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SCRAMBLED CANTOR SETS

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Abstract. We show that Li–Yorke chaos ensures the existence of a scrambled Cantor set.

Introduction

A *dynamical system* is a pair (X, f), where X is a metric space and $f\colon X \to X$ is a continuous function. Given such a system, we say that points $x, y \in X$ are *proximal* if $\liminf_{n\to\infty} d_X(f^n(x), f^n(y)) = 0$, and *asymptotic* if $\limsup_{n\to\infty} d_X(f^n(x), f^n(y)) = 0$. The pair (x, y) is *Li–Yorke* if x and y are proximal but not asymptotic, a set $Y \subseteq X$ is *scrambled* if (x, y) is Li–Yorke for all distinct $x, y \in Y$, and the system (X, f) is *Li–Yorke chaotic* if there is an uncountable scrambled set $Y \subseteq X$. In [LY75], Li and Yorke showed that every dynamical system on the unit interval with a point of period three is Li–Yorke chaotic.

The scrambled set constructed in [LY75] is indexed by an interval on the real line, and therefore has cardinality $2^{\aleph_0}$. Moreover, subsequent constructions of uncountable scrambled sets in the literature typically gave rise to sets of cardinality $2^{\aleph_0}$, or even *Cantor sets*, i.e., homeomorphic copies of the Cantor space $2^{\mathbb{N}}$. One example is the construction, in [BGKM02], of uncountable scrambled sets in dynamical systems of positive topological entropy.

A metric space is *Polish* if it is complete and separable, and a dynamical system (X, f) is *Polish* if X is Polish. At the end of [BHS08, §3], Blanchard, Huang, and Snoha asked whether every Li–Yorke chaotic Polish dynamical system admits a scrambled Cantor set. Here we utilize the descriptive set theory of definable graphs to obtain the following answer to their question:

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Theorem 1. *Suppose that (X, f) is a Li–Yorke chaotic Polish dynamical system. Then there is a scrambled Cantor set $C \subseteq X$.*

In §1, we establish an analog of the Kechris–Solecki–Todorcevic characterization of the existence of $\aleph_0$-colorings (see [KST99, Theorem 6.3]) within cliques. In §2, we use this to establish a similar analog of Silver's perfect set theorem for co-analytic equivalence relations (see [Sil80]). In §3, we use the latter to establish Theorem 1. And in §4, we discuss several generalizations.

1. Colorings in cliques

Endow $\mathbb{N}$ with the discrete topology, and $\mathbb{N}^\mathbb{N}$ with the corresponding product topology. A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^\mathbb{N}$, and *Polish* if it is second countable and admits a compatible complete metric. A subset of a topological space is *Borel* if it is in the smallest σ-algebra containing the open sets, and *co-analytic* if its complement is analytic. Every non-empty Polish space is a continuous image of $\mathbb{N}^\mathbb{N}$ (see, for example, [Kec95, Theorem 7.9]), thus so too is every non-empty analytic space, and a subset of an analytic Hausdorff space is Borel if and only if it is analytic and co-analytic (see, for example, the proof of [Kec95, Theorem 14.11]).

A *digraph* on a set X is an irreflexive binary relation G on X. A set $Y \subseteq X$ is *G-independent* if $G \upharpoonright Y = \emptyset$. An *$I$-coloring* of G is a function $c \colon X \to I$ such that $c(x) \neq c(y)$ for all $(x, y) \in G$, or equivalently, such that $c^{-1}(\{i\})$ is G-independent for all $i \in I$. A *homomorphism* from a binary relation R on a set X to a binary relation S on a set Y is a function $\phi \colon X \to Y$ for which $(\phi \times \phi)(R) \subseteq S$. We say that a set $Y \subseteq X$ is an *R-clique* if $x \mathrel{R} y$ for all distinct $x, y \in Y$.

