

ABSTRACT

Title of Thesis: **BEYOND PEMDAS: TEACHING STUDENTS TO
PERCEIVE ALGEBRAIC STRUCTURE**

Ethan Michael Merlin, Master of Arts, 2008

Thesis directed by: Professor Lawrence M. Clark
 Department of Curriculum and Instruction

Evidence shows that transforming expressions is a major stumbling block for many algebra students. Using Sfard's (1991) theory of *reification*, I highlight the important roles that the process of *parsing* and the notions of *subexpression* and *structural template* play in competent expression transformation. Based on these observations, I argue that one reason students struggle with expression transformation is the inattentiveness of traditional curricula to parsing, subexpressions, and structural templates. However, simply refocusing attention on these ignored aspects of algebra will not alone ensure that students avoid common pitfalls. After examining evidence that students are very prone to *overgeneralize*, I argue for a *connectionist* view of how people's minds work when they are learning algebra. Utilizing these additional insights, the instructional strategies I ultimately recommend are strategies that focus on structure, but in ways that will make structure a winning competitor for student attention.

BEYOND PEMDAS: TEACHING STUDENTS TO PERCEIVE ALGEBRAIC
STRUCTURE

by

Ethan Michael Merlin

Thesis submitted to the faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of

Master of Arts

2008

Advisory Committee:

Professor Lawrence M. Clark, Chair

Professor Daniel Chazan

Professor James T. Fey

© Copyright by
Ethan Michael Merlin
2008

TABLE OF CONTENTS

Introduction	1
What is expression transformation?	3
Evidence that students have difficulty with expression transformation	5
Simplifying numeric expressions and evaluating algebraic expressions	11
The non-notational precedence conventions: The hierarchy of operations	14
Notational precedence conventions: Grouping symbols	17
Parsing: Implicit activity versus explicit activity	19
Transforming expressions	20
Generalization, and the two “faces” of algebra	20
The rules of algebra	23
Transforming an expression: Matching given expressions to rule expressions	25
Subexpressions and structural templates	27
Transforming an expression: Using subexpressions	30
Using reference as support for expression transformation	33
Diagnosing student difficulties with expression transformation	35
Reification: Mathematical objects as compressed processes	36
Subexpressions and structural templates as mathematical “objects”	42
PEMDAS: Structure (or lack thereof) in the traditional curriculum	46
Diagnoses that support or overlap with mine	50
The role of overgeneralization	53

Matz: Overgeneralizing rule-revision strategies	55
Kirshner & Awtry: Overgeneralizing memorable visual sequences	59
The vicious circle of reification	63
The pragmatic value of reification	63
The vicious circle and algebra: A window for overgeneralization	65
Competing impulses: The connectionist view of mind	67
Kirshner on the role of spacing in parsing decisions	67
Kirshner on connectionism	71
Landy & Goldstone's research on formally irrelevant distractions	74
Other examples of competition in algebra performance	79
Instructional strategies	81
Instructional strategies for helping students achieve the process- perception of parsing	82
Instructional strategies for inducing reification of structure vertically	85
Instructional strategies for inducing reification of structure horizontally	88
Some supporting and overlapping curricular recommendations	92
Conclusion and implications	98
References	103

Introduction

Scholars of algebra education have long lamented that few students become competent in the subject despite years of exposure in school. For instance, Kieran (1983) states that algebra “is known to be a school subject which presents many cognitive obstacles to the student encountering it for the first time” (p. 162). Similarly, Booth (1984), after conducting a major study of student algebra proficiency, reports that “items representing the highest level of understanding were answered successfully by only a small percentage of children, and that in many cases the level of understanding ... improved relatively little as the child progressed” (p. 2). Herscovics (1989) surveys data about student achievement in algebra and concludes that “only a minority of pupils completing an introductory course achieve a reasonable grasp of the course content” (p. 60). Lee & Wheeler (1989) reach a similar conclusion, which they report with alarm: “It is tempting to describe high school algebra as it is unveiled in our research as a disaster area” (p. 53).

In this paper, I will propose some instructional strategies that I believe can modestly improve student performance on some algebra tasks. These proposals stem jointly from my experience as a classroom teacher and from empirical findings and theoretical frameworks in the scholarly literature.

I have deliberately chosen the word “modestly” to describe the hoped-for improvement in student algebra performance. Much evidence – both scientific and anecdotal – suggests that symbolic algebra will always be somewhat difficult for many students. There is no all-encompassing solution to the problems of algebra education.

Nonetheless, I am optimistic about the possibility of real, measurable improvements in student ability to master certain aspects of symbolic algebra. We are only just beginning to understand the cognitive tasks involved in learning algebra and doing algebra. Insofar as our understanding of human cognition is still evolving, it is perhaps not only possible but indeed likely that there are distinctly better ways to teach algebra that are yet to be discovered and implemented.

In this paper, I offer some instructional strategies to help students avoid common errors while performing the algebra task called *transforming expressions*. Evidence shows that transforming expressions is a major stumbling block for many algebra students. Using Sfard's (1991) theory of *reification*, I will highlight the important roles that the process of *parsing* and the notions of *subexpression* and *structural template* play in competent expression transformation. Based on these observations, I argue that one reason students struggle with expression transformation is the inattentiveness of traditional curricula to parsing, subexpressions, and structural templates – or, more generally, to *structure*. We will see, however, that simply refocusing attention on these ignored aspects of algebra will not alone ensure that students avoid the common pitfalls. After examining evidence that students are very prone to *overgeneralize*, I will argue for a *connectionist* view of how people's minds work when they are learning algebra. Utilizing these additional insights, the instructional strategies I ultimately recommend are strategies that focus on structure, but in ways that will make structure a winning competitor for student attention.

What is expression transformation?

In algebra, an *expression* is a single number, a single variable, or multiple numbers and/or variables linked by one or more arithmetic operations. If the expression contains at least one variable, then it is an *algebraic expression*. If it contains only numbers, then it is a *numeric expression*. These definitions are consistent with those used by many student texts in recent decades (e.g. Stein, 1956; Payne et al., 1972; Dolciani et al., 1983; Foerster, 1994). Note that expressions are not statements: they cannot be true or false. In particular, they do not contain relation symbols like the equal sign. Expressions are simply numbers, unknowns, and operations strung together. In this study, we will only consider expressions in which the operations are limited to the basic arithmetic operations of addition, subtraction, multiplication, division, exponentiation, and root-taking; we will not consider logarithms, trigonometric functions, or other more advanced “functional” operations.

Transforming an expression is a relatively complex task to describe precisely. It will be necessary first to describe two simpler tasks of algebra.

Two of the most basic activities of elementary algebra are *simplifying numeric expressions* and *evaluating algebraic expressions*. Simplifying a numeric expression consists of performing the indicated operations in a proper sequence to obtain a single number. Evaluating an algebraic expression consists of replacing each of the variables in the expression with given numbers and then simplifying the resulting numeric expression. Examples A and B below show typical instances of the sorts of exercises one encounters early in many algebra textbooks, as well as the sort of simplification work that a competent student might perform:

Example A

Simplify $-2 + 5(4 - 6)^2$.

$$-2 + 5(4 - 6)^2$$

$$-2 + 5(-2)^2$$

$$-2 + 5(4)$$

$$-2 + 20$$

$$18$$

Example B

Evaluate $5x^2 - 2xy$ using $x = -2$ and $y = -3$.

$$5(-2)^2 - 2(-2)(-3)$$

$$5(4) - 12$$

$$20 - 12$$

$$8$$

Two algebraic expressions are said to be *equivalent* if they yield identical numerical results upon being evaluated for any allowable value of the variable. For instance, the expressions $3x + 4y + 2x$ and $y + 5x + 3y$ are equivalent: both expressions yield the number 22 when evaluated with $x = 2$ and $y = 3$; both expressions yield the number -58 when evaluated with $x = -6$ and $y = -7$; and so on. This characterization of expression equivalence is the one in use in many textbooks (e.g. Payne et al., 1972; Dolciani et al., 1983).

Transforming an algebraic expression consists of replacing that expression with an equivalent expression. For instance, one might *transform* $3x + 4y + 2x$ into the equivalent $y + 5x + 3y$, or perhaps into the equivalent $5x + 4y$, which requires fewer symbols to write. Context or instruction will dictate what sort of equivalent expression is appropriate. *Simplifying* an algebraic expression and *factoring* an algebraic expression are two common types of transforming, but it is not necessary to elaborate here precisely what those terms mean. Note that thus far I have not described how to transform an expression but only what expression transformation is.

Evidence that students have difficulty with expression transformation

While teachers and researchers have identified algebra in general as an area of difficulty for many students, they have reported expression transformation as an area of particular difficulty. Because of the frequency and rule-like regularity with which students are prone to make certain errors, Sleeman (1984) and others refer to these “classic” errors as *mal-rules*. The central aim of this paper is to contribute research-

based instructional strategies to help students master the skill of expression transformation and avoid producing these well-known errors.

Marquis (1988) is a good example of someone who has collected some of these common errors together into one article. She describes the universality of a certain set of errors made by students who are attempting to transform algebraic expressions. She provides a list of twenty-two such errors:

$$1. |-3| = -3$$

$$2. 3^2 \cdot 3^3 = 9^5$$

$$3. a^2 \cdot b^5 = (ab)^7$$

$$4. x + y - 3(z + w) = x + y - 3z + w$$

$$5. \frac{r}{4} - \frac{(6-s)}{2} = \frac{r-12-2s}{4}$$

$$6. 3a + 4b = 7ab$$

$$7. 3x^{-1} = \frac{1}{3x}$$

$$8. \sqrt{x^2 + y^2} = x + y$$

$$9. \frac{x+y}{x+z} = \frac{y}{z}$$

$$10. \frac{1}{x-y} = \frac{-1}{x+y}$$

$$11. \frac{x}{y} + \frac{r}{s} = \frac{x+r}{y+s}$$

$$12. x\left(\frac{a}{b}\right) = \frac{ax}{bx}$$

$$13. \frac{xa + xb}{x + xd} = \frac{a + b}{d}$$

$$14. \sqrt{-x}\sqrt{-y} = \sqrt{xy}$$

$$15. \text{If } 2(2-z) < 12 \text{ then } z < -4.$$

$$16. \frac{1}{1-\frac{x}{y}} = \frac{y}{1-x}$$

$$17. a^2 \cdot a^5 = a^{10}$$

$$18. (3a)^4 = 3a^4$$

$$19. \frac{a}{b} - \frac{b}{a} = \frac{a-b}{ab}$$

$$20. (x+4)^2 = x^2 + 16$$

$$21. \frac{r}{4} - \frac{6-s}{4} = \frac{r-6-s}{4}$$

$$22. (a^2)^5 = a^7$$

Marquis emphasizes that the regular occurrence of such mistakes in algebra is strikingly independent of context: “After a few years of teaching mathematics courses in high school, teachers know which concepts and manipulations will cause difficulty for students. From year to year, class to class, students often make the same algebraic mistakes over and over” (p. 204). She also emphasizes the stubborn persistence of these errors. Student use of these mal-rules, according to Marquis, does not seem to be merely an appropriate developmental step on the path to eventual mastery; on the contrary, she explains that “in upper-level mathematics courses, students’ indication of mastery of the new concepts may be obscured by common algebraic errors” (p. 204).

Many teachers corroborate Marquis’ observations about the universality and persistence of the sorts of errors on Marquis’ list. Grossman (1924), for instance, provides the following examples of “cancellation” errors:

$\frac{\cancel{x}+7}{\cancel{y}+8}$	$\frac{\cancel{x}+\cancel{7}}{\cancel{7}+\cancel{7}}$	$\frac{8(x+\cancel{7})}{9(5x+\cancel{7})}$	$\frac{\cancel{3}(x+\cancel{7})+11}{\cancel{3}(x+\cancel{7})+2}$
-------------------------------------	---	--	--

Grossman describes such mistakes as commonplace:

Every teacher of experience knows that a great many of his algebra pupils all the way from the first year in high school up to college continue with almost comical regularity to make strange mistakes in the subject of “cancellation” in fractions—mistakes that show clearly that the essence of the matter has escaped them. (p. 104)

Schwartzman (1977) provides the following examples of “distribution” errors:

$$(a + b)^2 = a^2 + b^2$$

$$\sqrt{x^2 - y^2} = \sqrt{x^2} - \sqrt{y^2} = x - y$$

$$a(xy) = ax \cdot ay$$

He confirms that “these are three mistakes that my students frequently make” (p. 594), echoing others in noting the persistence and universality of such mal-rules.

Martinez (1988), too, describes common algebra errors of this sort. He provides a “list of students’ common errors” which he takes to be “representative of the kinds of difficulties that seem to be correlated with students’ misunderstanding of factors and terms” (p. 747). Nearly all of the errors on his list are very similar to the errors

Marquis included:

1. $x + x^2 = x^3$	2. $(x)(x) = 2x$
3. $3[2 + (x - 1) + 1] = 6 + (x - 1) + 1$	4. $(a + b)^2 = a^2 + b^2$
5. $x(x - 1) + (x - 1) = (x - 1)^2 x$	6. $\sqrt{a^2 + b^2} = a + b $
7. $\frac{x + 2}{2} = 10$ $x + 2 = 12$	8. $\frac{a + b}{a} = 1 + b, a \neq 0$
9. $\frac{2(x + y)}{xy} = 2$	10. $\frac{4(x + y)}{4x + y} = 1$

Algebra beginners are not the only students who fall victim to these mal-rules; Parish & Ludwig (1994) provide a list of twenty “typical mathematical errors made by high school and lower division college students” (p. 235). They write that “the

mathematics and science teaching professions are well aware of the fact that certain types of mathematical errors are continually repeated by students. Large numbers of typical errors are documented in the literature” (p. 235). This sampling of teacher observations typifies a broader awareness in the profession that algebra students are universally prone to persistently make “mal-rule” errors.

The authors of some major textbooks also seem to show an awareness that students are prone to making these sorts of errors when transforming expressions. Stein’s 1956 text is typical in this regard. In this text, each lesson begins with a “Procedure” that provides precise step-by-step instructions for performing the skill under consideration. For the lesson on “Reduction of Fractions” (p. 146), however, Stein departs from the usual format by including a step in the procedure for what *not* to do (emphasis mine):

I. Aim: To reduce algebraic fractions to lowest terms.

II. Procedure

1. Find the largest common factor of both numerator and denominator.

If the numerator, or denominator, or both are polynomials, factor them if possible.

2. Divide both numerator and denominator by the largest common factor.

3. *Do not cancel term with term.* See sample solutions 6, 7, and 8.

4. Check by going over the work again or by numerical substitution.

(p. 146)

Stein repeats the warning against canceling terms in his commentary (p. 146) to Sample Solution 6 (emphasis mine):

$$6. \frac{3x}{3(x+y)} = \frac{x}{x+y}$$

Divide numerator and denominator by 3.

Do not cancel x in answer.

Remove parentheses from $x + y$ in answer.

Answer, $\frac{x}{x+y}$

Foerster's (1994) *Algebra 1* text shows a similar awareness of student susceptibility to this and other common errors. In a lesson on "Simplifying Rational Algebraic Expressions," Foerster includes the following warning: "Do not read more into the definition of canceling than is there! For instance, in $\frac{x-7}{x+2}$, you *cannot* cancel the x 's" (p. 461). To reinforce the point, Foerster includes a set of exercises with the instruction "Can canceling be done? If so, *what* can be canceled?" (p. 462).

Elsewhere, Foerster includes the following exercise designed to draw student attention to a mal-rule frequently followed when squaring a binomial: "Explain the error in the work below: $(x+4)^2 = x^2 + 16$ " (p. 199). The newer Core-Plus

Mathematics Project texts also show awareness of common student algebra errors.

For instance, Coxford et al. (2003) includes the following reflection question (p. 205):

You've found that there are some common errors that occur when people operate on symbolic expressions in an effort to produce equivalent but more useful forms. What advice would you give to help someone understand why each of the following pairs of expressions is not equivalent?

- a. $a(x + b)$ is not equivalent to $ax + b$.
- b. $x - b$ is not equivalent to $b - x$.
- c. $(ax)(bx)$ is not equivalent to abx .
- d. $-(x - b)$ is not equivalent to $-x - b$.

Elsewhere, the same authors include the following writing prompt: "When beginning students are asked to expand expressions like $(x + a)^2$, one very common error is often made. What do you think that error is, and how could you help someone avoid making the error?" (p. 214). The Teacher's Guide's suggested response begins as follows: "Students often forget the middle term and expand $(x + a)^2$ to $x^2 + a^2$ " (p. T214). Thus, evidence from teachers and from textbooks makes clear that students are particularly prone to have difficulty with the task of expression transformation. In particular, they are universally prone to make certain predictable and persistent errors.

Simplifying numeric expressions and evaluating algebraic expressions

To help students overcome the transformation errors described above, we need to diagnose what *goes wrong* when students produce these common transformation errors. Before trying to diagnose the problem, however, I will

characterize *competent* performance of expression transformation. In order to give a precise account of competent performance of this skill, I must first give a more precise account of the two simpler algebraic skills mentioned earlier, namely simplifying numeric expressions and evaluating algebraic expressions.

Simplifying numeric expressions and evaluating algebraic expressions are not trivial tasks for novices. Reexamining Examples A and B above, we can isolate three distinct competencies involved in successful performance of these skills. First, students need to correctly perform arithmetic operations. For instance, in Example A, the simplifier needed to know that five times four equals twenty. Second, students need to successfully interpret and use algebraic *syntax*, that is, the symbolic notation of algebra. For instance, in Example A, the simplifier needed to know that the juxtaposition of 5 and $(4 - 6)^2$ indicates multiplication. Similarly, in Example B, the evaluator needed to introduce an appropriate notation for multiplication other than juxtaposition once x and y were replaced with negative numbers.

The third competency is determining an appropriate *order of precedence* for the operations in the expression. When multiple operations – or even multiple instances of the same operation – are indicated in an expression, different numerical results are sometimes obtained depending upon which operations are given precedence over which others. Consider, for instance, Examples C and D below:

EXAMPLE CSimplify $2 + 3(5)$.

$$2 + 3(5)$$

$$5(5)$$

$$25$$

EXAMPLE DSimplify $2 + 3(5)$.

