

ABSTRACT

Title of Dissertation: A TALE OF TWO COURSES; TEACHING
AND LEARNING UNDERGRADUATE
ABSTRACT ALGEBRA

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The abstract algebra course is an important point in the education of undergraduate mathematics majors and secondary mathematics teachers. Abstract algebra teachers have multiple goals for student learning, and the literature suggests that students have difficulty meeting these goals. Advisory reports have called for a move away from lecture toward investigation-based class sessions as a means of improving student understanding. Thus, it is appropriate to understand what is happening in the current teaching and associated learning of abstract algebra.

The present study examined teaching and learning in two abstract algebra classrooms, one consciously using a lecture-based (i.e., deduction-theory-proof, or DTP) mode of instruction and the other an investigative approach. Instructional data was collected in classroom observations, and multiple written instruments and a set of interviews were used to evaluate student learning.

Each instructor hoped students would develop a deep and connected knowledge base and attempted to create classroom environments where students were constantly engaged as a means of doing so. In the lecture class, writing proofs was the central activity of class meetings; nearly every class period included at least one proof. In the investigative class, the processes of computing and searching for patterns in various structures were emphasized.

At the end of the semester, students demonstrated mixed levels of proficiency. Generally, students did well on items that were relatively familiar, and poorly when the content or context was unfamiliar. In the DTP course, two students demonstrated significant proficiency with analytical argument; the remainder demonstrated mixed proficiency with proof and very little proficiency with other content. The students in the investigative class all seemed to develop similar levels of proficiency with the content, and demonstrated more willingness to explore unknown structures.

This study may prompt discussions about the relative importance of developing proof-proficiency, students' ability to formulate and investigate hypotheses, developing students' content knowledge, and students' ability to operate in and analyze novel structures.

A TALE OF TWO COURSES; TEACHING AND LEARNING
UNDERGRADUATE ABSTRACT ALGEBRA

By

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DEDICATION

To my wife Kate

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CHAPTER 1: RATIONALE AND SIGNIFICANCE

Rationale

The upper-division abstract algebra course is an important point in the undergraduate education of mathematics majors and pre-service high school mathematics teachers (Committee on the Undergraduate Program in Mathematics (CUPM), 1971; Mathematical Association of America (MAA), 1990; Conference Board of the Mathematical Sciences (CBMS), 2001). The course aims to develop student understanding and skill in work with mathematical structures such as groups, rings, and fields. But it is also expected to develop students' ability to analyze and construct mathematical proofs, to develop general habits of algebraic thinking, and to illuminate structures that underlie algebra in the school curriculum.

Unfortunately, while the abstract algebra course is one of great possibility, the literature suggests that many students are not meeting many of these goals (Findell, 2000; Hazzan & Leron, 1996; Leron & Dubinsky, 1995). Evidence from anecdotal reports and exploratory studies of student learning documents the gap between goals and results and suggests some explanatory factors (Edwards & Brenton, 1999; Findell, 2000; Larsen, 2004; Leron, Hazzan & Zazkis, 1995; Weber, 2001). But there has been little empirical research addressing the connections between instructional approaches implemented in abstract algebra courses and what students learn in those courses.

Abstract Algebra and the Undergraduate Mathematics Curriculum

Algebra is one of the preeminent disciplines in the mathematical sciences. Almost every undergraduate mathematics major is required to complete at least one semester of abstract algebra, and mathematicians and mathematics educators believe the course to be so important that the recent CBMS report on the *Mathematical Education of Teachers* (MET) explicitly argued for keeping the course in the pre-service teacher curriculum (2001).

In its current incarnation, “most such courses at the undergraduate level have a dual objective. Besides mastering the course content, students are expected to learn to write proofs that they have devised for themselves” (Edwards & Brenton, 1999, p. 122). Many researchers would argue that there is a third (often implicit) learning goal associated with abstract algebra courses—that the students should be improving their algebraic thinking skills (Cuoco, Goldenberg, & Mark, 1996; Smith, 2003). Others have suggested that the abstract algebra course is the “place where students might extract common features from the many mathematical systems that they have used in previous mathematics courses” (Findell, 2000, p. 12). In short, the abstract algebra course carries substantial expectations for student learning.

While an abstract algebra course may be associated with multiple learning goals, the literature suggests that students have difficulty meeting the goals their instructors may hold. The CBMS (2001) reported that most students fail to make effective connections between abstract algebra and other mathematics. Hypothesized explanations for student failure in abstract algebra focus attention on two main conjectures: instruction and student effort. In 1995, Leron and Dubinsky claimed, “The teaching of abstract algebra is a disaster, and this remains true almost

independently of the quality of the lectures” (p. 227). Advisory reports issued by the National Science Foundation (NSF, 1992) and the Mathematical Sciences Education Board (MSEB, 1991) have called upon faculty to move away from the lecture format and towards investigation-based class sessions in undergraduate mathematics courses as a means of improving student understanding. The appropriateness of these recommendations for abstract algebra in particular have been echoed by others writing on the topic (Burton, 1999; Edwards & Brenton, 1999; Hibbard & Maycock, 2001). However, in defense of the lecture method Wu (1999) carefully laid out his assumptions about collegiate education and used those to extrapolate some basic goals for his abstract algebra course. For example, he stated that it is critical to introduce students to all those topics in abstract algebra that are prerequisites for graduate study.

Not all faculty would lay the blame for students’ perceived failures at the feet of the instructors. For example, Wu (1999) has argued that students often fail because they have either an “unwillingness or inability to work on their own...” (p. 13) and that students are often “coming to class unprepared, or for that matter, leaving it without making an effort to understand it later” (p. 8).

Whatever the reason, students find abstract algebra very difficult and often fail to meet many of the important goals for the course (Dreyfus, 1999; Dubinsky, et. al, 1994; Hart, 1986; Hazzan, 1994; Hazzan, 1999; Hazzan & Leron, 1996; Leron, Hazzan, & Zazkis, 1995; Weber, 2001). Yet, there are also those who suggest that instructors’ beliefs about student failures are themselves contributors to student failure, and these beliefs certainly have an effect on what an instructor believes to be

reasonable goals for the course. “This conspiracy of expectations may lead to lowered goals and student achievement, and may be a contributing factor in the discrepancy between the “intended” and the “implemented” curriculum” (Francis, 1992, pp. 27-28).

Mathematicians and educators have long questioned whether the traditional abstract algebra course was meeting the needs of all of their students. For example, since 1964, the University of Maryland has taught a separate section of abstract algebra designed to meet the needs of pre-service teachers and mathematics majors not heading to graduate school in mathematics. In an abstract algebra textbook designed for that course, Davidson and Gulick (1976) wrote, “This book has grown out of our concern over the traditional method of teaching, accompanied all too frequently by passive learning for the student” (p. ix). Unfortunately, traditional teaching methods and concerns about their efficacy persist, as do concerns about student learning.

It is important to note that although the literature is filled with discussion of student difficulties, it is also filled with affirmations of the importance of the course. Recommendations call for the course to be improved rather than eliminated (Burton, 1999; Cuoco, 2001; CUPM, 1971; MAA, 1990). Increasingly these papers and presentations at professional meetings related to the teaching and learning of abstract algebra are exploring the understandings of abstract algebra that students do acquire, rather than describing how student understanding falls short of the ideal (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Edwards & Brenton, 1999; Leron & Hazzan, 1999; Iannone & Nardi, 2002).

Teaching and Learning Advanced Mathematics

There are few studies that describe traditional (lecture-based) teaching of abstract algebra. The most thorough of such descriptions characterize traditional teaching as primarily a recitation of content presented in a “definition-theorem-proof-corollary-example-application format” (Edwards & Brenton, 1999, p. 122). Much of the teaching based upon this format involves the instructor writing mathematical definitions, theorems, proofs, and step-by-step solutions to exemplar exercises on the board, with the instructor reciting what is being written. The instructor is also responsible for assigning homework problems from a text, creating exams, and grading homework. During classes with this format, the student’s responsibility is to copy everything written on the board as coherently and completely as possible and to spend time outside of the class working to understand material in the notes and the text.

William Thurston (1986), winner of the Fields medal in mathematics, gave a more thorough (if cynical) description of what such an undergraduate mathematics classroom might look like:

...we go through the motions of saying for the record what we think the students “ought” to learn, while the students are trying to grapple with the more fundamental issues of learning our language and guessing at our mental models. Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material “covered” in the course, and then grading the homework and tests on a scale that requires little understanding. (p. 343)

Weber (2004) noted his dissatisfaction with Thurston’s description when he stated, “It is widely accepted that advanced mathematics courses are frequently taught in what is colloquially referred to as a “definition-theorem-proof” (DTP) format... I am not

aware of a precise set of criteria that one can use to define DTP instruction” (p. 116). He continued by stating that he was unaware of any studies that actually described such teaching in an advanced mathematics class.

Although DTP is seen as the dominant mode of teaching in upper division undergraduate mathematics, it is also critiqued as intimidating and as misleading students about the nature of mathematics (Thurston, 1986; Cuoco, Goldenberg, & Mark, 1996), hiding much of the process used in mathematical thinking (Dreyfus, 1991), and ignoring the important role that mathematicians ascribe to ideas such as elegance, intuition, and cooperation (Burton, 1999; Dreyfus, 1991; Fischbein, 1987). The most fundamental critique that has been leveled against DTP is that it is not an effective way to promote student learning of the mathematics content (Leron & Dubinsky, 1995; MSEB, 1991; NSF, 1992). However, none of those making this last critique provide student data to substantiate their claims. Moreover, the same people who claim DTP is not an effective method do not describe their goals for student learning and the relative importance of those goals.

The critiques of DTP and the strength of faculty beliefs about students’ corresponding lack of success have given rise to a variety of class-level and program-level restructurings of the abstract algebra curriculum—each intended to improve student learning. In 2001, Hibbard and Maycock collected a large number of essays describing strategies for classroom change that had been tested at a variety of colleges and universities. There have been contributed paper sessions on classroom change strategies for abstract algebra at many recent Joint Mathematics Meetings. As an example of such a programmatic change, the University of Northern Colorado has

restructured its entire algebra experience for undergraduates with significant emphasis on student interaction as the focal point of the experience (Mingus, 2001). At the University of Northern Colorado, as described by Mingus, class meetings are problem-driven and feature students spending much of their time working collaboratively rather than taking notes. Another important feature of the approach at the University of Northern Colorado is the structure of out-of-class meetings. Tutoring for abstract and linear algebra is done at the same time and place in hopes of increasing students' awareness of the connections between their classes.

Changes like those at the University of Northern Colorado are based upon beliefs about student learning, not empirical research literature. In general, the changes involve transition from a teaching method that everyone can recognize (i.e., traditional lecture format) towards a type of teaching suggested by the MSEB (1991) and NSF (1992) advisory reports. Francis (1992) raised concerns about instructional change without evidence with an analogy to medicine:

The diagnosis of a problem must precede the prescriptions for its cure, and the thoroughness of the diagnosis and the quality of the information gathered prior to analyzing and subsequently forming conclusions leads to a more effective prescription in most cases. (p. 38)

The state of the field can be summarized as follows: Problems with the teaching and learning of abstract algebra have been identified and the field is proposing cures without agreement on the cause of the problem or even an accurate description of current practice. Before restructuring the teaching of abstract algebra in order to support student learning, it is necessary to understand what *is* happening with respect to current practices for teaching and learning abstract algebra.

Teaching and Learning Algebra

There have been a number of studies that describe individual students' learning and understanding of abstract algebra concepts (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996; Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997; Brown, DeVries, Dubinsky, & Thomas, 1997; Dubinsky, Dautermann, Leron, & Zazkis, 1994). There are studies that provide suggestions for classroom pedagogical improvement (Hibbard & Maycock, 2001; Leron & Dubinsky, 1995), and some describe formative assessments of these new pedagogies (Edwards & Brenton, 1999; Mingus, 2001), detailing programmatic changes undertaken to improve student retention and understanding (Mingus, 2001).

Descriptions of student learning and understanding of concepts

Studies that examine students' understandings of important ideas in abstract algebra and their abilities to make use of that content are the most common. For example, Dubinsky, and his colleagues (1994), offered a stage-theory perspective of student learning of group theory based upon the Piagetian process/object duality. Based upon student responses to various assessments, the authors articulated a genetic epistemology based upon the Action-Process-Object-Schema (APOS) model for the learning of the concept of group. This learning model was then refined and expanded by Brown, et al. (1997) and Asiala, et al. (1997).

The studies investigating student understanding of abstract algebra have focused solely on students, not on the instruction. For example, Dubinsky, et al. (1994) offer only a single paragraph descriptor of the classroom experiences of the students in this study.

Classroom strategies

Suggestions for innovative classroom and curricular strategies are delineated within the MAA volume, *Teaching Innovations in Abstract Algebra* (Hibbard & Maycock, 2001). Several papers in this collection offer (a) suggestions for assignments that will engage students more fully in investigating content, (b) new ways of structuring class time, and (c) problems that require more active thinking on the part of students. Other papers in this text describe technological innovations that are designed to assist with both student engagement and lowered computational barriers, thus allowing more time to be spent studying algebraic structure. However, authors of these papers generally presented only anecdotal data describing incidents of increased student learning.

Linking pedagogy and student outcomes

Studies that link the teaching and the learning of abstract algebra are rare in the literature. Moreover, located papers typically employ only rudimentary measures for student success. For example, one study's measure for success was to indicate the number of students earning a passing grade out of the total number of enrolled students. The authors then claimed, without support, that the passing ratio was higher than in more traditionally taught courses. Few of the papers include a description of the knowledge that students gained, a comparison with a traditional course on any but the most cursory of measures, or a thorough description of classroom interaction.

When writing about the reform initiative underway in calculus, Ganter (2001) noted:

...there is a great need for information on various instructional formats and their subsequent effects on learning.... Concrete information about what and how students learn, as well as decisions about program improvements, cannot be reasonably made with only anecdotes and success stories. The value of changes made to a course is very difficult to quantify; such decisions cannot be made without an enormous amount of data collected over a long period of time that include information on students before, during and after the course.... It is imperative to understand not only how student learn, but also the actual impact of different environments on their ability to learn. (p. 6)

Instructors are making changes in their abstract algebra courses; universities and colleges are making changes in their undergraduate mathematics programs based upon a set of beliefs about teaching and learning abstract algebra. But, instructors are making these changes in absence of a point of comparison for either traditional teaching or student learning, as they continue to publish success stories (Asia, et. al, 1997; Brown, et. al; 1997; Dubinsky, et. al, 1994; Edwards & Brenton, 1999; Hibbard & Maycock, 2001, Mingus, 2001). A needed study is one that describes the teaching of a DTP abstract algebra course and a reform-influenced abstract algebra course as well as the subsequent student learning in those courses.

Research Questions and Overall Study Design

The purpose of the study is to describe and analyze instances of different pedagogies used within abstract algebra courses. I offer a description of some of the different types of understandings and beliefs that students may develop while studying abstract algebra, and I attempt to describe some possible means of understanding the confluence of instruction and student development of understandings. As such, this study informs the conversation about pedagogy its alignment with goals for an abstract algebra course and students' mathematical development.

This study examined teaching and learning in two sections of an upper-division abstract algebra course—one using a lecture-based *definition-theorem-proof* style of instruction and the other using a more *investigative* approach to instruction. A variety of observation, interview, and student performance data was collected in order to address the following questions:

Teaching and Learning in the DTP Class

1. What are the defining characteristics of the teaching scripts collected in a *DTP* abstract algebra class?
 - a. What are the salient characteristics of teacher talk in the teaching scripts?
 - i. What types of declarative statements do teachers make?
 - ii. What types of questions (including rhetorical questions) do the teachers pose to the students?
 - iii. To whom do the teachers pose the questions (to the whole class, a group of students, or an individual student)?
 - b. What are students expected to do during the *DTP* class meetings?
 - i. What types of student action are encouraged by the instructor?
 - ii. What activities are the students engaged in during class?

Teaching and Learning in the Investigative Class

2. What are the defining characteristics of the teaching scripts collected in an *investigative* abstract algebra class?

- a. What are the salient characteristics of teacher talk in the teaching scripts?
 - i. What types of declarative statements do teachers make?
 - ii. What types of questions (including rhetorical questions) do the teachers pose to the students?
 - iii. How do the teachers pose the questions (to the whole class, a group of students, or an individual student)?
- b. What are students expected to do during the *investigative* class meetings?
 - i. What types of student action are encouraged by the instructor?
 - ii. What activities are the students engaged in during class?

Comparison of DTP and Investigative Class Teaching and Learning

3. Which, if any, of the characteristics in the collected teaching scripts seem to best differentiate an *investigative* abstract algebra class from a *DTP* abstract algebra class?
4. Which, if any, of the characteristics in the collected teaching scripts do *DTP* and *investigative* abstract algebra classes have in common?

Developed Mathematical Proficiency in DTP and Investigative Abstract Algebra Classes

5. What mathematical proficiency with the material of an introductory abstract algebra course is evidenced by students who voluntarily complete additional assessments during the *DTP* course?

6. What mathematical proficiency with the material of an introductory abstract algebra course is evidenced by students who voluntarily complete additional assessments during the *investigative* course?
7. What are the similarities and differences in mathematical proficiency developed by the students in the *DTP* class and the *investigative* class?

Limitations of the Study

There are no accepted theoretical principles or frameworks for teaching abstract algebra in either the traditional or reform models. While CBMS (1999) posited abstract algebra's importance in the mathematical education of secondary mathematics teachers, there was no published theoretical lens that highlighted connections between abstract algebra content, the knowledge of secondary mathematics teachers and the instruction offered by those teachers. Similarly, there is no theory or framework that specifies the generalized knowledge of mathematical structures or the reasoning underlying mathematical proof that are posited as key characteristics of an abstract algebra course. Thus, the defining characteristics of the teaching scripts in this study are not drawn from an a priori theoretical model. This is a recognized limitation of this study.

The methodology of the current study was also limited. First, because the two course sections were small, taught by autonomous faculty members, and peopled by different collections of students, each of the course offerings were unique. Due to the uniqueness of each context, the curricular pacing and the emphasis of the two course offerings were distinct. For example, one section spent considerably more time

studying the algebraic construction of roots of irreducible polynomials, whereas the other section spent almost 3 weeks more studying group-theoretic material. Time on task is an important predictor of student learning both at the macro and micro levels (Porter & Brophy, 1988). Reported differences in student understanding between the two course offerings are confounded by the amount of instructional time spent on a topic.

An additional methodological limitation of this study relates to the measure of participating student's mathematical proficiency. Each of the written assessments were taken home by students and completed with use of students' text and notes. As such, these measures did not directly assess the students' knowledge of definitions and typical examples of abstract algebra content.

This was a small-scale, exploratory study addressing the teaching of abstract algebra and the resulting mathematical proficiency of students as exemplified in single sections of an investigative and DTP style of teaching. Any conclusions are preliminary.

CHAPTER 2: LITERATURE REVIEW

The purpose of this study was to examine the characteristics of two instructors' abstract algebra courses that employed differing instructional methodologies, and to describe the mathematical proficiencies that students developed as a result of such experiences. Specifically, the study seeks to describe and characterize DTP and investigative teaching of abstract algebra. This literature review describes research on teaching and is organized into two sections, one that examines research on teaching and the other research on student understanding of abstract algebra.

The first section examines research on teaching with an emphasis on teaching undergraduate mathematics. Because there was little located research that focuses on teaching abstract algebra, the chapter will also review those papers that principally offer pedagogical suggestions. By far, most of the published papers about teaching abstract algebra describe pedagogical techniques. The dearth of located research about the teaching of abstract algebra indicates the importance of the current study. The section devoted to research on teaching has been further sub-divided into three subsections. The first subsection is focused on DTP teaching, first describing a learning theory which supports that type of instruction and then surveying the previous research on DTP teaching. The second subsection is focused on investigative teaching and is similarly organized. The third and final subsection is focused on the manner in which an instructor would decide upon pedagogical style.

A second purpose of this study was to describe the mathematical proficiencies that students developed while taking an abstract algebra course. As such, the second section of the chapter summarizes previous research on student understanding of abstract algebra topics and proof in abstract algebra.

Teaching Advanced Mathematics

The present study seeks to characterize two different instructional approaches to teaching abstract algebra. Before embarking on such an undertaking it is prudent to examine what literature exists that describes the two approaches to teaching as well as any literature that describes how instructors chose which style to implement. Because this section of the literature review examines previous work describing the two approaches to teaching, it will also include a survey of pedagogical suggestions made by abstract algebra instructors and any recorded research results.

The two most commonly caricatured teaching styles are lecture and something that might be referred to as ‘investigative,’ each of which can be seen as being supported by different learning theories. I will begin this examination with a summary of the theory underlying the two pedagogical approaches and then turn to studies and suggestions of class activities aligned with and investigative pedagogy.

DTP Teaching

The lecture style is commonly described as Definition-Theorem-Proof (DTP), although the actual order of presentation of the mathematics may vary, and the instructor may include lecture features that might be described as motivational or process-oriented (Edwards & Brenton, 1999; Thurston, 1994). In this case, students

are being asked to learn mathematics in a top-down manner or from the general case to the concrete example. That is, students are introduced to the abstract mathematical definitions and general theorems and then shown examples of those concepts and principles.

The belief in the ability to learn from general to concrete might be seen as aligned with the learning theory proposed by Lev Vygotsky (Kozulin, 1998). Vygotsky argued that schooling is not a time for direct application of cognitive resources, but rather, school “aims for a deliberate “denaturalization” of the students’ position, so that children can make their own actions a subject of their own deliberate analysis and control” (Kozulin, 1998, p. 47). Moreover, the idealized classroom based upon Vygotsky’s theory is one in which “instead of learning a particular task or operation the child acquires a more general principle applicable to different tasks” (Kozulin, 1998, p. 47). That is, the learner should be given a theory or general model and then learn to apply it in a variety of more concrete situations. While Vygotsky’s view provides some psychological support for a general to specific developmental approach, it is unlikely that most of the algebra instructors deciding to lecture and present a theoretical description first have read Vygotsky and taken the time to understand his positions. It seems more likely that the decision to lecture is not based in educational theory, but rather more aligned with the other concerns described below.

Instructors who hold the belief that students learn in a Vygotskian manner are probably strongly influenced by their own experiences (or the idealized recollection of their experience) as a mathematics learner. That is, instructors teaching abstract

algebra were generally very successful learners of both mathematics generally and algebra in particular. Thus, they have direct evidence that the traditional form of instruction (i.e., the one that they experienced as students) is effective. Additionally, they typically have no evidence (other than educational research, which is not widely read in the community of mathematicians) that any other style of teaching is effective.

An example of an analysis of lecture-style teaching of a real analysis class is seen in Weber's (2004) work. This analysis served to suggest some possible categorizations of lectures that may be found in an abstract algebra course.

Weber (2004) examined the instruction in an introductory real analysis course and identified three basic styles of teaching proof: *logico-structural*, *procedural*, and *semantic*. He characterized the *logico-structural* approach by its reliance on formal mathematical statement, the conspicuous lack of diagrams, lack of any semantic meaning for the concepts or the proof, and emphasis on careful use of definitions to both start and conclude a proof. Weber characterized the *procedural* style of teaching proof by its complete lack of semantic meaning; students were supposed to learn the structure of the argument and no more. The instructor (Dr. T in this study) would write the beginning and conclusion to the proof and "He [the instructor] would often remark about how one should always start the type of proof in the way he did" (Weber, 2004, p. 125). Then, given the incomplete proof, Dr. T. would demonstrate how to complete it. Lastly, the *semantic* teaching style is characterized by the instructor's use of intuitive descriptions of concepts and relationships. In each case, the instructor's teaching style was chosen to help students acquire some specific type

of knowledge or ability to construct proof (as well as demonstrate a large number of similar proofs).

What the faculty may likely believe, and what Weber's (2004) Dr. T stated, is that, "in the beginning of the course, students could only understand the topics as a string of words" (p. 128). In fact, Dr. T's teaching seemed particularly aligned with such a belief. His first two lecture styles made no effort to help students understand the ideas, the relationships between them, or the semantic meaning of the proofs. Instead, the students were supposed to be mimicking the actions they observed in class, a fact that Dr. T acknowledged by saying, "early in the course, you could get by with certain tricks and skills" (Weber, 2004, p.128). That is, Dr. T acknowledged that early in the semester, his students could complete nearly all of their work without any understanding. Yet, he continued by suggesting that at the end of the course students could not succeed without understanding the meaning of the ideas and semantic structure of the required proofs. In fact, Dr. T. believed that it is only after being able to perform the necessary symbolic manipulations that, "They can begin to understand what these words really mean" (Weber, 2004, p. 128).

Throughout the piece, Weber emphasizes the power that the instructor has in the situation: Dr. T. possesses the knowledge and hands it down to the students using different techniques of teaching, which center around exposition and giving examples. Exercises are performed to reinforce knowledge and practice its application; students first mimic the instructor and are eventually given additional tools to gain a better understanding of the meaning behind the examples. Throughout, all knowledge is provided to the students in the lecture-style classroom,

beginning with the general case and moving to the specific examples. This is in contrast with investigative teaching methods, which I will describe next.

Investigative Teaching

The ideal investigative teaching style likely includes a markedly increased amount of student conversation and interaction during class, with a corresponding decrease in the amount of time that the instructor spends making expository remarks (Davidson & Gulick, 196?). In investigative classes, students typically spend their class time working on mathematical tasks and sharing their findings in some way. Again, the way in which such teaching is enacted probably varies quite widely in the undergraduate classroom.

A second major theory of teaching advanced mathematics is defined by Freudenthal (1973) as moving from the particular to the general in a way that appears to be aligned with the historical development of mathematics. Freudenthal noted that, historically, formal definitions only appeared at the end of a long period of mathematical exploration with specific examples. He argued that mathematical instruction should mirror this process. Thus, the order of introduction is example-definition. Freudenthal's work is almost certainly informed by Piagetian stage theory. Piaget believed that students progress from "action to thought" (Kozulin, 1998, p. 52). The ideal Piagetian classroom is a rich environment for students where they come into contact and experiment with a variety of concrete expressions of an idea. It is through the students' interaction with these ideas (action) that they gradually acquire abstract understanding (thought). Piaget's work has been expanded upon by a variety of theorists, as will be described below. At the moment, it is

sufficient to show that Freudenthal's work, which Larsen (2004) argues is the basis for much of the reform effort in the teaching of abstract algebra, is aligned with that of Piaget.

Freudenthal's (1973) theory grew out of his beliefs about learning. Specifically, he claimed that mathematics moves from the particular to the general in both historical development as well as in individuals' minds. Freudenthal was even more specific regarding the teaching of group theory, writing that groups should be introduced through exploration of concrete examples of systems of automorphisms on structures. He felt that this approach has two major benefits. The first is that exploration of the collection of automorphisms is an activity similar to others that students had undertaken when exploring functions in previous classes. The second major benefit is that when introduced in this way, all such collections exhibit the group properties. Other theorists (e.g., Burn 1996; Dubinsky & Leron, 1994) have postulated similar theories of teaching that begin with asking students to explore concrete examples. Burn wrote that he began his course on algebra by having students engage with a multitude of tasks on geometric symmetry before even introducing the group axioms. He claimed that these axioms "were then immediately valued by the students" (p. Burn, 1996, p. 377). This view is mirrored by a passage in Dubinsky and Leron's statement to instructors on the proper way to use their text:

It should be noted that although it is assumed that each learning cycle begins with activities, the students are not expected to *discover* all the mathematics for themselves. In fact, since the main purpose of the activities is to establish an *experimental basis* for subsequent learning, anyone who spends a considerable time and effort working on them, will reap the benefits whether they have discovered the "right" answers or not." (1994, p. xvii)

The expository pages in Dubinsky and Leron's (1994) investigation-based abstract algebra text have the same mathematical statements of definitions, theorems and proofs, in the same order, as texts designed for use in a DTP class. The difference for students in a DTP class as compared with a class based upon a "constructivist approach to teaching" (Dubinsky & Leron, 1994, p. xvii) arises from the students' experience of the *process* of mathematics, by which authors commonly refer to the student's exploration of structures and the implication of a variety of rule-systems imposed upon those structures. Teaching aligned with Freudenthal's (1978) position requires that students engage in the process of attempting to make meaning from mathematical exploration and use that process to imbue the subsequent statement of definitions with meaning. This is contrasted with the DTP mode of teaching which insists that students, when presented with an abstract definition, can derive or make meaning from the subsequent exploration of examples, properties and logically derivable statements based upon that definition.

A case study in investigative teaching

One of the principle purposes of the current study is to describe what happens in an investigative abstract algebra class. In order to situate the current study in the research literature, it is appropriate to survey the previous work that has described investigative teaching. As noted with DTP teaching, not much research literature describes what actually happens in classrooms was located. This section details a description of an entire programmatic change at the University of Northern Colorado. The section that follows this one will provide a summary of other studies that address issues of teaching in abstract algebra, but none are as comprehensive in both

describing what happens in classroom and providing a description of the student learning outcomes.

The University of Northern Colorado study (Mingus, 2001) was based upon efforts to link a trio of courses (including abstract algebra) and to reform the pedagogy in those courses in order to bring it into greater alignment with what is known about the learning of advanced mathematics. She described their reform efforts as an attempt to create a curriculum that would:

- i) Inspire students to think abstractly and appreciate the need for abstraction.
- ii) Foster independence in the learning of abstract mathematics.
- iii) Enable understanding and value the need for mathematical proof.
- iv) Facilitate communication of student understanding to other people.

(Mingus, 2001, p. 28)

The faculty employed a variety of pedagogical approaches, including the use of small group work and, in the abstract algebra course, the software package *Exploring Small Groups*, which is specifically designed to help students develop understanding of group theoretic examples without heavy amounts of computation. The faculty also offered extended study sessions in which students from all three courses came together for 3 hours. Each class was allotted a single hour to ask questions for presentation at the board and the rest of the time was to be spent engaged in small group work or interacting with students in the other two classes. The faculty believed that this would help students see the mathematical connections between the courses as well as solidify prior knowledge.

Mingus (2001) continued, “another innovation used in abstract algebra was to take the chalkboard and overhead away from the instructors (and students). Students were asked to talk about normality and to describe quotient groups” (Mingus, 2001, p.

29). She asserted that these chalk-less talks often yielded insights about student misconceptions that would never have been apparent from more traditional modes of interaction. In order to make students more responsible for the production of original proofs and contributions to the class knowledge base, index cards were passed around at the beginning of the semester, each with a single theorem written on it. Students were expected to create a proof for the theorem on the card they received and present it to the class at an appropriate time during the semester. The students were expected to work closely with the instructor outside of class to develop this proof, an instructional strategy which had the added benefit of helping the students and instructor to develop a bond and to interact in a mathematically meaningful manner. Students' class work often involved working in small groups on problems that were of great importance for the continued mathematical development of the class. That is, the students were engaged in meaningful work, and the results they produced were, as Mingus describes them, "*structural* rather than pedantic, as the subsequent disciplinary development depended upon those results" (Mingus, 2001, p. 30).

The faculty believed that these pedagogical practices would help students to see and appreciate the connections between courses and concepts. Mingus (2001) stated,

The strength and depth of these connections can serve as a means for anchoring a student's understanding and enhancing their ability to recall that knowledge in problem solving situations. Students typically fail to make such connections on their own...as a consequence of their attitudes and beliefs about mathematics. Students develop negative attitudes and beliefs, including the view that it [mathematics] is an unchanging, disconnected discipline, as a result of the curricula to which they are exposed and the continued use of an absorption model of teaching. (p. 30)

Mingus' (2001) evaluation of the course sequence involved interviews with 12 students. She presented the students' comments about group work and technology as overwhelmingly positive and as enhancing their learning experiences. She argued that the use of technology enabled the students to develop a "conceptual understanding" rather than procedural fluency with co-sets and quotient groups (Mingus, 2001, p. 34). Should these courses have enabled students to develop a conceptual understanding (a working description of "conceptual understanding" is, in the language of Vinner (1991), to hold a concept image that is well correlated with a mathematically correct concept definition and be able to operate on both the image and the definition), they should be considered successful.

The description that Mingus (2001) offered, when coupled with the assessment that Grassl and Mingus (2004) provide, give a reasonably thorough description of students' reactions to the class. Grassl and Mingus (2004, ¶ 6) stated:

We observed together how students, once provided the assurance that their ideas would be listened to, can make great progress on resolving background deficiencies and moving forward. Half of the 25 students were women; evaluations and general discussion indicated that the presence of a female co-instructor tended to 'soften' classroom tone, creating a friendlier learning environment for them. The students reported experiencing a family-like atmosphere and its positive impact on their attitudes about the class and the subject. These improved attitudes translated into increased participation, willingness to take risks, decreased attrition (only one of 25 dropped), and increased attendance (On the average, only one person per week was absent).

Grassl and Mingus also claim that their pedagogical style improved students' senses of self-efficacy and responsibility for learning the class materials (something which Wu (1999) would clearly applaud). They stated, "the majority of the students took fuller responsibility for their own learning" (Grassl & Mingus, 2004, ¶ 6). Moreover, Grassl and Mingus claimed that this sense of responsibility had consequences for

student learning of the material as well as the students' sense of self-efficacy. They stated, "As a result of taking responsibility for their own learning, they were able to take responsibility for their success in the course; this increased their self-confidence..." (Grassl & Mingus, 2004, ¶ 5). Grassl and Mingus did not offer their findings as a contrast with a traditional abstract algebra class, but rather wrote about the effects of the modified class in isolation. Thus, their claims about increases in self-efficacy and responsibility on the part of the students should be viewed as statements about absolutes rather than as a comparison between two distinct courses.

Other studies that describe investigative teaching

What follows is a summary of a series of papers that offer pedagogical suggestions about teaching abstract algebra in an investigative style. The majority of the articles on teaching abstract algebra (i.e., Edwards & Brenson, 1999; Dechéne, 2001; Larsen, 2004; and Grassl & Mingus, 2004) describe pedagogical suggestions and then give some brief descriptor of the outcome. Additionally, there is a growing body of literature that looks specifically at student understanding of proof, including Dean (1996) and Weber (2004), which is not a principle focus of this study, but still merits inclusion. While none of these are as comprehensive as the Mingus study above, they provide insight that informs this paper.

Edwards and Brenton (1999) wrote about a teaching experiment in abstract algebra in which they engaged. Following the suggestions of Vinner (1991), they sought to structure their classroom in a manner that would help students to develop a concept image before acquiring a concept definition. In order to do so, they wrote that they made an effort to engage "students in activities with concrete examples"

(Edwards & Brenton, 1999, p. 123) and to generate conjectures and proto-definitions based upon their work with these examples. They continue, “we actively engaged students not only in the concrete activities themselves, but also in reflective discussion which followed those activities... By focusing discussion on the regularities or properties which they observed... students themselves were able to abstract the defining properties of group” (Edwards & Brenton, 1999, p. 123). Moreover, the reflective discussions were an attempt to help students understand ways of thinking mathematically, that is, to help students learn to think in ways that would advance their understanding of mathematics.

Edwards and Brenton (1999) described their work on an innovative course and then described the student-outcomes. The authors found that the experimental course resulted in greater levels of persistence; 24 of the initial 28 students persisted (87.5%), as compared with a 57% level of persistence over the prior 3 years in traditional abstract algebra courses. They also stated that, “students were able to form most of the constructions we intended, and active engagement in such constructions of knowledge apparently increased their confidence in their own ability to master the material” (Edwards & Brenton, 1999, p. 125). Here, they argued that they achieved good levels of student understanding, and that the course had a positive effect on student beliefs about self-efficacy.

Dechéne (2001) is one of a number of published studies that takes the position that students should first experience a structure via an accessible example, in this case the concrete and aural example of the British sport of change ringing (ringing all possible permutations of n bells) as a motivator for teaching students about the

permutation group in particular, and algebra in general. Dechéne described the context of change ringing, then gave a description of the mathematics that could be directly observed or quickly abstracted, explained how the mathematics could be connected to the Cayley graph, and explored the existence of a Hamiltonian cycle. Dechéne finished her piece by suggesting a variety of ways that the mathematics she described could be brought into the classroom both in an active manner (e.g., (actually ringing bells or moving in the order of ringing) or a more passive manner (e.g., using a collection of Java applets that demonstrated change ringing). There are a number of other such suggestions that were collected in Hibbard and Maycock's (2001) edited volume, a text that grew out of two sessions at a Joint Mathematics Meeting in which a series of reports were offered about innovative teaching (i.e., strategies that were primarily student-oriented rather than lecture) in abstract algebra classes. Additionally, others are using structures that might come from a computer program such as ISETL (Leron & Dubinsky, 1994) or ESG (Mingus, 2001) but more common are suggestions to use some physical manipulative or pictorial representations, as was the case in Dechéne's work.

Larsen (2004) provided significant detail in his description of pedagogical activities but he began with a similar goal content-goal; to help students learn algebra content without lecture. He gave the following prompt shown in Figure 1 to students.

Triangle Activity

1. Describe all of the symmetries of an equilateral triangle.
2. Can some of these symmetries be written as combinations of the other symmetries? What is the smallest number of “basic” moves that are needed to generate all of the symmetries?
3. What happens if two of these moves are performed consecutively? Is this combination the same as one of the original moves? If so, which one?

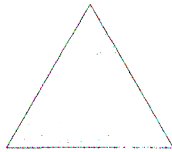


Figure 81. Sample Activity 1

Figure 1. (Larsen, 2004, p. 267)

Larsen, as did all the other authors noted above, asked the students to look across the different explorations that they had undertaken to look for commonalities and thereby create a definition of a group. The goal of these suggestions was to develop “an approach to the instruction of elementary group theory that supports the guided reinvention of the concepts of group and isomorphism as a result of the students’ own mathematical activity and informal knowledge” (Larsen, 2004, p. 252). Other curricular efforts have been designed with the goal that students develop an understanding of quotient groups, but the overall approach is the same: asking students to explore different situations to develop their understanding of the mathematical concepts.

Larsen (2004) continued by describing the class discussion that resulted as a means of providing other teachers with suggestions for eliciting important mathematical ideas such as asking students structured questions about their work. He noted that even after these explorations students do not identify the importance of an operation without prompting. He stated that he had to “elicit this observation by asking students what was needed before the concepts of identity elements, inverses,

and associativity made sense” (2004, p. 284) and even after this prompt, the students did not use the term operation but rather described “actions you can perform on those things” (p. 285). Moreover, he also noted that the students were likely to list properties that were not necessary into their definitions of groups. Larsen did not evaluate the student’s abilities to make use of the definitions in proof.

In a follow-up to the Mingus (2001) study, Grassl and Mingus (2004) evaluate their reformed abstract algebra course. Grassl and Mingus (2004, ¶ 6) stated:

We observed together how students, once provided the assurance that their ideas would be listened to, can make great progress on resolving background deficiencies and moving forward. Half of the 25 students were women; evaluations and general discussion indicated that the presence of a female co-instructor tended to ‘soften’ classroom tone, creating a friendlier learning environment for them. The students reported experiencing a family-like atmosphere and its positive impact on their attitudes about the class and the subject. These improved attitudes translated into increased participation, willingness to take risks, decreased attrition (only one of 25 dropped), and increased attendance (On the average, only one person per week was absent).

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The set of papers that relates to the teaching of proof with a focus on the content of abstract algebra are often similar to those suggesting new ways to teach

abstract algebra content in that they will describe a pedagogical approach or give suggestions and then offer very little in the way of evaluation of the approach. For example, Dean (1996) wrote a piece describing the pedagogical practices that she employed to help her students learn how to prove mathematics theorems. She articulated a six-phase model, which she claimed would assist students in developing proof competencies. The evidence that she offers to attest for the success of her model is anecdotal; she claimed that she expected students to complete novel proofs on exams and stated that they were able to do so; she also related a story of a student who employed the model in a later course.

There is also an ongoing set of work done by Weber (2004) designed to help students make proof-related decisions in an accurate manner; such decisions might include choosing the proof structure or the most appropriate knowledge to draw upon to craft the proof. Thus far, there have been three reported iterations of this project. In the first iteration, a computer was programmed to execute proofs using a heuristic and entering a set of facts that he claimed would be reasonable for undergraduates to know based upon his previous research. The computer was able to successfully create 13 of 16 direct proofs (ones which proceed as a set of linked logical statements from start to finish) but was unable to complete any of the indirect proofs (which require assuming a contradiction). In the second iteration, the researcher performed a similar experiment with undergraduate students by teaching them how to apply the heuristic and giving them a sheet of facts that they could use to write proofs. He noted that he made no efforts to teach meaning, but rather attempted to teach students to apply the procedure in a mechanistic manner. That is, he asked students to

construct a set of logically coherent and complete statements that demonstrated the truth-value of the proposition. The students were highly successful writing direct proofs, but did not develop any conceptual understanding in the form of definition (either concept image or concept definition). In the third iteration, there was an attempt to teach both proof-process and conceptual understanding. This last iteration again saw high rates of student success with proof (Weber, 2004). But, the students who had learned to apply his heuristic with meaning actually demonstrated lower rates of success with proof than either the computer or the students who were mechanistically applying the routine.

While there is a growing body of literature suggesting novel approaches to teaching abstract algebra content and proof-proficiency, the current literature does not adequately describe the mathematical proficiency that students will develop after completing an introductory abstract algebra course. Much of the current literature includes only cursory evaluations of student success, such as completion rates, or affective descriptions, rather than describing what content students know and how they are able to use their content knowledge in mathematical activity in areas including writing proofs. Moreover, those studies that do offer more description about the mathematical proficiency that students developed are small-scale teaching experiments that lasted a relatively brief amount of time and focused on teaching a specific set of knowledge or proof-writing skill rather than the entire body of material from an introductory abstract algebra course. The present study is an attempt to enrich the existing research literature by describing the mathematical proficiency that

students in an investigative introductory abstract algebra course develop and are able to demonstrate.

Teachers' pedagogical decisions

Abstract algebra instructors, prior to their course, must make a decision about whether to employ a DTP or reform pedagogical style. The research literature indicates that the choice of pedagogical style is most influenced by (i) beliefs about how students learn mathematics, (ii) beliefs relating to the goals for the class (including issues related to breadth of coverage versus depth of understanding), (iii) beliefs about the mathematical content to be covered (e.g., what are the most important topics to teach), and (iv) beliefs about evidence of student proficiency (Schoenfeld, 1998; Weber, 2001; Wu, 1999).

Time is viewed as an important constraint in advanced undergraduate mathematics courses, especially so in introductory abstract algebra classes. Wu (1999) captures this theme, writing, "I find the obstacle of the time-constraint almost impossible to overcome, and this constraint will be a recurring theme of this article" (p. 3). Similarly, Grassl and Mingus (2004, ¶ 5) wrote "*the timetable in abstract algebra is ferocious*" (emphasis in original) clearly communicating their frustration with the constraint of time. The introduction to an algebra text stated:

The book contains the material on groups, rings and fields usually covered in a one-semester course, though we would be happier if we could stretch it over 1.5 or 2 semesters. We feel that for many students, going beyond the material on group theory in one semester interferes with their ability to advance beyond a superficial understanding of abstract algebra (Dubinsky & Leron, 1994, p. xix).

Authors (Wu, 1999; Grassl & Mingus, 2004) included examples of major topics in abstract algebra (Euclidean Division and cosets, respectively) that they perceive as forced into a single class period by the constraints of time even though the concepts may have taken decades to evolve mathematically or, as Wu (1999) acknowledged, cannot be learned in such a brief time period. Wu conceded that learning the Euclidean division algorithm is difficult, and, after describing a “torturous” two-hour tutoring session with a student, he admitted that “it is likely that for most students this is the only way to learn [the topic]” (p. 3). Wu defended his decision to only spend half of a lecture (25 minutes) on Euclidean Division even if more time was warranted educationally by stating, “If I spent two hours to teach it, I would be fired for pedagogical turpitude, and rightly so” (p. 3). Therein lies the fundamental problem that Wu admits: It seems to be common belief that many students cannot learn the material in the time allotted and require substantial help from the teacher, but at the same time, the current structure of the abstract algebra class and university system does not allow for such flexibility.

Moreover, Wu (1999) is also making an assertion about the amount of content that can be covered via the different pedagogies. He argues that if he chose to engage in guided-discovery or an investigative teaching approach instead of a lecture-based pedagogy, “then the amount of materials that [could] be covered in each course would be reduced by half if not more.” He clearly believes that this outcome is incompatible with his stated goal of preparing students for graduate study (Wu, p. 6, 1999). It is likely that other authors would agree with Wu’s assertion.

All of these authors seem to be suggesting that they view the content of the abstract algebra course to be fixed – that there is a body of material that they must present to students. For example, Wu states, “after four years of college, students should be competent enough to start graduate work in their chosen disciplines” (Wu, 1999, p. 4) and that in a “junior level algebra course” students should have achieved mastery of the “most basic techniques and ideas in algebra: the concepts of generality and abstraction, the concept of mathematical structure, and certainly the basic vocabulary of groups, rings and fields” (Wu, p. 4, 1999). Grassl and Mingus (2004) argued that students would expect to see groups, subgroups, cosets and quotient groups in a typical first semester course. Each of the authors seems to be acknowledging that there is a tension between what they perceive to be the amount of material that an introductory course must cover and the amount of time it takes students to actually develop proficiency with material.

The authors cited above seem to be making an assertion about student learning. In particular, they are asserting that it is better to expose students to the entire scope of introductory algebra in order give students some opportunity to learn the material rather than spend more time on certain areas, to the exclusion of some topics. The instructor’s beliefs are somewhat supported by literature; specifically, the NRC (2001) discussed the importance of students’ opportunity to learn a concept and suggested that it was the most powerful predictor of student performance. It seems that most authors believe that students’ opportunity to learn a concept is directly tied to their exposure to the concept in a mathematics course. Moreover, there seems to be a belief among instructors that in order to prepare students for possible graduate

study in algebra (which in other venues authors claim caters to a dangerously small minority), it is more important to cover a great volume of material than to ensure any depth of student understanding (Dubinsky and Leron, 1994; Wu, 1999; Grassl & Mingus, 2004). It is possible to find writers taking well-argued positions on both sides of the issue. For example, Cnop and Grandsard (1998) are among many who have argued that it is preferable to present less material if it is better understood than to present more material that is poorly understood.

Lastly, there is also a set of beliefs about students' abilities to learn from different types of pedagogies that needs to be taken into account when teachers decide on their teaching styles. For example, in a DTP class, the order of introduction of new material is from the general to the particular, from abstract definition to concrete example. As suggested above, this time of learning in the DTP classroom is derived from Vygotsky's work. In the case of investigative teaching, the guiding theory is that students begin by investigating a series of discrete situations and abstracting and generalizing from those situations to the appropriate definitions and understandings, and is based on Piaget's theories on learning. While many mathematicians and mathematics teachers know about alternative pedagogies, they are less likely to know the learning theories that support those pedagogies; learning theory is not part of the typical preparation of a mathematician. As such, mathematicians are likely to make their decisions based upon implicit beliefs about teaching and learning mathematics rather than theoretical support.

These two theories of learning (i.e., Vygotsky and Piaget) seem to interact in complex ways within undergraduate mathematics faculty's perceptions of time

constraints and goals for their abstract algebra courses. Wu (1999) seems to be one of the most prominent and articulate defenders of the lecture style of teaching, yet even he admitted that it is likely not the most effective method for the promotion of learning.