We use $X^{<\mathbb{N}}$ to denote $\bigcup_{n \in \mathbb{N}} X^n$, (i) to denote the singleton sequence with value i, and $\sqsubseteq$ to denote extension. Following standard practice, we also use $\mathcal{N}_s$ to denote $\{b \in \mathbb{N}^\mathbb{N} \mid s \sqsubseteq b\}$ or $\{c \in 2^\mathbb{N} \mid s \sqsubseteq c\}$ (with the context determining which of the two we have in mind). Fix sequences $\mathbb{s}_n \in 2^n$ such that $\forall s \in 2^\mathbb{N} \exists n \in \mathbb{N}\ s \sqsubseteq \mathbb{s}_n$, and let $\mathbb{G}_0$ denote the digraph on $2^\mathbb{N}$ given by $\mathbb{G}_0 = \{(\mathbb{s}_n \frown (i) \frown c)_{i<2} \mid c \in 2^\mathbb{N} \text{ and } n \in \mathbb{N}\}$. A subset of a topological space is G_δ if it is a countable intersection of open sets.

Theorem 1.1. *Suppose that X is a Hausdorff space, G is an analytic digraph on X, and R is a reflexive G_δ binary relation on X. Then at least one of the following holds:*

1. *For every R-clique $Y \subseteq X$, there is an $\mathbb{N}$-coloring of $G \upharpoonright Y$.*
2. *There is a continuous homomorphism $\phi \colon 2^\mathbb{N} \to X$ from $\mathbb{G}_0$ to G for which $\phi(2^\mathbb{N})$ is an R-clique.*

Proof. Suppose that there is an R-clique $Y \subseteq X$ for which there is no $\mathbb{N}$-coloring of $G \restriction Y$. Then $G \neq \emptyset$, so there are continuous surjections $\phi_G \colon \mathbb{N}^\mathbb{N} \twoheadrightarrow G$ and $\phi_X \colon \mathbb{N}^\mathbb{N} \twoheadrightarrow \bigcup_{i<2} \mathrm{proj}_i(G)$. Fix a decreasing sequence $(R_n)_{n\in\mathbb{N}}$ of open subsets of $X \times X$ whose intersection is R.

We will define a decreasing sequence $(Y^\alpha)_{\alpha<\omega_1}$ of subsets of Y, off of which there are $\mathbb{N}$-colorings of $G \restriction Y$. We begin by setting $Y^0 = Y$. For all limit ordinals $\lambda < \omega_1$, we set $Y^\lambda = \bigcap_{\alpha<\lambda} Y^\alpha$. To describe the construction at successor ordinals, we require several preliminaries.

An *approximation* is a triple of the form $a = (n^a, \phi^a, (\psi^a_n)_{n<n^a})$, where $n^a \in \mathbb{N}$, $\phi^a \colon 2^{n^a} \to \mathbb{N}^{<\mathbb{N}}$, $\psi^a_n \colon 2^{n^a-(n+1)} \to \mathbb{N}^{n^a}$ for all $n < n^a$, and $\phi_X(\mathcal{N}_{\phi^a(s)}) \times \phi_X(\mathcal{N}_{\phi^a(t)}) \subseteq R_{n^a}$ for all distinct $s, t \in 2^{n^a}$. A *one-step extension* of an approximation a is an approximation b such that:

(a) $n^b = n^a + 1$.
(b) $\forall s \in 2^{n^a} \forall t \in 2^{n^b}$ $(s \sqsubset t \implies \phi^a(s) \sqsubset \phi^b(t))$.
(c) $\forall n < n^a \forall s \in 2^{n^a-(n+1)} \forall t \in 2^{n^b-(n+1)}$ $(s \sqsubset t \implies \psi^a_n(s) \sqsubset \psi^b_n(t))$.

Similarly, a *configuration* is a triple of the form $\gamma = (n^\gamma, \phi^\gamma, (\psi^\gamma_n)_{n<n^\gamma})$, where $n^\gamma \in \mathbb{N}$, $\phi^\gamma \colon 2^{n^\gamma} \to \mathbb{N}^\mathbb{N}$, $\psi^\gamma_n \colon 2^{n^\gamma-(n+1)} \to \mathbb{N}^\mathbb{N}$ for all $n < n^\gamma$, and $(\phi_G \circ \psi^\gamma_n)(t) = ((\phi_X \circ \phi^\gamma)(\mathbb{s}_n \frown (i) \frown t))_{i<2}$ for all $n < n^\gamma$ and $t \in 2^{n^\gamma-(n+1)}$. We say that γ is *compatible* with a set $Y' \subseteq Y$ if $(\phi_X \circ \phi^\gamma)(2^{n^\gamma}) \subseteq Y'$, and *compatible* with a if:

(i) $n^a = n^\gamma$.
(ii) $\forall t \in 2^{n^a}$ $\phi^a(t) \sqsubseteq \phi^\gamma(t)$.
(iii) $\forall n < n^a \forall t \in 2^{n^a-(n+1)}$ $\psi^a_n(t) \sqsubseteq \psi^\gamma_n(t)$.