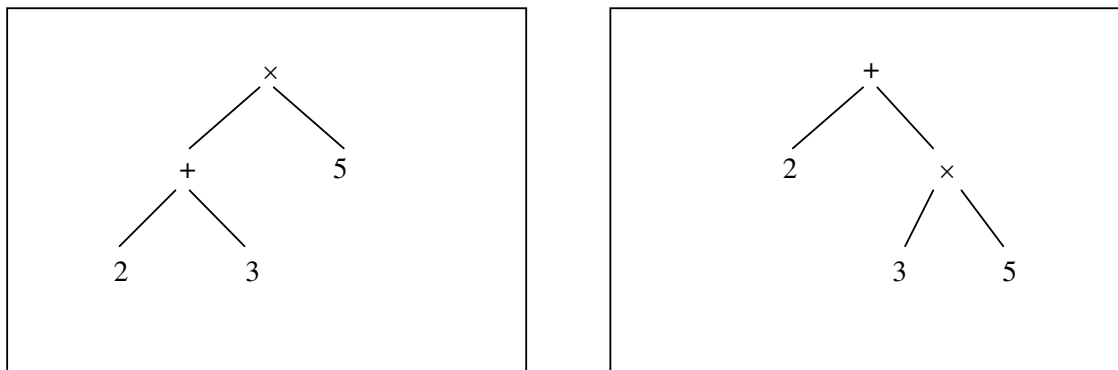
$$2 + 3(5)$$

$$2 + 15$$

$$17$$

In Example C, the simplifier treated the addition as more precedent than the multiplication. In Example D, the simplifier treated the multiplication as more precedent than the addition. Two different answers result. If expressions containing multiple operations are to have unambiguous meaning, then it is necessary for all users of algebra to make the same (or equivalent) decisions about operation precedence.

Several researchers (e.g. Sleeman, 1984; Ernest, 1987; Thompson & Thompson, 1987; Kirshner & Awtry, 2004) have illustrated the notion of operation precedence using *tree diagrams*, which do not allow for the ambiguity present in standard algebraic notation. In a tree diagram, a more precedent operation appears lower on the tree than a less precedent one. Here are tree diagrams illustrating the precedence decisions of the simplifiers in Examples C and D above:



For virtually all multi-operation expressions written in standard (i.e. non-tree) notation, mathematical conventions dictate an agreed-upon order of operation precedence, thereby eliminating ambiguity and determining a unique interpretation. Some of the conventions that determine operation precedence involve the use of notations that appear on the page, while others do not involve visible notations.

The non-notational precedence conventions: The hierarchy of operations

Some teachers (e.g. Schwartzman, 1977; Rambhia, 2002) describe the *non-notational conventions* for operation precedence by referring to the six basic operations (addition, subtraction, multiplication, division, exponentiation, and root-taking) as occupying levels in a hierarchy of operations. Kirshner (1989) provides the following table of the hierarchy of operations:

Syntactic Convention Using Hierarchy of Operation Levels

Level 1	Addition	Subtraction
Level 2	Multiplication	Division
Level 3	Exponentiation	Finding roots
(Level 3 is said to be higher than Level 2, which is higher than Level 1.)		

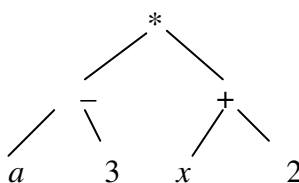
In this hierarchy, exponentiation and its inverse (root-taking) occupy the level of highest precedence; multiplication and its inverse (division) occupy the middle level; and addition and its inverse (subtraction) occupy the level of lowest precedence.

Revisiting the expression $2 + 3(5)$ from Examples C and D above, we see that according to this hierarchical convention, it is standard to interpret the multiplication as more precedent than the addition in the expression. Thus, Example C above is incorrect while Example D is correct, according to convention. This hierarchical convention is non-notational in that knowledge external to the written form of an expression dictates the order of precedence. (We will see later, however, that Kirshner (1989) identifies hints present in standard algebraic notation regarding the hierarchical precedence order.)

The core of the hierarchy – that exponentiation is more precedent than multiplication, which is in turn more precedent than addition – has deep mathematical underpinnings and is universally accepted. Miller (2006), in a study of mathematical notations, indicates that this basic convention was followed “in the earliest books employing symbolic algebra in the 16th century.” Similarly, Peterson (2000) affirms that the basic hierarchy of precedence “appears to have arisen naturally and without much disagreement as algebraic notation was being developed in the 1600s and the need for such conventions arose.” This history indicates that the order of precedence is not simply the sort of convention adopted willy-nilly that could have been otherwise; rather, it reflects something essential and deep about the operations themselves. Peterson posits that distributive relationships among the operations make this hierarchy the only natural one: exponents and radicals distribute over multiplication and division, while multiplication and division distribute over addition and subtraction. Both Peterson and Wu (2007) explain that since polynomials – primary objects of study in symbolic algebra – are sums of products, the easy

representation of polynomials in symbolic notation naturally motivated a convention in which multiplication is understood to be more precedent than addition. Thus, although teachers often present the hierarchy as a convention designed strictly to eliminate ambiguity (e.g. Rambhia, 2002), historical evidence belies this conclusion and points to something mathematically deep and essential about this “convention.”

While the core notion of exponentiation-precedes-multiplication-precedes-addition is uniformly accepted, its application becomes slightly more controversial when dealing with two related situations: the presence of any of the three inverse operations, and the appearance of multiple operations from the same hierarchical level in the same expression. Typically, the convention is stated in a manner similar to Kirshner (1989, p. 276): “In case of an equality of levels, the left-most operation has precedence.” In other words, perform all exponents and/or roots from left to right, then all multiplications and/or divisions from left to right, then all additions and/or subtractions from left to right. Yet a case can be made that this manner of stating the convention over-specifies the order. For one thing, in an expression such as $(a - 3)(x + 2)$, it makes no difference whether the subtraction or addition is performed first because the precedence of the intervening multiplication. In tree notation, it is easy to see the independence of these two operations in the separateness of the tree’s two main “branches”:



Moreover, since addition and multiplication are associative operations, it truly does not matter in expressions like $a + b + c$ (or abc) which addition (or multiplication) is performed first, even without the presence of an intervening higher-precedence operation. And, while Miller (2006) affirms that most “modern textbooks seem to agree that all multiplications and divisions should be performed in order from left to right,” he acknowledges that as recently as 1929 there was no agreement as to whether this was so or whether all multiplications should precede all divisions. Rather than overspecify unnecessarily, Wu (2007) prefers to avoid the question of what to do about multiple occurrences of operations from the same level of the hierarchy. His simpler formulation of the convention is “exponents first, then multiplications, then additions” (p. 2), but the cost of this economical formulation is the elimination of the inverse operations as independent operations, requiring a more sophisticated understanding of subtraction as addition of the opposite, division as multiplication by the reciprocal, and root-taking as raising to a fractional power. Thus, in formulating the details of the hierarchy of operation precedence, there is a trade-off between precision and accessibility to beginners.

Notational precedence conventions: Grouping symbols

Notational conventions for indicating operation precedence are syntactical indications in the written form of an expression. These include use of parentheses, brackets, and braces (often called *grouping symbols*); for instance, compare $2(x + 3)$, in which the addition is understood as precedent, and $2x + 3$, in which the multiplication is understood as precedent. Notational precedence conventions also

include use of the horizontal fraction bar and the radical symbol; for instance,

compare $\frac{x+3}{2}$ (addition more precedent) with $x + \frac{3}{2}$ (division more precedent), and

$\sqrt{x+3}$ (addition more precedent) with $\sqrt{x} + 3$ (root-taking more precedent).

Kirshner (1989) points out that raised notation also indicates precedence; for instance,

compare 2^{n+3} (addition more precedent) with $2^n + 3$ (exponentiation more

precedent). If there are multiple operations within a grouped or raised portion of an

expression, the conventional hierarchy of operations determines precedence.

Wu (2007) points out that there is a trade-off between utilizing notational and non-notational conventions to indicate operation precedence. On the one hand, a well-placed set of parentheses can help stave off parsing errors, and so grouping symbols are sometimes written even when formally unnecessary. Redundant parentheses, such as those in $(3 \cdot 10^2) + (4 \cdot 10) + 5$, can serve to reinforce the usual precedence hierarchy rather than to override it. Taken to the extreme, the use of notational precedence indicators could obviate the need for learning a hierarchy of operations. For instance, the expression $-2 + 5(4 - 6)^2$ from Example A above could be written as the formally-equivalent $-2 + (5((4 - 6)^2))$. But, as Wu points out, our current algebraic conventions make for a “notational simplicity” (p. 2) in comparison to such parentheses-laden expressions, despite the fact that these conventions place a certain demand upon the user to memorize and apply the operation hierarchy.

Parsing: Implicit activity versus explicit activity

We are now ready to define *parsing*. The term “parsing” comes from computer science and linguistics, where it is used to describe the process of breaking down a sequence of symbols or words into component parts. Many researchers (e.g. Sleeman, 1984; Kirshner, 1989; Jansen, Marriott, & Yelland, 2007; Landy & Goldstone, 2007a) have written about parsing as an important component of algebraic ability. In the context of algebra, parsing an expression means breaking the expression into pieces using the precedence conventions. We can therefore say that, along with performing arithmetic operations and interpreting and using algebraic syntax, parsing is the third component of competent performance of simplifying numeric expressions, as in Example A, and evaluating algebraic expressions, as in Example B.

Note, however, that in written performances like Examples A and B, the act of parsing is itself *invisible*. We can usually infer what parsing decisions were made from what is visible; at least one parsing decision is implicit in each step. But each step also involves performance of arithmetic and interpretation of syntax. Moreover, students sometimes perform more than one operation per step, in which case their parsing decisions are even less apparent.

We can contrast the implicitness of the parsing in Examples A and B with more explicit forms of parsing. One explicit form of parsing would involve inserting the redundant parentheses. Drawing an expression tree is another explicit way to parse an expression. Both inserting redundant parentheses and drawing expression trees involve indicating visibly the order of operation precedence.

Transforming expressions

Now that we have considered competent performance of the two more basic skills of simplifying a numeric expression and evaluating an algebraic expression, we are ready to return to the primary skill under consideration in this paper, namely expression transformation.

Thus far, I have only described the result of transforming an expression: when an expression is transformed correctly, an equivalent expression is produced. I have also shown examples of “classic” incorrect expression transformation. It is time to identify exactly what skills and concepts are involved in *correctly* transforming an expression. How does one go about finding an appropriate equivalent expression?

The following account builds upon writings in the information processing tradition, such as Matz (1980), Kirshner (1985), and Ernest (1987). Note that the following explanation of expression transformation is an idealization of algebraic behavior. I am not, in fact, making any claims about what actually transpires in the brains of users of algebra.

Generalization, and the two “faces” of algebra

To explain how one goes about the relatively complex task of transforming an algebraic expression, I must first say a bit about how algebraic symbolism is used for the purpose of *generalization*. In the process of describing algebra’s capacity for generalization, I will elaborate upon two mutually reinforcing and codependent aspects of algebra, which I will call the *referential* and *structural* aspects of algebra.

In contrast to expression transformation, which certainly resides within the realm of algebra, the two more simple skills discussed fall basically within the realm of arithmetic. True, the two do involve algebraic syntax for indicating operations, and they do involve algebraic conventions for operation precedence. Evaluating even involves variables, albeit briefly. However, we can characterize these activities as arithmetic rather than as algebra because they do not involve *generalization*. More than anything else, it is generalization – and not the presence of variables – that demarcates the territory of algebra.

Kaput (1995) describes two ways in which algebra functions as a tool for generalizing. First, algebra can generalize arithmetic facts. For instance, each of the arithmetic facts $5 + 0 = 5$, $12 + 0 = 12$, and $-7 + 0 = -7$ are instances of the algebraic generalization $m + 0 = m$. Second, algebra can generalize relationships among varying quantities. For instance, the relationship between the perimeter p of a rectangle and its base b and height h can be described by the generalization $p = 2b + 2h$.

When algebra is used to generalize arithmetic facts or relationships among quantities, its variables and expressions *refer* to numbers or quantities. These “referred to” numbers or quantities have an existence that is independent of the algebra symbolism used to express the generalizations on paper.

Yet algebraic symbolism is powerful not only because of its capacity for symbolically capturing generalizations. Algebraic symbolism’s power resides also in its ability to be considered *without* referential context. For instance, while the expression $2b + 2h$ can be considered in the context of rectangle and perimeter, it

also can be considered divorced from this context as a string of syntactically-related symbols. The task of transforming an expression is naturally performed while operating in such a decontextualized framework, as I will explain below.

We can idealize the importance of algebra in human activity as follows: first, a real situation about related quantities is generalized using algebraic symbolism; then, the symbolism is transformed into an equivalent form that is somehow different from the original form; and finally, the new form is usefully related back to the context that generated it. In the idealized description just provided of how people use algebra, the first and last phase of the process are alike in their focus on external referents, while the middle phase differs from the other two in its focus away from external referents and toward formal use of the symbolic language. Many researchers have therefore pointed out that it is possible to speak about two “aspects” (e.g. Kaput, 1995) or “faces” (e.g. Kirshner, 2006) of algebra, one focused on external referents and one focused away from them. Kirshner, for instance, describes the two faces of algebra as follows:

The *empirical face* points outward toward domains of reference, toward modeling phenomena in the world, toward application, toward number, quantity, and shape. The *structural face* points inward to the logical infrastructure, to the grammar of rules and procedures abstracted from external realms of interpretation. (p. 13)

Henceforth, I will describe these two aspects or faces of algebra as *referential* and *structural*.

Referential and structural algebra are co-dependent. Both are essential functions of algebra in human activity. Kirshner (2006) affirms the complementarity and necessity of the referential and structural aspects of algebra: “Internal structure and external reference are complementary and equally vital aspects of algebraic knowledge” (p. 14). Kaput (1995) agrees that the two aspects are complementary, touting both “the traditional power of algebra [that] arises from the internally consistent, referent-free operations that it affords” (p. 76) and the importance of the “semantic starting point where the formalisms are initially taken to represent something” (p. 76). Indeed, both algebra’s referential function and its structural function are powerful and necessary for algebra to be important for people.

The rules of algebra

Expression transformation is one of the tasks for which the structural aspect of algebra evolved. The process of transforming an expression can naturally be described by disregarding contextual referents and approaching algebraic symbolism formally. That said, in order for algebra to function in this decontextualized realm, one must have a supply of previously accepted generalizations of arithmetic facts. These previously accepted generalizations are often simply referred to as the *rules* of algebra.

There is no “official” list of the rules of algebra. Nearly all algebra texts would include any commutative, associative, identity, and distributive properties of the basic arithmetic operations as rules of algebra. They might also include properties

of exponents and properties of fractions. The rules are usually stated as equations, as shown here:

Examples of rules of algebra	
Commutative Property of Multiplication:	$mn = nm$
Distributive Property of Multiplication over Addition:	$m(n + p) = mn + mp$
Cancellation Property of Fractions:	$\frac{mp}{np} = \frac{m}{n}$

Note that each rule comprises two expressions, which we shall refer to as the *rule's expressions*.

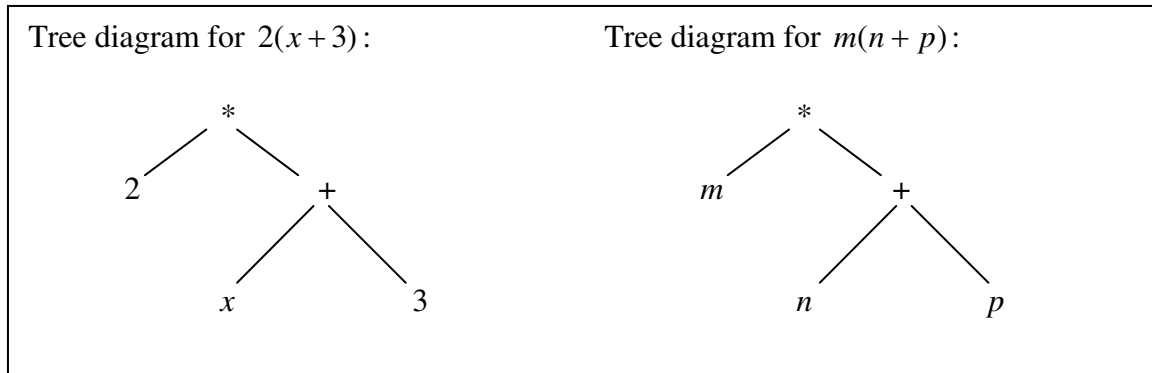
A thoroughly formal and axiomatic approach to algebra would designate a minimal number of these rules as *postulates* and then regard the other rules as proven *theorems*. Postulates are arithmetic generalizations accepted empirically and inductively without proof. Theorems, on the other hand, are derived from known rules. In practice, however, competent users of algebra need not concern themselves with the distinction between postulate and theorem. In fact, even expert users of algebra are not necessarily aware of the distinction: the rule $0 \cdot m = 0$ is typically regarded as a theorem in formal treatments of algebra, yet for most users of algebra this rule is simply an empirical generalization of known arithmetic facts. Since we are describing the use of algebra, we need not concern ourselves with any distinction between postulates and theorems, but can simply regard all previously and canonically accepted conclusions as rules.

Transforming an expression: Matching given expressions to rule expressions

The rules of algebra can be used to transform expressions in three distinct ways. I will describe the most basic way here; the other two will be described a bit later.

The basic way to use a rule to transform an expression involves matching the given expression to one of the expressions in a rule. Suppose, for instance, that a person encounters the expression $2(x + 3)$ and wishes to transform it. That person would compare the expression to each known rule of algebra, searching for a rule expression of which $2(x + 3)$ is an instance. That person might select $m(n + p)$, one of the expressions in the Distributive Property of Addition over Multiplication $m(n + p) = mn + mp$. In this case, 2 is playing the role of m , x is playing the role of n , and 3 is playing the role of p . After making this match, the person would turn to the rule's second expression – in this case $mn + mp$ – and substitute 2, x , and 3 for m , n , and p respectively, obtaining $2x + 2(3)$. The result, $2x + 2(3)$, is equivalent to the given expression $2(x + 3)$.

In the preceding example, $2(x + 3)$ was an instance of the rule expression $m(n + p)$ in the most basic of ways: each variable in the rule expression $m(n + p)$ corresponded directly with either a number or a single variable in the given expression $2(x + 3)$. Tree diagrams make the simplicity of the match even more apparent:



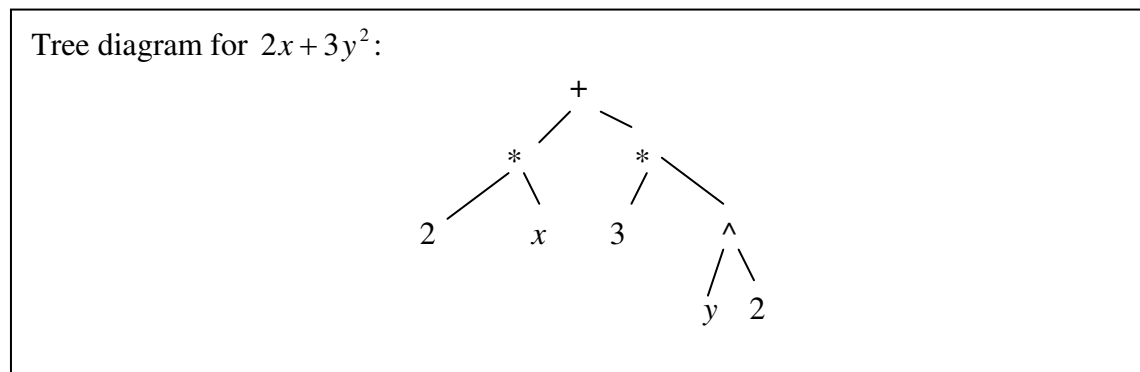
The trees of the two expressions have identical branch patterns and identical operations at the nodes of the branches; the only difference between the two expressions is that one's tree terminates in 2, x , and 3 while the other's tree terminates in a , b , and c .