Those who engage in a more investigative pedagogy, such as that which Dubinsky and Leron (1994) described in their curriculum, do not generally discuss their disciplinary beliefs in such an explicit fashion. Although to be fair, most of these studies take as their goal the description of student understandings of algebra concepts, and should not be faulted for failing to include a fuller description of instructor belief. Grassl and Mingus (2004) have offered one example of a course that was more investigative, and they clearly perceived the same time constraints as Wu. Grassl and Mingus (2004) seemed reasonably ambitious in their goals for the mathematical content of the course as they reported student discussion of quotient groups. Similarly, Findell (2000) reported that students had some experience with quotient groups (one question, out of 33, on the final examination required students to make use of the concept). Asiala, et al. (1997) also reported on student understanding of quotient groups, although again, it seemed to be the final concept covered in the course and lightly examined on assessments. As such, it is plausible to suggest that the authors cited above made a decision to cover less material than a typical one-semester course on group theory in an effort to increase student understanding. I might suggest that belief in the importance of student understanding might be more privileged in these authors' regard than the goal of complete coverage of the important mathematical content.

Summary

The literature on teaching cited above contains significant works on learning theory as well as a growing number of pieces which offer suggestions about teaching abstract algebra. Yet, there are very few located studies that offer a complete description of what happens in an undergraduate mathematics class, the most detailed of which is focused on analysis (Weber, 1999). As such, there is significant need in the field for a study that describes the teaching of undergraduate abstract algebra.

Studies that link teaching and learning are relatively rare in the research literature and those that do include some summary evaluation of the pedagogy generally only include measures such as course completion rate or satisfaction surveys. Two notable exceptions to this rule existed in the research literature. The first was a study in which the author was creating a pedagogy that would build student understanding of groups and group isomorphism (Larsen, 2004). Yet, this study was limited in scope both in terms of the content (only working through group isomorphism, not the entirety of a semester-long course) and only included two students in each iteration of the content. As such, it cannot be thought (nor was it intended to be) an evaluation of student learning in an introductory abstract algebra course. Similarly, there was a study on teaching student proof proficiency (Weber, 2001) that also included evaluation of learning. In that study, the students were able to make good use of a heuristic to produce proofs when making use of a list of definitions and results. However, the author noted that when he attempted to teach for conceptual understanding while also teaching proof proficiency, the students demonstrated less growth. Again, this study was focused on only one aspect of

mathematical proficiency and covered a limited set of content. As such, there is still significant question concerning how undergraduate mathematics courses are taught.

Student learning

The second major goal of the present research study is to describe the knowledge that students derive from an introductory semester of abstract algebra and what they are able to do with that content knowledge. There are a multitude of pieces of content knowledge and ways in which students are expected to make use of that knowledge during their abstract algebra course. In seeking to craft a readable yet also reasonably comprehensive description of students' abilities, it was necessary to make a number of choices that focused the study both in terms of the content of abstract algebra and also the manner in which students' abilities to make use of the content was studied and described. "No term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics" (NRC, 2001, p. 5), but as with the NRC's Mathematics Learning Study, I have chosen the term proficiency to describe how I will analyze the knowledge and abilities that the students demonstrated during the study.

The NRC Mathematics Learning Study report (2001) suggested five strands of mathematical proficiency as a means for organizing the understanding of mathematical learning of elementary students. These strands are as follows:

1. *Conceptual understanding* – comprehension of mathematical concepts, operations and relations.
2. *Procedural fluency* – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
3. *Strategic competence* – ability to formulate, represent, and solve mathematical problems.

4. *Adaptive reasoning* – capacity for logical thought, reflection, explanation, and justification.
5. *Productive disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy. (NRC, 2001, p. 5)

Although these strands were meant to describe the mathematical proficiency of elementary students, the writing of researchers in undergraduate mathematics education, and specifically, those working in the fields of abstract algebra and proof, suggest that there are analogous strands of mathematical proficiency for undergraduates. In undergraduate abstract algebra classes, students are expected to develop:

1. *Conceptual understanding* – comprehension of the concepts of set and operation form the basis for more advanced understanding of groups, rings, fields and the different relations between them.
2. *Procedural fluency* – skill in carrying out algebraic operations such as function composition and object permutation flexibly, accurately, efficiently, and appropriately.
3. *Strategic competence* – the ability to explore new mathematical contexts and categorize them into known examples.
4. *Adaptive reasoning* – the ability to create mathematical proof.
5. *Productive disposition* – a habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.

The pages that follow review the located research on each of these five strands of undergraduate mathematical proficiency.

Conceptual Understanding

In undergraduate mathematics study, the objects under study are given by definitions. As a result, in abstract algebra, and advanced mathematics generally, conceptual understanding is best understood in terms of students’ knowledge of definitions and examples (Tall & Vinner, 1985; Vinner, 1991; Hershkowitz, Schwarz,

& Dreyfus, 2001). As described above, a formal mathematical definition serves as the basis for much work in mathematics, and thus, the mechanism by which students learn mathematical definitions is of great interest. Vinner (1991) described two distinct structures that are necessary for advanced mathematical understanding: the *concept definition* and the *concept image*.

He claimed that a student's concept definition was the formal algebraic statement that the student associated with the term and that which the student can articulate in an (approximately) axiomatic manner. If a student only has a concept definition, but has no concept image, the student can do some work in mathematics and appear successful. A concept definition allows students to perform appropriate symbolic manipulations, but the student would be operating by moving symbols across a page following a set of rules without a deep understanding of why the manipulations are appropriate or what the manipulations show about the structures.

A concept image is "something non-verbal associated in our mind with the concept name" (Vinner, 1991, p. 68). This image often takes a visual or verbal form, although Vinner cautions that the verbal form was often acquired only after the learner had significant interaction with the idea. Thus, students who hold a correct concept image that is well-correlated with the concept definition, meaning that students can flexibly operate with both, are described as having successfully acquired conceptual understanding (Vinner, 1991).

Students in an introductory abstract algebra course are expected to develop a conceptual understanding of groups, rings, fields and isomorphisms as the primary structures. There are also a number of variants and sub-structures that students are

expected to understand. There is a large body of research into the manner in which students develop their understanding of groups, subgroups and quotient groups.

There is very little work on student understanding of other topics.

A theoretical perspective for describing student understanding

In their first study of student learning of group theory, Dubinsky, et al. (1994) presented a theoretical perspective characterizing student understanding, based on the work of a class of 24 in-service teachers enrolled in an abstract algebra class. The investigators drew their conclusions from student responses on a paper-and-pencil instrument and a collection of interviews with 10 of the students whom the researchers believed to be students in the process of learning the concepts. This theoretical perspective is termed Action-Process-Object-Schema (APOS). The APOS theory expands the Piagetian constructs of process and object into a four-stage theory that may then be used to create a genetic epistemology or learning trajectory for the major concepts in group theory and other mathematics. APOS is defined:

An action is any repeatable physical or mental manipulation that transforms objects in some way. When the total action can take place entirely in the mind of an individual, or just be imagined as taking place, without necessarily running through all of the specific steps, we say that the action has been interiorized to become a process. It is then possible for the students to use the process to obtain new processes, for example by coordinating it with other processes; that is, to combine two or more processes, connecting “inputs” and “outputs” appropriately so that another process is formed. Also, a process may be reversed to obtain a new process. When it becomes possible for a process to be transformed by some action, then we say that it has been *encapsulated* to become an object. (Dubinsky, et al., 1994, p. 270)

Since then, many studies relating to the teaching or learning of group theory have been written in response to or have built upon that work. While Dubinsky, et al. (1994) employed this theoretical perspective to explain how students learned the

concepts of groups, subgroups, cosets, and normality, the researchers did not consider how student understanding developed from one stage to the next.

Student understanding of groups

To better understand the type of work that the research program is attempting, Figure 2 presents the genetic epistemologies for group and subgroup that Brown, et al. (1997) described.

Group: is a schema that consists of three schemas: set, binary operation and axiom. The schemas of set and binary operation are coordinated through the schema of axiom. Axiom includes the notion that binary operations on a set may or may not satisfy a property, which is essentially the process of checking that property. It also includes four specific objects obtained by encapsulating the four processes corresponding to the four group axioms (Closure, Associativity, Existence of Identity Element, Existence of inverse element for each member of the set). Checking an axiom consists of coordinating the general notion of satisfying a property with the specific process for the axiom and applying it to a particular set.

The Group schema is thematized to form an object to which actions can be applied such as checking for isomorphisms. An important component of the group schema is the ability to consider a generic group as well as particular examples of groups.

Subgroup: Can be understood as a coordination of three schemas; group, subset and function. The function and subset schemas are coordinated to obtain the process of restriction of a function to a subset of its domain. This process is coordinated with the binary operation in the group schema to obtain the restriction of the binary object to a subset. Finally the axiom schema in the group schema is applied to the pair consisting of the subset and the restriction of the binary operation to that subset. In general, this articulation requires that the group concept be already established in the students' mind before they are able to understand subgroups.

Figure 2. (Brown, et al., 1997)

There are three important features of Brown, et al.'s (1997) analysis of students' understanding of groups according to Larsen (2004). The first important feature according to Larsen was that "students had the tendency to assume that features that hold in one part of an environment hold for the entire environment" (p. 23). Within the realm of algebra this is a belief that can cause significant problems for students.

MacLane and Birkhoff (1999) laid out the normal progression of algebra as defining a category, morphisms and then a subcategory. That means that the great majority of algebra requires the coordination of the notions of sets and subsets and their properties. This difficulty is probably one of the reasons for the prevalence of questions such as “Find a cyclic subgroup of order 4 in $U(40)$ ” and “Find a non-cyclic subgroup of order 4 in $U(40)$ ” (Gallian, 1994, p. 63).

The second feature of Brown, et al.’s (1997) analysis that Larsen (2004) identified as a particular barrier in learning abstract algebra was students’ limited understanding of sets and element inclusion relationships. Specifically, Larsen (2004) suggested that Brown, et al.’s (1997) analysis revealed that students are generally able to recognize that if an element satisfies all of the conditions for set membership then the element is a member of the set. However, students have great difficulty recognizing that membership in a set implies that the element satisfies all of the defining conditions of membership. This paired set of abilities is particularly important in advanced mathematics courses because, as Vinner (1991) described, mathematics textbooks and classroom practices (especially in DTP courses) are partly based upon several related assumptions:

1. Concepts are mainly acquired by means of their definitions.
2. Students will use definitions to solve problems and prove theorems.
3. Definitions should be minimal.
4. It is desirable that definitions be elegant.
5. Definitions are arbitrary. (pp. 65-66)

That is, to succeed in mathematics classes, students *need* to be able to reason from definitions.

A number of researchers have examined undergraduates' misconceptions of concepts about groups. For example, students frequently considered a "group as a special set" without recognizing the need to specify an operation with that set as they reviewed the properties of the operation as properties of the set of numbers. In addition, students frequently failed to verify that the operation defined on the set was associative (Iannone & Nardi, 2002). In short, for students to demonstrate conceptual understanding of the group structure it requires that they coordinate, and understand the importance of the linkages between the set, operation and properties schema. Hazzan (1999) considered Piaget's process-object duality and looked for a mechanism (to give more detail to the APOS work) by which students understanding develops. She suggested that students typically use unfamiliar concepts as processes and, as the students becomes more familiar, the conception shifts to object status. She then supported her claims by showing student thinking about the relationship between groups, subgroups and cosets. In each case, the students focused upon a process before moving to more general thinking (e.g., the process of finding inverses or the process of coset creation).

Students also have difficulty in understanding the structure of cyclic groups. This difficulty may result from non-mathematical generalization with the term 'cycle,' as well as the belief that all cyclic groups are finite because most of the examples that they see are finite (Lajoie & Mura, 2000).

The third major contribution of Brown, et. al (1997) as identified by Larsen (2004) demonstrated that for students to perform certain tasks relating to groups, students must have constructed the notion of a generic group (i.e., an abstract

understanding of groups). Given that the concept of group is taken as a starting point for abstract algebra and the study of groups builds to the notion of quotient group it is vitally important that students develop this type of abstract understanding of groups. Although Burn (1996) claimed that students can quite easily understand the idea of quotient groups, many researchers have found that students have great difficulty with the concept of quotient group (Asiala, et al., 1997; Brown, et al., 1997; Dubinsky, et al., 1994; Findell, 2000; Grassl & Mingus, 2004; Larsen, 2004; Weber, 2001). The concept of quotient group requires students to have the concepts of group and subgroup while also having the ability to consider, simultaneously, two different operations. For many students, the “crucial idea in calculating a quotient group may be constructing the binary operation, the importance of being able to choose appropriately between two binary operations defined on a set ..., and specific misconceptions such as the fact that some students believe Z_n is a subgroup of Z ” (Findell, 2000, p. 21).

Student understanding of other structures

Hazzan, Leron, and Zazkis (1995) published a landmark study that explored student understanding of isomorphism, as well as their ability to prove or disprove whether two groups are isomorphic. Assessing students enrolled in an ISTEEL-based abstract algebra course, their commentary on student understanding which was not as focused on creating an APOS decomposition, was the initial investigation of student understanding of isomorphism. Hazzan, Leron and Zazkis concluded that isomorphism is a difficult concept for students because it makes use of the constructs of group, function and quantifier. Further, students struggle to relate their intuitive

notions about equality and the formal definition of isomorphism as a correspondence. Research suggested that the concept of isomorphism is particularly difficult for students because it requires coordination of the concepts of group and function while also working with quantification, a difficult concept for students (Dubinsky, E., Elterman, F. & Gonc, C., 1998).

Procedural fluency

The research evidence for student proficiency with calculations is somewhat mixed and rather slim. Students seem to like to perform calculations on elements of sets, especially when compared to more abstract general calculations required for proof. For example, when confronted with a prompt asking if two rings are isomorphic, students can state that the commutative property is a more important characteristic in terms of isomorphism than orders of elements, but student will prefer to check orders of elements due to their local nature (Hazzan, Leron, & Zazkis, 1995). In a similar vein, when students are asked to prove the non-existence of an isomorphism, they will often point to a characteristic that precludes the existence of an isomorphism.

This tendency of students to think more frequently about operations on specific elements may indicate that they commonly hold a set of misconceptions related to the underlying concept of group and ring. Students often attribute properties of the operation to the elements (Ionne & Nardi, 1999). Additionally, when confronted with Cayley tables, students struggled to create strategies that would allow them to verify that all group properties hold. Generally, they preferred to operate on two elements and because of that never formally checked that the

operation was associative. Because of this tendency to generalize after checking a few discrete calculations, many students will accept that a given subset is a subgroup even if there are still unverified properties (Ionne & Nardi, 1999).

There are a number of possibilities that suggest themselves as a result of this research, but the most striking feature is how little research has been done. Searches for research literature on composition of functions, permutations, and factoring polynomials at the undergraduate level did not return any results.

Strategic competence

This strand might also be described as being about problem solving. There is a reasonable amount of literature about problem-solving, even at the undergraduate level, but none of it is specific to the types of questions that students confront in abstract algebra. Polya is the figure most connected to discussions of problem solving with his description of a problem solving heuristic and repeated arguments that problem solving should be an important part of the mathematical education of students.

Schoenfeld (1992) carried out a long program of research about problem solving that included exploring whether students can be taught problem solving, what the mental habits that contribute to successful problem solving and the types of habits that students acquire in “well taught” classes. While Schoenfeld did much of his work with undergraduate students, his emphasis was on non-routine problems that could be solved with relatively low-level mathematics (arithmetic, trigonometry, basic calculus). Thus, none of his work examined the types of explorations and habits

that lead to successful problem solving at the advanced undergraduate level inclusive of abstract algebra.

In abstract algebra, students are expected to explore new situations in a number of ways. There are multiple problem-archetypes in abstract algebra. When students are learning new concepts, such as rings, the text has an archetypical problem; the students are given an example and they need to determine whether the example satisfies the definition of a particular type of structure. For example, “Define $*$ such that $a*b = (2a+b)/2$ where a and b are integers. Determine whether $(\mathbf{Z}, +, *)$ forms a ring.” Similarly, students might be asked to verify that a group satisfies certain elementary properties such as, “If a and b are elements of group G , then $(ab)^{-1} = b^{-1}a^{-1}$.”

The research that does exist focuses primarily on students’ mechanisms for exploring unfamiliar situations. The research shows that when students are confronted with highly abstract concepts, they often revert to a canonical example that embodies the necessary qualities. For example, a set is replaced with one of its (familiar to the student) elements (Hazzan, 1999). This habit manifests itself in multiple situations. For example, students know (or can state) that the commutative property is a more important characteristic in terms of isomorphism than orders of elements, but students, instead of checking commutivity generally, will instead check orders of elements due to their local nature (Hazzan, Leron & Zazkis, 1995).

In instances where there is no canonical or immediately evident isomorphism, students struggle significantly more. When asked to prove the non-existence of an isomorphism, students would point to a characteristic that precluded the existence of

an isomorphism. If their strategy with the order of elements is unsuccessful they then start searching for some other property or local characteristic that precludes an isomorphism, such as the fact that one group is commutative and the other is not, rather than attempting a non-existence proof (Hazzan, Leron & Zazkis, 1995).

It is interesting that Weber (2001) saw students' identification of isomorphism-precluding properties differently. Rather than calling the student behaviors attempts to minimize work with abstraction, Weber described students using such strategies as exhibiting increased mathematical knowledge and understanding and as illustrating more strategic knowledge than students who attempted to disprove the existence of an isomorphism based upon more direct means.

Adaptive reasoning

The principal area in which students make use of their content knowledge within abstract algebra is in crafting proofs. Thus, helping students acquire the skills to create proofs is an important part of the training that is expected in undergraduate education (Wu, 1999). Content knowledge is not enough to enable students to write proofs; students must have and coordinate a variety of other types of proficiencies in order to make use of their content knowledge in crafting proof (Weber, 2001).

Dreyfus (1999) argues that teaching “mathematical justification conflicts with the pursuit of learning and teaching mathematical relationships, concepts and procedures in a flexible manner” (p. 104). Given the way in which mathematics is taught now, he continues, students have “few if any means to distinguish between different forms of reasoning and to appreciate the consequences for the resulting

knowledge; nor can they be expected to distinguish between explanation, argument, and proof” (Dreyfus, 1999, p. 104). Taking this assertion as true, it becomes unsurprising that so many researchers have written that students have little understanding of what constitutes mathematical proof.

With respect to improving the teaching of proof and student outcomes, Dreyfus (1999) suggested that it is not a question that can be fully addressed until mathematicians and mathematics educators can create a definition that characterizes what they wish students to do in their work with proof. It is in this tradition that Weber (2004) reported on the results of an iterative teaching experiment. Weber’s goal was to improve student proof abilities in abstract algebra classes. He schematized the structure of a collection of proofs and created a heuristic that could be flexibly applied and lead to proofs. He was building on his earlier work (Weber, 2001), where he reported that frequently students, even those with access to all of the knowledge needed to write the proof, were unable to do so.

Dreyfus (1999) continued, “Much of our students’ mathematical knowledge is tacit; and while tacit knowledge is likely to be used correctly in applications, it cannot be used explicitly in reasoning” (p. 104). One of the goals of mathematics textbooks (and instruction) is to help students acquire knowledge in a variety of forms, and to establish connections between and across forms. That is, students need some method for calling into their active memory the knowledge that will be appropriate in a given situation, and then sorting it based upon likely relevance. Weber (2001) termed this method strategic knowledge. He claimed that strategic knowledge would enable students to decide upon a proof structure and select the types of knowledge most

useful for writing the proof. For example, in abstract algebra, one canonical proof is to show that the ring of integers cannot be isomorphic to the ring of rational numbers. Weber noted that students with strategic knowledge would cite structural reasons that preclude the existence of an isomorphism between the two rings rather than attempting a non-existence proof while very novice students might propose possible functions and show that they are not isomorphisms.

Examinations of students' proof proficiency in group theory have found that students exhibited multiple proof-production strategies and that there were frequently occurring error types. One proof-production strategy that students make use of to lessen the need for strategic knowledge is to locate a worked example similar to the proof they are expected to write and change symbols as appropriate (Weber, 2004; Fukawa-Connelly, 2005). Other prominent proof-production strategies were guess-and-check, working backwards, and working forwards. Typically, guess-and-check was unsuccessful for students with low-levels of content knowledge and more successful for students with high levels of content knowledge. Working backwards was the primary success strategy of students with low levels of content knowledge, and working forwards was the primary strategy of students with high levels of content knowledge. One habit that separated expert proof-writers from novice proof-writers was the creation of new notation to assist in the work. Experts are very willing to create new, and appropriate, notation to help in a proof-attempt, whereas novices are not likely to create new notation.

Similarly, there are persistent error-types that have been noted in multiple studies. The two most fundamental mistakes that students make are incorrectly

determining what is to be proven or simply assuming that the desired result is true (Selden & Selden, 1987; Hart, 1994). In the domain of abstract algebra, students also habitually assume that inappropriate properties hold in abstract groups, especially the commutative property (Hart, 1994). It is also relatively common for students to fail to verify that the operation as defined on the set is associative (Iannone & Nardi, 2002).

Students are prompted to invoke a theorem (or its converse) on all examination problems by their naïve understanding of the subject (Hazzan & Leron, 1996; Weber, 2001). Named theorems or those that have a simple formulation are the most likely to be misapplied because they can be sloganized for easy recall and use (Hazzan & Leron, 1996). In abstract algebra, research has shown that these tendencies are especially pronounced in the case of Lagrange's Theorem. With that theorem, students often do one of three things: i) behave as if the converse is true, ii) use an incorrect converse, or iii) apply the theorem or its converse in an inappropriate manner (Hazzan & Leron, 1996).

The last major type of mistake the students make in proof-production happens after they have correctly identified relevant theorems. Students will apply the theorem without verifying that the hypotheses of the theorem have been satisfied (Selden & Selden, 1987; Hart, 1994).

Besides writing proofs, students are expected to read the proofs they write for correctness, that is, to check their work. In this context, this means that they are expected to verify the correctness of their proofs. To that end, a number of researchers have explored the proof-verification process. Research has shown that

proof-verification is a task that draws upon different types of knowledge than proof-creation and that it is a non-trivial task for students (Selden & Selden, 2003). There is some debate about whether proof-validation requires a subset of the knowledge required for proof-creation (Selden and Selden's position) or that it draws upon a different, but overlapping, set of knowledge (Weber's position). There is agreement that proof-verification tasks can provide teachers and researchers alike with a useful window into student understanding (Selden & Selden, 2003; Weber, 2001).

Knuth (2002) used proof verification tasks in interviews with in-service secondary teachers, and the teachers principally evaluated the proofs on the basis of methodology and mathematics (specifically, that each statement logically followed from the previous statement) (Knuth, 2002). Undergraduate students have exhibited many of the same habits in other studies (Selden & Selden, 2003; Weber, 2001; Weber, in press). The teachers distinguished among good and bad proofs by evaluating level of detail and knowledge dependent ideas (that is, quality). The teachers seemed to base their distinctions on knowing that a method is valid, without actually evaluating whether the method was used in a valid way in the supplied proof; that is, they decided based upon form (Knuth, 2002). Moreover, many of the teachers, even after seeing a proof, wanted to manipulate some of the figures to convince themselves of the validity of the statement, and that they were willing to believe that there might be a counter-example waiting to be found (Knuth, 2002).

Productive disposition

The importance of a productive disposition in mathematics education is relatively well documented at the macro-level. Undergraduate mathematics majors,

especially women, earn better grades and are more likely to complete the major when they feel connected to their classmates and teachers (Linn & Kessel, 1996). In this regard, students who feel a sense of belonging are more likely to persist and thus learn more mathematics.

There is some specific research on student dispositions in the case of abstract algebra, but there was no located research linking student achievement with disposition, nor with persistence in the course. What did exist in the research literature was a set of studies that examined how student dispositions related to their behavior in terms of computation and proof. The literature suggests that students prefer to work in less abstract settings and to employ a variety of strategies to do so (Hazzan, Leron & Zazkis, 1995; Hazzan, 1999). Hazzan articulated the definitions of levels of abstraction to describe “the quality of the relationships between the object of thought and the thinking person, abstract level as a reflection of the process-object duality, and abstraction level as the degree of complexity of the concept of thought” (Hazzan, 1999, p. 75). She noted that students find concepts less abstract when they have personal connections to the topic, and then suggested that the different documented methods of reducing abstraction are mental coping techniques that allow students to survive in a traditional course, but do not produce optimal levels of learning or understanding and are indicative of a level of discomfort with abstraction.

When confronted with highly abstract concepts, students often revert to a canonical example that embodies the necessary qualities (Hazzan, 1999). For example, a set is replaced with one of its (familiar to the student) elements. While this process is not problematic if students employ it in an exploratory fashion,

students often fail to return to the original level of complexity, instead believing that the specific example is a complete formulation of the original idea (Hazzan, 1999). For example, students might be asked to consider a universally quantified statement such as the non-existence of an isomorphism. Instead of crafting an argument that shows that an isomorphism cannot exist, the students would point to a characteristic that precluded the existence of an isomorphism, such as the fact that one group is commutative and the other is not. The same study also suggested that students become frustrated and experience difficulties deciding how to proceed when there is not a canonical (and fairly obvious) isomorphism, especially if they are forced to choose between a variety of non-canonical options (Hazzan, Leron & Zazkis, 1995).

Similarly, students prefer to do local calculations rather than global calculations. The authors state that this as a general coping mechanism, but suggest that it may be particularly endemic for proofs of statements regarding isomorphism. For example, students know (or can state) that the commutative property is a more important characteristic in terms of isomorphism than orders of elements, but students will prefer to check orders of elements due to their local nature (Hazzan, Leron and Zazkis, 1995).

Another example of the phenomenon of reducing abstraction is focusing upon surface features of a problem, such as in elementary school when students seize upon numbers in a story problem, or use a word to clue an operation. This strategy might manifest itself in multiple ways in abstract algebra. One way is a common misappropriation of Lagrange's theorem when students suggest that Z_3 is a subgroup of Z_6 because 3 divides 6 (Hazzan, 1999). Research also suggests that students first

think about the relationship between groups, subgroups and cosets by focusing on a process (e.g., the process of finding inverses or the process of coset creation) rather than the more general relationship that their teachers hope for (Hazzan, 1999).

Summary

There is much more literature exploring student learning of abstract algebra topics than there is describing the teaching of abstract algebra, but this literature is generally devoid of insight regarding how the students developed their understanding. Similarly, the literature is very focused upon student conceptual understanding of groups and quotient groups with some work also done on isomorphism. While these are important topics in an introductory course, they are hardly the only important mathematical concepts in an introductory course, and conceptual understanding is not the only important facet of mathematical proficiency. The other area in which there is a reasonably large and growing body of work is on students' proof creation and validation abilities, but again, these studies are almost entirely concentrated on students' work on groups and group isomorphism. The current study will significantly expand the research literature by offering a description of students' mathematical proficiency with rings, with a special emphasis on polynomial rings, a topic which seems untouched by other researchers even though it motivates the study of much of abstract algebra content. Moreover, the current study will also offer an initial description of the different types of mathematical proficiency that students might develop as the result of different pedagogical styles; that is, linking what happened in students' classes with what mathematical proficiency they developed. Thus, the aims of the present study, to describe what happens in an introductory

abstract algebra class and to describe what mathematical proficiency students develop after a semester of study, will make an important contribution to the research literature.

CHAPTER 3: CONTEXT AND METHODOLOGY

Introduction

The purpose of the study was to examine the characteristics of two sections of abstract algebra that each employed a distinct instructional methodology and the student learning that resulted from each. Initially, this chapter presents the context for the study describing the institution and mathematics department where the classes were taught. Subsequently, there is a detailed description of the two classes, including details regarding the student population, a few key volunteer student participants, and the faculty members responsible for each of the sections. This chapter also outlines the data sources accessed as well as the data analysis strategies employed.

Context

Midwestern State University (MSU) is a Doctoral I university in the Carnegie Classification system. It has been ranked consistently in the top 100 public colleges and universities by *U.S. News and World Report*. MSU is a large institution with approximately 20,000 undergraduate students and 5,000 graduate students that prides itself on the fine quality of the undergraduate education that it provides, and it is working to strengthen its graduate programs and research focus. It offers over 140 undergraduate majors and over 80 graduate degrees. There are nearly 1000 full-time members on the university faculty.

The Mathematics Department

Degree Programs

The mathematics department offers undergraduate majors in pure and applied mathematics as well as secondary mathematics teacher education and typically awards between 20 and 50 bachelor's degrees per year. The department awards master's degrees in applied mathematics and both masters and doctoral degrees in pure mathematics and mathematics education, including undergraduate mathematics education.

Faculty and Graduate Students

The mathematics department is made up of 31 tenure-line faculty members with a number of full- and part-time associated faculty. Faculty research areas include graph theory, algebra, analysis, applied mathematics, and mathematics education. Within the field of mathematics education there is ongoing faculty research in undergraduate mathematics education that supports departmental curriculum reform initiatives. For example, the mathematics department has recently finished a substantial change to their Introduction to Proof course which culminated in two members of the faculty writing a new textbook for the course. This introductory course in proof is a pre-requisite for the abstract algebra course under study.

During the time of the study the department supported 36 full-time, graduate students. Almost all of these full-time graduate students were pursuing doctoral degrees and most taught lower-level undergraduate courses, while a few were supported by research grants. During the course of this study, only one of the full-

time graduate students was pursuing a doctorate in undergraduate mathematics education.

The Abstract Algebra Course

The online catalogue of Midwestern State University gives the following description for Modern Algebra I:

This course introduces the abstract algebraic concepts of groups, rings, and fields, and shows how they relate to the problem of finding roots of polynomials. Topics include: Properties of the integers, congruences, the Euclidean algorithm, groups, subgroups, cosets, Lagrange's theorem, direct product, isomorphism, symmetric groups, rings, integral domains, polynomial rings, fields, field extensions, quotients of polynomial rings. Prerequisite: Mathematical Proofs. ("MSU Catalogue," 2004)

The course is offered each semester, often with multiple sections in the spring semester. Typically, the students in the spring semester course are juniors who have completed the calculus sequence and an introductory course on mathematical proof. The course is always taught by members of the tenure-line faculty. In this study all but one of the students had completed the introduction to proof course. The one student who had previously failed the proof course was concurrently retaking it with permission of the department.

Faculty Participants

The faculty participants were women in tenure-line positions at the time of the study, and both had previously taught introductory abstract algebra. The instructor teaching the investigative version of the course had an earned doctorate in mathematics education where her dissertation had focused on the reform of linear and abstract algebra courses, and she had already earned tenure. The instructor teaching the *DTP* course had an earned doctorate in mathematics with a dissertation in

representation theory, and she was granted tenure during the course of the study. Both faculty were mentoring doctoral students in their respective fields and had other departmental responsibilities. They were first approached about possible participation in the research project during the fall of 2004. They gave preliminary approval at that time, and the Human Subjects Review Board process was begun. The instructors were formally offered the opportunity to participate in March 2005, and both agreed.

Student Participants

All students enrolled in either section of Modern Algebra I at MSU were informed of the opportunity to participate in the study during the first course meeting of the semester. The students were formally informed of the benefits and requirements of participation and offered the opportunity to participate during the week of March 15. Of the 36 students across the two sections, 12 students (5 from the DTP class and 7 from the investigative class) agreed to participate fully, and all but one student in each section agreed to let their class activities be described. The students who chose to participate fully were asked to complete a written survey describing their educational background, as well as a brief mid-semester content assessment and lengthier end-of-the-semester measure. Each of those students was also asked to participate in an interview after submission of their final written assessment. Of the 12 participants, 10 were Caucasian. This ratio was reflective of the investigative class' apparent demographics. Of the 24 students in the investigative class, 22 were apparently Caucasian. Of the 12 students in the DTP

class, 3 were students of color (apparently, Black, Indian and Asian). Many of the participating students chose their own pseudonyms.

The students of the investigative class

There were seven students from the investigative class who agreed to participate in the study. Those students are:

Rebekah (AS) was a junior majoring in secondary mathematics education and secondary history education. Her overall GPA was 3.9, and she had earned an A in all previous advanced mathematics courses. During the study she was working on a secondary mathematics curriculum project as a student worker and the following year she was asked to be an Undergraduate Teaching Assistant.

Ned (JJ) was a junior secondary mathematics education major. His overall GPA was a 3.1, and he had earned a C/B in both of his previous advanced mathematics classes. He noted that he did not study much outside of class and generally did not work with other students.

James (CO) was a junior secondary mathematics education major. His overall GPA was a 3.2, and he had earned a B in the proof course and differential equations as well as a B+ in linear algebra. He claimed to not study much outside of class but was friends with Mark. He also professed interest in the carpenter and automotive trades. James was one of the two students of color; his background was Asian/Pacific Islander.

Mark (BSP) was a junior mathematics major. His overall GPA was 3.6, and he had earned an A in all previous upper-division math courses including Introduction to Proof, Linear Algebra, and Probability Theory. He reported that he typically spent an hour each week reviewing notes and practice problems. He was friends with James as they both lived on the same residence hall floor freshman year. He stated, "I hate theory, but love computation." By the end of the study, he believed that he would earn a B in abstract algebra, his first B in a mathematics course.

Johnny (PG) was a senior secondary mathematics education major who indicated that he aspired to earning a masters degree in mathematics education. Johnny indicated that he spent 2 hours a day studying mathematics, generally by reviewing his notes and working assigned problems.

Stephanie (NC) was a senior mathematics and English double major. She had a 3.3 GPA, and her mathematics grades were a mix of C's and B's. She was concurrently taking a geometry class. She indicated that she spent "multiple hours" each day studying mathematics, mostly with other students. She aspired to earning a master's degree, but had not yet decided on a field of study.

Kenny (VM) was a senior mathematics and history double major. His overall GPA was a 3.4 ,and he earned a B in Linear Algebra and A grades in his other previous mathematics classes. He intended to pursue graduate study in mathematics and was waiting on admission decisions at the start of the study. Eventually, he decided to continue his education at MSU. Kenny's class comments and questions indicated that he worked problems from the text that were not assigned.

The students of the DTP class

There were five students from the DTP class who agreed to participate in the study.

Jeff (DH) was a junior mathematics major with a 3.8 GPA. Jeff earned an A in all previous mathematics classes and indicated that he studied very little and never with other students.

Aurora (JA) was a junior mathematics major who had recently transferred to MSU. She had earned a 3.6 overall GPA. In her previous mathematics courses she had earned all A grades except in Introduction to Proofs in which she earned grade of B.

Steven (DS) was a junior secondary mathematics education and secondary history education major. He had earned a 3.8 overall GPA. His previous mathematics grades included a C in Linear Algebra, a B in differential

equations, and an A in Introduction to Proofs. He too indicated that he spent very little time outside of class studying.

Nathan (MC) was a senior secondary mathematics education major. His overall GPA was a 3.4, but his mathematics grades were very mixed. He had previously taken both Introduction to Proofs and abstract algebra and was retaking both classes because he did not pass. Nathan also earned a DC in geometry (passing). Nathan indicated that his racial ethnicity was “half-black/half-white.”

Lynn (MR) was a sophomore mathematics and Spanish double major. Her overall GPA was 3.9. The only class in which she had not earned an A was Graphs on Groups of Surfaces, an advanced course on graph theory. While enrolled in abstract algebra she was also doing independent research in graph theory with another professor in the department. She indicated that in the hour per week of study that she did for the abstract algebra course, she generally completed her homework and also did a number of unassigned problems. She had also won the departmental Freshman/Sophomore prize in mathematics as both a freshman and a sophomore.

The Role of the Researcher

I was introduced to students in each of the classes during the first course meeting as someone who was studying the teaching and learning of abstract algebra.

At this time I explained my goals, the purposes of the study and what participation in the study entailed. Over the course of the semester, I often observed the students in class as I made video recordings and took notes. I answered questions about my work as well as mathematical questions that the students posed. A few times throughout the semester during particular classes, the instructors encouraged me to interact with the students to better understand what was happening.

Data Sources

The first purpose of the study was to offer some characterizations of the teaching in two different types of abstract algebra courses. As such, the first primary data source was observations of abstract algebra course meetings. During the observations, particular attention was paid to: (i) what was written on the board; (ii) the manner in which the instructor presented the content; (iii) the motivation the instructor offered for the content; (iv) the type of tasks that the instructor posed for the students; either implicitly or explicitly; and (v) the explanation (or motivation) the instructor offered for proofs of theorems at two distinct grain sizes. These grain sizes were: global, noting the overall structure of a proof; and local, noting the purpose and justification of each statement in the proof. In addition, notes of student discourse were taken during the course meetings.

As argued in Chapter 2, faculty beliefs and goals have a strong and predictive relationship to their actions and decisions in the class. While classroom observation offered some opportunity to infer faculty beliefs, I conducted three interviews with each of the instructors, one prior to the start of the semester and again twice during the semester. The goal of each of these interviews was to elicit each instructor's

beliefs related to both content and pedagogy and to discern how those beliefs shaped the course. Topics included addressing course decisions (lecture or investigative approach; outline of course content) as well as class-level decisions (presentation of the proof of a particular theorem in a particular manner).

The second purpose of the study was to characterize understandings that students developed within an abstract algebra course offered under two differing instructional designs. In order to assess these understandings, I developed and administered two pencil-and-paper assessments, one delivered at mid semester and one at the end of the course. I also designed and conducted student interviews at the end of the course, after the students had completed the final written assessment. These two types of measures were employed with the intention of describing the depth and breadth of student understanding at a relatively fine grain.

The diversity of assessment instruments in this study allowed for triangulation when the data were analyzed. A summary of the data sources, timing and intent of the data collection in this study is presented in Table 1.

Paper-and-pencil instruments

Student Background Survey

Research suggested that prior achievement was the best indicator of students' future success in mathematics classes and that student feelings of inclusion within the major, as measured by perceptions of relationships with faculty and other students, predicted persistence and retention (Linn & Kessel, 1996). As such, the student background assessment presented prompts designed to access each of these constructs (see Appendix A).

Table 1: Data sources

Data Source	When Collected/Implemented	Purpose
Faculty Interviews	Before study of class began , multiple points during semester, at end of semester	Infer faculty beliefs, goals and knowledge
Student Background Survey	Beginning of semester	Describe students' prior knowledge and personal characteristics
Class observation	Periodically during the semester	Primary data source for characterization of a lecture-based and investigative approach in an abstract algebra course
Mid-Semester Instrument	During the semester, as soon as the students had seen the covered content	Measure student's developing understanding
End-of-semester content exam	Very near to final exams	Measure student understanding of mathematical content
Student Interviews	After exams	Examine of a select group of students' mathematical content knowledge in detail

Mathematics Content Assessments

These instruments were developed by the researcher and a mathematician in order to ensure mathematical relevance and accuracy. The purpose of the instruments was to measure students' group and ring theoretic proficiency, developing as a result of their introductory abstract algebra course.

Mid-Semester Assessment. This instrument presented a single task that assessed whether the participating students were able to determine if a proposed set with associated operations was a ring. This is a typical exercise within an introductory algebra course. While this task requires the students to write a proof, it was a proof whose structure should have been quite familiar. Verifying a structure is

a ring with all of the required properties is traditionally viewed as an important skill in algebra.

Let $(\mathbf{Z}, +, \times)$ be the ordinary ring of integers: $\mathbf{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$, $n + m$ is the sum of n and m , $n \times m$ is the product of n and m . If n and m are in \mathbf{Z} then $n \geq m$ if and only if $n - m$ is either positive or 0.

The algebraic system $R = (\mathbf{Z}, \oplus, \otimes)$ consists of the set \mathbf{Z} with the two binary operations \oplus and \otimes . The operation \oplus is ordinary addition: $n \oplus m = n + m$. The operation \otimes is defined by: $n \otimes m = \begin{cases} n & \text{if } n \geq m \\ m & \text{if } m \geq n \end{cases}$. Otherwise put, $n \otimes m$ is the maximum of n and m .

You are to determine if R is a ring. If it is a ring, show that it satisfies all of the defining properties of a ring. If it is not a ring, identify a needed property that R fails to satisfy, and demonstrate that this property is not satisfied by R .

Figure 3: Mid-Semester Assessment

This prompt assessed each student's ability to state and verify each of the ring axioms. All of the students were given this assessment on March 31, 2005, and they submitted their responses by April 5, 2005. The proposed structure's multiplication operation fails to distribute over the structure's additive operation. Checking all of the other properties required the students to write short proofs similar to those demonstrated in their text and class meetings. The only aspect of this task that required some innovation was supplying a counter-example that demonstrated the failure of multiplication to distribute over addition. Identifying a counter-example was a straight-forward task, as long as the students chose different-sized values to attempt.

End-of-Semester Written Instrument. This instrument (see Appendix B) was designed with one goal in mind: to describe the students' proficiency with group and ring theoretic ideas. Because of the difficulty scheduling a time and place for

administration, this instrument was completed by student on their own without proctoring. Three important constraints were considered:

- 1) the students would be completing the instrument during their final exam week and thus would not be able to devote more than a few hours to the task;
- 2) the students were enrolled in class sections that had spent quite different times studying rings; and
- 3) the students would have access to their text and notes when completing the instrument.

The participants were each given the instrument on April 8, 2005, and submitted a completed instrument by April 17, 2005. Upon completion of the three assessment instruments, each student was paid \$100.

The intent of the tasks on the end-of-semester instrument was to require the students to perform standard tasks in new settings. Each of the items on this exam asked the students to perform what should have been a familiar skill. These included deciding if a structure is a group; making a set equality argument; finding units and inverses; writing proofs; and working with polynomial rings and quotient fields. These items presented non-standard settings because students were allowed to access their text and notes when completing the exam. Use of a standard setting would have allowed the students simply to search through those materials until they located a similar example and then adapt the work (see Fukawa-Connelly, 2005).

Recall that the instructional goals of an abstract algebra class include teaching students the concepts of algebra, helping students to become proficient at writing

proofs (which includes understanding and making use of definitions), and developing increased understanding of school mathematics. This instrument was designed to assess the students' proficiency to do each of those things.

The end-of-semester instrument consists of two sections labeled Problem Set A and Problem Set B. The first section assesses proficiency with the concepts and structures of group theory, the ability to abstract and generalize, and proficiency with a variety of important proof-types. Much of the first section revolves around the important ideas of *unit* and *inverse*, among the most fundamental in algebra. Each of the five questions in the first section required the students to grapple with one of those concepts.

The proof-types that the students were asked to demonstrate in the first section include a proof that a proposed structure is a group, a completeness proof, and a set-equality proof. These three proof-types are some of the most common and important in *group theoretic* mathematics. Item One asked the students to write a proof of two of these types. First the students were asked to understand the definition of a new structure and to demonstrate that the set of units of the structure is a group. Then they were asked to perform a set-equality argument. Item Two required students to list all of the units in the Gaussian Integers (an important structure in number theory) and to demonstrate that the list is complete. Completeness arguments are important in algebra when attempting classification, such as with the Fundamental Theorem of Finite Abelian Groups.

Wu (1995) claimed that the ability to abstract and generalize is one of the hallmarks of modern mathematics. Item Three asked the students to examine an

unfamiliar structure and to describe the elements which have inverses. Item Four asked the students to abstract from that situation and to propose and prove a generalization of their results from Item Three. Item Five asked the students to demonstrate that their generalization has limits, as well as to propose and demonstrate the correctness of a set of qualifiers to generalize further the results from Item Four.

The second section of the end-of-semester assessment (Problem Set B) was designed to elicit information about student's proficiency with the ideas of ring and field theory and to relate those ideas more directly to school mathematics. In this section students were asked to consider the domain in which a polynomial is factorable, to offer a conjecture and proof about the greatest degree of an irreducible polynomial with real coefficients, and to work with elements from a quotient ring and to describe the multiplicative identity in that ring. The polynomial they were asked to consider was one that would be familiar to any high school student. Factoring polynomials is one of the roots of algebra (Kleiner, 1986), and the fact that all polynomials can be completely factored in the complex number system is known as the Fundamental Theorem of Algebra. This line of questions and the mathematical proficiencies were mathematically relevant and also important to pre-college mathematics.

The first item in this section asked students to grapple with polynomial factorization in the complex, real, and rational number fields. This required students to use the definitions of irreducible and reducible elements. Since the later concept is somewhat dependent on quantification, it can cause problems for students who want to equate it with having a root. The most direct means of solving this item required

the students to make use of the fact that the $\sqrt{2}$ is an irrational number. This is one of the most elementary facts of algebra and number theory (often it is one of the first proofs that undergraduate students are asked to complete). The second item required the undergraduates understand the quotient field structure and manipulation of elements in a quotient field. This item also asked students to demonstrate the completeness of a list of elements, an important recurring theme in algebra. The third and last item required students to create and prove a conjecture about irreducible polynomials with real coefficients. They were asked to formulate a conjecture about the largest degree polynomial that is irreducible in the field of real numbers (or, the ring of polynomials with real coefficients). This question directly relates to the algebra of school mathematics as it touches upon irreducibility (factorability) of higher degree polynomials, and it asked the students to generalize the results of the first item in the second section. These items assessed ideas, procedures, and skills within the range of expectations for students in their class. As such, this end-of-semester instrument was a reasonable test of students' understanding of the content and procedures of abstract algebra.

The Interview Protocol. Students were given the opportunity to participate in an interview at the end of the semester (after submission of the end-of-semester assessment). The student interview protocol was designed with two distinct purposes in mind. First, the interview protocol encouraged students to talk freely about mathematics in a format that had no correct response, and it gauged the student's ability to speak globally about abstract algebra. The interview questions addressing this purpose were:

- 1) What does it mean to study abstract algebra?
- 2) What do you think the ‘big’ ideas of abstract algebra are?
- 3) What might you suggest is the reason that both the stuff you did in high school and this is called algebra?
- 4) What about this class was most helpful to your learning?
- 5) What did you enjoy least/most?

Ideally, the first two of those questions elicited themes of algebra (such as exploring sets with operations and investigating implications constraints) rather than a laundry list of seemingly discrete topics (rings, field, groups and quotient structures). The third question intended to explore CBMS’ (2001) unsupported statement that most students fail to make a connection between abstract algebra and school algebra. The last pair of questions in the first section of the interview was intended to elicit description of the types of class activities the students engaged in which were helpful or enjoyable, as well as statements describing class experiences.

The second purpose for the interview protocol was to assess the student’s mathematical proficiency. Given a written definition of a ring, the students were asked:

- 1) What is an example of a ring?
 - a. Does your example have any other, more specific properties?
- 2) What other types of rings do you know?
 - a. Give a brief description of what needs to be ‘added’ to the definition of a ring to get one of these new types.
- 3) Give me an example of each of these types.
- 4) What is a homomorphism?
- 5) Can R be a homomorphic image of C ? By that I mean, is there a homomorphism from C to R which gives all of R as the image?
- 6) In general, if F and F' are fields, is there a homomorphism from F to F' ?
- 7) Is Z_3 a subgroup of Z_6 ?
- 8) If a group has an element of order 2 and an element of order 3 does the group have an element of order 6?
- 9) More generally, if a group has an element of order n and another of order m , what is true?

The first four questions required the students to give either a definition or an example of an algebraic structure. Given approximately 3 months of class time addressing rings and fields, these prompts were intended to be accessible to all the students and answered quickly. The fifth question was designed to assess student's understanding of and ability to make use of the definitions and properties of fields and homomorphism. The sixth question asked students to generalize their work from Question 5 in order to evaluate the student's ability to generalize and abstract from more concrete activities.

