ABSTRACT

Title of Document:	MATH AND MATH-IN-SCHOOL: CHANGES IN THE TREATMENT OF THE FUNCTION CONCEPT IN TWENTIETH CENTURY SECONDARY ALGEBRA TEXTBOOKS.
	Kyle M. Cochran, M.A., 2008
Directed By:	Associate Professor Daniel Chazan, Department of Curriculum and Instruction.

The exercises found in the chapters on function in five American textbooks (each taken from different twenty-year spans of the twentieth century) were analyzed using Mesa's (2000) coding scheme. Problems were analyzed based on the *context*, the *operations* needed to solve the problem, the *representations* used, and the *control structures* (or checks) available to the students. This analysis allowed for the identification of trends across time. These trends were compared to trends in the concept of function in the mathematics discipline and trends in recommendations for mathematics education.

This analysis was undertaken to address three basic research questions. First, is there evidence of change in the treatment of function in school algebra across time? Second, how do any changes that exist in the texts correlate with recommendations for mathematics education? And third, how do changes in the texts correlate with the developments of the function concept in the mathematics discipline?

MATH AND MATH-IN-SCHOOL: CHANGES IN THE TREATMENT OF THE FUNCTION CONCEPT IN TWENTIETH CENTURY SECONDARY ALGEBRA TEXTBOOKS.

By

Kyle M. Cochran

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Advisory Committee:
Associate Professor Daniel Chazan, Chair
Professor Emeritus James Fey
Professor Denny Gulick

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Chapter 1: Rationale

Where the Journey Began

A casual perusal of any contemporary Algebra One text will likely include the regular presentation of graphs and the explicit use of function for problem solving. From its pervasive nature in the textbooks of today, it is easy to imagine that algebra books have always looked this way. I was surprised, therefore, when I ran across Hall and Knight's *Elementary Algebra* (1965) in a used bookstore. The book, looking much older than it was, aroused my curiosity and I picked it up. I read the prefatory note which described Hall's contributions to the text since Knight's death in 1894. It contained a reference to the preface from their *New Edition* (1907) stating "The leading features are: (1) A full treatment of Graphs occupying more than 50 pages" (p. iii). I was curious to see what an algebra textbook would look like with only one section (50 pages) on graphing. I was even more curious about earlier editions of the book would look like – would there even be any graphs?

The idea that something I found to be so central to the notion of school algebra had not always been central really surprised me. Thus began my interest in examining how school curricula changes and the forces that prompt those changes.

The purpose of this paper is not only to examine the evolution of the function concept in secondary mathematics education (which would be a fascinating subject in its own right), but also to use my examination of this topic to look more closely at the nature of reform in mathematics education in general.

Background

As already indicated, there is a tendency to assume that the way things are is the way they have always been, and the content of mathematics courses seems particularly susceptible to this idea. Many people conceive of mathematics as a static field that is firmly established (Thompson, 1984). Despite the prevalence of this belief, educational curricula can (and do) change over time. These changes can be minor, like the inclusion or exclusion of a specific topic (like the binomial theorem). But they can also be more foundational changes – ones that can alter the fundamental understanding of the mathematics that is being taught. Function is such a notion.

The book *The Formation of School Subjects: The Struggle for Creating an American Institution*, edited by Popkewitz (1987), addresses the question of how our contemporary notion of subject-matter takes its form. Popkewitz contends, "The intense debate and social interests involved in the formation of American schooling is nowhere as clear as in the development of the subject-matter of schools" (p. ix).

Popkewitz (1987) also notes "the school subject-matter provides entrance to the interplay between social, cultural, economic and professional interests that give form to our contemporary school practices" (p. ix). The development of school subject matter is clearly complex and influenced by a number of different interest groups. One such group is the professionals in the discipline, whose use of the subject influences the perspective they advocate.

As far as the treatment of function in the discipline of mathematics is concerned, according to Kleiner (1989), "the evolution of the concept of function goes back 4000 years; 3700 of these consist of anticipation" (p. 282). The concept of function evolved from that of a table (e.g., for the Babylonians), to a curve (e.g., for Newton and Leibniz), to an analytic expression (e.g., Bernoulli and Euler), to a numerical correspondence (e.g., Dirichlet), and an arbitrary correspondence (e.g., Bourbaki). A change in the concept of function in mathematics could reasonably be assumed to produce a change in its treatment in education.

The fact that school subject matter is at the center of educational debates is most readily seen in the reports and recommendations – the products – of committees and individuals. Numerous articles have been written about the effectiveness of various reports and reform proposals, but most seem either to examine how broadly the recommendations of a single report were embraced, or to observe a trend in education and attribute it to the work of a specific committee's report (see Dexter, 1906; Châteauneuf, 1930; Butler, 1951; Roberts, 2001). If, instead, the proposals are viewed as evidence of the debate, then trends in the recommendations provide insight into the resolution of those debates.

There has been limited research on the evolution of the function concept in the context of school mathematics. Kennedy and Ragan (1969) examined the definition of function that was employed in twenty elementary algebra textbooks and fifteen college algebra textbooks. They contrasted the definitions used in textbooks published before 1959 with those published after 1959. Ten of the seventeen textbooks published after 1959 defined function as a set of ordered pairs, but none of the texts published before 1959 did.

Cooney and Wilson (1993) compared the definition of function presented in textbooks from the early 1900s to the 1990. In addition to the definitions, they described the way that the material was presented (e.g., was the coverage minimal, was the presentation graphical, did it emphasize sets and set notation, etc.). Cooney and Wilson presented a more complete picture of function was being used in the various texts, but the books that they compared were intended for different courses (some were elementary algebra texts while others were Algebra Two or Pre-Calculus texts).

As is easily noted, prior research on the history of function in secondary education has primarily dealt with the definition of function presented in textbooks from different eras. As well, the research has not always held constant which course (e.g., Algebra One, Algebra Two, or Pre-Calculus), the definitions were drawn from. My research is focused on the treatment of function only in Algebra One courses and is not only concerned with the definitions used in the various textbooks, but also with how those definitions are used by students in completing the textbooks' exercises.

Research Questions

These ideas led me to develop the following research questions:

> *Research Question 1:* By comparing the concept of function that is fostered by representative textbooks from different eras of the twentieth century, is there evidence for change in the treatment of function across time?

To address this question, I answered the following six questions about each of five textbooks that were selected to represent different twenty year spans of the twentieth century. The results of these questions were compared across the books to articulate changes that occurred in the treatment of function.

- How is function defined in each of the textbooks?

- Where is function introduced in the textbook? Is it addressed in multiple sections or is it isolated?
- In what context are exercises about function framed?
- What operations are required to solve problems about functions?
- Which representations (graphical, tabular, symbolic, etc.) are needed to obtain results for tasks about functions?
- What mechanisms are in place for students to determine if the answers they obtain are adequate and accurate?

Research Question 2: How do changes in representation of function in textbooks correlate with recommendations for mathematics education?

The entire twentieth century is filled with individuals and committees who have made recommendations about what should be taught in secondary mathematics and how that teaching should occur. I selected eight reports about mathematics education (from 1894 to 1989) to examine for recommendations about algebra more generally and the treatment of function specifically.

In addressing this research questions, I know that textbook writers are influenced by a number of different sources (the author's own education, the requirements of testing bodies, publisher demands, social pressures about the role of public education, etc.), and I am not attempting to claim causality between recommendations and material found in textbooks. Rather I am using the recommendations as evidence of a debate on a specific aspect of the treatment of function at that time in history. If the content of that recommendation ceases to be found in later reports, that suggests the debate about that topic has been resolved. If the proposed change is found in later textbooks, then the suggestion likely needed acceptance of the textbook authors to be resolved.

To illustrate this idea, the Committee on the Chicago Section of the American Mathematical Society (Young, et. al., 1899) recommended "at the close of each chapter or topic a synopsis in schematic form of its definitions, methods and results should be made" (p. 198). By suggesting the need for chapter synopses, it is reasonable to assume that many textbooks at that time did not include chapter reviews. Wentworth (1906), for example, lacks chapter summaries. However, reports after this time do not address the need for chapter synopses, though textbooks begin to include them. This indicates that textbook authors needed to include chapter summaries in their books for the debate over chapter synopses to be successfully resolved. To address my second research question, I ask the following underlying questions:

- What do individuals and committees recommend about the way that function should be taught in schools?
- Do these recommendations continue (despite already being articulated in prior reports), or are they only articulated in one of the reports?
- If a recommendation ceases to be made, is there evidence that it was heeded in the textbooks of that time?

> *Research Question 3*: How do changes in the texts correlate with the developments of the function concept in the mathematics discipline?

It is also possible that changes in the treatment of function in Algebra One textbooks reflect various stages in the evolution of the function concept in the mathematics discipline. To compare the narrative of the evolution of function with that of its treatment in textbooks, three underlying questions needed to be addressed.

- What significant changes in the concept of function occur in the discipline of mathematics? And when do they occur?
- Is there evidence of the direct influence of mathematicians on the content of these textbooks?
- What are the correlations between the treatment of function in the mathematics discipline and the recommendations for mathematics education?

Chapter 2: Theoretical and Analytic Frameworks

This chapter consists of four parts. To provide a filter for determining which aspects of the reports (and which aspects of the history of the function concept) to include in this study, it is important to know the way that function is being conceived. Therefore, the first part of this chapter describes the coding system that I employed in this study. In the second part, I identify eight reports or recommendations intended for mathematics education and identify the aspects of the reports that correlate to the categories used in my coding system. Many of these recommendations were found in *Readings in the History of Mathematics Education* (NCTM, 1970). The third section addresses the evolution of the function concept in the mathematics discipline, and the final section of this chapter identifies some of the underlying assumptions of the study.

Conception and The Coding of Data

The coding system that I used for this study is based heavily on the work of Vilma Mesa. Mesa's (2000) dissertation work examined the conceptions of function that were promoted by the exercises in textbooks of eighteen countries who participated in the Third International Mathematics and Science Study (TIMMS). To characterize the problems found in the textbooks, she used Balacheff and Gaudin's (2002) notion of conception.

To understand Balacheff and Gaudin's (2002) notion of conception requires a distinction between knowing and knowledge. Balacheff and Gaudin "follow the choice …to use *knowing* as a noun to distinguish the students' personal constructs from *knowledge* which refers to intellectual constructs recognised by a social body" (p. 1). Different situations can lead to different knowings for the individual. As Mesa (2000) notes, "Contradictory knowings can exist either at different times of a subject's history or because different situations enact different knowings" (p. 8). To resolve this issue, Balacheff and Gaudin propose the notion of conception.

According to Balacheff and Gaudin (2002), a conception is a quadruplet (P, R, L, Σ) in which:

- P is the set of problems;
- R is the set of operators;
- L is the a representation system;
- Σ is a control structure (p. 6).

The set of problems (P) is all the problems for which the considered conception provides efficient tools to obtain a solution. The set of operators (R) contains the tools for the resolution of the problem. These operators can be concrete or abstract and represent what actually needs to be done. The representation system (L) is an object that acts as the interface between the subject and the problem (possible representations include algebraic symbolism, natural language, geometrical drawings, etc). Finally, the control structure (Σ) acts as a check for the subject to determine adequacy and correctness of a result (pp. 6-7).

Newton, when working with problems about motion, employed techniques of measurement and computation, utilized drawings as a primary means of representation, and had control structures that were empirical in nature. In contrast, if function is conceived of as an analytical expression (e.g., Euler), then problems can be generated from the functions themselves (rather than a need to model some phenomenon). As well, in this situation, controls are mathematical proof and symbolic manipulation. Thus, different situations (i.e. differences in problems, operators, representations, or controls) may evoke different conceptions.

For Mesa's work (2000), she identified 10 different uses of function (problems), 36 operations, 9 representation systems, and 9 control structures. Using slight modifications, I employed her system of codification.

Recommendations and Reform

The reports or recommendations from eight individuals or committees (published from 1894 to 1989) are considered in this section. Only the recommendation in relation to the following six topics will be included: (1) the definition of function, (2) where function should be taught in the curriculum, (3) the context for problems, (4) methods of solving problems (specifically exercises about functions), (5) the representation systems that should be used, and (6) control structures (methods of verification) that should be used.

Felix Klein

No history of the reform movement of the function concept would be complete without addressing the contributions of Felix Klein. Hamley (1934) characterized Klein's participation in the reform movement as starting with a speech before the International Congress of Mathematicians in 1893 about the importance of functional thinking in school mathematics, and noted that he further developed this theme in various conferences in the years that followed. (p. 52).

Klein's role in the foundation of the movement was so profound that Hamley (1934) claimed that "the idea that the function concept should be made the central theme of school mathematics may be said to have originated with Klein" (p. 49). Despite Klein's indisputable influence, I did not catalog his recommendations. The voluminous extent of his writing and speeches made it impractical to do so. Additionally, by his exclusion, the remaining proposals share the condition that they were generated in the United States.

The Committee of Ten (1894)

The Report of the Committee of Ten was, by most accounts, a seminal work in the history of American Education. According to Dexter (1906), it "was officially contributed to by a larger number of persons than any other document of a similar character in the whole history of education, and persons, too, than whom there are none better fitted for the work in our country or any other" (p. 254). Despite the importance of the report in general terms, the intent was primarily to standardize the curriculum of schools (both elementary and secondary). The subcommittees were to address eleven questions, mostly dealing with the allocation of school time for mathematics, which topics should be covered, and recommendations for college entrance requirements for the subject.[1] The Conference did not make specific proposals about the treatment of function; but did recommend, "Illustrations and problems should, so far as possible, be drawn from familiar objects" (NEA, 1894, p. 105). Thus, for the purposes of this study, they recommended:

1. Problems – Familiar (contextualized).

[1] For a complete list of the questions of the committee, see NEA, 1894, pp.6-7

Committee of the Chicago Section of the American Mathematical Society (1899)

In 1899, the Committee of the Chicago Section of the American Mathematical Society prepared a report addressing the role of mathematics in secondary schools. Moore (1903) identified the report as "valuable work…in the formulation of standard curricula for high school and academies" (p. 408). As with the Report from the Committee of Ten, there is no specific reference to the teaching of function, but the committee did make recommendations that are important to my study. They recommended that arithmetic and geometry should be used as a source of illustration in algebra, "the pupil should be taught to test the accuracy of his results by applying a check whenever this is possible" (Young, et.al, 1899, p. 137), and "*not learning proofs, but proving* should be the pupil's principal activity in the study of mathematics" (p. 138)[2]. I summarize these recommendations as:

1. Problems – From arithmetic and geometry.
2. Operations – Proof should be emphasized.
3. Controls – Checks should be performed whenever possible.

Eliakim Hastings Moore (1903 and 1906)

According to Roberts (2001), "A strong claim can be made for Moore as a primary initiator of the use of graph paper in the American mathematical curriculum…. It is also likely that he paved the way for making the function concept more central to instruction, although widespread adoption would prove slow" (p. 695). I examined two of his works for recommendations pertinent to my research.

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[2] Italics in the original.

The first was his retiring presidential address to the American Mathematical Society in 1902. This speech was viewed by many as a pivotal speech in the mathematics education movement (see, for example, MAA, 1923). Though critical of Moore's decision to discuss matters of education in a speech before the AMS, D. E. Smith (1905) asserted, "Certain it is that the results of the presidential address have been such as to abundantly justify the assertion of some of the minority who heard it, that it would be much more epoch-making than most of its predecessors" (p. 206).

Moore's speech, though not containing specific mention of the function concept, did include pedagogical recommendations based on the notion of function. He emphasized the importance of graphical representations stating "they must enter largely into all the mathematics of the grades" (p. 409). He also spoke of the need to be able to interpret graphical representations, so that students might "know, for example, what concrete meaning attaches to the fact that a graph curve at a certain point is going up or is horizontal" (p. 409). Further, he called for the integration of the different fields of mathematics and of the pure and applied mathematics.

Moore's second work that I examined was "The Cross-Section Paper as a Mathematical Instrument," printed in *The School Review*, 1906. He began the article the same way that closed his presidential address, with a call for the correlation of arithmetic, algebra, and geometry and the unification of pure and applied mathematics. He calls for "the systematic use of cross-section [graph] paper as a unifying element in mathematics" (pp. 317-318) and ultimately proposed to "*Canonize the Cross-Section Paper*" (p. 338)[3].

[3] Italics in the original.

He argued that the use of cross-section [graph] paper leads directly to the concept of functionality. To Moore (1906), functionality was "*the relation or (mathematical) law of connection between two or more quantities or number subjects to simultaneous and interdependent continuous variation*" (p. 318)[4]. In addition to the call for graphical representations, Moore advocated "problems of practical and scientific nature" (p. 318-319). He dedicated the majority of the rest of his paper to explanations of nomography, linkage diagrams, and elimination tables. From these two sources, I summarize the recommendations of Moore as follows:

1. Problems – Practical and scientific problems.
2. Representations – Graphical.
3. Controls – Nomographic.

The Reorganization of Mathematics in Secondary Education (1923)

The National Committee on Mathematical Requirements (NCMR) published its report on secondary mathematics in 1923. The report was initiated in 1916 with the appointment of six professors to the committee. The committee was "instructed to add to its membership so as to secure adequate representation of secondary school interests" (MAA, 1923, p. vii). The committee added representatives of the three associations for the teachers of mathematics, three high school teachers, and the commissioner of Secondary Education from Sacramento, California. According to Butler (1951), "this report was destined to have what was probably a more powerful effect in shaping the course of mathematical education than any other report published either before or since that time" (p. 90). This sentiment is echoed by Jones and Coxford (1970), who identified the committee as "the most influential committee

[4] Italics in the original.

dealing with the teaching of mathematics appointed in the United States prior to the recent (1956-1959) Commission on Mathematics" (p. 40).

In contrast to some of the previous reports detailed, this report provided explicit recommendations about the treatment to the function concept in a ten-page chapter devoted to the topic. The committee cautioned against function notation and even use of the term function. "Indeed, it is entirely safe to say that the word 'function' had best not be used at all in the early courses. What is desired is that the idea of relationship or dependence between variable quantities be imparted to the pupil by the examination of numerous concrete instances of such relationships" (MAA, 1923, p. 64).

This stance is problematic for my research, because the exercises that I analyzed were found in sections of the textbooks that explicitly addressed function (see chapter 3). Although encouraging a more informal presentation of function, the committee made recommendations about the use of function, the representation systems that should be used, and the importance of verifying results.

The committee advocated that the problems relate mathematics to other courses "especially … courses in the sciences" (p. 28). And they promoted the connection between multiple representations for functions, arguing, "The idea that the three concepts, tables, graphs, and algebraic formulas, are all representations of the same kind of connection between quantities, and that we may start in some instances with any of the three, is a most valuable addition to the student's mental equipment" (p. 68). As well, the committee encouraged the use of control structures, stating "the ideals of accuracy and of self-reliance and the necessity of checking all numerical results should be emphasized" (p. 28). In short, their suggestions were:

1. Treatment – Function as a unifying element for secondary mathematics.
2. Problems – Increase in contextualized problems, specifically science examples.
3. Representations – Tabular, graphical, and symbolic.
4. Controls – Checks for all numerical results.