Were such one-to-one matching of given expression to rule expression the only method of expression transformation, then it would indeed be a limited activity, for it would only be usable if one were lucky enough to know a rule with an expression that constituted a one-to-one match to the given expression. Were one-to-one matching the only method of expression transformation, then in order to be prepared for all the possible expressions one might need to transform, one would need to memorize many, many rules.

Fortunately, such one-to-one matching is not the only way to transform expressions. The extraordinary power of expression transformation resides in the fact that it often can be carried out when the given expression and the rule expression do not match in this one-to-one fashion. There are two additional ways in which one can transform expressions. Both depend upon a careful characterization of *subexpression* and *structural template*.

Subexpressions and structural templates

Tree diagrams provide a useful medium for illustrating the notion of a subexpression. Consider, for instance, the following tree, which shows the expression $2x + 3y^2$:



A subexpression of $2x + 3y^2$ is an expression obtained by considering any of the tree's numbers, variables, or operations and the portion of the tree lying below it. Thus, if we take the leftmost of the two multiplication signs and everything below it, we obtain the subexpression $2x$. If we take the rightmost of the two multiplication signs and everything below it, we obtain the subexpression $3y^2$. If we take the exponentiation sign and everything below it, we obtain the subexpression y^2 . If we take just the 3, we obtain the subexpression consisting only of the number 3. The complete list of subexpressions of $2x + 3y^2$ is: 2 ; x ; 3 ; y ; y^2 ; $2x$; $3y^2$; $2x + 3y^2$.

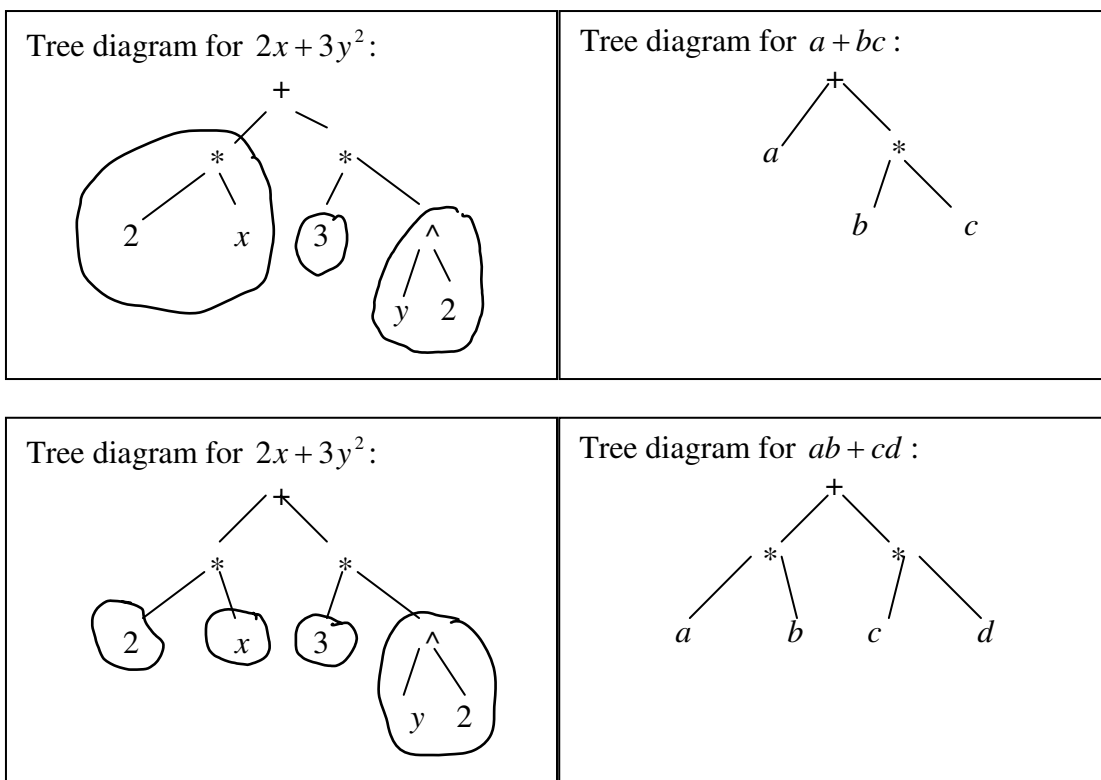
We come here to a crucial observation: to determine an expression's subexpressions, one must be able to parse that expression. Parsing, in other words, is a prerequisite skill for determining subexpressions. One cannot identify subexpressions unless one knows how to parse according to the conventions about operation precedence. y^2 is a subexpression of $2x + 3y^2$ because when the

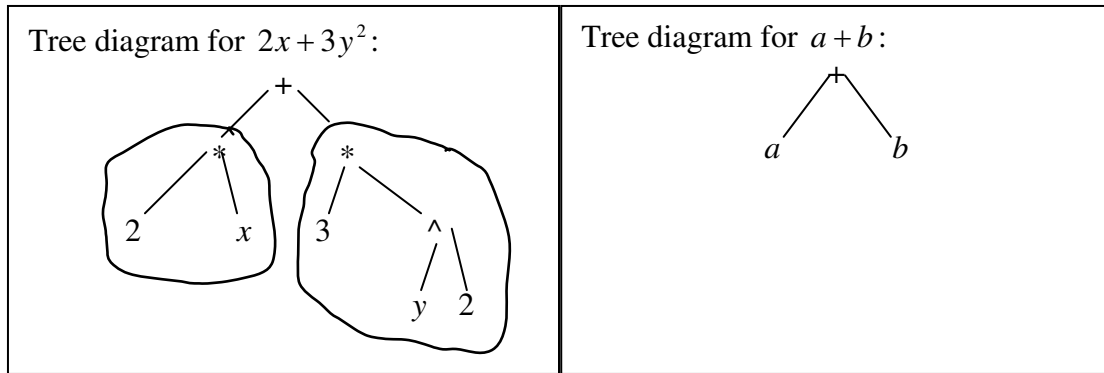
operations of $2x + 3y^2$ are performed according to the order of precedence, 2 and x are multiplied. On the other hand, even though $x + 3$ and $3y$ are perfectly good expressions in their own right, and even though these symbol strings occur in $2x + 3y^2$, neither is a subexpression of $2x + 3y^2$.

We also need to define a set of notions that are closely related to the notion of subexpression. The *dominant operation* of an expression (or of a subexpression) is the least precedent operation of that expression (or subexpression). For example, addition is the dominant operation in the expression $2x + 3$ while multiplication is the dominant operation in the expression $2(x + 3)$. When addition (or subtraction) is the dominant operation, then the subexpressions created by that addition (or subtraction) are called *terms*. So the terms of $2x + 3$ are the subexpressions $2x$ and 3 . When multiplication is the dominant operation, then the subexpressions created by that multiplication are called *factors*. So the factors of $2(x + 3)$ are 2 and $x + 3$. When division is the dominant operation, then the first subexpression is the *numerator* and the second is the *denominator*. When exponentiation is the dominant operation, then the first subexpression is the *base* and the second is the *exponent*.

We can also define the notion of the possible *structural templates* for an algebraic expression. For our purposes, a structural template for an expression is another expression containing only variables and operations (i.e. no numbers) that possesses a very particular sort of “top-down” identicalness to the original expression. More specifically, some of the two expressions’ least precedent operations and their indicated order of precedence must be exactly the same. For example, returning to the expression $2x + 3y^2$, we see that all of the following are possible structural

templates for $2x+3y^2$: $a+b$, $a+bc$, $a+bc^d$, $ab+c$, $ab+cd$, $ab+cd^e$, and even simply a . Again, tree diagrams provide a very useful medium for illustrating the possible structural templates of an expression. Consider again the tree diagram of $2x+3y^2$, shown several times below side-by-side with the tree diagram of one of its structures. In each $2x+3y^2$ diagram, the subexpressions playing the roles of the template's variables are circled. This circling makes evident the sense in which the original expression is "identical" to the structural template:





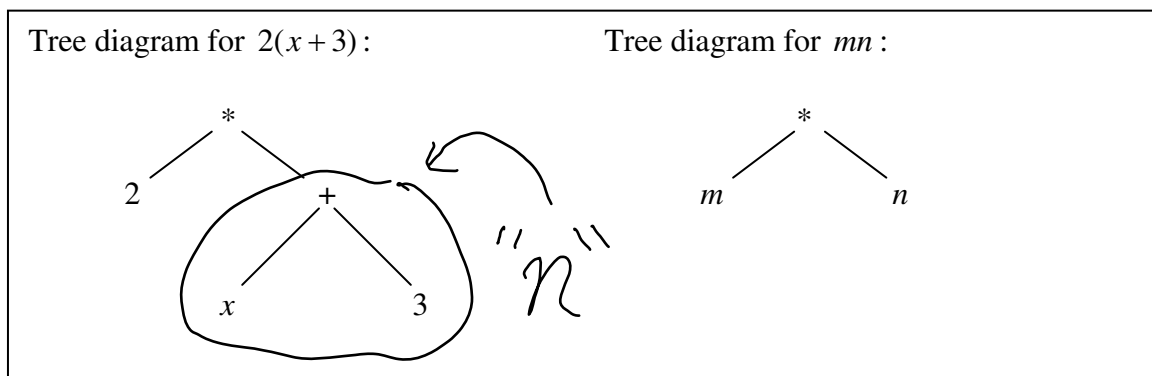
It is important to emphasize that most expressions have several possible structural templates. Depending upon the context, one might choose a structural template including either more or fewer of its operations, embedding fewer or more of the least precedent operations within variables.

Again, determining the possible structural templates for an expression depends directly upon one's ability to parse. For this reason, we might instead refer to a structural template of an expression as a *parse* of that expression, forming a noun out of the verb. Also, henceforth I will use the term *structure* to refer to the component of algebraic knowledge that concerns understanding of parsing, subexpressions, and structural templates.

Transforming an expression: Using subexpressions

Now that I have precisely defined all of the structural notions described above, I can proceed to describe the two remaining ways, along with straightforward matching of expression to rule, to transform an expression. Both of these ways to transform an expression depend heavily upon the notion of an expression's subexpressions.

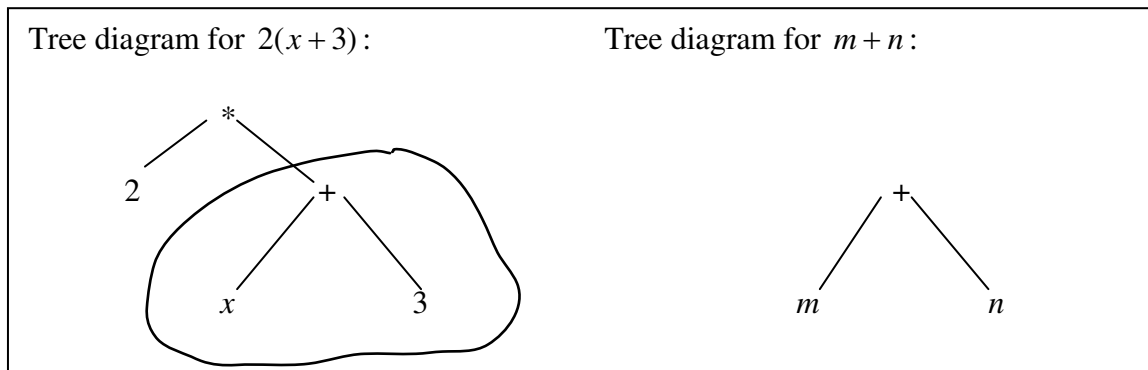
The second way to transform expressions utilizes the capacity of a variable in a rule to stand for an entire subexpression of a given expression. Consider again the expression $2(x+3)$. Suppose again that a person wishes to transform this expression. This time, suppose the person selects the expression mn from the Commutative Property of Multiplication $mn = nm$. Although mn is not a one-to-one match for $2(x+3)$ in the way that $m(n+p)$ was, mn is a structural template for $2(x+3)$. In this case, 2 is playing the role of m and the entire subexpression $x+3$ is playing the role of n . Here is the match in tree diagrams, with the subexpression $x+3$ circled to emphasize that we are treating it as a single entity matched with n :



After making this structural match, the person would consider the rule's second expression – in this case nm – and substitute 2 and $x+3$ for m and n , respectively, obtaining $(x+3)2$. The result, $(x+3)2$, is equivalent to the given expression $2(x+3)$.

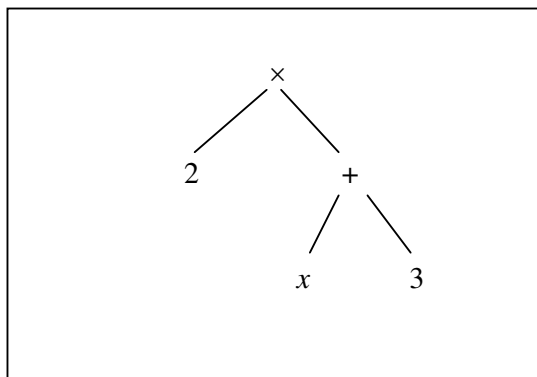
The third way to transform expressions involves finding a subexpression of the given expression for which a rule expression is a structural template. Consider once more the expression $2(x+3)$. Suppose again that a person wishes to transform this expression. This time, suppose the person selects the expression $m+n$ from the

Commutative Property of Addition $m + n = n + m$. Although $m + n$ is not a structural template for the given expression, it is a structural template for the subexpression $x + 3$. In this case, x is playing the role of m and 3 is playing the role of n . In tree diagrams, to see the match one must ignore part of one tree and focus on a subexpression only, which is circled here:

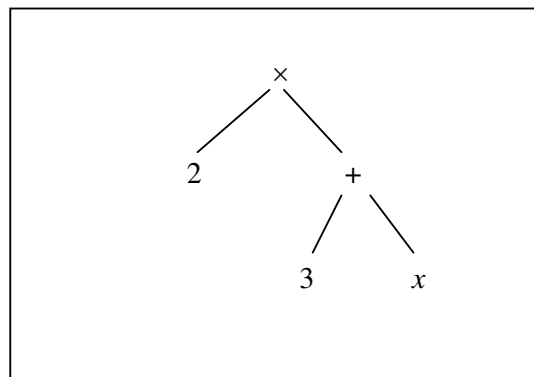


After making this structural match, the person would turn to the rule's second expression – in this case $n + m$ – and substitute x and 3 for m and n , respectively, obtaining $2(3 + x)$. The result, $2(3 + x)$, is equivalent to the given expression $2(x + 3)$. In tree diagrams, it is easy to see that only a subexpression is affected by the transformation; a portion of the tree is unchanged by the transformation:

Before the transformation:



After the transformation:



Clearly, expression transformation is extremely powerful. It allows us to transform all of an expression or just a part of an expression. Moreover, the capacity of variables to stand for subexpressions renders unnecessary the memorization of many individual rules like $m(n + p) = (n + p)m$, $(m + n)(p + q) = (p + q)(m + n)$, and so on; the single rule $mn = nm$ suffices to generalize these and infinitely other more specific instances. We can therefore transform all sorts of expressions while memorizing only a relatively limited set of rules.

Using reference as support for expression transformation

Before completing our description of expression transformation, we need to consider the role that referential algebra plays in the process.

In our idealized description of how people use algebra, we located expression transformation squarely within its structural aspect. Thus far in our description of expression transformation, we have regarded the rules of algebra as givens, and therefore the process of transforming an expression has been a purely formal process of using and following rules – the conventions of operation precedence and the rules of algebra – precisely. We have been regarding algebraic expressions as mere symbol strings to be interpreted and operated upon while ignoring their possible referential meanings.

It is also possible to incorporate referential interpretations into the process of expression transformation. These interpretations provide support for the formal structural decisions that occur during the transformation process described above.

Here is an example from the Core-Plus Mathematics Project of how referential interpretations can be used to support the structural transformation process. Coxford et al. (2003, p. 188) describe a situation involving profit and expenses for production of a CD:

Suppose that when a new band recorded its first album with a major label, it had to deal with these business conditions:

- Expenses of \$365,000 for the recording advance, video production, touring, and promotion (to be repaid out of royalties)
- Income of \$0.81 per CD from royalties
- Income of \$0.52 per CD for publishing rights

Letting x represent the number of CDs sold, students form a generalization: for any number x of CDs sold, the expression $(0.81x + 0.52x) - 365,000$ represents the profit made from these sales. Forming this generalization is clearly an act of referential algebra. Now, suppose the students want to transform this expression. Certainly students could transform the expression to the equivalent $1.33x - 365,000$ using the rules of algebra as described above. However, students can also reach the same conclusion by thinking about the situation and forming a new generalization. Specifically, it should be clear to students that while the positive component of the profit can be computed by individually multiplying the number of CDs sold by the royalty income and the publishing rights income and then adding, this profit can also

be computed by adding the royalty income per CD and the publishing rights income per CD and then multiplying by the number of CDs sold. Depending upon the experience of the student, such referential reasoning could inform the student's thinking about the symbol manipulations. The Teacher's Guide explains that students can reach the conclusion about the equivalence of $(0.81x + 0.52x) - 365,000$ and $1.33x - 365,000$ by applying "contextual knowledge to make sense of rearrangements of symbols" (p. T188). Elsewhere, the Teacher's Guide explains how empirical evidence from tables of values or from graphs can be used as referential support for expression transformation decisions.

Although referents can, as illustrated by these examples, be used as supports during expression transformation, this paper deals primarily with student learning of expression transformation as a structural task. Henceforth, whenever I refer to expression transformation, unless I specifically indicate otherwise, I will be describing a purely structural approach to expression transformation that utilizes the known rules of algebra without any further referential support.

Diagnosing student difficulties with expression transformation

Our overall goal in this paper is to derive some instructional strategies to help students overcome some common difficulties that they have when transforming algebraic expressions. In the previous section, I provided a careful description of competent performance of expression transformation. In particular, I uncovered the crucial roles for expression transformation of a procedure called *parsing* and of a set of structural concepts, especially *subexpression* and *structural template*.

Now we will begin to diagnose what goes wrong when students transform expressions incorrectly. In other words, we will begin to identify how the idealized description of competent expression transformation so often fails to become reality. Since an understanding of structure is so fundamental for competent expression transformation, I give the following initial diagnosis of student difficulties: *The traditional algebra curriculum fails to provide students with the necessary experiences to develop a full understanding of critical structural notions.*

To make this diagnosis credible, I will utilize Sfard's (1991) framework for the relationship between *processes* and *objects* in mathematics. We will see that Sfard's framework nicely captures the relationship between the process of parsing and the objects known as subexpressions. Her framework also provides insight as to how students can eventually come to understand mathematical objects. With Sfard's insights in mind, we will examine the traditional algebra curriculum and expose its superficial treatment of the very objects upon which expression transformation is performed.

