are accomplished, but the ZCZ width is only two. In this case, the results are not desirable and most MPI cannot be eliminated.

Remark 2: For $l=1$, the proposed result in *Theorem 1* reduces to $(p2^{n+1},2^m)$-$ZCCS_{2^{n+1}}^{p2^m}$ as in [12]. Therefore, the proposed construction is a generalization of [12].

Remark 3: In the same way as defined in [4, Remark1] Theorem 1. produces atleast $q^{m+1}\frac{m!}{2n!}(q-1)^{n(m-n)}q^{\binom{n}{2}}$ number of non overlapping ZCCSs. For lack of space, we have not included the proof in this letter. The reader is referred to [4, Remark1] for the proof.

Corollary 1: ([8]) Suppose G(h) is a path in which the edges have the same weight as $\frac{q}{2}$. Thus $h(\mathbf{V})$ can be written as

$$h(v_0,\dots,v_n)=\frac{q}{2}\sum_{\alpha=0}^{n-1}v_{\pi(\alpha)}v_{\pi(\alpha+1)}+\sum_{\alpha=0}^{n}u_\alpha v_\alpha+u$$

, where $u,u_0,\dots u_n\in\mathbb{Z}_q$. It is obvious from (11) that the kth column of $\psi(\Omega_r^{\mathbf{c}})$ is produced by setting $\mathbf{Y}$ at $\mathbf{k}=(k_0,\dots,k_m,\dots,k_{b-1})$, in the expression of $M^{\mathbf{V},\mathbf{r},\mathbf{c},h}$ where $(k_0,\dots,k_m,\dots,k_{b-1})$ is the binary representation of k. The kth column sequence of $\Psi(\Omega_r^{\mathbf{c}})$ is generated from a GBF whose graph is a path over $n+1$ vertices, as a result according to [8] the kth column sequence of $\Psi(\Omega_r^{\mathbf{c}})$ is a q-ary Golay sequence. Thus each column of $\Psi_{Trun}(\Omega_r^{\mathbf{c}})$ is Golay sequence. Thus the PMEPR of each column $\Psi_{Trun}(\Omega_r^{\mathbf{c}})$ is bounded by

TABLE I
ANALYSIS OF THE SUGGESTIVE CONSTRUCTION WITH EXISTING WORK

Source	Based On	Parameters	Conditions	Optimal
[4]	Direct	$(2^{k+p+1},2^{m-p})-ZCCS_{2^{k+1}}^{2^m}$	$k+p\leq m$	yes
[5]	Direct	$(2^{k+v},2^{m-v})-ZCCS_{2^k}^{2^m}$	$v\leq m,k\leq m-v,m\geq 3$	yes
[8]	Direct	$(2^{n+p},2^{m-p})-ZCCS_{2^n}^{2^m}$	$p\leq m,m\geq 2,n+p\geq 2,n\in\mathbb{Z}^+,p\in\mathbb{Z}^+$	yes
[11]	Direct	$(2^{k+v},2^{m-v})-ZCCS_{2^k}^{2^m}$	$v\leq m,k\leq m-v,m\in\mathbb{Z}^+,v\in\mathbb{Z}^+,k\in\mathbb{Z}^+$	yes
[13]	Direct	$(q^{v+1},q^{m-v})-ZCCS_q^{q^m}$	$v\leq m,q\geq 2,m\geq 2$	yes
[14]	Direct	$(2^{k+1},2^{m+1})-ZCCS_{2^{k+1}}^{3.2^m}$	$m,k\in\mathbb{Z}^+$	yes
[12]	Direct	$(p2^{k+1},2^m)-ZCCS_{2^{k+1}}^{p2^m}$	$m\geq 2$, $k\leq m$, p prime	yes
[16]	Indirect	$(2^{n+1},Z)-ZCCS_{2^{n+1}}^{N}$	$N\geq 3$, N is odd, $\lfloor\frac{N}{Z}\rfloor=1$	yes
[15]	Indirect	$(K,M)-ZCCS_M^K$	blue$M,K\geq 2$	yes
Theorem 1	Direct	$(\prod_{i=1}^{l}p_i2^{n+1},2^m)-ZCCS_{2^{n+1}}^{\prod_{i=1}^{l}p_i2^m}$	$l,m,n\in\mathbb{Z}^+,p_i's\ are\ primes$	yes

