ABSTRACT

Title of Dissertation:	AN INQUIRY INTO RELATIONSHIPS WITH MATHEMATICS: HOW IDENTITIES AND PERSONAL WAYS OF KNOWING MEDIATE AND RESPOND TO MATHEMATICS CONTENT EXPERIENCES
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The knowledge-base of teacher education contains substantial information and recommendations for effective professional development (both in- and pre-service), such as, for mathematics teacher education, a strong content focus grounded in the mathematics teachers will address within the curriculum. However these programs are only half the equation. The participants in any professional development are an unknown. We have little understanding of why and how different participants value different experiences. Because of the current challenges faced by middle school mathematics teachers, this inquiry uses a new research-based in-service master's program in middle school mathematics as a context to explore this unknown.

Challenging culturally embedded views of what it means to know mathematics, this research began with the assumption that, in part, what teachers know and how content is experienced are consequences of teachers' dynamic relationships with the discipline. This relationship represents the interaction, around the content, of identity formation and the personal process of coming to know mathematics. This exploration draws on Wenger's

(1998) constructs of identity—in terms of historical experiences/perspectives and participation in the school mathematics community of practice—and Handa's (2006) constructs of coming to know—as the processes of developing propositional knowledge, developing intimate understanding, or their combination—to develop rich, complex stories around two elementary-certified middle school teachers' relationships with mathematics.

The starting assumption behind this research was supported as identities and ways of knowing proved invaluable in understanding the growth that occurred, and the difficulties these teachers encountered, in the doing of mathematics. These relationships also evolved in response to their experiences. As courses were experienced and valued according to these two teachers' relationships rather than by program designers' intended purposes, this suggests that to facilitate growth, programs will benefit from incorporating and promoting explicit opportunities for each teacher to understand, respect, and critically examine her own relationship with mathematics. Supporting teachers as they engage in this challenging work also seems critical.

AN INQUIRY INTO RELATIONSHIPS WITH MATHEMATICS:
HOW IDENTITIES AND PERSONAL WAYS OF KNOWING
MEDIATE AND RESPOND TO MATHEMATICS
CONTENT EXPERIENCES

By

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Dissertation submitted to the Faculty of the Graduate School of the
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Dedication

This work is dedicated to the memory of my parents,

Eloise McCreery Murphy
and
F. Dean Badertscher.

It meant a great deal to both of them that I was working towards my doctorate,
and I am grateful they both saw me start the program.
I only wish that they were here to see me complete the work.
I share this accomplishment with them.

This work is also dedicated to my daughter

Cadence Arwen Badertscher Summers.

This small person changed my life just as I started this program.
She rode the ups and downs with me, smiled and hugged me whenever I was down,
weathered better then I could have hoped the times when Mommy was working,
and whose indomitable and loving spirit helped me to always see
what was most important. I hope only that
I and my work will serve as invaluable examples for her.

Lastly this work is dedicated to all the women in my family
who have made their own education and the education
of others the focus of their lives.
Their kindness, generosity of spirit,
limitless ability to love,
determination and quiet strength have been
a constant support and guide.

Acknowledgements

I am grateful to my husband Alec who really made this adventure possible through his support and time. He altered his own plans so that I could complete my doctoral work. It is only fitting that now, as I a finish, he embarks on his own doctoral degree in music.

I would not be the lover of mathematics that I am without having known and been a student of Elizabeth George Bremigan, Jim Fey and Sarah Sword. These three have not only shaped my enjoyment of mathematics, but have also shaped how I have come to view the teaching of mathematics. I will work to live up to their examples.

I will forever be grateful to Jeremy Price for his thoughtful guidance as I developed as a researcher. He challenged me to think hard about using research to empower and give voice to those that I work with, as well as about envisioning what educational research could be, not just what it is.

I want to thank all the teachers in the middle school mathematics program within which this research was based. This work is as much mine as theirs. Their willingness to participate, candor in discussing their experiences, and their engagement in mathematics are all greatly appreciated. I learned a tremendous amount from them, and I hope through this work I have given voice to their perspectives and experience.

I am grateful to all my fellow students and friends and others who supported me and worried with me, and cried with me—my dissertation meetings with Grace and Dara; my distressed, yet entertaining, phone conversations with Anne Marie; seminar meetings with Christy, Carolina, and Geoff; conversations with Whitney; and all the warmth and friendship offered by Sophie and her family at *The Daily Grind*.

Last but not least, I want to thank Dan Chazan. It is hard to put into words all he has done for me and how much I have enjoyed and learned while working with him. Whether in seminars, classes, on the dissertation or just in the multitude of conversations, your example will stay with me always. I hope to continue our conversations throughout our careers.

Thank you all!

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NCTM-   National Council of Teachers of Mathematics

NCLB-   No Child Left Behind

CBMS-   Conference Board of Mathematical Sciences

HQT-   High Quality Teacher

PSTM-   Professional Standards of Teaching Mathematics

PPT-   Pure Probability Theory

APT-   Applied Probability Theory

APBL-   Applied Probability of the Bottom Line

Motivating the Inquiry and the Research Questions

Overture

Progressive educational reform has been a small yet ever-present current in schooling in the United States (Cuban, 1993). As the discipline has grown and developed during the last twenty years, mathematics education in particular has been pushing on the boundaries of this ongoing movement. A particularly challenging issue with progressive mathematics education reform is that teachers are being asked to create for students experiences those teachers have never had themselves. Naturally a critical question has emerged as many mathematics teacher educators, math educators, mathematicians, professional developers, and researchers alike have been working to make such progressive mathematics learning environments a reality: how do we best prepare teachers to teach a mathematics radically different from that which emphasized drill, practice, and memorization of isolated facts and algorithms? Certain perspectives guided this work. First, regardless of the level they will teach, prospective (and current) teachers do not come to teacher education programs knowing all the mathematics they need to know (Cuoco, 2001; Conference Board of Mathematical Sciences—CBMS, 2001). Second, since reform challenges the culturally embedded views of mathematics, teachers potentially need opportunities to unlearn old views (Ball, 1996). Third, teachers need to explore alternative perspectives of mathematics, so they can justify their activities as mathematical (Hersh, 1979; National Council of Teachers of Mathematics—NCTM, 1991; CBMS, 2001). Finally, substantially different subject matter knowledge is required, not just for their own understanding, but also to enable teachers to recognize connections or to listen flexibly to students, hearing the mathematics and ideas within, and to see

where the student might be heading (Ball & Bass, 2000). Unfortunately teacher education has been ineffective in meeting these needs (Ball & Cohen, 1999).

To shape the development of teacher education, NCTM published the *Professional Standards for Teaching Mathematics—PSTM* (1991). In this document, several of these new stances were espoused. First, *knowing mathematics* was broadened to include discourse and mathematical perspectives in addition to content. In terms of these perspectives, this meant considering more philosophical, historical and social ways of knowing mathematics, particularly the nature of mathematics, how technology impacts the nature of mathematics, and the role of culture in the development of mathematics. Second, a more constructivist or situated perspective of learning was embraced that was coherent with the progressive mathematics education movement more broadly; "Teachers need opportunities to construct mathematics for themselves and not just experience the record of others' construction;" it is a human activity, "constructed through exploration and investigation" (p. 133).