Reification: Mathematical objects as compressed processes

As we have seen, knowing how *to parse* an expression is a prerequisite skill for simplifying numeric expressions and for evaluating algebraic expressions. Moreover, understanding the notion of *a parse* (or structural template) of an expression is a prerequisite concept for transforming algebraic expressions. For the simpler skills, parsing is a procedure to be performed. For the more advanced skill of expression transformation, a parse is a thing to be understood conceptually. What is

the relationship between knowing how to parse and understanding the notion of a parse?

This question really pertains to the relationship between the *procedural knowledge* of how to parse and the *conceptual knowledge* of what is a parse. To answer it, we will briefly examine a small part of the extensive literature about the relationship between procedural knowledge and conceptual knowledge in mathematics.

While current debates about curriculum might lead one to believe that the two are in opposition, scholars tend to agree that procedural knowledge and conceptual knowledge are both vital and necessary components of mathematical proficiency. For instance, Kilpatrick (1988) argues that “some balance needs to be found between meaning and skill” (p. 274). Similarly, Rittle-Johnson, Siegler, & Alibali (2001) conclude that “competence in a domain requires knowledge of both concepts and procedures” (p. 359) and that the two are mutually reinforcing: “The relations between conceptual and procedural knowledge are bidirectional and ... improved procedural knowledge can lead to improved conceptual knowledge, as well as the reverse” (p. 360). To cite one more example, Star (2005), in discussing procedural and conceptual knowledge, claims that “both are critical components of students’ mathematical proficiency” (p. 406).

Sfard (1991) goes one step further. Like the researchers cited above, Sfard regards procedures and concepts as necessary and mutually reinforcing. However, she takes the additional step of claiming that the two types of knowledge (she calls

them “operational” and “structural”) are “inseparable, though dramatically different, facets of the same thing” (p. 9).

Sfard’s claim – that procedures and concepts are two facets of the same thing – hinges upon her careful explanation of the notion of a mathematical *object*. In mathematics, Sfard explains, processes are usually performed upon one or another sort of “thing” or object. For instance, addition can be performed upon the objects known as natural numbers, composition can be performed upon the objects known as functions, and so on. According to Sfard, many of these fundamental mathematical objects are themselves simply processes which, through repetitive familiarity, become condensed in the mind of their users into static “things” or objects. For instance, Sfard argues that while rational numbers are mathematical objects, this object-perception of rational numbers as static entities grows out of and complements a process-perception of rational numbers as the comparing or dividing of two natural numbers. She provides many other examples of mathematical “things” that possess this process-object duality.

According to Sfard, process-perception necessarily precedes object-perception, both in the historical development of mathematics and for the student learning mathematics. One must, Sfard argues, learn how to perform a process before one can step back and look at the result of that process as an entity in its own right. The concept of a rational number as a static “thing,” for instance, emerges out of the process of forming ratios of natural numbers.

Sfard uses the term *reification* to describe the moment in which one shifts from a process-perception to an object-perception and forms a static, conceptual

understanding of a mathematical object. For example, the moment when a person shifts from only being able to regard 5 divided by 12 as a process to also being able to regard the ratio 5:12 as an object is a moment of reification. Sfard uses the terms *interiorization* and *condensation* to describe earlier stages in which the process gradually “condenses” for the learner. According to Sfard, interiorization and condensation are gradual processes; however, she regards reification not as a process but rather as a sudden change in perspective, in which a process is finally grasped all-at-once as an object.

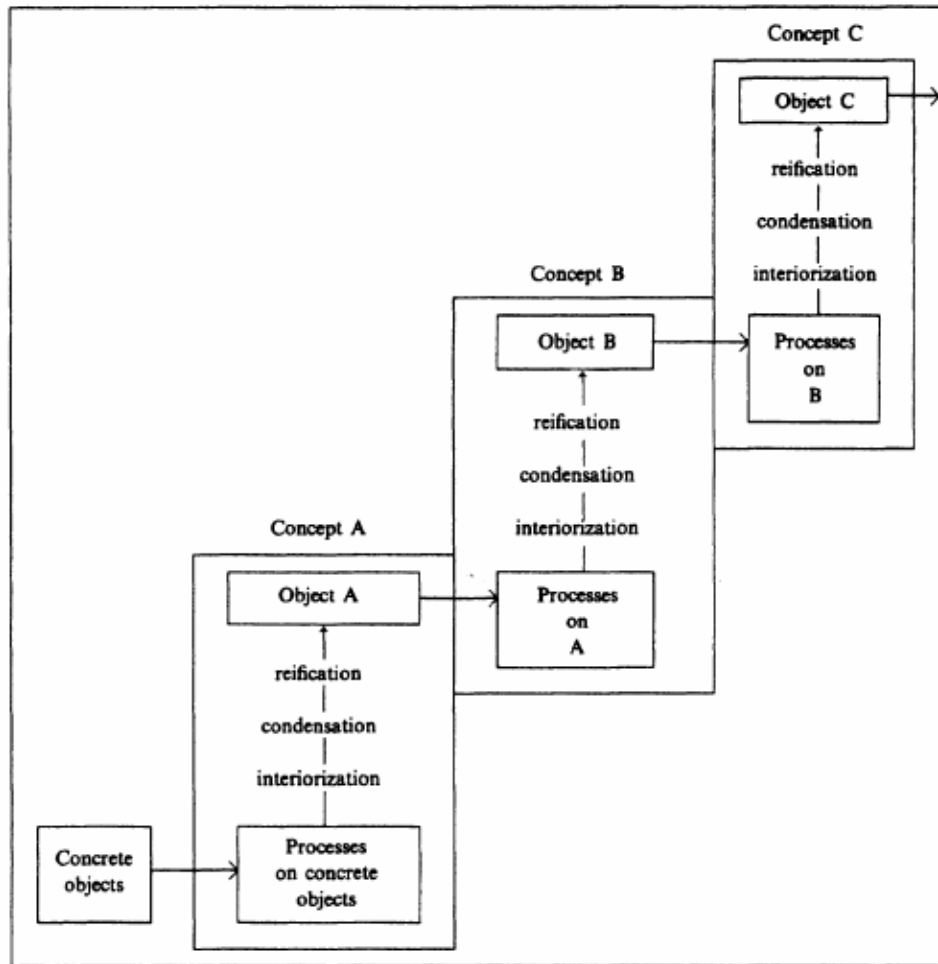
While Sfard regards the process-perception as necessarily preceding the moment of reification and its object-perception, she also regards the object-perception as necessary – for an understanding of even more advanced processes. From a historical perspective, Sfard explains, the mental demands of performing more advanced processes on the objects of simpler processes is what has necessitated reification and its ensuing object-perception. Let us continue to use rational numbers as an example. As long as simply comparing natural numbers to one another was the goal, there was no need to conceive of these ratios as objects in their own right. However, once people developed needs to compare, add, and multiply ratios, it became expedient to be able to treat individual ratios as objects – hence the reification of rational numbers into independent entities. Mathematical objects, therefore, serve as necessary “pivot points,” or, as Sfard says, “way-stations” (p. 29) between one procedure and a more advanced procedure.

Sfard regards this process-object-process sequence as repeating itself iteratively in humanity’s development of more and more advanced mathematics.

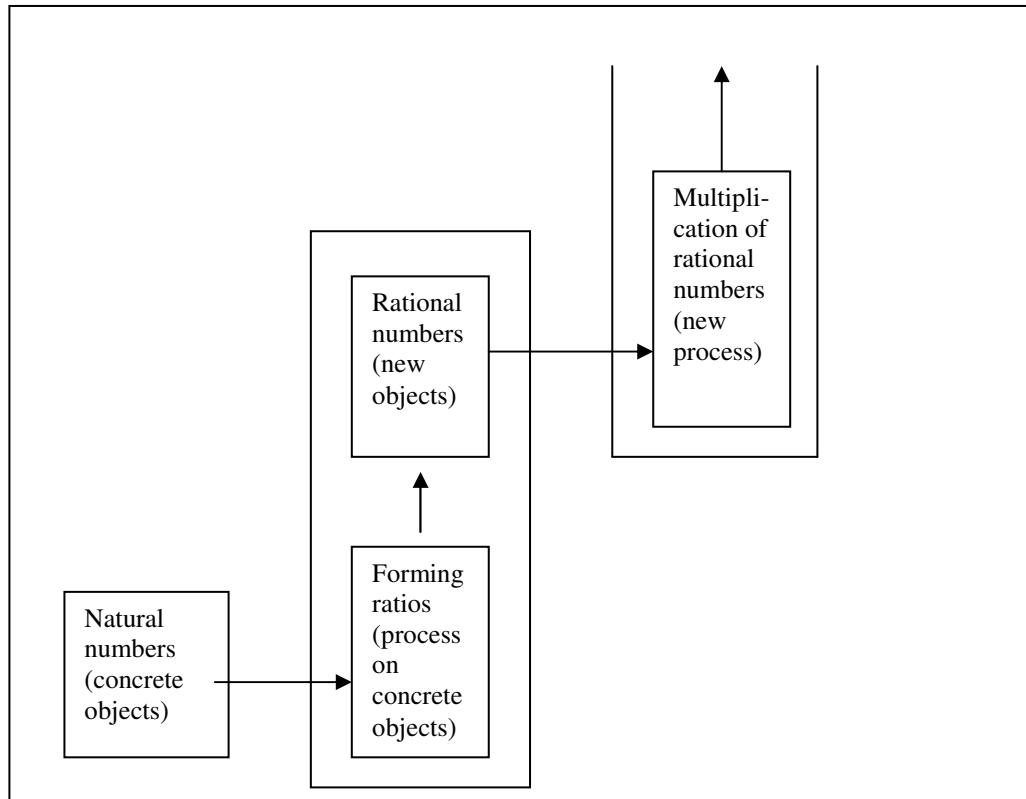
Over and over again, she explains, “various processes had to be converted into compact static wholes to become the basic units of a new, higher level theory” (p. 16). The process repeats itself, she explains, as one newly reified object itself becomes the object of more advanced processes:

When we broaden our view and look at mathematics (or at least at its big portions) as a whole, we come to realize that it is a kind of hierarchy, in which what is conceived purely operationally at one level should be conceived structurally at a higher level. Such hierarchy emerges in a long sequence of reification, each one of them starting where the former ends, each one of them adding a new layer to the complex system of abstract notions. (p. 16)

Sfard provides the following diagram to illustrate the process-object-process-object-process structure of mathematical knowledge:



Returning again to the example of rational numbers will help illustrate the meaning of this diagram. Here is how the diagram would depict the relationship between the object of rational number to the simpler processes that produce rational numbers and to the more complex processes for which rational numbers are the objects:



Sfard of course knows that she is oversimplifying the complex history of the emergence of mathematical concepts. Nonetheless, she regards this account as a good approximation of not only the historical development of mathematics but also of its development for the individual learner.

Subexpressions and structural templates as mathematical “objects”

Sfard’s theory of reification has profound implications for student learning of algebra. In a 1994 sequel entitled “The Gains and Pitfalls of Reification: The Case of Algebra,” Sfard & Linchevski explore the implications of the theory of reification for algebra in particular.

Sfard and Linchevski begin by showing that algebraic expressions possess the sort of process-object duality described by the theory of reification. They use the

expression $3(x + 5) + 1$ as an example. On the one hand, this expression describes a “computational process” to be performed upon a variable x : take x , then add 5, then multiply by 3, then add 1. On the other hand, this expression is an object in and of itself. Sfard and Linchevski mention three different ways to view this expression as an object. One involves viewing it structurally, in other words, viewing it “as a mere *string of symbols*” (p. 88). They note the power of this sort of object-perception for the tasks of structural algebra: “Although semantically empty, the expression may still be manipulated and combined with other expressions of the same type, according to certain well-defined rules” (p. 88). The other two ways to treat the expression as an object involve viewing it referentially, either as a name for a “certain number” or else as a “function.”

Interestingly, although Sfard and Linchevski mention the “semantically empty string-of-symbols” perspective of conceiving of an expression as an object – which is the way most suited for expression transformation – in the remainder of the article they focus on the “function” perspective. For example, they consider student understanding of relatively advanced algebraic tasks such as solving the inequality $x^2 + x + 1 > 0$, for which a function object-perception is particularly appropriate.

After reviewing a variety of empirical evidence, the authors conclude that reification of algebraic expressions into function-objects is particularly elusive. They regard the process-concept duality inherent in algebraic expressions to be very counterintuitive for novices: “Our intuition rebels against the operational-structural duality of algebraic symbols” (p. 199). In fact, they go so far as to suggest that this particular leap of reification is too difficult for some students to ever make:

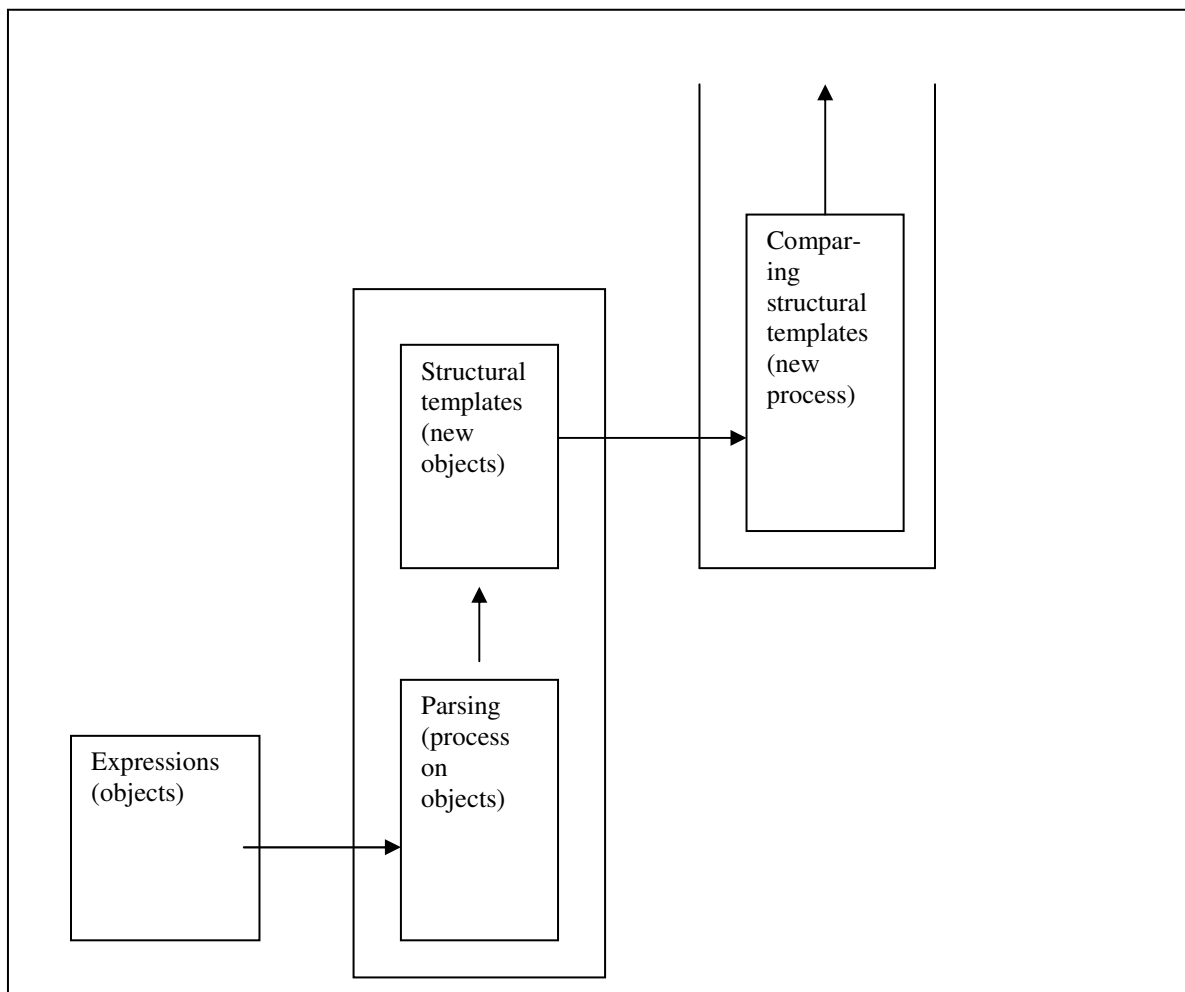
The data we collected up to this point provided sufficient evidence that reification is inherently very difficult. It is so difficult, in fact, that at a certain level and in certain contexts, a structural approach may remain practically out of reach for some students. (p. 220)

Thus, upon applying the theory of reification to the case of algebra, Sfard and Linchevski are discouraged by their findings regarding the potential for student understanding of algebra.

I maintain, however, that can draw more encouraging conclusions from an application of the theory of reification to the learning of algebra. This more optimistic perspective, however, involves applying the theory to algebra in a different way that Sfard and Linchevski apply it. Sfard and Linchevski have chosen to focus on the most advanced of all possible object-perceptions of an algebraic expression, namely the perception of an expression as a function. As Herscovics (1989) has shown, however, the concept of function brings along its own set of cognitive obstacles for the novice. Moreover, as we have shown above, expression transformation – one of the most central and most notoriously difficult algebraic tasks – depends solely on an object-perception of expressions as “strings-of-symbols” parsed according to known conventions.

Moreover, there is a second and perhaps equally important application of the theory of reification to algebra that Sfard and Linchevski do not identify. In addition to regarding algebraic expressions themselves as objects, it is important for students to be able to regard *structural templates* as objects. In fact, Sfard’s description of process-concept complementarity perfectly captures the relationship between the

process of parsing and the concept of structural template. An expression's structural templates are nothing but the results of parsing that expression. When one parses, one determines possible structural templates. Moreover, an expression's possible structural templates are themselves the objects of the more advanced process of comparing structural templates and forming matching with rule expressions. The mathematical objects known as subexpressions and structural templates therefore occupy the pivot point between two procedures in the sense described by Sfard. To illustrate:



This application of the theory of reification to algebra does not involve cognitively complex objects like functions. Instead, the objects here are “empty” structural

objects. Insofar as an understanding of an expression's possible structures is essential to the task of expression transformation, it is therefore worth exploring to what extent the traditional algebra curriculum provides the sort of experiences that are likely to foster reification of the concepts of subexpression and structural template.

PEMDAS: Structure (or lack thereof) in the traditional curriculum

Recall Sfard's contention that process-perception necessarily precedes object-perception. One consequence, therefore, of viewing the relationship between parsing and structural templates through the lens of Sfard's theory of reification is an implication for learning: students need to parse first in order to arrive later at a full understanding of structure. When we examine traditional algebra curricula, however, we find two striking absences. First, structure is considered only superficially; second, the parsing process that ideally reifies into an object-perception of structural template is also treated superficially.