2. Similarly the PMEPR of each column of $\Psi^{*}_{Trun}(\Lambda_r^{\mathbf{c}})$ is bounded by 2.

Example 1: Assume q=2, p_1=3, p_2=2, p_3=2, a=6, m=3, n=1, s_1=2, s_2=1 and s_3=1. Let's consider the following GBF $g:\{0,1\}^3\rightarrow Z_2$ as, $g=y_1y_2+y_0$, the graph corresponding to $g|_{y_0=0}$ and $g|_{y_0=1}$ forms a path with one of the end vertices being y_1. Let $h:\{0,1\}^2\rightarrow\mathbb{Z}_2$ defined by $h(v_0,v_1)=v_0v_1$ From (4) we have,

$$\begin{aligned}
&M^{\mathbf{c}}(\mathbf{Y})=3y_1y_2+3y_0+2c_1(y_3+2y_4)+3c_2y_5+3c_3y_6,\\
&N^{\mathbf{c}}(\mathbf{Y})=3\bar{y}_1\bar{y}_2+3\bar{y}_0+2c_1(y_3+2y_4)+3c_2y_5+3c_3y_6,
\end{aligned} \tag{19}$$

where $c_1\in\{0,1,2\}$, $c_2\in\{0,1\}$, $c_3\in\{0,1\}$. From (11) we have

$$\begin{aligned}
\Omega_r^{\mathbf{c}}=&\big\{M^{\mathbf{c}}(\mathbf{Y})+3(v_0v_1+v_0y_0+r_0y_0+v_1y_1):\\
&\qquad v_0,v_1\in\{0,1\}\big\}\\
\Lambda_r^{\mathbf{c}}=&\big\{N^{\mathbf{c}}(\mathbf{Y})+3(v_0v_1+v_0\bar{y}_0+r_0\bar{y}_0+\bar{v}_1y_1):\\
&\qquad v_0,v_1\in\{0,1\}\big\}
\end{aligned} \tag{20}$$

where $0\leq r_0\leq 1$. Therefore, the set

$$\mathcal{S}=\Big\{\Psi_{Trun}(\Omega_r^{\mathbf{c}}),\Psi^{*}_{Trun}(\Lambda_r^{\mathbf{c}}):0\leq r\leq 1,0\leq c_1\leq 2,\ 0\leq c_2\leq 1,0\leq c_3\leq 1\Big\},$$

forms an optimal $(48,8)$-ZCCS$_4^{96}$ over $\mathbb{Z}_6$ and the maximum column sequence PMEPR is at most 2.

A. Comparison with Previous Works

The proposed work is compared with the direct construction given in [4], [5], [8], [11]–[14] and indirect construction given in [15], [16] and provided these in Table 1. The constructions given in [4], [5], [8], and [11] produces optimal ZCCS of length power-of-two and constructions given in [12]–[14] produces optimal ZCCS of length non-power-of-two but do not cover all even numbers. By indirect approach [16] and [15] produces ZCCS of all possible odd lengths and all possible lengths respectively.The proposed construction uses GBFs, hence it is direct and produces all even length optimal ZCCS.

IV. Conclusion

In this work, a direct construction of optimal ZCCSs is proposed for all possible even lengths using GBFs. The maximum column sequence PMEPR of the proposed ZCCSs is upper-bounded by 2 which can be useful in MC-CDMA system to control high PMEPR problem. The proposed construction also provides more flexible parameters as compared to the existing GBFs based constructions of optimal ZCCS.

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