Mathematics in General Education (1940)

In 1932 the Progressive Education Association established a commission to examine the problems of secondary education. One of the committees was to deal specifically with mathematics. As a general framework, the committee held the view that education was to provide students with the personal characteristics essential to democratic living and address the basic educational needs. These needs, also called the *four basic aspects of living* were (1) Personal living, (2) Immediate personal-social relationships, (3) Social-civic relationships, and (4) Economic relationships (PEA, 1940, p. 20). The committee advocated that the mathematics being taught should be the mathematics that students were going to use. In this regard, they were not encouraging the use of real-life contexts for problems, they were advocating for real-life problems.

The committee found the concept of function to be sufficiently important to devote an entire chapter to it. However, they also felt the need to justify its inclusion. They acknowledged that the chapter was "more definitely mathematical than most of the preceding chapters" (p. 139) but argued that a study of functional relationships, though mathematical, was still important in general education.

The commission defined function as "any determined correspondence from one set to another, such that each object in the first set corresponds to a determined object in the second" (p. 142). At the end of the chapter they reiterate this definition by calling function a special sort of relation. Additionally, they caution against confining the study of function to just numerical cases. Instead they encouraged using examples from non-mathematical fields to illustrate the function concept, like capital cities as a function of state and performances at a specific theater as a function of time (pp. 161-163).

The report emphasized the importance of domain, stating "the identification of the domain is as much a part of the definition of a function as is the statement of the formula" (p. 143). Domain and range were explained and examples of functions with restricted domains and examples of piece-wise functions were presented. The report advocated that student be able to recognize four specific types of functions (linear functions, quadratic functions, exponential functions, and periodic or sinusoidal functions).

The final attribute of the report that relates directly to my work is the role of checking answers. The committee contended, "The student should be familiar with routine checking devices. Short-cuts, visual checks, scale drawings, slide-rule estimates, substitution of purported answers, applications of numerical data, examination of limiting cases, innumerable methods for testing and retesting each stage of his work should be part of his habitual practice" (p. 185). They argued that the less traditional the problem, the greater the need for checking. For the purposes of this study, the committee made the following recommendations:

1. Definition – A special type of relation, a determined correspondence from one set to another, such that each object in the first set corresponds to a determined object in the second.
2. Problems – Non-mathematical situations and situations that are important for the students outside school (or after schooling is complete).
3. Operations – The distinction between relation and function, identification of domain and range.
4. Representations – graphical, verbal.
5. Controls – innumerable methods of testing all stages of a response.

Report of the Commission on Mathematics (1959)

According to Fey (1978), the Report of the Commission on Mathematics, prepared for the College Entrance Examination Board "provided strong motivation and specific guidelines for a succeeding decade of vigorous research, curriculum development, and school innovation designed to bring major change in the form and substance of United States school mathematics instruction" (p. 339). The report was clearly influential in its day.

Despite its influence, the report had its critics. For example, Buck (1970) criticized the commission's treatment of function saying that it was not thorough (it was "confined to about a page" [p. 255]) and that it subsumed function to "emphasize the importance of set" (p.255). Buck seems to be referring to the report itself. But the commission also prepared appendices for the report which provided further explanations. There were three chapters in the appendices dedicated to the function concept, and the recommendations included in this study are those found in the appendices.

The commission provided the following definition of function: "A function in U is a set of ordered pairs belonging to $U \times U$, and having for each x at most one y" (CEEB, 1959, p. 17). They emphasize that relations need not be expressible in formulas, but all illustrations provided in the report are expressible as formulas. The report utilized numerical, tabular, or graphical representations in their presentation of relations and functions, but all of their examples were quantitative and none were drawn from non-mathematical fields.

The appendices also contain a chapter on linear and quadratic functions. In this chapter, they defined the terms *rise* and *run*, examined the role of m and b in the equation $y = mx + b$, and used those results to introduce a quick method for sketching linear equations (by finding the y-intercept and using the slope to place one additional point). The report did not address the role of verifying results in the chapters dealing with function.

1. Definition – A function in U is a set of ordered pairs belonging to $U \times U$, and having for each x at most one y.
2. Operations – identifying if a relation is a function, finding slope.
3. Representations – Symbolic, graphical, tabular.

Curriculum and Evaluation Standards (1989)

The final report that I examined for this study was the *Curriculum and Evaluation Standards* prepared for the National Council of Teachers of Mathematics (NCTM). This commission proposed reforms for many aspects of mathematics education for the entire K-12 curriculum. Consistent with the focus of this research, I only analyzed the recommendations intended for the treatment of function in grades 9-12. The report also contained recommendations for the treatment of function in grades 5-8 (see Standard 8 pp. 98-101), and introductory algebra could be taught in that grade band. But I felt that the recommendations for high school were most appropriate for my research.

It is important to note that the commission held as an underlying assumption that "scientific calculators with graphing capabilities will be available to all students at all times" (NCTM, 1989, p. 124). They advocated the use of functions to model real-world phenomena and the recognition that a variety of problem situations can be modeled with the same types of functions. They promoted the use of tables, verbal rules, equations, and graphs to represent functions and the ability to translate between these representations (including piece-wise functions).

For all students they recommended the use of "a graphing utility to investigate how the graph of $y = af(bx + c) + d$ is related to the graph of $y = f(x)$ for various changes in the parameters *a*, *b*, *c*, and *d*" (p. 155), and informal exploration of the concept of inverse (including its graphical representation). And for college-intending students, they additionally recommended fitting curves (for data presented in tables or found experimentally), exploring composite functions, and investigating informally surfaces generated by functions of two variables. Their recommendations, therefore, included:

1. Problems – modeling real world phenomena.
2. Operations – Comparison without computing.
3. Representations – Graphical, tabular, symbolic, and verbal rules.

A Brief History of the Function Concept

The notion of function has been extensively studied and its history well documented. Tables of values produced by the Babylonians some 4000 years ago provide the earliest known examples of what we now think of as functions. "Their use of tables like the one for $n^3 + n^2$, $n = 1, 2, \ldots, 30$, [suggested] the definition that a function is a table or correspondence (between n in the left column and $n^3 + n^2$ in the right column)" (Bell, 1945, cited by Kennedy & Ragan, 1969).

According to Kleiner (1989), a two hundred year span (circa 1450-1650) witnessed four developments that allowed for the next extension of the concept of function. The extension of the number system to include all real numbers, the development of symbolic algebra (e.g., the works of Viéte), the connection between algebra and geometry (e.g., Descartes), and the nature of the problems trying to be solved (problems of motion and curves and the works of Galileo, Kepler, and others) gave rise to a geometric understanding of function (p. 283).

The first formal definition of function was presented by Johann Bernoulli in 1718. He characterized a "function of a variable [as] a quantity composed in any manner whatever of this variable and of constants" (p. 721, Rüthing, cited by Kleiner, 1989, p. 284). Euler later modified this definition (1748) by replacing the term *quantity* with *analytic expression* (Youschkevitch, 1976, p. 61). Through debates surrounding the solution to problems about vibrating strings, this concept was extended to include analytic expressions that were defined differently on different intervals and curves that could be drawn but not a combination of analytic expressions. Function was now being considered algebraically.

In the early 1800s, Dirichlet revised the work of Fourier to "make it more mathematically respectable" (Kleiner, 1989, p. 290). Accompanying this work was the formulation of a new definition of function in 1829. Dirichlet stated that

> y is a function of a variable x, defined on the interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established. (Luzin, 1998b, p. 264)

The importance of this episode is not in the wording of the definition, but rather in the interpretation of those words. Many mathematicians, including Fourier (and as far back as Euler), had suggested that the correspondence that defined the function could be arbitrary. However, these mathematicians still thought of those functions as analytic expressions or curves. The Dirichlet function, $D(x)=\begin{cases} c, & x \text{ is rational} \\ d, & x \text{ is irrational} \end{cases}$ provided the first explicit example of a function that was not a curve and was not given by an analytic expression (Kleiner, 1989, pp. 291-292). This example freed the concept of function from being considered just an analytic expression and allowed it, for the first time, to be viewed as an arbitrary correspondence between numbers.

The concept of function was further extended to include arbitrary correspondences among sets in the second quarter of the twentieth century. This development is generally attributed to Bourbaki, who, in 1939 proposed the following definition of function:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a *functional relation* in y if, for all $x \in$ E, there exists a unique y $\in$ F which is in the given relation with x.
> We give the name of *function* to the operation which in this way associates with every element $x \in$ E the element y $\in$ F which is in the given relation with x; y is said to be the *value* of the function at the element x, and the function is said to be *determined* by the given functional relation. Two equivalent functional relations determine the *same* function. (Kleiner, 1989, p. 299)

With this definition, the extension of function to relations between non-numeric sets was established. Some accounts of the history of function treat this as the last step (Balacheff & Gaudin, 2002; Kennedy & Ragan, 1969). For those histories, the evolution of the concept of function is generally agreed to be function as table, function as curve (geometric), function as an algebraic expression (a rule or formula), and function as a relation (correspondence). Identifying the Bourbaki definition of function as the final development may stem from the fact that it is the last development to make its way into wide spread use in education.

There are, however, more recent developments in the concept of function. One of these is the use of category theory to define function as “an 'association' from an 'object' A to another 'object' B. The 'objects' A and B need not have any elements ….. In fact, the objects A and B can be entirely dispensed with” (Kleiner, 1989, p. 299). Kleiner describes this next step in the abstraction of the function concept as being “generalized out of existence” (p. 300). The inclusion of this aspect of the history of function is to illustrate that there are still developments being made in the mathematics discipline, and these developments may (or may not) be seen in the treatment of function in schools.[5]

A Hypothetical Enterprise

The textbook is just one of the sources of learning in a classroom and, even if it is followed to the letter, expresses the intended curriculum. I am not trying to identify trends in how the concept of function was taught (the enacted curriculum) or the concept of function that students actually learned (the experiences or learned curriculum). As Mesa (2000) notes, “an analysis of textbook content becomes in

[5] For a more thorough treatment of the history of function, see Youschkevitch, 1976, Kleiner, 1989, or Cooney & Wilson, 1993 or Luzin 1998a/Luzin 1998b.

some ways a hypothetical enterprise" (p. 7). Consequently, there are certain assumptions implicit in this study that deserve mention.

Assumption 1: All of the problems will be assigned. Of the five textbooks that I analyzed, four of them (Wentworth, 1906; Hart, 1935; Welchons & Krickenberger, 1949; and Dolciani, 1962) recommended not assigning all exercises. But, as there is no way to for me to know which exercises would be assigned, I had to give them equal weight. Again, this suggests the distinction between the intended curriculum (the focus of this study) and the enacted curriculum (which, for pragmatic reasons, cannot be the focus of my study).

Assumption 2: The definitions in the textbooks will be the ones used and the examples in the text will be followed. Stodolsky (1989) cites a study by Freeman and Porter which examined teachers' use of different sections of textbooks and found that when lessons were being used, 73.5% of the teacher-directed portions (e.g., introductory and developmental sections) were used (p. 167). This indicates that it is unlikely that all of the introductory material would be used. However, the only guide that I have for the methods that students were to employ in solving problems is to use the examples in the textbook.

Assumption 3: The whole textbook will get covered. The same study by Freeman and Porter found that teachers used 58% of the chapters in their texts and did not reach 29% of the chapters found at the end of the text (as cited in Stodolsky, 1989, p. 167). This is troubling because function is often treated at the end of Algebra One textbooks. But, again, without a clear way to decide what to omit, I chose to include all exercises that were intended for the first year of algebra study.

Chapter 3: Methodology

Textbook Selection

The Decision to use Textbooks

The decision to use textbooks as the primary source of data for this research project was dictated by the nature of the study that I wanted to undertake. As I was looking at the treatment of function over an extended period time, it was necessary to have a window into the last one hundred years of mathematics teaching. The only access that I had to the presentation of mathematics at the turn of the twentieth century was in the form of the textbooks that were used. Teacher guides or teacher's editions were generally not available for these early texts, and no reliable and comprehensive record of how teachers used these textbooks exists.

It is interesting to note that there is relatively limited research about mathematics textbooks; this is especially true of secondary education textbooks. There is also limited research about the connection between mathematics textbooks and mathematical content (Mesa, 2000; Conklin, 2004). This is all the more surprising considering the centrality of textbooks in education and that they "determine what school mathematics is (in a similar way to syllabuses and examinations)" (p. 93). (Dörfler & McLone, 1986, as cited in Mesa, 2004).

Stodolsky (1989), identifies three ways of analyzing textbooks: i) topics covered, ii) actual material in the textbooks, and iii) activities suggested in teachers' editions. I was concerned only with the treatment of function concept and was therefore not interested in other topic coverage. Additionally, as already noted, teachers' editions did not exist for the earliest texts in my study. So the actual textbook material (definitions, examples, problems, etc.) constitutes the primary corpus of my data. When I could access teachers' editions[6], I examined them to get a clearer picture of how the content was intended to be taught. These manuals were consulted to attempt to explain trends found in the data. They were not included in the original analysis.

How the Texts were Chosen

Since one of the goals of my research was to investigate the interaction that high school students would have with the functions concept, it was imperative that the texts that were chosen were intended for high school students. This distinction was particularly important for the earlier texts where the Algebra texts were as likely (or more likely) to be intended for collegiate use as they were for high school use.

The textbooks that were used for this analysis were the recommendations found in Eileen Donoghue's chapter "Algebra and Geometry Textbooks in Twentieth-Century America" in Stanic and Kilpatrick's *A History of School Mathematics: Volume One*. As the title suggests, the textbooks span the century from 1900 to 2000. The century was split into time spans of two decades, and a book deemed to be characteristic of the texts of each time period was selected. Though the only criterion for inclusion in my study was identification by Donoghue, there is additional evidence that the chosen textbooks were influential, widely circulated, or both.

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[6] UCSMP, 1990; Dolciani, 1962; and Hart ,1943 (A teacher's manual for the second edition [reprint of the first editon])

From the first twenty-year span, George A. ("Bull") Wentworth's *Elementary Algebra* (1906) was chosen. According to Donoghue (2003), "the Wentworth series and the subsequent Wentworth-Smith series were the dominant high school textbooks in the early decades of the twentieth century" (p. 331).

Châteauneuf (1930) produced a bibliography of all elementary Algebra texts copyrighted or registered from 1818 to 1928 (pp. 163-186). Wentworth is identified as writing six different texts from 1881 to 1906 (the year that he died). According to the Châteauneuf's bibliography, there were only 97 elementary Algebra texts written between 1880 and 1910. The only other authors to write more than three were Edward Brooks (four), William J. Milne (five) and Webster Wells (five). Not only did Wentworth write a number of texts, there is reason to believe that his texts were widely circulated as well. In his 1898 *New School Algebra*, Wentworth (writing in the third person) claimed "The remarkable favor with which his [Wentworth's] other Algebras have been received is shown by the fact that nearly a million copies have already been sold, and the sales continue to increase from year to year" (p. iv). The influence and widespread distribution of Wentworth's texts can hardly be questioned. Wentworth identifies himself as a "Professor of Mathematics at Phillips Exeter Academy" in his first algebra text (1881), but identifies himself as "an author of a Series of Textbooks in Mathematics" in his remaining texts.

For the years from 1921 to 1940, Donoghue identifies Walter W. Hart's *Progressive High School Algebra* (1935) as the representative text. Though this was the first edition of Progressive High School Algebra, Hart had already established himself as a textbook author at this time. The textbook was published by D.C. Heath as part of the Wells-Hart Mathematics Series.

By 1935, the Wells-Hart Mathematics Series had seen six different first year Algebra texts. This is in addition to the five books that Wells had written as a single author between 1885 and 1908 (Châteauneuf, pp. 163-186). Hart was an associate professor in the School of Education at the University of Wisconsin and Wells was a mathematics professor at Massachusetts Institute of Technology. Donoghue (2003) notes that despite Wells' death in 1916, "the publisher, D.C. Heath, retained his name as coauthor with Hart on subsequent versions of textbooks that they had coauthored" (p. 341).

As with Wentworth, Wells and Wells-Hart were the most prolific elementary algebra authors of their respective period. From 1880-1928, Wells and Wells-Hart wrote nine of the 163 texts (5.5%). William Milne wrote eight and Fletcher Durell (with various coauthors) wrote seven texts. The only other author of more than five texts from that period was George A. Wentworth with six.

For the years from 1941 to 1960 Donoghue recommends Alvin M. Welchons's and William R. Krickenberger's *Algebra, Book One* (1949). Both Welchons and Krickenberger were teachers of mathematics at Arsenal Technical High School in Indianapolis. Though this text was a first edition, they had both been authoring algebra textbooks for more than twenty years. Their first algebra texts was *Bobbs-Merrill Algebra: Book One* which they coauthored (with Whitcraft) in 1927.

Donoghue identifies the 1965 edition of *Modern Algebra: Structures and Methods, Book One* by Mary P. Dolciani, Simon L. Berman and Julius Freilich as a representative text for the twenty-year span from 1960 to 1980. She asserts, "The Dociani series was prominent in the American textbook market into the 1970s" (2003, p. 366).

Dolciani was a contributing author for the School Mathematics Study Group (SMSG), and SMSG's *Algebra* (1960) was her first textbook. *Algebra for Problem Solving, Book One* (1952), which was coauthored with Elsie Parker, provided Berman and Freilich with their first authorship experience.

Houghton Mifflin first published *Modern Algebra: Structures and Methods, Book One*, in 1962. The series continued (always featuring Dolciani, but with various coauthors) until the publication of *Algebra Structure and Method, Book One* in 1990. Though Donoghue identifies the 1965 edition of *Modern Algebra*, she does not distinguish it from other editions. I used the 1962 Teacher's Edition in this study as I had easy access to it.

In the final twenty-year span of the twentieth century, Donoghue recommends the University of Chicago School Mathematics Project (UCSMP) *Algebra*. UCSMP's *Algebra* was one of a six book series intended for grades 7 to 12. In 1989 the National Council of Teachers of Mathematics (NCTM) published their *Curriculum and Evaluation Standards*, and the authors of the UCSMP series described it as "the first full mathematics curriculum to implement the recommendations of the NCTM Standards Committee" (UCSMP, 1990, p. T5). As well, UCSMP (1998-99, p. 6) claimed that for the 1998-1999 school year "over three million students [used] UCSMP elementary and secondary materials" (as cited in Donoghue, 2003, p. 381). The popularity of the UCSMP curricula in the late 1990's does not mean that it was as influential in the early 1990's, but the importance of the series, and the decision to consider it representative of the texts of the 1980's and 1990's seems justified.