Traditional texts do not typically deal with structural notions explicitly. The word "subexpression" does not typically appear in a traditional algebra textbook. Nor do the words "structural template" in the sense we have used it here. Many texts do define specific structural notions such as "factor," "term," and so on. However, rarely do these texts provide exercises that require the student to discriminate among these structural notions. We have seen that structural notions play a critical role in competent expression transformation. The absence of attention to structural notions is all the more striking because expression transformation is perhaps the most central activity in the traditional algebra textbook. Foerster (1994), for instance, includes

chapters on “Distributing: Axioms and Other Properties,” “Some Operations with Polynomials and Radicals,” “Properties of Exponents,” “More Operations with Polynomials,” “Rational Algebraic Expressions,” and “Radical Algebraic Expressions.” Each of these six chapters (out of a total of fourteen) is devoted nearly entirely to expression transformation. Thus, despite the extensive focus of traditional textbooks on expression transformation, students learning algebra from those books might never encounter the structural notions the understanding of which is essential to transfer expressions competently.

Moreover, traditional algebra textbooks include only superficial exposure to parsing as a process, making it even more unlikely that students will attain the object-perception of structural template. On the one hand, typical algebra texts do very much require students to perform exercises that demand *implicit* demonstration of parsing abilities. Simplifying numeric expressions and evaluating algebraic expressions are standard fare in beginning algebra. On the other hand, algebra textbooks typically do not require explicit demonstration of parsing ability, as through insertion of parentheses or drawing expression trees. They also typically do not include exercises designed to push students from the process-perception of parsing to the object-perception of structural template.

The parsing instruction traditional algebra texts do include typically falls under the rubric of instruction in the “order of operations.” Foerster’s 1994 text is typical in this regard. Section 1-4 of the text is entitled “Order of Operations.” He introduces the topic as follows:

You have learned that symbols of inclusion can be used to tell which operation is to be performed first in an expression. If there are more than three operations, there would be so many parentheses and brackets that the expression would look untidy, like this:

$$(4 + [(9 \times 3) \div 6]) + [(5 \times 8) - 7].$$

To avoid all this clutter, users of mathematics have *agreed* on an order in which operations are to be performed. Parentheses are used only to *change* this order. (p. 18)

Foerster then goes on to state the agreed-upon “order of operations” as follows (p. 19):

ORDER OF OPERATIONS

If there are no parentheses to tell you otherwise, operations are performed in the following order:

1. Evaluate any *powers* first.
2. After powers, *multiply* and *divide*, in order, from left to right.
3. Last, *add* and *subtract*, in order, from left to right.

In the exercises following this presentation, Foerster’s text provides many examples of numerical expressions to be simplified and algebraic expressions to be evaluated.

Some teachers (e.g. Schrock & Morrow, 1993), and even some algebra texts, present a mnemonic device for helping students learn the order of operations. In the United States, this mnemonic is usually PEMDAS, which stands for Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction. To remember the

acronym, some students learn the phrase “Please Excuse My Dear Aunt Sally.” In some other English speaking countries, the acronym is either BEMDAS (with B for “Brackets”), BIMDAS (with I for “Indices”), or BOMDAS (with O for “Of,” as in “power of” or “root of”).

Many teachers, however, have discovered pitfalls of teaching students to rely on PEMDAS and similar mnemonics. Nurnberger-Haag (2003), for instance, notes that parentheses are only one type of grouping symbol that students encounter in algebra, and that the PEMDAS mnemonic overlooks brackets, fraction bars, and other notational parsing indicators: “Teaching students about only one special case of grouping symbols is analogous to teaching only a special case of exponents for the second step (such as squaring)” (p. 235). Moreover, several teachers (e.g. Rambhia, 2002; Nurnberger-Haag, 2003) point out that the PEMDAS mnemonic seems to imply that addition and subtraction occupy successive levels in the operation hierarchy (and similarly with multiplication and division), when in fact these operation pairs have the same degree of precedence. Rambhia cautions that as a result of PEMDAS-focused instruction, “many students come to believe ... that multiplication is done before division and that addition is more important than subtraction” (p. 194). Thus, PEMDAS and similar mnemonic devices hinder as well as assist the learning of order of operations.

In critiquing the traditional treatment of parsing, I am critiquing much more than a mnemonic device whose shortcomings are already famous. I have argued that understanding of structural template as a mathematical object is essential to the competent performance of a skill – expression transformation – that is the primary

content of traditional algebra instruction. The use of PEMDAS in classrooms across the United States is only a *symptom* of a much larger lack of attention to structure. I have argued that the necessary object-perception of structural template can only arise out of extensive experience parsing algebraic expressions, and I have argued that the traditional curriculum does not require students to parse other than implicitly. The net result is the following partial diagnosis for student difficulty transforming expressions: *The traditional algebra curriculum fails to provide students with the necessary experiences to develop a full understanding of algebraic structure.*

Diagnoses that support or overlap with mine

Some teachers and researchers who have written about student difficulty transforming expressions have arrived at diagnoses similar to the one I have argued for here.

Several teachers have blamed transformation errors on student confusion about the meaning of specific structural notions, especially the notions of *term* and *factor*. Martinez, for instance, in an article entitled “Helping Students Understand Factors and Terms,” concludes, after examining many instances of mal-rule behavior, that “in each instance the error is caused by students’ misunderstanding of factors and terms” (p. 747). Similarly, Laursen (1978) points out that some errors stem from the confusion of factors and terms:

Many of the theorems in elementary algebra relate specifically to either terms or factors, but not to both. For example,

$$\sqrt{(9)(16)} = \sqrt{9}\sqrt{16}$$

But

$$\sqrt{9+16} \neq \sqrt{9} + \sqrt{16} . \quad (\text{p. 195})$$

One more example comes from Grossman (1925), who diagnoses fraction cancellation errors as stemming from students' lack of understanding of what a factor is:

This is the real source of many of the mistakes in cancelling fractions. A pupil may understand all that has gone before and still cancel wrongly through not understanding how far the division effect of the cancellation of a factor extends. The essence of the idea is that it extends as far as the multiplying effect of the factor itself extends. (p. 107)

These teachers identify student confusion about factors and terms as responsible for many student transformation errors, consistent with my claim that the traditional curriculum does not attend sufficiently to structure.

Others have identified students' difficulty with the process-object duality of algebraic expressions as responsible for student transformation errors. Barnard (2002a), for instance, writes:

It can be argued that algebra starts when the things one is talking and thinking about have become mentally manipulable objects. At the heart of many errors is the failure to conceive the objects of manipulation (e.g. $\frac{3}{4}$, -7 , $2x+5$, $\sqrt{r^2+1}$) as meaningful 'things' in their own right. (p. 10).

He goes on to explain how student inability to perceive expressions as objects can

lead to mal-rules like $\sqrt{a^2 + b^2} = a + b$:

If pupils are able to see an expression like $\sqrt{a^2 + b^2}$ as one complete object, not only will they not feel the need to ‘work it out’ further (perhaps incorrectly replacing it with $a + b$), but also they will be able to move it around in an equation just as easily as they could move around a single letter or number. Stumbling blocks are often caused by the appearance of unsimplifiable expressions that have no meaning for pupils. (p. 11)

Booth (1984) also identifies student difficulty treating unsimplifiable expressions as objects. While these diagnoses are similar to mine, my diagnosis goes into more detail about the roles of subexpressions and structural templates in competent expression transformation and the failure of the traditional curriculum to help students understand these structural notions.

Still others have diagnosed student difficulties with expression transformation as stemming from students feeling that algebra is entirely lacking in “objects.” Sfard and Linchevski (1994), in their discussion of reification in algebra, make just such a diagnosis. After asserting that many students fail to grasp algebraic expressions as objects, they conclude that students will have difficulty learning to operate on expressions beyond what they can memorize from a purely procedural perspective:

The process of learning is doomed to collapse: without the abstract objects, the secondary processes will remain ‘dangling in the air’ – they will have to be executed ... on nothing. Unable to imagine the

intangible entities ... which he or she is expected to manipulate, the student [uses] pictures and symbols as a substitute ... In the absence of abstract objects and their unifying effect, the new knowledge remains detached ... from the previously developed system of concepts. In these circumstances, the secondary processes must seem totally arbitrary. The students may still be able to perform these processes, but their understanding will remain instrumental. (p. 221)

In other words, Sfard and Linchevski attribute student difficulty operating upon algebraic expressions to student failure to perceive these expressions as objects. Chazan (2000) also points to the lack of explicit “objects” in the traditional algebra curriculum as partly responsible for student difficulties with the subject.

Each of these prior diagnoses overlaps with and to some extent supports my diagnosis. However, my diagnosis specifically designates subexpressions and structural templates as the objects about which students are lacking conceptual understanding.

The role of overgeneralization

Thus far, we have identified expression transformation as an area of student difficulty in algebra. We have seen that understanding structure is critical for competent expression transformation. Using Sfard’s theory of reification, we have identified parsing as an activity whose performance naturally leads to an understanding of structure in algebra. We have diagnosed student difficulties with expression transformation as at least partly due to the traditional curriculum’s

superficial treatment of structure. Finally, we have collected some related diagnoses that support our diagnosis, although ours goes into more detail in its articulation of what it means to understand structure in algebra.

If this were the whole story, then I would now proceed to recommend a less superficial treatment of parsing and of structure. However, two related facts remain mostly unexplained. One fact is the striking uniformity of student errors: we have seen that teachers report predictable wrong responses, and lack of understanding of how to proceed correctly does not explain the uniformity in how many students proceed incorrectly in the same way. A second fact is that while some prior writings support my diagnosis, many other teachers and researchers identify a different culprit as responsible for student difficulties with expression transformation, namely a strong tendency for students to *overgeneralize*.

Schwartzman (1986) provides examples of overgeneralization. In an article entitled “The A of a B is the B of an A ,” he writes about how students have difficulty restraining themselves from overgeneralizing the notion of the distributivity of one operation over another. He provides a list of twenty non-equivalencies that students are prone to assume. These include $(a + b)^n \neq a^n + b^n$, $\frac{1}{a + b} \neq \frac{1}{a} + \frac{1}{b}$, and $\sqrt[n]{a + b} \neq \sqrt[n]{a} + \sqrt[n]{b}$. All of these examples are variations on true distributive statements, such as “the power of a product is the product of the powers” (p. 181). However, they are overgeneralizations in that the student who accepts them is assuming fewer constraints on which operations distribute over which others than actually exist.

In this section, we will examine what Matz (1980) and Kirshner & Awtry (2004) have to say about the role of overgeneralization in producing common transformation errors. In the following sections, I will argue that overgeneralization is not merely a strategy that students use in the absence of structural understanding but is something that algebra curricula need to address even if they give structure its due. In other words, I will argue that our initial diagnosis for why students make common algebra errors is only a partial diagnosis and that a full diagnosis needs to anticipate students' strong tendencies to overgeneralize.

Matz: Overgeneralizing rule-revision strategies

In her 1980 paper on common algebra errors, Matz sets out to provide an account of the processes that lead to the many universally predictable errors in algebra. Among the initial observable facts that demand explanation, she includes the “striking regularity of the answers produced” when students make mistakes while learning symbolic algebra. She strives for a theory of error with broad explanatory power: she laments that “previous studies of high school algebra errors have been essentially extensive lists” (p. 155), such as the one compiled by Marquis, and aims to provide an explanation for algebraic behavior that accounts for these diverse errors in a unified framework.

According to Matz, students encountering an algebra task often resort to a strategy that she calls *revising a rule*. Revising a rule involves taking one of the known rules of algebra and modifying it to fit a given problem to which it does not directly apply. Matz subdivides examples of revising a rule into two categories:

revising a rule “by generalization” and revising a rule “by linear application.” Matz claims that students are prone to resort to these strategies because these strategies have been successful in the past. In fact, sometimes when students use these strategies, they obtain correct results. However, while sometimes applicable, these inductive strategies are not part of the deductive process of transforming expressions using known rules of algebra. When students incorrectly revise a rule, they are overgeneralizing their applicability.

Revising a rule by *generalization* involves taking a known rule, deciding that one of its components (i.e. a number or an operation) is incidental rather than essential to the rule, and then replacing that component with another. Matz describes this technique as follows: “Generalization bridges the gap between known rules and unfamiliar problems by in effect revising a rule to accommodate the particular operators and numbers that appear in a new situation” (p. 105). Matz provides examples of rule generalization as a successful strategy, such as the fact that “‘minus’ can be substituted for the ‘plus’ operator in the distributive law” (p. 105). She also provides examples of incorrect rule generalization. These include many of the incorrect distributing errors discussed by Schwartzman (1986) and referred to above, but they also include instances of incorrectly generalizing specific numbers to arbitrary numbers, such as revising “ $(x - a)(x - b) = 0$ implies $x - a = 0$ or $x - b = 0$ ” to the incorrect “ $(x - a)(x - b) = k$ implies $x - a = k$ or $x - b = k$.”

Revising a rule by *linear application* involves assuming that if an expression can be transformed in a particular way, then its parts can be transformed in an identical way, regardless of what the parts are. Linearity, Matz explains,

describes a way of working with a decomposable object by treating each of its parts independently. An operator is employed linearly when the final result of applying it to an object is gotten by applying the operator to each subpart and then simply combining the partial results.

(p. 111)

Again, Matz emphasizes that linearly applying an operator or a procedure is often correct:

Most of a students' previous experience is compatible with a linearity hypothesis. In arithmetic, the immense number of occasions that students add and use the distributive law very likely reinforces their acceptance of linearity. This trend continues with early algebra problems. (p. 111)

However, Matz goes on to show how a linearity strategy can lead to incorrect results in algebra. For instance, all of the distributive errors discussed earlier reflect not only incorrect generalization but also an erroneous linearity assumption. Moreover, Matz regards fraction cancellation errors as also falling under the rubric of errors flowing from false linearity assumptions:

Cancellation errors also fit neatly into this theoretical framework.

Errors of the form

$$\frac{AX + BY}{X + Y} \Rightarrow A + B$$

can be reproduced using the extrapolation-by-iteration strategy. Here the (iterated) base rule is probably

$$\frac{AX}{X} \Rightarrow A.$$

According to this derivation, students notice two instances of the base rule in the new problem. This leads them to linearly decompose the expression, iteratively cancel, and then simply compose the partial results. (p. 118)

Thus, according to Matz, students frequently revise rules by assuming that they can be applied linearly in more sorts of situations than those in which they actually can be applied.

Over and over again, Matz emphasizes that students are drawn toward these two rule-revision strategies – generalization and linear application – because they have prior experience of these techniques yielding correct results. In arguing that these two strategies account for much errant algebraic behavior, she points not only to the fact that generalization and linear application describe the errors themselves but also to the fact that student successes with these strategies make their incorrect extension plausible:

They are descriptively adequate in that we can use them to reproduce common errors. But in addition to their purely descriptive value ... these techniques are the obvious ones *since they are methods that worked well for the student in prior mathematical experience*. Both linearity and generalization have this characteristic: they are useful, often encountered techniques that apply correctly in many situations. (p. 129)

Thus, Matz diagnoses common transformation errors as resulting from strong student tendencies to overgeneralize the context of application of two strategies – generalization and linear application – that worked successfully for them in the past.

Kirshner & Awtry: Overgeneralizing memorable visual sequences

Like Matz, Kirshner & Awtry (2004) explain many diverse common transformation errors with a single theory. Also like Matz, they identify overgeneralization as the heart of the problem. However, whereas Matz bases her account on overgeneralization of rule-rewriting strategies, Kirshner and Awtry base theirs on overgeneralization of particularly memorable visual sequences.

Kirshner and Awtry's study of transformation errors begins with an observation about the appearances of algebraic rules in standard printed notation. The authors notice that some rules possess a certain "visual coherence that makes the left- and right-hand sides of the equations appear naturally related to one another" (p. 229). Kirshner and Awtry borrow the term "visual salience" from the psychology of perception to describe this quality of some algebraic rules. They provide the following examples of rules which they deem visually salient and rules they deem lacking in this quality:

Visually salient rules:

$$x(y + z) = xy + xz$$

$$(xy)^z = x^z y^z$$

$$\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$$

$$(x^y)^z = x^{yz}$$

$$x^{y+z} = x^y x^z$$

$$\frac{w}{x} \frac{y}{z} = \frac{wy}{xz}$$

$$\frac{xy}{xz} = \frac{y}{z}$$

$$\frac{x + y}{z} = \frac{x}{z} + \frac{y}{z}$$

Non-visually salient rules:

$$x^2 - y^2 = (x - y)(x + y)$$

$$\frac{w}{x} \div \frac{y}{z} = \frac{wz}{xy}$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$\frac{w}{x} + \frac{y}{z} = \frac{wz + xy}{xz}$$

$$x^0 = 1$$

Kirshner and Awtry admit that visual salience cannot be defined rigorously: “The quality of visual salience is easy to recognize but difficult to define” (p. 229). They liken a visually salient rule to “an animation sequence in which distinct visual frames are perceived as ongoing instances of a single scene,” allowing us to “see the immediate connection between right- and left-hand sides as stemming from a sense that a single entity is being perceived as transformed over time” (p. 229). In other words, a visually salient rule is one for which the eye easily perceives a temporal narrative relating the left side of the rule to the right side of the rule: “the x was

distributed,” “the fractions were smushed together,” and so on. The visually salient rules have a narrative “sense” to them apart from their truth as generalizations of arithmetic. In contrast, the non-visually salient rules appear to connect two expressions with little obvious visual relationship; their only “sense” comes from the semantics of arithmetic, which is not visually obvious.

After classifying the rules of algebra as either visually salient or non-visually salient, Kirshner and Awtry observe that virtually all of the common transformational mal-rules themselves possess the quality of visual salience. Moreover, they observe that the mal-rules are very similar in appearance to correct visually salient rules. For instance, the mal-rule $\frac{a+x}{b+x} = \frac{a}{b}$ possesses visual salience (‘the x ’s were cancelled’), and it is very similar in appearance to the correct visually salient rule $\frac{ax}{bx} = \frac{a}{b}$. They provide the following table to show the visual similarity between visually salient mal-rules and correct visually salient rules:

Mal-rules	Correct Rules
$(a + b)^c = a^c + b^c$	$(a + b)c = ac + bc$
$\sqrt[c]{a + b} = \sqrt[c]{a} + \sqrt[c]{b}$	$\sqrt[c]{ab} = \sqrt[c]{a} \sqrt[c]{b}$
$a^{mn} = a^m a^n$	$a^{m+n} = a^m a^n$
$a^{m+n} = a^m + a^n$	$a(m + n) = am + an$
$\frac{a}{b + c} = \frac{a}{b} + \frac{a}{c}$	$\frac{b + c}{a} = \frac{b}{a} + \frac{c}{a}$

Thus, unlike Matz, who observes that many transformation errors can be described as overuse of generalization or linearity, Kirshner and Awtry observe that many transformation errors can be described as *visual mimicking* of correct rules.