Item 7 "Is Z_3 a subgroup of Z_6 ?" (Dubinsky, et al. (1994); Findell, 2000; Hazzan & Leron, 1996; Hazzan, 1999) is particularly interesting because the statement is purposefully ill-formed and requires students to make a variety of mental accommodations in order to approach it (Dubinsky, et al., 1994). Findell (2000) described this task as having the operation purposely omitted due to the belief that students struggle to see groups as sets requiring an operation and subgroups as subsets of the original that are closed under the operation *inherited* from the original group. The difficulty of the problem results from students not understanding that the operation of the original group must be restricted to a subset of the original set, but at the same time must remain invariant.

There is also difficulty coordinating an understanding of the elements of the sets associated with Z_3 and Z_6 (Burn, 1996). One way of conceptualizing the elements of Z_3 is as equivalence classes that have as members the integers 0, 1 and 2 (integers that are also used to denote those equivalence classes). This conceptualization requires understanding and managing multiple levels of abstraction;

another way of conceptualizing the elements of Z_3 , that also requires multiple levels of abstraction is as the elements of a cyclic group of order three, commonly denoted with the integers 0, 1, and 2. Providing a complete and mathematically correct answer to this question requires a simultaneous balancing of both conceptualizations of the elements of Z_3 as well as the relationship between groups and subgroups with respect to the group operation. As such, the question evaluated a student's understanding of the relationship between groups and subgroups with a relatively low threshold of technical knowledge.

Items 8 and 9 in the interview assessed student thinking about an abstract group structure (Brown, et al., 1997) as students' responses to these problems indicated the level of student understanding about groups in general and student understanding of order in particular.

Data Analysis

To explore the central research questions of the study, I began by analyzing the body of data from each data source separately in order to generate initial hypotheses. Then I undertook a global analysis of the data to confirm, refute or refine the initial hypotheses. This type of analysis is commonly called *grounded theory* (Glasser, 1992; Glasser & Strauss, 1967; Strauss & Corbin, 1990, 1998) and has been used when describing student understanding of abstract algebra (Findell, 2000).

The two principal goals of the study were to offer preliminary characterizations of teachers' and students' actions (interactions) within two different types of instructional approaches within an abstract algebra course and to describe the mathematical proficiencies of volunteer students enrolled in those sections. The ideal

method of data collection and analysis within grounded theory is an iterative process in which initial data collection is followed by analysis, and the emergent theory is allowed to inform and guide subsequent data collection (Glaser, 1992). Given that both of the principal research questions were approached via the collection of longitudinal data, analysis was ongoing and early stages informed the later stages of data collection.

Analysis of abstract algebra instruction

In the investigation of the teaching of DTP and investigative abstract algebra classes, I observed 16 hours of the DTP class meetings and 13 hours of investigative class meetings. While observing classes, I began to formulate some descriptions or categorizations of the types of lecture or investigative instruction that I observed. My observations of classes led to some initial hypothesis that were later examined by thorough reviews of the class transcripts and my notes. Each of these initial descriptions or categorizations served as data for an analysis of slightly larger grain-size. I frequently returned to the data and initial analyses to judge the faithfulness of the emerging findings and the accompanying explanation. In this manner, I strived to arrive at an empirically grounded analysis.

Findell (2000) argued that such a method of analysis was well aligned with Glaser and Strauss' (1967) description of grounded theory. He stated:

I realized that the detailed summaries functioned as codes, the preliminary observations served as initial categories and hypotheses, and the synthesis of the working hypotheses formed the core of an emergent theory. It is now

apparent that ... the method is consistent with the constant comparative method of Glaser and Strauss (1967) (Findell, 2000, p. 122).

Schoenfeld (1998) gave a detailed description of how to analyze a single class, which I believe is also aligned with the proposed interpretative framework. Schoenfeld called his method of analysis “lesson parsing” and described it as an iterative process. He stated, “the parsing, which proceeds in stages, consists of the iterative decomposition of a body of instruction, which we shall refer to generically as a chunk, into smaller chunks, each of which coheres on phenomenological grounds” (Schoenfeld, 1998, Lesson Parsing and Model Building, ¶ 2). Schoenfeld described this decomposition as goal based. By that, he meant that each chunk would have at least one highly activated goal. He asserted that there are often always-activated goals, but for the purposes of analysis, those might be omitted as background. To create a decomposition, Schoenfeld suggested that the first step is to search for break points. He wrote:

A break point represents a change in the character of the instruction that is significant at the current level of grain size—that is, a change in focus, direction, emphasis, etc., that is notable with respect to the chunk of instruction being parsed. (Break points might correspond to the end of the discussion of a particular topic and the introduction of a new one, to the discussion of a problem, to a shift in classroom organization from whole-group to small-group, etc.) (Schoenfeld, 1998, Lesson Parsing and Model Building, ¶ 2)

This type of analysis allowed a variety of characterizations of class types to surface in my study. That is, the categories of the data arose from a close reading of appropriately-sized chunks of data. Then, reading across the class meetings and globally reading the types of categories that emerge from the data gave rise to some broad descriptions that define a small number of types of classes.

Prior research has characterized categories of lecture as formalist and intuitive (Weber, 2004). A formalist lecture offers no discussion of meaning, but presents logical listings of symbolic manipulation according to a specified set of rules to arrive at a specified set of ends. This type of mathematical activity is aligned with Hilbert's school of formal mathematics. Weber described a real analysis instructor who engaged in this form of teaching because he believed that proficiency with symbolic manipulation had to precede a more intuitive understanding of either the mathematical concepts or the structure of the proof. An intuitive lecture focuses upon the meaning of the mathematical concepts and how those meanings are used to shape arguments. The technical aspects of the proof are seen as secondary to (or perhaps deriving from) understanding. No preliminary schemes for describing and categorizing types of investigative teaching at the collegiate level were located.

Analysis of student learning

A similar iterative process was used to analyze the student proficiency data. Initial analysis of each data source (e.g., mid-semester assessment) began soon after the data was collected. As such, interpretation of student responses to individual assessments lead to initial hypotheses that were revised after analysis of later data sources. Descriptive characterization or analysis that derived from individual data

sources were re-examined to yield a broad narrative addressing student understanding and proficiency with the content of an introductory abstract algebra course.

This analysis was distinct from that proposed by Glaser (1992) in that it addressed research literature and utilized conceptual categories within the discipline of abstract algebra (e.g., group, ring and field concepts). In addition, this analysis focused on the teaching that occurred within the DTP and investigative classes, and the kinds of understandings that students derived from the DTP and investigative abstract algebra classes. Thus, the focus was on the breadth and depth of instruction and students' understandings compared to mathematical ideals. This is contrary to Glaser's (1992) perspective that the meaning and seeds of research subjects is the primary lens for analysis.

CHAPTER 4: LEARNING ABOUT TEACHING

A number of studies have attempted to better document how student understanding of specific content in abstract algebra develops (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996; Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997; Brown, DeVries, Dubinsky, & Thomas, 1997; Dubinsky, Dautermann, Leron, & Zazkis, 1994; Findell, 2000). Similarly, a number of papers have been written and conference presentations given that provide suggestions for pedagogical changes to the course (Edwards & Brenton, 1999; Hibbard & Maycock, 2001; Mingus, 2001). Together these works indicate that there is a general unease within the field about the teaching and learning of abstract algebra. While these are not new concerns, recent work has targeted the teaching of abstract algebra as a focal point for change much more frequently than in the past. Yet, while the field may have diagnosed teaching as a potential problem source, the diagnosis was based upon anecdote and, while instructional solutions are being proposed, there is no agreement on the cause of the problem or even an accurate description of current practice with respect to the teaching and learning of abstract algebra. As a step towards developing a better understanding of abstract algebra instruction, this chapter describes the classroom activities of one traditional DTP course and one investigative course as offered during a single semester. This description is meant as a description of one such instructional offering of each approach to an introductory undergraduate course in abstract algebra, not a thorough description of DTP or investigative teaching. It is meant to be read as a description of what is possible in such a class.

Methodology

These data were primarily collected through classroom observation. As noted in Chapter 3, I observed 16 meetings of the DTP course and 15 meetings of the investigative course, always taking field notes and making a video recording. Because each of the classes was generally teacher-centered, the video captured the image at the front of the room. All of the classroom dialogue was later transcribed.

Review of the video data and transcripts revealed that break points (Schoenfeld, 1998) sometimes existed within class meetings but very often there were multiple classes that were better understood as a single “chunk” and to that end I have referred to a *teaching episode* to describe the large “chunks” of data which stretch across multiple classes. In my analysis I have used the term *teaching script* to refer to smaller “chunk” contained in a single class. Teaching scripts were generally repeated in multiple class sessions with similar purpose and methods. After identifying teaching scripts associated with each of the two instructors, I focused on describing the important characteristics of these scripts including: the type of questions that the instructor asked; the kinds of statements that she made; and the expected and actual behaviors of the students during each of these scripts. All of this was done through the repeated reading of transcripts and descriptions, as suggested by Glasser (1992).

Due to the importance of proof in the DTP class, this chapter begins with a description and analysis of the ways that Dr. Hedge used and created proofs in class. Then, this chapter presents and characterizes three different teaching scripts that Dr. Hedge was observed enacting in the DTP class. Dr. Hedge’s scripts were primarily focused on proof-writing. Similarly, this chapter offers a description and

characterization of the teaching scripts that Dr. Parker used during her investigative class. Lastly, I will offer a comparison of the teaching scripts used in the two classes in order to describe similarities and differences, both within and across teaching scripts.

Proof in the DTP class

There were three principal styles in which proofs were presented during the class meetings of the DTP abstract algebra course. During the time that I observed the class, either the students or the teacher wrote 29 analytical verifications of properties or other types of results. Seven of these proofs were given entirely by students, and each of these was a property-verification argument. The other 25 proofs included verification that particular properties of a ring held for a set and operations, that a function preserved an operation, or that a function was injective, surjective, or well-defined. In order of frequency, the three styles of proof writing I observed Dr. Hedge use during the class were participatory, student-authored, and teacher-authored.

Participatory proof

The first style of proof, participatory proof, was the most common, representing 21 of the 29 observed proofs. In each case where Dr. Hedge used a participatory proof strategy, she took responsibility for the overall structure of the proof, but she asked the students a number of questions whose answers were integral to completing the proof. None of these proofs involved maneuvers that the students would not have seen and practiced before. As such, Dr. Hedge was likely requiring student participation as a way to check for understanding about various topics and

proof structure and using the questions to model good mathematical thinking.

Consider the following complete example of the participatory-style of proof which

Dr. Hedge enacted in her class.

A proof that the kernel of a ring homomorphism is an ideal.

Dr. Hedge: Let's see why the thing called K , which has a name, it's the kernel, is an ideal all the time. So, we need to get back to this ring homomorphism. If we have any ring hom f from R to S , let's show K is an ideal. What do we have to do to show it's an ideal? [pause] You have to show it's closed under addition, closed under multiplication, it's non-empty, every element has an additive inverse. What do those four things tell us?

S: Subring.

Dr. Hedge: Subring. And then, I need what? I multiply an element in R , I take anything in K and I multiply by any element in R , and that product comes back into K . That's the ideal condition. That last condition actually includes, like we said, that multiplication is closed. So, then we just need to check addition, that it's non-empty, that additive inverses work out, and the ideal condition. So, let's do that.

S: [inaudible]

Dr. Hedge: That would tell me?

S: [inaudible]

Dr. Hedge: Yes. But, I want to stick with this for just a minute because I want to emphasize something about homomorphisms, so this is not the shortest thing we could do. How could I show it's non-empty? How do I show that there's something that goes to zero?

S: [Inaudible]

Dr. Hedge: Yeah. So, we know that $f(0r)$ equals $0s$, so there's something there. So, $0r$ is in K . So, it's got something in it. That may be all that's in it, and if so, that tells us something very special about that map. Okay, let's take two things, not r and s , how about a and b . If a and b are in K , I want to show their sum is in K . How do I show their sum is in K ? You have to use the definition of big K . The only thing you know about big K is, well, it consists of stuff that gets mapped to zero. So, what do I have to show about $a + b$ to show it's in K ?

S: It gets mapped to zero.

Dr. Hedge: Ok, so, let's look at what f does to $a + b$. So, Tr, what can I say about $f(a + b)$?

S₂: It equals $f(a)$ plus $f(b)$.

Dr. Hedge: Is there anything I know about $f(a)$ now?

S₂: It equals zero.

Dr. Hedge: Great, and $f(b)$? And what do I know about zero plus zero? That's zero. Great. So, $a + b$ meets the condition it needs to be in K .

[pause] So, that's the property of f preserving addition that we just used, and that gives us that the kernel is closed under addition. In a little bit we'll be moving to groups and we won't have two operations, we'll just have one, so it's nice to see what you can get with just that one operation. What about $f(ab)$? C ? What can you say?

S₃: $f(a)$ times $f(b)$.

Dr. Hedge: Great, if it's a ring homomorphism, you can split it up. So, what can you say about $f(a)$? It's zero. So, I actually have zero times, and, does it matter what $f(b)$ is? [students shaking head]. And, let me change something... Also, how about we make this for any a in K , well, we already said a was in K , and how about for any r in R ? Well, if I change b to be r , actually we could have just left this as b , but then we'd get zero. And that includes a and b being in K , as we were just talking about. Likewise, $f(ra)$, what is that going to equal?

S: $f(r)$ times $f(a)$.

Dr. Hedge: Great. And what is that going to equal?

S: $f(r)$ times 0, zero.

Dr. Hedge: So in other words, it doesn't matter, they're both the same. So, ra and ar are in K . So, what do I need left to check that this is an ideal?

S: Inverses

Dr. Hedge: Good, additive inverses. And otherwise the things we've already checked. [erases board] So, again, we had to use that f preserved multiplication. Finally, let's look at what f does to negative a . For all a in K , $f(-a)$, what can I say about that? What does f do to negative a ?

S: [inaudible]

Dr. Hedge: Yeah, this is another one of our properties of homomorphisms. f carries an additive inverse to the additive inverse of the image, to negative $f(a)$, and that equals, negative zero! Yeah! And what's the additive inverse of zero? Zero, so we get zero. So, K is an ideal. [Underlines: K is an ideal.] So as a consequence, any time you have a homomorphism of rings, you get an ideal.

Consider a second, partial example of a participatory proof. This is the beginning of a proof that a function $f\text{-tilde}$ is an isomorphism. Dr. Hedge had begun the proof the previous day but ran out of time in the class period before proving the homomorphism property. This is a fairly unique set of questions in that Dr. Hedge prompted students by name to participate in the proof writing. Note how she began by stating the fact to be proved and then asking the students questions to advance the proof:

A proof that the function f -tilde is a preserves addition.

Dr. Hedge: So, if it's a bijection and a homomorphism, then it's an isomorphism. So, how should I show that it's a homomorphism? Lynn, let's start with you. How should I show that it's a homomorphism?

Lynn: Show that it preserves operations.

Dr. Hedge: Okay, let's take a couple of cosets... Okay, so we can do that. So, our two cosets are r plus the ker of f and t plus the kernel of f . We're not saying these are the same coset, they were a minute ago, but now they're just two cosets. What two operations do I have to check S?

S: Addition and multiplication.

Dr. Hedge: So, what's it going to look like when I check addition?

S: $f r$ plus kernel of f plus t plus the kernel of f .

In both of these proofs the dialogue is always a teacher-initiated question and a student response that is relatively short and can quickly be judged to be either appropriate or inappropriate. In the first participatory proof shown above, Dr. Hedge asks a question of the class as a whole, but she expects some individual student to respond: " $f(r)a$, what is that going to equal?" This question has multiple correct responses, but, in the context of the proof, there is only one appropriate response, and that is given by the student, " $f(r)$ times $f(a)$." The appropriateness of this response is determined by the context. In this case, the proof is about a function that is a homomorphism. Dr. Hedge uses the circumlocution when she wants the student(s) to give a response making use of the homomorphism property, and, it is clear which operation the proof is currently describing. Consider the similar question and answer pattern from the second of the proofs exhibited above:

Dr. Hedge: So, what's it going to look like when I check addition?

S: $f r$ plus kernel of f plus t plus the kernel of f .

Similarly, Dr. Hedge also asked students to derive results via questioning.

Returning to the first proof, note how she immediately followed the student's

response with the question, “And what is that going to equal?” This question also had a response that was clearly appropriate. The classroom norms regarding proof required that each utterance comprise only one logical step. Thus, the student was expected to determine that a is a member of the kernel so $f(a)$ is zero. She correctly did so and gave the response, “ $f(r)$ times 0, zero.”

The last type of question that Dr. Hedge repeatedly asked students to respond to might be characterized as asking about the status of the proof. For example, she asked, “So, what do I need left to check that this is an ideal?” All of the students were supposed to know what had been verified and what was left to verify. In this situation there was only one property left to verify, and the student correctly identified it, “inverses.”

In some way, each of these questions may be thought of as requiring some strategic knowledge about proof archetypes to articulate a correct response (Weber, 2001). The student must have the ability to read the proof and also have a structure and logical framework in mind. But, these questions were generally only factual questions. Dr. Hedge used very similar phrasing each time she wanted the students to make use of the homomorphism property in their response, and she gave other similar verbal cues when asking about proofs of other properties. The remainder of Dr. Hedge’s questions were quite clearly simple factual questions, such as “Why do I know that f splits things up like that?” with a correct response, of “Because it’s a homomorphism.” Thus, while I have labeled these participatory proofs because they include a large amount of dialogue, Dr. Hedge was the principal author of the proofs, and her questions could be seen as ways of checking for understanding and modeling

the type of questioning that an expert-proof writer uses (such as she did when writing a teacher-authored proof). The students were not responsible for knowing the type of proof (direct, indirect, contraposition), the properties to be verified, or the structure of the argument, but rather were required to answer factual prompts. Thus, while participatory, the level of student intellectual engagement with the proof tasks required was still rather low.

During participatory proofs student participation was varied. There were two students, Lynn and BS who were, by far, the most vocal during proof writing.

Typically each would contribute one or more responses during each proof. On the other extreme, there were also multiple students who never contributed to participatory proofs. It is unclear whether Dr. Hedge considered or presumed that an appropriate response from one student, even when repeatedly from the same pair, indicated that the majority of the class understood the situation or could have offered the same correct response. Dr. Hedge's commentary during other class meetings, such as when she was responding to questions about the homework or addressing the student's exams, indicated a realization that not all of the students were developing the type of knowledge that she hoped in terms of proof proficiency, but her on-line commentary did not allow for such inference.

Student authored proof

The student-authored proofs were interesting in that, although a student was presenting a proof, there was minimal dialogue. The proof presentation was generally either a monologue on the part of the student author, or did not involve any talking at all (in the case of one of Jeff's proofs). During the 16 class meetings that I observed,

8 different students came to the board or overhead projector to either give a complete proof of a theorem or to give a proof that a property held in a given situation (a partial proof of a theorem). These eight students only proved seven different results because two students came to the board to give a proof of one result.

For four of these proofs, the students wrote proofs on transparencies and presented them via the overhead projector as they read through their work. In general, each of these proofs was an adaptation of the proof of an earlier result that generally only required a change in notation. The in-text directions for one of the presented proofs read, “Copy the proof of Cor. 2.4 with obvious notational changes.” These student-authored proofs were somewhat different than any of the other proofs given in class in that they were completely prepared significantly in advance of presentation.

The other four proof attempts were given during a class period without advance preparation on the part of the student. In each case Dr. Hedge called the student to the board and asked him or her to give a proof that a property held in a given situation. In two of these cases, the student was asked to write a proof that was essentially the same proof that Dr. Hedge had just written. Consider this case where Dr. Hedge asks Jeff to demonstrate that a function preserves multiplication. She had just completed a proof that the function preserves addition:

Dr. Hedge: Yes, f tilde preserves addition is what we just showed. Good.
Jeff, why don't you come up and write down the next bit.

Jeff comes to the front and writes without talking:

$$\begin{aligned}
& \overline{f((r + \ker f)(t + \ker f))} \\
&= \overline{f(rt) + \ker f} \\
&= f(rt) = f(r)f(t) \\
&= \overline{f(r + \ker f)}\overline{f(t + \ker f)} \\
&\therefore \overline{f} \\
&\quad \text{preserves multiplication}
\end{aligned}$$

Dr. Hedge: Okay, what do you guys think? Is it good? Any questions?
 [pause] So, for this part, does it look like it matters that we're using the kernel of f for the ideal here? [pause] Just coset operations and f is a homomorphism. So, why do we need the kernel of f part? Or, do we need the kernel of f part? In other words, could we use any old I here, for this whole set up?

In this case Jeff altered the proof that Dr. Hedge had just demonstrated. This situation was nearly identical to the situation in which Nathan was called to the board and asked to demonstrate that a function preserves multiplication. Once again, Dr. Hedge had just completed a participatory proof that the function in question preserves addition and the symbolic argument was still on the board.

Nathan comes to the front and writes without speaking:
 $f(a,b)(c,d) = f(ab, cd)$
 [Nathan then looks directly at the addition, pauses 30 seconds]
 Nathan: Where would this go next?
 [Pause 30 seconds]
 Nathan: [inaudible] Can I do? Oh, that would be zero, and that would be bd . And that's [inaudible]...
 Writes:
 $= bd = f(a,b)f(c,d)$

Dr. Hedge: So how was that so far? [Pause, 10 seconds] So what's left?

In both of these cases Dr. Hedge asked the students to demonstrate that a given property (multiplication) held immediately after she had given a very similar demonstration for addition that was still recorded on the board. The requested proof could have been derived by students simply via a change in notation.

There was one other student-authored proof that I observed. Aurora was asked to prove that R/I was associative given that R is a ring and I is an ideal. What made this unique is that it was the first proof in the general setting of R/I . Thus, while Aurora should have been quite proficient at verifying that a given structure is associative, the first property verified as part of the proof, this was the first time that any such work was done in the general structure, and she struggled to write the proof.

Dr. Hedge: So, let's check a couple of the properties of the general $R \text{ mod } I$. If we take any cosets, $a + I$, $b + I$, $c + I$, let's check associativity of addition. What do you have to look at to do that? Aurora, if you start to check that, that would be like this piece here [indicated appropriate section of polynomial proof on overhead], what do you have to do?

Aurora: You want to $a + I$, plus parenthesis $b + I$ plus $c + I$ and then close parenthesis, and, I don't know, okay, yeah, then you put big parenthesis around all of them...

Dr. Hedge: The whole bit? Why don't you come up and show me?

At this point it seems that Dr. Hedge made an on-line decision to switch from a participatory-proof to a student-authored proof due to Aurora's response. Hedge's initial question seemed ask Aurora to describe how to begin a proof that associativity holds, but in all previous student-authored proofs the student was called to the board without Dr. Hedge asking any questions about the proof. A proof from the text of a similar result in a polynomial ring was displayed on the overhead when Aurora came to the front. She seemed to base her work on the displayed proof. In the dialogue that follows observe that although Aurora's text is analogous to that displayed from the book when Dr. Hedge asked if anyone had questions, Lynn, who observations indicated was a very strong student, asked for clarification and refused to accept Aurora's assertion that her actions were equivalent to those in the text.

Aurora comes to the front. She does not say anything while writing.

Written: $(a + I) + ((b + I) + (c + I))$

$$\begin{aligned}
&=(a + I + b + I + c + I) \\
&=(a + I + b + I)+(c + I) \\
&=((a + I)+(b + I))+(c + I)
\end{aligned}$$

Dr. Hedge: Okay, so we're looking for associativity, we want to show that when we're adding any three cosets, it doesn't matter, we can add any two of them first. Do you guys have questions or comments about what she's got?

Lynn: Is she justified going to her second thing?

Dr. Hedge: What do you think?

Aurora: It's the same idea as that one. [indicating polynomial proof]

Lynn: I don't know.

Aurora: It's the same idea.

Lynn: You took out a lot of parenthesis.

Here there is a whole-class discussion in which the students, prompted by Dr.

Hedge's earlier questioning, attempt to help Aurora add detail to her proof to better indicate the collapse and addition of parenthesis.

Aurora: I don't understand what you're asking.

S: She doesn't like your parenthesis is what she's saying.

S: Shouldn't you have parenthesis around the $b+c$?

Aurora: [adds them] Why?

S: Take say, $b+c$ and we'll label it as b or something like that, so you're trying to say that the $b+c$ go together and you need to put parenthesis around those to make it clear that they go together.

Aurora: [inaudible]

Dr. Hedge: Why don't one of you finish, from the back?

At this point Jeff came to the front and wordlessly completed the proof by erasing all but the first line and writing work detailing the changes in parenthesis, turned and sat back down. Dr. Hedge then asked the whole class "Are there any questions?" No one said anything and then Dr. Hedge transitioned to the next part of the proof.

This proof could be thought of as simply requiring a change in notation from previous proofs the students had seen, but, the exact model was not extant, either on the board or immediately at hand for the students as with previous student-authored proofs. Thus, this proof could be thought of as significantly more cognitively

challenging for the students. During the class period Dr. Hedge's actions and speech made it seem that she believed the students would find this proof rather trivial and expected a reaction similar to the one Aurora gave in presenting her work, "It's the same idea as that one." Instead, it seemed that the other students, and perhaps Aurora (she may have been transcribing without understanding) were struggling to understand how parenthesis can be removed and added in coset notation and because of that Dr. Hedge made an on-line decision to follow the student's line of questioning and have more detail added the proof.

The governing characteristic of the types of proofs that Dr. Hedge selected for student authorship during my observations seemed to be that the proof required only a change in notation from previous proofs. It could be that she asked students to author different types of proofs during class meetings that I did not observe. It could be that copying proofs except for a change in notation is an important aspect of the learning of a new proof-type (for example, the proof of the homomorphism property) or in authoring a known proof-type using new a new set of symbols (such as in the ring R/I).

Dr. Hedge was fairly consistent in her response to student-authored proofs. In all of the cases she asked the other students in the class to evaluate the proof. After Aurora's attempted verification of associativity in R/I she asked, "Do you guys have questions or comments about what she's got?" Dr. Hedge asked a nearly identical question after Jeff's attempt to demonstrate that a function preserve multiplication, "Okay, what do you guys think? Is it good? Any questions?" Finally, after Nathan's proof that a function preserves multiplication Dr. Hedge asked, "So how was that so

far?" In each case she asked a neutrally phrased question which did not intrinsically indicate whether she believed the proof attempt to be valid. Although Dr. Hedge asked for questions after each of the eight students came to the board, it was only Aurora's work that garnered any comments or questions. All of the other proofs by the students were substantially mathematically correct and it could be that the students recognized this fact. But again, it is worth noting that Aurora's proof attempt was the only spontaneous proof writing that required more than a notational change.

Teacher-authored proof

Although I observed some 15 class meetings, I only saw one example of a proof where Dr. Hedge did not interject questions or ask for student contributions as she worked her way through the proof. Considering that I observed her giving four complete proofs, each of which involved verifying between two and five properties, not to mention the property verification proofs that she or the students did while discussing homework, it seems that in this traditional classroom the students were much more likely to be participating in, or to be lead through, proof creation rather than acting as the passive observers caricatured in more cynical descriptions of traditional teaching. This suggests that the vision of a lecture-based class as one where the teacher does all of the talking and the students almost none is not always accurate. While this style of teaching does exist, it was not the norm in Dr. Hedge's traditional DTP instruction.

The single teacher-authored proof that I observed was a verification that multiplication distributes over addition in R/I if R is a ring and I an ideal. Dr. Hedge only asked two questions both of which could be interpreted as rhetorical.

Dr. Hedge: If we want to prove, there's distributivity, what do you think we'll use? Operations and... the fact that there's distributivity in R . Let's write that one down real quick and then the others are simple. [erases previous work] Let's make sure that everyone's okay. I think once you get the associativity then life is good. So, let's check distributivity. $(a + I)$ times $[(b + I) + (c + I)]$ that's our set-up. What should I do first? The sum, then I multiply the sum. So, let's take the sum first. So, what's the sum of $b + I$ and $c + I$, yeah, $b + c$, so now I have

$$\text{Written: } (a + I) [(b + I) + (c + I)] = (a + I)((b + c) + I)$$

Dr. Hedge: So, now let's multiply. How do I multiply cosets? I multiply the representatives. So, I'll have a times b plus c is grouped together just by definition of multiplication. So, now what am I going to use?

Nathan: well, you can say that since it distributes in R , that you've got $ab + ac$.

Dr. Hedge: Great. Because it distributes in R since R is a ring. Okay, now what do I do? The coset of ab plus ac is the same as what? The coset of ab plus the coset of ac , and that's by the definition of addition, and now what? The coset of the product ab is the same as the coset of a times the coset of b so that $a + I$ times $b + I$ plus the coset of $a + I$ times the coset of $c + I$, and that's distributivity. How's that? [pause]

This is the expected type of proof, where Dr. Hedge wrote the entire proof on the board while also stating it aloud. On the board Dr. Hedge wrote only the symbols and text necessary to constitute a complete proof. This was her general practice, even when she was working in a more participatory style. While Dr. Hedge only wrote the minimal necessary symbolic argument, her spoken argument also included substantially more information and contained a description of her thought process. It modeled the type of questions that a proof writer may ask during the writing process. Consider her dialogue during the proof, "What should I do first? The sum, then I multiply the sum. So, let's take the sum first," or "Okay, now what do I do? The coset of ab plus ac is the same as what?" Each of these can be understood as Dr. Hedge thinking aloud as a means of showing off the thought processes that lead to a complete and correct proof.

Summary of proof-writing in the DTP class

Teacher-authored proofs were, as stated above, uncommon during the class meetings I observed. It is quite possible that they were more common at the beginning of the semester when the students were still learning the form of the various new proof types such as the homomorphism proof or the verification of properties. The one example of this type of proof offered little insight into this style. Although, while it featured only the necessary symbolic argument on the board, Dr. Hedge did model aloud her thought process. This style of proof is similar to the student-authored proofs in that there was no discussion, although both Aurora and Nathan did engage in the same sort of thinking aloud process as Dr. Hedge, whereas Jeff was much more likely to work silently.

The use of student-authored proof seems to be relatively frequent with 7 instances during the 16 class meetings, thus making it far more common than teacher-authored proofs. The great majority of these student-authored proofs were participatory in the sense that they featured a large amount of dialogue between Dr. Hedge and the students. Dr. Hedge would ask a series of questions about facts, the direction of the proof, and the progress of the proof, either directed at a particular student or, more commonly, at the class, and she expected a quick response. The important characteristic of these questions is that they all had an appropriate or correct response. In the case of the factual or proof-progress questions, these all had a correct response, such as “inverses remains to be checked,” or the fact that “ f is a homomorphism” means that it preserves operations. The proof-direction questions were all phrased in such a way as to give rise to an appropriate response such as

making use of the homomorphism property to move from one symbolic statement to the next. Thus, while they are participatory in the sense that student comments are involved, the students have very little authority or responsibility in the creation of these proofs. Dr. Hedge was very much the author and seemed to make use of the questions to check for student understanding and to model the proof-creation strategy of an expert.

Characterizing DTP Teaching

The caricature that we have of a DTP teacher is one of a “Sage on Stage.” In this caricature the teacher stands at the front of a classroom while talking and writing on the board. Students are expected to sit quietly and take notes documenting what is on the board and what the teacher has said aloud. Finally, students are expected to practice skills in their homework and to demonstrate their learning on exams. In reality, most classrooms are far more complex than presented in any caricature.

The DTP teacher in this study did not fit this caricature of “Sage on Stage.” One of the most important characteristics of her class was that she encouraged active classroom participation and engagement with the material. She often asked if the students had questions and would answer a question whenever it was asked. She gave extra credit points for catching mathematical mistakes in work that she presented. She required each of the students in the class to present a proof during the course of the semester, and she asked many questions during the course of a class meeting. There were also multiple class meetings where the students were working problems in class or performing some other activity designed to improve their understanding of the content. Thus, while students were always expected to take

notes and to copy text from the board into their notebook, Dr. Hedge expected them to be more than passive consumers. Perhaps half of the students would answer at least one of her questions in a class meeting, but there were a few students who only answered questions when called on by name. In general, Dr. Hedge seemed to assume that Wu's (1999) pedagogical contract held, as she principally used class time to outline the content that the students were expected to master and to demonstrate specific skill-sets that students needed to develop. But her teaching scripts also included more on-line checking of student understanding, factual mastery, and class engagement than might be expected in a pure lecture class.

Teaching Scripts

Dr. Hedge made use of three principal teaching scripts during the class meetings I observed. While it is important to consider the narrative arc of an individual course meeting, I do not believe this to be the proper unit of analysis. Each course meeting had a particular rhythm or structure, but in terms of the analysis of the mathematical arc, ideas were generally developed over multiple class meetings. Examples and proofs often took days to develop—a pattern that could not be captured well in a class-by-class analysis.

The first typical teaching script in Dr. Hedge's class was an *introductory dialogue* that occurred in each class period. The second principal teaching script that Dr. Hedge used was the *participatory proof*, and the last was the *exemplar dialogue*.

The Introductory Dialogue

At the level of a single class meeting, Dr. Hedge's class was fairly standard. The meeting opened with Dr. Hedge addressing administrative and bureaucratic

details. Of the 15 class meetings I observed she began 3 of them by asking for student questions first.

Dr. Hedge entered the room, said hello and took off her bag. She opened it, took out her text and started class.

Dr. Hedge: Any questions from your homework?

S: Shouldn't there be a bracket in number 4 in five two?

Dr. Hedge: So, this is what it says. [Writes text on the board.]

Dr. Hedge: You are correct, there should be a bracket here because this is an equivalence class. Other questions?

S: Can you do five point three, number 10?

Dr. Hedge reads the problem aloud and writes the symbolic portions on the board and starts working through a proof by contradiction. She approaches the point where she needs to derive a contradiction relating to the homomorphism property.

For the most part though, she moved directly from addressing administrative details to a very brief participatory, introductory dialogue that served to recall previous work and to launch the day's class. This introduction almost always featured a few factual statements that began the recall process followed by a set of direct or implied factual questions addressed to the whole class. Only one time did she address these questions to individual an student. A typical introductory dialogue is shown below.

Dr. Hedge: Okay, we had, and we're going to be talking about more examples today, we've got a ring R with an ideal in it [writing]. We said that working with congruence mod I was an equivalence relation and that gives us the set of equivalence classes R modulo I . How did we realize last time that we could write these equivalence classes? What did they look like? [pauses 15 seconds while students flip through their notes]

S: $a + I$ where a in R . [writing all of this in symbolic form]

Dr. Hedge: $a + I$ where a is in R ... Excellent. We would have written this as square brackets a before. These congruence classes have another name, we would have called these...

S: Cosets

Dr. Hedge: Beautiful. Cosets are equivalence classes. What else do we know about this set? What do we claim to know about this set? We have a theorem about this set...

Once this initial dialogue was complete, Dr. Hedge immediately transitioned to new mathematical material for the remainder of the class session, rejoining an ongoing *episode* that had begun in a previous class meeting.

The participatory proof

The second principal type of teaching script that Dr. Hedge made use of was the participatory proof. Participatory proofs were the most common type of proofs in the *DTP* algebra class and one of the most common teaching scripts that she employed. Proof happened on an almost daily basis in Dr. Hedge's class. Moreover, these proofs were the setting for the great majority of the questions that Dr. Hedge asked the students. Thus they offer a useful window into the types of questions that Dr. Hedge asked.

Prior text in this chapter presented two examples of the participatory-style of proof which Dr. Hedge enacted in her class. Both of these proofs involved crafting a proof that the students should already have some proficiency with. In the first instance, Dr. Hedge wrote a proof that a given set K is an ideal. This proof included demonstrating that the set K is also a subring of the ring R and then demonstrating the ideal property. By March 17, more than half-way through the semester the students should have been quite proficient at demonstrating a given set is a subring of a ring R . As such, a participatory proof would not make major cognitive demands upon the students and might be thought of as a way to check for understanding about the various topics and proof-structure.

In these proofs the dialogue was always a teacher-initiated question followed by a student response that was both relatively short and easy to judge as appropriate

or inappropriate. As with her questions during the daily introductory dialogue, the great majority of Dr. Hedge's questions during proofs were addressed to the whole class. She seemed to have three different types of questions that she asked, but there seemed to be a set of purposes that underlay all three question types. Specifically, Dr. Hedge seemed to use all of her questions to: (1) keep the students engaged with the proof-writing task; (2) stimulate the student's thinking about proof; and (3) assess the student's developing proof proficiency.

The first type of question that Dr. Hodge made use of was fact-checking. This is the same type of question that Dr. Hedge used in the introductory dialogue. These questions prompted the students to state a specific fact. Consider the examples below:

Example 1:

Dr. Hedge: Let's see why the thing called K , which has a name, it's the kernel, is an ideal all the time. So, we need to get back to this ring homomorphism. If we have any ring hom f from R to S , let's show K is an ideal. What do we have to do to show it's an ideal? [pause] You have to show it's closed under addition, closed under multiplication, it's non-empty, every element has an additive inverse. What do those four things tell us?

S: Subring.

Example 2:

Dr. Hedge: Okay, let's take two things, not r and s , how about a and b . If a and b are in K , I want to show their sum is in K . How do I show their sum is in K ? You have to use the definition of big K . The only thing you know about big K is, well, it consists of stuff that gets mapped to zero. So, what do I have to show about $a + b$ to show it's in K ?

S: It gets mapped to zero.

These questions have answers that are either correct or incorrect without any reference to the proof-context and, in the case of Example 2, Dr. Hedge actually told the students the answer to the question just before she asked it. During participatory

proofs Dr. Hedge seemed to use these questions to help students cement their knowledge of definitions and theorems.

The second type of question can be thought of as a complete-the-sentence task. Such questions typically had many possible correct responses. But, in the context of the proof, there was usually only one appropriate response. In order to respond to these types of questions, the students need to use the proof context and Dr. Hedge's phrasing of the question to determine how to correctly finish the proof-step that Dr. Hedge had started. Consider the example below where the class is working on a proof about a ring homomorphism:

Dr. Hedge: Ok, so, let's look at what f does to $a + b$. So, S, what can I say about $f(a + b)$?
S: It equals $f(a)$ plus $f(b)$.
Dr. Hedge: Is there anything I know about $f(a)$ now?
S₂: It equals zero.
Dr. Hedge: Great, and $f(b)$. And what do I know about zero plus zero? That's zero. Great. So, $a + b$ meets the condition it needs to be in K . [pause] So, that's the property of f preserving addition that we just used, and that gives us that the kernel is closed under addition.

The appropriateness of this response is determined by the context. The proof is about a function that is a homomorphism. The circumlocution is that which Dr. Hedge used when she wanted the student(s) to give a response making use of the homomorphism property, and, it was clear which operation the proof was currently describing.

The last type of question that Dr. Hedge repeatedly asked students to respond to might be thought of as proof-strategy questions. One version of this type of question required the students to determine what must be shown in the proof-archetype that they were working on, what had already been completed, and what remained to be shown. Consider the example:

Dr. Hedge: So in other words, it doesn't matter, they're both the same. So, ra and ar are in K . So, what do I need left to check that this is an ideal?

S: Inverses

Dr. Hedge: Good, additive inverses.

Another way that Dr. Hedge used this type of question was to ask the students to describe proof archetypes. The students had just verified that a function f preserved multiplication and Dr. Hedge asked the students to give a description of the proof archetype for verifying that the function preserved addition:

Dr. Hedge: So, what's it going to look like when I check addition?

S: f of r plus kernel of f plus t plus the kernel of f .

While Dr. Hedge may have intended to ask three different types of questions with different purposes for each type, the manner in which she asked them actually made most of the questions factual questions. For example, in the complete-the-sentence questions above, whenever Dr. Hedge asked the students a question where they were expected to use the homomorphism property, she used very similar phrasing and gave other verbal cues when asking about proofs of other properties. Even the questions that I have called proof-strategy questions were basically factual. In the first example above, Dr. Hedge had stated at the beginning of the proof what needed to be verified and at each step she clearly labeled what was being verified. All that was required to give a correct response was to read a list of properties and state those that had yet to be verified. Thus, I suggest that the great majority of Dr. Hedge's questions were factual in nature.

Throughout a typical proof, Dr. Hedge's statements served as structural controls. Dr. Hedge principally made three types of declarative statements during these proofs. The first type validated student's responses to her questions. As shown

above Dr. Hedge would often respond to correct statements by saying, “Good,” and then repeating whatever the student had said, sometimes rephrased to be more mathematically complete or correct. The second type of declarative statement that Dr. Hedge made during participatory proofs was a statement of fact needed in the proof. In the second example shown above Dr. Hedge stated a fact that was needed for the proof. She stated, “The only thing you know about big K is, well, it consists of stuff that gets mapped to zero.” The last type of statement that Dr. Hedge made during the course of proofs was to state proof-goals. There were times where she would describe the outline of the proof, such as in the first example above, and places where she would make statements that organized the verification of individual properties. An example of Dr. Hedge organizing a verification of an individual property is when she stated, “Ok, so, let’s look at what f does to $a + b$.” In this case she was telling the students the next part of the proof to work on.

The exemplar dialogue

As noted earlier the most profound difference between Edwards and Brenton’s (1999) DTPE caricature and the actuality of Dr. Hedge’s class seems to be the use that Dr. Hedge made of examples in her teaching. Dr. Hedge included a large number of examples in her classes, often one to introduce a definition, one to situation a theorem, and it seems fairly clear that this increased number of examples was an attempt to help students develop deeper understanding of the topics.

Her practice was to give some number of examples of structures and to ask students to consider them before giving students the formal definition of the structure. This seemed to be an effort to help the students develop a concept image before

developing the concept definition, to use the language of Vinner (1991). Yet, rather than give the students multiple examples of a structure and then ask them to discern the commonalities, she then stated the definition, and then, in both episodes I saw, asked the students to consider another example. This second example took on a dual role, besides serving as an example of the structure. Dr. Hedge also chose to make this an example from which new mathematical generalizations were motivated. Most of her examples served these two purposes: examples of and examples from which to generalize (except the ones that she used between a statement of a theorem and the proof).

That is, her pedagogical move was to ask the students to work an example, often for homework, and then to use that example to launch new mathematics, especially new definitions and theorems. She used examples to help students develop increased understanding of a new structure, and then she used examples to motivate new mathematics by asking students to consider specific features of her examples. At the end of these exemplar dialogues Dr. Hedge would usually give the students some understanding of the coming direction and flow of the course —often explaining how the next developments would be a generalization of that which the students had already done.

This dual use of examples is illustrated by the example below. The first day that Dr. Hedge introduced ideals she asked the students to consider a number of examples of ideals. The last one that they discussed before class ended was the very familiar structure $\mathbf{Z}/(n)$ where \mathbf{Z} represents the integers and (n) represents the multiples of n . The students concluded that (n) is an ideal in the integers. Dr. Hedge

then ended class by saying, “In Chapter 6 we’re going to look more closely at ideals in an abstract setting, and we’ll redo many of these results using ideals instead of irreducible polynomials.” The next day she took questions on the homework at the start of the class period and then began her introductory dialogue:

Dr. Hedge: Ok, so, let’s move on. Let’s recall...

Dr. Hedge wrote: for $n \in \mathbb{N}$ we had $\mathbb{Z}/(n)$. For F , field, and $p(x) \in F[x]$, we had

$$F[x]/(p(x)).$$

Dr. Hedge: And we found that these were rings whose elements were equivalence classes. Well, how did we define those equivalence classes?

S₁: By $ax + by$ equals n .

Dr. Hedge: Ok, was there another way?

S₁: ax is n minus by ?

Dr. Hedge: Ok, what about $a - b$, what can we say about that?

S₂: It’s a multiple of n ?

Dr. Hedge wrote: $a \equiv b \pmod{n} \Leftrightarrow a - b = kn$ for some $k \in \mathbb{Z}$

Dr. Hedge: Then we can say in terms of ideals that if a minus b is in I , then a and b are equivalent mod I ? What if we say that a minus b equals an element in I ?

S₂: That a minus b is in I .

Dr. Hedge wrote: $a - b = i$ for some $i \in I$.

Dr. Hedge: How about we take that to be our definition, what does that get us?

Dr. Hedge wrote: Goal: If this relation, congruence modulo I , is an equivalence relation; then we’ll try considering equivalence classes modulo I & look at possible ring structures on the set of equivalence classes.

During this interaction Dr. Hedge made three factual statements (two written but not spoken) and asked a number of questions. Dr. Hedge would first state the example, often preceded by the word *recall*. Then she would indicate the particular aspect of the example that she wanted to focus on for the discussion. For example, in the script above she wanted to focus on the fact that $\mathbb{Z}/(n)$ is a set of equivalence classes and that (n) could be defined in terms of an actual multiple of n , the element that generated (n) . Dr. Hedge’s last statement informed the students what the goal of

the coming classes was. It was a written version of her statement from the previous class that, “In Chapter 6 we’re going to look more closely at ideals in an abstract setting and we’ll redo many of these results using ideals instead of irreducible polynomials.” In each case Dr. Hedge’s statements were intended to frame the manner in which the students would be engaged with the material and to direct them to a particular explicit goal.

All of the questions were factual questions directed at the class as a whole. Dr. Hedge was seeking a specific manner of defining a congruence class, and she kept asking questions until the students had stated that definition. In other exemplar dialogues, Dr. Hedge asked factual questions about the results of computations, functions with particular characteristics, and other topics that should have been very familiar to the students—either because they had seen the example numerous times or had worked with it in the immediate past (most often the previous class or their homework).

Salient characteristics of the observed teaching scripts

The observed teaching scripts each had a number of different characteristics but there were many commonalities. As noted above, in each of the teaching scripts described above Dr. Hedge asked a large number of questions and expected students to respond as a way to participate in class. As such, the next level of analysis is to look for patterns across the teaching scripts as a way to better understand the characteristics of *DTP* teaching writ large.

Declarative statements

Across the three teaching scripts, Dr. Hedge made consistent use of

declarative statements. Her primary use of declarative statements was in making factual statements about the content of abstract algebra. This type of statement took a slightly different form in each of the three teaching scripts, but it was always present. In the Introductory Dialogue she would prompt the students to recall the previous day's material by use of phrases such as "we had..." or "we saw..." in each case following closely with a mathematical statement giving the context of the previous material.

Dr. Hedge's declarative statements in the Exemplar Dialogue were similar in both nature and purpose. These dialogues often began with the statement, "Recall..." followed by a description of a mathematical structure that the students had studied previously, this might also include the statement of a mathematical definition, as in the case of the kernel of a homomorphism as shown above. Subsequent declarative statements would structure the coming class by either stating a goal or a particular aspect of the example to focus on. For example, in the example shown above, Dr. Hedge concluded the Exemplar Dialogue with the following:

Goal: If this relation, congruence modulo I , is an equivalence relation; then we'll try considering equivalence classes modulo I & look at possible ring structures on the set of equivalence classes.

In both of these teaching scripts declarative statements are used for two principle purposes. The first is to prompt the students to recall previous mathematical content that Dr. Hedge will reference in subsequent parts of the dialogue of class meeting. The second purpose of Dr. Hedge's declarative statements was to give the students directions for future work. In many ways, these two purposes are closely tied in that Dr. Hedge would establish the context for the mathematical work and then

state how it would be carried out. Clearly, these two teaching scripts featured very similar declarative statements. On first inspection, it seems that those declarative statements Dr. Hedge employed in the Participatory Proof had a different purpose entirely.