The analysis was limited to just these five textbooks from the twentieth century. The decision not to extend the analysis back into the 1800's was due in part to the low high school enrollment levels in the nineteenth century. According to James and Tyack (1983) only 6.7% of all 14- to 17-year olds were enrolled in high schools in 1890 (p. 401). In addition to relatively small percentage of the population that would have been exposed to high school algebra before the turn of the twentieth century, most accounts of the history of the teaching of function in school mathematics identify the turn of the twentieth century as the start of the movement (see, for example, Hamley 1934).

The decision not to extend the analysis to the present day was primarily a result of need for consistency. As the other texts were chosen to represent a twenty-year period, I was hesitant to pick a text from the abbreviated time span from 2000 to 2008.

How the Sections were Chosen

The first criterion for retaining a section was that it was intended for the first year of algebra study for high school students. The texts had already been selected in such a way that I knew they were intended for high school students. The distinction that it was intended for the first year of study proved more difficult. The three most recent texts (UCSMP, Dolciani, Berman, & Frielich [Dolciani], and Welchons & Krickenberger) were labeled as first year texts. Hart's *Progressive High School*

Algebra (1935) was intended for a three semester course, but the preface stated that Chapters I to XII were intended for the first year (p. iii).

All of this is consistent with the Graham's observation (1954) that the secondary school algebra curriculum had, by the 1920s, generally been separated into introductory topics intended for the first year of study and more advanced topics for a third semester or a second full year (as cited in Donoghue, 2003, p. 341).

For Wentworth's text I selected the treatment of quadratics as the indicator of what the first year of algebra was to cover, as this seems to be a distinction that was often made (for example, among colleges for entrance requirements).

According to Hanus (1896), "Nearly all colleges require a knowledge of Algebra through Quadratics" (p. 535). He goes on to state that that some colleges require additional coverage (e.g. logarithms, the binomial theorem, and permutations and combinations), but notes that these additional requirements are not common. He concludes his section on college entrance requirements by stating, "the phrase *Elementary Algebra through Quadratics* practically covers the admission requirements in Elementary Algebra for nearly all colleges whatever the text-book used may be" (pp. 535-536)[7].

I have chosen to only examine Wentworth's explicit treatment of function in the chapter on graphing (Chapter XIII), as this proceeds the treatment of quadratics. Consequently, neither his treatment of function in the chapter on ratio, proportion, and variation (Chapter XXII) nor his treatment of function in the chapter on variable and limits (Chapter XXV) will be included.

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[7] Formatting in the original.

I decided to focus my analysis on the parts of the texts that explicitly referenced function. Therefore, a chapter was retained if the title of the chapter contained the word *function*, function was defined in the chapter, or the word *function* was used as a way of classifying problems or curves. There were two basic reasons for the decision to use these restrictions.

The first is a pragmatic reason. As noted by Hedrick (1922), "functional relations – that is, relations between quantities – will occur on every page of every book on mathematics unless we suppress them" (p. 195). The variety of settings in which functional thinking can be addressed is practically limitless. Wilson and Cooney paraphrase Breslich (1928) as saying "equations, polynomials, ratio, proportion and variation, relationships stated in words, relationships in tabular representations of numerical facts (including tables used in graphic representations), and relationships represented by formulas all provide opportunities for emphasis in functional thinking" (cited in Wilson & Cooney, 1993, p. 137). This list seems to cover the majority of Algebra One topics.

The pervasive nature of the function concept is captured by Dreyfus and Eisenberg (1982), when they observe "it appears and reappears like a thread throughout school mathematics from grade 1 (e.g., addition as a function from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$) to grade 12 (e.g., calculus)" (p. 361). Despite the fact that addition is a function, it seemed inappropriate to identify problems involving addition as part of the treatment of function unless the author was trying to use addition as a way of developing the function concept.

The second, and more important, reason was an attempt to more accurately represent the concept of function that the author(s) intended students to obtain from the use of his/their text. I felt that the only way that I could accurately identify where the author was talking about function was to see where he said he was talking about function. For me to decide where (and if) the author was talking about function seemed to undermine the focus of my research. In short, I did not want to use my 2008 understanding of function to decide when the author intended to be talking about function. Once the chapters were obtained, the definitions were noted and the problems contained in the chapters were analyzed.

The Coding System

In order to characterize the individual exercises in the textbooks, I first worked each of the problems. Careful attention was paid to the examples in the textbook so that I would perform the task in a way most similar to a student who had been using the text. The coding system that I employed was (in essence) the system developed by Mesa (2000). As already noted, she used Balacheff and Gaudin's (2002) notion of conception (a quadruple consisting of the problem, operations, representation system, and control structure) to analyze exercises in mathematics textbooks. Some modifications to Mesa's (2000) coding system were needed for my analysis, two codes were added, and 11 codes were not needed[8]. Ultimately, the

[8] Omitted operation codes were: Find average, Find composite of two functions, Find inverse relation, Function is <certain characteristic>, Find percentage or number, Give period, Operation is <certain characteristic>, Trace identity line, and Trace regression line.
Omitted representation system codes were: Arrow diagram and Number line.

coding scheme I used contained 11 problem types, 27 operations, 8 representation systems, and 9 control structures. The list of these codes is provided in Table 1. A working definition for some of the codes is provided in the examples that follow, and a more thorough treatment of all of the codes can be found in Appendix A.

P (Set of Problems)	R (Set of Operations)	L (Representation System)	Σ (Control Structure)
Use of the function	All of the operations that were needed to solve the task	Representations that were needed to solve the task	Structures for the student to verify adequacy and correctness
-Cause and effect relationship -Constructed relationship -Data reduction relation -Geometric relation -Graph defined relation -Ordered pair -Pattern relation -Proportional relation -Rule relation -Time relation -Other	-Calculate -Compute with different units -Change between symbolic forms -Carry out experiment -Comparison without calculating -Determine domain or range -Describe shape in a graph -Determine type of relationship -Find element of the range or domain of a relationship -Find non-explicit characteristic -Find relation between two (sets of) numbers -Find slope -Fill table -Give cardinal -Give definition -Give example or counterexample -Give unit -List the elements of a relation -Locate points on a graph -Measure -Name point on axis -Name variables -Prove -Relation is Function? -Read point from graph -Apply trigonometric identity or formula -Use proportionality within entries	-Calculator/Computer -Graph (two axes) -Numerical -Pictorial -Semi-symbolic -Symbolic -Tabular -Verbal	-Compare with previous examples -Continuity is assumed -Double check -Look for likely or unlikely results -More than one point -Use alternative representation -Use calculator or computer -Use checkpoints -Use given information

Table 1. The Codes Used in Categorizing Problems

Examples of the Coding System

This section will include three exercises (each taken from a different textbook). The examples are included to illustrate the coding system and to foreshadow some of the results of the study by showing that solution methods have changed. Each exercise was assigned a single use (or problem) code and as many codes as were necessary for operations, representations, and control structures. For purposes of illustration, a semicolon will be used to separate each element of the conception and a comma to separate the codes within an element. The codes were recorded in an Excel spreadsheet to manage the data and facilitate analysis.

> Example 1: Hart (1935) p. 190, Problem #4
>
> Draw the graph by the short method taught in §146 above.
>
> 4. $x + y = 8$

This particular problem is presented as a symbolic rule (it is given symbolically without any context). To solve this problem, the student is supposed to choose three points for *x* and find the resulting value of *y* (without rearranging the equation to solve for *y*). According to the example, these points may be written in a table or graphed directly. Two of the points should be located on the graph and a straight line drawn between them. Then the third point should be located on the graph. The process is confirmed if this point falls on the line. As with most exercises, the results can be verified by doing the problem again. This problem was coded RR; FIP, FT, LPCP, NPOX; G, T, S; UCP, DC.

RR	*Rule relation* - Used to code content in which an input is transformed by certain procedure to obtain an output and in which a context is not provided[9].
FIP, FT, LPCP, NPOX	*Find element of the range or of the domain of a relationship* – The student needs to find in the range of the relation a value (or element) associated with a given element of the domain, or find a domain element associated with a range element, or both. *Fill table* – The student needs to either create or complete a partially filled table of values. If a relation is given or asked for (via any representation), this operation has to go together with FIP because the student will need to find images and pre-images through a relation in order to fill out the table. FT goes by itself when it is a step inside a data collection process: the relation is to be determined afterwards, using the information in the table. *Locate points in graph* – The student needs to locate points in a graph; a graph can be any of the types defined in the section on representations. Whenever a Cartesian plane is involved, the code must be applied if both elements of the ordered pair are known and need to be located. If that is not the case (e.g., the time at which the temperature is 50°C), then use the operation FIP. LPCP always requires NPOX when a Cartesian plane is involved. *Name point on axis* – The student needs to determine a number on an axis, either by reading it from the scale given or by doing an interpolation. It may or may not require a calculation (by means of adding a certain number a needed number of times).
G, S, T	*Graph in two axes* – The task uses a Cartesian plane, frequency diagram, histogram, broken line (time series), or scatter-plot. *Symbolic* - The task uses expressions with only symbols. This includes arithmetical notation, sets $\left(\text{e.g.}, \{x \mid x > 0, x \in \boldsymbol{N}\}\right)$, ordered pairs, equations $\left(\text{e.g.}, f(x) = x + 1, y = x + 1\right)$, mappings $\left(f : x \to x + 1\right)$, or intervals. *Tabular* – The task uses a table. The table can be given, asked for, or a requisite for the process of keeping track of the entries.
UCP, DC	*Double check* – The student either repeats the process used to obtain the answer (e.g., relocates points in the Cartesian plane) or reverses the process to get something that is given in the statement of the task (undoes the sequence of operations). *Use checkpoints* – The statement of the task offers answers to previous questions, warns the student about what is not a correct answer, suggests looking at the following tasks, or suggests checking with a partner who is doing the same task. Or the task is checked by solving it a different way or additional data is used to verify that the formula is correct.

This example also serves to illustrate the importance of using the textbook as a guide in the coding system. It is equally plausible that the student could solve the equation $x + y = 8$ by first solving the equation for *y* and identifying the *y*-intercept of the equation and locating that point on the graph. The slope could be used to find one (or more) other points and the line drawn. This approach would utilize different operations, different representations, and different controls. Whenever possible, the textbook was used as a guide for solving the exercises.

[9] The wording for the descriptions of these codes is identical to that of Mesa (2000) except for the codes that I modified (as will be articulated in the next section).

Example 2: Welchons and Krickenberger (1949) p. 492, Problem #3

Here is a diagram of a bicycle pump filled with 20 cubic inches of air. The air pressure is 14.6 pounds per square inch.

[The text contains a picture of a pump here]

The pressure and volume of this of the air inside the pump obey the law expressed by the formula $pv = 292$ when no air is allowed to escape. Using the formula, copy and complete the following table:

v	1	2	4	8	10	16	50	100
p	?	?	?	?	?	?	?	?

Tell whether *p* and *v* vary directly or inversely.

This problem deals with a physical phenomenon from a cause-and-effect perspective. The student is to use the formula to find values of *p* and put them in the table. Upon completing the table, the student is to determine the type of relationship that exists between the two quantities. This question utilizes five representations: pictorial (by including the picture of the pump), numeric (for example, answering "what times 1 equals 292?" to fill entries in the table), tabular, symbolic (in presenting the equation $pv = 292$ and likely for the student to solve for some of the entries like $50p = 292$), and verbal (in describing the relationship). The student can tell that he or she is correct by redoing the calculations, reasoning that the behavior exhibited by the first few entries would continue for the remaining entries, and/or by looking at other tables that had been generated by similar equations earlier in the chapter. The problem was coded CER; DTR, FIP, FT; N, P, S, T, V; CON, CPE, DC, LFLUR.

CER	*Cause/effect relationship* – Used to code content that refers to physical phenomena other than time related and in which the behavior of one variable is an effect of the behavior of the other (it is a directional relationship)
DTR, FIP, FT	*Determine type of relationship* – The student needs to determine whether the relation between two sets of numbers exhibits direct, inverse, joint, or combined variation, is linear or nonlinear, or whether there is no relation. *Find element of the range or of the domain of a relationship* – See Example 1. *Fill table* – See Example 1.
N, P, S, T, V	*Numerical* – The task does not require operations with number s any symbols; instead, it requires numbers. *Pictorial* – The task uses drawings of machines, maps, geometrical shapes and figures, photos, or pictograms (frequency diagram where the *y*-axis is not present), pies (only one variable is sketched) or any other kind of drawing. *Symbolic* - See Example 1. *Tabular* – See Example 1. *Verbal* – The task uses a description of a situation using natural language (e.g., a pound of apples costs 30 cents) or requires the student to interpret a situation with natural language.
CON, CPE, DC, LFLUR	*Continuity is assumed* – The student uses the fact that a function is continuous to determine the likeliness or unlikeness of a result. Also the situation is such that if continuity is not assumed, the problem cannot be solved (e.g., the student has a table of values—of a polynomial function—and he or she needs to find the image of a value that is not in the table). *Compare previous examples or problems* – The student has performed similar operations and can use them in this particular situation. *Double check* – See Example 1. *Look for likely or unlikely results* – The student can use indicators in the statement of the task (e.g., the student obtains a number too big or too small for a given scale in a Cartesian plane, or he or she is getting decimals or negative numbers when whole or positive numbers are expected, or a set of points in a Cartesian plane are not aligned on a line) or can use previous knowledge (e.g., the sides of a square have the same length).

The problem was not coded UPWE (use proportionality within entries). The method of finding the values of *p* does not depend on the proportionality of the entries, so the proportionality of the variables was inconsequential to the problem. The exercise, with only minor modifications (i.e. changing the context), could be presented without even being proportional (see Welchons & Krickenberger p. 491 problem #2).

Example 3: UCSMP (1990) p. 641, Problem #6

In 3–6, two first coordinates of the relation are given. **a.** Find all corresponding second coordinates. **b.** Is the relation a function?

6. Latitude-high temperature relation on page 639; latitude = 6, latitude = 56.

This exercise refers to a relation that was presented graphically in the text. The average high temperature in April (in degrees Fahrenheit) was plotted against the latitude (in degrees north) for various cities. Each point on the graph was labeled with a city's name. The relation was not a function due to the fact that two cities at the same latitude (56°N) had different high temperatures.

This problem type for this exercise was a constructed relation (latitude versus mean high temperature). To solve the problem, the student needed to identify elements in the range for two given values of the domain. Since no formula is given, the student must read the points from the graph provided. The scale for latitude is every 5° and the scale for temperature is every 10°. Since the values asked for in the problem are 6 and 56 (i.e. not multiples of 5) the student must also calculate where these points are. Then, using the definition of function, the student must determine if the graph represents a function. The student can compare results with earlier examples (the text says that this relation is not a function on page 639) or perform the "vertical line test" to determine the correctness of a solution. This problem was coded CR; CALC, FIP, NPOX, RIF; G, V; CPE, MT1P.

CR	*Constructed relationship* – Used to code content that refers to "real life" situations other than cause/effect, time, data reduction, and geometrical. In these relations it is somehow arbitrary which variable is called dependent and which one independent. An interchange of the roles of the variables produces equally valid—for the context—relationships.
CALC, FIP, NPOX, RIF	*Calculate* – The student needs to operate with numbers (e.g., add, subtract, multiply, divide, square a number, or find the square root, take log). *Find element of the range or of the domain of a relationship* – See Example 1. *Name point on axis* – See Example 1. *Relation is function?* – The student needs to determine if a relation is a function or not.
G, V	*Graph in two axes* – See Example 1. *Verbal* – See Example 2.
CPE, MT1P	*Compare previous examples or problems* – See Example 1. *More than one point (vertical line test)* – The student has to determine if an element of the domain of a relation has one and only one element assigned from the co-domain of the relation.

Modifications to the Coding System

For the purposes of this research project, the coding system used by Mesa (2000) needed slight modifications. Some codes needed to be clarified, others needed to be modified to include more (or different) cases, and two codes needed to be created. Most of the modifications seemed to stem from changes in the content of exercises across time.

There were three alterations for the codes about problem use. The code *Set of Ordered Pairs* (SOP) evokes the image of function as an abstract relation between two sets. To avoid that implication, this category was renamed simply *Ordered Pairs* (OP). This may seem like a trivial semantic change, but it is actually quite important. This change allowed for the classification of problems from early texts (i.e. Wentworth (1906) that dealt with graphing points in the Cartesian plane, but was not defining or presenting function as a set of ordered pairs.

The category *Direct Proportion/Proportional Relation* (DPPR) modified to include other types of proportion (i.e. inverse, joint, and combined variations), and was renamed *Proportional Relation* (PPR). And a category, *Other* (OTH) was necessitated to code exercises that addressed the nature of function without falling into one of the established categories. For example, "When no domain is given for a function, what can you assume about the domain?" (UCSMP, 1990, p. 654, #6)

There were four operation codes whose use in my research seemed to differ from the interpretations given by Mesa (2000). *Determine Type of Relation* (DTR) was extended to include joint and combined variation. *Give Definition* (GD) was expanded to include recitation of facts that are not definitions *per se*. The exercise asking about the assumed domain of a function if not domain is explicitly given acts to illustrate this change as well. *Name Variable* (NV) was only used if the student was asked to provide the variables that were used or instructed to name variables as part of the solution process. Finally, *Apply Trigonometric Identity* (TRIG) was used to indicate a step in the setup of a problem (e.g., identifying the ratio to use). Finding a specific value, like tan(15°), was labeled as *Finding Element of the Range or of the Domain of a Relationship* (FIP).

For my research, one additional representation system was necessary, *Calculator/Computer* (CC). Balacheff and Gaudin (2002) identify mathematical software and calculators as examples of representation systems (p. 7). And though I was hesitant to include this representation (as it could likely be used for many of the exercises), I chose to only code for it when the student was instructed to use a computer or calculator to obtain a result. This proved to be helpful for coding some of UCSMP's "Exploration" problems which often asked the students to use a graphing device. Consider the following example from UCSMP (1990):

> Let $f(x) = \dfrac{12}{x-a}$. Using an automatics grapher, graph the function from $x = -5$ to $x = 5$ when $a = 1$, $a = 2$, and $a = 3$. What do the graphs have in common? How are they different? (p. 646, #28)

The use of some graphing device (i.e. a calculator or computer) greatly facilitates the solution of this problem. Note that the graph of these functions are hyperbolas, and great care must be taken in graphing these functions if the role of *a* as a parameter is to be determined. It is indeed possible to graph these functions without a calculator, but the exercise presupposes that a calculator will be used. Again, this code was only used if the student was asked to use a calculator to *solve* the problem (the use of a calculator as a means of verification was coded CLC in the control codes, consistent with Mesa's (2000) usage).

Since almost all exercises use natural language the representation *Verbal* (V) was reserved for exercises that utilized natural language in both the presentation of the problem and the solution. In like fashion, due to the general dependence of mathematics on numbers, the representation *Numerical* (N) was generally only used in situations where symbols were not used. Therefore a problem asking to verify a proportion would be coded as Numerical, but a problem asking to solve a proportion $\left(\text{e.g.,} \dfrac{x}{36} = \dfrac{5}{9}\right)$ was coded as symbolic.