Kirshner and Awtry conduct an empirical research study to test the hypothesis that students tend to overgeneralize visually salient rules. In the study, students with no previous algebra schooling were taught ten algebra rules, including five visually salient rules and five non-visually salient rules. They were taught the rules purely structurally, without any reference to contextual situations. After instruction, students were given “recognition tasks” that tested their “ability to identify routine applications of the rules” and “rejection tasks” that tested their “ability to constrain overgeneralizing the context of application of the given rules” (p. 242). (All of these tasks involved only simple one-to-one matches and did not require the use of subexpressions.) The results of the study confirm Kirshner and Awtry’s hypothesis that visually salient rules are relatively easy for students to remember but also relatively easy for students to overgeneralize:

Percentage correct scores for recognition tasks were significantly higher for visually salient rules than for non-visually-salient rules.

Such scores for rejection tasks were significantly lower for the visually salient rules. (p. 242)

As a control, other groups of students were taught the same rules using tree diagrams that uniformly lack any of the visual narrative sense that is sometimes present in standard notation. Significantly, the absence of standard algebraic notation affected the results: unlike their peers who learned the rules using standard notation, students

who learned the rules using tree notation did not find the visually salient rules easier to recognize or easier to overgeneralize.

The vicious circle of reification

Competent expression transformers certainly know not only the rules of algebra but also know which of their components are essential and which can be generalized away. What educational implications, then, can be derived from Matz's account and from Kirshner & Awtry's account of transformation errors? Should we regard these accounts of students overgeneralizing merely as explanations of how students produce "filler" in the absence of the necessary structural understanding? If so, then educators could ignore these tendencies to overgeneralizing and assume that they will go away once they have remedied the lack of attention to structure in the traditional algebra curriculum.

I now present two reasons why it would be a mistake to dismiss these overgeneralizing behaviors as lacking in educational significance. The first reason stems from a situation that Sfard calls the "vicious circle of reification." The second reason flows from Kirshner's argument for a connectionist view of mind.

The pragmatic value of reification

Recall, for a moment, Sfard's diagram showing the progress of mathematics toward ever-more abstract objects. Recall, too, that mathematical objects, in Sfard's scheme, serve as pivot points between a more basic process and a more advanced process. We have already discussed Sfard's contention that a student must first

understand the simpler process before reification of the process into an object can occur.

One might naturally assume, given the layout of Sfard's diagram, that learning can and should proceed in a neat, stepwise fashion: process, reified object, new process, new reified object, and so on. If that neat alternation prevailed, then the educational implications for algebra would be that students should first learn how to parse, next attain an object-perception of structural templates, and only then begin to learn to transform expressions by comparing and matching structural templates with rule expressions.

However, Sfard posits the existence of something she calls the "vicious circle of reification," a situation that makes reification inherently difficult and renders neat sequential learning nearly impossible. According to Sfard, there is an inherent difficulty in advancing up the hierarchy of mathematical understanding. She describes this difficulty as stemming from the circularity that obtains between understanding a mathematical object and understanding the higher processes performed upon that object. On the one hand, reification and its object-perception is a prerequisite for fully understanding the higher process: one cannot truly understand a process if one does not first understand the objects upon which one is performing that process. On the other hand, engagement with a higher process is precisely what motivates reification and its attendant object-perception: the higher process provides the pragmatic value for the object-perception. In other words, Sfard regards understanding a mathematical object and understanding the processes performed

upon that object as prerequisites of one another, hence the “vicious” circularity.

Crucially, then, the moment of reification is typically difficult for students to attain:

On the one hand, a person must be quite skillful at performing algorithms in order to attain a good idea of the ‘objects’ involved in these algorithms; on the other hand, to gain full technical mastery, one must already have these objects, since without them the processes would seem meaningless and thus difficult to perform and remember.

(p. 32)

One implication for student *learning* is that understanding of mathematical objects must be encouraged simultaneously from two directions: the object-perception can only develop from sufficient experience performing both the more basic process (of which the object is the result) and the more advanced process (which is performed upon the object). For instance, understanding of rational numbers as objects must be encouraged by simultaneously engaging students in the more basic process of dividing two natural numbers (from which rational numbers result) and the more advanced process of comparing ratios (which takes rational numbers as its objects). For convenience, I will – based on Sfard’s diagram – speak of the need to induce reification *vertically* and *horizontally*.

The vicious circle and algebra: A window for overgeneralization

How does the vicious circle of reification play out in the learning of algebras?

Recall that we have been regarding structural templates as mathematical objects.

Recall further that we have been regarding these objects as the results of the simpler

process of parsing and the objects of the more advanced process of comparing and matching structural templates for expression transformation.

The vicious circle of reification implies that conceptual understanding of structural template, on the one hand, and procedural ability to find matches in structure and transform expressions, on the other hand, are prerequisites for another. It is impossible, according to Sfard's theory, to proceed sequentially from the process of parsing to the concept of structural template to the process of comparing structure for transforming expressions. As a result, students must necessarily engage not only in the more basic process of parsing but also in the more advanced process of comparing structures *before fully attaining an object-perception of structural template*.

This situation creates an opening – a window – for student tendencies to overgeneralize to interfere with student learning of expression transformation. During the messy time before reification of structural concepts has occurred, students will be engaged in processes like expression transformation requiring a full understanding of those very structural concepts! Lacking this full understanding but still of necessity engaged in these more advanced processes, students will tend to overgeneralize, making common errors. The vicious circle of reification is one factor that limits the possible improvement in student algebra performance: Understanding the objects of algebra is inherently difficult, and strong tendencies to overgeneralize – whether in response to past successes, or in response to memorable visual sequences, or both – are likely to interfere even given a curriculum that attends adequately to structure.

Competing impulses: The connectionist view of mind

There is a second reason that instruction cannot ignore student tendency to overgeneralize. In a series of papers, Kirshner (1989, 1993, 2001, 2004, 2006) builds a case for looking at the mind of the algebra learner from the perspective of a school of thought in cognitive psychology called *connectionism*. Connectionism regards the mind as inherently ill-suited to formal reasoning tasks, like those involved in expression transformation, and stubbornly inclined to incorporate formally irrelevant information, such as visual patterns, into its decision-making process. In another series of papers, cognitive scientists Landy & Goldstone (2007a, 2007b, 2007c) support Kirshner's connectionist perspective on algebra learning. Their research indicates that as novices work toward an understanding of algebraic structure, that understanding will necessarily be in competition with non-rational impulses, such as overgeneralizing tendencies.

Kirshner on the role of spacing in parsing decisions

When Kirshner makes his fullest case for connectionism in a 2006 paper, he cites both his and Awtry's 2004 study discussed above and a 1989 study about parsing.

Kirshner's 1989 study is motivated by an observation about algebraic notation: in standard printed algebra, there is a correlation between the degree of precedence of an operation and the type of spacing used to indicate that operation. While students are encouraged to memorize PEMDAS as a rule, they likely get a

silent “assist” from the spacing around operations in standard algebra notation. Specifically, Kirshner observes that the least precedent operations (addition and subtraction) are indicated by “wide spacing”; the next least precedent operations (multiplication and division) are indicated by a closer “horizontal or vertical juxtaposition”; and the operations with the highest precedence (exponentiation and root-taking) are indicated by “diagonal juxtaposition” (p. 276). He provides the following table to illustrate the distinctive spacing conventions of each operation level:

Alternative Characterization of Operation Level

Level	Visual characteristics	Examples
1	Wide spacing	$a + b, a - b$
2	Horizontal or vertical juxtaposition	$ab, \frac{a}{b}$
3	Diagonal juxtaposition	$a^b, \sqrt[n]{b}$

Thus, as mentioned parenthetically earlier in this paper, Kirshner challenges the notion that the “exponents before multiplication before addition” convention is completely non-notational; rather, spacing provides clues about an operation’s level in the precedence hierarchy.

Kirshner’s observation that “operation levels correspond with distinctive visual characteristics” (p. 276) causes him to question commonsense assumptions about student parsing abilities. As we have seen, parsing is a prerequisite skill for evaluating algebraic expressions. Common sense would seem to indicate that a student who repeatedly demonstrates success at evaluating algebraic expressions must therefore know the rules of operation precedence. Kirshner, however, hypothesizes that while some students who can evaluate algebraic expressions correctly may

actually know the rules of precedence, others may depend upon the visual spacing cues of standard notation to make correct parsing decisions. This latter group may use spacing cues in the same manner that parentheses are meant to be used: as visually present indicators of how to parse.

To test this hypothesis, Kirshner conducts an experiment involving a nonstandard “nonce” notation. In this nonce notation, capital letters indicate operations in place of the usual symbols $+$, $-$, and so on. For example, 3A5 means $3 + 5$. The experiment involves both a “spaced nonce” and an “unspaced nonce” notation. The spaced nonce notation is designed to mimic standard notation by correlating proximity of symbols with precedence of operation. Kirshner’s table illustrates these alternative notations:

Standard	Unspaced nonce	Spaced nonce
$a + b$	aAb	$a \ A \ b$
$a - b$	aSb	$a \ S \ b$
ab	aMb	$a \ M \ b$
$\frac{a}{b}$	aDb	$a \ D \ b$
a^b	aEb	aEb
\sqrt{b}	aRb	aRb

Kirshner hypothesized that if students spontaneously use spacing cues to make parsing decisions, then many students would correctly parse expressions presented in the spaced nonce notation, which mimics the visual assist of standard algebra notation, yet be unable to parse expressions presented in the unspaced nonce notation, even though both nonce notations technically contain all the information algebraically needed to parse:

It was reasoned that the ability to correctly parse algebraic expressions presented in the unspaced nonce notation would indicate the presence

of propositionally based syntactic knowledge. Conversely, inability to transfer competent behaviors from standard notation to the nonce setting would indicate a dependence on the surface cues of ordinary notation. (p. 277)

Indeed, Kirshner's results did show that a significant number of students had more difficulty with the unspaced nonce notation than with the spaced nonce notation:

Almost all the subjects participating in the study were able to evaluate expressions such as $1 + 3x^2$, for $x = 2$, when presented in standard notation. It proved, however, to be significantly more difficult to transfer this ability to the unspaced nonce notation than to the spaced nonce notation. These two notations differ only in the spacing of the symbols, the latter notation having been devised, specifically, to mimic spacing features of ordinary notation. Thus it seems necessary to conclude that for some students the surface features of ordinary notation provide a necessary cue to successful syntactic division. (p. 282)

Kirshner therefore infers that the way operations are spaced on the printed page in standard algebraic notation functions as a notational parsing cue for some students. Moreover, he infers that for some students, spacing is a *necessary* cue: their ability to order operations according to convention depends not upon declarative knowledge of the conventional rules but rather upon having this visually present spacing cue, and they are unable to parse correctly without it. Because the visual crutch is embedded in the way students usually encounter algebra problems, there is no way to

discriminate between the student who really understands the structure and the student who is using this crutch.

Kirshner on connectionism

Although Kirshner's two research experiments (1989, 2004) pertain to different skills, he draws similar conclusions from the two studies. In both studies, Kirshner concludes that successful performance of a skill (evaluating expressions in the one, transforming expressions in the other) does not necessarily indicate mastery of the formal rules that constitute true competence. In both cases, novices rely upon visual features of standard written algebra (spacing in the one, memorable animation-like visual sequence in the other) to make successful decisions. In both cases, introducing new notations that lack these helpful visual features (the unspaced nonce in the one, the tree notation in the other) is shown to reduce student ability to perform the skill, even though all of the technical information needed to perform the skill is still present in the alternate notation.

In reflecting upon these findings, Kirshner adopts a *connectionist* view of cognition that rejects the analogy of the mind to computer. Connectionism does not view the mind as a neat and orderly machine with a centralized rule-processing apparatus: "Connectionist psychology posits dramatic redundancy and a superabundance of active elements, in contrast to the neat, linear processes of rule-based systems" (2001, p. 90). Connectionism, Kirshner explains, considers cognition to be spread out rather than centralized: "In analogy to the neurology of the brain, connectionism asserts that cognition is parallel and distributed, rather than serial and

digital” (p. 90). Kirshner uses an analogy to inputs and outputs to explain the connectionist view of how the mind does work: “Typically connectionist systems include input nodes corresponding to features of the domain to be mastered and output nodes related to actions that can be taken or decisions that can be reached, as well as hidden units that intermediate between input and output nodes” (2006, p. 7). According to the connectionist view, these different inputs are “competing” at all times, and the relative weight of their input – not a formal rule process – determines which input or inputs win out and result in an output:

When a certain threshold of activation is reached, the node sends signals to those other nodes to which it is connected. ... Connectionism models cognitive skills as weighted correlations among a large number of input, output, and intermediate nodes. No centralized rule based program runs the show. (p. 7)

Connectionism therefore sees the mind as ill-suited for sequential rule-processing tasks and well-suited for tasks involving making judgments based on many related and competing sets of input criteria. Kirshner (2001) explains the connectionist view of what the human mind does best:

The primary cognitive functions are pattern matching and associative memory, not logic or rule following. Connectionism notices that the long chains of extended reasoning that serial digital computers do best, are hardest for humans. Things that humans do best, like recognizing faces in different situations and from different angles, are the most

difficult feats to simulate on serial computers, but the easiest to implement in connectionist architectures. (p. 90).

A connectionist view does not regard a person's understanding of formal rules as non-factors in the person's cognition, but merely as one of many parallel factors competing to produce action: "It is too extreme to argue that rules play no role in competent performance, but it is an ancillary role informing cognition rather than constituting it" (p. 95).

In particular, Kirshner sees connectionism as dovetailing nicely with his empirical observations about how students learn algebra. Kirshner's two research studies both suggest that some student algebra behavior can be explained as responses to visual features of standard printed algebra, despite the fact that the visual appearance of algebraic expressions is a mere accident of our notation and not inherent to the structural content of algebra. For Kirshner (2001), connectionist theories incorporate such observations naturally: "The connectionist framework seems, in general terms, to afford the possibility of an alternative account of algebraic symbol skills that is more faithful to our observation as educators that students' work in algebra is non-reflective and pattern-based" (p. 90). Moreover, connectionism explains the stubbornness with which students cling to visual approaches to algebra skill acquisition, for it asserts that "learning always is grounded in perception and pattern matching as embedded in practices, not in abstraction and rule following" (p. 95).

Landy & Goldstone's research on formally irrelevant distractions

Kirshner, as we have seen, posits a connectionist understanding of cognition. Connectionism helps to explain Kirshner's findings that students seem to spontaneously utilize visual regularities and memorable visual features of algebraic notation when learning algebra, despite the fact that such visual cues are not part of the formal, rule-based apparatus for making decisions in algebra.

In a series of recent papers, Landy & Goldstone (2007a, 2007b, 2007c) describe a set of experiments designed explicitly to test the role of formally irrelevant visual cues in the performance of algebraic tasks. In particular, their paper "How Abstract Is Symbolic Thought?" (2007a) has substantial implications for our consideration of how students learn to transform expressions. In this paper, Landy and Goldstone describe four research experiments, each designed to measure the effect of a formally irrelevant visual "distracter" on a person's ability to determine whether an expression has been transformed correctly or incorrectly.

The first of these four experiments gives a sense of the gist of their work. In this experiment, spacing was the manipulated visual feature. Subjects were asked to judge the correctness of equivalences like the following:

$$\begin{aligned}
 h + q^*t + n &= h + t^*q + n \\
 u + p^*k + x &= k + u^*x + p \\
 g^*m + r^*w &= m^*g + w^*r \\
 y^*s + f^*z &= s^*f + y^*z \\
 b + h^*v + k &= k + h^*v + b \\
 t + j^*n + e &= n + e^*t + j \\
 q^*r + h^*c &= c^*h + r^*q \\
 w^*g + x^*j &= j^*g + x^*w \\
 t+p^*m+f &= t+p^*m+f \\
 w+n^*r+k &= r+k^*w+n \\
 s^*n+e^*c &= n^*s+c^*e \\
 g^*a+w^*j &= w^*g+j^*a
 \end{aligned}$$

Note that in some of these equations, such as the very first one, spacing has been manipulated so that the very wide spacing is correlated with less precedence (as in standard algebra notation), while in other equations, such as the very last one, spacing has been manipulated so that the very wide spacing is correlated with more precedence. Subjects were timed on their responses, and subjects were informed immediately of any incorrect responses. Landy and Goldstone found, like Kirshner, that subjects tended to use wide spacing as an indication of lower operation precedence, even when the wide spacing was around multiplication. In other words, they found that spacing, while irrelevant from a formal perspective, nonetheless influenced subjects' syntactic judgments: "The physical spacing of formal equations has a large impact on successful evaluations of validity" (p. 724).

While the first study involved manipulating spacing – a formally irrelevant factor that authentically plays a role in standard algebra notation – the remaining three studies involved manipulating more contrived visual factors. While also formally irrelevant, these other visual factors do not typically arise as distracters in actual algebra usage. The manipulated visual feature in the second experiment was an oval-shaped region in the background of the equations:

$$\begin{array}{l}
 h + \boxed{q * t} + n = h + \boxed{t * q} + n \\
 u + \boxed{p * k} + x = k + \boxed{u * x} + p \\
 \boxed{g * m} + \boxed{r * w} = \boxed{m * g} + \boxed{w * r} \\
 \boxed{y * s} + \boxed{f * z} = \boxed{s * f} + \boxed{y * z} \\
 b + h * v + k = k + h * v + b \\
 t + j * n + e = n + e * t + j \\
 q * r + h * c = c * h + r * q \\
 w * g + x * j = j * g + x * w \\
 \boxed{t + p} * \boxed{u + f} = \boxed{t + p} * \boxed{u + f} \\
 \boxed{w + n} * \boxed{r + k} = \boxed{r + k} * \boxed{w + n} \\
 s * \boxed{n + e} * c = n * \boxed{s + c} * e \\
 g * \boxed{a + y} * j = w * \boxed{g + j} * a
 \end{array}$$

The manipulated visual feature in the third experiment was the internal structure of the rearranged terms:

$$\begin{array}{l}
 (d*d*d) + (4/q) * (9/m) + (p+2*x) = (d*d*d) + (9/m) * (4/q) + (p+2*x) \\
 (t*t*t) * (m*m*m) + (5/h) * (2/v) = (m*m*m) * (t*t*t) + (2/v) * (5/h) \\
 (8/g) + (6a-n) * (3m-p) + (a*a*a) = (3m-p) + (8/g) * (a*a*a) + (6a-n) \\
 (f*f*f) * (u*u*u) + (y+3*k) * (a+2*q) = (u*u*u) * (y+3*k) + (f*f*f) * (a+2*q) \\
 (g+4*v) + (u+1*b) * (c+9*f) + (r+8*v) = (r+8*v) + (u+1*b) * (c+9*f) + (g+4*v) \\
 (t+6*h) * (y+2*q) + (a+5*x) * (k+9*r) = (k+9*r) * (a+5*x) + (y+2*q) * (t+6*h) \\
 (j+7*t) + (u+1*b) * (w+9*y) + (r+8*v) = (w+9*y) + (r+8*v) * (j+7*t) + (u+1*b) \\
 (t+6*h) * (y+2*q) + (a+5*x) * (k+9*r) = (t+6*h) * (a+5*x) + (y+2*q) * (k+9*r) \\
 (8/h) + (3/n) * (w+4*c) + (h+1*b) = (8/h) + (3/n) * (w+4*c) + (h+1*b) \\
 (3/b) * (j*j*j) + (z*z*z) * (7u-y) = (j*j*j) * (3/b) + (7u-y) * (z*z*z) \\
 (d*d*d) + (q*q*q) * (4/q) + (9/m) = (4/q) + (9/m) * (d*d*d) + (q*q*q) \\
 (v*v*v) * (3r-h) + (8e-u) * (2/s) = (8e-u) * (v*v*v) + (2/s) * (3r-h)
 \end{array}$$

The manipulated visual feature in the fourth experiment was alphabetical proximity of the variables:

$$\begin{array}{l}
 a + p * q + z = a + q * p + z \\
 u + 2 * 3 + e = 3 + e * u + 2 \\
 k + m * n + t = m + k * t + n \\
 w * x + q * r = x * q + w * r \\
 a + u * j + m = m + u * j + a \\
 r + w * q + b = q + b * r + w \\
 r * f + w * n = n * w + f * r \\
 4 * u + s * e = 4 * s + u * e \\
 r + s * b + c = r + s * b + c \\
 a + b * 3 + 4 = 3 + 4 * a + b \\
 4 * b + p * r = b * 4 + r * p \\
 c * 4 + 5 * r = 5 * c + r * 4
 \end{array}$$

Thus, Landy and Goldstone go quite a bit further than Kirshner. They consider the effects of a variety of formally irrelevant factors on people's parsing decisions.