In the Participatory Proof, Dr. Hedge made a number of declarative statements that were noted above. Each of the different types seemed to be used as a structural control in the proof-creation process. Consider that Dr. Hedge would validate student's responses to questions with "good" or other similar phrases, she would restate student's mathematical statements in ways that made them more mathematically complete and correct and lastly, she would state proof-goals. In stating the goals of a proof Dr. Hedge is clearly directing the subsequent class. The other two types of declarative statements can also be understood as performing the function of factual recall. In each case, Dr. Hedge has solicited a student response to a prompt and the student has made a statement that either is complete and correct or needs correction. Dr. Hedge's statements either confirm the correctness of the statement (that is, validate the factual recall) or reformulates the statement so that it is correct, serving the exact same purpose as stating the fact herself using the term "recall." As such, it seems that the two principle uses which Dr. Hedge made of declarative statements in her observed teaching scripts are stating facts for use during the class and stating goals for future work.

Question types

Dr. Hedge made consistent use of questions in each of her teaching scripts. In the Introductory Dialogue she asked questions such as, "How did we realize last

time...?” In the Participatory Proof she asked questions such as, “What do we have left to show?” or “What do I know about ...?” Finally, in the Exemplar Dialogue she asked questions such as “Well, how did we define those equivalence classes?” In all of these instances Dr. Hedge’s questions were very factual in nature. As described above, although a surface reading might indicate some of the questions were open-ended, the context and circumlocutions that she used constrained students’ responses and effectively made all questions factual in nature. In all of these teaching scripts her questions seemed to have three primary purposes: (1) to engage the students with the given task; (2) to stimulate the student’s thinking about mathematics; and (3) assess the student’s understanding and recall of the content. In short, she seemed to use questions as a means of on-line checking on students’ understanding and engagement.

Posing questions

Dr. Hedge asked a large number of questions and also used incomplete sentences with hanging pauses as a second means of soliciting student participation. Although the vast majority of her questions were factual in nature, they were generally intended for students to respond to them. During proof-writing, as noted above, she did ask some on-line questions that seemed more rhetorical in nature. In those cases she seemed to be modeling her internal dialogue while writing proof. As such, even these questions had a pedagogical purpose.

The most obvious commonality of Dr. Hedge’s questions is how she directed them. The overwhelming majority were addressed to the class as a whole. Unless there was a discussion between Dr. Hedge and a student in progress, she almost never

directed a question at a specific student. In her observed Participatory Proof teaching scripts although she would ask numerous questions throughout, only a total of two of those questions were directed at an individual student, the rest were asked of the entire class. Similarly, in the observed Exemplar Dialogues and Introductory Dialogues she directed only one question at a student with the remainder being asked of the whole class. It was unclear from her manner of asking questions whether she intended for one student or some chorus-type response to these whole-class questions, but the responses were almost always by an individual student. Periodically, two students would respond to the same prompt.

Student Scripts

The teaching scripts described above capture the majority of both Dr. Hedge's expected student actions and actual student scripts. Most of the students participated in class discussions fairly regularly, usually by answering the questions that Dr. Hedge posed during the teaching scripts. The students generally asked very few questions of Dr. Hedge or each other. The majority of their questions were requests for clarification, usually about the purpose of a particular symbol or the meaning of a particular word. Students did, when afforded the opportunity, ask Dr. Hedge to do problems from their homework on the board.

There were a few instances where it was particularly easy to infer what the students expected to gain from their classroom actions. There were multiple instances where a student would catch Dr. Hedge in a mathematical error. By pointing out the error, the student expected some extra-credit points to be added to an exam score. Similarly, when students asked for clarification or to see a homework

problem on the board it seemed that they were hoping for increased understanding of the content or use of the proof archetype. In terms of their normal participation and note taking, I can only surmise that they too believed in Wu's pedagogical bargain: if they took notes, did what was asked, and studied a bit, they would be able to demonstrate enough proficiency with the content to earn a passing grade in the course. None of the students claimed to know, prior to enrollment, what the course content was. During the end-of-semester interviews there were still a few who could not articulate any overarching course themes. As such, it is unclear if the students actually hoped to gain a deep understanding of the content of abstract algebra or if their only goal in taking the course was to fulfill departmental requirements.

Summary of DTP teaching

In almost all cases, Dr. Hedge's statements and questions seemed very purposeful. In each of her principal teaching scripts she expected students to be active participants, and she asked the class numerous factual questions. She likely used these questions to check for understanding. But another possible use of the questions was to explicitly model the type of thinking that a mathematician uses in the different tasks. This later theory is supported by her actions when she gave a proof lecture. There was an instance where Dr. Hedge delivered a standard lecture which included a proof. She did not direct any questions at students, but, as she was writing the proof she verbalized the same types of questions that she asked the students during more participatory proof writing. Consider this dialogue from that lecture, "What should I do first? The sum, then I multiply the sum. So, let's take the sum first," or "Okay, now what do I do? The coset of ab plus ac is the same as

what?” Each of these can be understood as Dr. Hedge showing the thought processes that lead to a complete and correct proof.

Similarly, Dr. Hedge’s questions and statements during the exemplar dialogue seemed intended to give the students insight into the process by which mathematicians abstract and generalize. She asked questions about the conditions, but phrased them in a way so that a generalization seemed natural. Consider the manner in which she introduced the generalization of R/I for R a ring and I an ideal:

Dr. Hedge: Ok, so, let’s move on. Let’s recall...

Written: for $n \in \mathbb{N}$ we had $\mathbb{Z}/(n)$. For F , field, and $p(x) \in F[x]$, we had $F[x]/(p(x))$.

Dr. Hedge: And we found that these were rings whose elements were equivalence classes.

Dr. Hedge then asked the students to think about how the two sets of equivalence classes were defined and by doing so lead them to the generalization. That is, she asked the students to look at two seemingly very different structures, and she modeled the type of questions that a mathematician might ask in leading to the generalization that both the known examples are instances of R/I .

Dr. Hedge used declarative statements relatively sparingly. Her principal uses of declarative statements were to state facts that they would soon be using, to explain the structure of a proof or portion of a proof, and to validate student responses to her questions. Generally these uses served to organize the class interactions and activities (proofs and examples) either by giving structure to proofs, specifying the important knowledge to be used in a proof, or stating the future direction for mathematical work.

It is clear from her comments and actions that Dr. Hedge wanted her students to develop a deep and connected understanding of the mathematical material and to develop significant proficiency with a number of proof-types, especially property proofs and homomorphism proofs. She encouraged student engagement and participation via frequent questions, encouraging students to ask questions, giving extra points for catching mathematical mistakes and her pedagogical choices, especially the copious use of examples.

She expected that the students would frequently answer her questions, attempt to make sense of the class activities, present at least one proof during the semester, and otherwise maintain the pedagogical contract. She expected that they would take, maintain, and study their notes in order to master facts like vocabulary and statements of theorems. Moreover, she expected the students to regularly complete homework that included practice with a number of skills—both computation and proof-writing. In general, this is exactly what the students did. Almost all of them were active participants during course meetings, and, although they did not ask many questions, they did actively answer Dr. Hedge's questions. Moreover, they all seemed to take notes good notes from which they could quickly access information, and they all seemed to study outside of class hours. What is most likely true is that the students had very mixed goals for the course with some students hoping to gain a lot of mathematical understanding and others hoping for a passing grade.

Analysis of Teaching in the Investigative Class

Although DTP is seen as the dominant mode of teaching in upper division undergraduate mathematics, it is also critiqued as intimidating and as misleading

students about the nature of mathematics (Thurston, 1986; Cuoco, Goldenberg, & Mark, 1996), hiding much of the process used in mathematical thinking (Dreyfus, 1991), and ignoring the important role that mathematicians ascribe to ideas such as elegance, intuition, and cooperation (Burton, 1999; Dreyfus, 1991; Fischbein, 1987). The most fundamental critique that has been leveled against DTP is that it is not an effective way to promote student learning of the mathematics content (Leron & Dubinsky, 1995; MSEB, 1991; NSF, 1992). However, none of those making this last critique provide student data to substantiate their claims. The critiques of DTP and the strength of faculty beliefs about students' corresponding lack of success have given rise to a variety of pedagogical approaches intended to improve student learning. These new pedagogical approaches involve transition from a teaching method that everyone can recognize (i.e., traditional lecture format) towards the type of teaching suggested by the MSEB (1991) and NSF (1992) advisory reports which might be called *investigative*. Although there is a growing body of literature describing different ways of structuring classroom interaction and the way that students encounter the mathematical material, thus far there have been no descriptive studies of the actual teaching of an investigative abstract algebra course. One aim of the current study was to give such a description of one instance of an investigative abstract algebra class and to begin describing the range of pedagogies and teaching scripts that are used in actual classrooms.

Teaching scripts

The tone of the investigative class was significantly different than that of the traditional class in that students were much more likely to ask questions and to make

comments. Moreover, what happened in the classroom on a given day was, to a great extent, reflective of the student's questions and comments. Because the class activities were varied and reflected the student's input, I was able to discern very few repeated teaching scripts in Dr. Parker's work. There were three that I observed and noted. The first was her repeated assertion that mathematics is about making meaning. The second occurred during her repeated teaching of computation, especially composition of permutations. The last teaching script occurred while Dr. Parker taught proof.

Throughout the course, the vision of mathematics that Dr. Parker promoted was a humanistic one, in which she emphasized that mathematics was about making meaning. She wanted the students to develop understanding of the underlying logic of mathematics. Multiple pieces of evidence for that exist, but perhaps the most interesting was her attempt to help the students understand the reason that order is used to describe both the number of elements in a group and the least power to which an element can be raised to return the identity.

The purpose of mathematics; Introducing a new concept

Let us now consider the manner in which Dr. Parker introduced cyclic subgroups. The students had just begun their study of groups but, because they had studied rings previously, were realizing that terms generally meant the same thing. Thus, they already knew the definition for the order of a group and the order of an element. One of the students gave both definitions and Dr. Parker wrote them on the board. A student asked to see an example from the homework about the order of an element in a permutation group. This question inspired a short lesson and

conversation about composition of permutations. Moving back to the side of the board where she had written the definitions of the order of a group and element, Dr. Parker asked, “Why would they use the same word to mean two different things?” Dr. Parker’s question here is asking the students explicitly to make connections about nomenclature in mathematics. A student responded, “Because that’s how they do us.”

Even though a student has indicated the common understanding that there is often similar language throughout mathematics (integers and integral domain in abstract algebra) they did not seem to have recognized that mathematicians purposefully made these choices. Dr. Parker continued to insist that the students wrestle with this relationship, and she gave a brief explanation about one goal of mathematics as a field.

Mathematics is all about making meaning, so it must be meaningful, it must be about making sense, so, it must be about making sense that they called both of these things orders.

It was at this point that, while the students were struggling with these concepts that Dr. Parker introduced two new ideas—a subgroup and a cyclic subgroup—and she asked the students to recognize these new concepts as analogous to previous work in rings. At this point Dr. Parker asked the class to consider the cyclic subgroup of S_3 generated by (123) . She taught an explicit lesson on how to compose permutations and answered a series of questions. She finished by stating:

Dr. Parker: This is the subgroup that’s generated by that element. What’s the order of that subgroup? It’s three. So, when you look at this as a set, it’s order is three. What’s the order of this as an element? It’s three. So, actually, so calling that order does make sense. You could also say that the order of an element is the cardinality of the subgroup generated by that element. So, they are actually talking about the same thing, it just happens to be that the way

that we find the order is that we take successive powers until we reach the identity. Does that make sense? [pause] Why they call it the same thing? [pause]

Finally, Dr. Parker had answered the question that she posed near the beginning of the hour, a question about the logic of naming mathematical objects. In fact, this question, about why two seemingly dissimilar mathematical objects carry the same name was used to motivate the introduction of two other mathematical structures; subgroups and cyclic subgroups.

This episode offers a strong contrast to the manner in which new mathematical structures were introduced in the DTP class. For example, in the DTP class a kernel of a homomorphism was presented as an example of an ideal. The new structure was an example of an existing structure whereas here, these new structures were brought into play as a set of tools that students could use to make sense of the mathematical concepts and the relationship between them. This means of introducing mathematical structures affords students more understanding of the process by which a mathematician would invent new structures—a process otherwise mysterious to students. In the DTP class Dr. Hedge introduced a mathematical concept and then asked questions about it. This contrasted with the practice in Dr. Parker's class where they typically started with a question and introduced new mathematical concepts in order to answer that question.

The class meetings of Dr. Parker's investigative class were directed towards substantially different goals than the meetings in the DTP class and could not be analyzed in the same way. Attempting to categorize them based upon the presentation of mathematics—such as Definition-Example-Example-Theorem—

would not give much insight into either Dr. Parker's vision for the lesson or the actual activities of the classroom. Because one of Dr. Parker's goals was to help the students understand that mathematics is a human endeavor, in this case it seems more important to analyze Dr. Parker's goals for the students and to explain the manner in which she expected the students to develop mathematical understanding. It is worth noting that I have little basis for analyzing the manner in which Dr. Parker used proof in her class, because I only observed two instances where she wrote formal proofs. Because of the paucity of proof, the analysis of the teaching of the investigative class cannot exactly parallel the analysis of the teaching of the DTP class. Yet, perhaps this is actually more appropriate because the vision of mathematics that the two teachers communicated to the students was sufficiently different that developing an understanding of the two classes requires thinking about them in substantially different ways.

Teaching Computation

The most important repeated script I noted in Dr. Parker's teaching was her insistence that mathematics is about making meaning, and this insistence shaped the manner in which she structured her classroom. The second important script came about because she wanted theorems and definitions to arise from work with specific groups and rings. This insistence on context meant that a large amount of Dr. Parker's class time was spent teaching computation.

I had the opportunity to observe 11 sessions of the investigative class, and Dr. Parker demonstrated, discussed, or explicitly taught computations in specific groups during 8 of those class meetings. In two of the other class meetings the class

discussion was about the possible existence of an isomorphism between two groups, with most of the talk centered upon mapping specific elements in one group to the other and the order of each of the elements (in this case, discussing a computation but not as the focus of the conversation). It was through these explicit and concrete discussions that she expected the students to derive questions and thoughts about mathematics, and it was in the concrete that she tried to keep all of her discussions grounded. She explicitly stated that this is how she operated and how she wanted the students to be able to operate. She said, “(It) is great when you know, well, the theoretical side, but I want to know in a group and given a subgroup you can actually find these things.”

During this conversation about computation Dr. Parker asked a large number of questions, with most being factual in nature. These ranged from factual questions in the same style as in the DTP class, but she also prompted students to consider issues of nomenclature as described above, and questions such as “what do you notice” which do not have a defined correct response. Consider her factual questions in the following exchange with a student:

Dr. Parker: So, this isn't a field. How do I know?

S: Well, in a field everything must have an inverse.

Dr. Parker: Right, and $x + 1$ doesn't have an inverse, in fact, it's worse than that because $x + 1$ times itself is zero. That makes it a zero divisor, and we can't have a field if there's a zero divisor. Have we had another structure where we've had an entry appear multiple times in the same row or column?

S: Yeah, Z_6 .

Dr. Parker: Ok, so what kind of structure was that?

S: A ring with identity

Dr. Parker: So, not a field, not even an integral domain, bummer. But, we've got an identity, anything else?

S: It's commutative.

Dr. Parker: Ok, so, when we don't have an irreducible polynomial it looks like the best we can hope for is a commutative ring with a one.

Similarly, she would ask, "So, how would I know if (a) that's my inverse and (b) if my inverse is in there, in my subset?" But, many of her other questions were asking about motivations or observations, such as, "Does everybody believe $x^2 + 1$ is a reducible polynomial in $\mathbf{Z}_2[x]$? So, how did I come up with $x^2 + 1$?" And lastly, she would ask questions which asked for alternatives methods like, "But, how else can you tell...?" and "Is there another way ...?"

Because so much of Dr. Parker's class was focused on computation in a number of different rings and groups, most of what Dr. Parker said was fairly explicit directions related to computing. She spent a significant amount of time telling and showing students how to compute with cosets, permutations, functions, and polynomials with coefficients from a finite field among others. But, the difference between Dr. Parker's statements and Dr. Hedge's seems to be that Dr. Parker was much more explicit about what she expected students to do during class. For example, "Take a couple of minutes and check if you have all the right cosets," or "Take 5 minutes and figure out what goes in the chart."

There was one teaching script that Dr. Parker returned to repeatedly—teaching and talking about computations. As I have noted above, she was quite explicit that theoretical work be grounded in some concrete example. Because of that, once the content of the class turned to groups Dr. Parker spent a lot of class time talking about the permutation groups, and she explicitly discussed how to compose permutations six times on five different days that I observed. Each time I observed

her she did nearly the same thing. Either she or a student would nominate an example and then Dr. Parker would write it on the board and talk through the example while also tracing a path between the permutations with her finger. In my observations she would do one example and then move on to whatever other topic she wanted to talk about.

She first introduced the permutations on the set of three elements (S_3, \circ) and composition of permutations on March 18 near the end of class. She showed the students how to compose two of the elements and then asked them to compose two. For homework she asked them to complete a Cayley table for the group. The next class period, on March 22, she spent time talking about the conventional notation and then started writing the Cayley table. After a student nominated an element as the result of a composition, another student disagreed and a class discussion resulted. Dr. Parker told the class that the easiest way to resolve the question would be to actually do the composition, and then she did it as an example. She told the students again that their homework was to complete the Cayley table for the group. Dr. Parker also showed examples on March 24, March 31 and April 1. Her presentation of each of these examples was nearly identical. Consider the two examples from March 24, one was near the beginning of the hour and the other near the end.

Example 1:

Dr. Parker: Let's look at a simple one.

Written: $\alpha = (123)$

Dr. Parker: What's the order of a three-cycle? [pause] Well, what's alpha squared? You've got to be a little careful because there's baggage with that word, when we say alpha cubed or alpha squared or alpha to the fifth, there's baggage with that word that it's alpha times alpha times alpha, but when I write that what you have to picture in your mind is that it's the star operation, because that's the only operation that we have.

And, so this represents, in this case, alpha composed with alpha. And so, what would that mean?

Written: $(123)(123)$

Dr. Parker: 1 goes to 2 and in the other one 2 goes to 3 so 1 went to 3, so it's almost like I'm leaping over. And 3 goes to 1 goes to 2. So, 2 goes to 3 goes to 1. So, alpha squared is 1, 3, 2. [Dr. Parker points at each number when she says it aloud.]

Written: $\alpha^2 = (132)$

Example 2:

Mark: It says the $(124)(23)$ is (1234) , I didn't know how you combine those.

Dr. Parker: So, these two are essentially the same. I'm gonna go right to left, so I'll show you how to combine them. So, first we apply this one, then we apply this one. So, I'll start with 1, 1 goes to 2 and 2 goes to itself, so 1 goes to 2. So, I applied this piece first.

Written: $(13)(12)$

Mark: So, if it's missing then we just assume that...

Dr. Parker: It goes to itself. I'm at 2, so 2 goes to 1, and then 1 goes to 3 so 2 goes to 3. I'm at 3 so 3 goes to itself and 3 goes to 1, so I've closed it back up. So, this is a way of writing (123) . If you went the other way, did you get (132) ?

A week later, on March 31, she was again talking about permutations and again she did an example for the students:

Dr. Parker: So, there's one of them, there's a second one...

Written:

$$(23)(1)(23) = 1$$

$$(23)(123)(23) = (132)$$

$$(23)(132)(23) = (123)$$

S: Can you please run through the second one just to help me remember?

[crosstalk]

Dr. Parker: Okay, so I'm working right to left. I start with 1. 1 goes to 1, 1 goes to 2, 2 goes to 3, so 1 goes to 3. I'm at 3. 3 goes to 2, 2 goes to 3, 3 goes to 2, so it must've been 2. I'm at 2, 2 goes to 3, 3 goes to 1 and 1 goes to itself, which is good, I was hoping so.

Finally, she used almost the exact same language and actions the next day in class:

Dr. Parker: Let's double check. 1 goes to 1, 1 goes to 3. 3 goes to 2, 2 goes to 1. Let's just double check, 2 goes to 3, 3 goes to 2.

If Dr. Parker has any teaching scripts, one is certainly the manner in which she does examples of composition of permutations.

During this teaching script I believe that she expected the students to take notes and to ask questions if they did not understand (as she did on March 24). She also expected that her examples, especially the way that she deliberately talked through the permutation of each element, would help the students develop proficiency with composing permutations. It seemed that most of the students did take notes during this part of the class. Given the number of times that the students asked about it, it seems that they had not achieved the level of proficiency that Dr. Parker had hoped after nearly 2 weeks spent considering permutations.

Teaching proof

The final teaching script that Dr. Parker enacted arose in those instances where she wrote a proof. I believe that this is a place where Dr. Parker's vision for the class was most in conflict with the actual events in the course. As I noted in the section describing the proof proficiencies that the students displayed, Dr. Parker intended the students to have significant opportunity to learn how to write algebraic proofs. For example, the syllabus stated that students' "facility with reading and writing proof will be used and extensively enhanced," such that proof will be a means "for demonstrating and explicating their understanding." To that end, the students were expected to read and understand the text and were given a reading guide (for at least the first two chapters) that asked them to consider proof development. Moreover, on assessments (homework and exams) the students were responsible for making proof-based arguments. For example, on the final exam for the course, the students were asked to demonstrate that a given set and operation form a cyclic group, to show that a given group is metabelian, and make a series of small

arguments that made use of Sylow- p subgroups. I believe that this speaks to Dr. Parker's intentions in that she wanted the students, even expected the students, to be developing significant skill with proof throughout the course.

What actually transpired during course meetings is that, as noted above, there was a class discussion about computation more than half of the days that I observed (principally in response to student's questions). When days spent in the computer lab using *Exploring Small Groups* are also included, that meant that a significant amount of time was used to think about specific rings, groups, calculations, and other localized discussions. This use of class time mitigated against spending significant amounts of time discussing, teaching and demonstrating proof during class meetings. I only saw Dr. Parker's class talk about proof during 4 of the 12 class meetings that I observed. Moreover, two of these discussions derived entirely from student's questions about the homework that she assigned and were basically demonstrations of the proofs students were required to complete the homework correctly.

It seems that Dr. Parker compromised on her vision for the amount of proof that happened in class. I believe that Dr. Parker had a different ideal for how proof would be written than what actuality unfolded in her course. But I will give a description of what happened when Dr. Parker wrote proofs in class and how we can understand it as an enacted teaching script. Afterwards, I will briefly note how I believe the actuality of class contrasted with her vision for teaching proof.

One of the proofs that Dr. Parker demonstrated was the proof of Lagrange's Theorem. Dr. Parker's presentation was a nearly uninterrupted lecture in which she presented the classical argument. She wrote a set of symbols on the board and used

her commentary to tell students an outline of the proof, to explain how it would be built out of three lemmas, and then to tell students the logic of each step that she was doing. During this presentation the students all seemed to be copying down the text on the board, but I was unable to tell if they were also summarizing her comments about proof structure. This presentation had little in common with the other proofs that Dr. Parker wrote during class.

In each of the other observed cases of proof in the investigative class, Dr. Parker was standing at the front board and all of the students were sitting in their seats. Dr. Parker did all of the writing and the dialogue was very teacher centered. The students never talked directly to each other, but rather always talked to Dr. Parker who would respond to individuals or the class as a whole. The general pattern of the dialogue was teacher-student-teacher-student. Periodically, two students would make concurrent or sequential comments, but not actually reacting with each other.

Dr. Parker: Again, it's just intuitively what you're thinking. If these two groups are actually identical in their structures, then we should be able to make this association between elements and if we have this association between elements, then we should have this association between subgroup structures.

Je: [inaudible]

Dr. Parker: So?

Je: Say the element little g has an order, whatever, n , and then operate on g with the [inaudible].

Dr. Parker: So, I'm gonna stick with k , okay? [inaudible]

Je: If it mapped to an element of a different order, then wouldn't it fail to be a homomorphism?

Dr. Parker: Okay, so, what you want to do is... [pause ,writing] Consider that, then you know that $\rho(g^k)$ would equal $\rho(e)$. [pause]

Written: Suppose $g \in G$ has order k and consider $g^k = e$, then

$$\rho(g^k) = \rho(e).$$

Dr. Parker: Now what? [pause]

Je: Suppose, well, suppose $\rho(g)$ has a different order.

Dr. Parker: Can we try to do it directly first rather than contradiction?

Je: I'm just feeling contrary today.

Dr. Parker: Well, contrary right back at you. So, what do you want to do with that?

Je: Break it up, phi of g, phi of g, phi of g...

Dr. Parker: So, the is phi of g k times, which is phi of g, phi of g, phi of g, which is phi of g to the k. Yeah, I know I'm writing a lot of details right now, but bear with me. So, this is the identity element in the group G, but down here, phi of g is an element that is in the group H. So, if we're trying to find the order of the element, we want to find the least integer so that when we raise the element to that power we get the identity, but, it's the identity in H, so it's not the same. I'm going to patch this.

$$\begin{aligned}\text{Written: } \rho(g^k) &= \rho(gggg\dots g) = \rho(g)\rho(g)\dots\rho(g) = \rho(g)^k \\ &= \rho(e) = \bar{e}\end{aligned}$$

Dr. Parker: And, we add the little box at the end... Are you convinced?
[pause] The next natural statement is the one that I want you to deal...
Someone can just come in on Thursday and just put it up there.

Note that this entire dialogue-driven proof was basically a conversation between Dr. Parker and one student, here noted as Je. At the time of the class, Je was a senior mathematics major recognized by the department as an outstanding undergraduate, and she had been selected as one of only four Undergraduate Teaching Assistants further illustrating her already proven success in math. All of the other proof attempts are similar in that Je was often the only student giving responses to Dr. Parker's prompts that moved the work forward.

Let us more closely examine the types of questions that Dr. Parker asked here and in other proofs. In the proof above, Dr. Parker began by asking, "Now what?" as a prompt to ask the students to supply the beginning of the argument. This was a question that Dr. Parker asked multiple times while working through proofs at the board. She expected students to provide the first line of a proof (and thus shape the structure) and to tell her how to move on to the next line. Other variations on this question included, "So, how would you suggest that I do this?" at the start of a proof, and "So, now I stare" as a prompt for student participation. Dr. Parker was willing to

take the student's suggestions when she believed the step reasonable, even if it was not her preferred method. In the proof shown above, Je suggested a proof-by-contradiction, and although Dr. Parker tried to dissuade her, the demonstration was done by contradiction.

Dr. Parker had other uses for questions as well. She used questions as a means of indicating to students that their suggestion was incorrect factually or invalid. Moreover, the students recognized this rhetorical move. They seemed quite attuned to the difference between Dr. Parker's validation of correct proof moves and incorrect moves. When a student made a correct proof move Dr. Parker would restate the suggestion and often elaborate on the suggestion by providing more detail. When a student made an incorrect proof move Dr. Parker would ask a question. Consider the manner in which she responded to the student below who made incorrect statements. This proof was part of a homework assignment and multiple students had requested that Dr. Parker demonstrate it during class.

$G = \{a_1, a_2, \dots, a_n\}$ group of order n . G is abelian. Prove that $x^2 = e$ when $x = a_1 a_2 \dots a_n$

Dr. Parker: Okay, so, what's problem 16 say? It says, G is a group so I know I'm going to have closure, identity, associativity, inverses for all.

Further, it's abelian, so all the elements commute. And, I need to show that the order of this element has order, well, that this element squared gives me the identity. Now, one of these guys is the identity, but that doesn't matter. So, how would you suggest that I do this?

S: We know all the subgroups are order 2.

Dr. Parker: We do?

S: Well, in a subgroup we're going to have the subgroup and the element, because the element times itself gives you the identity.

Dr. Parker: Are you talking about this element or every element?

S: Every element.

Dr. Parker: Every element has order two?

Dr. Parker began by asking how to begin, a very open question, but once a student made an incorrect factual observation her questions changed significantly. In fact, her questions were thinly veiled statements indicating that students had made factual errors. She would often repeat the question. Consider this exchange:

Je: I thought we proved that a long time ago.

Dr. Parker: Ah, but did we? Do we have that, have we proven that the image of the identity is the identity? That would seem the natural step. Do we have that?

Written: Do we have, aka have we proven, that the image of the identity is the identity?

Je: Usually when you ask that many times the answer is no.

In this case Je explicitly named what Dr. Parker was doing, using a question as a statement that an assertion was not currently warranted.

Dr. Parker also asked factual questions during proof-writing. Consider the set of interactions below:

Dr. Parker: It's a bijection.

Je: Yeah, a bijection.

Dr. Parker: Which gives us?

Je: 1-1 and onto.

And, while starting a different proof she asked:

Dr. Parker: Ah, it means that you're a subset that's a group. So, in order to show that a subset of a group is a group, what do you have to show?

In each case she used these questions to further the proof, but they served to solicit the exact information that she wanted in order to begin or continue the proof. In the cases where Dr. Parker asked a factual question, she already had decided upon the next step and was asking a question of the students so that they could fill in the details of the proof structure upon which she had already decided.

It is worth discussing the frequency of the two different types of questions that Dr. Parker asked. She asked no questions of the students while proving Lagrange's Theorem, but when she was writing each of the other proofs she asked questions of the students. In one proof-writing activity she asked only two questions, both open-ended. In another she asked nine questions and made one statement that was interpreted as, "What next?" Six of the questions were factual and the others were open-ended; asking students "What next?" or "How should I begin?" In general, Dr. Parker seemed to expect that the students would talk, even if she did not ask a question, but her questions were more likely to be open-ended than factual.

While Dr. Parker's use of questions was interesting, what was perhaps more interesting is what she would do in response to the student's answers. Specifically, as noted above, when a student gave an incorrect response to a question she would repeat the question or some variation as an indication that the response was incorrect. When the students gave a correct response to a question she would often repeat what was said, then reformulate the response to include more mathematical details than the students gave (possibly moving beyond what the students intended). Almost every correct statement by a student was received this way. Consider the interaction below where the student said 11 words and Dr. Parker repeated and expanded to nearly a paragraph.

Je: Break it up, phi of g , phi of g , phi of g ...

Dr. Parker: So, the is phi of g k times, which is phi of g , phi of g , phi of g , which is phi of g to the k . Yeah, I know I'm writing a lot of details right now, but bear with me. So, this is the identity element in the group G , but down here, phi of g is an element that is in the group H . So, if we're trying to find the order of the element, we want to find the least integer so that when we raise the element to that power we get the identity, but, it's the identity in H , so it's not the same. I'm going to patch this.

$$\begin{aligned} \text{Written: } \rho(g^k) &= \rho(gggg\dots g) = \rho(g)\rho(g)\dots\rho(g) = \rho(g)^k \\ &= \rho(e) = \bar{e} \end{aligned}$$

In this case, Dr. Parker has repeated the student's statement, written down the original statement, and reformulated it while writing it down so that, besides incorporating the student's statement, it surpasses it to include the end of the proof as well. Moreover, she also explained the rationale that supported each of the logical moves. In that way she expanded the scope of the comment and stated the logical underpinnings when the student had said nothing about them.

Je: And we just order it so that it's near it's inverse or by it's inverse. We just rearrange it, the operation.

Dr. Parker: So, here you want me to rearrange it so that a^{-1} is next to its inverse?

Mark: You have two copies of x .

Dr. Parker: Oh, you didn't tell me that. Well, at least you didn't tell me to write that down. Okay. So, you're saying that another way to write this down is to write it as... a^{-1} times a^{-2} time blah blah blah, and then, someplace in here is a^{-1} -inverse and a^{-2} -inverse and a^{-3} -inverse. And, because it's commutative I can write that down and say a^{-1} -inverse, a^{-2} -inverse, and a^{-3} -inverse. Even better than that, that's kind of a bad order, I could write it as a^{-n} -inverse, a^{-n-1} -inverse, blah and a^{-1} -inverse. I can do that...

S: Because the group's abelian.

Dr. Parker: Because the group's abelian, so I can swap elements and I can order them in any order I want to. And so, now, I can go through a process of association and I can take off the parenthesis and put them around those two and then it kinda drops out and then this one's going to be paired with it's inverse and all the way down the line. [inaudible]

Je: I was thinking that you could just take every element in G , and if you multiply every element in G together that you'll get the identity.

Written:

Proof: Let $x = a_1 a_2 \dots a_n$. Since G is a group then each element has an inverse among the elements of G .

Consider $x^2 = (a_1 a_2 \dots a_n)^2 =$

$(a_1 a_2 \dots a_n) (a_1^{-1} a_2^{-1} \dots a_n^{-1})$ [Dr. Parker crossed out this second parenthesis]

$(a_1 a_2 \dots a_n) (a_n^{-1} a_{n-1}^{-1} \dots a_1^{-1}) =$

$(a_1^{-1} a_2^{-1} \dots a_2 a_1)$

In this case Dr. Parker took the kernel of an idea that Je and Mark have suggested and gave it the notation and structure that was necessary to make it a proof of the claim. Moreover, later conversation gave evidence that the students did not actually intend the claim to be understood the way that Dr. Parker decided to write it. Her reformulation actually gave the students the key step in this proof without them having yet thought through all of the details.

In summary, this proof script seems to have been enacted differently than Dr. Parker intended. In the script we see above, Dr. Parker used questions and similar prompts to solicit student comments and thoughts, often about the structure of the proof. If the students gave a correct response, she would repeat it and then expand upon it. If the students gave an incorrect response or one that she thought unhelpful, she would ask a factual question of the student(s) until the offending student retracted the assertion (“We do?” “Do we have ...?”). In this script the students were envisioned to be active participants, making contributions to the proof and asking questions, but the reality is that only three or four students out of the class of 24 made any meaningful contribution to the proof writing. The students were also expected to take notes on the proof and to record both the text on the board and some sketch of the logic that Dr. Parker said aloud. It is unclear how much the students actually did take notes on the proof. Moreover, because participation levels were very different, it is unlikely that there is any small set of expectations that the students might hold for this teaching script. Most likely they expected to learn enough about proof to pass the class but the mechanism for this learning was probably not obvious to the students.

Salient characteristics of the observed teaching scripts

In each of the teaching scripts noted above, Dr. Parker was the primary actor. She did the majority of the talking and directed the conversation. All of the conversation, if it existed, was very teacher-centered. These scripts have relatively little in common, given that in the first teaching scripts described above Dr. Parker is presenting or telling information, whereas in the proof script she is attempting to solicit the ideas from the class and transcribe them. But, in a deeper examination of what actually transpired in the proof script, Dr. Parker took amorphous suggestions from the students and added detail and expanded the explanation for each of the statements that the students made so that, in essence, she was telling the student the correct steps in the same way that she told the students the idea or procedural description in the other two scripts. The major difference between the scripts was the number and type of questions. In the first two scripts Dr. Parker asked almost no questions. In the proof script she repeatedly asked questions, and those questions were often very open-ended with the (unrealized) potential to allow the students significant control over the direction of the proof.

Declarative statements

Dr. Parker made extensive use of declarative statements in each of her three observed teaching scripts. In her introduction of new mathematics Dr. Parker repeatedly emphasized that one of the goals of the field of mathematics is that “mathematics is all about making meaning, so it must be meaningful.” She repeated this sentiment at other points in the class as well. That is, she used declarative statements to communicate to students how she viewed the field of mathematics as a

means of helping them develop an understanding of the field. Similarly, Dr. Parker used declarative statements to communicate which aspects of mathematics were most important to her in terms of student proficiency. For example, during one Teaching Computation script she stated, “(It) is great when you know, well, the theoretical side, but I want to know in a group and given subgroup you can actually find these things.” That is, she was communicating to her students that she valued their ability to carry out computation, perhaps above their knowledge of definitions, theorems and proof abilities.

In all of her observed teaching scripts Dr. Parker used declarative statements to communicate mathematical facts. She made statements such as “You could say that the order of an element is the cardinality of the subgroup generated by the element” during an Introducing Mathematics teaching script. The Teaching Computation script featured very direct statements about the processes of computing. She made explicit statements about the process for combining permutations within a given group on multiple instances. These statements would tell students what order to combine permutations, how to follow an individual element through a permutation and a validation of results.

In the observed Teaching Proof script Dr. Parker’s statements were almost entirely stating the question that she would answer and restating student’s suggestions. Dr. Parker’s restatement of student suggestions seems very similar to Dr. Hedge’s in that she would she could add detail to a student’s statement in order to make it more mathematically correct and complete. As such, it seems that Dr. Parker made different uses of her declarative statements in the different teaching scripts. In

the first two teaching scripts Dr. Parker's statements were generally giving students context as in the Introductory teaching script or direction as in the Computation teaching script. Her statements in the Proof teaching script were factual statements. This differentiation of statement-types and uses is not unreasonable as the first two teaching scripts are primarily information-dissemination whereas the Proof teaching script seemed intended to be conversational or generative where responsibility is distributed between teacher and class. Yet, as noted above, during the Proof script, Dr. Parker was, in essence, also telling students the correct steps in a writing a proof. In that sense, her statements during the Proof script were similar to the process-directions in the Computation teaching script.

Question types

Dr. Parker made very different uses of questions in the three observed teaching scripts. In the first two teaching scripts Dr. Parker made very similar use of questions; they were factual or on-line checks for student understanding. In the Introducing Mathematics teaching script Dr. Parker asked a number of factual questions such as, "How do I know," and, "What kind of structure was that?" In this case the students were supposed to recall and state previous facts. During the Teaching Computation script students were expected to state previously learned facts, "What's the order of a three cycle?" In both of these teaching scripts Dr. Parker would also solicit student questions by asking questions such as, "Does that make sense?" Dr. Parker also used questions in this script to give students direction for their subsequent investigation. "What happens..." "What can we say...?" She would use these question types in place of the more explicit directions for student

work and frame them with, “For homework I want you to explore...” As such, this last type of question might be seen as framing the student’s work. This question type seemed intended to help the students develop more understanding of the process of mathematical discovery in that they would engage in semi-directed exploration with no clear answer.

Dr. Parker’s questions in the Teaching Proof teaching script seemed different in type than those in the other two teaching scripts. She asked a greater number of questions in this teaching script and, on the face, they seemed to be of different types as well. Dr. Parker asked questions to solicit student participation in proof creation at multiple points. First, she asked the students for how to structure the proof, and then she asked multiple iterations of the question, “Now what?” requesting that the class supply the next step in the proof. Neither of these question types should be seen as factual in that they were legitimately open-ended and Dr. Parker did allow student’s suggestions to structure the proof creation. She was also observed asking a very closed, almost rhetorical, type of question during proof creation. Dr. Parker would use questions as a means to indicate the factual correctness or validity of a student’s response to a previous question. This use of questions was recognized by students as a thinly veiled statement of error.

Posing questions

Dr. Parker’s questions were varied in style and purpose. As noted above, one type of question was actually a thinly veiled statement to indicate the correctness of a student’s response. All of the questions Dr. Parker was observed asking during class were directed at the entire class. She never addressed a question to an individual

student unless it was a direction question, “Would you please...?” Again, it was unclear from the context whether Dr. Parker hoped for individual students to respond to her prompts or whether she preferred a chorus-type of response.

Student Scripts

It does not seem that the students were expected to participate in the same way in each of the teaching scripts. In the first two, the students were expected to be passive participants, only asking questions for clarification, whereas in the proof script the students were expected to supply much of the direction for the proof writing. But the student’s participation actually seemed to be rather similar across the scripts in that generally they were passive, perhaps taking notes on the presented material. As noted above, even in the proof writing script, where Dr. Parker expected active participation, only a few students actually did participate. Because of that we can safely conclude that despite Dr. Parker’s best intentions, the students were generally fairly passive during the three teaching scripts described above. This is not to say that the students were passive throughout the class, but rather that once one of these scripts was initiated they became passive. In fact, their questions were very likely to initiate one of the above scripts, especially the computation script. Basically, there was a large range of student behavior. At one extreme were the few students who took notes all of the time and were a nearly continuous part of the conversation. At the other extreme were those students who asked and answered questions, but whom I never observed taking notes. As with the students in the DTP class, it is very likely that the students had very mixed goals for the course with some students

hoping to gain a lot of mathematical understanding and others hoping to earn a passing grade.

A Comparison of the Observed Teaching Scripts

As described above, one of the major goals of the current study was to offer a researched description of the teaching of one DTP abstract algebra course and one investigative abstract algebra course. As such, the current study offered a unique opportunity to compare the teaching of two sections of an introductory abstract algebra course at the same university. Because the students from Dr. Hedge's class and Dr. Parker's class were expected to be ready to take a common second semester of algebra meaning, the two instructors needed to cover approximately the same content. Because the two sections were to cover the same content during the same semester I believed it likely that it would be possible to look for similarities and differences in the teaching scripts that the two instructors employed. The goal of this comparison was to use the similarities and differences to better understand teaching abstract algebra generally and to better describe the characteristics that might differentiate the two pedagogical approaches.

However, the reader is cautioned that these two classes, as represented by their observed teaching scripts, represent just two possibilities in the broad spectrum to teaching. Thus, this analysis which compares and contrasts the teaching of the two classes should be read as particular to these two classes. Further, there were limited observed teaching scripts in each of the two courses. As a result, the comparison is limited.

Dr. Hedge and Dr. Parker both encouraged active participation on the part of their students. For example, Dr. Hedge often asked if the students had questions and would answer any question whenever it was asked. She gave extra credit points for catching mathematical mistakes in work that she presented. She required each of the students in the class to present a proof during the course of the semester and asked many questions during the course of a class meeting. There were also multiple class meetings where the students were working problems in class or performing some other activity designed to improve their understanding of the content. Thus, while students were always expected to take notes, copying text from the board into their notebook, which they all did, Dr. Hedge expected them to be more than passive consumers. Perhaps half of the students would answer one of her questions in a class meeting, but there were a few students who only answered questions when called on by name. Dr. Hedge principally used class time to outline the content that the students were expected to master and to demonstrate specific skill sets that students needed to develop. Her teaching scripts also included more on-line checking of student understanding, factual mastery and class engagement than might be expected in a lecture class.

The tone of the investigative class was significantly different than that of the traditional class in that students were much more likely to ask questions and to make comments. Although Dr. Hedge encouraged questions, for the most part students did not ask many. In the investigative class the students asked many questions, and what happened in the classroom on a given day was, to great extent, reflective of the students' questions and comments. As noted, one of the most common teaching

scripts in Dr. Parker's class was a script where she taught computation. She always started this script in reaction to a student question. In comparison, none of Dr. Hedge's teaching scripts were ever initiated in response to a student question.

The actual teaching scripts revealed both similarities and differences. Consider the teaching scripts which most distinguish the two sections. Dr. Hedge used three teaching scripts regularly but the one that most distinguished her from Dr. Parker was her exemplar dialogue. While this script had an analogue in Dr. Parker's insistence on grounding abstractions in concrete example, the manners in which they did so was quite different. Dr. Parker's most unique teaching script was her insistence on the human nature of mathematics.

The evidence suggests that there are differences both in emphasis and tone between the two classes. There were also differences between the characteristics of the observed teaching scripts in each of the classes. As such, we recall the first of the research questions comparing the two pedagogical styles:

Which, if any, of the observed teaching scripts seem to best differentiate an *investigative* abstract algebra class from a *DTP* abstract algebra class?

In this case, our analysis of declarative statements and questioning suggests obvious choices.

Teaching scripts which differentiate the sections

Dr. Hedge's exemplar dialogue was a way to introduce new concepts prior to stating a formal definition. She would give the students several examples of a structure and ask them to consider them, to perform some calculations in the structure, or to write a proof about the structure. It was only after the students had worked with an example or two of the structure that Dr. Hedge would give the formal

definition. Moreover, these examples also served as a means for Dr. Hedge to motivate her generalizations. Dr. Hedge generally described the new mathematics as a generalization of prior work, while emphasizing the fact that it was built on mathematics that the students had seen multiple times. Dr. Parker had a similar pedagogical move in that she expected students to perform many calculations. The significant difference between the two teachers comes in the next step, the stating of generalizations or theorems for proof. Dr. Hedge gave every generalized statement that was proved in class whereas Dr. Parker allowed the students to make all of the general claims. This seems a significant distinction, but it should be noted that Dr. Parker carefully crafted the student's work so that they would be confronted with clear patterns that would lend themselves to many of the traditional group and ring theorems. Because of this lesson crafting, the distinction between having students make the claims and the teacher stating them outright seems somewhat less important than might otherwise be the case.

Because of the differences in the use of examples and the manner in which they were used to launch new mathematics, Dr. Hedge stated and proved more theorems and general claims during my observations. Dr. Parker's class only stated and proved only three generalizations and spent more time discussing computation. Dr. Hedge's class was always cycling between examples, theorems and proofs.

The teaching script that most differentiated Dr. Parker's class from Dr. Hedge's was her insistence that mathematics is about making meaning. Although this was a relatively brief teaching script, it was important to Dr. Parker's class. She would consistently call mathematics a human endeavor that was an effort to make

meaning from patterns they had seen. This script served to emphasize Dr. Parker's point that students were learning mathematics by engaging in the process of making meaning and that by doing calculations, making and testing conjectures, and always grounding general statements in concrete examples.

Dr. Parker's teaching script offered a strong contrast to the presentation of mathematics in the DTP class. Dr. Parker always insisted that mathematics was about making meaning, about asking questions. For example, Dr. Parker's means of introducing mathematical structures afforded students more understanding of the process by which a mathematician would invent new structures. In Dr. Parker's class they started with a question, and she introduced new mathematical concepts in order to answer that question. This contrasted with the practice in the DTP class where Dr. Hedge introduced a mathematical concept and then asked questions about it. For example, Dr. Hedge introduced a kernel of a homomorphism as an example of an ideal, that is, the new structure was an example of an existing structure. In Dr. Parker's class new structures were brought into play as a set of tools that students could use to make sense of the mathematical the relationship between mathematical concepts. Dr. Hedge wanted the students to understand mathematics as a logical, ordered and interrelated body of knowledge, but I never observed her making any statement to the class about the development of mathematics as a human endeavor or about the role of questions in furthering mathematics as a field.

Shared teaching scripts

The second of the comparative research questions focused on the shared teaching scripts in the two pedagogical styles:

Which, if any, of the observed teaching scripts do *DTP* and *investigative* abstract algebra classes have in common?

As suggested in the analysis above, while the two professors presented different beliefs about the purpose of mathematics, Dr. Parker and Dr. Hedge also presented very different visions for teaching: One claimed to be a traditional, *DTP*-style teacher, and the other claimed to do very different things and to teach in an investigative manner. Thus, the next piece of analysis addresses what teaching scripts the different teaching styles have in common.

Even though these two teachers did present rather different visions of mathematics in class, they also both made use of a proof-writing script that had many common characteristics. The two proof-writing scripts make for an interesting point of comparison because proof writing constituted one of the most common activities in Dr. Hedge's class and the only one which had significant overlap with the daily activities in Dr. Parker's class.

In Dr. Hedge's class the majority of the observed proofs were written in a participatory style. In this style Dr. Hedge would generally begin the proof, providing the structure, and then, while writing the proof, ask factual questions that she intended the students to answer. She would generally direct these questions to the whole class but was also observed calling on individual students. Dr. Hedge seemed to use her questions to: (1) keep the students engaged with the proof-writing task; (2) generally stimulate the student's thinking about proof; and (3) give her another means of assessing the student's developing proof proficiency. Her questions were almost always of two types. She either was asking the students to state a specific fact, for example in asking, "And what do I know about zero plus zero," or she was asking the

students to complete a sentence with an appropriate factual statement. When Dr. Hedge asked a question of the class there were a small number of students who were very likely to respond, and there were a number of students who were unlikely to respond unless directly called upon.