Even with these professional standards, as well as growing research in mathematics teacher education and mathematics education, there were few concrete teacher education policies that attempted to lay out what was required by these changing needs. In 2001, CBMS released *The Mathematics Education of Teachers.* This presented in great detail recommendations for the preparation of mathematics teachers in the K-12 system. More content was recommended at every level, and the authors asserted that content specialists should teach all students in grades 5 and above. This document made general recommendations, such as focusing on the development of mathematical habits of mind; content recommendations by level, such as focusing on *Algebra and Functions*,

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Number and Operations, *Measurement and Geometry*, and *Data Analysis, Statistics and Probability* for middle school teachers; and recommendations around program structure, such as detailing that there should be a capstone sequence for high school teachers that addressed not only each of the previously learned content areas but also focused on historical and common conceptions of the material. The authors further described the types of experiences useful for this type of learning: collaborating, investigating, and communicating to solve problems, thereby supporting the work of NCTM. In these recommendations, learning content through active construction is essential. Experience with mathematical ways of thinking also is stressed. Additional recommendations include examining mathematics from a variety of perspectives, considering the nature of mathematics (which reflects PSTM), emphasizing connections and developing a course to connect school mathematics with undergraduate mathematics coursework. This document emphasized that the goal was not just preparation; rather "college mathematics courses should be designed to prepare prospective teachers for life-long learning of mathematics, rather than to teach them all they will need to know in order to teach mathematics well" (CBMS, 2001, ¶ 12 of chapter 1). Of course there is no way to teach a teacher everything she will need to know. Even if it were possible, as the landscape changes, what teachers need to know changes; teachers need the tools and dispositions to meet this type of change head-on. Inquiry in mathematics is one avenue to potentially support these needs. As a tool for learning mathematics as in the discipline, it facilitates agency and life-long learning as well as understanding (Lampert, 1993; Cuoco, 2001; Davis, 1997). Teachers must have plentiful opportunities to investigate their own

mathematical questions, make their own connections, develop their own knowledge, and reflect on this process.

There have been several more recent shifts in the landscape of schooling that further influence what is needed in the preparation of teachers. At the middle school level, one of these changes relates to content. At least two years (now with murmurs of a third) of typical high school mathematics content, Algebra I/Data Analysis and Geometry, have been moving to the province of middle schools, and not just for honors students. As middle school teachers are generally elementary-certified and until recently were not expected to teach this content, their typically limited content preparation has made successful and effective teaching of these subjects difficult. They have had to learn the material while teaching, without being prepared as life-long learners who learn in and from their practice. This of course bolsters the CBMS recommendations that from fifth grade on, mathematics should be taught by content specialists, and that teachers should be prepared as life-long learners.

Another dramatic change in the landscape occurred with the 2001 passing of *No Child Left Behind* (NCLB). Schools and teachers are now to be held accountable for the learning of each child (though many reform advocates would argue that preparing teachers to teach mathematics from a progressive perspective is the best way to ensure meeting all students' needs), through the use of standardized tests; moreover, schools are required to have high-quality teachers[1] (HQT) in every room. Thus while middle school mathematics teachers are being asked to teach content for which they have not been

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[1] A high-quality teacher is one that holds a bachelor's degree from an accredited institution of higher education, holds valid teaching certifications, and satisfies more particular requirements as defined by teaching level and time hired. Such requirements might include completing an academic area major for pre-service teachers or a graduate degree for in-service teachers. Source: www.marylandpublicschools.org/MSDE/programs/esea/docs/TQ_Regulations/general.htm

prepared, they are simultaneously being held increasingly accountable for student learning of that content. This is creating a dire situation in the preparation of both pre-service and in-service teachers.

Not surprisingly, we are in the middle of an unprecedented call for program development aimed to better prepare teachers to teach through drawing on research-based recommendations for effective professional development, while simultaneously helping them to achieve HQT status as described in NCLB. Regarding effective professional development, research-based recommendations include program coherence, a strong content focus, teachers going through a program together as a cohort in order to establish a community of practice, and that programs should be of long duration (Garet, Porter, Desimone, Birman, & Yoon, 2001; Little, 1993). One local school district has partnered with a state university's mathematics and mathematics education departments[2] to develop just such a program to prepare highly qualified teachers in middle school mathematics. The design of this master's program in middle school mathematics sought to explicitly incorporate much of what has been learned about effective professional with many of CBMS' pre-service recommendations. In addition to the three-year duration, the cohort structure, and the emphasis on an inquiry approach to teaching through activities such lesson study and a culminating action research project, this program provides varied, extensive content experiences (with an emphasis on three of the four middle school content foci recommended by CBMS), explicit courses to connect these content foci with practical issues of pedagogy, and opportunities to explore mathematical questions through inquiry. Though comprehensively designed to respond to a multitude of needs,

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[2] Such partnerships are a keystone of the CBMS recommendations.

the unknown in this situation is of course the group of teachers who would participate. How would they experience this program? Because of the urgency of the middle school mathematics education situation in particular, compounded at a minimum by the joint pressures of NCLB and shifting content, using this setting to explore and understand how teachers engage with such progressive content experiences and how these mediate continued learning seems to offer a tremendous learning opportunity that can inform mathematics teacher education. It is in no small part for these reasons that I have chosen this as the setting in which to explore a group of teachers' engagement with and responses to the varied content experiences.

Architecture of an Exploration

In this dissertation, I use this program as a context to explore what roles teachers' mathematical engagement has in what teachers know of mathematics and how they teach it. In so doing, I hope to both challenge some traditional perspectives that have shaped our understanding, while also helping to build on our understandings of what should be considered to create similar experiences more broadly for teachers and students of mathematics. This exploration is organized in three main sections: an initial section that sets up and motivates the purposes of this inquiry that I have undertaken, an exploratory section providing an in-depth examination of two teachers, and a discussion section focused on what has been learned from these cases and how this might help us shape continued explorations around the teaching of mathematics and teacher education. The word inquiry will figure prominently in this dissertation, and it is a word that rather than define, I choose to demonstrate on two main levels. First I consider this whole work as an inquiry into knowing mathematics. Second, regarding mathematical inquiry specifically, I

will demonstrate what I mean by inquiry through rich descriptions of a variety of mathematical inquiry opportunities in which groups of teachers and individual teachers engaged. Demonstration rather than definition is essential in this exploration as there are many different interpretations of the activity in this document; providing an image will give any reader something specific to hold onto rather than an abstract notion through which to consider alternative perspectives. Certainly my own perception of what inquiry is did not often coincide with how those I worked with interpreted it, yet we all engaged in roughly the same type of activities.

In the first section, my main focus is on motivating the fundamental questions at the focus of this inquiry. In order to do so, I follow a more organic structure. I begin with an autobiographical account of my own mathematics learning and my journey towards wanting to explore relationships and identity. This will in part give the reader a better sense of what I bring to this work. But more importantly, as I will demonstrate that understanding teachers' histories is crucial to understanding what they know and draw on in their work, I must highlight this for my own work. It would be inconsistent to not acknowledge the role these have played in what I bring to this research. I do not believe that I would have come to a point where I see teachers' histories as important to shaping their work if I did not feel that my own history was instrumental in shaping mine.