Their overall findings support Kirshner's connectionist perspective.

Repeatedly, they conclude that formally irrelevant features can distract people who otherwise make correct parsing decisions into making incorrect ones. They conclude that "a reasoner's syntactic interpretation may be influenced by notational factors that do not appear in formal mathematical treatments" (p. 721). Like Kirshner, they deem these findings significant because of how they challenge standard assumptions about how people make decisions in rule-based mathematical environments. It is standard, they explain, to assume that when students operate with good faith in a rule-based environment like structural algebra, they make all decisions based only on their understanding of the rules of the domain: "Cognitive conceptions of abstract formal interpretation generally follow formal logics by assuming that reasoners explicitly represent rules of combination, and apply those rules to symbolic expressions" (2007c). Those who assume students learn algebra solely by learning and applying rules will also, by implication, regard mistakes as evidence of misunderstandings of the rules. Landy and Goldstone see their results as disproving this assumption that people who perform algebra tasks in good faith make their decisions based only on their understanding of the rules:

Fundamentally these results challenge the conception that human reasoning with formal systems uses only the formal properties of symbolic notations, and that errors are driven by misunderstandings of those properties. Instead, people seem to use whatever regularities—formal or visual, rule-based or statistical—are available to them, even on an entirely formal task such as arithmetic. The engagement of visual features and processes indicates that formal reasoning shares mechanisms with the diagrammatic and pictorial reasoning processes with which it is normally contrasted. (2007c)

Put another way, Landy and Goldstone join Kirshner in concluding that student performance on algebra tasks is best modeled not by computer-like rule-following machines but rather by the sort of associative reasoning captured by connectionist frameworks of mind.

More impressively, Landy and Goldstone demonstrate that formally irrelevant features can persist in influencing algebraic decision-making even when the person making the decisions actually knows the correct rules. Their interviews with study participants reveal “that some participants realized that they were affected” by formally irrelevant features and that “participants knew that responding on the basis of space, alphabetic formality, and similarity of notation were incorrect, but they continued to be influenced by these factors” (2007a, p. 730). Furthermore, participants persisted in using formally irrelevant features in their decision-making even while receiving feedback during the experiment itself:

One might have argued that participants were influenced by grouping only because they believed that they could strategically use superficial grouping features as cues to mathematical parsing. However, constant feedback did not eliminate the influence of these superficial cues. This suggests that sensitivity to grouping is automatic or at least resistant to strategic, feedback-dependent control processes. Grouping continued to exert and influence even when participants realized, after considerable feedback, that it was likely to provide misleading cues to parsing. (p. 730)

Landy and Goldstone cite other psychological research on non-mathematical rule-based domains that also shows “that people may use perceptual cues instead of rules even when they know that the rules should be applied” (p. 731). Ultimately, their findings indicate that a person’s knowledge of algebraic structure competes with other inputs during algebraic decision-making, even when those other inputs are irrelevant from a formal perspective. Transformation errors are not necessarily symptoms of lack of structural understanding but rather of the fact that structural understanding competes for attention alongside formally irrelevant visual features. Their research therefore implies that students’ strong tendency to overgeneralize is not just in play when students do not understand necessary structural concepts.

Other examples of competition in algebra performance

Other instances of “competition” between formally relevant and formally irrelevant features support these conclusions. We will examine two such instances.

Wong (1997) provides one example of structure in competition with other factors. She observes that students who can perform a transformation task involving only variables sometimes have difficulty performing a structurally identical task involving both numbers and variables. For instance, Wong observes that some students who successfully transform $(hk)^n$ into $h^n \cdot k^n$ will consistently err when h is replaced with a number, transforming $(2a^m)^n$ into $2a^{mn}$, $(2x^3)^4$ into $2x^{12}$, and so on. Her general conclusion is that students who have learned “to transform algebraic expressions according to some standard procedures” will sometimes “fail to do the transformation correctly when the familiar letters are replaced by numbers,” despite the fact that the replacement leaves the structure of the expression unchanged (p. 286). Wong explicitly links her findings to a connectionist framework, noting “the importance of the degree of strength between the connections of information items in learning situations,” and concludes that students sometimes “fail to activate the appropriate information items in their mind” (p. 289).

Linchevski & Livneh (2002) describe situations in which structure competes with specific number combinations for student attention. In a study, they found that certain biasing number combinations can override student structural knowledge and lure students into parsing errors. For instance, they find that students who repeatedly parse expressions of the form $m - n + p$ correctly are somewhat more prone to parse this expression incorrectly when presented with $267 - 30 + 30$. In this example, the repetition of 30 draws student attention to the addition first, despite what students “know” about how to parse expressions with this structure generally. The authors conclude that “the particular number combination in the expression competes with the

algebraic structure” for the student’s attention. While from a structural point of view, the particular numbers in an addition expression are irrelevant, in practice the particular numbers involved can lead to a greater or lesser frequency of particular parsing errors. Student knowledge and understanding of structure competes with other stimuli for student attention.

Instructional strategies

Earlier, we attributed student difficulty transforming expressions to their insufficient experience parsing and to insufficient exposure to the structural notions that underlie the expression transformation process. However, we then saw that several teachers and researchers attribute the uniformity and persistence of common transformation errors to student tendencies to overgeneralize. We asked whether overgeneralization is merely a strategy that students adopt when they lack structural understanding or whether it has deeper educational implications. We have now seen two reasons why overgeneralization merits educational consideration: (1) Because of the vicious circle of reification, students should start operating on subexpressions and structural templates before they attain a reified object-perception of those objects, opening a window for overgeneralizing tendencies to influence student decision-making; (2) Connectionism suggests that student understanding of structure is not all-or-nothing but rather in competition with other impulses, especially with the impulse to incorporate formally irrelevant visual cues into algebraic decision-making.

What would instructional strategies for improving student ability to learn expression transformation look like? From the conclusions about the centrality of

parsing reached earlier in this paper, we can derive the following instructional principle: Algebra curricula need to give explicit attention to parsing and to structure. However, from the conclusions about overgeneralizing and associative reasoning reached more recently in this paper, we can modify the instructional principle so as to incorporate our findings about student receptivity to the visual: *Algebra curricula need to give explicit attention to parsing and to structural notions in ways that will make structure a strong competitor for perceptual salience among the many impulses competing for student attention.*

Instructional strategies for helping students achieve the process-perception of parsing

As we have discussed repeatedly, the process-perception of parsing necessarily precedes the object-perception of parsing: students need to parse before they can attain an object-perception of parsing as creating structural template. Since students ultimately need to attain the object-perception of parsing in order to transform expressions, curricula ought to make certain that students learn how to parse. However, as discussed earlier, traditional algebra texts treat parsing superficially, presenting the order of operations and then testing student ability to parse only *implicitly* through activities like numeric expression simplification.

My first instructional proposals, therefore, are for activities and exercises that ask students to parse expressions explicitly. For instance, students could be required to draw expression trees:

Exercise Set A

Draw an expression tree for each expression.

1. $2x + 5$

2. $2(x + 5)$

3. $3x^2 + 5x + 2$

4. $4(x + 2)(2x + 3) + 3(x + 1)^2$

Such exercises reveal explicitly for the teacher the extent to which students understand the parsing conventions.

While Exercise Set A requires students to resort to an alternative tree notation to parse an expression, an instructional strategy that I call “surgery” provides students with the opportunity to parse expressions physically and visually right on their standard worksheets. Here is how the surgery approach to parsing involves students in actively breaking expressions into their component parts. First, I tell students that they have two “knives”: the Addition Knife and the Multiplication Knife. The Addition Knife is the primary one and is used for underlining, and the Multiplication Knife is secondary and is used for “slashing.” The rule for using the Addition Knife is as follows: start underlining from the beginning of the expression, and start a new underline for each plus or minus sign that is outside of grouping symbols. For instance, if I told students to underline the expression $3x^2 + 5x + 2$ using their Addition Knife, I would expect the following result:

$$\underline{3x^2} + \underline{5x} + \underline{2}$$

Similarly, if I told students to slash apart the first “underline” using their Multiplication Knife, I would expect the following result:

$$\underline{3/x^2}$$

In essence, the underlining provides visual support to the role of addition and subtraction as the least precedent of all operations, and the slashing provides visual support to the role of multiplication as more precedent than exponentiation.

Performing “surgery” on an expression reinforces correct parsing decisions and helps students avoid defaulting to a left-to-right order of operations. For instance, recall Example A, which asked students to simplify $-2 + 5(4 - 6)^2$. Here is how a student might proceed to write the work using underlining and slashing to parse the expression visually:

Example A

Simplify $-2 + 5(4 - 6)^2$.

$$\begin{array}{r} \underline{-2} + \underline{5/(4-6)^2} \\ \underline{-2} + \underline{5/(-2)^2} \\ \underline{-2} + \underline{5/(4)} \\ \underline{-2} + \underline{20} \\ 18 \end{array}$$

The underlines effectively serve as grouping symbols, making it very unlikely for students accidentally to proceed left to right and obtain the incorrect $3(4 - 6)^2$.

Instructional strategies for inducing reification of structure vertically

Thus far, I have proposed activities for helping students achieve procedural mastery of parsing. Procedural mastery of parsing necessarily precedes an object-perception of structural notions. In this subsection and in the next one, I propose instructional strategies for helping students achieve this object-perception. Here I propose activities that lead toward object-perception of structural template and subexpression by gradually helping students to “compress” the results of the parsing processes into objects. Earlier, we have called this *inducing reification vertically*. In the next subsection I will propose activities that urge students toward reification by illustrating the pragmatic value of an object-perception of structure. We have called this *inducing reification horizontally*.

First, it is worth noting that the “surgery” approach to parsing discussed above already goes a long way toward helping students transition from parsing as process to parse as object. When students simplify numeric expressions, as in Example A, without underlining and slashing, parsing for them is just a decision-making process used to determine what to do first, second, and so on. When students underline and slash in the process of the same simplification task, they transform parsing into an activity that makes subexpressions visible. Students gradually are able to transition from thinking of underlining as something they do to thinking of the “underlines” or terms as things they can see. Underlining and slashing, therefore, not only help

students make correct parsing decisions; they also move students toward seeing subexpressions as objects.

Exercises that explicitly use names of various types of subexpression can further this process-to-object transition even further. Consider, for instance, Exercise Set B:

Exercise Set B

1. What is the second term of $3x(2x + 5n) + 3n + 4xy(n + 2)$?
2. What is the second term of the third factor of $4n(m + 2n)(x + 3)$?
3. What is the first factor of the second term of the second factor of the first term of $3(4x + (m + n)(2m + n) + 5)(m + n) + 4mn$?

These questions explicitly engage students with structural notions and structural vocabulary. However, if students do not yet have a solid understanding of factors and terms as objects, they can continue to use the “surgery” method to parse the expressions and locate the desired subexpression. For instance, for the third question in Exercise Set B, I would instruct students first to underline to identify the first term of the expression, after which they can ignore the part of the expression that is not the first term. Then I would instruct students to slash to find the second factor, and so on until they finally locate the requested structural entity. Here is how I would model this process:

$$\begin{array}{c}
 \frac{3(4x + (m+n)(2m+n) + 5)(m+n) + 4mn}{3(4x + (m+n)(2m+n) + 5)(m+n)} \\
 \frac{4x + (m+n)(2m+n) + 5}{(m+n)(2m+n)} \\
 m+n
 \end{array}$$

Such exercises, which ask the students to hunt for various subexpressions by name, use the parsing process to guide students toward an object-perception of the various sorts of subexpressions as static “things.”

While the surgery strategy, combined with exercises like the previous ones, push students toward an object-perception of particular subexpressions, the questions in Exercise Set C push students toward an object-perception of structural template. This exercise set also utilizes Kirshner’s alternative notation for operations, for such occasional forays into alternative notations can combat students’ tendency to lean on visual features of standard notation for behavioral cues:

Exercise Set C

Lefty is confused about the order of operations. He believes that all operations should be performed from left to right unless parentheses indicate otherwise. Rewrite each expression with just enough additional sets of parentheses so that Lefty will perform the operations in the same order as someone following the order of operations. If the expression does not need to be rewritten, write *fine*. (Note: E indicates exponentiation, M indicates multiplication, and A indicates addition.)

1. $3MxA2$

2. $5A3Mx$

3. $(4M3Ex)Mx$

4. $(2AxA5)E2$

This exercise set gently encourages students to see the result of parsing an expression as an object in its own right. By thinking about how to preserve the answer resulting from the process of evaluating the expressions, students are forced to think about parsing in a condensed way – to think about how to preserve *the structure* of the expression.

Instructional strategies for inducing reification of structure horizontally

Now I will proceed to outline some instructional strategies that seek to induce reification of the object-perception of structural notions by making students aware of the pragmatic value of this perspective. Ultimately, the object-perception is needed for expression transformation. However, the traditional curriculum leaps quickly into expression transformation, and students are strongly prone to overgeneralize if they transform expressions without understanding structure. I therefore will propose instructional strategies to induce reification horizontally that are more resistant to overgeneralizing tendencies and that make structure a visually salient competitor for student attention.

As we have seen, comparing structural templates and determining matches is a prerequisite skill for expression transformation that the traditional curriculum does not isolate as a skill in its own right. Typically, this skill is taught only implicitly by requiring students to transform expressions. Exercise Set D isolates this activity as a skill in its own right:

Exercise Set D

A list of structural templates is provided below. For each expression, select all of the structural templates from the list that describe the expression.

List: $(M + N)P$, $MN + P$, $M^N P$, $M(N + P)$,
 $M + N$, MN , $M + NP$, $M + N + P$

1. $2(x + 3)^2(x + 3 + w)$

2. $(3 + x)5 + w$

3. $x + m^2w + 2$

4. $(4a)^2(a + m + f)$

The capital letter variables are meant to be suggestive of the fact that these variables might stand for subexpressions rather than just numbers. This exercise set demands an object-perception of structure. For the student who has not yet achieved this object-perception and is somewhere in the vicious circle of reification, this exercise set can motivate reification, yet it does not tempt the student to fall victim to competing impulses to overgeneralize like traditional expression transformation exercises do.

Expression transformation exercises also have the potential to induce reification horizontally. However, if we instruct students in expression transformation tasks prior to their achieving a fully-reified object-perception of structure, then we need to take care to make structure visually and conceptually

salient so as to win the competition for student attention against their strong overgeneralizing tendencies.

One instructional strategy is to explicitly study the rules of algebra from a structural perspective. For instance, let us consider the Cancellation Rule for

Fractions, $\frac{mp}{np} = \frac{m}{n}$, whose linear misapplication accounts for notorious fraction

cancellation errors. We can make the structure of this rule salient by inviting students to perform “surgery” on the rule itself:

$$\frac{\cancel{m}p}{\cancel{n}p} = \frac{m}{n}$$

This active, visual parsing of the rule can lead students to formulate a verbal description of the requirements for cancellation: one of the factors of the numerator needs to be the same as one of the factors of the denominator.

Then, having invited students to visually parse this rule, I would next present students with a variety of expressions and invite them to determine the applicability of this rule by visually parsing the given expressions. Exercise Set E, for instance, invites students to determine whether or not cancellation is possible:

Exercise Set E

Simplify each fraction by dividing the numerator and denominator by a common factor. If the numerator and denominator do not have a common factor, write *can't*.

1. $\frac{3a}{8a}$

2. $\frac{x+3}{7x}$

3. $\frac{m+4}{5+m}$

4. $\frac{7w+2}{8wx}$

Students could “slash” numerators and denominators to determine whether or not

they possess the correct sort of identity in structure to $\frac{mp}{np}$. Here is what I would

expect students to write:

1. $\frac{\cancel{3}a}{\cancel{8}a}$

2. $\frac{\cancel{x}+3}{\cancel{7}x}$

3. $\frac{\cancel{m}+4}{\cancel{5}+m}$

4. $\frac{\cancel{7}w+2}{\cancel{8}wx}$

The visual parsing supports the correct structural interpretation and helps students avoid erroneous cancellation. Only in the first question is there an identical expression in numerator and denominator between two factor slashes. In the fourth question, for example, student recognition that w is a factor of the term $7w$ but not of the entire numerator is supported by the visual appearance of the slashes. Exercises such as these can help students come to understand the importance of engaging with transformation tasks as structural tasks rather than as visual tasks. They can help students resist the urge to overgeneralize spontaneously. Perhaps most importantly,

by revealing to students the pragmatic value of the ability to determine identity in structure, such exercises can help to induce reification of structural notions, lift the student out of the vicious circle of reification, and install structural understanding in the student's mind so that it will permanently be a viable and salient competitor among the many behavioral impulses influencing algebraic behavior.

Some supporting and overlapping curricular recommendations

Finally, I will examine some other teachers' proposed instructional strategies that overlap to a certain extent with those I have presented above.