In terms of actually writing the proof, Dr. Hedge was always the author, in the sense of actually writing the text, in this proof-writing script (although there were a number of observed instances where the students were the authors). At least as important though was the fact that Dr. Hedge also created the structure for the proof. In all of the participatory-proof-writing scripts I observed Dr. Hedge undertake, she always started the proof. These were also all direct proofs, none involved contradiction. It was not until after she had begun a proof that Dr. Hedge started asking the students questions. In summary, Dr. Hedge asked factual questions that advanced the proof, but did not allow students the opportunity to structure the proof.

Let us now turn to Dr. Parker's version of the proof-writing script and see how it was similar to and different than Dr. Hedge's. As noted above, Dr. Parker also had a very participatory proof-writing script, but she did not write many proofs. In the proof-writing script Dr. Parker was always the center of the discussion. While writing she asked many questions of the class as a whole and of individual students responded, but there was one student who gave the overwhelming majority of all the responses. The students who responded always spoke to her, never to each other. In this regard the two instructor's proof writing scripts were very similar. Both were the actual writers, both directed the conversation, and both were at the focus of the conversation. Moreover, in each of the classes the majority of the students' responses

were from a small group of students with a larger group not part of the conversation. I believe that both teachers envisioned a class where most or all of the students would be active participants, perhaps even to the point where they, the instructors, would act more as transcribers for the students' responses in crafting the proof.

Let us now turn to the types of questions that Dr. Parker asked the students. Dr. Parker asked three principle types of questions. In addition to factual and open-structural questions she asked a third type that was actually a thinly veiled statement that the student had just made an incorrect statement. Dr. Parker asked factual questions with some frequency, but less than Dr. Hedge. Almost all of Dr. Hedge's proof writing consisted of an almost call-and-response interaction where nearly every utterance she made was in the form of a question. Dr. Parker made many more direct statements and, because of that, her question frequency was lower. Similarly, she also asked different types of questions, and because of that, factual questions were not as prevalent in her proof-writing script. Consider the question, "So, in order to show that a subset of a group is a group, what do you have to show?" While Dr. Parker asked this question, it was a question that would also have been consistent with Dr. Hedge's questioning style. Their factual questions were all simply stated with easily verifiable correct responses.

One of the principle ways that Dr. Parker's proof-writing script was different than Dr. Hedge's was that she asked two types of questions that Dr. Hedge did not. Dr. Parker asked a significant number of open questions that were generally asking students to supply structural elements of the proof that she was writing. She asked questions such as, "How should I begin?" at the start of proofs and took the students'

suggestion. She asked questions such as, “Now what?” in the middle of proofs and waited until a student suggested a direction. Dr. Parker asked open questions nearly as often as she asked factual questions. These questions seemed to indicate Dr. Parker’s expectation that the students would bear primary responsibility for proof writing and that she would almost act as a transcriber rather than as an author. She actually wanted and expected the students to provide the structure and impetus for writing the proof. But, they also realized that she was still the authority on proof within the classroom and was controlling the flow of the proof. Her third type of question, and the way that students responded to it, showed that they recognized her role as authority and may have never actually felt or assumed the hoped-for proof-writing authority.

Dr. Parker’s third type of question was always in response to an incorrect statement by a student. When a student made an incorrect statement such as, “We know...” Dr. Parker would respond with a question such as, “We do?” or “Have we ...?” She responded to correct statements very differently. She would repeat them and add the type of detail needed to use them to substantiate a proof. This repetition and expansion was also mirrored by what she wrote on the board during proof writing. Sometimes Dr. Hedge’s restatement of the student’s idea would go beyond the student’s thought or even be a slight alteration that better fit the needs to the proof. In this way she was very much exercising authority over the direction of the proof, even while asking open questions of the students that demanded answers which directed the structure of the proof.

Dr. Parker's students had learned that she reacted differently to their correct and incorrect statements. In doing so, Dr. Parker actually ensured that only correct mathematical statements that advanced the proof were written on the board. That is, she validated warranted statements and kept unwarranted statements from the board, acting in the capacity as a mathematical gatekeeper. This also contrasted with Dr. Hedge's class. In Dr. Hedge's class the students made almost no mathematical inappropriate statements. Because of that Dr. Hedge did not have to filter their responses to ensure the correctness and completeness of the proof. But, this fact should not be taken to indicate that Dr. Hedge's students were necessarily more competent with proof or had a better command of mathematical facts. It was a function of the different types of questions that the two instructors asked. When Dr. Parker asked a factual question, the student's response was almost always correct. It was when Dr. Parker asked an open question that students were likely to make misstatements. Thus, it seems that Dr. Parker assumed the responsibility as mathematical gatekeeper in reaction to her use of open questions.

Summary

Comparison of the observed teaching scripts revealed a number of similarities. Both instructors expected the students in their classes to be active participants during meetings. Both of the instructors expected the students to ask and respond to questions throughout the meetings. But, although both teachers encouraged active participation there were some distinctions in terms of the types of participation that were observed. For example, the students in the investigative class were much more likely to ask questions than were the students in the DTP class, and these questions

covered a wide range of relevant algebra expanding beyond homework problems. In contrast, the students in the DTP class all presented a proof at the board or overhead projector and were much more likely to participate in a proof-writing script. There were students in the investigative class who did not speak as part of the teacher-centered dialogue in any of the class periods that I observed (although everyone seemed to be an active participant during computer-lab sessions).

The two instructors both made use of a number of teaching scripts that reveal how the two sections were similar and different. For example, the two teachers each had a participatory proof-writing script. For Dr. Hedge it was one of at least three proof-writing scripts that she made use of and it was the only proof-writing script that I observed Dr. Parker use consistently. In this script each of the instructors would stand at the front of the classroom and ask a series of questions of the students. The students' responses to these questions were incorporated into the proof that the teacher was writing. Both teachers made significant use of factual questions which had a correct answer. Both of the teachers asked either all or the majority of their questions to the whole class and waited for individual students to respond.

The proof-writing scripts were different in the level of responsibility for proof structure that the two teachers gave the students, the frequency of open questions the teachers asked, and the different manners in which the teachers reacted to the students' responses to the questions. In all of the observed proofs, Dr. Hedge set the structure for the proof whereas Dr. Parker gave the students more responsibility for the structure, going so far as to write a proof-by-contradiction because a student wanted to. This increase in responsibility for the structure of the proof also meant

that Dr. Parker asked more open questions than Dr. Hedge. Dr. Parker asked multiple open-ended questions during proof writing whereas almost all of Dr. Hedge's questions were factual in nature.

Lastly, because Dr. Hedge's questions were almost always factual, they had correct responses which she then repeated. In all of her uses of the proof writing script I never observed the students make an incorrect or unwarranted assertion in response to her question. That is, the students always gave the response that she expected and thus, I never observed her react to an incorrect student statement. In the case of Dr. Parker's class, because she asked a larger number of open-ended questions her students made a reasonably large number of incorrect or unwarranted assertions. Dr. Parker reacted to these unexpected responses very differently than to correct responses. When a student gave a correct response she repeated it and added detail. When a student made an incorrect response she would ask a question, often repeating the question. The last difference in the proof-writing scripts was the frequency with which the two teachers enacted them. Dr. Parker only wrote a handful of proofs in the observed class meetings while Dr. Hedge enacted this script in nearly all of the observed class meetings.

There were also teaching scripts which showed that the two classrooms were quite different. Dr. Parker repeatedly emphasized the human nature of mathematics and the importance of understanding mathematics as a way to answer questions. She emphasized that mathematics was about making meaning of experience. Dr. Hedge also wanted the students to develop an understanding of mathematics as a meaningful and logical discipline, but she did so via an exemplar dialogue in which she would

ask students to think about a structure that they knew very well in a new way. Most frequently, she wanted the students to understand that the new concept was actually a generalization of an idea they had long known. In short, the teachers had both a shared teaching script and, more importantly, a shared goal for their students, but attempted to achieve that goal by different means.

CHAPTER 5: STUDENT PROFICIENCY WITH ABSTRACT ALGEBRA

Abstract algebra is a pivotal point in the educational trajectory of mathematics majors and future teachers. Students' understanding of the fundamental concepts of identity, inverse, unit, and polynomial are the key building blocks from which they develop their proficiency with the basic structures of the discipline— groups and rings. A number of studies have attempted to document the ways that student understanding of specific algebraic content develops (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996; Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997; Brown, DeVries, Dubinsky, & Thomas, 1997; Dubinsky, Dautermann, Leron, & Zazkis, 1994; Findell, 2000). To that end, one of the central goals of this study was to describe the level of understanding that a typical student might develop after a semester of study of abstract algebra. This chapter responds to that research goal using data from two written instruments and interviews with individual students.

After a brief description of the proposed and implemented methodology of the study, this chapter provides a description of the proficiencies that the students developed after one semester studying abstract algebra. These data were analyzed by content strands. First, the analysis examines the students' proficiency with the concepts of identity, inverse and units, followed by consideration of the students' proficiency with polynomials and the basic structural concepts of groups and rings. Finally, there is some analysis of the students' proficiency with crafting algebraic proofs. Each of these data and analysis sections includes a brief description of the relevant mathematical content, the expectations that an instructor might hold after an

introductory course, the students' opportunity to learn the content, and an analysis of data gathered by the different assessments.

Introduction and Methodology

The original intent of the study was to write a description of the proficiency with group theory that students develop after an introductory semester of abstract algebra. At the start of the course, the two instructors believed that they would spend approximately one-third to one-half of the semester on group theoretic content. To that end, I proposed a study in which I would assess the student's proficiency with group theoretic topics via two written instruments—a short mid-semester survey and a longer end-of-semester survey—and interviews with individual students. Almost all of the questions on the proposed surveys probed the student's proficiency with group theory—including questions about Lagrange's Theorem, cyclic groups, cyclic subgroups, group homomorphisms, group isomorphisms, and quotient groups. The assessments were fairly comprehensive with respect to introductory group theory.

These assessments of knowledge of group theory were designed to permit a comparative description of the proficiency with group theory that students in the DTP and Investigative classes demonstrated. Unfortunately, the DTP class spent only four class meetings on group theoretic content, while the investigative class spent approximately one month on group theory. Thus the DTP students had very limited opportunity to learn specific group theoretic content, while the investigative students were exploring quotient structures and the characteristics necessary in the modulus to produce a quotient group. This discrepancy of course content meant that it was

impossible to craft a group theory assessment that would fairly infer the impact of the instructional approach on students' proficiency with those topics. Students in the DTP and investigative sections not only experiences differing instructional approaches, they had different opportunities to learn.

As a result the research goal was modified and a new set of assessments were developed which focused on ring theory and fundamental algebraic concepts such as identity, inverse, and unit. Thus, this study has assessed students' proficiency with concepts that underlie the study of both groups and rings and contributes information not provided by earlier research that had been limited to group theory.

Because there had been no prior published research regarding student understanding of ring theoretic content, there were no readily available written assessment instruments— other than an interview addressing students' proficiency with homomorphism proofs (Weber, 2001). As a result, this study's assessments have had limited validation and refinement. Nonetheless, the assessments still elicited student responses that permit meaningful description of the proficiencies that these students developed after an introductory semester of abstract algebra.

This study of student's learning was also compromised by limited student participation in the written assessment and interview phase. Only 5 of the 13 students in the DTP class and 7 of the 24 students in the investigative class agreed to complete the two written surveys of algebraic knowledge. Only six of those students completed an individual interview—one from the DTP class and the other five from the investigative class. This small sample size, the differences in content covered, and discrepancy in rate of interview participation made it inappropriate to craft a

comparative description of the proficiency that the DTP and Investigative students developed from their experience in a semester of abstract algebra. As a result, the following reports of student performance should be interpreted as exploratory probes into student learning, not as comprehensive descriptions or comparative evaluations of effects from the two teaching approaches.

I first attempted to analyze the data in terms of the students' responses to the individual assessment items. This reading allowed for a thorough description of the students' ability to answer a single question but this type of analysis obscured the patterns of responses that illuminated students' understandings of the important content strands. While this first reading of the data generated some interesting patterns (notably about student response rates to individual items) the most important result of note was the lack of complete exams. The type of analysis that would have resulted would have been a description of what the students could not do. But rather than writing a study framed in the negative I wanted to focus on what the students showed that they can do. As such, I changed the manner in which I was reading the data and instead of focusing on the responses to individual questions, I separated the responses into groups based upon content strands. This yielded the structure and analysis that follows. The analysis is based on content strands. Within each strand the students' responses are organized by question, yielding patterns in the responses of individual students within each of the strands as well as larger patterns in responses. The analysis that follows initially presents an examination of the students' proficiency with the concepts of identity, inverse and units that inform students'

proficiency with both groups and rings, followed by a characterization of their understanding of polynomial and proof.

Identities, Inverses, and Units

The concepts of identity and inverse elements are among the most important in abstract algebra. A group is a set with an associative binary operation for which there is a unique identity element and every element has an inverse. Rings are algebraic structures with two binary operations, one of which satisfies the properties of a commutative group. Fields are special kinds of rings that have identity and inverse elements for the second operation as well.

Understanding Identities, Inverses, and Units

Whether an abstract algebra course begins with groups or with rings, the concepts of identity and inverse are almost certainly introduced early. The first theorems and proofs almost always involve use of those ideas, and they are used throughout the developments that follow. There are some proficiencies in use of those concepts and factual knowledge about the concepts common to every abstract algebra course.

Identity Element

In a set S with binary operation $*$ an element e is called an *identity* element for the algebraic structure $(S, *)$ if and only if $e*x = x*e = x$ for any x in S . Successful students in an abstract algebra course need to be able to apply this definition to recognize identity elements in specific examples of algebraic structures. They need to

use the definition of identity element in proofs of properties for specific examples and for general classes of algebraic structures.

Inverse Elements

In an algebraic structure $(S, *)$ with identity element e , two elements a and b are said to be *inverses* of each other if and only if $a*b = b*a = e$. If we know only that $a*b = e$ then a is called a *left inverse* of b and b is called a *right inverse* of a . Successful students in an abstract algebra course need to be able to apply the definitions to see which elements of an algebraic structure have inverses and to use the connection between inverse and identity elements in proving properties of specific or general algebraic structures.

Units

In an algebraic structure $(S, *)$ with identity element e , whenever $a*b = e$, we say that a and b are *units*. The concept of unit is, in some sense, a bridge between the notions of identity and inverse elements. It is of particular interest when only some pairs of elements in a system can be combined to produce the identity. Students in an abstract algebra course that deals with two operation systems such as rings need to be able to identify units and to use the concept of unit in analyzing structure of an algebraic system because, it is an important aspect of understanding the mathematical systems and it is a good habit to ask, “What has an inverse?”

Opportunities to Learn

Although I was not present in the classes when the concepts of identity, inverse, and unit were introduced, subsequent observations made it clear that those concepts were presented and used often in both classes. Students were asked to

describe the identity elements in particular rings, to demonstrate that inverses exist in particular structures like extension fields, and to use the concepts of identity and inverse in proofs of group and ring theorems. In work with Cayley tables for groups, students were frequently asked to determine identity and inverse elements and to describe the order of invertible elements. Lastly, the students in the investigative class had a refrain “Inverses for all” they used to describe fields. These tasks required the students to manage concepts related to both identity and inverses.

Assessment of Student Understanding

The principal measure of student learning involving the concepts of identity, inverse, and unit was five items on the end-of-semester assessment. There were two principal types of items. The first assessed the student’s ability to make use of identity and inverse in proof (items 1 and 3). The second type of item asked the students to determine the identity or elements with inverses in a given structure (items 2 and 3). Items 4 and 5 expected students to generalize their work from item 3. Because only three students successfully responded to item 3, this analysis does not address items 4 and 5.

Identity Elements

The students were able to state and use the definition of identity elements in appropriate ways. Almost all of the students, 10 of 12, did cite the identity as a member of the set (U, \bullet) on Item 1a. For example, Lynn wrote, “Since S contains an identity e , and $e * e = e$, e is a unit and U is non-empty and has an identity.” This is a type of proof and a specific statement that the students had all practiced a number of times. Because of that, it is not surprising that the students were proficient at such

work because throughout each of the courses, whenever students were expected to show that a set was non-empty, they would show that the identity is an element of the set.

While students displayed skill in determining and verifying identity elements for familiar sets, the students had great difficulty in determining the identity in an unfamiliar setting as evidenced by their work on Item 3. But, this difficulty was probably caused by an interaction of the context (functions of a discrete variable) and the students' difficulty with notation rather than an actual lack of ability to determine an identity element.

Elements with Inverses, Also Called Units

Similarly, the students were able to identify elements with inverses in a reasonably familiar context but struggled to do so in an unfamiliar context. The students were generally able to make use of the formal definition of inverse elements when writing a proof. However, in their proofs, many of the students had difficulty giving a complete proof of the closure of the set U under the operation. It seems likely that this difficulty was more derived from problematic proof proficiencies rather than problems with the content. Students stated that the product of units is a unit but did not supply a proof for this statement.

The students wrote proofs which, when the misstatements were taken as true, were logically complete in terms of the structure, but many students made factual misstatements which indicated that they did not make use of basic facts about inverses to monitor their proof-production. They also struggled to interpret or

manage the additional cognitive complexity which came with adding quantification to the concept of inverse.

Identification of elements with inverses

Item 2 and Item 3 both asked the students to identify elements of a structure that have inverses, that is, to identify the units. Item 2 asked the students to determine all of the units of the Gaussian integers and to demonstrate that their list is complete. The students should have been reasonably familiar with the Gaussian Integers (or at least the complex numbers) through their work on ring theory in which they studied the complex numbers as a field and demonstrated and made use of the Fundamental Theorem of Algebra. Item 3 asked the students to determine all of the units in a less-familiar structure, specifically, the set of all functions that map a set of three elements back to itself. While the students did work with functions of a finite variable during their study of ring homomorphisms, the use of functions as a context was unfamiliar and challenging with notation that seemed to deter many of the students.

In their work on Item 2, students attempted one of two different paths. There was a group of 8 students who simply listed the units that they knew from their work with the complex numbers, 1 , -1 , i , and $-i$. (Three of the students listed incorrect units and one of the students gave an incomplete list.). These students established that each of their listed elements had an inverse by demonstration; two students ‘showed’ incorrect elements had ‘inverses’ in this manner. The other group of students attempted to solve the item more analytically by writing two arbitrary Gaussian integers, multiplying them together and writing a pair of equations in four

variables, which they then attempted to manipulate such that the units are derived from the equations. Only one of the students who attempted this path, James, was able to carry it far enough to be successful, the others all stopped in mid-process. But, these manipulations did not draw upon the content knowledge of inverse or identity and thus, his response is not treated here in detail.

Five (out of twelve) students (James, Mark, Stephanie, Rebekah, Kenny) were able to list all of the correct units of the Gaussian integers. All but one (Rebekah) of these five students attempted to show that their proposed elements are units. One of these students, Stephanie, had complex arithmetic errors leading her to ‘demonstrate’ that three non-units were actually units. Three of the students attempted to justify the completeness of their list but do not do so correctly. Mark’s work is an example of this type of work.

Since Gaussian integers have to be integers there are no fractions. i.e. 2 ’s inverse would be $\frac{1}{2}$ and $2i$ ’s inverse would be $-1/2i$. 1 and -1 are there [*sic*] own inverse. i and $-i$ are each others [*sic*] inverse.

All of these students knew the definition of unit and recognized what it meant in the context of the Gaussian integers. Except for Stephanie, they listed all of the correct units and no others.

Moreover, two of these five students (Kenny and Mark) attempted to justify the completeness of their lists by relying on reasonable premises, albeit with incorrect language. For example, Mark stated that fractions are not allowed and that for any other elements to have an inverse, a fraction would be necessary. But, Mark did develop this good idea into an actual formal justification. Kenny stated that all units in \mathbf{C} are on the unit circle, which is incorrect. Yet again, this is a reasonable

assumption given that these are the only elements that have a vector length of 1. If Kenny had stated that the units in the Gaussian integers must be on the unit circle, he would have made a correct, but unsupported, assertion. In short, both of these students seem to have the fundamental proficiencies with inverse and identity necessary to do this problem but were lacking the necessary proof skills.

Two of the students, Steven and Aurora, derived a list of candidate units and then demonstrated that their candidates are units, sometimes employing faulty arithmetic. One of the students, Steven, listed a correct unit. Initially, he wrote the expected formal statement for multiplying units to arrive at the identity. From that statement, he seems to intuit that the only unit is the identity element.

We already have the identity of G , which is $1+0i=1_G$.

Thus, we need to find all elements that have an inverse in G .

Also, because G is a squadron we know that there is at least one inverse.

For $a, b, c, d \in \mathbf{Z}$ we have $(a+bi)(c+di)=1+0i$

$(ac-bd)+(ad+bc)i = 1+0i$

The units are therefore such that $b, d = 0$ and $a, c = 1$.

$(1 \cdot 1 - 0 \cdot 0) + (1 \cdot 0 + 1 \cdot 0)i = 1 + 0i$.

It seems that Steven knows what a unit is, recognizes that he has at least 1 unit, and then tries to decide what the other units must be. His unwarranted assumption in this case is that $b, d = 0$ and $a, c = 1$. He does offer proof that 1 is a unit. The only possibility that he recognizes for $(ac-bd) = 1$ is that $a=c=1$. He proved in both group and ring theory that $-1 \cdot -1 = 1$. As such, he has simply failed to draw upon his existing knowledge. He does not even seem to recognize that -1 would be a unit. It may be that the unfamiliar context has caused him difficulty, and that he is unable to make use of long-held knowledge. His shortened list seems at least partially derived from not thinking more creatively about arithmetic.

The other student, Aurora, offered a general candidate for a unit and then showed, via faulty arithmetic, that her candidate is a unit.

The units in \mathbf{G} are [boxed]

$$\pm(-a-1) + i\sqrt{a}, \forall a \in \mathbf{Z}$$

$$\begin{aligned} (+(-a-1) + i\sqrt{a})(-(-a-1) + i\sqrt{a}) &= -(-a-1) + (-a-1)i\sqrt{a} - (-a-1)i\sqrt{a} + i^2\sqrt{a}^2 \\ -(-a-1) - a &= a+1 - a = 1 \end{aligned}$$

Why or how she derived this element as a ‘unit’ is really quite puzzling. It seems unlikely that she spontaneously wrote it down. It seems that she tried experimenting with $\pm 1 + i\sqrt{a}$ but found that she had an extra a term and wanted some way to eliminate it (she included some scratch work, but had crossed it out). As such, she thought that she had determined how to make it go away, and she did not recall that she needed to square terms. She provided incorrect justification that her candidates are all units as her approach relied on faulty distribution, but, in terms of this strand, her mistaken multiplication was not as relevant. She does not make any argument to explain why her list is complete, and she likely does not recognize a need to do so.

These two students at least made an attempt to generate a list of units, and in fact, did offer some result. There was one other student who also listed units.

Nathan’s work:

Since $n, m \in \mathbf{Z}$, the units of G are $1+i$, $-1+i$, $1-i$, and $-1-i$. I’m not quite sure how to demonstrate that they are all the units.

Nathan recognized that he was being asked to list the units of G and understood that he should demonstrate the completeness of his list. His assertion that he is “not quite sure how to demonstrate that they are all the units” is problematic. It seems to include two discrete and related statements: He is unsure how to a) demonstrate that his listed elements are units, and b) demonstrate that there are no other units. This

second task, demonstrating that his list is complete, is fairly complex and loads at least as heavily on proof proficiency as it does content knowledge regarding inverses. Yet, he does not demonstrate that his proposed elements are, in fact, units and he implies that he does not know how to do so. Yet, in *Item 1a*, he wrote, “Let a, b be units in U . Then by definition, there exists some $u \in U$ such that $au=1 \dots$ ” This indicates that he is able to write the formal definition of unit. As such, he should be able to apply this definition to a specific context. If he had attempted to demonstrate that his candidates are units, he may have discovered that none of the elements on this list are actually units in the Gaussian integers. Fundamentally, this response is troubling both in the candidates that Nathan lists and in his implications regarding his ability to provide justification. This response suggests that Nathan’s proficiency with the concepts of inverse and identity is low, perhaps limited to writing the formal statement in appropriate contexts and deriving quite basic conclusions from that statement.

Lastly, four students in the class (Johnny, Ned, Lynn and Jeff) were unable to make any real progress on the item. These students did indicate that they are able to express the form of inverse elements in a new context even if they are unable to determine what the units are. As such, they do demonstrate some level of proficiency, just not the expected level. All of these students know the definition of unit and recognize what that means in the Gaussian integers. None of them make use of the idea that G has more constraints on the elements than the students made use of. All of the students but Ned set up the usual equations and completed some derivations, but they did not know how to proceed from there. Ned’s work included

odd use of notation, which in this case seems to indicate a lack of understanding of the problem situation. He wrote that $o,p \in G$, which is incorrect. Specifically, $o,p \in \mathbf{Z}$ and as a pair form an element of G .

Item 3 required the students to list all of the elements of a squadron and state which are units. As described in the analysis of symbolic fluency, it seems that this item had a high barrier for entry related to the symbolic notation used, but three students (Kenny, Lynn and Jeff) all gave complete and correct responses, and three additional students (Johnny, Mark and James) gave the correct list of elements but did not correctly identify the units. Thus, there is some small amount of information to be gleaned regarding the students' understanding of unit and identity from their responses to this item.

Kenny, Lynn and Jeff clearly recognized that, in the context of functions from a finite set to itself, the only functions that have inverses are the permutations. They each gave a correct list of functions and found some means of indicating the permutations. For example, Kenny wrote, "This function is a permutation, and so is a bijective function. Therefore all elements of S must be mapped to all elements of S , but in any order." Similarly, Lynn, "The only elements of S^S with inverses are the bijective elements, i.e., those belonging to $S(S)$."

Three additional students from the investigative class were able to give the correct list of triples, but did not give a correct list of units. One of the students, Johnny, identified two candidates and the other two students did not list any. Johnny listed all of the correct functions and then stated:

The elements of S^S that have inverses are the elements where $f : S \rightarrow S$ generate the entire set of S . (The range of $f=S$).

He later crossed out this characterization, wrote “this is wrong” next to it and added:

$$\begin{pmatrix} abc \\ bca \end{pmatrix}, \begin{pmatrix} abc \\ cab \end{pmatrix} \leftarrow \text{These are the elements with inverses.}$$

He never stated exactly why he changed his list of units. He did, initially, seem to identify the six elements with inverses as the permutations on three elements; he had written 1, 2, and 3 in pencil below a , b , and c respectively on a number of his listed elements. Yet, this made his response even more difficult to analyze. He has the appropriate level of proficiency to answer the question correctly and even makes the appropriate connections to his prior content knowledge, but then doubts himself and changed his response.

The next two students (Mark and James) did not offer any unit-candidates. Instead, they both stated that they were unable to determine which elements have inverses. Mark wrote, “Depending on what the identity is, the function \circ is, and the nature of the group determines who has inverses.” James’s statement regarding his inability to determine which elements have inverses, “I’m not sure what the identity is here, but if I did, I’m sure I could determine the inverses.”

Both Mark and James believed that if he had known what the identity was, they would have been able to identify the elements with inverses. In this case, other evidence suggests that the students’ assertions of proficiency, if given the identity, are somewhat questionable. If they were proficient with functions, they should have recognized that the function $\begin{pmatrix} abc \\ abc \end{pmatrix}$ is the identity, as each element is mapped to itself. In effect, I told them which function was the identity. I stated:

The identity for the squadron |

is the function e defined by $e(s) = s$ for all $s \in S$.

That is, the identity is the function which returns the input. The only function that does this is that listed above. What this suggests is that the students had great difficulty in making sense of the function notation used, and were unable to use this to state the identity.

All of the remaining students (Stephanie, Ned, Rebekah, Aurora, Steven, Nathan) did not sufficiently understand the context of the problem, and their work offered very little insight into their proficiency with identities, inverses, or units. In terms of what these students have demonstrated, I believe that Nathan best summarized the situation that these students find themselves in. He wrote, "I don't quite understand this question. I'm not sure how to find the sets."

Summary of Student Identification of Units

On the end-of-course assessment 11 of the students were able to list elements in a set that they believed had an inverse either on Item 2, Item 3, or both (all but Ned). Moreover, Ned did make some progress on Item 2 and wrote a pair of equations indicating that he recognized the form of a unit in the Gaussian integers. Because of this level of success, it is reasonable to argue that the students in the study had some ability to determine which elements have inverses in a given structure. In general, they were more successful with the Gaussian integers, a somewhat familiar structure with relatively little symbolic notation, than they were at working with functions, a relatively abstract and unknown structure with a large amount of symbolic notation.

Use of Inverses in Proof

The students demonstrated a number of levels of proficiencies on the proof task of Item 1a, but there were definite themes that emerged. The students generally fell into two groups. The first group recognized that it was necessary to prove that the product of two units is a unit; the other group asserted it without proof. All of the students were able to state and use the definition of unit in a formal proof.

There are four different components of this item wherein the students needed to make use of the definitions of identity and inverse. The students were asked to demonstrate that a structure is a group. This required the students to show that the structure is closed, has an identity, that each element has an inverse and that the elements are associative. This last requirement, associativity, has no relation to the fact that the elements of the structure are units. As such, the students' work on that section of the task is not presented.

All of the students, except one in each class, stated that the identity of the squadron is also the identity of the set of units. Two of the students did not make any statement about the identity—it is very likely that each of the students recognized the essential fact. All of the students were able to state that if an element is a unit, then by definition, it has an inverse. Lynn and Jeff from the traditional class and James from the modern class all gave responses that indicated that they had all the appropriate proficiencies. Lynn's work is shown below.

Since S contains an identity e , and $e * e = e$, e is a unit and U is non-empty and has an identity.

Suppose $a, b \in U$. Then a, b are units, so $\exists a^{-1}, b^{-1} \in S$ st $aa^{-1} = e, bb^{-1} = e$.

So, $a^{-1}, b^{-1} \in U$. Then $(ab)(b^{-1}a^{-1}) = abb^{-1}a^{-1} = aea^{-1} = aa^{-1} = e$. So, $ab \in U$, and U is closed.

Since (S, \bullet) is associative, U inherits this property.

For $a \in U$, $\exists a^{-1} \in S$ st $aa^{-1} = e$. Then a^{-1} is a unit and $a^{-1} \in U$, so every element of U has an inverse.

Thus, (U, \bullet) is a group.

This response indicates that Lynn is proficient with the definition of identity, inverse and unit and can make use of each in a proof as appropriate. That is, Lynn, as well as James and Jeff, demonstrated all of the proficiencies required to complete this item. Steven did not submit a correct response, but the manner in which he used the definition of unit also placed him within this group of students. Steven also made a mistake in this proof, but his mistake was based upon faulty proof proficiency rather than content-related difficulties. That is, Steven incorrectly chose his elements x and y in (U, \bullet) such that $xy=1$. He then claimed that, since $1 \in U$, the set satisfies the closure requirement. That is, he set a condition on x and y rather than choosing arbitrary elements; and as such, his argument was incorrect.

While other students were able to make use of the definitions of identity and inverse in appropriate ways, they also included much more questionable assertions in their responses. The group with these problematic responses was larger than the group with correct responses. This group principally had difficulty in demonstrating closure. For example, both Ned and Kenny stated that by definition of the set it is closed. Consider Ned's work:

Since U is the set of units in S and (S, \bullet) is a squadron, then we know by definition that it satisfies closure for multiplication, associativity and identity. Also, since U is the set of units in S , then by def or a unit $u \in U$ has an inverse. Thus, U satisfies the four axioms needed to be a group.

While it is true that closure will derive almost directly from the definition of a unit, there is nothing inherent in either the definition of the set of units or in the definition of unit that would guarantee that the set of units would be closed. The students seem

to have read too much into the problem. Other students offered incorrect responses relating to closure that were less instructive. Johnny, Stephanie and Rebekah also asserted that the set is closed without any sort of rationale, although it seems likely that they made the same assumption as Kenny and Ned.

Nathan's work also contained problematic statements about closure, but, it also included statements indicating he did not have all the basic facts about units.

Suppose that (S, \bullet) is a squadron. Let U be the set of Units in S .

Prove that (U, \bullet) is a group.

Let a, b be units in U . Then by definition, there exists some $u \in U$ such that $au=1=bu$. Then $ab=aubu=u(ab)=1$. So, U is closed. Now observe that multiplication in S is associative, so U is associative. The identity of S is a unit, so U has an identity element. Finally, U has inverses by the definition of unit. Therefore U is a group.

Nathan's response is problematic for multiple reasons. First, Nathan uses idiosyncratic language in his work; he wrote, " U has inverses by the definition of a unit." It seems likely that he meant that the elements of U all have inverses, but this is unclear from his work.

He exhibits some interesting understandings of inverse as he allowed both a and b to have the same inverse. He does not use the fact that inverses must be unique to actively monitor his proof production. There are two possible explanations for his mistake. He may not know that inverses must be unique. Second, he may not be able to monitor his proof production or to check his work.

Nathan also used commutivity in an inappropriate setting and in a non-standard way when he wrote $ab=aubu=u(ab)=1$. This statement makes use of the commutative property of the operation even though this has not been established. Further with, " $aubu=u(ab)=1$ " he has claimed that u is the inverse of a and that u is

the inverse of b , but then he claimed that u is the inverse of the product ab . Again, Nathan did not monitor his proof production: If u is the inverse of a , then u^2 is the inverse of a^2 (or in this case ab). While Nathan wrote the definition of unit in an appropriate context, he only evidenced a rudimentary, almost rote, proficiency with inverses and units. He could not reliably use what should have been basic definitions and modes of thought about units.

Both Aurora and Mark wrote responses that were somewhat problematic to analyze. Consider Aurora's work:

By definition we know a squadron is a group so $\exists a, b \in G$ st $ab \in G$, and also a and b have an inverse, namely a^{-1} and b^{-1} st $aa^{-1} = e$ and $bb^{-1} = e$ so a^{-1} and b^{-1} are units, and $a^{-1}, b^{-1} \in G$, but also to H . If $a^{-1}, b^{-1} \in G$, $a^{-1}b^{-1}$ must belong to G . Since $ab \in G$ and $a^{-1}b^{-1} \in G$ $(ab)(a^{-1}b^{-1}) = aa^{-1}bb^{-1} = e$, so $a^{-1}b^{-1}$ must be a unit and $a^{-1}b^{-1} \in H$. For $a^{-1} \in H$, $a^{-1}a = e$, so a is also a unit and belongs to H . Therefore, H is a subgroup of G and is a group.

This response begins with a fundamental error that seems to derive from a misunderstanding of the context of the problem, but Aurora's faulty assumptions allowed her insight into the problem. She did give a correct statement of the definition of an inverse, but she never stated or made reference to the definition of the inverse. This response indicates that Aurora was fundamentally confused in the context and calls into question her understanding of groups. As such her response does not allow much insight regarding her ability to use the definitions of inverse and identity in creating a proof. In addition she did not properly write the inverse of an arbitrary product of units; instead she assumes commutivity. She should have known that the inverse of a general product (ab) is $(b^{-1}a^{-1})$ or $(ab)^{-1}$, but she either did not know to make use of this fact here or she did not know the fact.

Similarly, the work of Mark is difficult to interpret given the limited evidence.

It is shown below:

Proof: Let x, y, z be units in the squadron S .

S1: $x*y$ will be in S and U because the result will be x, y or z (let z be an arbitrary unit). If x or y is the identity, then the result will be the opposite. If neither is the identity, x or y (being a unit) will “divide” z because they “divide” every element.

S2: $(x*y)*z=x*(y*z)$

Following the same steps as above, A unit “divides” every element and $x, y,$ and z are all units.

S3: The identity is always a unit. Therefore it is included in U . Thus, U is a group.

Mark’s attempts at closure and associativity suggest a fundamentally unique notion of unit. While he indicated via quotation marks that he realized that division is not one of the available operations, he seems to lack any other language to approach the problem. More interesting is his assertion that xy will be equal to some arbitrary unit z . He suggests this must be true because a unit “divides” all other elements. I believe what he was attempting to argue is that ‘the multiplicative inverse of x times z is an element in the squadron.’ Yet, this argument is either conceptually incomplete or circular. He may mean the ‘in the squadron’ idea, but then his ‘proof’ fails to make the critical argument that the product of xy is a unit. Or, his argument is that $x^{-1}z \in U$ and this assumes what he wants to prove.

Item 1b required students to make use of the concept and notation for inverse elements in proof. This item also added some cognitive complexity by asking the students to consider right inverses and left inverses, that is inverses with quantification. Research shows that quantification is difficult for students to understand and manage (Dubinsky, Elterman & Gong, 1988; Bills & Tall, 1998). Because of that, this task was more demanding. In terms of the students’ actual work,

there were only a few completely correct responses. Most of the students made some sort of factual misstatement that may have been motivated by difficulty understanding the quantification.

While some students displayed proficiencies, the overwhelming majority of the students made similar errors indicating a substantially incomplete understanding of the problem situation. Two of the traditional students submitted work that was fundamentally complete. All of the other students submitted work in which they assumed that an element with a left inverse and a right inverse must be a unit.

Two students, Lynn and Jeff, submitted responses that were essentially correct. Moreover, both of them correctly made use of the left inverse and right inverse as appropriate. Lynn's work is shown below.

Suppose $x \in L \wedge R$. Then $yx=e \exists y \in S$ and $xz=e \exists z \in S$. So, $yx=e$. If we multiply on the right by z we get $(yx)z=ez=z$. Since our operation is associative $(yx)z=y(xz)=ye=y$, and thus, $y=z$. So, $y = x^{-1}$ and x is a unit. Thus, $L \wedge R \subseteq U$. Suppose $x \in U$. Then $\exists x^{-1}$ st $xx^{-1} = x^{-1}x = e$, so $x \in L \wedge R$. Then $U \subseteq L \wedge R$. Thus, $U=L \wedge R$.

Most importantly to her argument, she recognized that asserting that x is a unit actually required demonstrating that a left inverse is a right inverse.

The next three students (Kenny, Ned, Stephanie) wrote the correct pair of equations (e.g., " $yx=e$ and $xz=e$ ") and from this concluded that x has an inverse, leaving an unproven assertion. Consider Ned's work:

Let $x \in L$ have a left inverse of y if $yx=e$. Let $x \in R$ have a right inverse of z if $xz=e$. Let's say $L \cap R$, so, $yz=e \cap xz=e$, since $yx=e$ and $xz=e$, then we and conclude that $y=z$, so we'll call it t and label it so, $y=z=t$. Thus, $t \in U$ and $tx=xt=e$, thus t is the inverse of x and t is a unit and \therefore in U .

This implies an insufficient strategy for proof rather than specific content knowledge.

They simply did not know how to proceed, although they knew what they needed to

demonstrate. The only thing that these students were missing was the application of one of the inverses to both sides of their equation. This has implications for proof proficiency because it loads more heavily on students' cognitive actions in the writing of a proof than any particular aspect of content knowledge.

Five students (Rebekah, James, Johnny, Steven, Aurora) all assumed that if an element x has both a left inverse and a right inverse, then those are the same element. Of the students in the investigative class, three out of seven asserted this. Only one of the DTP students did so. Rebekah's formulation is the most succinct:

$L \cap R$ gives us all of the elements with both left and right inverses.
 $L \cap R = \{x | nx = xn = e\}$ where e is the identity. Therefore, $L \cap R = U$.

These students all failed to recognize that an element in $L \cap R$, while having both a left and a right inverse, does not come with a guarantee that those two elements are necessarily the same. It seems most likely that these students are having difficulty with the quantification of left and right inverses. Yet, consider the definition of left inverse and right inverse that the students were given as part of the assessment:

Definition: Let (X, \bullet) be a squadron with identity e and suppose that $x \in X$. An element $y \in X$ is called a *left inverse of x* if $yx = e$. An element $z \in X$ is called a *right inverse of x* if $xz = e$. An element $t \in X$ is called an *inverse of x* if $tx = xt = e$. An element $x \in X$ that has an inverse is called a *unit*.

Had the students simply written down the definitions of left inverse and right inverse that they were given, they would have exhibited a more correct proof. Instead, it is worth asking why these students did this. The students had to actually make an effort to write a new symbol string that includes this error. In effect, that required the students to make more cognitive effort. In considering why they would do this, I believe they did not see an immediate way to proceed from the more correct formulation, " $yx=e=xz$," and they attempted to locate a cognitive short-cut. That is, it

has been shown that students often attempt to minimize the abstraction of a problem (Hazzan, 1999) via a number of means, and students expect their faculty to make problems easier for them (Fukawa-Connelly, 2005) in a number of ways. This seems to be some combination of the two, perhaps a learned behavior.

It is possible that the students were actively looking for a hint in the problem that would make it less challenging. They have likely developed a number of strategies that they use to reduce cognitive difficulty. It appears that, in this case, the students assumed that because of the language of the problem, making use of elements that have both a left inverse and a right inverse implied that the two elements must be the same. This would make sense if you were to assume that the problem must contain a hint to make it easier. The wording suggested that the problem is about inverses of a sort, and the requirement that an element have a left and right inverse could be taken as assurance that those elements will be the same. Now, it is important to note that these two elements are, in fact, the same, but it requires some proof to demonstrate this fact. The students relied on the suggestive power of language and symbols to derive this conclusion without proof.

Summary of Student Use of Identity and Inverse in Proof

All of the students made use of the definition of unit and identity in the context of this problem. Students in each class demonstrated that they were able to write " $aa^{-1} = e$ " or some other iteration. As such, the students could write a formal statement that an arbitrary element of a structure is a unit. Moreover, they were able to recognize an appropriate context for doing so (although the problem should be considered fairly standard).

In terms of more advanced levels of proficiency, such as using facts to monitor proof production, the students were more varied in their levels of proficiency. All of the students in the traditional class attempted to prove that the product of arbitrary units is a unit. That is, they attempted to show closure. Three of the students made fundamental errors that have been discussed above. Four students simply asserted that (U, \bullet) is closed without explanation. Two more asserted that (U, \bullet) is closed by definition of a unit. While it is likely that the four who asserted this without explanation have misconceptions, the two students who asserted closure by definition are not distinguishing between properties that were proven from the definition and the definition itself. That is, they seem to not be using the formal definition in their monitoring of their proof production.

Lastly, there were multiple instances of students not using basic facts about inverses and units in their monitoring of proof production. Most of the students had not developed proficiency with units, inverses and identity to the extent that they were able to ensure that all of their statements in a proof were warranted.

There were three students who offered complete and correct results. Lynn is, unusual as she is a sophomore in abstract algebra, indicative of advanced mathematical proficiency and above average ability. While these students evidenced strong knowledge there is no indication that the majority of the students have that proficiency.

Summary of Demonstrated Proficiency with Identity, Inverse, and Unit

The students demonstrated mixed levels of proficiency with the concepts of identity, inverses, and units. They generally seem to have mastered the notation

mathematicians use to denote an identity, an inverse of a given element, and a unit. Moreover, most of the students were able to make appropriate use of that notation in writing the proofs for Item 1. All of the students but one, Nathan, were able to write the formal definition of a unit in context or to list pairs of elements that are inverses. As such, it is reasonable to claim that the students have a flexible enough proficiency with the formal definition to be able to apply it in a reasonably familiar setting.

The students struggled to manage the notation when an additional quantifier (left or right) was added to the notion of an inverse. Previous research indicates that students have difficulty with quantification generally (Bills & Tall, 1998), and because of these difficulties it was expected that the students in the current study would display a wider range of proficiencies on an item that included unusual quantifiers. But, the student's work on Item 1b indicated that many of the students immediately concluded that any element with both a left inverse and right inverse must have an inverse. Approximately half of the students simply made this assertion without proof. While this may be a valid conclusion, in the context of this assessment it required proof. It seems likely that this difficulty was more derived from problematic proof proficiencies rather than problems with the content.

The students offered proofs which, when the misstatements were taken as true, were logically complete in terms of the structure, but many students made factual misstatements which indicated that they were or could not make use of basic facts about inverses to monitor their proof production. They also struggled to interpret or manage the additional cognitive complexity which came with adding quantification to the concept of inverse.

In terms of their ability to identify the identity element and units in different structures, the students were more capable in more familiar structures and less so in less familiar structures. Item 2 and Item 3 required students to identify units. They should also have demonstrated that their proposed elements were units and then shown that their list was complete.

Item 2 required students to apply the formal definition of a unit in the context of multiplication in the Gaussian integers. Three students were able to give a complete list of units and to give either a justification or to state that all units in the Gaussian integers must be on the unit circle. One other student was able to give a correct list of units but she did not offer any justification of the completeness of her list.

There were three students who included incorrect candidates on their unit list but, due to lack of proficiency with complex arithmetic, were unable to rule them out. That is, these students seemed to have the correct understanding of unit, had the ability to apply it in context, and knew how to demonstrate that their candidates were units, but they lacked proficiency with arithmetic. Steven seemed to exhibit a similar tendency. Although, instead of listing incorrect unit candidates, he failed to list obvious candidates, simply stating that the only unit is the identity. Lastly, there were four students who made almost no progress on the problem other than writing the formal definition of unit in the context of multiplication in the Gaussian integers. These students did not even list the obvious candidate of the identity.

In short, it appears that all of the students could apply the definition of unit in a reasonably familiar setting, and that most could identify unit candidates and then

use their applied definition to demonstrate the appropriateness of candidate choices. One of the students seemed unable to check his candidates; he may not have known how to apply the definition of unit in this context.

Two of the strongest students in the study, Lynn and Jeff, did not list any candidates. The other two students who did not list candidates were mid-level students. It seems that the strongest students did not want to hazard a guess without analytic support, whereas the other students were more willing to offer partial answers or to make informed guesses. This is somewhat aligned with Jaffe and Quinn's (1993) exploratory mathematics.

The students were less successful at determining either the identity element or elements with inverses in an unfamiliar setting. They were not very successful at identifying the identity element for the set of functions of a discrete variable in Item 3. In fact, only four of the students were able to do so. But, it is likely that a significant portion of this difficulty was attributable to their difficulty making sense of the notation and the use of functions as the context of the problem. There were only six students who were able to correctly list the elements of the set, meaning that half of the students had no opportunity to demonstrate their proficiency at determining either the identity or units in the set. Of the six students who were able to list all of the elements in the set, four of them were able to identify all of the units (although Johnny later changed his mind and crossed out a number of his candidates). The other two students, Mark and James, claimed that they were unable to determine the identity of the set and because of that were unable to make more progress.

In summary, it seems that the context was more problematic than the students' proficiency with identity and units, but there was not enough data to claim that the students had proficiency identifying the identity element and units in unfamiliar contexts. As with all of the other content strands, the students demonstrated more proficiency with more familiar tasks and struggled when either the task or the context of the task was less familiar.

Polynomials

Polynomials are a major object of study of abstract algebra, and the search for solutions and methods for solving to polynomial equations is a root of the discipline of abstract algebra (Kleiner, 1986).