For the codes about the control structures, I felt the need to clarify three of the codes. *Use Check Point* (UCP) was written to include cases where a true check point could be used (e.g., verifying that a liner equation developed from two points works for a third point) and cases where the solution could obtained another way (e.g., solving a proportion then verifying that the fractions are equivalent). *Compare with Previous Examples or Problems* (CPE) was extended to include exercises where the textbook modeled the solution in an example (rather than only using it if multiple problems of the same type are asked).

Finally, for the code *Look for Likely or Unlikely Results* (LFLUR), I had to make decisions about what a student could reasonably be expected to anticipate as a result. So, for problems about inverse variation or direct variation the situations were deemed to be transparent enough that the student could predict if the value of the variable should increase or decrease (and in many cases estimate what that should be). However, for problems of joint variation or combined variation, it was supposed that these situations were sufficiently complex that it would not be obvious if the value of the variable should be greater or less than the original.

Chapter 4: Results

These results are guided by the following three research questions:

> *Research Question 1:* By comparing the concept of function that is fostered by representative textbooks from different eras of the twentieth century, is there evidence for change in the treatment of function across time?
>
> *Research Question 2:* How do changes in representation of function in textbooks correlate with recommendations for mathematics education?
>
> *Research Question 3*: How do changes in the texts correlate with the developments of the function concept in the mathematics discipline?

To address these questions, an accurate representation of the treatment of function in each of the textbooks must first be established. Following a characterization of the treatment of function in the texts, each of the research questions is addressed in turn.

Treatment of Function in the Textbooks

The characterization of the textbook's treatment of function found in this section is based on the six underlying questions given in the rationale section: How is function defined in each of the textbooks? Where is function explicitly treated in the textbook? And is it addressed in multiple sections or is it isolated? In what context are exercises about function framed? What operations are required to solve problems about functions? Which representations (graphical, tabular, symbolic, etc.) are needed to obtain results for tasks about functions? What controls are in place for students to determine the adequacy and accuracy of their responses? Due to the complexity of the coding system and sheer volume of the data (970 exercises were coded), only certain aspects of the treatment of function will be addressed in this section.

Wentworth's *Elementary Algebra* (1906)

Wentworth defined a function of a variable as "an expression that changes in value when the variable changes in value. In general, any expression that involves a variable is a function of that variable" (p. 199). Wentworth's only treatment of function before quadratics was found in his chapter on graphing.

It is important to note that Wentworth uses the term *expression* in his definition. By doing so, Wentworth is emphasizing the importance of symbolic representations of functions. In fact, 43 of the 47 exercises in the chapter (91%) involve symbolic representations.

He uses the terms *abscissa* and *ordinate*, but does not use the term *ordered pair*. His use of ordered pairs is strictly for plotting points (and lines). The presentation of function seems to focus on the dependence between the variable, and he does define *dependent* and *independent variable*. The textbook contains no reference to *domain* or *range* of a function. Additionally, despite the chapter dealing with the graphical representation of linear equations and systems of linear equations, neither *slope* nor *y-intercept* is addressed.

All of the exercises dealing with function are framed in a mathematical context. Of the 47 problems in the chapter, 37 (79%) were classified as *Rule Relation* (RR) problems. Only two of the remaining categories for problem type were represented, *Ordered Pairs* (OP; 3 tasks, 6%) and *Graph Defined Relations* (GDR; 6 tasks, 13%).

Hart's *Progressive High School Algebra* (1935)

The treatment of function in Hart's (1935) *Progressive High School Algebra* is complicated. In the preface he claims, "Formulas, equations, problems, functional thinking, numerical trigonometry, and simplified algebraic technique are the immediate objectives of the first year course" (p. iii). He goes on to say, "Functional relationship receives merited treatment in the text. It begins informally early in the text; it grows through repeated reference to it; it culminates in a summarizing chapter devoted to a new treatment of Variation, Chapter XXI" (p. v). This preface indicates that function and functional thinking should be pervasive in the text, and frequently referenced. In the material intended for the first year of algebra (Chapters I-XIII), there is only one specific reference to function.

In Chapter VIII (Equations having two Unknowns), there are two sections that explicitly reference function: §158 **The function *ax***, and §159 **The function *ax + b***. Function is not defined in these sections – or in the entire book for that matter. The closest that Hart comes to defining function occurs in Chapter XXI (intended for the second year of algebra) in a discussion about the formula $W = 25n$ (describing the wages earned in n hours at 25 cents per hour). He states, "W is said to be a function of n because, for each value of n, a value of W is determined by a definite law" (p. 466).

Hart's is the first textbook in my study that clearly identifies the difficulty of its exercises. It labels some problems as X or Y (Y indicating more advanced material and X indicating the most difficult problems in the text) and "recommends that the study of such material be required of only the abler pupils" (p. iii). The sections on function in the first year of study are all identified as X problems. So the explicit treatment of function was not perceived by Hart as something that all students needed. This perspective is made even more clearly in the Teacher's Manual for the Second *Edition of Progressive High School Algebra*.

Chapter VIII (the chapter explicitly dealing with function) from the first edition (1935) is identical to Chapter VII of the second edition (1943). In fact, it seems that the Second Edition is just a reprinting of the First Edition. The accompanying Teacher's guide (*Teacher's manual*, n.d.), describes the sections on function this way:

> Some of your brighter pupils will like to play with these pages, particularly if you tell them that they are getting an introduction to analytic geometry, which is usually studied in college. These pages are included in response to a demand from certain progressive teachers. (p.10).

So despite Hart claiming that functional thinking is one of the immediate goals of first year algebra, he clearly feels that function is not an appropriate topic for all students in first year algebra.

Like Wentworth, Hart introduces function in a graphical context. But unlike Wentworth, he does use the terms *slope* and *y-intercept*. Despite the use of these terms, the method taught for graphing lines does not take advantage of these ideas.

Also like Wentworth, the notion of *domain*, *range*, *relation*, and *set* are completely absent from Hart's treatment of function (none of these terms are addressed anywhere in the textbook). And, like Wentworth, function is viewed as a dependence between quantitative variables.

Hart's text is the first in my study to provide exercises that are drawn from non-mathematical contexts (12% are Constructed Relation problems). As well, Hart's is the first text to require students to find the slope of a line (required in 6% of the tasks). The vast majority of the problems in Hart's text use a symbolic representation (93%). And, much like Wentworth, the use of tabular representations and graphical representations coincide (as a table of values needed to be computed to produce the graph).

The other unique attribute of Hart's text is the surprising frequency of Using a Check Point as a control structure (87% of the tasks could use this control). This seems to reflect the nature of the exercises. Most of the problems in the chapter I analyzed required the student to graph a line (where an additional point could be used to verify that the line was right) or to solve a system of equations (where the solution could be checked in both equations).

Welchons and Krickenberger's *Algebra, Book One* (1949)

Welchons and Krickenberger address function in their chapter on Proportion and Variation (Chapter XVII). They never define the term, but emphasize the dependence of the value of one variable on the value of another. They illustrate the terms *variable*, *function*, *independent variable*, and *dependent variable* with the formula $A = \pi r^2$, and state, "The area *A* is said to be a *function* of the radius *r*, since its value depends upon the value of *r*" (p. 485).[10]

By the use of the word "value" when describing function, and the fact that all of the examples and exercises in the chapter deal with quantitative variables, it seems that Welchons and Krickenberger frame function as a dependence of one quantitative variable on another.

[10] Italics in the original

The treatment of function in Welchons and Krickenberger's texts seems almost cursory. Only 9 exercises (6%) specifically use the term function, and although it is the first text in my study to include word lists in the chapter reviews, the terms function, variable, dependent variable, and independent variable are not included (though terms like extremes and means are).

The chapter does not address methods for graphing linear equations, and only two pages on graphing first degree equations exist in the entire book. Even there, neither *slope* nor *y-intercept* is mentioned. The text does not use the terms *domain*, *range*, or *set* anywhere. But it does colloquially use the term *relation* and contains two questions that ask the student to determine if a particular situation involving two variables is a function.

The treatment of function in Welchons and Krickenberger occurs in their chapter on variation and proportion, so it is not surprising that over a quarter (26%) of their exercises were coded as Proportional Relations exercises. This text provides the first exercises drawn from the sciences (accounting for just over 3% of the problems). Using Proportions with Entries and Determining the Type of Relation (direct, inverse, etc.) played a prominent role in the chapter, being used in 36% and 23% of the exercises respectively.

Welchons and Krickenberger did not emphasize graphing in their treatment of function, which helps to explain the near absence of three codes in the data. Few problems (less than 1%) were classified as Ordered Pairs, Graphical representations were only used in 3% of the exercises, and only 2.5% of the exercises were coded with the control Use Alternative Representation. This textbook also provides the first exercises that were coded as using Pictorial representations. In this text, for the first time in this study, illustrations occur in the body of the text and photographs begin each chapter. Geometric figures had been present in Hart's text, but not in the chapter dealing with function.

Dolciani's *Modern Algebra: Structures and Methods, Book One* (1962)

Dolciani defines function as "a relation which assigns to each element of its domain one and only one element of the range" (pp. 438-439). A relation had been defined three pages earlier as "any pairings of two sets of number" (p. 435). And the Teacher's Manual proposes an alternative definition of relation as "any set of ordered pairs of numbers. Function is treated in Chapter 12 (Functions and Variation) and Chapter 14 (Geometry and Trigonometry). Dolciani's text is the first to contain a list of symbols used in the text, but the list does not contain function symbols, like $f(x)$.

Terms relating to function surface in this text that did not exist in previous texts. Just in the definition, we see that *domain*, *range*, and the distinction between a *relation* and a *function* are likely to play a role in the treatment of function in this textbook. In addition to these differences, Dolciani's text begins with a chapter on sets (described as any collection of objects, p. 10).

Dolciani has, by far, the smallest percentage of problems coded as a Rule Relation (only 27%) of the textbooks in my study. This seems to be partly a consequence of function being introduced in the chapter on variation (proportional relations accounting for 10% of the exercises) and a deliberate focus on providing a context for the problems. Social contexts accounted for almost a quarter of the problems, and physical representations accounted for an additional 11%.

Three operation codes were used for the first time with the Dolciani text. Two dealt with domain and range, and the other dealt with trigonometric functions. The operations Listing the Elements of a Relation (all elements in the domain or range of a relation [i.e. finite]) and Determining the Domain and Range of a Relation (examples where the domain or range are intervals) were used in 4% and 8% of the exercises respectively.

The code TRIG (used to code situations where the student must use a trigonometric identity or set up a trigonometric ratio to solve the problem) was used for 13% of the exercises. Clearly this is a result of the explicit use of function in relation to trigonometry. All of the preceding texts had a chapter on trigonometry, but, as they did not identify the ratios as functions, they were not included in my study. Another consequence of the chapter on trigonometry is an increase in the use of tabular representations. Nineteen percent of the exercises in Dolciani were coded as using tables. But the vast majority of these were problems that required the use of a trigonometry table.

The use of Semi-symbolic representations occurs for the first (and only) time in the data in Dolciani's treatment of joint and combined variation. As an oral exercise, students are supposed to describe proportions like $\frac{t_1}{t_2} = \frac{n_1 v_2}{n_2 v_1}$ with descriptions like *t* varies directly as *n* and inversely as *v* (p. 454, #13). The final observation from the data collected for Dolciani's text is that it is the first to use the control structure MT1P (More than One Point, the "vertical line test"). This seems to be consistent with the emphasis for determining if a relation is a function (noted earlier).

UCSMP's *Algebra* (1990)

UCSMP claims, "The idea of function is one of the unifying themes in mathematics. Similarly, in this chapter functions summarize many of the ideas that have been covered in this book" (p. 636C). As that synopsis indicates, the chapter on function is the final chapter in the book. They define function as "a set of ordered pairs in which each first coordinate appears with exactly one second coordinate" (p. 638). UCSMP illustrated the notion of function in many different contexts, including relations that cannot be expressed analytically (like the relation between mean high temperature and latitude).

UCSMP contains the terms *domain* and *range*, and asks students to determine the domain and/or range for different function. For the first time in the study, students were also asked to identify restrictions on the domain of a function (e.g., they were asked to identify that –1 is not in the domain of the square root function, and that 3.5 is not in the domain of the factorial function). Like Dolciani, this book contains a table of symbols. But, for the first time $f(x)$ appears in the list. Seven additional "calculator function keys" are also included $\left(x^2, \sqrt{\ }, x!, \tan, \sin, \cos, \log\right)$.

Even though UCSPM's Algebra is chronologically the final book in my study, there are aspects of the text that are not present in any of the preceding textbooks. USCMP is the first text to utilize DRR (Data Reduction Relations) as a problem type. Almost all of these problems came from statistical situations found in the section on probability functions. Two operations codes COE (Carry Out Experiment) and GECE (Give Example or Counterexample) occur for the first time in this text. With these codes, we see an increased emphasis in having students interact with the mathematics that they are learning.

This text necessitated a new code for representation systems (one that was both new to my data, and one that Mesa (2000) had not needed either). This was the code CC (Calculator/Computer). This code was only used if the student were asked to use a calculator or computer to complete the problem. For example, exercise #17 (p. 669) asks the students to write a computer program to determine to values of LOG X (for x = 1,2,3,…10). Clearly, for problems like this, the computer provides a necessary interface for successful completion of the problem. For much the same reasons, UCSMP also had the first tasks that were labeled CLC (Use Calculator or Computer to Check the Answer).

Presentation of Results

In this section, the definitions of function that the textbooks used will be compared, and the exercises from the texts will be compared based on the problems type (use), the operations necessary to complete the exercises, the representations used, and the control structures in place for the students to determine in their answer is correct. Not all aspects of data can be addressed here. To help draw attention to the most important and interesting aspects of the data, some figures and tables will not include all of the codes. Complete figures and tables for each of the elements of the quadruples (problem type, operations, representation systems, and control structures) can be found in Appendix A.

Definition of Function and Placement in the Textbook

Table 2 compares the definitions of (or examples used to illustrate) function. A comparison of these definitions indicates a shift in the presentation of function from one that requires an expression or definite law to a function as a special type of relation or a set of ordered pairs. But I am interested in much more than just the words that are used to describe what a function is. Do the problems in the textbooks reflect the definitions they provide? What else can be learned about the changes in treatment of function by looking at the problems in the textbooks?

	Definition	Chapters about Function	Chapter Title
Wentworth (1906)	A *function* of a variable is an expression that changes in value when the variable changes in value. In general, any expression that involves a variable is a function of that variable.	Chapter 13 (of 19 before the treatment of quadratics)	Graphs
Hart (1935)	*W* is said to be a **function** of *n* because, for each value of *n*, a value of *W* is determined by a definite law.	Chapter 8 (of 13)	Equations having two unknowns
Welchons and Krickenberger (1949)	The area *A* is said to be a *function* of the radius *r*, since its value depends upon the value of *r*.	Chapter 16 and 17 (of 20)	Quadratic Equations Proportion and Variation
Dolciaini (1962)	A **function** is a relation which assigns for each element of the domain one and only one element of the range.	Chapter 12 and 14 (of 15)	Functions and Variation Geometry and Trigonometry
UCSMP (1990)	A function is a set of ordered pairs in which the first coordinate appears with exactly one second coordinate	Chapter 13 (of 13)	Functions

Table 2. Comparison of the Definition of Function in the Textbooks

Problems

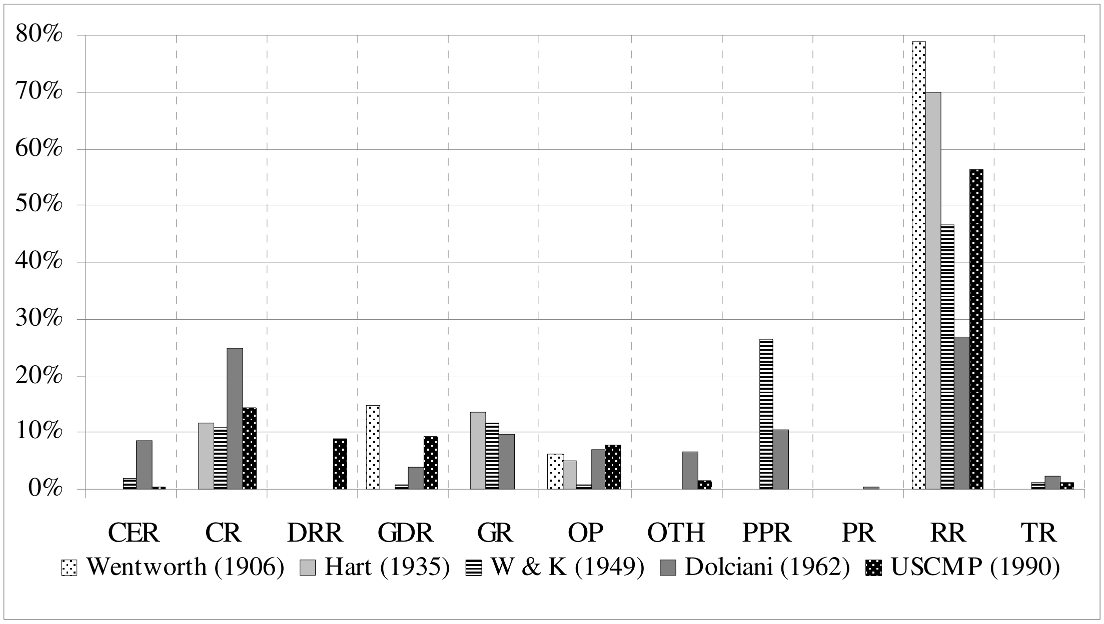


Figure 1. Problem type Across Time

Figure 1 shows the distribution of problem types across the textbooks. Four of these problem types warrant further discussion. As is evidenced by the figure, *Rule Relation* (RR) is the most prevelant problem type across the entire span (accounting for 78% of the exercises in Wentworth's text down to 27% of the exercises in Dolciani's text). It is also evident that UCSMP is the only text that contains *Data Reduction Relation* (DRR) problems. These problems came exclusively from the section on probability functions, so it appears that using statistical bases for exercises about function is a modern trend in the presentation of function.

Proportional Relations (PPR) has a drastic spike in the middle of the period which can be explained by the fact that Welchons and Krickenberger and Dolciani both introduced function in their chapter on variation. The other texts include proportional relations, but not in their treatment of function. The last aspect of this figure that needs to be examined is the category *Ordered Pairs* (OP). From examining the figure, it appears that roughly 8% of the problems were classified as Ordered Pairs problems and that this proportion was fairly uniform across time. Despite the similarity in percentage across time, the treatment of these problems is very different.