Following the autobiographical account which will help bring to the fore my motivation to engage in mathematical work with a group of teachers, I describe the first inquiry experience that I organized. From the description of this experience, you as a reader will come to understand how I see inquiry and the issues that were important to me in the designing of inquiry opportunities. It also will serve to highlight the varying

degrees to which this set of teachers re-interpreted what was meant by inquiry. These sometimes divergent interpretations of and reactions to inquiry provide the opportunity to begin to shape my research questions at the heart of this study through asking specific questions related to this inquiry experience. As standard theoretical perspectives did not feel satisfactory in trying to make sense of what took place in this inquiry experience, I will share my concurrent inquiries regarding knowledge, which culminates in a description of the theoretical perspectives that frame this inquiry. However, further literature and descriptions of methods will be interwoven into the exploration at appropriate times. This will allow the reader to have access to the literature that shaped the progress of this inquiry in a natural and more coherent manner—when it is useful to make sense of the process, the data, or the discussion. I will conclude the first section by stating more formally the research questions on which this exploration is based.

A Select Personal Autobiography of Mathematics—The Bermuda Triangle!

> Jay: *What do they call that ... when everything intersects?*
> Sam: *The Bermuda Triangle.*
>   —*Sleepless in Seattle*, 1993, Sony Pictures

While I did not fully realize this until years later, I had the luxury from very early on of having mathematics teachers who, while they may have seemingly taught traditionally, encouraged creative mathematical thought. In fact, I only recall two lasting negative experiences in my K-12 experience. In third grade, I absolutely loathed the timed multiplication tests, because, being a precise person, I always checked my answers, and thus never finished. This removed me from consideration for mathematical enrichment (gratefully my mom had enough spit and to have me pulled out finally, though there were only 2 more sessions left). I also had trouble in 8th grade Algebra I. No

matter what I did, I could only ever get a B on the tests. Strangely, I never felt confused or lost, but I certainly felt unsuccessful. Overall though, I loved math and I loved science. In Algebra II, I remember going home to work on homework with square roots, and just from the understanding I had developed in class, I invented my own way of finding square roots that just made sense to me based on what I knew a square root was. I was able to calculate an irrational square root out about 10 to 15 decimal places correctly before stopping. My teacher had me demonstrate my method in class the next day and explain how I had thought of it. In my senior year, I took AP Calculus and Physics simultaneously. I was always actively connecting the two, and consequently they made sense. Our teacher knew us so well (I had her for Algebra II and Pre-Calculus as well) that she was able to present material one way and then alter her presentation for others who would make sense of it better a different way. I remember hearing her say, "It will make more sense to you this way." She also encouraged my investigations into fractals, allowing me to borrow and read James Gleick's *Chaos: Making a New Science* for the first time in 1989. Maybe it will surprise no one that this teacher went on herself to get a doctorate in mathematics education. I want to say that math just clicked, but I know even during lectures I was always making connections and trying to think about the math in different ways, so while I may have sat passively, I was still actively involved in the mathematics, and my questions were always welcomed. Then I went to college.

I thought I would be an astrophysicist. It sounded wonderful, combining my love of math and science. I even worked at an observatory before starting college, working to map photographic plates of the sky to search for black holes and quasars. This was to be my life. "My life" only lasted one semester. Math and physics changed. I loved physics

lectures because we had interesting demonstrations, but class and problem sets seemed to floor me. I just did not seem to be able to make any sense of math, and yet it was only the next semester of Calculus and I had received a 5 on the AP! Maybe that had been a mistake. I hated going to class, and so I didn't always. Osmosis had stopped working. Certainly I tried to memorize, but that did not work so well either. I remember struggling before the final to try to teach myself the whole textbook, since I had not successfully taught myself previously. How I escaped with a B I don't know since I am pretty sure I got a 47/100 on the final and was told it was a C. The grade did not matter though. I knew I was not good at math. I was scared of math and science.

What I still don't understand is why 3 years later, I missed math (frankly I would have thought I would have missed the science first). I unfortunately did not have any time to take it—that may have been a good thing. So I decided that I would find a way to get back into math when I was working and that I would become a teacher. Teaching always had been high on my list of professions, up there with astrophysicist and restaurateur; all the women on my mom's side of the family over the previous three generations had been teachers of various sorts, though none of mathematics or science (after all, these were not considered feminine subjects). So I found a way to work at a state university beginning in the fall of 1997, and I started math classes in the spring of 1998. I took Linear Algebra and Differential Equations (seven years exactly after receiving that B in Calculus). I basically still had to teach myself, because lecture was not a place of connections and creativity, but I missed math, and no one was going to stop me from learning it. I received As in both classes, and I knew with a lot of effort, this was something I could do—I wanted to do. This confidence was a product of desire and interest.

In the fall of 1998, I took a course in Euclidean and Non-Euclidean Geometries. Unlike the previous college courses I had taken, suddenly I felt like I was in a creative environment; I never recall being lectured at in that class, but I was given problems, for which we would work together to construct our own proofs; we did not write down proofs we had memorized but actually talked about ways of thinking about the problems, trying things to see if they worked, revamping our approach, and so on. What was so amazing to me was how I felt in this environment. I thought I really was learning the geometry. When I would go home, I was not trying to teach myself the math in order to do the problem set; I was using my experiences to develop more understanding. Everything just came together, and while there was a lot of work, it was enjoyable and interesting and motivating. What is so funny is that I still remember more from that class than almost any class I had ever taken. I even remember the nature of the exam problem that I flubbed, though the precise question escapes me. So I knew at that point that active engagement in mathematics really made a powerful difference in learning. I knew this was how I wanted to teach!

Certainly my master's program facilitated this perspective, particularly as I learned about constructivism and the importance of story, and as I learned about Dewey, Freire and Deborah Ball, but what I did not learn at that point was, How the heck do I actually do this? Honestly, I did not think it would be so hard; after all I had experienced it and thought I knew what it looked like. Of course now I realize that what I had witnessed was what teaching looked like after years of crafting. Guess what? I couldn't do it—not really. I fell consistently short of where I wanted to be and I lectured often, though I hope I allowed room for creative thought. It was a constant battle. Sometimes I

would develop wonderful creative lessons, such as using TI-Interactive to investigate average and instantaneous rates of change, but most of the time, I was trying to stay afloat. I did luckily get to teach a reform-oriented Calculus class my second year. Other than that though, I had a hard time teaching the conceptual, active way that I wanted. It is in part for this reason that I opted to go back to get my doctorate in math education. I wanted to be able to figure out how I could teach like my Geometry teacher had.

Amidst all our other classes, it was as a doctoral fellow that I was able to take a course called Fundamentals of Mathematics, or FM for short. It was basically a course designed to provide research experiences in mathematics for doctoral students in mathematics education. I was like a fish in water. I was back in the realm of active engagement with mathematics, where I remembered almost everything that went on. I learned in class. I would think about the math all week until the next class. This class had two (actually many) advantages that no other math class I had ever taken had: there was no set content we had to learn and I did not have to chase a grade. Thus this course was able to take me where Geometry had left off. Of course math is creative. Of course the way I think about a problem is important. Of course doing math and developing mathematical thinking are more important than any formula, than any *truth*. Don't get me wrong, I still puzzle over much of the math in those classes, but that was the beauty of the class. For everything I learned, I felt like it opened a new idea or new direction, leaving me with the feeling that I needed (and wanted) to do more of this. This course was followed up by a course in the Philosophy of Mathematics, where I was able to consider questions, such as What is the role of truth in mathematics? What does it mean to know mathematics? Is mathematics empirical like science, or is it absolute? And How does

mathematical knowledge develop? I subsequently took yet another FM course. What I found was that the Geometry and first FM experience shaped my perspective going into Philosophy of Mathematics, but then my experience in Philosophy of Mathematics further shaped my activity in FM. In particular, I took more risks to pursue off-the-wall questions or ideas the second time in part because I felt like it actually was important to mathematics, not just to me. It was hard to do, and I did not always believe what I was asking, but it seemed more important now—it was actually part of mathematics, not just my own weird way of looking at the world. What all these classes did for me mathematically is help me to become a doer and thinker of mathematics, one who does not need a course to move forward. I could initiate and pursue my own mathematical interests, or work with someone else at any time. I could continue to learn without classes (though I still like taking courses for the exposure to ideas).