Understanding polynomials

The definition of a polynomial

In algebra, a polynomial is a mathematical expression involving a sum of powers of a variable (indeterminate) multiplied by coefficients. In algebra we require that the coefficients all belong to the same ring. Generally, we discuss expressions of the form $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$ where all a_i are elements of the same ring.

Factors of polynomials

A factor of a polynomial

$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$ is another polynomial (of degree less than n) $Q(x)$ which can be multiplied by another polynomial $R(x)$ such that $P(x) = Q(x)R(x)$. Just as with numbers, a factor divides the original polynomial

evenly. We often require that all of the coefficients of the factors are elements of the same ring as the coefficients of the polynomial. Linear factors of polynomials are often considered the most important, they are polynomials of the form $Q(x) = a_1x + a_0$ which give rise to roots of the polynomial. A root is a value of x that the polynomial maps to zero, that is $P(x) = 0$.

Idealized Student Proficiency with Polynomials

Each of the observed course sections devoted a number of hours to the study of polynomial rings, polynomial equations, and constructing roots algebraically. Developing proficiency with polynomials is one of the major expected outcomes of an introductory course covering ring-theoretic topics. Moreover, the students should be able to draw upon their previously developed proficiencies with the topic. By this point in their mathematical careers, students should be able to write polynomials with real coefficients, to factor and find roots of quadratics with real coefficients, and to write polynomials with given roots. In their abstract algebra class, students should start thinking about polynomials from a structural perspective, as they consider the ring of coefficients, determine the ring where the polynomials are irreducible, reducible, and completely factorable, and practice factoring and determining roots. The students also learn how to construct a root of an irreducible polynomial algebraically via the creation of an extension field. That is, the students should develop the ability to:

- recognize and write polynomials with coefficients from a specified ring.

- decide if a polynomial is reducible or irreducible in a given ring (which means that students must know what it means to be reducible and be able to distinguish this from having a root).
- factor a polynomial with integer coefficients in the rational numbers, the real numbers, and the complex numbers.
- algebraically construct a root of a polynomial irreducible in a given ring/field.
- make and demonstrate the truth of conjectures about polynomials (with real number coefficients).

Opportunities to Learn

In both sections, instruction addressed the definition of a polynomial and a polynomial ring, as well as what it means to have coefficients from a specific field. In both courses the students were responsible for demonstrating that a given polynomial is irreducible in a given ring, although it seems that most, or possibly all of the polynomials that the students examined were quadratics or cubics. Students in both courses were taught and asked to use the rational root test.

Additionally, both of the teachers demonstrated an algebraic construction of the square root of two. Both teachers repeatedly stated that if F is a field, x is an indeterminate, and $p(x)$ is an irreducible polynomial element of $F[x]$, then

$F[x]/(p(x))$ is a field. Students from both classes constructed a number of Cayley tables of extension fields created by this process; students in the investigative class constructed more of these tables. While both instructors were careful to use both bracket notation for cosets and the expanded form $f(x) + p(x)g(x)$ during classes to

indicate that they were working in the quotient field, I observed that students never explicitly articulated either form.

The DTP class was asked to determine whether $\mathbf{Q}[x]/(x^2 + 2)$ is isomorphic to $\mathbf{Q}[x]/(x^2 + 3)$. Because the students seemed to struggle with this, on March 22, Dr. T conducted a teaching episode on the question that lasted approximately 20 minutes. The students in that class were also asked to find all of the roots of the polynomial $x^2 + x + 1$ in the field $\mathbf{Z}_2[x]/(x^2 + x + 1)$ for homework and they briefly discussed this problem in class.

The students in the investigative class constructed the Cayley tables for $\mathbf{Z}_2[x]/(x^2 + x + 1)$ during class. This was done partly in groups and partly during a whole-class teaching episode in which Dr. Parker discussed how to determine which element in the field represented $[x^3]$. During the construction of these Cayley tables for the field $\mathbf{Z}_2[x]/(x^2 + x + 1)$ the students did not consider which of the elements of the field were roots of the polynomial $x^2 + x + 1$.

The students in the investigative class were repeatedly asked to state the form of elements in polynomial quotient rings and regularly articulated possible degrees and elements.

In another instance the students in the investigative class were to consider what Dr. Parker called “two weakenings” of their hypotheses for the theorem that $F[x]/(p(x))$ is a field; they were asked to construct the Cayley tables of two rings

$\mathbf{Z}_6[x]/(p(x))$ where $p(x)$ is an irreducible polynomial in the ring $\mathbf{Z}_6[x]$ and

$\mathbf{Z}_2[x]/((x+1)^2)$ and “see what happens.” In the following class meeting, the students

worked in groups to construct the Cayley tables for the first ring, and they determined that the structure was a commutative ring with identity that had zero divisors. Then

in the next session the class had a discussion in which they agreed that if the

polynomial is reducible then there will be zero divisors in the quotient structure.

Later, the students in the investigative class also saw a demonstration and proof that

all elements of the fields $\mathbf{Q}[x]/(x^2 + 2)$ and $F[x]/(p(x))$ have inverses.

Assessment of Student Understanding

While I believe that students should learn how to identify a polynomial as an element of a given ring and to write a polynomial with coefficients from a specific ring, these skills were not directly assessed on my final exam. Instead the exam contained items which assumed that the students were proficient at these skills and would make use of these skills in their work on the item. Because of the limits of the design, allowing students to complete the exam while having access to their text and notes, such questions did not seem appropriate.

The second set of questions on the final exam was designed to assess students' proficiency with the polynomial content strand (see Appendix B). In this section, students were asked: (1) to demonstrate that the domain in which a polynomial is factored leads to very different factorization possibilities and to show fluency at factoring in different domains; (2) to offer a conjecture and proof related to the

greatest degree of an irreducible polynomial with real coefficients; and (3) to construct roots of a polynomial via construction of an extension field. The polynomial that they were asked to consider would likely be familiar to any advanced high school student. Because the students were not as proficient as I assumed, the prompts that I wrote actually assessed a different set of proficiencies than I intended. Specifically, the proficiencies I actually assessed were:

- Demonstrate fluency in writing polynomials with real and complex coefficients;
- Demonstrate fluency in factoring polynomials in the rational numbers, real numbers, and complex numbers (including showing when a polynomial cannot be factored);
- Demonstrate the ability to construct all of the roots of a polynomial by constructing the minimal extension field; and
- Demonstrate an ability to make and prove conjectures about polynomials.

Four weeks elapsed between the time the students studied polynomials and this exam. During this time, students studied different topics and likely only completed one homework assignment about polynomials. Upon reflection, it is not surprising that the students exhibited very low levels of proficiency with polynomials.

Evidence of Student Proficiency

On the assessment, there were many students who made little or no progress on the three items assessing learning in the polynomial strand, suggesting that these items were significantly beyond their abilities. Because of that, this exam does not

allow for much meaningful differentiation between the students' abilities. This should not be read as a critique of either the instruction or the students, but rather as a point for beginning a discussion about the nature of expectations for students and the structure of the introductory abstract algebra course.

*Demonstrate fluency in writing polynomials
with real and complex coefficients*

On Item 3 the hint suggested that the students write a polynomial with two complex roots, a complex number z and its conjugate. Of the 12 students in the study, only 7 gave a response which allowed analysis of their ability to write polynomials with specific characteristics. Two of those students, Jeff and Lynn, showed that they were proficient at writing a polynomial with an arbitrary complex root. One student, James, provided a response that suggested that he conceptually understood the requirements of the problem, but seemed to encounter difficulty in managing the necessary symbolic systems. The other four students gave responses which suggested that they were not able to write a polynomial with an arbitrary complex root. In general, it seems that the students were not able to write a polynomial with an arbitrary complex root, but at least four of them were able to write a quadratic polynomial with real coefficients which they correctly identified as irreducible over the real numbers.

Six students submitted work that did not allow for interpretation with respect to this proficiency, and two students, Nathan and Rebekah, simply did not attempt the item. The other four students, Steven, Stephanie, Kenny and Mark, submitted responses which provided little basis for discussing their proficiency writing

polynomials with either real or complex coefficients. Steven claimed that “any polynomial of the form $x^a + 1$ is irreducible in $\mathbb{Q}[x]$,” a response that failed to address the question. Mark made no progress on the item, and Stephanie seemed to be trying to make sense of the term irreducible: “If $p(x)$ has a root then it’s reducible and if $p(x)$ had degree greater than 1 then it has no real roots, thus its [sic] irreducible. If $\deg p(x) \leq 3$ and $p(x)$ has no roots in the field then $p(x)$ is irreducible.”

Three students demonstrated that they were able to write a polynomial with an arbitrary complex root; two of those students further showed that they were able to correctly expand the polynomial into one with real coefficients. Here is a portion of Lynn’s response to Item 3 in which she demonstrated her proficiency with polynomials:

By Gauss’ theorem there exists at least one root, say $z = a + bi$. You provided us that $\bar{z} = a - iy$ is also a root of $p(x)$. So, in $\mathbb{C}[x]$,

$$\begin{aligned} p(x) &= (x - (a + bi))(x - (a - bi))q(x), \text{ there is } q(x) \in \mathbb{C}[x], \\ &= (x^2 - (a + bi)x - (a - bi)x + (a^2 + b^2))q(x) \\ &= (x^2 - 2ax + (a^2 + b^2))q(x) \end{aligned}$$

James’s work suggests that he was able to write a polynomial with an arbitrary complex root, but had difficulty managing the notation associated with functions and complex numbers when used together. He began by writing a polynomial with a variable of a but then claimed that this polynomial would divide another with a variable of x . James:

$$\begin{aligned} x - yi &\Rightarrow x + yi && \text{so } (a - x + yi)(a - x - yi) \mid p(x) . \\ a^2 - ax - ayi - ax + x^2 + xyi + ayi - xyi + y^2 \\ a^2 - 2ax + x^2 + y^2 &\mid p(x) \end{aligned}$$

His confusion seems to have resulted from the fact that the hint presented an arbitrary complex number as $z = x + yi$, but he was asked to consider a polynomial with a variable of x as well. He successfully wrote a polynomial with a variable of a , but then seemed unsure how to relate the new polynomial that he had written to the requirements of the item. In retrospect, the more standard notation of $a + bi$ may have been preferable. Some of the cognitive complexity that James, and perhaps others, encountered may have resulted from my use of less familiar notation.

One other student demonstrated that she could write a polynomial that she knew to be irreducible in $\mathbf{R}[x]$ but reducible in $\mathbf{C}[x]$. Aurora wrote, “ $x^2 + 1$ is an irreducible polynomial $p \in \mathbf{R}[x]$.” She then demonstrated that the polynomial is reducible in $\mathbf{C}[x]$.

Ned and Johnny attempted to write an appropriate polynomial. For example, consider Ned’s:

Note the complex roots $z = x + iy$ and its conjugate $\bar{z} = x - iy$ multiply together to get $(x + iy)(x - iy) = x^2 - y^2i^2 = x^2 + y^2$ which will divide $p \in \mathbf{R}[x]$.

Both he and Johnny attempted to write a polynomial in $\mathbf{R}[x]$, but both used two variables, x and y , in their work. This inclusion of a second variable raised serious concern that these students understood the context of the problem as polynomials.

In general the students did not display an ability to write a polynomial with an arbitrary complex root or an ability to write a polynomial that is irreducible over the real numbers, but factors over the complex numbers. This lack of progress was surprising.

Fluency in factoring polynomials

Item 1 asked students to factor a polynomial over the complex numbers and the real numbers and then to demonstrate that the polynomial is irreducible over the rational numbers. In effect, this required the students to demonstrate two different proficiencies with factorization. The first required them to demonstrate fluency in factoring over two different fields. The second required them to write a proof demonstrating that a polynomial is irreducible and therefore does not factor.

The students were asked to factor $x^4 + 1$ over the complex and real numbers. Of the 12 students in the study, only two gave a correct factorization in the complex numbers. These two were also the only students who were able to give a factorization over the real numbers. It should have been possible for a student to give a correct factorization over the real numbers without giving one over the complex numbers, because the factorization over the real numbers could have been done by making use of knowledge of polynomials from high school. The students could have written two quadratic factors with an indeterminate for a coefficient (a here) and solved the resulting equation for a : $(x^2 + ax + 1)(x^2 - ax + 1) = x^4 + 1$.

The other 10 student's responses showed varying levels of proficiency with finding a factorization of a polynomial. One student gave a linear factorization which demonstrated good fluency with polynomial skills but not complex numbers. One student essentially gave a proof of the Fundamental Theorem of Algebra to claim that $x^4 + 1$ must have four linear factors but seemed unable to actually give a factorization of the polynomial in either $\mathbf{R}[x]$ or $\mathbf{C}[x]$. The rest of the students either did not give a

response or gave a response which suggested more conceptual problems than proficiencies with factorization or arithmetic in $\mathbf{R}[x]$ and $\mathbf{C}[x]$.

Upon reflection, it is not surprising that the students had difficulty in finding a correct factorization because they did not have much practice factoring over the complex or real numbers. Instead they primarily factored polynomials over the integers.

The students who demonstrated the highest level of proficiency with factoring polynomials over the complex and real numbers were Lynn and Kenny. Both of them wrote a complete factorization of $p(x)$ in $\mathbf{C}[x]$ and multiplied pairs of linear terms to produce a pair of quadratic polynomials with real coefficients.

On the first item, Lynn and Kenny each gave a complete factorization of $p(x)$ in $\mathbf{C}[x]$ thus demonstrating their fluency with complex polynomials. Jeff's work was somewhat different. Instead he argued analytically that all polynomials split in $\mathbf{C}[x]$ and that four unique linear factors must exist. He did not show that $p(x)$ is reducible in $\mathbf{R}[x]$. Lynn's work was as follows:

For $p(x) = x^4 + 1 \in \mathbf{C}[x]$, since $\mathbf{C}[x]$ is a field, we may have no more than 4 roots of $p(x)$. Notice that the following four elements of \mathbf{C} are roots of $p(x)$:
 $a = \sqrt{1/2} + \sqrt{1/2}i$, $b = \sqrt{1/2} - \sqrt{1/2}i$, $c = -\sqrt{1/2} + \sqrt{1/2}i$, $d = -\sqrt{1/2} - \sqrt{1/2}i$.
Thus, by the factor theorem, $p(x) = (x-a)(x-b)(x-c)(x-d)$, four linear factors in $\mathbf{C}[x]$.
For $p(x) = x^4 + 1 \in \mathbf{R}[x]$, $p(x) = x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$. Since these are irreducible in $\mathbf{R}[x]$, $p(x)$ is the product of these.

I believe that in the first line, Lynn meant \mathbf{C} is a field and not $\mathbf{C}[x]$. As part of her response, she cited a result that was shown in class, the factor theorem. She made appropriate and correct use of the result in this context. Lynn made an important statement that the other students with similar work did not, she argued that she had

exhibited all of the roots. Secondly, Lynn exhibited a factorization of $p(x)$ in $\mathbf{R}[x]$ that satisfies the problem. VH's work was similar.

Aurora also gave a linear factorization, but instead of expressing each root in the form $a+bi$, she left her roots in the form $\sqrt{\pm i}$ and wrote the polynomial as $p(x) = x^4 + 1 = (x + \sqrt{i})(x - \sqrt{i})(x + \sqrt{-i})(x - \sqrt{-i})$. With this factorization Aurora demonstrated that she was able to factor creatively, including factoring $x^2 + i$ as a difference of squares although in a highly unusual manner. She was the only other student who demonstrated a high level of fluency with polynomial factorization. Aurora was not able to write a pair of quadratic factors of $p(x) = x^4 + 1$ with real coefficients.

Jeff's work is also complete and correct. He, in effect, crafted a proof of the Fundamental Theorem of Algebra from a collection of theorems that are given in his text and used his proof to argue that the polynomial splits in $\mathbf{C}[x]$. Jeff demonstrated that he was capable of writing an analytic proof that $p(x)$ must factor in $\mathbf{C}[x]$ but he did not actually demonstrate a factorization of $p(x)$ in $\mathbf{C}[x]$, moreover his work on other parts of the exam indicated that he likely could not. His proof is as follows:

$p(x)$ is a product of four first degree polynomials in $\mathbf{C}[x]$:
By Thm 4.13, $p(x)$ is a product of irreducible polynomials in $\mathbf{C}[x]$. By Corollary 4.26, each of these polynomials is of degree 1. By thm 4.2, the number of these first degree polynomials is equal to the degree of $p(x)$, and thus, $p(x)$ is the product of four first degree polynomials in $\mathbf{C}[x]$.

Jeff seemed to have a very high level of proficiency with analytic reasoning about polynomials. All of the hypotheses are met when he made use of a result and he used the results correctly. In effect, he argued that $\mathbf{C}[x]$ is a unique factorization domain and that polynomials will factor completely by making use of some very powerful

results. He then showed that these two facts are sufficient to demonstrate that p is the product of four first degree polynomials. That is, he seemed to identify the theorem that he needed to prove, and he was then able to demonstrate a very marked ability to reason about the ring of polynomials with complex coefficients by building an analytic proof of that theorem.

The remainder of the students made a collection of errors. Two made mistakes in complex arithmetic. Many of the students seemed to search for polynomial factorizations by writing plausible factorizations and then multiplying them out. Two students (Rebekah and Nathan) made no attempt at the item whatsoever.

Two additional students, Steven and James, made a reasonable beginning on the item. They each indicated that they were searching for roots of $p(x)$ and recognized that, should they find those roots, they could construct an appropriate polynomial. In James's scratch work he wrote, " $x^4 + 1 = (x^2 + i)(x^2 - i)$." He then attempted to factor in a number of unsuccessful ways, until he seems to have realized that he needed to find the square root of the imaginary number.

As his response James wrote: " $\pm\sqrt{i} = x$. But \sqrt{i} isn't in \mathbf{C} I don't think."

The difference between James and Aurora was small based upon this set of evidence. Both recognized the correct form of the first factorization, but James did not continue and thus did not demonstrate the ability to think about polynomials with the same level of fluency as Aurora.

James did offer a brief response to the prompt relating to real numbers. He wrote, " $-1 = x^4$, x to an even power, where $x \in \mathbf{R}$ will always be ≥ 0 , same applies to

Q.” This response is fairly consistent with those offered by other students on this section. James seems to be indicating that he believes that p cannot be factored over $\mathbf{R}[x]$ because it does not have any roots in \mathbf{R} . This implies that he does not fully understand the relationship between reducible, irreducible, and roots.

Two students, Ned and Stephanie, both gave the same incorrect proposed complex factorization. Johnny wrote a few factorizations, expanded the products, and decided that none were giving the appropriate polynomial. None of the students made any meaningful progress in demonstrating that $p(x)$ is the product of two irreducible polynomials in $\mathbf{R}[x]$. Both Ned and Stephanie understood that they were asked to demonstrate that $p(x)$ factors completely in $\mathbf{C}[x]$, and realized that exhibiting four linear factors would be a complete answer to the question. Stephanie showed in the first problem set that she did not have much proficiency with complex arithmetic, and her lack of fluency may mean that she did actually expand her factorization. She gave no indication on her materials that she did so. Similarly, Ned gave no indication on his materials that he has actually expanded his factorization. He did not attempt the second item in the first set, thus, I have no means to evaluate his proficiency with complex arithmetic. Ned’s work:

Notice $(x+1i)(x-1i)(x+1i)(x-1i)=x^4 + 1$. Thus, p is the product of four first degree polynomials from $\mathbf{C}[x]$.

$$\text{Notice } x^2 + x + 1 \quad \left. \begin{array}{l} x^2 - x + R(x+1) \\ \end{array} \right) x^4 + 1.$$

He gave no indication why he chose to do polynomial division or what he believed he had demonstrated; he did not indicate where he derived $x^2 + x + 1$.

The last student to attempt a response on this item was Mark. His response was particularly troublesome, as he did not seem to even understand what a response to the item requires. Mark's work:

Since y is attached to i it takes 4 polynomials to turn i into 1. $i \cdot i = -1$, $-i \cdot i = 1$.
 Since x is not attached to i it will be multiplied by itself 4 times to get x^4 .
 $1^4 + 1 = 2$, $2 \bullet 1 = 2\sqrt{1}$
 $x^4 + 1 =$ if x is odd, the function will be a prime times 2, both irreducible. If x is even, the function will be a prime times 1, both irreducible.
 $2^4 + 1 = 17$ $17 \cdot 1$
 $3^4 + 1 = 82$ $41 \cdot 2$
 $4^4 + 1 = 258$ $257 \cdot 1$
 $5^4 + 1 = 626$ $313 \cdot 2$

Mark's first assertion seems to be the claim that $(x+i)^4 = x^4 + 1$ and he has therefore found a correct factorization of $p(x)$. It seems highly unlikely that he thought through the implications of his claim. He provided no indication that he was thinking about what his proposed solution implies for the problem. He seemed to claim that $p(x) = (x+i)^4$, but then p has a root at $-i$. He does not seem to recognize that his implied solution is incorrect. That is, should he substitute $-i$ into $p(x)$ he will not get 0, moreover, this is not what he does to verify that his work is incorrect. Instead, his subsequent assertion is that, " $1^4 + 1 = 2$, $2 \bullet 1 = 2\sqrt{1}$ " It seems that he was not thinking about what a root means when he answered this question. Instead he was simply experimenting with complex arithmetic in search of a plausible-seeming factorization. Moreover, the statement that y is attached to i was curious. It is likely that he was not thinking of polynomials with complex coefficients, but rather thinking of ' $x+iy$ ' as a complex polynomial, as James did.

Mark stated in his interview that he did not know the definition of such terms as group and ring. In his response, he focused his attention on the values that $p(x)$ can assume when evaluated at integers. This seems a fundamental misunderstanding of the problem and indicates that Mark does not know what he has been asked to do. It seems highly likely based upon his work and his admission that he does not know the meaning of the term reducible.

Showing that a polynomial is irreducible

The students were unable to show when a polynomial is irreducible in $\mathbf{Q}[x]$, and their efforts indicated that many have an incorrect definition of *irreducible*. Instead of meaning no factors, they take it to mean no roots; 6 of the 12 students made a claim indicating that they believed this incorrect definition. Two students gave a complete and correct proof that $p(x)$ is irreducible in $\mathbf{Q}[x]$ and a third student wrote the most important fact but did not give an actual proof.

Both Jeff and Lynn gave a complete proof that $p(x)$ is irreducible in $\mathbf{Q}[x]$ whereas Kenny explained in a sentence why $p(x)$ could not have factors. Jeff's work is very similar to Kenny's in execution and level, but, he adds slightly more detail to his result, correctly arguing that for p to factor in $\mathbf{Q}[x]$ it must also factor in $\mathbf{Z}[x]$.

Suppose, to the contrary, that $p(x)$ is reducible in $\mathbf{Q}[x]$ so it can be factored as the product of two non-constant polynomials in $\mathbf{Q}[x]$. If either has degree 1, then $p(x)$ has a root in \mathbf{Q} . But, the rational root test shows $p(x)$ has no roots in \mathbf{Q} (the only possible roots are ± 1 and neither is a root of $p(x)$). Thus, if $p(x)$ is reducible, the only possible factorization is as a product of two quadratics; by thm 4.2. By Thm 4.22, there is such a factorization in $\mathbf{Z}[x]$. Furthermore, $p(x)$ can be factored as a product of monic quadratics in $\mathbf{Z}[x]$, say

$$(x^2 + ax + b)(x^2 + cx + d) = x^4 + 1, \text{ with } a, b, c, d \in \mathbf{Z}.$$

We get $x^4 + (a + c)x^3 + (ac + b + d)x^2 + (bc + ad)x + bd = x^4 + 1$. Equal polynomials have equal coefficients so $a + c = 0$, $ac + b + d = 0$, $bc + ad = 0$, and $bd = 1$. We see that $a = -c$, so $ac + b + d = -c^2 + b + d = 0$ or $c^2 + b + d = 0$.

But, $bd=1$, so either $b=d=1$ or $b=d=-1$.

Thus, either $c^2 - 1 - 1 = 0$ or $c^2 + 1 + 1 = 0$

$$c^2 = 2 \qquad c^2 = -2$$

There is no integer whose square is 2 or -2, so a factorization of $p(x)$ as a product of quadratics in $\mathbf{Z}[x]$, and hence in $\mathbf{Q}[x]$, is impossible. Thus, $p(x)$ is irreducible in $\mathbf{Q}[x]$.

Lynn and Jeff gave a complete and correct response, both indicating that they understand what it means for a polynomial to be irreducible in a given domain and how to demonstrate this. Additionally, they demonstrated that they are able to write two arbitrary polynomials and reason generally about polynomials via algebraic manipulation. Moreover, both of these students recognize that the important contradiction to derive is the fact that the square root of two is irrational. Lynn and Jeff both displayed quite high levels of proficiency on this item; they are the only two students to give a complete and correct argument.

In comparing this portion of Jeff's response to that of the first part of Item 1, it is important to note that he has, in fact, derived enough knowledge about the necessary coefficients in the factorization of p into two quadratics to give a factorization in $\mathbf{R}[x]$. He has stated that " $(x^2 + ax + b)(x^2 + cx + d) = x^4 + 1$," and he has determined that:

$$a=-c$$

$$c^2=2$$

$$b=d=1 \text{ or } b=d=-1$$

However, none of his submitted work provided evidence that he substituted these derived values into the general quadratics that he had written. Given the level of work that Jeff exhibited, it seems quite obvious that he would have been capable of such substitution. Yet, on his submission he wrote, " $p(x)$ is the product of two irreducible polynomials in $\mathbf{R}[x]$:" and then wrote nothing below that. It seems that he

does not realize that he has all of the necessary information to write these two polynomials. He certainly realized that he left that part of the problem incomplete. His analytic argument that p factors into linear terms, and inability to list inverse on Item 2 in the first problem set, may indicate one of two things: It may be that that he does not have great facility with computation, or, that he simply does not fluidly switch between formal proof (and abstract manipulation) and explicit values in the appropriate systems. This is definitely a case that would merit further exploration, as this hypothesized set of proficiencies appears to be quite uncommon.

Kenny, another student who gave a correct response in the first part of this item, seemed to know the kernel of the argument that he needed to give, but did not supply enough detail to have a response that can be considered correct. He wrote:

Coefficients such as $\sqrt{2}$ are not in \mathbf{Q} , so the polynomial is irreducible in \mathbf{Q} .

Here he was referencing his earlier work on the problem, and his statement is correct in that the polynomials that he wrote do not have coefficients from the rational numbers. Had he argued that this is the only possible factorization of p , his response would have been complete. It seems likely that he believes this to be a unique factorization, but it is unlikely that he has learned that $\mathbf{R}[x]$ is a unique factorization domain. As such, his response should be judged incomplete.

The rest of the responses to this item were far less complete, but did allow students to display a misconception relating to polynomials. Six students argued that because $p(x)$ has no roots in $\mathbf{Q}[x]$ it is irreducible. For example, James wrote:

$-1 = x^4$. x to an even power, where $x \in \mathbf{R}$ will always be ≥ 0 , same applies to \mathbf{Q} .

[Scratch work] If there is a solution, there's a factor and is reduc. But, $x^4 \neq 1$ in $\mathbf{Q}[x]$. So, it's irreduc in $\mathbf{Q}[x]$.

For each of these five students, this is the first time that they had to demonstrate what it means for a polynomial to be irreducible as opposed to simply not having roots in the proposed domain. Stephanie submitted the work that was the most difficult to parse. She showed some correct reasoning around irreducibility in $\mathbf{Q}[x]$ and, I believe, indicated that she had an incorrect definition of irreducible.

p is the product of 2 irreducible polynomials in $\mathbf{R}[x]$ if the two polynomials in $\mathbf{R}[x]$ are of degree 2 or less then its irreducible.

Since p is irreducible in $\mathbf{R}[x]$ then it must be irreducible in $\mathbf{Z}[x]$ and therefore irreducible in $\mathbf{Q}[x]$.

Specifically, Stephanie correctly asserted that if p is irreducible in $\mathbf{R}[x]$ it is then irreducible in both $\mathbf{Q}[x]$ and $\mathbf{Z}[x]$. She seemed to believe that she has demonstrated that p is irreducible in $\mathbf{R}[x]$ in the first part of this item—in light of this belief, her work here should be judged correct. In fact, she correctly asserted a relationship between elements of the three rings. Yet, it also seems that in the first part of the item she claimed that p could both be factored and irreducible.

Summary of student proficiency with reducible and irreducible

Reading across both parts of this item, the majority of the students did not exhibit a good working definition of irreducible. Jeff, Kenny and Lynn were the only three students who exhibited a mathematically correct definition. Six of the students believed that irreducible means that a polynomial has no roots in a given ring. Three students did not exhibit any understanding of the item. Yet, as all of the polynomials that the students examined during my observations were either quadratics or cubics, reducibility was equivalent to having a root. Thus, the students could quite

reasonably form a naïve understanding in which not having a root is logically equivalent to irreducible.

Another very curious aspect of the students' work was the statements they made which indicated beliefs about polynomials. Johnny and Steven both made remarks that raise substantial questions about their understanding of the domain from which coefficients come. Johnny was attempting to show that p is the product of two irreducible polynomials in $\mathbf{R}[x]$ and that p was irreducible in $\mathbf{Q}[x]$. To demonstrate this he wrote plausible factorizations and then expanded in an effort to arrive at $x^4 + 1$. What was most interesting is that all of his factorization candidates drew all of their coefficients from the ring of rational numbers. He never wrote a candidate with any irrational coefficients. Steven's work was also quite interesting. He wrote that complex polynomials "look like $ax + bi$." This is a possible form of a complex polynomial, but it seems to indicate that he was not able to write a linear polynomial with two complex coefficients. These two students showed a very low level of proficiency with managing the relationship between the polynomials and the domain from which the coefficients are drawn. In general, there were two students whose work was of a much higher caliber than the others, Lynn and Jeff, and two students, Mark and Aurora whose work was highly problematic. The great majority of the students demonstrated that they had some misconceptions, but also had at least some fluency with polynomials.

As noted in discussion of student work in other content strands, this problem did not allow for much differentiation between students. Because the students performed so poorly in terms of writing the factorization of p in $\mathbf{C}[x]$, they had almost

no ability to write p as a pair of irreducible polynomials in $\mathbf{R}[x]$. The responses that the students submitted for that part of the item provided almost no useful information about their proficiencies. As such, observations about the learning were limited by problematic item design.

Constructing the minimal extension field

This proficiency was assessed by Item 2 where the students were asked to determine all of the roots of the polynomial $p(x) = x^4 + 1$ in the field. Both teachers repeatedly demonstrated the construction of extension fields in this manner, and students in the DTP class had been asked a similar question as part of their homework. It was probably unreasonable to expect any of the students to demonstrate that they had found all of the roots. I expected that, at minimum, the students would all state and demonstrate that $[x]$ is a root of the polynomial $p(x)$ in the field $\mathbf{Q}[x]/p(x)$ and that many students would make more progress on the item by finding two or even all four of the roots. In fact the students seemed to experience significant difficulty with the item.

Two students, Jeff and Lynn, stated that $[x]$ is a root of the polynomial. These were the only two students who demonstrated that they were working in the correct field. Consider Jeff's work:

We search for all functions $t(x)$ such that $t^4 + \bar{1} = \bar{0}$. This is similar to the initial function $p(x) = x^4 + 1$. $[p(x)] = [x^4 + 1] = [0] = \bar{0}$. So, we look for all functions $t(x)$ so that $[t^4 + \bar{1}] = [p(x)]$.

$$\text{So, } [t^4 + \bar{1}] = [p(x)] = [x^4 + \bar{1}]$$

$$[t^4] + [\bar{1}] = [x^4] + [\bar{1}]$$

$$[t^4] = [x^4]$$

$$[t] = [x].$$

Thus, all functions $t(x)$ that leave a remainder of x when divided by $x^4 + 1$ satisfy $t^4 + \bar{1} = \bar{0}$.

Both Lynn and Jeff correctly identified one of the roots of p in the field $\mathbf{E}[x]$, but, this should not necessarily be taken as demonstrating understanding of the problem situation. While these two students spent significant classroom time on similar questions, their teacher gave the students so much explicit help that they did not have to make sense of the constructs, but only manipulate the symbols. Thus, the fact that these two students can state that $[x]$ is a root of the equation is unsurprising. The fact that neither of them were able to find any other roots demonstrates that neither of them had developed substantially more proficiency with the concepts than the ability to re-write material they saw during the lectures.

The rest of the students were not able to make any progress on the item whatsoever. Yet, some of them did give responses that are informative. Four of the students (Rebekah, Mark, Kenny and Johnny) provided responses indicating that they were not operating in the correct field. Johnny's work is shown:

$\bar{1}$ is multiplicative identity and $\bar{0}$ is the additive identity. The field $E = \mathbf{Q}[x] / p[x]$ with $p(x) = x^4 + 1$. We must find $t \in E$ st $t^4 + \bar{1} = \bar{0}$.

$\bar{1} = \pm 1$ in $\mathbf{Q}[x]$

$\bar{0} = 0$ in $\mathbf{Q}[x]$

So, we want $t^4 + 1 = 0$ for $t \in \mathbf{Q}$. This seems very similar to the previous problem that I could not solve.

Each of these students recognized they are being asked to find the roots of the polynomial equation $x^4 + 1 = 0$. Yet, all of them indicated that they were writing a root in the field $\mathbf{C}[x]$ rather than in the field $\mathbf{E}[x]$.

Unlike the other three students who stated that the root is the fourth root of negative one, Johnny made use of coset notation, but also stated that the problem “seems very similar to the previous problem that I could not solve.” This implied that he did not understand how to construct the roots of an irreducible polynomial.

The investigative class also covered the construction of roots via extension fields. All of these students also made use of coset notation as part of their class activity. Thus, this work should be taken as an indication that they most likely saw the material as a series of symbolic manipulations and never developed any meaning for the process. They were unable to recall the fact that $[x]$ is a root of the polynomial.

Perhaps because he did not know how to make much progress on the item as it was stated, James chose to create an entirely new mathematical structure that operated according to the rules of the rational numbers:

$$E = \frac{Q[x]}{p[x]} \quad t^4 + \bar{1} = \bar{0}; \quad \bar{0} = \frac{0}{x^4 + 1}; \quad \& \quad \bar{1} = \frac{x^4 + 1}{x^4 + 1}$$

$$t^4 = -\frac{x^4 + 1}{x^4 + 1}$$

$$t = \pm \sqrt[4]{-\frac{x^4 + 1}{x^4 + 1}}$$

Which again, I have no idea how to reduce.

He failed to recognize quotient fields as a construct, but, did construct a structure involving quotients. He then looked for a root of the polynomial in this newly constructed structure. He relied on older understanding of quotient (fractions) and attempted to find a way to make the problem situation (i.e., creation of quotient

fields) conform to his understanding of quotients. Finally he gave the same response as his classmates above, writing that the root of the polynomial is the fourth root of -1 . This response does, at the very least, show a willingness to do significant mathematical exploration, even when he is quite clear that he did not really understand the problem. This is indicative of a quite strong affective response. Most students, when confronted with a seemingly impossible problem will simply omit it.

Lastly, there were three students who simply restated the problem; Ned, Stephanie and Steven and one student, Nathan, did not attempt the item.

Summary of student proficiency with constructing roots

Many of the students did not seem to understand what a quotient field is, as most of the students who submitted a response attempted to find roots in either \mathbb{C} or a newly created field. Even the two strongest students, Lynn and Jeff, seemed to be merely copying mathematics that they had seen in their class.

This item afforded very little information about those students who did not immediately recognize the construct of a quotient field. The item exceeded the students' proficiencies.

Making and proving conjectures about polynomials

In order to demonstrate significant levels of proficiency on this item, the students needed to write polynomials with an arbitrary complex root. Given that only a few of the students were able to do so, it was therefore unlikely that they would create a reasonable hypothesis about polynomials and a proof of that conjecture. Only three students gave a conjecture, and two of them offered a complete proof.

The remainder of the responses were not sufficiently developed to allow any insight into the student's abilities to develop and prove a conjecture about polynomials.

Lynn and Jeff are the two students who made and proved a conjecture about polynomials. Both of them provided similar responses, although Lynn's included a small error that seemed typographical. Jeff's response is shown below.

We know that every polynomial of degree 1 is irreducible in $\mathbf{R}[x]$, so we suppose $f(x)$ is irreducible in $\mathbf{R}[x]$ and $\deg(f(x)) \geq 2$. Then, since $f(x)$ is a non-constant polynomial in $\mathbf{C}[x]$ it has complex roots $z = a + iy$ and $\bar{z} = a - iy$. So, by the factor theorem:

$$f(x) = (x - (a + iy))(x - (a - iy))h(x), \text{ for some } h(x) \in \mathbf{C}[x].$$

We let $g(x) = (x - (a + iy))(x - (a - iy)) = x^2 - 2ax + a^2 + y^2$ and so the coefficients of $g(x)$ are real numbers. The Division Algorithm shows that there are polynomials in $q(x), r(x) \in \mathbf{R}[x]$ such that

$$f(x) = g(x)q(x) + r(x), r(x) = 0 \text{ or } \deg(r(x)) < \deg(g(x)).$$

In $\mathbf{C}[x]$, we have $f(x) = g(x)q(x) + 0$. Since $q(x), r(x)$ are also in $\mathbf{C}[x]$, the uniqueness part of the Division Algorithm in $\mathbf{C}[x]$ shows that $q(x) = h(x)$ and $r(x) = 0$.

Thus, $h(x) = q(x) \in \mathbf{R}[x]$. Since $f(x) = g(x)h(x)$ and $f(x)$ is irreducible in $\mathbf{R}[x]$, and $\deg(g(x)) = 2$, $h(x)$ must be a constant...

So, $f(x)$ is a quadratic polynomial... and the largest possible degree of an irreducible polynomial in $\mathbf{R}[x]$ is 2.

Jeff's work demonstrated a very high level of proficiency with proof, and also a high level of proficiency with polynomials. He correctly used the factor theorem to write a polynomial with complex roots, was able to make use of Euclidean Division in a formal argument, and correctly made use of the two possible domains from which the polynomial might draw coefficients. This demonstration of proficiency was unmatched by any of the other students, although Lynn's work was also quite good with only minor errors.

Kenny is the only other student who made significant progress on the item in terms of creating an appropriate and reasonable conjecture. Instead of making use of the hint that I provided, he relied upon a more intuitive understanding of polynomials.

Every polynomial of degree 0 is obviously irreducible. Also we know that every polynomial of degree 1 is irreducible since it cannot be expressed as the product of two polynomials of lesser degree, in this case 0. From the quadratic theorem we know that we do not always have roots in \mathbf{R} . Thus, polynomials of degree two or less are sometimes irreducible. However multiplying a polynomial of degree 1 by a polynomial of degree 2 gives a polynomial of degree 3. By our theorem, this $p(x)$ is reducible. Thus, 3 is the lower bound for all polynomials $p(x) \in \mathbf{R}[x]$ to always be reducible.
 $(x + 1)(x^2 - 2)$.

His argument can be summarized in the following way: We know that there are polynomials of degree 2 that are irreducible, but, even if we have one of those, multiplying it by a linear polynomial gives us a degree three polynomial that is reducible. His response does not include an argument that the greatest degree of an irreducible polynomial is 2 and it does not seem likely, based upon the work that he has shown, that he would be able to craft a proof.

Summary of Demonstrated Proficiency with Polynomials

The final assessment did not give students much opportunity to show what proficiencies they had with polynomials. Instead it produced a rather negative reading which described what they could not do. Because the students were able to make use of their class materials, these items were slightly non-standard while assessing skills that they should have developed during their semester of study.

In terms of allowing students to show their skill with writing polynomials, the context of the complex number system caused difficulty, because the students were not proficient enough with complex arithmetic to write the fourth roots of unity, and

those who attempted the item were very likely to make arithmetic errors. The other unanticipated difficult point for the students was that those who attempted to factor $p(x)$ in $\mathbf{R}[x]$ without having a correct factorization in $\mathbf{C}[x]$ only attempted factorizations with integral or rational coefficients, almost as if they had not read the last part of the question which stated that there is no factorization in $\mathbf{Q}[x]$.

Only three students demonstrated that they had a correct definition of irreducible, and most gave indications that they believe irreducible to be equivalent to has no roots. Given that most of their previous experience was with polynomials of degree two or three, it is understandable that the students would have no basis for distinguishing between the two concepts, because with polynomials of degree two or three the terms are equivalent.

The most surprising result of the assessment about polynomial proficiency was that only two of the students in the study gave any indication that they knew how to construct the roots of an irreducible polynomial by constructing an extension field. Specifically, the students were presented with an extension field created by modding $\mathbf{Q}[x]$ out by an irreducible polynomial and asked to determine the roots. Two students correctly listed $[x]$ as a root in the new field. The great majority of the students gave responses which did not indicate that they recognized this new construct, instead attempting to give roots in $\mathbf{C}[x]$ or some other field. One student attempted to write the roots as rational functions (in an attempt to make sense of the notation of quotient field). In general, the students demonstrated a low level of proficiency with root construction or even quotient fields as a construct that was lower than expected, given that each of the sections spent multiple class meetings

discussing the construct and each had a homework assignment requiring work with the construct.

Lastly, the students were generally unable to make and prove conjectures about the polynomials with real coefficients. The students were generally unable to articulate the greatest degree at which a polynomial might be irreducible over the real numbers. The students did not have sufficient proficiency at writing a polynomial with complex roots. Many of them attempted to write a polynomial with an arbitrary complex root and used notation suggesting they were unable to parse the difference between a polynomial with a complex root and a complex number. This is not surprising given the low level of fluency with complex factoring that the students demonstrated in their responses to the first item in the set.

Structure/substructure and what is inherited

Groups, rings, and fields are three of the fundamental and most studied structures in mathematics; these are the principal objects of study in an abstract algebra class. Often, the study of these three structures defines the organizing principles of the abstract algebra class, and the classes of MWU were no different. Students in the investigative class spent nearly two-thirds of the semester studying rings and fields and one-third of the semester studying groups. Students in the DTP class spent only eight class periods on group theoretic material, and the rest studying rings and fields. In both classes, the students were also repeatedly asked to look for substructures and morphisms between structures.

Understanding groups, rings and fields

The definition of a group

A group is a set with an operation that satisfies four properties, (1) closure, (2) associativity, (3) existence of an identity, and (4) existence of inverses for each element.

One common example of a group is the set of integers under the operation of addition. The integers under addition possess the additional property of commutativity— regardless of the order in which integers are added the sum is the same. This makes the integers under addition a commutative or *abelian* group.

A subgroup is defined in relation to a group. For a given set and operation that form a group, any subset which, under the same operation, satisfies the four group properties is a subgroup of the original group. For example, the even integers form a subgroup of the integers under addition.

The definition of a ring

A ring is a set together with two operations that satisfy eight properties. The set and first operation must form a commutative group. The second operation must satisfy three properties: (1) it must be closed on the given set; (2) it must be associative; and (3) it must distribute over the first operation. One common example of a ring is the set of integers with the operations of addition and multiplication. Given a ring, if there is a subset of the original set that itself forms a ring under the two operations, it is called a sub-ring. Again, the even integers under the operations of addition and multiplication form a sub-ring of the ring of integers.

The definition of a field

A field is a set with two operations that satisfies all of the ring properties along with three further properties: (1) the second operation has an identity, (2) the second operation is commutative; and (3) each element of the set has an inverse under the second operation. The set of rational numbers under the operations of addition and multiplication is a field.

A brief summary of other structures that the students study

Students in both the DTP and Investigative abstract algebra classes studied specific examples of each of the above structures and learned about additional properties that such structures might have. For example, as noted above, a group might be commutative. Many of the further properties give rise to additional structures that are given names which describe their behavior. For example, both classes studied normal subgroups, quotient groups, quotient rings, ideals (a type of sub-ring), kernels and images (which can be either a subgroup or a sub-ring depending on the context).

Idealized Student Proficiency with Structure

After a semester in a typical abstract algebra course, students should be able to define each of the primary algebraic structures and to give multiple examples of each. They should know the structural properties of the classical number systems and be able to explain why those properties hold. Within each of the specific algebraic topics, the students should be able to:

- explore examples and perform computations.
- identify elements of these examples.
- give basic classifications based upon properties.

- identify and create homomorphisms that map from one structure to another, and be able to use isomorphisms to identify identical structures.
- reason from the definitions about properties of elements and the entire structure.
- create new structures by use of cosets.
- evaluate candidates to determine whether they satisfy the appropriate axioms.
- recognize elements (and elements with particular characteristics), make arguments about the elements, and make conjectures and demonstrate the truth of those same conjectures.

Opportunities to Learn

The two abstract algebra classes in this study addressed the definitions of a ring and a group. Both teachers demonstrated a number of proofs showing how given sets and operations satisfied the properties for groups rings, fields, integral domains, and ideals. Similarly, both classes repeatedly engaged in class discussion about isomorphism and homomorphism. Thus, the students could be expected to have mastered those definitions as well. I observed multiple instances in each class of the instructors asking the students to list the properties that a given structure must satisfy. Moreover, multiple students in the DTP class (both those in the study and others) were asked to come to the front of the class and present a proof (either prepared or constructed on the spot) that a structure satisfies a given property. In each class, the

students were also given homework tasks where they were expected to make an argument that a structure either satisfied or failed to satisfy the appropriate properties.

The students also spent time studying sub-groups, sub-rings, and extension fields. As a result, it was reasonable to expect that they could state the definitions of these structures, to determine whether a proposed structure satisfied the appropriate axioms, and to give examples of sub-structures when asked.

The students in both classes saw multiple examples of each type of algebraic structure as both instructors used examples with great frequency in their teaching. Moreover, the students' homework included questions about additional examples of each type of structure.

The students in the DTP class were also expected to develop an understanding of the first isomorphism theorem, and Dr. Hedges showed them how to use that result in discerning relationships between structures. The DTP students were consistently encouraged to look for similarities between structures they knew and the new constructs that Dr. Hedges asked them to consider. In the investigative class the students were repeatedly encouraged to describe groups based upon their subgroup lattice, to study cosets, as well as orders of elements and, in general, to make use of thinking about structure to understand groups.

Assessment of Student Understanding

The students' ability to state the definitions of group, ring, and field, as well as their ability to give examples of any relevant structures were not directly assessed as part of the quiz or exam. They were assessed as part of the interview. Because only six students were interviewed, the findings described in this section are preliminary.

Definitions and examples

During the interview each of the students was asked to give an example of a ring, to describe its properties, and then to give the definition of a ring. They were asked to describe any other types of rings that they knew. Most of the students who were interviewed were asked to state the definition of a group and an isomorphism. When many of the students struggled to give the definitions of particular types of structures, I suggested an example, such as the complex numbers, and asked the students to identify the properties that the example satisfied.

The students were expected to demonstrate their ability to use the definition of a group on the exam. In their work on Item 1a of Set 1, students needed to determine whether a proposed set and operation was a group.

1. Suppose that (S, \bullet) is a group. Let U be the set of units in S .
 - (a) Prove that (U, \bullet) is a group.

The proposed set is a subset of a given structure. The students should have been able to determine that associativity is inherited and then to demonstrate proficiency in verifying the other properties that a group must satisfy.

Determining if a structure is an example

During the interview I asked many of the students to decide whether Z_3 is a subgroup of Z_6 . Specifically, the difficulty with this problem results from students not understanding that the operation in a subgroup must be the same as that in the original group, albeit restricted to a subset of the original group. This difficulty offers a lens that reveals the student's proficiency with groups and subgroups.