All of the instances of *Ordered Pairs* in Wentworth's text and in Hart's text are in the context of plotting points in the Cartesian Plane. In contrast, *Ordered Pairs* problems in Dolciani often involve questions about the domain or range of the relation and questions asking to determine if the given relation is a function. Hoewever, in Dolciani, the ordered pairs are all numerical. In UCSMP, the *Ordered Pairs* problems are used to represent both numerical and non-numerical data. For example, question #27 (p. 673) asks the students to "determine whether or not the set of ordered pairs (student, age) for students in your classroom is a function." UCSMP contains other problems where the domain of the function is non-quantitative (e.g., eye color or days of the week). The final observation about *Ordered Pairs* as a problem type is that it is more prevalent than this figure suggests. The coding convention was to label each exercise with one problem type, so some exercises were coded in other categories despite the fact that they utilized ordered pairs, due to problem context.

Operations

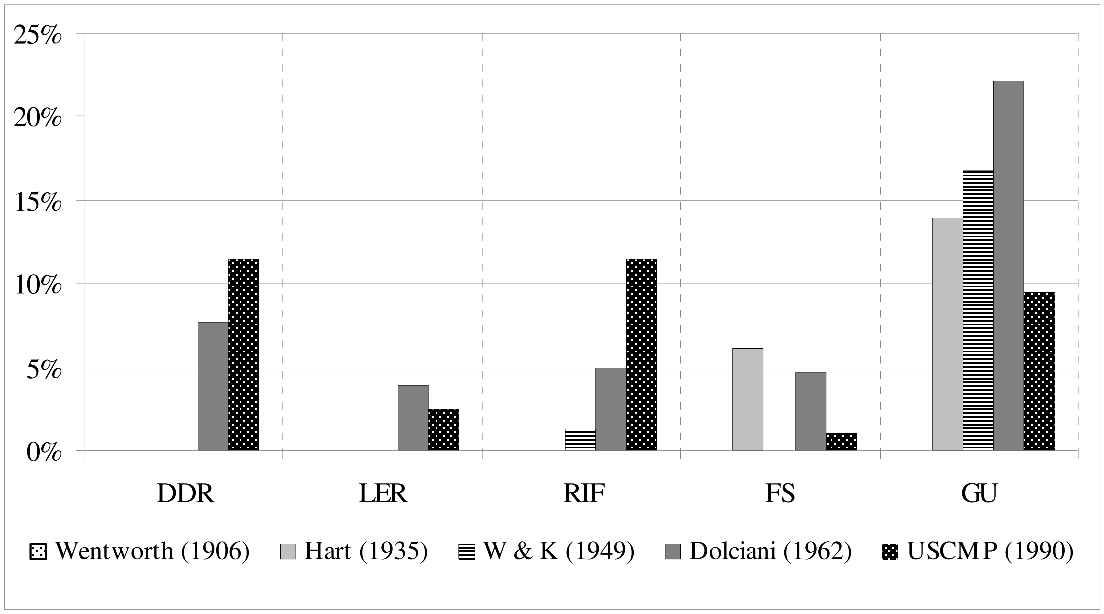


Figure 2. Select Operations across Time

Figure 2 only presents the data for five of the operation codes. There were 27 operations coded from the five textbooks, but these are the five that show the most interesting developments. Tables and figures with all of the data for operations can be seen in Appendix B. It is readily observed that Wentworth's (1906) textbook does not utilize any of these operations. The other aspect of this figure that merits immediate attention is the fact that these operations are all used in less than 25% of the exercises in any of the texts. There are other operations that were more prevalent, but these operations seem to tell the most about changes in the treatment of function.

Two codes, DDR (*Determine Domain or Range of a Relation*) and LER (*List elements of a Relation)* occur for the first time in Dolciani's (1962) *Modern Algebra*. This is also the first text to define domain or range. The rise of these codes indicates that there is something new or special about the identification of the domain or range of a function. This could be related to the presentation of function as a set of ordered pairs. At least the operation LER (*List Elements of a Relation*) requires a finite and discrete representation. Functions represented as a set of ordered pairs are clearly conducive to this type of task.

The operation RIF (*Relation is Function?*) was used to code two exercises in Welchons and Krickenberger's textbook (roughly 1%), but was more common in Dolciani's textbook (5%) and even more prevalent in UCSMP's textbook (11%). This indicates that both the role of relations and the distinction between relation and a function have heightened importance in the presentation of function in the latter half of the twentieth century.

The importance of slope in the presentation of function is captured in code FS (*Find Slope*). We see that Hart is the first textbook in the study to require students to find slope. Welchons and Krickenberger's *Algebra, Book One* did not even contain the word slope. But both Dolciani and UCSMP contain tasks which require the student to determine slope. As well, both of these texts utilize slope and *y*-intercept in graphing linear functions (earlier texts in the study relied on the development of a table of values to graph linear functions).

The code GU (*Give Unit*) was included in Figure 2 because it acts as another indicator of the use of context for the exercises (a perennial recommendation from the reports). In general, students do not need to provide units for problems that lack context. So, we see that starting with Hart's (1935) *Progressive High School Algebra* the students need to provide units for some (roughly 10% to 20%) of their answers.

Representation Systems

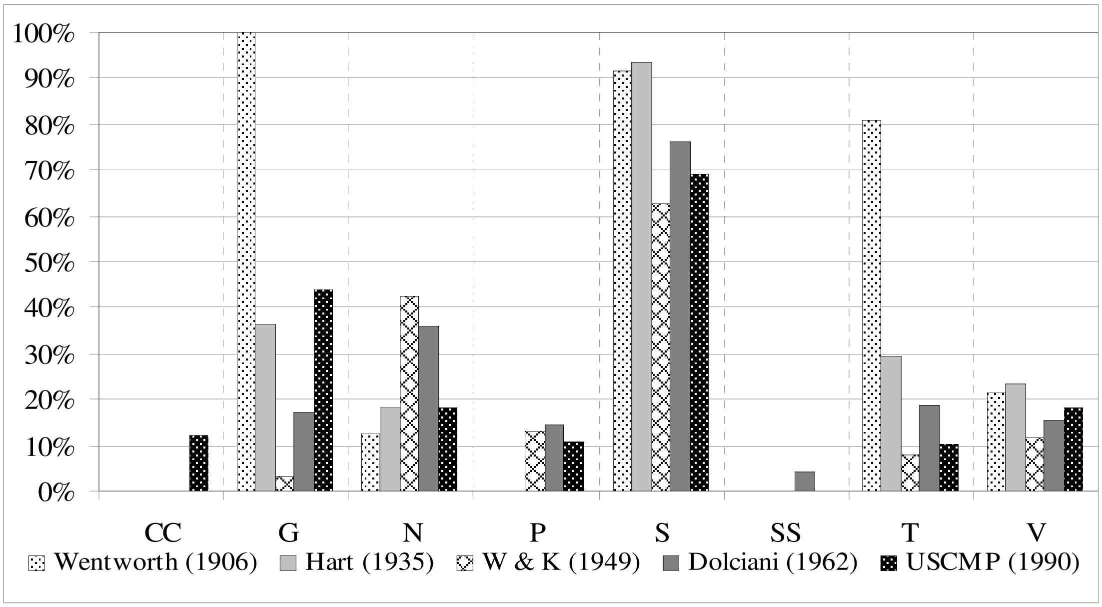


Figure 3. Representation Systems across Time

In much the same way that *Rule Relations* (RR) were found to be the dominant problem type, *Symbolic representation* (S) was clearly a preferred means of representation. It is true that Wentworth (1906) presented all of his tasks in a way that utilized *Graphical representations* (G), but 742 of the 970 exercises (76%) from the textbooks used symbolic representations.[11] In addition to the regularity of symbolic representations, I would like to address trends in the use of *Tabular representations* (T).

The use of tabular representations across time is interesting not in numerical trends, but through variation in how it was to be used. For the treatment of function in Wentworth's (1906) textbook and Hart's (1935) textbook, function was introduced in conjunction with graphing. For these exercises, a table of values was presented as

[11] To see this point made more clearly, please see figure B7 in Appendix B.

a necessary step in graphing equations that represented functions. For Welchons and Krickenberger (1949), the table was a means of representing the value of two variables that could be examined for the existence of a direct or inverse variation. The table was used in Dolciani's (1962) textbook to represent ordered pairs and as a presentation of values of trigonometric functions. Finally, in UCSMP's (1990) textbook, the table was used in the presentation of data and as a step in graphing absolute value functions. Other mechanisms, including analysis of parameters in equations and using graphing utilities were presented for graphing other functions.

It is believable that textbooks written since 1990 might have even less emphasis on tabular representations (as graphing calculators make the use of trigonometric tables obsolete and can provide students with graphs without requiring tables as an intermediate step). I would also conjecture that the use of calculators and computers as representation systems would increase if this study were extended to the present day.

The last observation that I would like to make about the representation systems that were used in these textbooks is the advent of *Pictorial representations* (P), and its fairly stable use in the final three textbooks of my study (being used in roughly 10% of the exercises in each of the textbooks). This code was used for exercises that had illustrations and for those that recommended drawing a diagram of the situation. I find it surprising that all three textbooks that use pictorial representations do so in roughly the same proportion. I wonder if this 10% is consistent throughout the entire textbook (rather than just the chapter on function), and if this is somehow an ideal proportion.

Control Structures

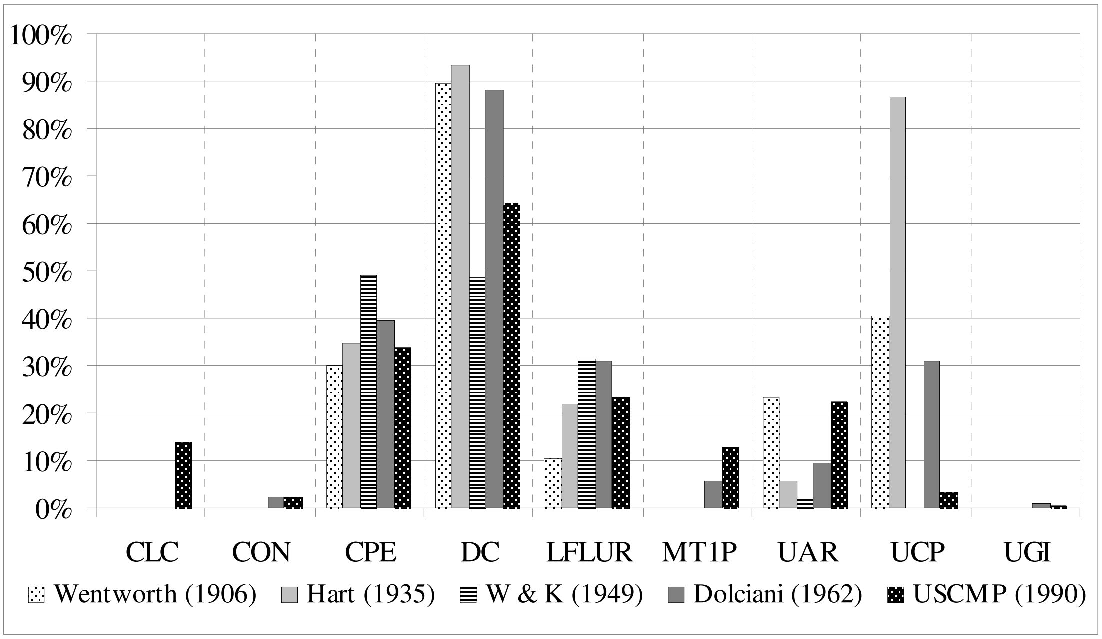


Figure 4. Control Structures across Time

Not surprisingly, the code DC (*Double Check*) is the most common control available for students for all of the textbooks in this study. Even if there are not other control structures, it is likely that they can redo their work. And since graphing calculators were not marketed until 1985 it is not surprising that CLC (*Use a Calculator or Computer to Check*) does not occur until the UCSMP text. But this is more a testament to technological advances than to changes in presentation of function. The one control structure that signals a change in the concept of function is MT1P (*More than One Point*). This control is only found in the final two texts in the study. The final two texts are also the only to explicitly define relation, and the control MT1P is generally used in these two textbooks to determine if a relation is a function.

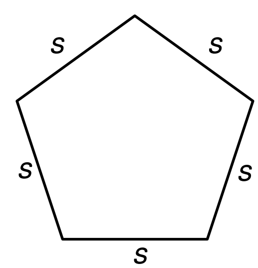
Is p a function of s?

A consideration of each of the elements of the conception individually has shown that there have been changes in what problems types are asked in textbooks, what students must do to complete the exercises, which representation systems are employed, and what mechanisms are in place for the student to assure the correctness of the solution. But what if these elements are examined simultaneously? Would such a comparison lead to a clearer picture of the changes in the treatment of function? That is the subject of this section.

When examining the data for trends, I expected that the operation RIF (*Relation is Function?*) would always be accompanied with the control structure MT1P (*More than One Point*). I was surprised to find that this was not the case. There were two exercises from Welchons and Krickenberger's textbook that were code as RIF (*Relation is Function?*) but not MT1P (*More than One Point*).

Welchons and Krickenberger (1949) Exercise #1, p. 491

Write the formula for the perimeter p of the polygon below. Then copy and complete the following table.



s	1	2	3	4	6	8	10	20
p	?	?	?	?	?	?	?	?

Do p and s vary directly or inversely? Is p a function of s? How does p change when s increases? What is the ratio of p to s? of s to p?

I would like to draw specific attention to the question “Is *p* a function of *s*?” For a student using this textbook, and being assigned this question, the answer would be, “Yes, *p* is a function of *s*.” But why is *p* a function of *s*? In Welchons and Krickenberger’s (1949) text, function was introduced as the dependence of the value of one variable on the value of another variable (p. 485). So a student could reason that *p* is a function of *s* because the value of *p* depends on the value of *s*. But the student could not use the reason “each value of *s* is paired with exactly one value of *p*.” To determine if a relation was a function, one had to show dependence of the value of a variable on the value of another. It is implicit that both of these variables were to be quantitative, and that the dependence could be expressed by a rule.

A student using Wentworth’s *Elementary Algebra* could say, “The problem asked me to find the formula for perimeter and I got $p = 5s$. Now, since a function is just an expression of a variable, then *p* is a function of *s*.” Both Hart and Welchons and Krickenberger identify function as the (numerical) dependence of one variable on another. So a student using either of these texts could say “Since the value of *p* is dependent on the value of *s*, *p* is a function of *s*.” But a student using either Dolciani or UCSMP could say “*p* is a function of *s* since each value of *s* is paired with exactly one value of *p*.”

The reason that this exercise is important to my study is that it illustrates a fundamental difference in how function is being conceived in the textbooks. It shows that the same question could be asked, and the same answer obtained, but with different justification and different implications about what a function really is.

Trends in the Recommendations

To address the second research question (about the correlation between changes in the textbooks and the recommendations for mathematics education), general trends and specific proposals from the recommendations needed to be identified. As indicated in Chapter 2, I was only concerned with the aspects of the reports and recommendations that related directly to my research. For this reason, only recommendations about the problems, operations, representations, or control structures were recorded. Table 3 contains a summary of these recommendations.

	Problems	Operations	Representations	Controls
Klein	Omitted			
NEA (1894)	- drawn from familiar objects			
AMS (1899)	- Geometric examples	- Proof should be emphasized		- Checks should be performed whenever possible
Moore (1903, 1906)	- Practical and scientific problems		- Graphical	- Nomographic
MAA (1923)	- Increase in practical problems (especially scientific)		- Tabular, graphical, and symbolic	- Checks for all numerical results
PEA (1940)	- Non-mathematical situations	- Identifying if a relation is a function - Identification of domain and range.	- Graphical, verbal	- Checking all stages of a response
CEEB (1959)		- Identifying if a relation is a function - Finding slope	- Symbolic, graphical, tabular	
NCTM (1989)	- Modeling real world phenomena.	- Comparison without computing	- Graphical, tabular, symbolic, and verbal rules - Access to graphing utilities (i.e. calculator)	

Table 3. Summary of the Recommendations

From this table, we see that there is an almost universal call for placing exercises and tasks in a context. In most cases this call is for contexts from outside of the mathematics classroom, but some (like the AMS, 1899) recommend using other mathematical disciplines (e.g., geometry) as a context for exercises in algebra. An almost equally consistent recommendation is for students to verify their results. Despite the regularity of that suggestion, few examples are given of how students are to perform the verification.

These reports also indicate a call for a greater variety of representations to be used when dealing with exercises about functions. As for operations, these reports make some specific recommendations. Those which warrant the greatest attention are the recommendations of the Progressive Education Association (1940) and the College Entrance Examination Board (1959). Both of these reports emphasize the distinction between a relation and a function, and the first also emphasizes the role of domain and range when describing a function.

Chapter 5: Discussion

Research Questions

This study was undertaken to answer the following research questions:

> *Research Question 1:* By comparing the concept of function that is fostered by representative textbooks from different eras of the twentieth century, is there evidence for change in the treatment of function across time?
>
> *Research Question 2:* How do changes in representation of function in textbooks correlate with recommendations for mathematics education?
>
> *Research Question 3*: How do changes in the texts correlate with the developments of the function concept in the mathematics discipline?

Each will be addressed separately in this section.

Research Question 1: Trends in the Treatment of Function across Time

The results of this study show that there is a difference in the way that function is defined in textbooks throughout the twentieth century; from an expression, to a dependence among variables, to a numerical set of ordered pairs, to a not-necessarily numerical set of ordered pairs. More importantly, than a difference in words, there is a difference in the concept of function that is being presented. Some of these differences were shown by examining the problem type, the operations used, the representation systems, and the control structures for exercises found in Algebra One textbooks.

For problem type, there seemed to be a greater emphasis on "real-world" contexts in the later books than in the earlier books. Table 4 divided the problems from each textbook into those that provided no context, those that used a mathematical context and those that used a "real-world" context[12]. Some might be surprised at the relatively low percentage of problems from the USCMP (1990) textbook that were coded as having "real-world"context. This can be explained, in part, by the nature of the chapter on function in that text. A primary emphasis of the chapter is to introduce the student to the scientific calculator. Exercises that asked students to approximate log(5) or 10! (Problem #20 and Problem #14, p. 673) provide no context, but use the notion of function (e.g., the logarithm function and the factorial function) to familarize the student with the calculator.

	Percentage of Problems with no Context	Percent of Problems with only Mathematical Context	Percent of Problems with "Real-World" Context
Wentworth (1906)	100%	100%	0%
Hart (1935)	75%	88%	12%
Welchons and Krickenberger (1949)	74%	86%	14%
Dolciani (1962)	54%	58%	42%
UCSMP (1990)	75%	75%	25%

Table 4. Percentage of Problems Generated from Mathematical and Non-Mathematical Situations

As for operations, we see an interest in identifying the domain and/or range of a relation in Dolciani and UCSMP that did not exist in earlier textbooks. There was also an interest in determining if a relation was a function beginning with Welchons and Krickenberger. So the importance of relation, domain, and range for inclusion in Algebra One textbooks all occur in the middle of the twentieth century.