Looking back, again I am surprised at how natural and at home I felt in these experiences. They seemed to all converge and reinforce the same understandings, and, almost without my input, to pull me down a path of doing mathematics rather than just learning mathematics. The power of what I learned and did was almost indescribable. I knew in my heart that this type of empowering experience should be accessible to all doers of mathematics, in no small part so they could shape their learning experiences, so that they would be free to pursue any avenues that felt relevant, and in part so that when they did have to learn through lecture (there is far too large a repertoire of mathematics for it all or even much of it to be learned exclusively through investigative experiences), they would have substantial resources on which to build, and to fall back on when necessary. They would have more opportunity to be seized and propelled by their

interest in math, to be able to see their ideas grow, develop, change, and lead to some profound understandings. I simply had to share this experience with others. Perhaps what made these experience more urgent to share is how they resonated with other perspectives I had come to cherish, such as Dewey's (1938) emphasis on experience and democracy, and Freire's (1998) emphasis on critical learning and liberatory pedagogy. It was not just the math experiences that converged, but my other educational perspectives and my developing philosophy of mathematics as well, that all seemed suddenly to intersect. The reinforcing, coherent nature of these experiences and ideas had to mean something. I was drawn to this Bermuda Triangle, and I needed to make better sense of it.

As I knew my dissertation would somehow focus herein, I still had to figure out what, how, and where to do it. Using the PROMYS institute (Cuoco, 2001) run by the Education Development Center and Boston University as a model, what I had hoped to do was find a way to teach a course that integrated inquiry mathematics with the philosophy of mathematics so that we would be able to consider questions about mathematics while we worked in mathematics. Strongly discouraged from such a risky venture—after all, what if no one signed up for the course—I was encouraged to find a way to integrate these types of experiences into a new master's program in middle school mathematics for elementary-certified teachers, which was to be starting shortly. This seemed like a wonderful opportunity. Not only did I not have to worry about issues of whether anyone would sign up for my course because of the cohort structure, but it worked with teachers who did not have significant college-level mathematics experiences, but who were facing increasing demands of mathematical knowledge, as the high school content of algebra and geometry was shifting down to middle school.

Working with these teachers to investigate and make sense of their own (and students') mathematical ideas and questions would help them to continue learning mathematics in and from their practice (Ball and Cohen, 1999) of mathematics teaching. Given that the professional demands placed on them were changing and likely would continue to change, it seemed particularly important to not just help their formal math knowledge to develop, but to develop the tools that would help them face continuing change.

While I knew in part from experience in the FM class that not everyone was going to find this enjoyable, I still felt it important to share these inquiry experiences. I expected that it would be difficult for some because they would not readily recognize the practices we would be engaging in as mathematics. Thus, I also had this notion, from my own experiences and from reading about other teachers who had engaged in significant change or who had not quite managed significant change, that considering questions in the Philosophy of Mathematics could facilitate making these experiences positive for teachers. It could help them find meaning in the practices in which they were engaging, or at least recognize them as mathematics. While I will speak at the end about what I learned regarding how these inquiry opportunities were structured, I was very excited to begin the first inquiry opportunity with this a group of teachers in the fall of 2005. I did not know what I was going to experience or learn at that point in time. I did not know if the reception would be positive, negative, or neutral. I did not know whether these teachers would willingly engage in this opportunity when they would be receiving no grade for their work. I was still excited. The big question, of course, then is: What happened?

What follows is an abridged account of the first inquiry experience that these elementary-certified mathematics teachers engaged in during their algebra course. The topic of the investigation needed to reflect the content of the course itself, and so it was related to algebra. The contrast in what I saw versus what some of the students felt seems, at times, remarkable. I will present the assignment, what the work looked like over the three days, and then how some teachers responded to the activity in their journals. I am presenting this account here to characterize inquiry, to help illustrate some of the issues of which I hoped to make sense in my dissertation, and to demonstrate the importance of making sense of these issues as we consider how to facilitate knowing mathematics as an empowering, liberating, and cognitively exciting content area.

Explore the Set of Points Equidistant...

For the first investigation, I modified an activity I had done as a lesson study for a previous class. The exact assignment given to the group follows.

There are many sets of points that can be defined in terms of a distance relationship. For example, a circle is that set of points that are all equidistant from the same point (center). Many functions as well can be defined by a distance relationship. For example, the function **y = x** can be described as the set of points that are equidistant from both the **x** and **y** axes.
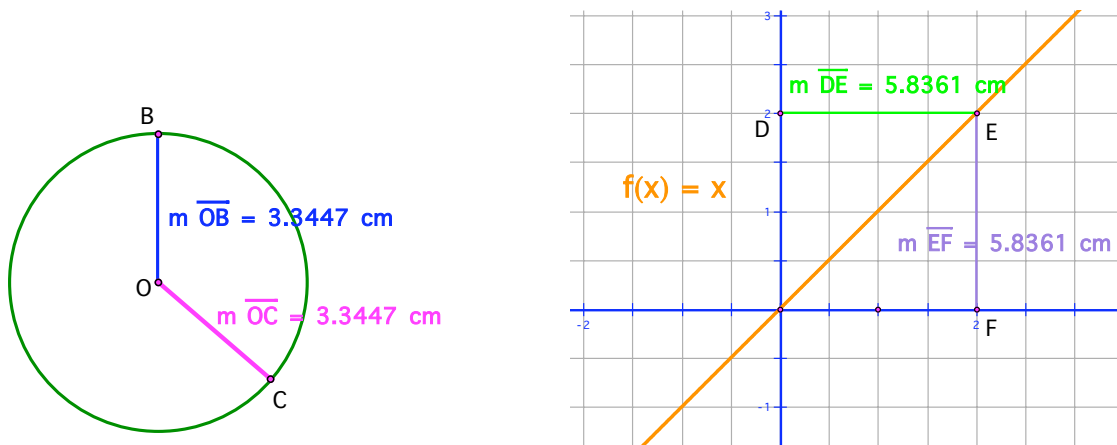
We are going to examine another set of points defined by a distance relationship: the set of all points that are equidistant from a line *l* (red line)[3] and a point[4] J off the line.
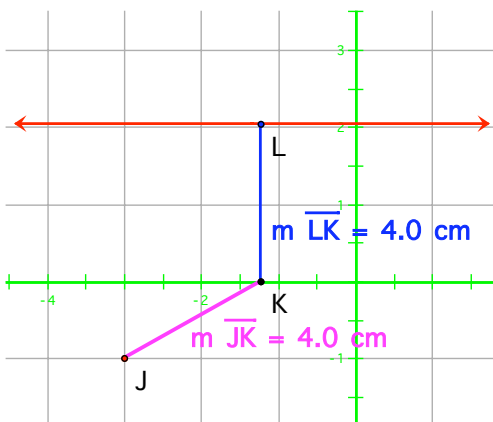
Working in groups of 2 or 3, use compass, meter stick or other tools at your disposal to begin locating this set of points on the graph paper provided, on which the given line and point pictured. You may want to develop a strategy to do this. As you find these points, begin making conjectures about this set of points. While you should have a sound reason for your conjectures, keep this in mind: **if all your conjectures are correct, you are probably not being bold enough!!!!!!!**

Please keep track of any strategies you develop, your conjectures and your reasoning for them. We will share the conjectures.