Evidence of Student Proficiency

Definitions and examples of groups.

During the interview the students were directly asked to state the definitions of a group. Rebekah and Johnny both gave correct formulations. As stated by Rebekah, “A group is a set of elements that is closed under whatever the operation is, it’s associative, it has inverses and it has identity.” Then she added, “Groups seem to me to be half a ring because it only has to satisfy those things for one operation whereas rings have to satisfy those things for two operations.” This is a reasonable interpretation of the relationship between a group and a ring for a first semester algebra student. She understands that they both must satisfy a collection of properties, and she recognizes that rings have to satisfy more properties because they have two operations. Kenny also gave a correct definition and noted that many groups have a fifth property, specifically, “and it could be abelian, that’s commutative, that’s a fifth property, just like with a ring you have the original eight properties and then you add on the special types.”

Johnny’s understanding seemed to be quite similar. Johnny stated his understanding of the relationship between a group and a ring in his initial statement of the definition.

I look at a group as a little bit less than a ring. I mean it has the four properties that a ring has and you only have to worry about one binary operation.

He then listed the four properties that a group must satisfy. But, then on the exam he and three other students (Ned, Kenny, Stephanie) claimed that the closure of a subset was inherited—raising questions about their ability to make use of the definition of a group. Consider Johnny’s work as exemplary of the four:

A) we want to prove that (U, \bullet) is a group. Since $U \subseteq S$, and since S is a squadron, (U, \bullet) is preserved under the binary operation, is associative and has identity e . This is all by the definition of a squadron. Since U consists of the units of the set S , by definition of a unit, for each $u \in U$, $\exists u^{-1} \in U$ st $uu^{-1} = e$. Therefore, (U, \bullet) is a group.

I believe that by “preserved” Johnny meant closed. Multiple times during the course meetings, students in the investigative class (Johnny included) misused the word “preserved” to mean “closed” and claimed that the set and operation was preserved when the context indicated that “closed” would be the appropriate term. These students correctly identified associativity as inherited from the super structure, but they assumed that too much is inherited in claiming that closure is also inherited by any subset.

When asked the definition of a group James stated that the operation must have, “Closure, non-empty, and inverses.” He omitted associativity in his response, which was consistent with his work on *Item 1a* on the exam as shown:

$e \in G$ and $ex = xe$ so e satisfies the conditions to be a unit in G , so $e \in U$. Thus, U is non-empty. By DEF, a is a unit if $au = e = ua$ for some unit u . So, u is the inverse of a . So, all units have inverses that are units in U . For $a, b \in U$, $ab \in U$ since $(ab)(b^{-1}a^{-1}) = e$ and so ab has an inverse. $(b^{-1}a^{-1})(ab) = e$ also, so ab satisfies the definition of a unit.

He correctly demonstrated all of the necessary properties except associativity. Thus, it seems that his working definition of a group does not depend upon associativity.

Yet, he also stated that he has difficulty with definitions, “I’m not a very good memorizer ... on a lot of our homework and exams I would always have to go back and find the definitions because, well, they’re not complicated definitions but there’s just so many of them.” This suggests that, while James did not know the definitions well, he was quite proficient at using them and did so correctly after looking them up.

When I asked Mark to give the definition of a group, he paused for approximately 10 seconds and then stated, “When I look at groups, it’s like, [pause] oh, wait, that’s abelian. And, you want me to do it without looking in here [indicates text] right?”

I replied, “I want to know if you can.” He replied, “There’s an identity, there’s inverses, see I’ve been dealing with normality and commutativity lately, so those are the sticking out in my head, so, transitivity, is that part of it? Where a times, b times c equals a times b, times c.” Here it is obvious that Mark does not know the definition of a group and it was only after looking in his text that he was able to correctly give the four properties necessary for a structure to be a group. This conclusion was only reinforced by Mark’s work on Item 1 of the exam. He made two fundamental errors: First he did not prove the correct list of properties, and second he tried to discuss division. He did not attempt to show that all elements of U will have an inverse in (U, \bullet) .

Proof: Let x, y, z be units in the squadron S .

S1: $x*y$ will be in S and U because the result will be x, y or z (let z be an arbitrary unit). If x or y is the identity, then the result will be the opposite. If neither is the identity, x or y (being a unit) will “divide” z because they “divide” every element.

This willingness to make use of an undefined operation raised serious questions about Mark’s understanding of groups. Mark’s work suggested that he had very little understanding of groups even after many weeks of study.

Summary

The students’ ability to state the definition of a group during the interview was generally good. The students showed a fairly wide range of proficiencies from Mark

not knowing the definition of a group to Lynn who was able to state and use the definition without attributing any additional properties to the structure.

Definitions and examples of rings

Just as the students displayed different proficiencies in stating and applying appropriate limits to the definition of a group, they displayed different abilities in describing and checking properties of examples for rings.

I began each of the interviews by asking the student to describe an example of a ring. All of the students stated that their canonical ring is the integers, except for Johnny who suggested the real numbers. Because all of the students gave an example with more properties than necessary, I asked them to state the additional properties that their example satisfied. Rebekah, Johnny, and Kenny quickly identified the correct additional properties that their example satisfied. Rebekah stated, “It’s commutative, it’s got identity and doesn’t have zero divisors so that makes it an integral domain.” Similarly, Johnny quickly identified the additional properties that his example satisfied. He stated, “The real numbers have everything, it’s an integral domain, it has, every element has inverses. I think of the real numbers as basically the real thing, everything that a ring can have.”

The other students struggled to state these properties and the type of ring. For example, Mark was not able to give the properties that the integers satisfy that are not part of the ring definition, “As far as properties? [pause, 20 sec] Honestly, I couldn’t tell you right now.” In my questioning of James he never directly responded to this question.

When I asked the students to identify examples of rings with fewer properties, they started to struggle more. Kenny, Lynn, and Rebekah gave nearly identical responses. Rebekah said:

The reals are a field so that gives you inverses. $Z \text{ mod } 6$ has zero divisors so that's just a commutative ring with identity. ...[pause, 10 sec] I'm trying to picture some of the rings that we've dealt with. The even integers doesn't have multiplicative identity, but it is commutative. 2×2 matrices are not commutative but they do have identity... I can't think of anything off the top of my head that's just a ring.

She correctly identified a field, a commutative ring with identity, and a non-commutative ring with identity. She did not seem to have any example of a ring without an identity, either commutative or non-commutative. In general though, she demonstrated a good ability to state examples of different types of rings. Moreover, she gave her response fairly quickly with little prompting from me.

Johnny gave a similar set of responses to my questions and also listed “matrices” as a non-commutative ring. In this regard, Kenny, Rebekah, Lynn, and Johnny demonstrated a good ability to give examples of particular types of structures with the difference being that the first three were able to give a correct example of a field while Johnny was not. He suggested that the rational numbers were “between a ring and an integral domain.”

The remaining two students struggled to identify either an example with specific properties or to identify the properties of their stated examples, and they ended up relying on the book to make progress. Both James and Mark stated that they did not know the definitions and that they relied upon the text to recall properties. The interview bore that out. The following interaction with James illustrates his struggle with definitions and properties without using his text:

TFC: What's your favorite example of a commutative Ring?
 James: I'm pretty sure Z still fits for that. It's a nice one.
 TFC: What more do you need to be an ID?
 James: Shoot, this again goes to the definition thing. I know it has commutativity, and I want to say that it has units. I know there's two or three steps in between and I know commutative ring was one of them. Shoot. I want to say it's units, but I don't think that's right because I don't feel like the one with units... But, I'd have to use a book for that because, well...
 TFC: What do fields have?
 James: What do fields have that integral domains don't?
 TFC: Or, are any of these [indicating examples on the handout] fields?
 James: As we went up, I know that we'd have different sets. I know we went from Z to R , but I think we did go to Q , from Z to Q and Q to R . Shoot. Uhm... [pause, 30 sec] Again, I'm really bad with definitions.
 TFC: Let me give you a hint. Q is a field.
 James: So, if Q is a field, then Z is a ... oh, it's inverse right, to get you back to the identity. I mean 1 goes to 1, but, no other number would go back to 1, which is the multiplicative identity. So, we need, so if Q is a field, in n/m has the inverse m/n which would get you back to 1, so, fields have units, no, inverses. Z might be an integral domain.
 TFC: Is R a field?
 James: Yes, I would think so, because R is the same idea, you can put a number over a number, I mean they don't have to be integers, but if you have r , then you can put 1 over r and that gets you back to 1.

While James did not know the definitions of the structures, he was able to determine at least some of the different properties that the structures have after I told him that the rational numbers are a field. Moreover, he was then also able to correctly assert that the real numbers are a field. It seems that he had good ability to think about the differences between structures. He has good mathematical habits but has not developed the level of content knowledge that he needs in order to correctly give examples of specific types of structures.

Mark did not display the same level of ability to reason about the properties that structures have, although he was quick to identify the rational numbers as having

inverses for all elements. Instead he relied on his text and his ability to find descriptions of each of the types of rings in his text.

TFC: If you go to the rationals, what does that give you the integers don't have?

Mark: It gives you fractions. We never had a ring dealing with rational numbers, it was either integers or complex. As far as rationals, granted, it was, [pause] we never did that, which surprises me now that I look at it or real numbers. We did Z or Z adjoin x . [pause, 10 seconds] More elements.

TFC: What about properties?

Mark: As far as commutative and, uhm... [pause, 15 seconds] I'm taking out my book. I always look in my book for these.

Once Mark opened his text, he was able to start to identify the different properties that the examples have, although he was still making errors. For example, he stated that he did not believe the integers to be a field, "I really don't think that Z is unless it's cyclic, but we don't have cyclic in rings." The integers are cyclic, and the rational numbers are not. Moreover, he has certainly seen a proof of this fact.

Summary

In terms of the content knowledge that the students displayed, there was a substantial difference between the students who knew and could state definitions and examples, and those who could not. Lynn, Rebekah, Johnny, and Kenny were all able to give a number of examples of rings and to discuss the properties that their examples satisfied. They all struggled to name a ring without any additional properties, and all suggested that some collection of matrices would likely satisfy the requirements. In terms of their ability to identify commutativity, identity, and fields, they were quite good and gave a diverse list of examples. On the other hand, James and Mark demonstrated significantly less proficiency, as neither of them were able to indicate the properties a ring might satisfy or to offer examples of rings. They both

stated that they generally make use of their textbook. This difference also seemed to underlie Mark's problems with Item 1a on the exam as he did not list the correct set of properties for proof and made use of "division" as an operation.

While the students generally seemed able to state the definition of a group, they were also very willing to assume too much of a group in proofs. Nearly half of the students either suggested that closure is inherited by a subset or gave the group additional properties (such as commutativity). The fact that they did this during proof, but not when asked directly, seems to suggest that they had difficulty separating what a group is (meaning a structure that satisfies some minimal requirements) from other properties that a group might satisfy.

Determining if a structure is an example

The next type of proficiency that the students were asked to demonstrate was to determine if a given set and operation(s) is an example of some particular algebraic structure. That is, there were asked to determine if it satisfied all of the required properties. In essence, this required the students to list the necessary properties and then to demonstrate some proficiency at verifying them. In order to assess students' proficiency with structure rather than proof, I made use of an interview task that did not ask for proof, but rather asked students to deal with ambiguous language and the lack of a specified operation. The prompt was "Is Z_3 a subgroup of Z_6 ?" This prompt has been used in a number of research studies and the responses of the students in this study were consistent with those reported earlier (Findell, 2000; Brown, et al., 1997). James and Lynn both gave responses which indicated substantial understanding. James's is shown below:

Over addition right? Well, I mean, most of the other problems we've done with Z_6 we've always done adding. So, I guess, I ran into this problem on the test because I was thinking we were talking about regular addition. I guess if we were going to do subgroup, then we'd have to ... I guess I'd just, I guess, it's because in class we always jump to conclusions so I'd say yes. [pause, while writing—1 minute] Oh, I got no. Well, like, Z_3 would be 0, 1, and 2. But, $2+2$ is 4 and so, it's not closed. So, no.

He correctly identified the necessary operation and then noted that the set $\{0, 1, 2\}$ would not be closed if the operation from Z_6 was carried over. This is an ideal response for novice students according to previous research (Findell, 2000). Rebekah began by indicating that she believed Z_3 to be a subgroup of Z_6 . I began probing further.

TFC: Is Z_3 a subgroup of Z_6 ?

Rebekah: [pause, 90 sec] I know Z_3 is a group, and it's elements are a subset of Z_6 . Yeah...

TFC: When you take your two favorite elements in Z_6 , when you put them together, what do you get?

Rebekah: Ok, I got 2 and 3, so if I do multiplication with them, I end up with 6 and that's zero. With addition, that's five.

TFC: If you did 2 elements in Z_3 , how would you put them together?

Rebekah: So, 0 is boring, so we'll do 1 & 2. 1 times 2 is 2 and 1 plus 2 is zero.

TFC: You reduced in both cases, but you reduced by a different number. Is that okay when you move between a group and a subgroup?

Rebekah: Hmm... [pause, 30 sec] I was just thinking about creating an isomorphism, and mapping 6, well, 6 doesn't exist in the other one [draws chart on page] and, well, there's some way that you can make that association there.

It seems that rather than confront her misconceptions, she changed the subject. She never returned to the question of the operation, nor did I force her to revisit her answer to the original question. The other three students, Johnny, Mark, and Kenny all made a common mistake. Each of them was willing to allow Z_3 to be a subgroup of Z_6 even though they understood that they reduced by different values when working in the different sets. Consider Johnny's argument that because Z_3 is a group

and its elements are also elements of Z_6 , then Z_3 is a subgroup of Z_6 . He has not considered the operation at all.

Yes, I would say it is, because Z_3 has an identity, any time you combine an element in Z_3 with another in Z_3 , it's gonna be Z_3 , say you combine 1 with 2 that'll give you zero. And that's Z_3 . So, Z_3 itself is a group, and, yes, cause any element that's in Z_3 is in Z_6 , but I guess that the problem I'm thinking of right now is that if you were to combine an element of Z_3 with an element of Z_6 , I can't think of where you would put it. I was using addition as the operation, and when you add 4 and 4 you like, you'd think that would be 8, but since it's Z_6 you'd go back to 2, so you reduce it because these are all the congruence classes...

TFC: I want to make sure it's not a problem to reduce by a different number?
Johnny: Nope, in Z_3 you look at numbers that are multiples of three and that's the important part. I guess I'd have to think about that more because I don't see where it would be based on the subgroup lattice.

I explicitly asked Kenny if this change of base was acceptable:

TFC: Is it okay to mod out by different things when you move between a group and a subgroup?

Kenny: Yeah. On the surface, yeah, I just want to make sure. Yeah, I would say so.

But, Kenny was also operating on Z_6 under the operation of multiplication which does not form a group. Although each of these students was able to state that a group includes a set and an operation, it seems that none of them has connected the idea that the cosets which make up Z_3 are not the same cosets that make up Z_6 nor is the operation on these sets the same. Yet according to previous work (Findell, 2000) this is the most common response, and thus it is not unexpected even after a semester of study of groups rather than the few weeks that these students had.

Summary

The strongest two students on this item were Lynn and James who correctly responded that Z_3 was not a subgroup of Z_6 . None of the other students correctly asserted that Z_3 cannot be a subgroup of Z_6 . In fact, Rebekah, Mark, Johnny and

Kenny all gave responses which indicated that they did not consider the change in modulo (or the means of reducing) to be a problem. As noted above, this is an expected response for novice students, but it indicates that they had not fully developed their understanding of groups as a set and operation that satisfies given properties.

Summary of Demonstrated Proficiency with Structure

Generally, the students all knew the definition of a group. They were able to give an example of a ring, to identify additional properties that a ring (or group) might satisfy, and to state a variety of examples of rings with a range of properties. They showed a good depth of knowledge here—citing matrices as non-commutative rings, the real numbers and $\mathbf{Z} \bmod p$ (p prime) as fields, the integers as an integral domain, and the even integers as a commutative ring without a multiplicative identity. None of the students was correctly able to identify a ring without any additional properties. Two of the students, Mark and James, stated that they did not know definitions or examples and needed to use their textbooks to respond to any such questions. The students were generally able to make use of their definition of a group in proof.

The students' responses to the prompt "Is Z_3 a subgroup of Z_6 ?" also suggested that they have not yet fully developed their understanding of the sub-group concept. The majority of the students made an expected assertion that Z_3 is a subgroup of Z_6 , meaning that although they recognized that the modulo is different, they did not recognize that the operation must be the same in a group and subgroup. Iaonnone and Nardi (2002) noted that novice students often understand a group as a

set and do not see the operation as integral. Their research gives a good explanation for this tendency of students and helps researchers better describe the development of student understanding of groups.

In general, the students had some proficiency at stating definitions and examples, but their understanding of groups and rings was still rather tenuous and developing as is appropriate after one semester of study. Their ability to state a range of examples with different properties suggested good familiarity with the basic concepts and will provide them a base from which to grow. But it is fairly clear that at the end of one semester they had not yet developed enough understanding to not over-attribute properties to structures, nor had they fixed into their understanding the fact that the set and operation(s) together form a structure.

Proof proficiency

Developing students' proof proficiency is one of the major foci of most abstract algebra courses. Many students arrive in an abstract algebra course with some exposure to proof, such as epsilon-delta arguments in calculus, but they often have not developed significant proficiency. Abstract algebra is often the first course where students are exposed to proof as the primary means of developing the mathematical content of the course. Moreover, at many institutions, this is the first course where students are expected to regularly produce proofs on their own. In the specific context of this study, the students had completed a course on proof writing prior to enrolling in abstract algebra.

Idealized Student Proficiency with Proof

From a logical perspective, proofs in all branches of mathematics have a common underlying logical structure. Beginning from axioms, prior theorems, and definitions, they provide chains of logical inferences leading from the hypothesis to the conclusion of the conjectured result. However, successful application of this overall logical scheme to proving results in specific mathematical topics requires more nuanced understanding of subject-specific concepts, techniques, and reasoning strategies.

In a semester of abstract algebra students are exposed to and expected to develop proficiency with a large number of proof archetypes (Rossi, 1997) as well as a number of unique proofs of named results, such as Lagrange's Theorem, that they are expected to memorize. These proof archetypes include proving properties such as that an operation in a set is closed or assembling logical arguments such as that a function is a homomorphism from one structure to another.

Crafting a proof implies that the students possess a number of other proficiencies as well. For example, deciding whether to construct a proof by direct argument or to assume the opposite of the result and derive a contradiction. In addition, students should be able to take a newly proposed structure and to explore the features of that new structure, ensuring that hypotheses are satisfied and that conclusions are meaningful.

Finally, the students should be able to assess both the completeness and correctness of their arguments. Most often, this arises as proof validation. Students are expected to read their own work critically to ensure that the proof covers all the

possible cases, that the proof gives an argument for all necessary assertions, and that all the statements in the proof are valid and warranted (Weber, 2005).

After a single semester studying abstract algebra, no one expects students to have achieved the same level of proficiency as experts, but most teachers believe that their students should be able to make some progress on the above types of proofs using appropriate knowledge. In general, students will be far more successful at crafting those types of arguments that they attempt more frequently and less successful at crafting those types of arguments that they attempt less frequently (this seems to be the analogue of time-on-task from process-product research).

So what is a reasonable expectation for student proof skills resulting from a semester of abstract algebra? In almost every abstract algebra course, students repeatedly see and create arguments about properties on operations. Thus, I expected that the students would demonstrate proficiency with proving and offering counter-examples related to the group and ring axioms.

The local situation

The situation at Midwestern State University is different from that of many other institutions in that abstract algebra has an introduction to proofs course as a prerequisite. This proofs course was intended to mitigate the overwhelming nature of the abstract algebra class by teaching students many of the proof archetypes before they begin studying the content of algebra. The catalog description of “Mathematical Proofs” reads:

The prime objective of this course is to involve the students in the writing and presenting of mathematical proofs. The topics in this course will include logic,

types of proof, sets, functions, relations, mathematical induction, proofs in an algebraic setting such as divisibility properties of the integers, proofs in an analytic setting such as limits and continuity of functions of one variable.

Additional topics may include elementary cardinal number theory, paradoxes and simple geometric axiom systems.

(MSU Math Department Page, <http://www.MSU.edu/math/coursedesc/>, 4/18/06).

Opportunities to Learn

Students in the DTP class saw a proof of basic properties in nearly every class meeting and during class they were often called upon to supply a part of a property proof or a homomorphism proof. In her syllabus, Dr. Hedge wrote that “proofs form the backbone of this course,” and that she expected students to develop the “ability to conceive of and write up proofs.” To that end, she demonstrated and expected students to take part in demonstrations of proofs in class. Moreover, the questions that students asked about homework and exams indicated that proof was one of the principle components of each of those types of assessment as well. Thus, as is to be expected, the students in the DTP class saw many proof models, took part in proof creation, and wrote a number of proofs of properties of operations and functions.

The students in the investigative class saw and made many fewer proof-arguments as part of their class meetings, but there still were some proofs presented in class. As part of class discussions, the students were regularly asked to state the necessary properties that a specific structure must fulfill, with the implication that they should be able to demonstrate those properties. The syllabus for the class stated that the student “facility with reading and writing proof will be used and extensively

enhanced,” such that proof will be a means “for demonstrating and explicating their understanding.” To that end, the students were expected to read and understand the text, and were given a reading guide (for at least the first two chapters) that asked them to consider proof development. Moreover, on assessments the students were responsible for making proof-based arguments. For example, on the final exam for the course, the students were asked to demonstrate that a given set and operation formed a cyclic group, to show that a given group was metabelian, and to make a series of small arguments that made use of Sylow-p subgroups. In short, while they did not seem to see proof or practice it in class as much as the DTP students, they were still expected to have developed quite high levels of proficiency with proof.

Assessment of Student Understanding

Due to limited time and the constraints of the exam, I decided to only assess the students’ proficiencies on two types of proofs: (1) proofs about properties of operations, with a special emphasis on inverses; and (2) proofs about polynomials, with a special emphasis on factors and roots. (See Appendix B)

Evidence of Student Proficiency

Group and ring axiom proofs

As expected, the students proved to be quite proficient with these proof types. All but one of the students gave a correct response to the quiz question and demonstrated an understanding that a single counterexample is sufficient to show that a property does not hold. Similarly, all of the students did quite well with the proof form on the item which asked them to demonstrate a set with an operation formed a

group. On each of these items some of the students did demonstrate some questionable understanding of both content and logical structures, but on balance they showed they were capable of creating the appropriate proof structure and giving a reasonable proof for most group and ring axioms.

Proofs and counterexamples

On the quiz I proposed a structure and asked the students to determine if the structure was a ring. The students had to determine that the distributive property did not hold, and thus demonstrate that the structure was not a ring. To demonstrate that the distributive property did not hold in the structure, the students needed to show a counterexample. Interestingly, this item also gave the students the opportunity to demonstrate proficiency in writing proofs of the ring axioms. Because the students were not told whether or not the proposed structure was a ring, many of them started by writing proofs for each of the properties until realizing that the distributive property did not hold.

Only 1 student of the 12 incorrectly concluded that R is a ring; the other 11 students correctly stated that R is not a ring and concluded that the distributive property was the problem. The one student who concluded that R is a ring fell victim to overgeneralization and thus, incomplete reasoning. Two students claimed to demonstrate that distribution failed, but did not actually do so. All but one of the students demonstrated that they knew that a single instance is all that is necessary to demonstrate that a property does not hold, and that a single property not holding is sufficient to confirm that a candidate is not a ring. Lynn's response was correct and also the most succinct:

No, R is not a ring, because it doesn't satisfy ~~associativity~~ distributivity. We provide here a counterexample.

$$3*(2+2) = 3*(4) = 4, \text{ but}$$

$$(3*2)+(3*2) = 3+3 = 6.$$

The distributive property does not hold so, R is not a ring.

Although she does not state why she only evaluated whether associativity and distribution hold, Lynn demonstrated that she understood exactly what was required to show that the candidate is not a ring, and she demonstrated that she knew that the property which did not hold was the distribution of the maximum operation over addition.

All of the 11 students who wrote that R is not a ring attempted to show counterexamples to demonstrate that the distributive property did not hold. Due to the number of attempts that the students made to show a counterexample, it is reasonable to suggest that all of the students understood that showing a counterexample is sufficient to demonstrate that a conjecture is not true.

All of the students but one wrote down the eight properties that a ring needs to satisfy and then proceeded to demonstrate that each holds. While working through the eight properties, six of the students [Bob, Ned, Kenny, Aurora, Jeff, Mark] realized that all of the addition properties were inherited from the integers and stated such. For example, Mark, a student in the investigative class, wrote:

Addition Closure \Rightarrow Same as $+$ \checkmark
Associative (Add.) \Rightarrow Same as $+$ \checkmark
Commutative (Add.) \Rightarrow Same as $+$ \checkmark
Zero Element = 0 \checkmark

Most of the students, when they arrived at distribution, wrote something like Ned, “ $3(1+6) = 3(7) = 7 \neq (3)(1) + (3)(6) = 3 + 6 = 9$. Thus $a(b+c) \neq ab + ac$ and (dots) the

distributive axiom fails.” There were two students who claimed that they showed that distribution did not hold, but did not do so correctly. Consider Steven’ work:

$$m \cdot \text{op} \cdot (n+l) = \max(m, n) + \max(m, l) \text{ or } \max(m, n+l)$$

$$(m+n) \cdot \text{op} \cdot l = \max(m, l) + \max(n, l) \text{ or } \max(m+n, l)$$

This property fails because say $m = 1, n = 2$ and $l = 3$
Then $\max(1, 2) + \max(1,3) = 2 + 3 = 5$
Max $(1, 3) + \max(2, 3) = 3 + 3 = 6$.

Similarly, James did not actually demonstrate what he claims that he did.

Consider his work:

$$a \cdot (b+c) = a \cdot (b+c) = (a \cdot b+c)$$

$$(a+b) \cdot c = (a+b) \cdot c = (a+b \cdot c)$$

So, $a = a+b$ or $a = c$
 $B+c = a+b$ or $b+c = 0$
 $B = 0$
Conditional a must be c
Conditional b must be 0
Not always true
So not a ring

It was exactly at this point of generalizing from the properties of the integers under addition that Jeff made his error. Let us now consider the student with the incorrect response.

Suppose, wolog, $a < b < c$, then $a(bc) = ac = c = bc = (ab)c$. So, multiplication is associative in R .
Since multiplication is distributive in Z , it is distributive in R as well.
Therefore, R is a ring.

His assumption regarding the ordering of the elements would actually allow a quick-check of distributivity to seem as if it behaved as appropriate

$[b+c=a(b+c)=ab+ac=b+c]$. He apparently did not realize that his assumption leaves unchecked the case where $a > c$ in which case distributivity fails. That is, he was unable to determine that his reasoning was incomplete in the given situation.

All of the students gave evidence on the quiz that they had good proficiency with the proof archetypes for the group and ring properties. They gave more evidence for this with their work on Item 1a of the exam. On this item the students were given a candidate structure and asked to show that it is a group:

1. Suppose that (S, \bullet) is a squadron. Let U be the set of units in S .
 - (a) Prove that (U, \bullet) is a group.

The first line references a structure (squadron) defined as part of the introduction to the exam. This problem assesses students' ability to give a proof that a set and associated operation form a group. On this item the students needed to demonstrate that they knew the correct proof archetype and could complete an argument for each of the four properties of closure, identity, inverses and associativity.

The students demonstrated a number of levels of proficiency on Item 1a, and definite themes emerged. For example, there were multiple students who demonstrated that they possessed all the proficiencies necessary to craft a proof about the group properties. Lynn is an example of such a student and her response is below.

Since S contains an identity e , and $e * e = e$, e is a unit and U is non-empty and has an identity.
 Suppose $a, b \in U$. Then a, b are units, so $\exists a^{-1}, b^{-1} \in S$ st $aa^{-1} = e, bb^{-1} = e$.
 So, $a^{-1}, b^{-1} \in U$. Then $(ab)(b^{-1}a^{-1}) = abb^{-1}a^{-1} = aea^{-1} = aa^{-1} = e$. So, $ab \in U$, and U is closed.
 Since (S, \bullet) is associative, U inherits this property.
 For $a \in U$, $\exists a^{-1} \in S$ st $aa^{-1} = e$. Then a^{-1} is a unit and $a^{-1} \in U$, so every element of U has an inverse.
 Thus, (U, \bullet) is a group.

Two other students demonstrated a similarly high level of proficiency; Jeff and James. Both Jeff and James were missing a proof of one of the properties, but their omission had no impact on the validity of their proofs. Both submitted proofs that

were correct in details and logically complete. Jeff never explicitly stated that (U, \bullet) is non-empty. But he demonstrated that (U, \bullet) has an identity and thus is non-empty.

James never demonstrated that the operation on the elements of (U, \bullet) must be associative. This seems a larger omission than that of Jeff. When showing that a given structure is a subgroup, associativity is omitted because it is inherited. This is true in this case, and it could be that James recognized this and simply did not mention it.

Six students (Kenny, Ned, Stephanie, Johnny, Rebekah and Nathan) crafted proofs which would have been complete and correct had they not made content-based mistakes (as discussed in the identity strand or the structure strand as appropriate). All of these six students made an error in their demonstration that the operation is closed; typically they simply asserted that the operation was closed. Johnny's response was representative of the mistake, and it also included idiosyncratic language, which may indicate that he had a misconception related to basic facts.

A) We want to prove that (U, \bullet) is a group. Since $U \subseteq S$, and since S is a squadron, (U, \bullet) is preserved under the binary operation, is associative and has identity e . This is all by the definition of a squadron. Since U consists of the units of the set S , by definition of a unit, for each $u \in U$, $\exists u^{-1} \in U$ st $uu^{-1} = e$. Therefore, (U, \bullet) is a group.

When read without interpretation, Johnny's work does not demonstrate that (U, \bullet) is closed. Yet, observation of his classroom sessions indicated that he and other students incorrectly use "preserved" when they meant "closed." Thus, if we assume that he is likely to make the same errors in his written work as he does in his spoken work, he, meant to argue that (U, \bullet) is closed. Assuming that Johnny did mean closed, he has made a content-based error in his proof, which has already been

discussed above. Thus, Johnny should be considered as part of the group that has demonstrated the appropriate proof proficiencies on this item.

The three remaining students all made significant errors that indicated problems with proof-proficiency. For example, Steven incorrectly chose his elements x and y in (U, \bullet) such that $xy=1$. She then claimed that, since $1 \in U$ the set satisfies the closure requirement.

Proof: Since S is a squadron, $\exists x, y \in U \subseteq S$ such that $xy = 1_S$. We have closure:

$$xy = 1 \in U \subseteq S$$

For, $x, y, z \in U \subseteq S$ we have

$$(xy)z = x(yz) \text{ and thus we have associativity.}$$

According to S3, there is an identity element, say e such that $ex = x = xe$, $\exists x \in U \subseteq S$.

Now we need an inverse.

Because x, y are units, we have

$$xy = 1_S \text{ and thus,}$$

every element is an inverse.

Therefore (U, \bullet) is a group.

Steven needed to choose arbitrary elements x and y in the set U and show that the product xy is an element of the set. That is, he set a condition on x and y rather than choosing arbitrary elements; thus, his argument was incorrect. This is the type of argument with which he should have developed significant proficiency during his proofs class. Moreover, he certainly had opportunity to improve his proficiency during his semester of abstract algebra.

Aurora made a number of serious errors. First, she believed that (S, \bullet) is a group and that she needed to demonstrate that (U, \bullet) is a subgroup, although her notation makes this interpretation somewhat unclear. Consider her work:

By definition we know a squadron is a group so $\exists a, b \in G$ st $ab \in G$, and also a and b have an inverse, namely a^{-1} and b^{-1} st $aa^{-1} = e$ and $bb^{-1} = e$ so

a^{-1} and b^{-1} are units, and $a^{-1}, b^{-1} \in G$, but also to H . If $a^{-1}, b^{-1} \in G$, $a^{-1} b^{-1}$ must belong to G . Since $ab \in G$ and $a^{-1} b^{-1} \in G$ $(ab)(a^{-1} b^{-1}) = aa^{-1} bb^{-1} = e$, so $a^{-1} b^{-1}$ must be a unit and $a^{-1} b^{-1} \in H$. For $a^{-1} \in H$, $a^{-1} a = e$, so a is also a unit and belongs to H . Therefore, H is a subgroup of G and is a group.

Even if we accept that she intends to show that (U, \bullet) is a subgroup of (S, \bullet) , her proof is still incomplete. If we ignore her content mistakes, she seems to have demonstrated that each element has an inverse and that (U, \bullet) is closed. Her proof of closure seems complete, although when she demonstrated that the product of two arbitrary elements is a member of the set U , she chose to make use of inverse notation. At no point does Aurora demonstrate that the set U is non-empty, nor does she argue that e is an element of U . Thus, although she has demonstrated some proficiency at showing that certain properties hold, she did not demonstrate that she knows what must be shown in order to write a valid proof that a structure is a subgroup.

The last student, Mark, may have submitted the most problematic response to the item. He made numerous content errors, and he did not demonstrate that he knows what he needs to prove in order to demonstrate that (U, \bullet) is a group or that he has the ability to correctly verify the properties that he has attempted to show. His work:

Proof: Let x, y, z be units in the squadron S .

S1: $x*y$ will be in S and U because the result will be x, y or z (let z be an arbitrary unit). If x or y is the identity, then the result will be the opposite. If neither is the identity, x or y (being a unit) will “divide” z because they “divide” every element.

S2: $(x*y)*z = x*(y*z)$

Following the same steps as above, a unit “divides” every element and x, y , and z are all units.

S3: The identity is always a unit. Therefore it is included in U . Thus, U is a group.

In each of the cases above, Mark has, rather than writing a proof, given a description of the intuitive understanding he sees as necessary to write a proof. In analyzing his attempted proof, it seems that he attempted to show that (U, \bullet) is closed, associative, and that (U, \bullet) has an identity. He has not attempted to show that each element of (U, \bullet) has an inverse. When we examine his attempt to show that (U, \bullet) is closed, we see that he has made errors in terms of both content and proof-proficiency. He attempted to claim that xy must be x , y or z where z is an arbitrary unit and then gave some explanation. In terms of his proof proficiency, because he did not give an argument it is impossible to evaluate his fluency with symbolic argument and, further, his use of non-standard phrasing makes it unclear whether he has mastered the concepts.

Summary of proofs and counterexamples for property arguments

All of the students knew the properties that they needed to check in order to demonstrate that a proposed structure is a ring. They all knew how to check whether the properties were satisfied. Most were able to recognize that they did not need to address the additive properties, because, in this case, they were inherited from the integers. Of the 11 students who determined that R is not a ring, all of them recognized that the distributive property was problematic and attempted to present a counter-example. This implies that they realize that a single counterexample is sufficient to show that a proposed structure is not a ring. This was a case where the students should have been quite proficient with this type of problem, and, in general, they were.

There were two students who believed that they showed a counterexample of the distributive property, but instead showed something else. One of the students was

from each class and thus, it seems that there is not a class effect that needs to be explored. The data from the quiz are fairly consistent and suggest that the students developed proficiency with this type of exercise through practice.

The proof proficiencies that the students demonstrated on the exam question about sub-groups were much more mixed. The great majority of the students knew the properties they needed to demonstrate in order to prove that (U, \bullet) is a group. Except for a few content-based errors, they demonstrated appropriate proof proficiencies in carrying out these arguments. This is a proof type that the students had practiced many times, as reflected in their substantial proficiency.

Polynomial Proofs

There were three types of proofs related to polynomials on which I was able to evaluate the students. The students needed to demonstrate that a given polynomial factored in one domain and was irreducible in another. To show the first of these, the students should have constructed the factors of the given polynomial and then demonstrated that their factorization was correct via polynomial multiplication.

In demonstrating that a polynomial is irreducible, the students should recognize that this is a non-existence proof and choose an argument by contradiction. They should make this choice because almost all non-existence proofs are done by contradiction. A proof should begin with the assumption that some factorization exists, derive facts about this factorization and then conclude that the facts contradict some part of the hypotheses or known facts.

Lastly, I asked the students to craft a conjecture and proof about the existence of irreducible polynomials with real number coefficients. Unfortunately, not many

students made much progress on crafting a conjecture, so their responses did not give much evidence about their proof proficiency.

Let us first consider the proof proficiency that the students demonstrated with factoring a polynomial. In their responses to Item 1 of the second set many of the students gave a plausible factorization of the polynomial $x^4 + 1$. But only one, Aurora, gave a possible factorization and then expanded her factorization. Neither of the other students who gave a correct factorization, Kenny and Lynn, showed that their factorization expanded correctly. The two students who gave incorrect factorizations, Ned and Stephanie, also did not attempt to actually expand their factorizations. Instead, each of these other four students simply asserted that their factorization was correct without any demonstration. For example, consider Ned's assertion, "Notice $(x + 1i)(x - 1i)(x + 1i)(x - 1i) = x^4 + 1$. Thus, p is the product of four first degree polynomials from $\mathbf{C}[x]$."

In fact, Aurora was the only student who gave a factorization and then expanded to show that her work was correct. Besides Aurora though, there was another student who showed exceptional proof-proficiency on Item 1. Jeff did not factor the polynomial, but instead he crafted a proof of the Fundamental Theorem of Algebra from a collection of theorems that are given in his text. Jeff demonstrated that he was capable of writing an analytic proof that $p(x)$ must factor in $\mathbf{C}[x]$ but he did not actually demonstrate a factorization of $p(x)$ in $\mathbf{C}[x]$. His work:

$p(x)$ is a product of four first degree polynomials in $\mathbf{C}[x]$:
By Thm 4.13, $p(x)$ is a product of irreducible polynomials in $\mathbf{C}[x]$. By Corollary 4.26, each of these polynomials is of degree 1. By thm 4.2, the number of these first degree polynomials is equal to the degree of $p(x)$, and thus, $p(x)$ is the product of four first degree polynomials in $\mathbf{C}[x]$.

Jeff seemed to have a high level of proficiency with analytic reasoning about polynomials. All of the hypotheses are met when he made use of a result (which is not common) and he used the results correctly. In effect, he argued that $\mathbf{C}[x]$ is a unique factorization domain and that polynomials will factor completely. He then showed that these two facts are sufficient to demonstrate that p is the product of four first degree polynomials. That is, he seemed to identify the theorem that he needed to prove, and then was able to demonstrate a very marked ability to reason about the ring of polynomials with complex coefficients by building an analytic proof of that theorem.

The remainder of the students made a collection of errors in their responses or did not respond at all because of incomplete knowledge of polynomials or complex arithmetic. These errors meant that the students could or did not progress far enough to then demonstrate any fluency with polynomial proof on this part of Item 1.

Showing that a polynomial is irreducible

The students were unable to show that a polynomial is irreducible in $\mathbf{Q}[x]$; their efforts indicated that many have an incorrect definition of irreducible. But the students also did not seem to use the correct type of argument. Two students gave a complete and correct proof that $p(x)$ is irreducible in $\mathbf{Q}[x]$. A third student wrote the most important fact, but did not give an actual proof. The other students failed to make significant progress on the item or demonstrated that they confused irreducible with has no roots. The students were quite good at showing that the polynomial did not have roots in the field $\mathbf{Q}[x]$.

Both Jeff and Lynn gave a complete proof that $p(x)$ is irreducible in $\mathbf{Q}[x]$ whereas Kenny explained in a sentence why $p(x)$ could not have factors. Jeff's work was very similar to Kenny's in execution and level, but, he added slightly more detail to his result, correctly arguing that for p to factor in $\mathbf{Q}[x]$ it must also factor in $\mathbf{Z}[x]$.

Jeff wrote:

Suppose, to the contrary, that $p(x)$ is reducible in $\mathbf{Q}[x]$ so it can be factored as the product of two non-constant polynomials in $\mathbf{Q}[x]$. If either has degree 1, then $p(x)$ has a root in \mathbf{Q} . But, the rational root test shows $p(x)$ has no roots in \mathbf{Q} (the only possible roots are ± 1 and neither is a root of $p(x)$). Thus, if $p(x)$ is reducible, the only possible factorization is as a product of two quadratics; by thm 4.2. By Thm 4.22, there is such a factorization in $\mathbf{Z}[x]$. Furthermore, $p(x)$ can be factored as a product of monic quadratics in $\mathbf{Z}[x]$, say

$$(x^2 + ax + b)(x^2 + cx + d) = x^4 + 1, \text{ with } a, b, c, d \in \mathbf{Z}.$$

We get $x^4 + (a + c)x^3 + (ac + b + d)x^2 + (bc + ad)x + bd = x^4 + 1$. Equal polynomials have equal coefficients so $a + c = 0$, $ac + b + d = 0$, $bc + ad = 0$, and $bd = 1$. We see that $a = -c$, so $ac + b + d = -c^2 + b + d = 0$ or $c^2 + b + d = 0$. But, $bd = 1$, so either $b = d = 1$ or $b = d = -1$.

Thus, either $c^2 - 1 - 1 = 0$ or $c^2 + 1 + 1 = 0$

$$c^2 = 2 \qquad c^2 = -2$$

There is no integer whose square is 2 or -2, so a factorization of $p(x)$ as a product of quadratics in $\mathbf{Z}[x]$, and hence in $\mathbf{Q}[x]$, is impossible. Thus, $p(x)$ is irreducible in $\mathbf{Q}[x]$.

Lynn and Jeff gave complete and correct responses, both indicating that they understand what it means for a polynomial to be irreducible in a given domain and how to demonstrate this. Additionally, they demonstrated that they are able to write two arbitrary polynomials and to reason generally about polynomials via algebraic manipulation. Moreover, both of these students recognized that the important contradiction to derive is the fact that the square root of two is irrational. Lynn and Jeff both displayed quite high levels of proficiency on this item—they were the only two students to give complete and correct arguments.

In comparing this portion of Jeff's response to that of the first part of Item 1, it is important to note that he has, in fact, derived enough knowledge about the necessary coefficients in the factorization of p into two quadratics to give a factorization in $\mathbf{R}[x]$. He has stated that " $(x^2 + ax + b)(x^2 + cx + d) = x^4 + 1$," and he has determined that:

$$\begin{aligned}a &= -c \\c^2 &= 2 \\b = d &= 1 \text{ or } b = d = -1\end{aligned}$$

However, none of his submitted work provided evidence that he substituted these derived values into the general quadratics that he had written. Given the level of work that Jeff exhibited, it seemed that he would have been capable of such substitution. Yet, on his submission he wrote, " $p(x)$ is the product of two irreducible polynomials in $\mathbf{R}[x]$:" and then wrote nothing below that (this line is on the same page as the above work). It seems that he does not realize that he has all of the necessary information to write these two polynomials. He certainly realized that he left that part of the problem incomplete. His analytic argument that p factors into linear terms and his inability to list inverses on Item 2 in the first problem set may indicate one of two things. It may be that that he does not have great facility with computation. It may be that he simply does not fluidly switch between formal proof and explicit values in the appropriate systems. This is a case that would merit further exploration, as this hypothesized set of proficiencies appears to be quite uncommon.

Kenny, another student who gave a correct response in the first part of this item, seemed to know the kernel of the argument that he needed to give, but he did not supply enough detail to have a correct response.

Coefficients such as $\sqrt{2}$ are not in \mathbf{Q} , so the polynomial is irreducible in \mathbf{Q} .

Here he was referencing his earlier work on the problem, and his statement is correct in that the polynomials that he wrote do not have coefficients from the rational numbers. Had he argued that this is the only possible factorization of p , his response would have been complete. It seems likely that he believes this to be a unique factorization, but it is unlikely that he has learned that $\mathbf{R}[x]$ is a unique factorization domain. As such, his response should be judged incomplete.

Stephanie also gave a logical chain that demonstrated correct logic with regard to proof. She wrote:

Since p is irreducible in $\mathbf{R}[x]$ then it must be irreducible in $\mathbf{Z}[x]$ and therefore irreducible in $\mathbf{Q}[x]$.

Her response correctly claimed that a polynomial which is irreducible in $\mathbf{R}[x]$ is thus irreducible in both $\mathbf{Z}[x]$ and $\mathbf{Q}[x]$. Her only logical problem was that she relied on an incorrect premise, although she believed it to be true based upon her misunderstanding of irreducible. Thus, in terms of proof proficiency, this is a very reasonable demonstration of proficiency.

The rest of the responses to this item were far less complete, but did allow students to display a misconception relating to polynomials. Six students argued that because $p(x)$ has no roots in $\mathbf{Q}[x]$ it is irreducible. For example, James:

$-1 = x^4$. x to an even power, where $x \in \mathbf{R}$ will always be ≥ 0 , same applies to \mathbf{Q} .

[Scratch work] If there is a solution, there's a factor and is reduc. But, $x^4 \neq 1$ in $\mathbf{Q}[x]$. So, it's irreduc in $\mathbf{Q}[x]$.

For each of these five students, this is probably the first time that they had to demonstrate that they understand what it means for a polynomial to be irreducible as

opposed to simply not having roots in the proposed domain. Thus, their lack of proficiency with this type of proof is somewhat understandable. Moreover, the number of non-existence proofs that they wrote in a semester of abstract algebra is probably also relatively small. Thus it is not surprising that the students did not demonstrate high levels of proficiency at actually showing that the given polynomial was irreducible.

On the other hand, almost all of the students who submitted work were quite capable of showing that the polynomial did not have rational roots. Most of the students correctly applied the rational root test to the polynomial and then concluded that it could not have rational roots. This demonstrated that they knew when and how to use the test, they knew all of the hypotheses to fulfill, and they knew how to correctly interpret the results of the test. That is, the students demonstrated some proficiency with polynomial proof, although of a less developed nature than anticipated.

The last item on which a reasonable number of students made any proof-attempt was Item 3 in the second problem set. Students' poor performance on this item was directly related to their inability to access the item. But some of the students who made an attempt also made significant errors that were directly related to their proof proficiencies. On this item, Jeff submitted a response that was nearly ideal, and Lynn's work also exhibited a very high level of proficiency with proof. Jeff's work:

We know that every polynomial of degree 1 is irreducible in $\mathbf{R}[x]$, so we suppose $f(x)$ is irreducible in $\mathbf{R}[x]$ and $\deg(f(x)) \geq 2$. Then, since $f(x)$ is a non-constant polynomial in $\mathbf{C}[x]$ it has complex roots $z = a + iy$ and $\bar{z} = a - iy$. So, by the factor theorem:

$$f(x) = (x - (a + iy))(x - (a - iy)), \text{ for some } h(x) \in \mathbf{C}[x].$$

We let $g(x) = (x - (a + iy))(x - (a - iy)) = x^2 - 2ax + a^2 + y^2$, and so the coefficients of $g(x)$ are real numbers. The Division Algorithm shows that there are polynomials in $q(x), r(x) \in \mathbf{R}[x]$ such that $f(x) = g(x)q(x) + r(x)$, $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.

In $\mathbf{C}[x]$, we have $f(x) = g(x)h(x) + 0$. Since $q(x), r(x)$ are also in $\mathbf{C}[x]$, the uniqueness part of the Division Algorithm in $\mathbf{C}[x]$ shows that $q(x) = h(x)$ and $r(x) = 0$.

Thus, $h(x) = q(x) \in \mathbf{R}[x]$. Since $f(x) = g(x)h(x)$ and $f(x)$ is irreducible in $\mathbf{R}[x]$, and $\deg(g(x)) = 2$, $h(x)$ must be a constant...