An approximation a is *Y'-terminal* if no configuration is compatible with both Y' and a one-step extension of a. Let $A(a, Y')$ denote the set of points of the form $(\phi_X \circ \phi^\gamma)(\mathbb{s}_{n^a})$, where γ varies over all configurations compatible with a and Y'.

Lemma 1.2. *Suppose that $Y' \subseteq Y$ and a is a Y'-terminal approximation. Then $A(a, Y')$ is G-independent.*

Proof. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, both compatible with a and Y', with the property that $((\phi_X \circ \phi^{\gamma_i})(\mathbb{s}_{n^a}))_{i<2} \in G$. Fix a sequence $d \in \mathbb{N}^\mathbb{N}$ with the property that $\phi_G(d) = ((\phi_X \circ \phi^{\gamma_i})(\mathbb{s}_{n^a}))_{i<2}$, and let γ be the configuration given by $n^\gamma = n^a + 1$, $\phi^\gamma(t \frown (i)) = \phi^{\gamma_i}(t)$ for all $i < 2$ and $t \in 2^{n^a}$, $\psi^\gamma_n(t \frown (i)) = \psi^{\gamma_i}_n(t)$ for all $i < 2$, $n < n^a$, and $t \in 2^{n^a-(n+1)}$, and $\psi^\gamma_{n^a}(\emptyset) = d$. Then γ is compatible with a one-step extension of a, contradicting the fact that a is Y'-terminal. ⊠

Let $Y^{\alpha+1}$ be the difference of Y^α and the union of the sets of the form $A(a, Y^\alpha)$, where a varies over all Y^α-terminal approximations.

Lemma 1.3. *Suppose that $\alpha < \omega_1$ and a is a non-$Y^{\alpha+1}$-terminal approximation. Then a has a non-Y^α-terminal one-step extension.*

Proof. Fix a one-step extension b of a for which there is a configuration γ compatible with b and $Y^{\alpha+1}$. Then $(\phi_X \circ \phi^\gamma)(\mathbb{s}_{n^b}) \in Y^{\alpha+1}$, so b is not Y^α-terminal. ⊠

Fix $\alpha < \omega_1$ such that the families of Y^α- and $Y^{\alpha+1}$-terminal approximations coincide, and let a_0 be the unique approximation for which $n^{a_0} = 0$ and $\phi^{a_0}(\emptyset) = \emptyset$. As $A(a_0, Y') = Y' \cap \bigcup_{i<2} \mathrm{proj}_i(G)$ for all $Y' \subseteq Y$, we can assume that a_0 is not Y^α-terminal, since otherwise there is an $\mathbb{N}$-coloring of $G \upharpoonright Y$.

By recursively applying Lemma 1.3, we obtain non-Y^α-terminal one-step extensions a_{n+1} of a_n for all $n \in \mathbb{N}$. Define $\phi', \psi_n \colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\phi'(c) = \bigcup_{n\in\mathbb{N}} \phi^{a_n}(c \upharpoonright n)$ and $\psi_n(c) = \bigcup_{m>n} \psi_n^{a_m}(c \upharpoonright (m-(n+1)))$ for all $c \in 2^\mathbb{N}$ and $n \in \mathbb{N}$. Clearly these functions are continuous.

To see that the function $\phi = \phi_X \circ \phi'$ is a homomorphism from $\mathbb{G}_0$ to G, we will show the stronger fact that if $c \in 2^\mathbb{N}$ and $n \in \mathbb{N}$, then $(\phi_G \circ \psi_n)(c) = ((\phi_X \circ \phi')(\mathbb{s}_n \frown (i) \frown c))_{i<2}$. It is sufficient to show that if U is an open neighborhood of $((\phi_X \circ \phi')(\mathbb{s}_n \frown (i) \frown c))_{i<2}$ and V is an open neighborhood of $(\phi_G \circ \psi_n)(c)$, then $U \cap V \neq \emptyset$. Towards this end, fix $m > n$ such that $\prod_{i<2} \phi_X(\mathcal{N}_{\phi^{a_m}(\mathbb{s}_n \frown (i) \frown s)}) \subseteq U$ and $\phi_G(\mathcal{N}_{\psi_n^{a_m}(s)}) \subseteq V$, where $s = c \upharpoonright (m-(n+1))$. As a_m is not Y^α-terminal, there is a configuration γ compatible with a_m, so $((\phi_X \circ \phi^\gamma)(\mathbb{s}_n \frown (i) \frown s))_{i<2} \in U$ and $(\phi_G \circ \psi_n^\gamma)(s) \in V$, thus $U \cap V \neq \emptyset$.