Our schedule for engaging in this inquiry activity was as follows:

1. Day 1: **What about this set of points?** Work on finding points, making conjectures, raising questions, etc. Send journal to me before next class.
2. Day 2: **What about this set of points?** 20 minutes working on large Post-its and finalizing their questions/conjectures. Rest of time spent with groups presenting various methods of finding the points and noting questions or conjectures that came up. Send journal to me before next class.
3. Day 3: **What about this set of points?** Sharing their attempts at proving the set of points was a parabola, in addition to demonstrating arguments for other conjectures—end of this segment. Send final journal response to me.
4. Day 4: **What is a proof?** What are our own ideas? Let's define for ourselves, share, and discuss each other's arguments from last week by that definition
5. Day 5: **What is a proof?** External perspectives. Review triangle has 180 degrees proof and its reliance on axiomatics. Discuss Thurston and Lakatos' description of proof and the implications for mathematical work.

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[3] This line is called the *directrix*. These footnotes were not included in the assignment.
[4] This point is called the *focus*.

What is this set of points?

Figure 1 depicts the resulting set of points. Three points, P1, P2, and P3 are demonstrated to be the same distance from the focus and the directrix using same-colored dashed lines. If this set of points looks familiar, it is because this definition is actually the definition of a parabola. However, the assignment did not ask for or require a demonstration that this set of points was in fact a parabola, though I was expecting that some of them might make this connection.
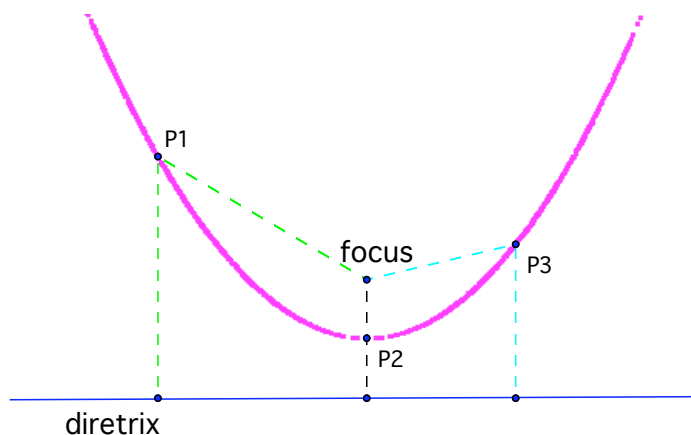
Figure 1: Demonstration of the distance definition of a parabola

Day-by-Day Account of Inquiry Activity

Day 1: Before introducing the first situation, I took about five minutes to talk to the full group about the kind of work that we would be doing in this inquiry strand—that we would be exploring open-ended situations for which the path was not laid out in advance and there was not a pre-defined piece of mathematics that was to be learned; the path would arise through their mathematical thinking. I then handed out the assignment. I had them individually read the task first and then we briefly talked about it. One item that I had learned to highlight in my previous work with this set of points was that we needed

to talk about why, when measuring from the down to the directrix from a point in the set, the measurement had to be done perpendicular to the directrix (shortest distance, only unique distance).

Following these conversations, I provided each group with a large piece of Post-it graph paper that had a pre-determined focus and directrix labeled, which they were to use to locate the set of points. Very quickly, different groups began to ask me for help. They wanted to clarify the directions—my diagram had been confusing because the point K on my diagram was in the central position, which seemed to them to indicate that this was the focus. They were having difficulty reconciling the directions with the diagram. In addition, once most groups were underway, they asked me to verify that they were doing the work properly and that their points were correct. My standard response was the following set of questions: *Can you show me that the distances are equal?* (Typically the response was "yes.") *Does that meet the definition that the distances are equal?* (Again the answer was "yes".) *So what does that mean?* I was trying very hard to remove myself as the authority figure from the beginning, but I found this very difficult as the various groups continued to look for my verification and approval. I was struggling with how to help them look to each other for this support. During the class, these teachers were focused on finding the set of points that met the definition. Various methods emerged that turned out to all be variations of the same approach. The first involved using a compass and a ruler. One group of three simply picked a measurement, such as 5 inches, to start. They set the compass to that distance and drew a circle around the focus (see Figure 2); they then drew a line parallel to the directrix (blue line) that was that same distance above

the directrix and on the same side as the focus. The intersection of this circle and this line were points in the set circle (see b1 and b2).
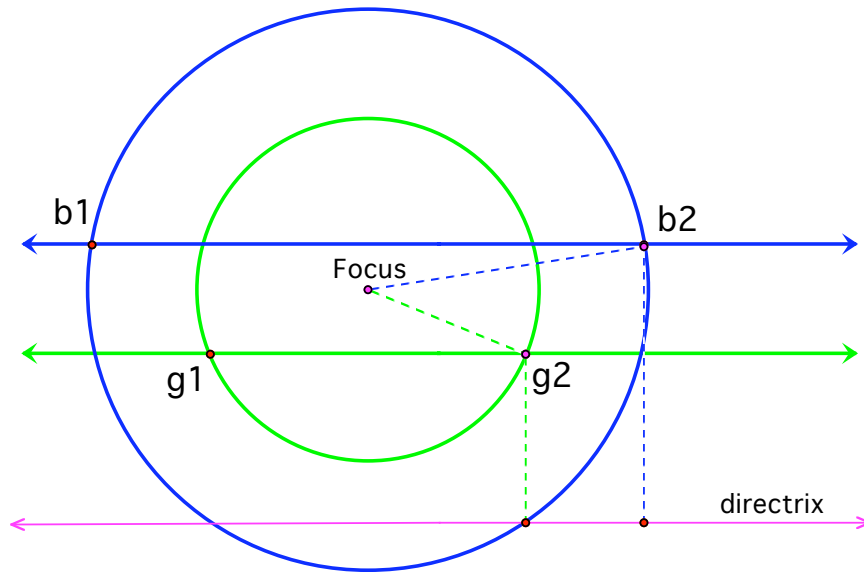
The blue circle has a radius equal to the height of the blue line above the directrix (pink line); their intersections (b1 and b2) are thus the same distance from both the focus and the directrix. The green circle has a radius equal to the height of the green line above the directrix so its intersections (g1 and g2) are also the same distance from both the focus and the directrix.

Figure 2: The intersection of certain lines and circles forms the set.

One variation of this method involved drawing the horizontal lines first, then using the compass to make arcs that intersected the line. The second variation involved using only one measuring device: one group drew the horizontal line a particular distance from the directrix; then, placing the 0 of the meter stick (or other measuring device) on the focus, they swung the meter stick until the desired distance intersected the line. This basically involved using a measuring device as a compass. The journals from the first day tended to detail the procedures, as well as issues that individuals had with the task.

> *Amelia*: At first, I didn't realize that the points we were supposed
> to locate had to be the same distance from both the point and the line (at

the same time). We also tried to use the compass to make arcs that had an equal distance from the point and the line. It reminded me of constructions where the compass must be more than half the distance in order to have a point of intersection. I think that the points I found had to be half, or more than half of the radius. There seems to be symmetry between the points that we found. The farther the points are from point P, the farther they are from the line. . . . Then I thought that anything that fell on that parallel line to the given line that was equidistant from the point and the line would work. There were some that seemed to work (with the compass) and some that did not. I used the distance formula to try to test the point (-1, 1) and it did not work (Points Journal, September 18, 2005).