I am not the first to advocate making structure visibly present for students. Teachers offer a variety of strategies for helping students "see" the structure of an expression, much like my "surgery" approach. Rambhia (2002), for instance, teaches students that when confronted with a numeric expression to simplify, they "can separate the problem into parts by drawing lines" (p. 194). Here he describes this strategy in detail:

A slightly more difficult problem would be the equation:

$$6(3 + 5) - 4(8) + 5(6 + 1) = \underline{\quad}$$

Some students feel overwhelmed by such problems until I remind them to separate the problem into parts. They start to understand that because addition and subtraction are done last, those operations are the keys to breaking down the problem. Specifically, the addition or subtraction signs that are not enclosed in grouping symbols partition

the problem. The problem above, for example, has three parts, as shown below:

$$\begin{array}{r|c|c|c|c}
 6(3+5) & - & 4(8) & + & 5(6+1) = \\
 6(8) & - & 4(8) & + & 5(7) = \\
 48 & - & 32 & + & 35 = \\
 48 & - & 32 & + & 35 = 51
 \end{array} \quad (\text{p. 194})$$

This advice is clearly along the lines of my proposal to underline terms. Barnard (2002b) offers the following suggestions for helping students to perceive structure visually:

- Write different parts of expressions in different colors.
- Use highlighters to shade different parts of expressions.
- Close brackets [parentheses] into bubbles: $(2x + 5) \rightarrow \textcircled{2x + 5} \dots$
- Always draw boxes around terms in an equation. This stresses that the important sign is the one in front of the term. Students are very happy that missing signs are positive.

$$\boxed{+3x} \quad \boxed{-5} \quad \boxed{=} \quad \boxed{+7} \quad \boxed{+9x} \quad (\text{p. 41})$$

Pierce & Stacey (2007) also recommend helping students focus on visual structure:

Students must learn to read the clues to the structure of symbolic expressions and equations. Putting the spotlight on the *structure* of the

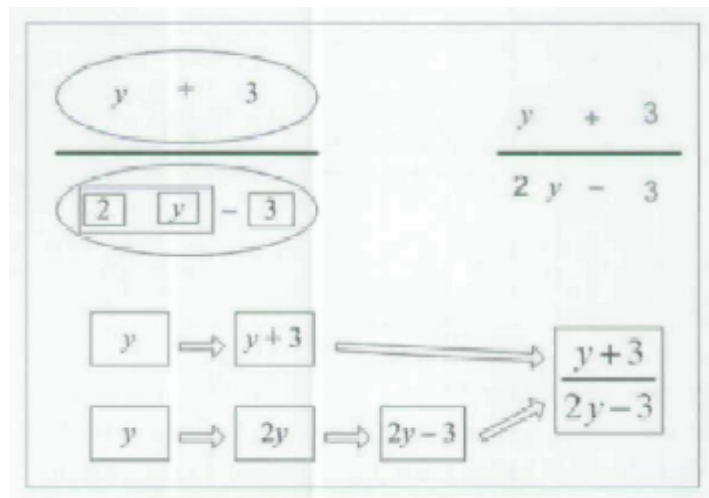
expression $\frac{y+3}{2y-3}$ highlights the division of two linear expressions,

which hence have to be treated as units and cannot be broken up and

$$\text{cancelled out like this: } \frac{y+3}{2y-3} = \frac{y}{2y} + \frac{3}{-3} = \frac{1}{2} - 1.$$

Simple techniques, such as writing the basic substructures (in this case, numerator and denominator) in different colors, can draw attention to the structure and in time minimize errors of this nature. (p. 14)

Pierce & Stacey then go on to show “three simple graphical devices for highlighting the structure of an expression” (p. 14):



In different ways, all of these teachers are recommending student-driven annotations to standard algebraic notation that can help make structure visually salient for students.

Kirshner (2006) proposes a curricular approach to structural algebra called the Lexical Support System (LSS), which shares with my curricular proposals a focus on structural awareness. The LSS is a program for involving students in structural discourse – that is, in a setting in which precise structural language is used by teacher and student to describe the pieces of algebraic expressions. According to Kirshner, the LSS’s goal is “providing a structural vocabulary that enables more rigorous

description of algebraic rules and procedures” (p. 13). The LSS first introduces students to the rules of operation precedence. Then, Kirshner explains, “upon the foundation of order of operations is erected the basic lexical elements” (p. 15) of the structural discourse of algebra. He goes on to explain how he would rigorously define “principal operation,” “principal subexpression,” “next-most principal subexpression,” “factor,” “term,” and other structural notions. Kirshner cites personal experience that such an explicit immersion in structural discourse can help students not only parse expressions but also talk intelligently about the pieces of those parsed expressions.

Kirshner provides an extended example of a hypothetical interaction between student and teacher in an LSS curriculum classroom. I will include this example in its entirety because it vividly illustrates Kirshner’s understanding of how the LSS would function. Here is the example in full:

The following contrived episode, similar to many I’ve engaged in when using the LSS approach, illustrates the sort of communicational possibilities opened up by these more rigorous discursive practices. This interaction involves a student’s erroneous cancellation of the 3s in

$$\frac{3x^2 + 1}{3y - 2} = \frac{x^2 + 1}{y - 2}$$

Teacher: What rule are you using in this step?

Student: The cancellation rule for fractions.

Teacher: Can you remind me what that rule is?

Student: It's the rule that allows canceling a common factor of the numerator and denominator of a fractional expression.

Teacher: Okay, let's take a look at it. What have you canceled?

Student: The threes, because they're factors, they're multiplied.

Teacher: Good, they are indeed factors, but are they factors of the *numerator and denominator*? Let's check. What is the principal operation of the numerator?

Student: Let's see, there's an exponentiation, a multiplication, and an addition. So the principal operation is addition, the least precedent one according to the hierarchy of operations.

Teacher: Good, now what are the principal subexpressions called in this case?

Student: They're called terms. ...Oh, I see, it has to be a factor of the *whole* numerator and denominator to be canceled; not just part of it.

Such communicative possibilities can be contrasted with traditional algebra instruction in which students and teachers talk past each other as they use words like "term" and "factor" without structural grounding. (p. 18)

In this way, Kirshner shows the ways in which he imagines that the LSS would help students avoid common algebra errors.

While my curricular proposals share Kirshner’s structural emphasis, mine also take advantage of the very thing that Kirshner’s research uncovers: student receptivity to the visual. The success of the teacher’s intervention in the previous discussion depends upon the student already having access to fully reified structural notions like “factor of the numerator.” As we have seen, however, reification is difficult to achieve and needs to be induced horizontally as well as vertically. While the above intervention could succeed for a student who has achieved the object-perception of these structural notions, it might not succeed if the student making the cancellation error is still negotiating Sfard’s vicious circle. I propose the following sort of intervention as more likely to succeed for students whose structural understanding is still in formation and who are therefore very susceptible to overgeneralizing tendencies:

Teacher: What are you doing in this step?

Student: I am cancelling.

Teacher: Can you remind me what we do before we cancel?

Student: We slash the top and bottom?

Teacher: Good. Okay, so let’s do that. Take out your Multiplication Knife and parse the numerator and denominator.

Student writes
$$\frac{\cancel{3}\cancel{x}^2 + 1}{3y - 2}$$
 then stops.

Student: Oh, I see. We can't slash all the way because of the plus. Really it's like this:

Student writes

$$\frac{\cancel{3}x^2 + 1}{\cancel{3}y - 2}$$

Teacher: That's correct! 3 is a factor of the first term of the numerator, but 3 is not a factor of the entire numerator.

Student: So we actually can't cancel at all because when you slash the numerator you just get the numerator, and that's not the same as what you get when you slash the denominator?

Teacher: Exactly. The numerator and denominator have no common factors.

This discussion illustrates how the teacher can use visual parsing techniques to communicate structural content to the student while student understanding of structure is still solidifying. While this student lacks the precise formal language for describing structure that Kirshner's student possesses, this student's understanding of structure is nonetheless progressing, aided by the visual parsing cues that make structure visually salient.

Conclusion and implications

In this paper, we have considered student learning of the algebra skill known as expression transformation. We have examined evidence that students are

universally prone to make certain common and persistent errors while transforming expressions, and I have diagnosed student difficulties as stemming in part from insufficient attention on the part of the traditional curriculum to the activity of parsing and to important structural concepts. We have also seen that students are strongly prone to err by overgeneralizing, and I have argued that these tendencies to overgeneralize are not just strategies adopted due to lack of structural understanding but impulses that are likely to compete with correct structural understanding at all times when the student is doing algebra. I have proposed a variety of instructional strategies and exercises designed to help students perceive structure in algebra – conceptually and visually. I believe that sustained use of such strategies in the algebra classroom can modestly improve student learning of expression transformation and help students avoid notoriously persistent common errors.

Before concluding, I want to acknowledge the presence in the literature of another diagnosis for these difficulties with expression transformation. This is the view that student difficulties with structural algebra stem from lack of sufficient referential support for symbol manipulations. There are two ways to frame this set of views. One way is to frame it around content: students find structural algebra hard because decontextualized symbol manipulation without frequent review of its referential meaning is inherently hard. That is, students find it hard because they lack good answers to the question “What is this about?” Another way is to frame this set of views around motivation and affect: students lack interest in algebra because they have insufficient exposure to its applications. That is, students find it hard because

they lack good answers to the question “What is this good for and why should I care?”

This view is popular and represents something of a consensus among contemporary researchers of algebra education. Kaput (1995), for instance, expresses this view that difficulties with algebra stem from insufficient attention to referential connections:

Acts of generalization and gradual formalization of the constructed generality must precede work with formalisms – otherwise the formalisms have no source in student experience. The current wholesale failure of school algebra has shown the inadequacy of attempts to tie the formalisms to students’ experience after they have been introduced. It seems that, ‘once meaningless, always meaningless.’ (p. 76)

Similarly, Resnick, Cauzinille-Marmeche, & Mathieu (1987) argue that if students were better able to “understand algebra expressions as having referential as well as formal meaning,” those students would then “be in a position to use what they already know about the semantics of situations and of fundamental mathematical concepts to constrain their formal constructions” (p. 201) and avoid common algebra errors. This supposition leads these authors to criticize the lack of referential content in the traditional curriculum: “It seems likely that if algebra is to be well learned by children, algebra expressions and laws of transformation must be related to the reference situations that might generate them, as well as to the mathematical constructs that they represent” (p. 201).

I do not deny that affect and motivation play a critical role in determining student behavior, particularly in a subject that involves decontextualized reasoning and abstraction, and particularly when the students are adolescents. However, were motivation the sole problem, we might expect a random assortment of varying errors. The universality of these errors, the striking identicalness of these errors across settings and decades (during which algebra has been taught at different grade levels), and the persistence of these errors among students taking higher-level math courses all suggest that affect and motivation alone do not account for them.

Moreover, the research of Kirshner and of Landy & Goldstone makes a formidable case that visual cues and other formally irrelevant factors influence decision-making spontaneously and persistently, even for individuals who are motivated to learn, intent on success, and aware of the rules. In other words, even if one believes, as I do, that referential approaches can contribute to improved performance on structural tasks, this research suggests that there are other reasons besides lack of referential context that students have difficulty with structural tasks – *reasons that will not go away no matter how much referential context students take in.*

When Kirshner invites us to consider “A New Curriculum for Structural Understanding of Algebra,” as in the title of his 2006 paper, he is not downplaying the importance of referential algebra. Rather, he is simply asserting that referential and structural goals are sufficiently independent so as to sometimes warrant separate attention: “It is sensible for us to focus curricular attention on this face independently, to ensure that algebraic structure is properly represented for our students” (p. 14).

Indeed, the National Council of Teachers of Mathematics, in its *Principles and Standards for School Mathematics* (2000), retains decontextualized symbol manipulation as a desired outcome for students:

Students should be able to operate fluently on algebraic expressions, combining them and reexpressing them in alternative forms. These skills underlie the ability to find exact solutions for equations, a goal that has always been at the heart of the algebra curriculum. (p. 301)

Insofar as decontextualized symbol manipulation is prized as *one* of the outcomes of algebra education, the research discussed here is relevant to curriculum and instruction. Therefore, although this paper focuses in on acquisition of structural skills, only one face of a subject with two faces, it ought to be of interest to all who are in the business of teaching algebra.

Perhaps the most significant implication of this research is the notion that the difficulties of algebra are inherent to the human minds that learn it. Sfard points out the inherent difficulty of achieving reified understanding of mathematical objects because of the vicious circle. The psychological research of Landy & Goldstone suggests that humans naturally consider formally irrelevant visual features while making decisions in rule-governed domains, even when they know they should not do so. It seems clear that there is no easy “answer” for student struggles to master symbolic algebra. But any effective instructional strategies that do constitute a partial answer will, like those offered here, help make structure visually salient for students.

References

- Barnard, T. (2002). Hurdles and strategies in the teaching of algebra: Part I. *Mathematics in School*, 31(1), 10-13.
- Barnard, T. (2002). Hurdles and strategies in the teaching of algebra: Part II. *Mathematics in School*, 31(4), 41-44.
- Booth, L. R. (1984). *Algebra: Children's strategies and errors*. Windsor, Berkshire, UK: NFER-Nelson.
- Chazan, D. *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom*. New York: Teachers College Press.
- Coxford, A. F., Fey, J. T., Hirsch, C. R., Schoen, H. L., Burrill, G., Hart, E. W., et al. (2003). *Contemporary mathematics in context: A unified approach: Course 3 Part A: Teacher's guide*. New York: Glencoe McGraw-Hill.
- Dolciani, M. P., Wooton, W., & Beckenbach, E. F. (1983). *Algebra I* (Rev. ed.). Boston: Houghton Mifflin Company.
- Ernest, P. (1987). A model of the cognitive meaning of mathematical expressions. *British Journal of Educational Psychology*, 57, 343-370.
- Foerster, P. A. (1994). *Algebra I: Expressions, equations, and applications* (3rd ed.). Menlo Park, CA: Addison-Wesley Publishing Company.
- Grossman, A. (1924). An analysis of the teaching of cancellation in algebraic fractions. *Mathematics Teacher*, 17, 104-109.
- Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and*

teaching of algebra. (pp. 60-86). Reston, VA: National Council of Teachers of Mathematics.

Jansen, A. R., Marriott, K., & Yelland, G. W. (2007). Parsing of algebraic expressions by experienced users of mathematics. *European Journal of Cognitive Psychology*, *19*(2), 286-320.

Kaput, J. J. (1995). A research base supporting long term algebra reform. In D. T. Owens, M. K. Reed, & G. M. Millsaps (Eds.), *Proceedings of the Seventeenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 71-94). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kieran, C. (1983). Relationships between novices' views of algebraic letters and their use of symmetric and asymmetric equation-solving procedures. In J. C. Bergeron & N. Herscovics (Eds.), *Proceedings of the Fifth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, 161-168). Montreal, Quebec, Canada: Universite de Montreal.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra*. (pp. 33-56). Reston, VA: National Council of Teachers of Mathematics.

Kilpatrick, J. (1988). Editorial. *Journal for Research in Mathematics Education*, *19*(4), 274.

Kirshner, D. (1985). A linguistic model of algebraic symbol skill. In S. K. Damarin & M. Shelton (Eds.), *Proceedings of the Seventh Annual Meeting of the*

North American Chapter of the International Group for the Psychology of Mathematics Education, 153-164. Columbus, OH.

Kirshner, D. (1989). The visual syntax of algebra. *Journal for Research in Mathematics Education*, 20(3), 274-287.

Kirshner, D. (1993). *The structural algebra option: A discussion paper*. Paper presented at the Annual Meeting of the American Educational Research Association, Atlanta, GA.

Kirshner, D. (2001). The structural algebra option revisited. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on School Algebra*, 83-98. Dordrecht, Netherlands: Kluwer Academic Publishers.

Kirshner, D. (2006). A new curriculum for structural understanding of algebra. *Journal of the Korea Society of Mathematical Education Series D*, 10(3), 169-187.

Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education*, 35, 224-257.

Landy, D., & Goldstone, R. L. (2007a). How abstract is symbolic thought? *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 33(4), 720-733.

Landy, D., & Goldstone, R.L. (2007b). How space guides interpretation of a novel computational system. *Twenty-Ninth Annual Meeting of the Cognitive Science Society*.

Landy, D., & Goldstone, R.L. (2007c). The alignment of ordering and space in arithmetic computation. *Twenty-Ninth Annual Meeting of the Cognitive Science Society*.

Laursen, K. W. (1978). Errors in first-year algebra. *Mathematics Teacher*, 71(3), 194-195.

Lee, L., & Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics*, 20(1), 41-54.

Linchevski, L., & Herscovics, N. (1994). Cognitive obstacles in pre-algebra. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (Vol. 3, 176-183). Lisbon, Porgugal: PME.

Linchevski, L., & Livneh, D. (2002). The competition between numbers and structure: Why expressions with identical algebraic structures trigger different interpretations. *Focus on Learning Problems in Mathematics*, 24(2).

Marquis, J. (1988). Common mistakes in algebra. In A. F. Coxford & A. P. Shuttle (Eds.). *The ideas of algebra, K-12: 1988 yearbook*. (pp. 204-205). Reston, VA: National Council of Teachers of Mathematics.

Martinez, J. G. R. (1988). Helping students understand factors and terms. *Mathematics Teacher*, 81, 747-751.

Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Mathematical Behavior*, 3, 93-166.

Miller, J. (2006). *Earliest uses of symbols of operation*. Retrieved June 7, 2008, from <http://members.aol.com/jeff570/operation.html>

Nurnberger-Haag, J. (2003). Order of op hop. *Mathematics Teaching in the Middle School*, 8(5), 234-236.

Parish, C. R., & Ludwig, H. J. (1994). Language, intellectual structures and common mathematical errors: A call for research. *School Science and Mathematics*, 94(5), 235-239.

Payne, J. N., Zamboni, F. F., & Lankford, Jr., F. G. (1956). *Algebra one* (2nd ed.). New York: Harcourt Brace Jovanovich, Inc.

Peterson. (2000). *History of the order of operations*. Retrieved June 7, 2008, from <http://mathforum.org/library/drmath/view/52582.html>

Principles and standards for school mathematics. (2000). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Rambhia, S. (2002). A new approach to an old order. *Mathematics Teaching in the Middle School*, 8(4), 193-195.

Resnick, L. B., Cauzinille-Marmeche, E., & Mathieu, J. (1987). Understanding algebra. In J. A. Sloboda & D. Rogers (Eds.). *Cognitive processes in mathematics*. (pp. 169-203). Oxford, United Kingdom: Clarendon Press.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93(2), 346-362.

Schwartzman, S. (1977). Helping students understand the distributive property. *Mathematics Teacher*, 70, 594-595.

Schwartzman, S. (1986). The *A* of a *B* is the *B* of an *A*. *Mathematics Teacher*, 79(3), 180-183.

Schrock, C., & Morrow, J. Wright or wrong: Teaching the order of operations. *School Science and Mathematics*, 93(1), 28-30.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 1-36.

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification: The case of algebra. *Educational Studies in Mathematics*, 26(2/3), 191-228.

Sleeman, D. (1984). An attempt to understand students' understanding of basic algebra. *Cognitive Science*, 8, 387-412.

Star, J. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, 36(5), 404-411.

Stein, E. I. (1956). *Algebra in easy steps* (3rd ed.). Princeton: D. Van Nostrand Company, Inc.

Thompson, P. W., & Thompson, A. G. (1987). Computer presentations of structure in algebra. In J. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the Eleventh Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 248-254). Montreal: University of Quebec at Montreal.

Wong, M. P. H. (1997). Numbers versus letters in algebraic manipulation: Which is more difficult? *Proceedings of the 21st Conference for the Psychology of Mathematics Education*, Lahti, Finland, 4, 285-290.

Wu, H. (2007). "Order of operations" and other oddities in school mathematics. Retrieved June 7, 2008, from <http://math.berkeley.edu/~wu/order4.pdf>