To see that $\phi(2^\mathbb{N})$ is an R-clique, observe that if $c, d \in 2^\mathbb{N}$ are distinct and $n \in \mathbb{N}$ is sufficiently large that $c \upharpoonright n \neq d \upharpoonright n$, then $\phi(c) \in \phi_X(\mathcal{N}_{\phi^{a_n}(c\upharpoonright n)})$ and $\phi(d) \in \phi_X(\mathcal{N}_{\phi^{a_n}(d\upharpoonright n)})$, so $\phi(c) \mathrel{R_n} \phi(d)$. ⊠

2. Separability in cliques

The following well-known fact rules out the existence of a Baire-measurable $\mathbb{N}$-coloring of $\mathbb{G}_0$:

Proposition 2.1. *Suppose that $B \subseteq 2^\mathbb{N}$ is a non-meager set with the Baire property. Then B is not $\mathbb{G}_0$-independent.*

Proof. Fix a sequence $s \in 2^{<\mathbb{N}}$ for which B is comeager in $\mathcal{N}_s$ (see, for example, [Kec95, Proposition 8.26]). Then there exists $n \in \mathbb{N}$ for which $s \sqsubseteq \mathbb{s}_n$. Let ι be the involution of $\mathcal{N}_{\mathbb{s}_n}$ sending $\mathbb{s}_n \frown (0) \frown c$ to $\mathbb{s}_n \frown (1) \frown c$ for all $c \in 2^\mathbb{N}$. As ι is a homeomorphism, it follows that $B \cap \iota(B)$ is comeager in $\mathcal{N}_{\mathbb{s}_n}$ (see, for example, [Kec95, Exercise 8.45]), so $B \cap \iota(B) \cap \mathcal{N}_{\mathbb{s}_n \frown (0)} \neq \emptyset$. As $(c, \iota(c)) \in \mathbb{G}_0 \upharpoonright B$ for all $c \in B \cap \iota(B) \cap \mathcal{N}_{\mathbb{s}_n \frown (0)}$, it follows that B is not $\mathbb{G}_0$-independent. ⊠

The following corollary is also well known:

Proposition 2.2. *Suppose that E is a non-meager equivalence relation on $2^{\mathbb{N}}$ with the Baire property. Then E is not disjoint from $\mathbb{G}_0$.*

Proof. By the Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]), there is a sequence $c \in 2^{\mathbb{N}}$ for which $[c]_E$ has the Baire property and is not meager, so Proposition 2.1 ensures that $[c]_E$ is not $\mathbb{G}_0$-independent. ⊠

A *partial transversal* of an equivalence relation E on a set X is a set $Y \subseteq X$ that does not contain distinct E-related points.

Theorem 2.3. *Suppose that X is a Hausdorff space, E is a co-analytic equivalence relation on X, and R is a reflexive G_δ binary relation on X for which there is an R-clique $Y \subseteq X$ intersecting uncountably-many E-classes. Then there is a Cantor set $C \subseteq X$ that is both a partial transversal of E and an R-clique.*

Proof. Set $G = \sim E$, appeal to Theorem 1.1 to obtain a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G for which $\phi(2^{\mathbb{N}})$ is an R-clique, and let F be the pullback of E through ϕ. As $\mathbb{G}_0 \cap F = \emptyset$, Proposition 2.2 implies that F is meager, thus Mycielski's Theorem (see, for example, [Kec95, Theorem 19.1]) yields a continuous injection $\psi \colon 2^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}$ sending distinct elements of $2^{\mathbb{N}}$ to F-inequivalent elements of $2^{\mathbb{N}}$, in which case the set $C = (\phi \circ \psi)(2^{\mathbb{N}})$ is as desired. ⊠