Robin: We got off to a pretty good start by finding the distance between point P and the line. The distance is 5 cm. We decided that two of the points in our set would be 5 cm to the left of P and 5 cm to the right of P. Those two points are the same distance from point P as they are to the original line. Then we began to think about circles and the fact that any point on the circle was the same distance from the center of the circle. The circle that connected our new points (A and B), with point P at the center, was a circle with a radius of 5 cm. . . . As we made larger and larger circles, we kept on finding two of the set of points per circle. . . . Something that I am still not sure about is how to actually label the points. They do not all fall "neatly" on the grids of the graph paper. How can I

find the exact location to name the points in the set? (Points Journal,

September 18, 2005).

While their notes focused on procedures, my reflective notes from the first day highlight

the types of conjectures that I overheard, as well as my own issues with what took place:

I explained to the class the nature of open-ended investigation and what

their role would be. Some seemed really disbelieving that I would not

have an ultimate answer/goal in mind with this project. They kept wanting

to check, perhaps to make sure that I was not going to drop something on

them later. . . . I heard some conjectures come up as people were working

on the points: it is a parabola (Eric, Emma, and David; Kathleen and

Libby); it is symmetrical (Rick and Amelia); [the focus] lies on line of

symmetry (Libby and Kathleen). While I am not sure who it was, at least

one group remarked that the points had to keep going out, and another

group verged on suggesting that no points would be on the side of the line

opposite [the focus] (Personal Reflective Notes, September 13, 2005).

Day 2: For this class I had two goals. I wanted to provide time for the groups to

continue their work (given what I had read in the journals following the first class) of

finding points in order to put their work on large Post-it graph paper for everyone to see,

and to allow them to share their work and/or discuss conjectures and questions. The full

group expressed their desire to hear different small groups' approaches to locating points

and what their thinking was. Each of the five groups presented their basic methods, while

I asked them to write on a Post-it at the front any conjectures that seemed to emerge from

their presentations and discussions. In the first excerpt below, a group makes use of the strategy that was detailed above with the circles.

Rick: All right. We started out thinking there was only three points. [Indiscernible] so at about the halfway point, they were about equal distance from here and the line; and we felt we were stuck for awhile. And then we thought a circle within a circle, and drew a circle through another concentric circle. And we kept getting more points, right? I'm not sure whether you see the orange points, these two points here, two points here [using] concentric circles gave us a lot more points of [indiscernible] when we stopped, all right? Then we started noticing when we got like a point over here, there was another matching point over here. There was like symmetry. And then when we got enough points, we sort of noticed the curve. All right. So after finding that curve, then I plugged in points [on the calculator], and some of the points I had to estimate a little bit. You know, like these right here. Once I plugged in the points, then I used a graphing calculator and did the quadratic regression[5] sort of like we do linear regression for [pause]. And then we made an equation [indiscernible] curve. It wasn't perfect. It would be off a little bit, but, you know [pause]. It showed the curve for these points and

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[5] A quadratic regression basically examines the points that have been given and approximates the best-fitting quadratic function for that set of points.

then gave me an equation, which is close [indiscernible] pretty

close (Class Transcript, September 27, 2005).

This group had located the points, noticed that the points were symmetrical, and then

noticed the shape of the curve. What is not explicit in Rick's description is that his group

had hypothesized that the curve was in fact a parabola, which is why they decided to

enter the points in the calculator and do a regression—to see if they would get a curve

that looked like the one they were graphing by hand.

In the following excerpt, teachers from three different groups are engaged in a

conversation about methods and conjectures:

Robin: We're the second one over. . . where the circles are. The orange

shows, you know, the radius and [pause] we dragged the focus

over until it hit the circle—

Heidi: See, this – that's our starting point– so we started with – here is a

good examples: This is our 7 centimeter radius circle. So there

are 7 centimeters from a given point, as you can kind of tell. So

all of these circles are the same type—5cm, 7, 9… and then we

did—

Robin: The halfway point is at 2.5 in general.

Rick: So yours opens upward, right? Ours was downward.

Heidi I think they're all the same –

[Overlapping conversations.]

Eden: Any questions?

Rick: One of my questions, I know they all did what ours did and I thought like they would all be the same curve like I could plop them on top of each other and they would be the same curve, but it didn't seem to be that way.

Emma: Because they are different distances from the lines. So like if you look at Heidi's and ours is the one right next to Heidi's with the pink equation on it, ours would be exactly the same because they are 2 spaces away from the given line. [Emma tries to put the conjecture into words about parabola relating to distance.] (Class Transcript, September 27, 2005).

At the time of this exchange I was both personally excited and intrigued by the conjecture of "sameness." Parabolas are in fact self-similar,[6] but this is typically counter-intuitive. As Heidi continued to assert that she thought they were the same, I was wondering if there was something about building or locating this set that made this abstract or bizarre attribute of parabolas accessible. However, her terminology seemed to contradict Emma's conjecture that the distance from the focus to the directrix determines the set of points. I hoped we would spend some time exploring this conjecture.

As the class went on, it became increasingly clear not only that most groups thought it was a parabola, but that demonstrating this was the focus of their energies.

Emma: We did the next one with the compass, the next two with the compass, and I think the next few we could probably figure out

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[6] Self-similarity implies that any two parabolas can be overlaid one on top of the other simply by dilating one.

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with no problem; and we started to see a pattern, and we were trying to prove our pattern, trying to figure out, one, if it was parabola and made a quadratic function. We did not figure out for the life of us, um [pause], made the table, and we found the second difference. We did find the second difference, you could use to prove it was a quadratic equation; and it didn't come up right at all. I don't know if that's because points were estimated or not, I have no idea.

So I was convinced it wasn't a perfect parabola and [pause] trying to come up with the equations here. David was trying to find the equation for it with the points we had, it took quite a while to figure out. Then we went to the Pythagorean Theorem. You gotta be able to use Pythagorean Theorem somehow. I don't know how much you all know about the Pythagorean Theorem—a squared plus b squared equals c squared—one triangle worked. This way, the first point worked; and then we did 3,4,5 and it worked. (Class Transcript, September 27, 2005).

This group was drawing on the Pythagorean Theorem, which relates the side-lengths of a right triangle, to locate specific points that met the distance definition of the set of points. In the last 10 minutes, at least half the class expressed that their primary interest was in the parabola conjecture. We briefly talked about how such a proof could be explored, and I stressed that they needed to work with what they knew—that the distances were equal.

It was at this point that, while they wanted to prove the points made a parabola, it became clear that they not only did not think they knew how to pursue a proof, but that they did not know where to end with such a proof:

> **Robin:**   What sort of definition of parabola are we working towards [indiscernible]]? . . . But then we're trying to say that it is a parabola. What do I have to find to be like, "woo-hoo, isn't that good?" (Class Transcript, September 27, 2005).

Another teacher then added that she did not think that they had enough knowledge to prove it was a parabola. Sensing the worry and frustration, and feeling unsure myself why it had become so important to pursue this proof, I tried to emphasize no one had to focus on the parabola proof; we were trying to make sense of their questions and conjectures, not whether or not it was a parabola. I said, "If you don't want to focus on whether or not this is a parabola, that's not the task that is important to you, pick another one. Okay? . . . I want you to pick something, a question that I want you to try to answer or a conjecture that I want you to try to justify." (Class Transcript, September 27, 2005). The main questions and conjectures that had arisen in class were:

Questions
1. Is it a quadratic function?
2. Is there a way to derive the equation?
3. Is it really a parabola?
4. Does Eric's way (PT) always work?
5. What happens if l is vertical?
6. Is l a reflection line?
7. Does this have to do with absolute value?
8. Can you use the Pythagorean Theorem to analyze second differences?