[12] For the creation of this table, Rule, Ordered Pairs, Proportional Relations were classified as problems without context. Graph Defined, Geometric, and Pattern Relations added to the above list to create the category mathematical context only. The category "real-world" context consisted of those problems coded Cause-and-Effect, Time, Constructed, or Data Reduction Relation.

We find that for representation systems, symbolic representations remain one of the most dominant representations across time, but that there is a general trend to include more representation types across time (see Table 5). It was also found that there were changes in the way that some representations systems (i.e. tabular) were used over time.

	Number of Representation Systems	Representation Systems used (in order of decreasing frequency)
Wentworth (1906)	5	Graphical, symbolic, tabular, verbal, numerical
Hart (1935)	5	Symbolic, graphical, tabular, verbal, numerical
Welchons and Krickenberger (1949)	6	Symbolic, numerical, pictorial, verbal, tabular, graphical
Dolciani (1962)	7	Symbolic, numerical, tabular, graphical, verbal, pictorial, semi-symbolic
UCSMP (1990)	7	Symbolic, graphical, verbal, numerical, calculator/computer, pictorial, tabular

Table 5. Number of Representation Systems used in the Textbooks

For control structures, the inclusion of the vertical line test (MT1P) in the final two textbooks of the study indicates a change in the importance of the univalent nature of functions. Though changes in any one of these attributes can be observed, examining them together provides additional insight into the changes that have occurred. Exercises like Welchons and Krickenberger (1949), exercise 1, p. 491 ("Is *p* a function of *s*?") help illustrate that different reasoning might be necessary to answer the same kind of question depending on when in the century the question was asked.

This study has shown that there are some questions which could not be asked in all of the textbooks. For example, Wentworth's (1906) text does not use the terms domain, range, relation or slope. So questions asking for the student to determine the domain or range of a relation, to determine if a relation is a function, or to calculate the slope of a linear function could not have been asked in that book.

Some questions would have been answered differently. USCMP (1990) asks the students to identify if the set of ordered pairs represented by (student, age) for the students in the class is a function (Exercise #27, p. 673). Wentworth's (1906) presentation of function required an expression, so a pupil using his text would say "no." Both Hart (1935) and Welchons and Krickenberger (1949) emphasize dependence of value in their presentation of function, and a student using either of these textbooks would require the variable be numerical, and thus say "no." A student using Dolciani's (1962) text would know that each element of the domain was paired with exactly one element in the range, and be tempted to say yes. But all of the exercises about function in *Modern Algebra* used quantitative variables and the student would likely say "no." But, the student using UCSMP's textbook would say "yes."

Finally, some questions would be answered the same, but the method for obtaining that answer would be different. For example, the reasoning required to determine if a relation is a function (see Chapter 4 [Is p a function of s?]) or the method for graphing an equation like $y = \frac{2}{3}x + 5$ (see Chapter 2 [Examples of the Coding System]). From this study we find that some questions about function could not always be asked; some questions would be answered differently; and some questions, though producing the same answer, would require different solution methods.

Though the research for this study was conducted on only a small sample of the Algebra One textbooks from the twentieth century, they were specifically chosen for their representative nature. It is believed, therefore, that the trends noted in this study reflect trends in Algebra One textbooks of the twentieth century in general.

Research Question 2: Correlations between the Textbooks and the Recommendations

Two of the textbooks in this study explicitly referenced the influence of reports or recommendations on the writing of their texts. One of these texts was Dolciani's *Modern Algebra*. The Teacher's Manual (1962) claims, "In preparing this text, the authors have considered the recommendations of many groups (such as the Commission on Mathematics of the College Entrance Examination Board, the School Mathematics Study Group, and the several Cambridge Conferences)" (p. 3 of the Teacher's Manual).

The influence of these reports on the text is evident. For example, Dolciani's use of slope and *y*-intercept to graph linear equations is clearly illustrated in the appendices to the report (see CEEB, 1959, pp. 42-44). Also the definition of function that Dolciani uses, the emphasis on set, domain, range, and the distinction between relations and function are all recommended in the Commission's report (see CEEB, 1959, Chapters 2 and 4).

UCSMP's *Algebra* (1990) identified the NCTM recommendations as being influential in the formation of the textbook and claimed to be "the first full mathematics curriculum to implement the recommendations of the NCTM Standards

Committee" (p. T5). Both UCSMP and the recommendations of the NCTM articulated the assumption that all students would have access to a scientific calculator with graphing capabilities at all times (NCTM, 1989, p. 124 and UCSMP, 1990, p. T15). Beyond claiming that the report was influential UCSMP clearly included some of NCTM's recommendations.

A striking similarity is found between NCTM's recommendation "to investigate how the graph of $y = af(bx + c) + d$ is related to the graph of $y = f(x)$ for various changes in the parameters *a*, *b*, *c*, and *d*" (p. 155) and exploration questions found in UCSMP (see exercises #28 p. 646, #22 and 23 p. 651, and #23 p. 655). It is clear that UCSMP was influenced by the recommendation found in *Curriculum and Evaluation Standards for School Mathematics*.

The *Reorganization of Mathematics in Secondary Education* (MAA, 1923) seemed to have had an effect on the presentation of material in Hart's *Progressive High School Algebra* and Welchons and Krickenberger's *Algebra, Book One*. The report recommended an informal treatment of function, which is consistent with the fact that neither text provides a definition of function. As well, the report emphasizes "dependence between variable quantities" (p. 64) which is the language and spirit of the treatment of function found in these two texts. Finally, questions dealing with the formula $F = \frac{Wv^2}{32r}$ (for centrifugal force) were used as an illustration in the report, and as exercises in Welchons and Krickenberger's text (see Exercises 10 and 11, p. 493 and Exercise 10, p. 504). The formula is not extremely common in the sciences, and the units are not metric. Both of these reasons indicate that recommendations of the

National Committee on Mathematical Requirements were considered by Hart and Welchons and Krickenberger in the writing of their texts.

Lest we assume that all recommendations are heeded, we see that the Chicago Section of the American Mathematical Society called for an increased emphasis on proof in their 1899 report. And only 5 questions out of the entire study (4 from Dolciani and 1 from UCSMP) were coded as requiring the student to prove something. As well, we find that there are recommendations by the Progressive Education Association (1940) about determining domain and range of a function and distinguishing functions from relations. These recommendations do seem to be integrated into the Dolciani text (1962), but are completely absent in Welchons and Krickenberger's text (1949).

All of this suggests that there is a correlation between the recommendations of committees and the content of textbooks. Clearly not all recommendations are integrated into the textbooks (and even fewer recommendations are *immediately* integrated into textbooks). As well, not all changes in textbooks are called for in recommendations. For example, I have no record of a proposal to include illustrations and diagrams in the texts even though there is a clear point in time when pictorial representations begin to be used. Despite these differences, the story that is told by the recommendations about trends in the treatment of function is consistent with the story that is told by examining the textbooks.[13]

[13] The intent of the study was to use the reports and recommendations as an alternative primary source for the narrative of the evolution of the function concept. I was unable to examine enough reports in enough detail to feel confident in the story that the reports generated. The story that these reports tell is consistent with the story that the textbooks tell. But they are not identical. I still believe that reports and recommendations can be used as a valid primary source for this kind of study, but found it to be too extensive for this research. A companion study that just addressed the recommendations would provide interesting insight into evolution of the treatment of function.

Research Question 3: Correlations between the Textbooks and the Mathematics Discipline

Initially I thought that I would be able to identify instances where a development in mathematics was directly observed in textbooks that I analyzed. Finding instances of direct influence from the mathematics discipline on textbooks proved to be a rather fruitless endeavor. From the textbooks in the study, there is only one reference to individuals influence on a textbook. The following is from Wentworth's (1906) preface, "The author is indebted to Professor Eugene K. Miller, Jackson, Louisiana, for the second method of factoring found on page 272, and to Professor B. F. Yanney, Alliance, Ohio, for reading the proofs and giving suggestions" (p. vi).

Preliminary research indicates that Yanney was indeed a professor (as we would now define professor). Yanney was a professor of mathematics at Mount Union College then The College of Wooster. I have no evidence that Eugene Miller was a professor, and it is likely that he was a teacher. This is consistent with the Webster's (1913) definition of professor as "one who professes or publically teaches any branch of learning; esp. a lecturing or teaching officer in a university, college, or other seminary" (p. 545). As well, Wentworth referred to himself a professor even though we would now call him a teacher.

In any case, a specific reference to the contributions of individuals seems to be one of the only ways that evidence of a direct influence could be argued. This indicates that the reports and recommendations might be a necessary step in the process for influencing textbooks. So, a more appropriate question might be: What are the correlations between the treatment of function in the mathematics discipline and the recommendations for mathematics education?

There is evidence that some of the recommendations promote an understanding of function that correspond to different times in the history of function in the mathematics community. For example, the National Committee on the Reorganization of Mathematics (1923) recommended teaching function in the secondary school as a dependence between variable quantities (p. 64). This is consistent with the pre-Dirichlet idea of function. And the Commission on Mathematics (1959) say that "a function in U is a set of ordered pairs belonging to $U \times U$, and having for each x at most one y." This is similar in formulation to the Bourbaki definition of function given in 1939.

My study of the history of function in the mathematics discipline dealt primarily with the definition of function and the classes of problems that such definitions could address. Consequently, there are many aspects to the relationship between the history of function in mathematics and the treatment of function in recommendations about mathematics education that I cannot address in this paper.

Nonetheless, the implication from the textbooks and recommendations that I have included in this study is that part of the history of function is mirrored in the treatment of function in Algebra One in the twentieth century. Function was presented as an expression (Wentworth, 1906) akin to the notion of Bernoulli and Euler, then as a correspondence or dependence (Hart, 1935; Welchons & Krickenberger, 1949), and as a set of ordered pairs (Dolciani, 1962; UCSMP, 1990) similar to the Dirichlet-Boubaki definition.

Note that this correlation is easier to see in the "reverse direction" – that is by considering the history of function in the mathematics discipline in light of the observed changes in the textbooks. In some regards, this really makes sense. There is likely to be some source (like the mathematics discipline) for the changes that are found in the textbooks, but it is possible (even likely) that not all changes in the discipline would be expressed by changes in the textbooks.

As well, it is important to note that this is not the whole story of the development of function in mathematics. For some in the mathematics community the conception of function is based in category theory rather than set theory, and it remains to be seen if the idea of categories will taken up by secondary education. I would imagine that it will not since there is research indicating that a less abstract notion of function is preferable for an introduction to function (see Cooney & Wilson, 1993). But, there does seem to be a correlation between the history of function in mathematics and the treatment of function in Algebra One.

Primary Results of the Research

Up to this point, I have been very comprehensive in my treatment of my research questions and the sources of my data. I would like to deviate from that approach in this section to help emphasize the more important results of this study. I became interested in this line of research because for me, like many others, there is a strong temptation to believe that the content of mathematics is static. Sure, some ideas, like how to teach factoring or whether to include joint variation might change. But other topics, like slope or graphing or the quadratic formula, seem too central to our notion of Algebra One. It is difficult even to imagine what Algebra One would look like without them.

I chose to look specifically at function because I knew that the concept of function in the mathematics community had undergone revision, and it was likely that these changes might be seen in textbook presentations. I found that changes in the presentation of function did occur, and that the changes fell into the following three categories:

1. Changes in what can be asked.
2. Changes in what the "correct" answers are.
3. Changes in how the answer is obtained or justified.

Some questions could not be asked in certain textbooks because the textbook lacked the terminology to answer them. Questions about the domain and range of functions or the slope of a line could not be asked in all parts of twentieth century because the textbooks lacked the language to answer those questions. The introduction of new terms suggests an expansion of the influence of function. A more sophisticated vocabulary was needed.

Some questions would be answered differently depending on when in history they were asked. Specifically questions that related non-numerical sets of data would not universally be identified as functions. A change in how questions would be answered reveals an evolution of topic itself – a change in the fundamental understanding of what a function is.

And some questions, though receiving the same answer, would require different justification. This can also be illustrated by a question asking to identify if a relation is a function. In some time-periods this could be answered by a reference to the vertical line test. In other time-periods a display of the dependence of the value of one variable on another was required. And in the earliest part of the twentieth century, an expression relating those variables was needed.

Considering these three categories of change suggests another question: Are these three changes specific to the concept of function or would similar trends be found for other subject matter? In thinking about this question in the context of Algebra One topics, I found that many themes (solving of equations, graphing, variation, etc.) had such a dependence on function that these changes seen in the treatment of function would likely be reflected in those topics as well.

I would expect that, to a greater or lesser degree, within a well-chosen time span similar changes (in what can be asked; in what the "correct" answers are; in how the answer is obtained or justified) could be found in any topic in any field of education (sciences, humanities – any field). I find this to be the most central result of the study. But I would like to address some of the other results in the implications section that follows.

Implications of the Study

Implications for Teachers and Curriculum Writers

One of the most difficult aspects of the coding of this data was determining what control structures were available for the students. In this regard, I echo Mesa (2004) who observed, "Very few textbooks provided explicit indications for the students to control their activities" (p. 280). If the recommendations for mathematics education are any indication, then verification is important in mathematics. More emphasis should be placed on helping students assess the accuracy of their answers. It is likely that textbook writers should be more deliberate in presenting controls in their books. But, at a minimum, teachers should consider what control structures are available to students and encourage them to determine if their answers are correct. Some questions have no real control structures. A question like "a function is a special type of _____" can only be verified by appealing to some other authority (likely the teacher, a fellow classmate, or the book). If the questions are written with an attention to the available control structures, the students are bound to benefit.

More important than simply the role of verification in exercises about function (or more generally in mathematics), is the need to consider the picture of function that is being presented. This is true for textbook authors and teacher alike. The conception of function that students are likely to obtain is not solely a result of the wording of the exercises (or the definition for that matter). An analysis of the presentation of function, in the aggregate, seems to be the best predictor of the conceptions that students are likely to form. To the extent that developing certain student conceptions is important, this kind of analysis needs to be conducted by textbook writers.

Implications for Understanding Reform

This research also has implications for those who want to reform mathematics education, and those who want to analyze reports and recommendations for reform. One of the results of this study was the discovery that direct influence of the mathematics discipline on the content or presentation in textbooks is difficult to discern. It is possible, therefore, that the means by which mathematicians succeed in influencing textbook writing is through their participation in committees whose purpose is to make recommendations to textbook authors.

I acknowledged in the treatment of my third research question, there are other ways that mathematicians can try to influence textbook authors. For example, textbook authors may learn specific problem solving methods from mathematicians (possibly as student in their classes). But it appears that a much more effective way to influence the content of textbooks is through participation in committees whose purpose is to make recommendations for textbook authors.

This implication is further supported by the observation that the voices of individuals play no part in this study after Moore's recommendations in 1903 and 1906. The formation of such organizations as the College Entrance Examination Board (CEEB) in 1900, the Mathematical Association of America (MAA) in 1915, and the National Council of Teachers of Mathematics (NCTM) in 1920 seem to provide a better forum for making recommendations.

This study also has implications for the study of recommendations and reports. By considering a large body of recommendations, a narrative of the issues that are in debate can be derived. This provides a way of developing the story of trends in education that does not depend on the material that is ultimately written in textbooks. A comparison of this narrative with the narrative which is told by examining changes in textbooks provides a means of verification that the observed changes were actually occurring. And that it was not just a consequence of the specific textbooks that were chosen.

Implications for Further Research

There are a number of possible directions that research derived from this study might take. One likely direction is to use the coding system to examine the treatment of function across different grades. If differences were found in the conception of function that is promoted at different grades, reasons for those differences could be examined. Are there more sophisticated conceptions of function that should be reserved for upper grades? Could there be a correlation between student misconceptions about function and the richness of their initial introduction to function?

This research also suggests a framework for studying the evolution of other topics in education. How has the treatment the derivative (or of multi-digit addition) changed in mathematics education? Or how has the treatment of genetic drift changed in the teaching of biology? Questions like this would require identification of the problems, operations, representations, and control structures for that topic, but applications of this framework to other topics (or fields of study) will help to strengthen our understanding of, and ability to discuss, conception.

There are also implications for additional research on the correlation between recommendations and changes in textbooks suggested by this research. Recommendations can be understood to represent unsettled aspects mathematics education. Changes in what the recommendations propose can be an indication that there is a shift in what aspects of mathematics education are being debated. In this regard, the narrative derived from trends in recommendations can be compared to the changes observed in textbooks, and each can act to verify the other. To perform this kind of comparison of narratives from different sources (e.g., textbooks, recommendations, or the discipline) it is possible that correlations will be more evident in one direction than the other. To better understand correlations in the "forward" direction, more detailed accounts may be necessary.

The correlation between the mathematics discipline and changes in the textbooks would benefit from further research. Constraints for this paper (i.e. constraints on time and scope) prohibited me from pursuing this question with the depth that it deserves. To pursue this question with more rigor, a more thorough description of the history of function in mathematics would be required. What were the positions of individual mathematicians about the concept of function? Were these mathematicians members of any recommending committees? Did these individuals produce writings before their participation in the committee that would indicate the position they would advocate as a member of that committee? Are there writings produced after their involvement that would indicate a change in their position?

As far as this study itself is concerned, additional research is warranted by extending the study to include textbooks written since 1990. Much research has been conducted on the effectiveness of teaching function from a set-theoretic perspective (see Cooney & Wilson, 1993), and generally a more concrete approach is recommended. I know that it might take years before representative texts from the time period since 1990 are identified, but I would imagine that their inclusion in such a study would present a wrinkle in some of what has been presented here. What does it mean if the textbooks no longer present function as a set of ordered pairs? If a more historic approach, like defining function as a correspondence is used, what are the implications of intentionally distancing school mathematics from the discipline?

Finally, review of this study in terms of conception quadruples (as conceived by Balacheff & Gaudin, 2002) would provide insight into the development of various conceptions of function. The current study examined the coordinates separately, but a fuller picture may be seen when looking at each exercise's quadruple. In essence this was the approach used for the discussion of the problem of perimeter and side length (see again Chapter 4).