So, $f(x)$ is a quadratic polynomial... and the largest possible degree of an irreducible polynomial in $\mathbf{R}[x]$ is 2.

In this instance, Jeff has again demonstrated that he was able to craft an argument that supported his hypothesis. For example, he made use of the necessary results such as the division algorithm in appropriate ways. After he noted that one complex root of a polynomial gives rise to a quadratic with real coefficients, he made use of the division algorithm to demonstrate that h must then be a constant in the ring $\mathbf{R}[x]$. Since he had assumed that f was irreducible, he realized that he had then demonstrated that the power of f is two. This proof is nearly identical to the proof that I gave the students as a solution, including assuming that f is irreducible to start. The reason to assume f is irreducible is to be able to declare, without further argument, that f must be a quadratic. Lynn did not do so, and was then forced to argue that any polynomial which had a degree larger than 2 was reducible. This seems minor, but could be read as indicative of a difference in their levels of proof proficiency. Jeff seems to have slightly more strategic knowledge when it comes to proof-construction than Lynn (Weber, 2001). Yet, in comparison, none of the other students in either class demonstrated nearly their level of proficiency with analytic argument.

Kenny stated that there are irreducible polynomials of degrees 0, 1, and 2 and claimed that the quadratic theorem supports his assertion.

Every polynomial of degree 0 is obviously irreducible. Also we know that every polynomial of degree 1 is irreducible since it cannot be expressed as the product of two polynomials of lesser degree, in this case 0. From the quadratic theorem we know that we do not always have roots in \mathbf{R} . Thus, polynomials of degree two or less are sometimes irreducible. However multiplying a polynomial of degree 1 by a polynomial of degree 2 gives a polynomial of degree 3. By our theorem, this $p(x)$ is reducible. Thus, 3 is the lower bound for all polynomials $p(x) \in \mathbf{R}[x]$ to always be reducible.
 $(x + 1)(x^2 - 2)$.

I interpreted Kenny's work as meaning that the quadratic formula shows that not all quadratic equations have roots in \mathbf{R} . While he made a good beginning of an argument, in an attempt to show that there are irreducible polynomials of degree two, he then encountered difficulty in completing his argument that all polynomials of degree greater than two must be reducible. He may have been drawing on his previous knowledge of functions in claiming that "multiplying a polynomial of degree 1 by a polynomial of degree 2 gives a polynomial of degree 3." He knew that all cubic polynomials have at least one real root and can thus be factored over \mathbf{R} . But, he did not recognize that this was a non-existence argument and thus required a proof by contradiction. This may be because he had not mastered polynomial content, but it seems more likely that he had not yet developed enough fluency in determining situations appropriate for argument by contradiction. This item, and proof generally, is the place where the differences between the two most proficient students, Jeff and Lynn, and Kenny become apparent. Both Jeff and Lynn gave nearly perfect responses to all of the proof items while Kenny struggled, submitting more logically incomplete work and proof attempts in which he failed to identify the correct proof archetype.

Aurora made some progress towards a proof. For example she began by demonstrating that irreducible polynomials with real coefficients of degree two exist.

$$\text{In } \mathbf{C}[x] \quad x^2 = -1$$

$$x = \pm \sqrt{-1}$$

$x = \pm i$ -> Both roots in $\mathbf{C}[x]$.

$$x^2 + 1 = (x - i)(x + i)$$

In general, $(x + iy)(x - iy) = x^2 - i^2 y^2 = x^2 + y^2$, the highest degree in $\mathbf{R}[x]$ is 2.

The work that followed this initial step was less helpful in her attempt to make and demonstrate a claim about polynomials. It seemed that her lack of fluency with complex numbers prevented her from making significant progress on the item, but what she did do is still logically problematic. She demonstrated that multiplying two linear terms with complex conjugates as roots gives rise to a real quadratic. She then claimed to have demonstrated that the greatest degree of an irreducible polynomial in $\mathbf{R}[x]$ is two. She did indicate the highest degree of an irreducible polynomial, but her attempt at proof highlighted her lack of proficiency with polynomials with complex coefficients. Reading past her incorrect knowledge about polynomials, she did note that multiplying complex conjugates always gives rise to real numbers and used this idea to support her claim. Her work is a reasonable use of the material that she has mastered and at least acknowledged the task.

James also made an attempt at the problem and his efforts included stating the existence of irreducible polynomials in $\mathbf{R}[x]$. He showed an irreducible quadratic, but he did not give a greatest degree for irreducibility. His work did not include any mistakes, but it also did not allow any real insight into his proof proficiency other than the fact that he recognized that he should exhibit an irreducible quadratic.

Seven students, (Nathan, Rebekah, Ned, Johnny, Mark, Stephanie , Steven) either made no attempt or their work did not give any indication of their proof proficiencies. These students did not seem to have the level of fluency with polynomials or complex numbers that they needed to demonstrate proof proficiency on this item.

Summary of polynomial proof proficiency

In general, the level of difficulty of this item did not allow seven of the students any meaningful opportunity to demonstrate proof proficiency. The five students who made some progress on this item all correctly noted that there are irreducible polynomials of degree two and most exhibited such a polynomial. This is a good first step for proving a conjecture about the greatest degree of an irreducible polynomial. It was the ability to make more progress on the item that truly distinguished those students with highly developed proof proficiencies from all of the others, even those with quite high levels of content proficiency. In this case Aurora, James and Kenny all made some further attempt on the item. Kenny and Aurora also both asserted that, “multiplying a polynomial of degree 1 by a polynomial of degree 2 gives a polynomial of degree 3.” In this case they were both likely claiming that a cubic polynomial must always have a real root which is useful in the context of the problem.

None of the three students (Aurora, James or Kenny) made use of an argument by contradiction, the type of argument that was most likely to help them make real progress on the item. It was their correct choice of an argument by contradiction that gave Lynn and Jeff the ability to give a complete proof. The most curious aspect of

the item was that Kenny offered a non-existent theorem in support of his claim. He did this in other places on the exam as well such as on Item 2 in the first problem set.

Jeff crafted an argument that is nearly identical to that which I wrote as a solution and Lynn's work showed nearly the same level of proficiency, except that she did not explicitly state her assumption that her polynomial was irreducible. Thus, when she reached the end of her proof, her conclusion of contradiction was not completely warranted.

While it is certain that all but two of the students lacked the content knowledge to make real progress on this item, the proof proficiency that they demonstrated was still somewhat less than expected. The students made unsupported assertions, made assertions that were not logically supported by the justification that they did offer and, in one case, did not even attempt to prove the correct result. Overall, the students' work on this non-traditional problem was really quite ineffective.

Summary of Demonstrated Proof Proficiency

There were three significant types of proficiencies related to proof that this section assessed; (1) the student's ability to create proofs about group and ring properties, (2) the student's ability to write proofs about polynomials, and (3) the student's ability to select the correct proof-archetype for a given proof challenge.

Overall, the students were most proficient at the types of proofs that they practiced most often and demonstrated less proficiency in crafting non-routine proofs. On two different items the students were asked to prove or disprove that a set and operation

- is associative
- distributes over a second operation \circ
- has an identity
- gives rise to inverses for elements (some, all, none)
- is closed

On the quiz the students needed to demonstrate that the distributive property did not hold. All but one of the students correctly did so by exhibiting a counter-example, thus demonstrating that they recognized that a single counter-example is sufficient to prove that a property does not hold. All but one of the students correctly determined that because the maximum operation did not distribute over addition, the proposed structure could not be a ring. The last student seemed to not be paying sufficient attention to his proof, as he demonstrated a very high level of analytic proficiency on all of the other items. As such, it seems that most of the students are capable of demonstrating that a property does not hold, and they understand that if a single property does not hold, then the structure cannot be a group or ring.

Similarly, in their work on the quiz and test item, the students all demonstrated good proficiency, barring difficulties derived from their content knowledge, in proving that group or properties hold in particular structures. Moreover, the majority of the students also demonstrated that they knew which properties they needed to verify in order to show that a set and operation formed a sub-group.

Not surprisingly, the students demonstrated less proficiency with polynomial proofs than with the group and ring property proofs. Most students were quite good

at showing when a polynomial does not have rational roots, and they were also very good at demonstrating the existence of irreducible polynomials with real number coefficients. Almost all of the students were able to determine when a given polynomial does not have rational roots by correctly using the rational root test. Similarly, the majority of the students seemed to know that a polynomial such as $x^2 + 1$ is irreducible over the real numbers and they cited this as evidence that the maximal degree of an irreducible polynomial over the real numbers must be at least two.

However, while the students were quite proficient at demonstrating some facts about polynomials, they struggled to create more advanced or non-routine proofs about polynomials. For example, the students were generally unable to demonstrate that the particular fourth degree polynomial $x^4 + 1$ is irreducible over the rational numbers. Part of this difficulty likely stemmed from an incomplete understanding of the term irreducible, but part of the problem was likely due to the fact that the students did not recognize that a non-existence proof needed a contradiction proof-archetype. That is, they lacked the appropriate strategic knowledge. Similarly, the students struggled to demonstrate that the given polynomial had four linear factors over the complex numbers. Only one of the students actually offered an analytic proof. The other students who made an attempt all exhibited four linear factors (some of which were actually correct).

The most surprising aspect of the student's difficulties with polynomials and polynomial proof was their lack of progress on Item 2. The item asked them to construct a root of an irreducible polynomial by creating an extension field. They

were expected to cite the fact that $[x]$ is a root of the polynomial in the new field. To complete the claim that $[x]$ satisfied the conditions of the problem, the students should have shown that $[x]$ is a root of the given polynomial. Only one of the students both stated and demonstrated that $[x]$ is a root of the polynomial. None of the other students demonstrated that they could show a given value is a root of a polynomial.

Comparing the Demonstrated Proficiencies of the Two Classes

During meetings of the two different classes of abstract algebra, the students saw and helped write very different amounts of proof. The students in the DTP class saw and helped write at least one proof per class period, whereas the students in the investigative class saw one proof a week or less. Because of this difference in classroom experience, it is plausible to expect to see very different types of proof proficiencies on the part of students in the two classes. The current data were not rich enough to provide a means for comparison, with only one item actually assessing student's proficiency with property proof and such a small student sample from each of the two classes

A very preliminary reading of the differences between the classes would note that Lynn and Jeff were both students in the DTP class. They exhibited the most proficiency with proof of all of the students in the study. However, Nathan and Aurora were also students in the DTP class, and they exhibited the least proficiency with proof of the students in the study. It is also interesting to note that almost all of the students in the investigative class exhibited a willingness to create statements that had the correct conclusions, to match hypotheses appropriately to what they had shown, and then to claim that they had written a complete proof. Kenny was the most

prominent example, although Stephanie, Rebekah, and Mark did so as well. For example, the investigative students all attempted to show that their candidates for units in the Gaussian integers were integers. Taken together we can assert that in writing proof, the students demonstrated good proficiency with property-verification arguments, struggled with quantification (as expected), and did not have much opportunity to demonstrate real proficiency with polynomial arguments due to the unfamiliar context.

Conclusion

This chapter addresses the mixed levels of proficiencies that the students demonstrated on the content strands of identity, inverse and unit, polynomials, structure, and proof. On the end-of-semester assessment, none of students gave complete and correct responses to all of the items and the responses that were given showed significant variation in quality. In terms of the proficiency that the students demonstrated it seems that Dr. Kenneth Berg's assertion, "I find that students generally learn what they've been taught" provides a succinct summary. Generally, the students did quite well on items in which both the question type and the context were relatively familiar, and they did quite poorly when both the question type and the context were unfamiliar. Because I needed to design an assessment on which access to class notes and a text would not be an aid, more of the items relied on an unfamiliar context or type of question. For that reason the results were generally lower than they would have been on a more traditional abstract algebra exam.

This use of unfamiliar context and questions meant that, in general, the students' submitted rather incomplete results. Yet, while the responses were

incomplete, there was also significant variation in terms of the types of proficiencies that the students did display. While some students revealed very little proficiency with any of the assessed concepts, two students were extremely capable with almost all of the concepts. For example, there were two students, Rebekah and Nathan, who did not submit any response to any of the last four items giving no basis to describe their proficiency with polynomials. In contrast, Lynn and Jeff submitted bodies of work that demonstrated extremely high levels of proficiency with all of the concepts under study and gave responses to all but one of the items.

Lynn and Jeff were the only students who made meaningful progress on Item 4 and Item 5 in the first problem set (proficiency with proof about inverses and fluency with quantification) and similarly the only students who made meaningful progress with multiple of the items in the second problem set. Both gave a mathematically correct (although incomplete response) to Item 2 when no other students did, gave a complete and both gave a correct response to Item 1 and Item 3 when no other students did. Moreover, during her interview, Lynn was the only student who was able to correctly respond to all of the prompts, including the prompts about group theoretic concepts which she had never studied. Interestingly enough, neither of these two students identified a single unit in the Gaussian integers. It was almost as if they were so focused on the symbolic-proof aspects that they were not able to make use of previous knowledge about complex numbers. Lynn and Jeff were strong students who were in the DTP class and had many opportunities to develop significant proficiency with proof, proficiencies that no other student was able to demonstrate. Thus, while there was significant variation when viewing the student's

responses across the whole of the tests, their responses also showed somewhat different proficiencies when analyzed in light of specific content strands.

Consider the proficiency that the students demonstrated with the concepts of identity, inverses, and unit. The students generally seem to have mastered the notation mathematicians use to denote an identity, an inverse of a given element, and a unit, and the students had a flexible enough proficiency with the formal definition to be able to apply it in a reasonably familiar setting. Moreover, most of the students were able to make appropriate use of that notation in writing the proofs. But, they struggled to manage the notation when an additional quantifier (left or right) was added to the notion of an inverse. It seemed that many of the errors that the students made on the items with the familiar content and context of identity and inverse derived from the cognitive complexity of quantified inverses or resulted from problematic proof-proficiencies rather than actual difficulties with the concept of inverse. For example, the students gave proofs which, when the misstatements were taken as true, were logically complete in terms of the structure, but they made factual misstatements which indicated that they were or could not make use of basic facts about inverses to monitor their proof-production.

In terms of their ability to identify the identity element and units in different structures, as expected, the students were more capable in more familiar structures and less so in less familiar structures. Half of the students were able to give a complete and correct list of units in the Gaussian integers but all of them struggled to justify the completeness of their list of units. There were three students who did include incorrect candidates on their list but, due to lack of proficiency with complex

arithmetic, were unable to rule them out. That is, generally, the students seemed to have the correct understanding of unit, could apply it in the context, and knew how to demonstrate that their candidates were units, but they lack proficiency with arithmetic.

It appears that all of the students could apply the definition of unit in a reasonably familiar setting. Most could identify unit candidates and then knew to use their applied definition to demonstrate the appropriateness of candidate choices. One of the students seemed unable to check his candidates, meaning that he may not know how to apply the definition of unit in this context. It is also worth noting that two of the strongest students in the study, Lynn and Jeff, were the students who did not list any candidates. It seems that the strongest students did not want to hazard a guess without analytic support, whereas the average students were more willing to give partial answers or make informed guesses.

The students were less successful at determining either the identity element or elements with inverses in an unfamiliar setting. They were not very successful at identifying the identity element for the set of functions of a discrete variable. In fact, only four of the students were correctly able to do so. But, it is likely that a significant portion of this difficulty was attributable to their difficulty making sense of the notation and the use of functions as the context of the problem as only six students gave a correct list of the elements of the set. This task required the students to operate in an unfamiliar context and to manage complex functional notation. This combination presented too high a barrier for entrée for the majority of the students to manage. Generally, the students were able to recognize and list elements with

inverses in a familiar structures and were able to apply and manage the notation of identities, inverse and units in proof.

The students did not have as much opportunity to show their proficiency with polynomials as they did with inverse, identity and units. The end-of-course assessment used a fourth-degree polynomial as its principle context. I never saw the students study a fourth degree polynomial during class, only quadratic and cubic polynomials. Moreover, I asked them to consider factorization in the rational and real numbers. Again, these were unfamiliar domains for the students. Most of the students who attempted to factor $p(x)$ in $\mathbf{R}[x]$ without having a correct factorization in $\mathbf{C}[x]$ only attempted factorizations with integral or rational coefficients, almost as if they had not read the last part of the question which stated that there is no factorization in $\mathbf{Q}[x]$. Because factoring with irrational coefficients was not often practiced, it is no surprise that the students did not think to make use of them. But, that also meant that they had no possibility of completing the item without first finding a correct factorization in the complex numbers. Because of the unfamiliar context for the items, the assessment produced a rather limited reading of their understandings.

Only three students demonstrated that they had a correct definition of irreducible; most students gave indications that they believed irreducible to be equivalent to “has no roots.” Given that most of their previous experience was with polynomials of degree two or three, it is understandable that the students had no basis for distinguishing between the two concepts, because with polynomials of degree two

or three the terms are equivalent. It is not until students study polynomials of degree four or more that the more complex definition of irreducible becomes necessary.

The most surprising result of the assessment addressing polynomial proficiency was that only two of the students in the study gave any indication that they knew how to construct the roots of an irreducible polynomial by constructing an extension field. Specifically, the students were presented with an extension field created by modding $\mathbf{Q}[x]$ out by an irreducible polynomial and asked to determine the roots. Only two students correctly listed $[x]$ as a root in the new field. The other students did not give any indication that they understood the goal of the construction of a quotient field. In general, the students demonstrated a very low level of proficiency with root construction or even quotient fields as a construct.

The students did not demonstrate much proficiency at writing a polynomial with complex roots. Instead many of them attempted to write a polynomial with an arbitrary complex root and used notation suggesting they were unable to parse the difference between a polynomial with a complex root and a complex number. This meant that, for the most part, they were not able to demonstrate any proficiency with making and proving conjectures about polynomials. It seemed that, generally, the students had poor fluency with complex numbers meaning that they had difficulty factoring a polynomial in the complex plane or writing a polynomial with arbitrary complex roots.

While the students were not able to demonstrate much proficiency with polynomials, the mid-semester instrument and interview did allow them to demonstrate that they had strong knowledge of and ability to state definitions and to

offer examples of groups and rings. Generally, the students all knew the definition of a group and a ring. They were able to give an example of a ring, to identify additional properties that a ring (or group) might satisfy, and to state a variety of examples of rings with a range of properties. They showed a good depth of knowledge here—citing matrices as non-commutative rings, the real numbers and \mathbf{Z} mod p (p prime) as fields, the integers as an integral domain, and the even integers as a commutative ring without a multiplicative identity. None of the students was correctly able to identify a ring without additional properties.

The majority of the students were quite proficient at determining when properties are inherited from a super-structure to a sub-structure. They demonstrated this both on the quiz and on Item 1a of the exam. Five of the twelve students did overstate what properties could be inherited by a sub-group or sub-ring and over-attributing properties that a group or ring possesses generally. That there were multiple students with a tendency to attribute additional properties to a structure is not surprising, because most of the examples of rings and groups that the students worked with during their semester were commutative. Thus, their experience had taught them that commutativity is often a valid assumption. The most significant problem that the students had was differentiating properties that a group (or ring) must satisfy from those that it might satisfy and managing that distinction in the context of proof.

Four students, Mark, James, Nathan, and Aurora, did not adequately demonstrate knowledge of definitions or examples for rings and groups. During their interview both Mark and James stated that they had to look in their text when working on problems. Nathan's work also demonstrated significant problems with

abstract computation. He attempted to operate in a way that violated rules of uniqueness for both identity and inverses. He exhibited fundamental misunderstandings of structure that indicated minimal concept development around groups.

In general, the students had some proficiency at stating definitions and examples, but their understanding of groups and rings was still rather tenuous and developing as is appropriate after one semester of study. Their ability to state a range of examples with different properties suggested familiarity with the basic concepts and will provide them a base from which to grow. But it is fairly clear that at the end of one semester they had not yet developed enough understanding to not over-attribute properties to structures, nor had they fixed into their understanding the fact that the set and operation(s) together form a structure.

Lastly, in terms of proof, the students were most proficient at the types of proofs that they practiced most often and demonstrated less proficiency in crafting non-routine proofs. On two different items the students were asked to prove or disprove that a set and operation is associative, distributes over a second operation \circ , has an identity, gives rise to inverses for elements (some, all, none), and is closed. It seemed that most of the students were capable of demonstrating that a property does not hold, and they understand that if a single property does not hold, then the structure cannot be a group or ring. Similarly, in their work on the quiz and test items the students all demonstrated proficiency, barring difficulties derived from their content knowledge, in proving that group properties hold in particular structures. Moreover, the majority of the students also demonstrated that they knew which

properties they needed to verify in order to show that a set and operation formed a sub-group.

Not surprisingly, the students demonstrated less proficiency with polynomial proofs than with the group and ring property proofs. Most students were quite good at showing when a polynomial did not have rational roots, and they were also very good at demonstrating the existence of irreducible polynomials with real number coefficients. However, while the students were quite proficient at demonstrating some facts about polynomials, they struggled to create more advanced or non-routine proofs about polynomials. For example, the students were generally unable to demonstrate that the particular fourth degree polynomial $x^4 + 1$ is irreducible over the rational numbers. Part of this difficulty likely stemmed from an incomplete understanding of the term irreducible, but part of the problem was likely due to the fact that the students did not recognize that a non-existence proof needed a contradiction proof-archetype (that is, they lacked the appropriate strategic knowledge).

Across all of the content strands the students repeatedly showed that they were quite proficient at those things they practiced frequently. They generally knew the definitions of different types of structures, could state the definitions using the appropriate symbols, and use the definitions in writing proofs. Moreover, the students seemed to have a ready store of examples of different types of structures and could generally state if and what additional properties their example satisfied. The students showed good fluency working in specific examples of structures, especially verifying that properties hold. They were also very capable of listing units in the ring

of Gaussian integers. Similarly, they also demonstrated good proficiency at using the rational root test to demonstrate that a given polynomial has no roots in the rational numbers. Yet, just as the students were quite proficient at familiar problem types they were less proficient at less practiced problem types. For example, the students demonstrated very little proficiency with demonstrating that a fourth degree polynomial is irreducible or at constructing the root of an irreducible polynomial via a quotient field.

In short, the students displayed a very wide range of proficiency no matter whether we analyzed their proficiency by content strand or by class. Future studies should take this fact into account in instrument design by creating instruments with a low barrier for entry and high ceiling for exit. Despite the tendency to write a single, general, description of the proficiency that a student will develop as a result of an abstract algebra course, it may be more credible to give a description of the range of proficiencies and to determine how students distribute along that range.

CHAPTER 6: SUMMARY, IMPLICATIONS AND DIRECTIONS FOR THE FUTURE

The present study examined teaching and learning in two sections of an upper division abstract algebra course, one consciously using a *DTP* style of instruction and the other intentionally using an investigative approach to instruction. The first primary goal of the study was to describe, compare, and contrast instruction within an abstract algebra course under these two different pedagogical approaches. The second primary goal of the study was to describe the understandings and proof proficiencies that students developed during these offerings.

Classroom observations were conducted in order to collect instructional data. These entailed observing meetings of both classes, making video recordings, and transcribing the classroom discussions. To develop descriptions of the students' mathematical proficiencies I drew upon the classroom observations as well as a brief mid-semester written instrument, a longer end-of-semester written instrument, and a set of interviews that were administered to those students who consented to the interview and testing.

Teaching

While both of the instructors hoped that their students would develop a knowledge base that was deep and connected. The two instructors described their intended classes quite differently. However, they both envisioned a participatory classroom where students were actively engaged with the material, asking and answering questions. Both teachers repeatedly asked the students if they had

questions and answered any questions that were asked. Moreover, during instruction both teachers asked many questions and expected the students to answer them. Lastly, both teachers used many examples in class and expected the students to calculate within these structures, write proofs about these structures, or to use an example to create entirely new structures. Both teachers used these examples as the impetus for study of new mathematical content. Thus, in many ways, instruction as delivered within the two approaches was very similar.

Both instructors used a participatory proof-writing script, emphasizing questioning of the class as a whole. In the DTP class, the observed data as recorded in teaching scripts consisted entirely of factual questions, with students always giving correct responses that were represented by Dr. Hedge. In the investigative class, the observed data recorded in teaching scripts included substantially more open questions with students often offering unexpected or incorrect answers. Dr. Parker responded differently to correct and incorrect answers. When a student gave a correct response, she would repeat the statement; when students offered an incorrect response, Dr. Parker would ask a question which indicated in a thinly-veiled way that the student was incorrect.

Dr. Hedge used an exemplar dialogue to introduce new mathematical structures by connecting them to structures that the students had seen and worked with before. She seemed to do this in order to help the students develop an understanding of the interrelated nature of mathematics. Dr. Parker had a script which was also intended to help the students develop some understanding of the discipline of mathematics. She repeatedly invoked the role of mathematics as a tool

for furthering human understanding, as a way to make meaning out of the patterns they saw and experiences they had during their work computing in specific examples of structures.

The actual teaching of the DTP class and the investigative class did not actually enact the stereotypes. In particular, the DTP class did not actually proceed in a repetitive sequence of Definition-Theorem-Proof-Example, as Dr. Hedge made much more frequent use of examples. For example, in one teaching episode where Dr. Hedge introduced a new concept, she then followed the following pattern: DEETPETPCETPEE (where C is a corollary). While Dr. Hedge did repeat TPE in multiple instances, it is important to note that she used the examples in two different ways. She used an example to illustrate the ideas of a theorem, but then she also used an example to introduce the next generalization, to give the students a context for the next theorem. In this case, the example might be seen as preceding the theorem and the sequence might be better understood as DE-(ETPE). Moreover, the anticipated “sage on stage” approach was clearly not the manner in which Dr. Hedge operated in class. While she controlled the content and direction of the class, she also demanded significant student participation as illustrated within her participatory proof-writing script, her near constant prompt for questions, and by her requirement that each student give a formal proof at the board or overhead during the semester. Each of these practices departed from the expected model for a DTP class.

Dr. Parker’s actions were similarly unexpected. I anticipated a class in which students were often at the board presenting computations, conjectures, and proofs. But, students were more likely to be at the board in Dr. Hedge’s DTP class. In all of

my observations of the investigative class I only observed two class meetings in which students wrote on the board. One of them was when they were to display the results of their explorations with the software package *Exploring Small Groups*. The students who came to the board listed a number of different subgroups. I also observed one student write a proof on the board before the beginning of class that Dr. Parker then referred to during class time. Thus, while much of the daily activity of the class was driven by student's questions (especially about computation) Dr. Parker generally served as the principle author of board work and filtered what was written on the board so that only correct mathematics appeared. The two significant departures that Dr. Parker made from traditional pedagogy was the use of the software *Exploring Small Groups* as a teaching tool and a decrease in the number of proofs presented in class.

Students' Demonstrated Proficiencies

Students demonstrated mixed levels of proficiency on the content strands of identity, inverse and unit, polynomials, structure, and proof. The end-of-semester assessment showed that no student gave complete and correct responses to all of the items. Generally, the students did quite well on items in which both the question type and the context were relatively familiar, and they did quite poorly when both the question type and the context were unfamiliar. Because students had access to class notes and a textbook, most of the assessment items relied on an unfamiliar context or a question that required a transfer of knowledge or an application in a new context.

There was significant variation in terms of the types of proficiencies that the students did display. Some students demonstrated very little proficiency with any of

the concepts failing to respond to four or more items while and two students demonstrated high levels of proficiency with all of the concepts under study, offering responses to all but one of the items.

Students generally seemed to have mastered the notation mathematicians use to denote an identity, an inverse of a given element, and a unit. They had sufficient proficiency with the formal definitions to be able to apply them in familiar settings. Moreover, most of the students were able to make appropriate use of that notation in writing proofs. But they struggled to manage the notation when an additional condition (left or right) was added to the notion of an inverse. The students were less successful when determining either the identity element or elements with inverses in an unfamiliar setting.

All of the students could apply the definition of unit in a familiar setting. Most could identify unit candidates and then use their applied definition to demonstrate the appropriateness of candidate choices. It is worth noting that two of the strongest students in the study, Lynn and Jeff, were the students who did not list any unit candidates. It seems that the strongest students did not want to hazard a guess without analytic support, whereas the other students were more willing to give partial answers or make informed guesses.

The students had limited opportunity to display their proficiency with polynomials. The end-of-course assessment used a fourth-degree polynomial as its principle context. Because of the unfamiliarity of this context, the assessment did not establish what the participants did understand.

Only three students demonstrated a correct understanding of irreducible, as most indicated that irreducible was equivalent to “has no roots.” Given that most of their previous experience was with polynomials of degree two or three, it is understandable that the students would have no basis for distinguishing these two concepts. For polynomials of degree two or three the terms are equivalent.

Only two of the students in the study gave any indication that they knew how to construct the roots of an irreducible polynomial by constructing an extension field. Specifically, the students were presented with an extension field created by modding $\mathbf{Q}[x]$ out by an irreducible polynomial and asked to determine the roots. Two students correctly listed $[x]$ as a root in the new field. In general, the students demonstrated a very low level of proficiency with root construction or even quotient fields as a construct.

While the students were not able to demonstrate much proficiency with polynomials, the mid-semester instrument and interview did allow them to demonstrate their knowledge of and ability to state definitions and examples of groups and rings. In general, the students had proficiency at stating definitions and examples, but their understanding of groups and rings was still rather tenuous and developing. Their ability to state a range of examples with different properties suggested good familiarity with the basic concepts and will provide them a base from which to grow. But, at the end of one semester, the students had not yet developed enough understanding to not over-attribute properties to structures, nor had they established an understanding that a set and operation(s) together form a structure.

Lastly, in terms of proof, the students were most proficient at the types of proofs that they practiced most often and demonstrated less proficiency in crafting non-routine proofs. Most of the students were capable of demonstrating that a property did not hold, and they understood that if a single property does not hold, then the structure cannot be a group or ring. Similarly, in their work assessments, the students all demonstrated proficiency, barring difficulties derived from their content knowledge, when proving that group properties hold in particular structures. Moreover, the majority of the students also demonstrated that they knew which properties needed to be verified in order to show that a set and operation formed a sub-group.

Not surprisingly, the students demonstrated less proficiency with polynomial proofs than with the group and ring property proofs. Most students could show when a polynomial did not have rational roots, and they could demonstrate the existence of irreducible polynomials with real number coefficients. However, they struggled to create more advanced or non-routine proofs about polynomials.

Across all of the content strands, the students repeatedly demonstrated proficiency with those things they practiced frequently, no matter whether their proficiency was assessed by content strand or by class. This analysis did not present a description of the mathematical proficiency that a set of students developed after a semester of abstract algebra, rather it offered a description of the range of proficiencies that a set of students demonstrated. There will always be substantial differences in student proficiencies. Future studies should create instruments with a low barrier for entry and a high ceiling. Similarly, future studies will need to be

mindful of this variation in terms of research goals. Rather than provide a single, general, description of the proficiency that a student will develop with algebra content, it may be informative to describe the range of proficiencies and to describe how students distributed along that range.

Limitations of the Current Study

The current study should be interpreted as an initial exploration of the teaching and learning in two sections of an abstract algebra course. This abstract algebra course was unique as all of the students had completed an introduction to proofs course prior to enrollment in abstract algebra. This prerequisite likely had a substantial impact on expectations instructors had for their students, the classroom activities, and the types of proficiencies that the students were able to demonstrate. Similarly, this was a course that followed a less standard content sequence as the students first studied ring theory and then group theory; the reverse is more common. Finally, while MSU is a doctoral granting university, it is not the flagship campus in mathematics, mathematics education, or teacher training. The enrolled students viewed this school as a regional institution, with a caliber of students substantially different from that at a flagship campus or a selective liberal arts college. The students in both sections were predominately Caucasian, with only two students of color in the DTP section.

While the above are context-based based limitations to the current study, there were also a number of structural limitations. Due to the limited number of student participants, it was not possible to make a true comparison between the mathematical proficiencies that the students in the two classes developed and demonstrated. Of the

12 student participants, 5 were from the DTP class and 7 were from the investigative class. There were only 13 students in the DTP class, so 5 participants from the DTP class represented a significant proportion of the total students enrolled in that section. But, there were 25 students enrolled in the investigative section, and there is no reason to believe that the seven participating students were at all representative of the class as a whole. Because interviews were scheduled at the end of the semester concurrent with final exams, only six students completed the interview, including only one from the DTP class.

Lastly, and more importantly, I envisioned this study would focus on group theory but the two instructors did not progress as rapidly through the material as they had intended. Because of that the classes spent significantly different amounts of time studying group theoretic material with the DTP class spending only the last week of the semester studying groups.

This change was significant for a number of reasons. First, in terms of a comparative study there is an assumption of approximately equal opportunity to learn. In the case of group theoretic material this assumption would have been fundamentally violated. Thus, this study was recast to address student learning and proficiency with ring-theoretic material and the more general content topics such as identity and inverse. This need to change the content focus of the study raised significant problems because almost all of the research into student understanding of abstract algebra content has focused on group theory, and all of the published assessment tools focus on understanding of group theory. This meant that an entirely new assessment instrument had to be crafted. Moreover, because the instrument was

completed by students at home where it would be assumed that they had access to class notes, textbooks, and the internet, the written instruments had to consist of items that would not be compromised by access to these resources. For this reason, the assessment asked students to consider novel algebraic structures. This required substantial use of set-theoretic, quantification, and functional notation which seemed to create a very high unintended barrier for student entrée. Many of the participating students made minimal progress on a number of the items. Since the pool of participants was already small and many of the items yielded very little useful data, the conclusions about student proficiency after a semester of algebra are based on a very small sample indeed. This study should not be read as any attempt to describe what students can do after a semester of algebra but rather as a localized description of what these students demonstrated.

Implications for the Field

Undergraduate abstract algebra instructors have flexibility in course design and pacing. There is no common exit exam, nor is there even a common curriculum for such classes even within the same institution. While at MSU students study rings before groups, this is the less common order for mathematical content, and individual instructors have a great deal of autonomy in selecting the content they will cover. Dr. Hedge, in the DTP class, spent all but a few weeks of the semester covering rings whereas Dr. Parker had essentially finished ring theory with two months left in the semester. In this study these two teachers were both preparing students to enter the same section of second semester abstract algebra. Thus it is reasonable to presume that similar or greater variation exists across abstract algebra offerings at differing

institutions. This level of variation complicates any discussion about what students learn in an introductory abstract algebra class.

The differences in proficiencies and mathematical habits that the two groups of students demonstrated raises significant questions about goals for an introductory abstract algebra course. There has been work in which modules of abstract algebra instruction were designed to help students meet very specific goals such as developing proof proficiency, strategic knowledge or specific types of content knowledge (Weber, in press; Larsen, 2004). Yet, there is no general discussion about the relative importance of any of the goals that might be accomplished in an introductory abstract algebra course. The different teaching scripts that the two instructors employed in this study illustrate two possible sets of goals for an abstract algebra course, and both have mathematical validity. As such, this study may prompt discussion about the relative importance of developing proof proficiency, students' ability to formulate and investigate hypotheses, developing students' content knowledge, students' ability to operate in and analyze novel structures. Without agreement about the relative importance of each of the possible goals of an introductory abstract algebra course, it seems impossible to determine either how to determine what approach is most effective, or, a curricular approach that gives students the greatest chances of success.

There is substantial discussion addressing the importance of recruiting and retaining more mathematics majors. Yet, the different levels of proficiency that the DTP students developed suggests that this goal may be significantly more difficult than previously anticipated, without also reforming the process of undergraduate

education. In the traditional course there were two students who developed the level of proficiency with analytical argument which is necessary for success in graduate study. But the other students from the DTP course in the study performed very poorly on the assessment instrument. If the most important goal of an undergraduate mathematics preparation is to separate the most able students and to develop their analytical skills to a very high level, the DTP course seems to do just that. But, when contrasted with the stated desire to increase the percentage of mathematics majors, it seems that such an approach would imply that a significantly greater number of students must enroll in mathematics courses so that the same winnowing process can occur. The investigative course seems to present at least one alternative, although it implies a compromise in terms of the proficiencies that students develop. The investigative students in the study all seemed to display similar levels of proficiency and could thus serve as a model for increasing the percentage of mathematics majors by not filtering the students so severely.

Taken together the three assessment instruments (the quiz, the end-of-course assessment and the interview) yielded significantly less data and ability to discriminate between the students' proficiencies than hoped. Many of the questions did not allow the weaker students any entrée to the material. Thus their responses contained almost no data from which to draw inferences about their proficiency. The overwhelming majority of the useful data came from only four of the eight items on the end-of-course assessment. The interview protocol significantly over-estimated the students' proficiency and was therefore adapted during the study in favor of one that more directly assessed knowledge of facts. In general this study illustrated the

tensions inherent in instrument design for assessment. In attempting to create a set of assessments that students could complete at home, with the working assumption that they would use their text and notes, the items made use of non-standard structures and also drew heavily upon proficiency with functions of a discrete and finite variable, all written in symbolic form. The students made little progress on those items. Yet, any other choice also seemed likely to yield little useful information as then students would have been able to locate extremely similar items in their text and notes and would have all made excellent progress. While this would have provided more data, it would have created a different problem with analysis, specifically, disentangling the student's proficiency with the content from their ability to adapt work in their text or notes. Moreover, it seems likely that this approach would also have yielded little ability to discriminate between the students in terms of their proficiencies.

Fundamentally, the most important conversation that this study can inform is that about the relationship between goals and pedagogy and assessment. While there is conversation about new classroom activities and different pedagogies, there is no agreement on the goals towards which any classroom activities should be directed, nor are there currently means of assessing student's progress in achieving these goals.

Directions for Future Study

The present study was a first exploration of the teaching and learning in two instances of an introductory abstract algebra course; it is neither comprehensive nor exhaustive. Fundamentally, it was designed to explore what happened in abstract algebra classrooms and what students gained from instruction. Yet, it examined only one section each of DTP and investigative courses. These descriptions of classroom

activities should not be interpreted as either exhaustive of the range of activities that took place or might take place in these two instructional approaches. Moreover, because the study examined only a single version of each type of class, it offered no means to determine characteristics which might be shared by all such courses. The same research questions could quite productively be applied to other instances of traditional and investigative courses in order to give more depth to the initial sketches of classroom activity presented in the current study.

Similarly, the student proficiencies described in this study were preliminary and raised more questions than they answered. As became clear through the course of the study, there are no assessment instruments that enable researchers to assess student proficiency in the manner needed to capture an accurate picture of student abilities. As such, if the field hopes to arrive at some consistent means of assessing students and courses, significant work is needed in this area, both in terms of written assessments and interview protocols. Existing items only assess student's proficiency with limited group theoretic material. There is no pool of items from which to draw to assess student's proficiency with rings and fields and, as needed, non-standard structures. More work is clearly needed in this area.

The present study drew upon a small sample of students from a single institution who had similar mathematical preparations. This study should not be generalized to all introductory abstract algebra students. Thus, additional studies using these research questions could develop a more broad-based understanding of the range of proficiencies that students develop across the undergraduate setting. This work would certainly be helpful in grounding any conversation about

appropriate goals for an introductory abstract algebra course and may indicate that different types of instruction would be necessary in order to meet the needs of different groups of students. Finally, the current study could serve as a base for a research program investigating and describing the development of algebraic thinking at the undergraduate level. This could include the different uses to which students put such thinking after they graduate, including how secondary teachers make use of the algebraic thinking developed during their abstract algebra course in their K-12 classrooms.

8. Please indicate the mathematics classes you have taken, and if you can remember, your approximate grade in each class

Calculus and Analytic Geometry (Calc 1)		Second semester calculus	
Vector calculus (or third semester calculus)		Linear or Matrix Algebra at the 200 level	
Real Analysis		Number Theory	
Introduction to Proof		Introduction to Differential Equations	
Geometry		History of Math	
Other? Please indicate class as well as the grade			

9. Please evaluate the importance of each of the following items in your choice of major

[1 not important at all; 2 of little importance; 3 very important; 4 essential]
(Mark one answer for each possible reason)

- A parent, mentor or friend suggested this area of study
- The person(s) paying for my education insisted that I major in this area
- A parent, mentor, or friend pressured me into this major
- My father, mother, a close family member, or friend has a career in this field
- A good math teacher inspired me to pursue this degree
- A bad math teacher inspired me to pursue this degree
- I enjoy studying mathematics
- I am good at math and science
- Not many people are pursuing this degree
- I want to get a high paying job
- I want a highly respected career
- Mathematics is a useful subject to study
- Other (Please explain) _____

10. Relating to your experience in past mathematics courses, please briefly describe:

a) How frequently you study.

b) The types of activities that you engage in while studying.

c) How frequently you study with others.

11. Have you been asked to be a tutor or grader for any mathematics course? If so, did you accept? Please briefly explain below the circumstances.

12. What, if any, other classes are you taking this semester? Which course do you expect to be the most difficult? The easiest?

13. Have you ever thought of leaving your current major? If so, why?

14. Please evaluate the following statements regarding this class:
 - I wanted to take this class regardless of who taught it because I am interested in abstract algebra
 - I am taking this class because it is required
 - I wanted to take a class from this instructor
 - I know and like the instructor
 - A friend suggested I take this class with this instructor
 - I am friends with at least one person in this class
 - I already have plans to study for this class with someone
 - I think this course will be well taught
 - I expect to have to work hard
 - I expect this course to be relatively easy

APPENDIX D: END-OF-SEMESTER WRITTEN ASSESSMENT

We set the following notation:

\mathbf{N} consists of the positive integers: $\mathbf{N} = \{1, 2, 3, \dots\}$.

\mathbf{Z} consists of all integers: $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$.

\mathbf{Q} consists of all rational: $\mathbf{Q} = \{x \mid x = \frac{n}{m} \text{ for some } n, m \in \mathbf{Z}, (m \neq 0)\}$.

\mathbf{R} consists of all real numbers.

\mathbf{C} consists of all complex numbers: $\mathbf{C} = \{x + iy \mid x \in \mathbf{R}, y \in \mathbf{R}\}$. ($i^2 = -1$)

Definition: A *squadron* (X, \bullet) consists of a set X and a binary operation \bullet satisfying:

S1 $x \bullet y \in X$ for all x and y in X .

S2 $(x \bullet y) \bullet z = x \bullet (y \bullet z)$ for all x, y and z in X .

S3 There is at least one element $e \in X$ such that $e \bullet x = x \bullet e = x$ for all x in X . (It is not necessary that the element with this property be called e .)

We often describe the above properties by saying that \bullet is closed, associative, and has an identity e . We often write xy as an abbreviation for $x \bullet y$. Notice that if \bar{e} satisfies $\bar{e}e = e$ then $\bar{e} = \bar{e}e = e$. In particular, a squadron has only one identity.

Here are two examples:

Ex1 Let $(R, +, \times)$ be a ring having a multiplicative identity. Then (R, \times) is a squadron. The identity e for the squadron is the multiplicative identity for the ring.

Ex2 Let S be any set. The notation $f : S \mapsto S$ means that f is a function whose domain is S and whose range is a subset of S . Following standard notation we let $S^S = \{f \mid f : S \mapsto S\}$. If f and g are in S^S then $f \circ g$ is the function defined by $(f \circ g)(s) = f(g(s))$ for all $s \in S$. With these definitions, (S^S, \circ) is a squadron. The identity for the squadron is the function e defined by $e(s) = s$ for all $s \in S$.

Definition: Let (X, \bullet) be a squadron with identity e and suppose that $x \in X$. An element $y \in X$ is called a *left inverse of x* if $yx = e$. An element $z \in X$ is called a *right inverse of x* if $xz = e$. An element $t \in X$ is called an *inverse of x* if $tx = xt = e$. An element $x \in X$ that has an inverse is called a *unit*.

All of the above can be taken as granted. Now we set some tasks for you.

Problem Set A:

1. Suppose that (S, \bullet) is a squadron. Let U be the set of units in S .
 - (a) Prove that (U, \bullet) is a group.
 - (b) Let $L = \{x \mid x \text{ has a left inverse}\}$ and $R = \{x \mid x \text{ has a right inverse}\}$. Prove that $L \cap R = U$.
2. Let $(\mathbf{C}, +, \times)$ be the field of complex numbers. Let $\mathcal{G} = \{n + im \mid n \in \mathbf{Z}, m \in \mathbf{Z}\}$. The elements of \mathcal{G} are called Gaussian integers. The Gaussian integers, with the usual operations of $+$ and \times , form a commutative ring with identity $1 = 1 + 0i$. As noted in the first example above, (\mathcal{G}, \times) is a squadron. Find the units of (\mathcal{G}, \times) and demonstrate that your list is complete.
3. Let $S = \{a, b, c\}$. List all of the elements of S^S . (Notice that there are 3^3 of them, which suggests the reason for the notation S^S .) Determine which elements of (S^S, \circ) have inverses.
4. Suppose that S is a finite set. Is it true for every x in the squadron (S^S, \circ) that a left inverse is necessarily a right inverse? Explain your answer.
5. Let $S = \mathbf{N}$. Find an element f of $(\mathbf{N}^{\mathbf{N}}, \circ)$ that has a left inverse but not a right inverse, and an element g that has a right inverse but not a left inverse. Using standard function terminology, such as injective, bijective and surjective, determine when an $f \in \mathbf{N}^{\mathbf{N}}$ has a left inverse, a right inverse, and an inverse. Give a proof for your results for the left inverse. The following construction may be of use to you: Suppose that $f : \mathbf{N} \mapsto \mathbf{N}$. For each y in the range of f , let $g(y)$ be the smallest integer in the set $\{n \in \mathbf{N} \mid f(n) = y\}$.

The next set of problems will deal with irreducible polynomials. Let F be a field and let $F[x]$ denote the polynomials in the variable x with coefficients in F . If F_1 is a subfield of F_2 then a polynomial p in $F_1[x]$ is also in $F_2[x]$. A polynomial $p \in F[x]$ is said to be irreducible in $F[x]$ if it cannot be written as the product of two polynomials in $F[x]$, each of lesser degree.

Problem Set B:

1. Let $p(x) = x^4 + 1$. Prove that p is the product of four first degree polynomials from $\mathbf{C}[x]$, that p is the product of two irreducible polynomials in $\mathbf{R}[x]$, and that p is irreducible in $\mathbf{Q}[x]$.
2. Let $\bar{1}$ be the multiplicative identity and $\bar{0}$ be the additive identity in the field $E = \mathbf{Q}[x]/p[x]$, with $p(x) = x^4 + 1$. Find all elements t of E such that $t^4 + \bar{1} = \bar{0}$.
3. The fundamental theorem of algebra (this theorem was proved by Gauss as a teenager and we will accept it as true for the purposes of this problem) asserts that every polynomial in $\mathbf{C}[x]$ of degree at least one has a root in \mathbf{C} . This is equivalent, via the factor theorem, to saying that an irreducible polynomial $p \in \mathbf{C}[x]$ has degree 1 or less. In the same spirit, the fundamental theorem of algebra limits the degree of an irreducible polynomial $p \in \mathbf{R}[x]$. Discover the largest possible degree of an irreducible polynomial $p \in \mathbf{R}[x]$, and prove that you are correct. Suggestion: An irreducible polynomial $p \in \mathbf{R}[x]$ of degree greater than one can have no real roots (why?). If $z = x + iy$ is a complex root of a polynomial $p \in \mathbf{R}[x]$ then the conjugate $\bar{z} = x - iy$ is also a root. Make use of these roots to construct a polynomial $q \in \mathbf{R}[x]$ that divides p .

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