Conjectures
1. It is symmetrical.
2. It is a parabola.

3. If P is above *l*, it opens up.
4. If P is below *l*, it opens down.
5. Halfway point (between P and *l*) is top/bottom of curve.
6. The curve never crosses the given line.
7. *The dilation depends on the distance between P and *l*.*
8. The curves are all similar.
9. The points can be defined using isosceles triangles.
10. The curve has an infinite number of points.
11. The curve continues moving out and away from the point and line (e.g. no asymptotes) (Summary notes, September 28, 2005)

Interestingly, to have this similarity conjecture included on the main list I had to pointedly ask Heidi to restate her thinking at the end of class, because it was not something that she had chosen to include on the list, despite her repeating this belief several times. While I forgot that Rick, too, had suggested such a conjecture, in asking Heidi to describe what she meant, the vocabulary shifted from *sameness* to *similarity*. Looking back, I do not sense that Rick was using *sameness* to imply similarity.

Over the course of the next few days, I began to receive e-mails that were full of frustration around how to proceed. So I decided to send a mass e-mail to everyone trying to convey that I understood their frustration with the work, and that they should not expect themselves to have readily available answers. For some teachers, this seemed to spark a feeling of responsibility for me, especially among those who were less bothered by frustration; Beth, who had not yet written a journal response, responded:

*T*his may be the first time that a math teacher/instructor has actually made me feel like I have something to offer. Not just that I can do it, but that my thinking might be valuable. Anyway, I am fascinated by the geometric relationships of circles and triangles. I learned during the summer that Pythagoras used only tools to figure out the right triangle relationships... I have watched my students "discover" how overlapping circles and

connecting various intersecting points leads to perpendicular lines, so I was wondering if there was some way—and I haven't really tried it yet—that circles and triangles could be used to "build" the set of points, whether it's a parabola or not (Points Journal, September 29, 2005).

Julie: Eden, thanks so much for such a thoughtful e-mail. I appreciate how you've handled the questions and frustrations as we've worked through this ambiguity. I think the people who say the math is too hard don't have any experience with quadratic equations and the relationship to a parabola so they're feeling lost. I have a lot more math background than some and I still find the process challenging and uncomfortable at times because I always feel like I should know more than I do. Your email put me more at ease about that. My only frustration in the class (and it has nothing to do with you) is someone in the class who calls out constantly and indirectly puts people down by her statements that they're wrong about something. There's always at least one in every class and I can handle it—just find it unpleasant sometimes, and doesn't give me processing time when I want to think through something. (Points Journal, September 30, 2005)

Day 3: This was our final day working with the mathematics in class. The following two days were devoted to issues of proof and proving that came up during this investigation. During this final hour, various teachers demonstrated to the class what they worked on the previous week relating to conjectures and questions, which at times

sparked other longer discussions. Emma demonstrated her exploration of a vertical[7] rather than horizontal directrix. While on the one hand, the shape of the curve remained the same, a curve opening sideways felt problematic to her in that if the shape was a parabola, it could no longer be defined by a function. This led into discussions about what a function is, and in what cases might the sideways curve be defined as a function.

Emma: So I mean, I developed another conjecture from it, or another question from it basically is what I did. I said, I'd see if you make a given line vertical versus horizontal, that all points are rotated 90 degrees. It still forms the same shape, but it brings up another question. When a parabola is turned on its side, is it no longer a function 'cause there's one more value made for each point in the range?

And since it is no longer a function and a parabola's graph is the quadratic equation, it is no longer a quadratic equation. Can you get an equation from it? What would it be? How would you derive it? So that's what I came up with and punched it in the calculator to see if you got an equation from it and you didn't get an equation from the calculator—

My math background goes through functions so I know that for a function, you can write an equation fairly easily. If you don't have a function, can you write an equation for it? You [Eden] then wrote back to me, "If you can write an equation for

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[7] When the directrix is vertical rather than horizontal, the parabola just rotates 90 degrees and opens sideways, rather than up and down.

a circle and a circle's not a function. . ." So is there a way to write an equation for something that's not a function?

Marjorie: But can it be a function relative to the other axis?

Eden: What are you –

Unknown: Could you write it in reverse?

Marjorie: An *x* equals equation instead of a *y*, you know, we live in a *y* equals world, right? But can it be an *x* equals where it's gonna be –

Unknown: Switch the *x* and the *y*?

Emma: If you do it in reverse instead of saying a function of *x* you say a function of *y*.

Unknown: What about the vertical line test?

Eden   Okay.

Emma: But then it would be a horizontal line test.

Unknown: It would become horizontal—

Matt: ertical line test. But then it—

Emma: But you reverse everything...
(Class Transcript, October 11, 2005).

Regarding students who had worked on trying to prove it was a parabola, one teacher did not want to come up and show her work because her numbers were not *friendly*; she was coaxed up when we asked her to show her thinking rather than her specifics. She demonstrated how she used the distance formula to show that 3 points were in the set of points that met this distance definition. She verified this using the

Pythagorean Theorem; she also knew these points were on a parabola. In the course of this discussion, though, she asked an important question that guides future discussions.

> **Kathleen:** I used the distance formula first to prove my distances and then I used Pythagorean theorem to get my hypotenuses to make sure everything was equal. So then I was able to prove these points existed. So I didn't know if those were enough points to say it was a parabola or not. But -
>
> I did a combination – I used the distance from the corners for the points I knew existed, and then I used the Pythagorean Theorem to get my diagonals to prove that this diagonal was the same distance from point P as the point was from the line. But my question was how many points to have to –
>
> (Class Transcript, October 11, 2005)

While Kathleen asked this question about how many points did she need to prove it was a parabola, another student began to question how Kathleen knew the points were connected. This discussion, sometimes at my instigation, led some of the class to consider whether demonstrating that specific points both met the definition and were on a parabola was a valid proof. However, Kathleen repeatedly returned to her question about how many points she needed. She seemed to get frustrated when I simply didn't tell her how many points she needed, and suggested that I had been disregarding the question: "Alright. I just didn't know if those points were enough to make it a parabola and to say it was a parabola. So that was my question. I e-mailed that to you too, I think on Sunday night actually."

(Class Transcript, October 11, 2005). As this continued, I started trying to push at her argument a bit, and some of her peers began to get involved; however, Kathleen continued to want to have a simple answer to how many points were needed.

Eden: Why would what you've done for those four points tell me that that point your pointing to now it also on the curve?

Kathleen: Because I just connected the dots. That's why I found this question for you _____.

Eden Okay, alright, so –

Kathleen: Because I just assumed, being that, you know, we were talking about functions, that I should be able to connect all the dots.

Matt: Did you pick any point on there and prove – use the Pythagorean Theorem… if it's a parabola?

Kathleen: That's what you asked me.

Matt: If it's a parabola.

Kathleen: Right, I could pick any of these points and continue using the formulas I was using. That's right. But that's why I said to you, I knew it was a function. Right? So I knew that the dots had to be connected – the points had to be connected, but I just wanted to know how many I needed to show.

Eden: Well, I think a question that I would ask is if I had any two points, how many different ways could I connect them?