3. Li-Yorke chaos

We say that a dynamical system (X, f) is *analytic* if X is analytic. As every Polish dynamical system is analytic, Theorem 1 is a consequence of the following result:

Theorem 3.1. *Suppose that (X, f) is a Li–Yorke chaotic analytic dynamical system. Then there is a scrambled Cantor set $C \subseteq X$.*

Proof. Note first that the set

$$\begin{aligned} R &= \{(x, y) \in X \times X \mid x \text{ and } y \text{ are proximal}\} \\ &= \textstyle\bigcap_{\epsilon>0} \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} \{(x, y) \in X \times X \mid d_X(f^m(x), f^m(y)) < \epsilon\} \end{aligned}$$

is G_δ, and the equivalence relation

$$\begin{aligned} E &= \{(x, y) \in X \times X \mid x \text{ and } y \text{ are asymptotic}\} \\ &= \textstyle\bigcap_{\epsilon>0} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \{(x, y) \in X \times X \mid d_X(f^m(x), f^m(y)) \leq \epsilon\} \end{aligned}$$

is Borel. As the fact that (X, f) is Li–Yorke chaotic yields an R-clique intersecting uncountably-many E-classes, Theorem 2.3 yields a scrambled Cantor set. ⊠

4. Further generalizations

Suppose that S is a semigroup, d_S is a metric on S, X is a metric space, and $S \curvearrowright X$ is an action by continuous functions. The notions of proximal and asymptotic, and therefore of Li–Yorke pair, scrambled, and Li–Yorke chaotic, generalize naturally to such actions, as does Theorem 3.1 and its proof.

Given $\delta \geq 0$ and a dynamical system (X, f), we say that points $x, y \in X$ are *δ-proximal* if $\liminf_{n\to\infty} d_X(f^n(x), f^n(y)) \leq \delta$. The above argument also yields the generalization of Theorem 3.1 to the analog of Li–Yorke chaos where proximality is replaced with δ-proximality.

Given $\epsilon \geq 0$ and a dynamical system (X, f), we say that points $x, y \in X$ are *ϵ-asymptotic* if $\limsup_{n\to\infty} d_X(f^n(x), f^n(y)) \leq \epsilon$, which gives rise to an analogous notion of *ϵ-scrambled*. By replacing Theorem 2.3 with its generalization from equivalence relations to extended-valued quasi-metrics in the proof of Theorem 3.1, one can establish the generalization of the latter in which the uncountable set is ϵ-scrambled and the Cantor set is $(\epsilon/2)$-scrambled.

However, when $\epsilon > 0$, the strengthening in which $\epsilon/2$ is replaced with any $\epsilon' < \epsilon$ follows from the analog of Theorem 3.1 in which asymptoticity is replaced with the strengthening of ϵ-asymptoticity where one requires that $\limsup_{n\to\infty} d_X(f^n(x), f^n(y)) < \epsilon$, which was originally established by Blanchard, Huang, and Snoha (see [BHS08, Theorem 16]). As the corresponding analog of Li–Yorke pair is a G_δ condition, their result follows from the special case of Theorem 2.3 where E is equality, which is far simpler to establish (see [She99, Remark 1.14]).

At the end of [Aki04, §6], Akin noted that the analog of Theorem 3.1 for complete perfect metric spaces is open. While our approach does not fully resolve this problem beyond the separable case, it does generalize to show that if κ is an infinite cardinal, X has a dense set of cardinality κ, and there is a scrambled set of cardinality strictly greater than κ, then there is a scrambled Cantor set. To see this, endow κ with the discrete topology, and $\kappa^{\mathbb{N}}$ with the corresponding product topology. We say that a topological space is *κ-Souslin* if it is a continuous image of a closed subset of $\kappa^{\mathbb{N}}$. The proof that every Polish space is analytic easily adapts to show that every complete metric space with a dense subset of cardinality κ is κ-Souslin, and the proof of Theorem 1.1 easily adapts to show the analogous result in which the digraph G is merely κ-Souslin and the $\mathbb{N}$-coloring in condition (1) is replaced with a κ-coloring. The proof of Theorem 2.3 therefore adapts to show the analogous result in which E is co-κ-Souslin, the pullback of E through every continuous function $\phi\colon 2^{\mathbb{N}} \to X$ has the Baire property, and Y intersects more than κ classes. But this can be plugged into the proof of Theorem 3.1 to obtain the desired result.

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