Appendix A: Examples and Definitions of the Codes[14]

Table A1. Problem Codes

Code	Name and description	Examples
CER	*Cause and Effect Relationship* Used to code content that refers to physical phenomena other than time related and in which the behavior of one variable is an effect of the behavior of the other (it is a directional relationship)	- Boiling point vs. altitude - Force associated with circular motion $\left(F = \frac{mv^2}{r}\right)$ - Heat lost (through window) vs. temperature differential, area, and thickness - Heat radiation vs. distance - Heat vs. resistance, current, and time - Ideal gas law $(PV = nRT)$ - Intensity (luminosity) vs. distance - Mass vs. volume - Newton's Law of Universal Gravitation $\left(F = \frac{Gm_1m_2}{r^2}\right)$ given as $\left(F = \frac{k}{r^2}\right)$ - Ohm's law $(V = IR)$ - Pressure (of liquid) vs. speed and diameter - Pressure vs. altitude - Pressure vs. depth (under water) - Resistance vs. cross-section of wire - Resistance vs. length of wire - Wind pressure vs. area - Wind pressure vs. area and velocity - Work exerted vs. force applied $(W = F \cdot d)$

CR	*Constructed Relationship*
	Used to code content that refers to "real life" situations other than cause/effect, time, data reduction, and geometrical. In these relations it is somehow arbitrary which variable is called dependent and which one independent. An interchange of the roles of the variables produces equally valid—for the context—relationships.
		- Batting Average (hits vs. plate appearances)
		- Conversions
		  o Centimeters vs. inches
		  o Centimeters vs. meters
		  o Fahrenheit vs. Celsius
		  o Feet vs. meters
		  o Inches vs. centimeters
		  o Italian lira vs. US dollars
		  o Kilograms vs. pounds
		  o Meters vs. feet
		  o Miles vs. nautical miles
		  o US dollars vs. British pounds
		- Cost of a job vs. number of workers and the amount of time
		- Cost per person vs. number of people
		- Cost vs. gallons of gasoline
		- Cost vs. wattage, hours of operation, and

[14] The wording for the descriptions of these codes is identical to that of Mesa (2000) except for the codes that I modified (as will be articulated in the next section).

		kilowatt-hour costs - Distance from cliff vs. height above water - Height vs. distance from an object (i.e. angle of elevation/depression) - Height vs. shadow length - Hydrochloric acid vs. lye - Latitude vs. temperature - Number of accidents vs. age of driver - Number of UN members vs. year - Number of votes vs. year - Number of workers vs. amount of work and time to do a job - Number of workers vs. time to do a job - Population vs. year - Power vs. speed - Profit vs. Number of items sold - Scale drawings (blueprints) - Scale drawings (maps) - Scale models (three dimensional) - Simple interest formula $(i = prt)$ - Size of pipe vs. time to fill a tank - Slaked lime vs. sand in mortar - Visibility vs. height above the earth's surface - Weight (Earth) vs. weight (Neptune) - Weight and speed vs. kinetic energy (for football players) - Weight vs. distance (see-saw) - Weight vs. length and cross sectional area - Weight vs. length of wire - Weight vs. price of diamond - Winning Olympic times vs. year - Yard line vs. yards walked (on a football field)
PPR	*Proportional Relation.* Used to code content where there is an explicit reference to a proportion or a direct proportion without context.	- Determine if two (or more) ratios are in proportion - Fill a table to make the entries proportional - Solving given proportions

DRR	*Data Reduction Relation*
	Used to code statistical situations; in situations involving two variables it may be possible to have more than one outcome for a given value of a variable.
		- Estimate vs. error
		- Percent of vote vs. error
		- Probability vs. day of arrival for a letter
		- Probability vs. eye color of children
		- Probability vs. face on a fair die
		- Probability vs. face on a weighted die
		- Probability vs. number of correct multiple choice responses
		- Probability vs. region of a *n*-sectioned spinner
		    o All sections equal
		    o Unequal sections (30°, 60°, 90°, 90°, 90°)
		    o Unequal sections (60°, 120°, 180°)
		- Probability vs. sum of three dice
		- Probability vs. Sum of two (fair) dice

GDR	*Graph Defined Relation* Used to code content where the relation is presented in a graph and no outside context is given	- Angle of a line with the positive *x*-axis - Comparing the graph of inverse and direct variation - Does the given graph represent a function?

		- Does the graph of $ax \pm by = 0$ pass through the origin? - Where must a point lie if the ordinate is zero?
OP	*Ordered Pairs Relation* Used to code content where a list of ordered pairs is given or requested.	- Brothers and sisters - Is this set of ordered pairs a function? - Locate points in a Cartesian plane
OMR	*Other Mathematical Relation* Used to code content that refers to mathematical relations (in general these will be geometric).	- Area vs. altitude and bases (trapezoid) - Area vs. base and altitude of a triangle - Area vs. length and width (rectangle) - Area vs. radius (and square of the radius) - Area vs. side length (equilateral triangle) - Circumference vs. radius - Diameter vs. radius - Digit problems - Finding the angle formed by the diagonal of a rectangle - Height vs. area (of parallelogram) - Number of triangles vs. number of diagonals in an *n*-gon. - Perimeter vs. side length (regular figures) - Proving trigonometric identities - Ratios of triangle sides (setting up trigonometric ratios without context) - Similar figures - Sum of interior angles vs. number of sides in an *n*-gon - Surface area vs. radius - Systems rising from dimensions of figures - Systems rising from sum and difference problems - Volume vs. basal area and height - Volume vs. height and square of the radius - Volume vs. radius of a sphere
PR	*Pattern Relation* Used to code content in which given a sequence the question is to find the general term (or an expression for the *n*th element)	- Number of diagonals from a vertex in an *n*-gon - Number of triangles inside an *n*-gon
RR	*Rule Relation* Used to code content in which an input is transformed by certain procedure to obtain an output and in which a context is not provided.	- Absolute value - All polynomials - Computer programming work: the student needs to write a computer/calculator program to produce particular outputs - Exponential functions - P(x)/Q(x), where P and Q are polynomials with coefficients in **R**. - Piece-wise functions - Radical functions - Systems of equations - Trigonometric (without context)
TR	*Time Relation* Used to code content that refers to physical phenomena where time is involved and the variable is treated continuously.	- Distance from checkpoint vs. time - Distance vs. time travelled - Free-fall distance vs. time $\left(d = \frac{1}{2}gt^2\right)$ - rpm for a motor vs. diameter

		- rpm for gear vs. number of teeth (cogs) - Speed vs. distance (for stopping)
OTH	*Other Situation* Used to code content that could not be coded in other problem types. These were generally verbal problems that asked for recitation of a fact.	- What is the domain assumed to be if not explicitly mentioned? - A relation may be shown by ___, ____, or ____. - Other vocabulary exercises

Table A2. Operation Codes

Code	Definition
CALC	*Calculate* The student needs to operate with numbers (e.g., add, subtract, multiply, divide, square a number, or find the square root, take log).
CDU	*Compute with different units* The student needs to convert units by applying a proportional relationship between measures or a formula to express a result (e.g., speed given in m/s and answer is needed in k/h)
CF	*Change between symbolic forms* The student needs to perform algebraic manipulations on a symbolic expression to obtain another one.
COE	*Carry out experiment* The student needs to perform a series of steps to collect data (either statistical or physical). Applies also to the case in which the student needs to write a computer program.
CWC	*Comparison without calculation* The student needs to produce a conjecture based on the observation of information available in task. There is no proof or calculation required (e.g., segments are proportional, lines with a slope of 0 are horizontal, one group outperforms another one).
DDR	*Determine domain or range* The student needs to find the domain, the range, or both of a relation. This can be part of a process and might not be explicitly stated in the task.
DSCP	*Describe shape in a graph* The student needs to describe the shape obtained after joining certain points in a Cartesian plane (e.g., squares, stars, and triangles). This includes examining the graph as a whole and also looking at particular intervals (e.g., observe minimum and maximum values, intercepts, sections where the relation is increasing or decreasing and so forth) with the aim of describing those features.
DTR	*Determine type of relationship* The student needs to determine whether the relation between two sets of numbers is direct, indirect, linear, or nonlinear, or whether there is no relation.
FIP	*Find element of the range or of the domain of a relationship* The student needs to find in the range of the relation a value (or element) associated with a given element of the domain, or find a domain element associated with a range element, or both. This includes finding one more ordered pair of the relationship, where the student might need to choose an element of the domain and find its corresponding value in the range through the relation. It includes algebraic manipulations that involve solving for x in $f(x) = k$, where k is a given value, or finding $f(m)$ where m is an algebraic expression, finding the solution of $f(x) = f—1(x)$, finding asymptotes. This code is also used when the student needs to find the function that results from the operation of two given functions; the process can be carried out by operating component by component in a table or by operating on the expressions that define the relation. This includes finding, for example, the image when $x = 0$ and the pre-image when $y = 0$, with all algebraic manipulation that may be required. There is no restriction on the representation used for the pair.
FNEC	*Find non-explicit characteristic*

	The student needs to demonstrate or prove that a particular object in the situation has a certain characteristic (e.g., a parallelogram is a square, intercept with *y*-axis has an abscissa of 0, there is a rectangle with maximum area, or $f^{-1}(3)$ is the solution of $f(x) = 3$).
FR2N	*Find the relation between two (sets of) numbers* The student needs to explain or produce an expression or a description of the relation between two given numbers or between two sets of numbers. The operation includes the variable identification and the process of writing down the expression (using any representation). To find the relation the student may recall previous knowledge (e.g., formulas for areas, volumes, perimeters) or base the solution on the information given (e.g., use proportionality). The relation can be given in any representation as described in the section on representations.
FS	*Find slope* The student needs to find (by a calculation, by a formula, by inspection) or locate (e.g., in a symbolic expression) the slope of a straight line.
FT	*Fill table* The student needs to either create or complete a partially filled table of values. If a relation is given or asked for (via any representation), this operation has to go together with FIP because the student will need to find images and pre-images through a relation in order to fill out the table. FT goes by itself when it is a step inside a data collection process: the relation is to be determined afterwards, using the information in the table.
GC	*Give cardinal* The student needs to count the number of elements of a set.
GD	*Give definition* The student needs to produce a definition based on the reading of the text (e.g., define *argument*, *ordered pair*, *abscissa*).
GECE	*Give examples and counterexamples* The student needs to find examples of cases that satisfy a given situation, cases where a proposed situation does not hold, or both. Used also when the student needs to make up a story about a particular situation (e.g., a bath in a bathtub, or changing speeds of racers).
LER	*List the elements of a relation* The student needs to produce a listing of all the elements of the relation. Note that this applies to relations where the domain is a finite set (e.g., a family tree with grandparents, parents, and children).
LPCP	*Locate points in graph* The student needs to locate points in a graph; a graph can be any of the types defined in the section on representations. Whenever a Cartesian plane is involved, the code must be applied if both elements of the ordered pair are known and need to be located. If that is not the case (e.g., the time at which the temperature is 50°C), then use the operation FIP. LPCP always requires NPOX when a Cartesian plane is involved.
M	*Measure* The student needs to apply a measurement procedure (e.g., in the Cartesian plane, variables in an experiment).
NPOX	*Name point on axis* The student needs to determine a number on an axis, either by reading it from the scale given or by doing an interpolation. It may or may not require a calculation (by means of adding a certain number a needed number of times).
NV	*Name variables* The student needs to identify the given variables in a situation (or representation) or to establish them. It is not necessary to use this code if FR2N is used.
P	*Prove* The student needs to produce a proof of a statement either given in the text or produced by the student.
RIF	*Relation is function?* The student needs to determine if a relation is a function or not.
RPCP	*Read points from graph*

	The student needs to read the coordinates of a point or a set of points from a graph. A graph can be of any of the types defined in the section on representations. Whenever a Cartesian plane is involved, the code must be applied if both elements of the ordered pair have to be determined (e.g., the coordinates of the maximum value of a relation). If that is not the case, then use the operation FIP (e.g., the time at which the temperature is 5oC). RPCP always requires NPOX when a Cartesian plane is involved.
TRIG	*Apply trigonometric identities/formulas* The student needs to use basic trigonometric relations between angles and sides of triangles to find unknown values of sides or angles.
UPWE	*Use proportionality within entries* The student needs to use the fact that a given relation is proportional.

Table A3. Representation System Codes

Code	Description
CC	*Calculator/Computer* The task specifically asks the student to use a computer (like write a computer program) or a calculator (like graph) to obtain an answer
G	*Graph in two axes* The task uses a Cartesian plane, frequency diagram, histogram, broken line (time series), or scatter-plot.
N	*Numerical* The task does not require any symbols; instead, it requires numbers.
P	*Pictorial* The task uses drawings of machines, maps, geometrical shapes and figures, photos, or pictograms (frequency diagram where the *y*-axis is not present), pies (only one variable is sketched) or any other kind of drawing.
SS	*Semi symbolic* The task uses expressions that contain both words and symbols. (e.g., cost of *x* pounds of apples = 0.3 x *x*).
S	*Symbolic* The task uses expressions with only symbols. This includes arithmetical notation, sets $\left(\text{e.g.,} \{x \mid x > 0, x \in \boldsymbol{N}\}\right)$, ordered pairs, equations $\left(\text{e.g.,}\ f(x) = x + 1, y = x + 1\right)$, mappings $\left(f : x \to x + 1\right)$, or intervals.
T	*Tabular* The task uses a table. The table can be given, asked for, or a requisite for the process of keeping track of the entries.
V	*Verbal* The task uses a description of a situation using natural language (e.g., a pound of apples costs 30 cents) or requires the student to interpret a situation with natural language.

Table A4. Control Structure Codes

Code	Description
CLC	*Use calculator or computer to check the answer* The student is told to create a program so that he or she can check his or her answers, or to observe a graph in the calculator.
CPE	*Compare previous examples or problems* The student (or the textbook) has performed similar operations and can use them in this particular situation.
CON	*Continuity is assumed* The student uses the fact that a function is continuous to determine the likeliness or unlikeness of a result. Also the situation is such that if continuity is not assumed, the

	problem cannot be solved (e.g., the student has a table of values—of a polynomial function—and he or she needs to find the image of a value that is not in the table).
DC	*Double check* The student either repeats the process used to obtain the answer (e.g., relocates points in the Cartesian plane) or reverses the process to get something that is given in the statement of the task (undoes the sequence of operations).
LFLUR	*Look for likely or unlikely results* The student can use indicators in the statement of the task (e.g., the student obtains a number too big or too small for a given scale in a Cartesian plane, or he or she is getting decimals or negative numbers when whole or positive numbers are expected, or a set of points in a Cartesian plane are not aligned on a line) or can use previous knowledge (e.g., the sides of a square have the same length).
MT1P	*More than one point (vertical line test)* The student has to determine if an element of the domain of a relation has one and only one element assigned from the co-domain of the relation.
UAR	*Use alternative (given or not given) representations* The student can use other representations (e.g., results in a table vs. results with a formula or a graph, a set of ordered pairs as an arrow diagram). These can be explicitly given in the statement of the task or can be result of something the student was asked to do.
UCP	*Use checkpoints* The statement of the task offers answers to previous questions, warns the student about what is not a correct answer, suggests looking at the following tasks, or suggests checking with a partner who is doing the same task.
UGI	*Use given information.* The statement of the task gives additional information that can be used to test a result and that might not be relevant to the solution of the problem (e.g., if there is only $100 to spend—then the domain of the relation has to be restricted).

Appendix B: Tables and Charts of the Textbook Analyses

Problem type

	Wentworth (1906)	Hart (1935)	W & K (1949)	Dolciani (1962)	USCMP (1990)
CER	0	0	2	9	<1
CR	0	12	11	25	14
DPPR	0	0	26	10	0
DRR	0	0	0	0	9
GDR	15	0	1	4	9
GR	0	13	12	10	0
OP	6	5	1	7	8
OTH	0	0	0	6	1
PR	0	0	0	1	0
RR	79	70	46	27	56
TR	0	0	1	2	1

Table B1. Distribution of Problem Types (as a percent). Totals may not add to 100 as a result of rounding.

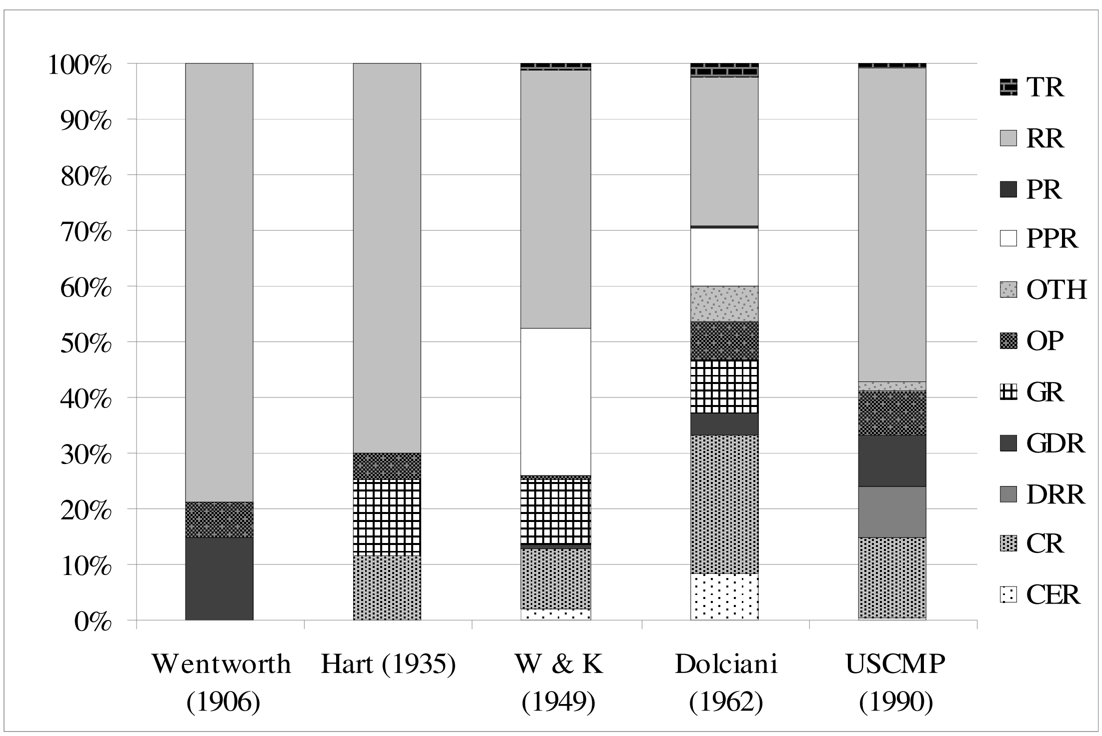


Figure B1. Relative distribution of problem type across time.

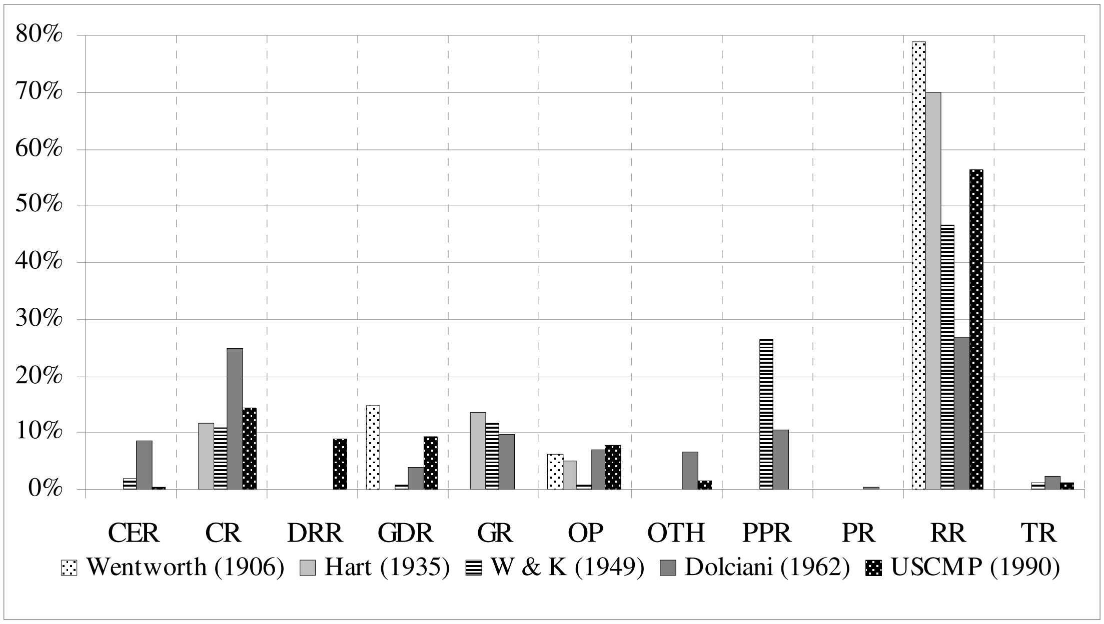


Figure B2. Distribution of problem type across time.

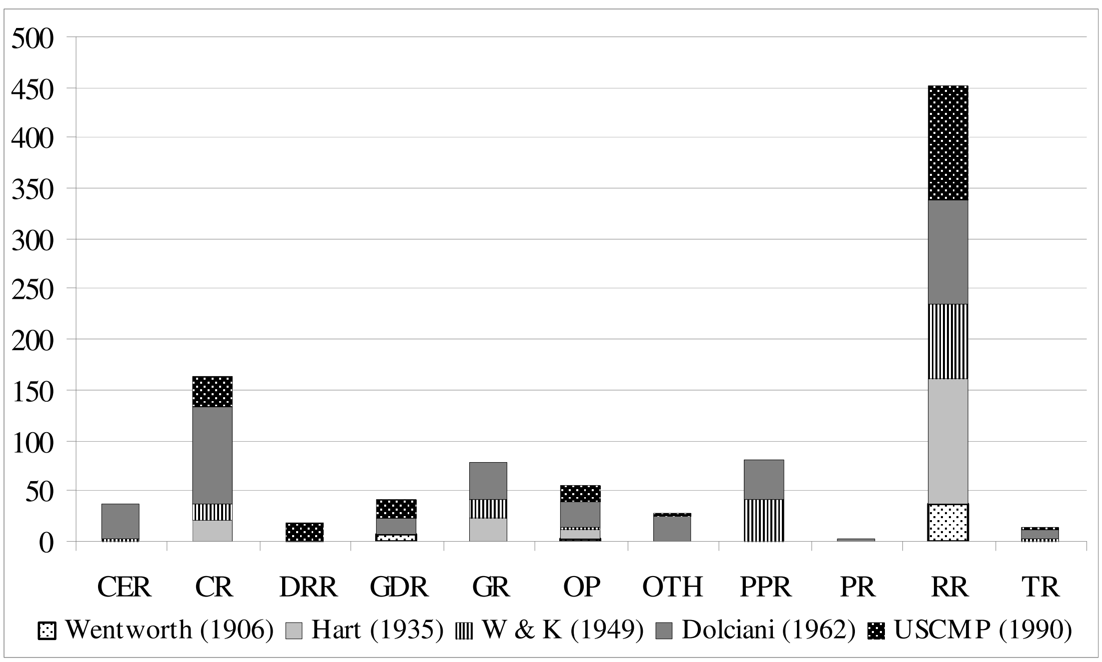


Figure B3. Total number of problems coded for each problem type.

Operations

	Wentworth (1906)	Hart (1935)	W & K (1949)	Dolciani (1962)	USCMP (1990)
CALC	40	14	41	32	22
CDU	0	0	0	<1	0
CF	66	45	8	4	2
COE	0	0	0	0	1
CWC	2	0	2	0	4
DDR	0	0	0	8	11
DSCP	0	1	0	0	2
DTR	0	0	23	8	4
FIP	83	94	11	47	58
FNEC	13	0	0	0	0
FR2N	2	4	2	26	3
FS	0	6	0	5	1
FT	81	25	6	3	15
GC	0	0	0	0	<1
GD	0	0	5	6	5
GECE	0	0	0	0	2
GU	0	14	17	22	9
LER	0	0	0	4	2
LPCP	91	32	3	14	25
M	0	0	3	0	0
NPOX	96	36	3	14	33
NV	0	27	0	3	0
P	0	0	0	1	<1
RIF	0	0	1	5	11
RPCP	47	14	0	0	9
TRIG	0	0	0	13	6
UPWE	0	0	36	38	0

Table B2. Percentage of exercises that were coded for each operation type. Totals will not add to 100 as each exercise could have multiple codes.

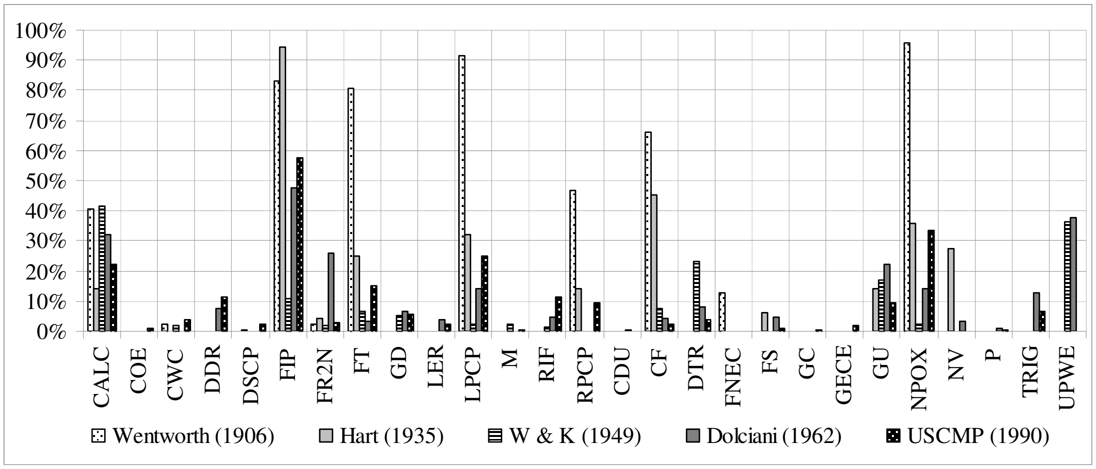


Figure B4. Distribution of operation type across time.

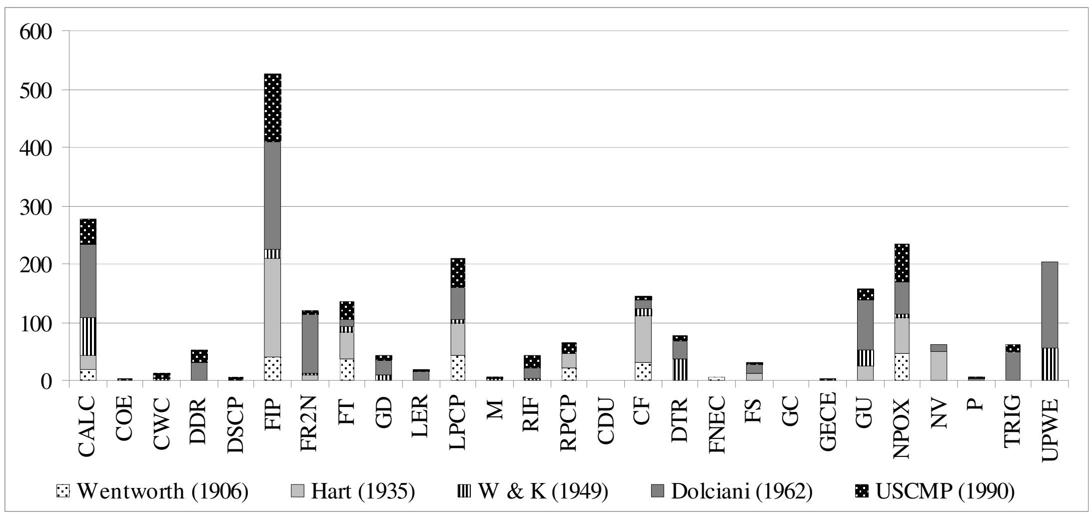


Figure B5. Total number of problems coded for each operation.

Representation Systems

	Wentworth (1906)	Hart (1935)	W & K (1949)	Dolciani (1962)	USCMP (1990)
CC	0	0	0	0	12
G	100	36	3	17	44
N	13	18	43	36	18
P	0	0	13	15	11
S	91	93	63	76	69
SS	0	0	0	4	0
T	81	30	8	19	10
V	21	23	12	15	18

Table B3. Percentage of exercises that were coded for each representation system type. Totals will not add to 100 as each exercise could have multiple codes.

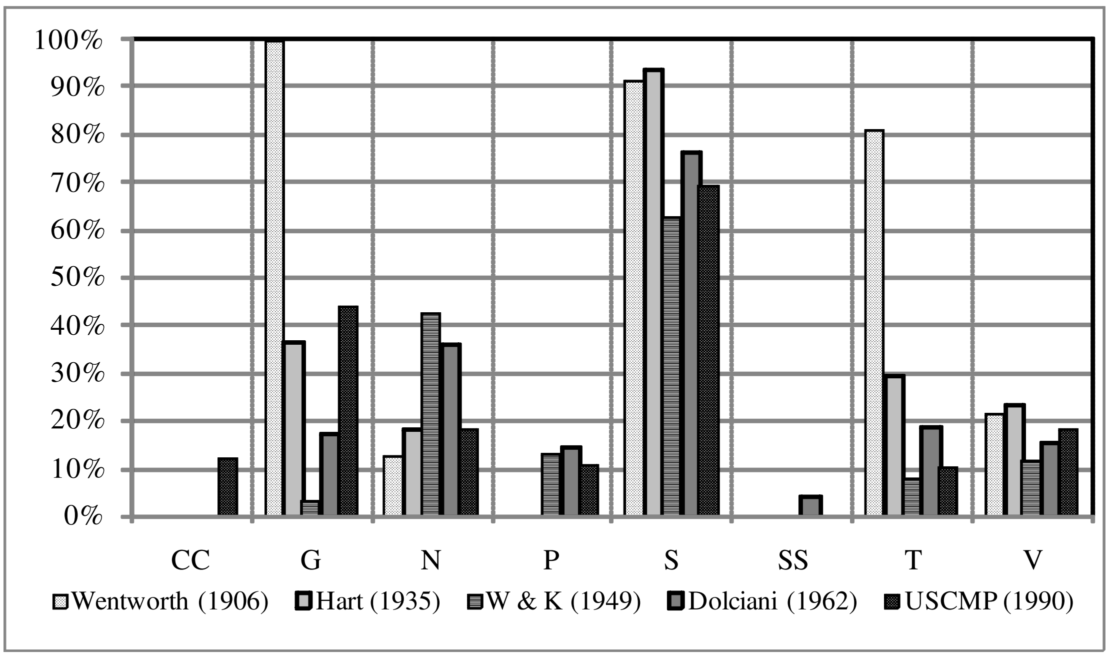


Figure B6. Distribution of representation systems across time.

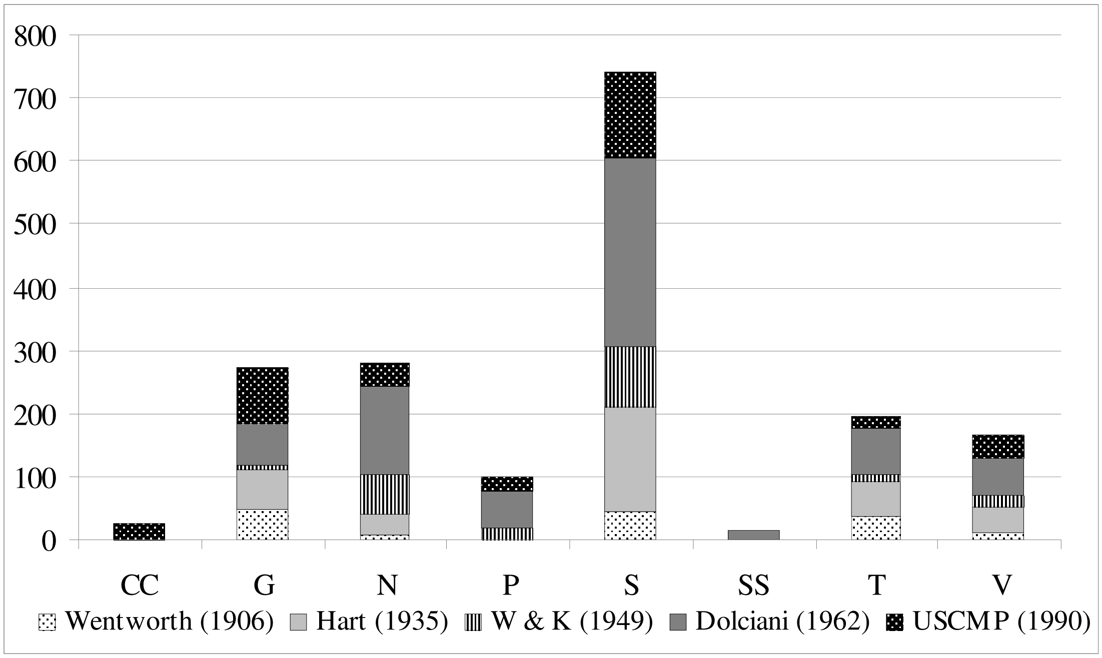


Figure B7. Total number of problems coded for each representation system.

Control Structures

	Wentworth (1906)	Hart (1935)	W & K (1949)	Dolciani (1962)	USCMP (1990)
CLC	0	0	0	0	14
CON	0	0	0	2	2
CPE	30	35	49	40	34
DC	89	93	48	88	64
LFLUR	11	22	32	31	23
MT1P	0	0	0	6	13
UAR	23	6	3	10	22
UCP	40	87	0	31	3
UGI	0	0	0	1	<1

Table B4. Percentage of exercises that were coded for each control structure type. Totals will not add to 100 as each exercise could have multiple codes.

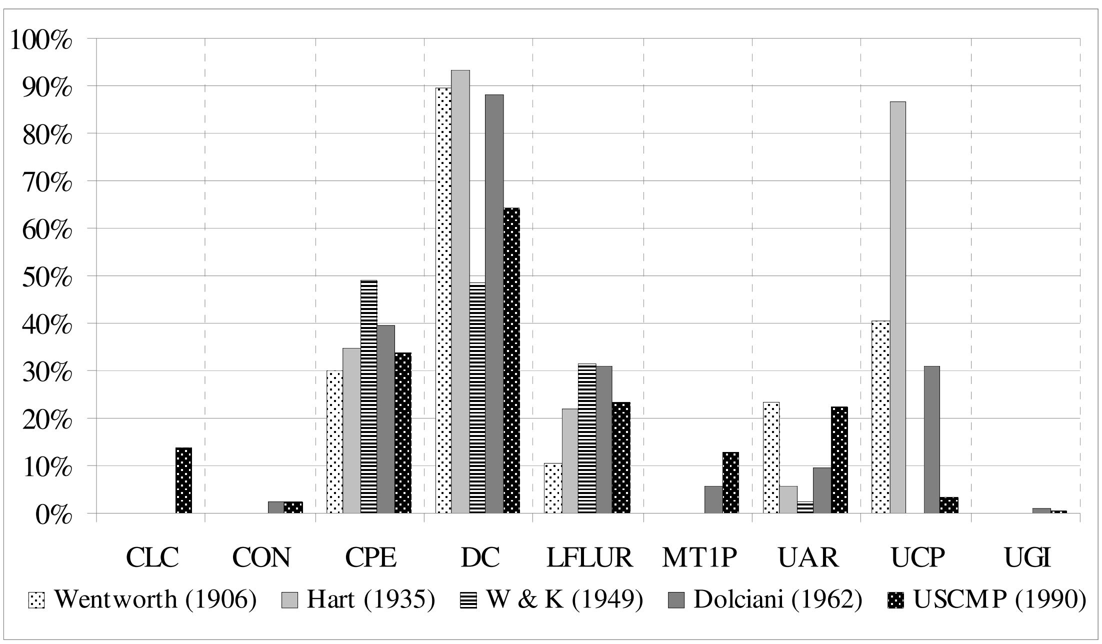


Figure B8. Distribution of control structures across time.

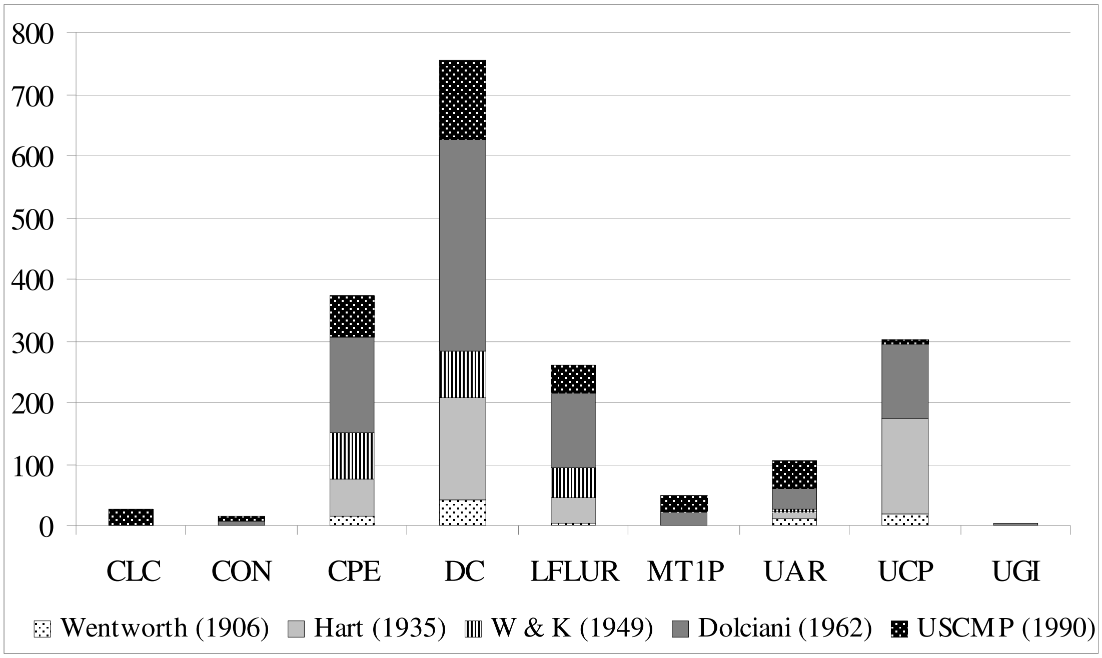


Figure B9. Total number of problems coded for each control structure.

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