Unknown: Even if you think they're all on the parabola?

Eden: Even if I can't – my assumption is they're all on the same

	curve.
Unknown:	Right.
Rick:	But the curve might not be a parabola.
Matt:	So you're saying she could connect those points a different way?
Libby:	You could make straighter lines.
Unknown:	Straight lines—
Eden:	If you connected them with straight, lines would it still be a function?
Matt:	Each side [inaudible] be open to a function [inaudible]
Kathleen:	Well here to here would be if it went like that.
Unknown:	Right.
Eden:	Okay, so I think the hole for me would be how do I then take what you've done to show that it applies to the other points as well? That would make—and I don't think that's a trivial question. I mean when I—
Kathleen:	And see, I think that relates to this exactly what I was asking you. Like I think we're asking the same—
	(Class Transcript, October 11, 2005)

Susannah had tried something similar, and even tried to work backward, but this felt unsatisfactory. She said, "And it's real close, but, you know, and I don't know that that really proves it or if it really is correct?" (ibid). Thus the theme of what was a valid proof and how you could prove it was a parabola kept coming up. Their work strongly

suggested to them that it would work, but they were looking for some final piece. Interestingly, two students, Libby and Amelia worked with their mothers, both of whom are math teachers, to demonstrate that the distance definition yielded a quadratic equation, but they would not present their work, perhaps because they had not done it themselves or were unsure of it. I must confess that I was very sorry that nobody, not even Heidi, had chosen to explore the similarity issue. Closing this class was very difficult for me for another reason. I knew that we were not going to explore this curve more, and I was concerned that it might be difficult for some to get a sense of the purpose of this type of activity when we did not keep moving forward until we had developed something firm, though I hoped next week's proof discussion might help. I was, however, hopeful that we could find a way to bring this work into other classes and explorations.

The final journal response was meant not to explore where they currently were in their thinking, but more importantly how this open-ended inquiry experience had felt to them. Two such final responses follow:

> *Heidi*: Over the past few weeks while working on Investigation 1,
> there were several moments when I felt overwhelmed and challenged to
> say the least. It was difficult for me to get started and solely work with the
> given line and point. I depended on my partner to help me begin to plot
> points on our grid. I relied upon Robin to explain to me why some of the
> points were being plotted where they were. After plotting a few points, I
> felt confident enough to find a few more points on my own. What was
> difficult about the investigation at that point was not really knowing if I
> was right or wrong with the location of the points I plotted. I felt like I had

no idea if I was on track or completely off! By the second day of the investigation, I was still feeling iffy on the location of my points, but felt better about my work because there were others in the class feeling the same way. Part of me wondered if I would ever make sense of the given line and point, and the other part of me felt like no matter what I thought, I was going to be correct because it was an open investigation. . . . Since I was unfamiliar with what defines a parabola, I felt as though I was clueless and at a loss. . . . By not knowing what I was searching for, I was deterred from the investigation. (Points Journal, October 17, 2005)

Susannah: It is with great pleasure that I have abandoned the search for whether or not this figure was indeed a parabola. I know that it was a parabola, but how to prove it escaped me. I am not actually sure whether or not I needed more guidance, more parameters, more time, or if it is just that my mind is not used to blindly exploring a conjecture. The last time I remember feeling this uncomfortable in math was when I was in 9th grade geometry. Whenever we were expected to come up with a proof, I was completely lost. I never could figure out how to make the leaps from one thought to the next. Once I had seen it demonstrated, I understood where it was coming from, but I could not come up with my own proofs. Perhaps that was part of the problem here. While my mathematical knowledge has increased, I still find it difficult to make those leaps.

Was this a challenging assignment? Absolutely. I had to search for concepts I knew that would actually help me and try to apply them in a way that I had not had to do before. Was it difficult? Parts of it were difficult and parts were easy for me. The beginning of the assignment came very easily to me. I remember thinking that it had come too easily. It had to be harder than that. And then it got harder. Once I had to explore whether or not my conjecture was true and to find a way to prove that, I almost feel as if I shut down. I just could not figure out how to do it. . . . Of course, I feel that some of these things that I believe would have helped me were purposely left out or done as part of the assignment to encourage us to think a little bit more. (Points Journal, December 19, 2005)

Days 4/5: It was because of this challenge posed by proving that over the following two weeks, these teachers and I considered what constituted a proof, what it is and what it was not. We also talked about how, while their work did not strictly constitute a formal proof, they could use their exact work to do a proof. The class developed their own definition of proof and used it to critique each other's work; then we looked at expert perspectives of a proof from a philosopher and a mathematician, which I then connected to the work these teachers had done.

My Lingering Questions and Concerns

Throughout this investigation, I struggled with deciding what was the best way to proceed. Certainly every decision I made felt justified, but somehow I felt I was missing some important information that would have facilitated my decision-making. It was quite

clear to me throughout this work that many of those involved hated the lack of direction in this exploration and were anxious when I would not answer their questions directly, or worse, when I answered with another question. While I was continually trying to remove myself as the authority on mathematical issues, those who were more active participants seemed to want to push me towards that role. While I chose not to do this to help teachers develop their own agency, could it have been more important to prevent frustration and facilitate engagement? Why was frustration continually present in most sentiments around this problem, but in two cases, this frustration did not seem detrimental to progress? Why did one teacher not look for external direction? Why did the majority of these teachers react by effectively creating a specific assignment for themselves that I purposefully had tried to avoid? Why did one teacher not seem to care about the group-defined parabola proof task? Why did these teachers feel frozen and unable to choose a path to pursue? Why was one teacher able to recognize the learning of the group, when others just felt their own frustration? Is it coincidence that the two teachers who had positive reactions worked together initially? Why did two teachers go to their mothers, who were math teachers, for help on this assignment? Why did only a few of the people go to the board, and those who had parental help to complete proofs chose not to share their work? Why did some people never respond in journals?

One way that people might typically answer such questions is tied up in perspectives of knowledge and beliefs. Certainly one ready-made answer is that these teachers did not have enough knowledge (as a few teachers themselves suggested) to pursue this problem. However, it is interesting that Robin and Susannah, two of the few teachers who had taken Calculus (and done well), found this work quite frustrating, while

Beth (whose highest math was Algebra II/trig, which she failed the first time) and Julie (who has extensive educational experience, but not advanced mathematics training) found this quite valuable? Robin and Heidi were elementary teachers, and the others who expressed frustration above were all middle school teachers, as was Beth. There is little observable rhyme or reason as to who did, and who did not, respond well to this investigation, at least according to how we might readily differentiate people's math capabilities. [8] This suggests that to make sense of such reactions we must problematize[9] what we consider mathematical knowledge. In order to address these questions of interest, I need to sharpen my focus and consider how I want to think about knowledge.

Is There a Problem With Knowledge?

In my Philosophy of Mathematics course, for our first activity we were asked what does it mean *to know*? We generated a long list of different ways we make use of this word, although none of them was precisely the formal definition of propositional knowledge that we were then given from Philosopher Israel Scheffler, which has its roots in Plato's *Theaetetus*. Our list contained descriptions of *to know,* such as in knowing a fact or knowing of something. But we also had elements on the list that included far more subjective types of knowledge, such as knowing something in your gut, a type of feeling or an intuitive type of knowing; or knowing a person really well, which involves a type of intimate relationship. While these types of knowing might feel familiar, they are in conflict with the formal philosophical definition, which has three criteria: a belief, a

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[8] There were some teachers who never seemed to write journals and who never expressed feelings to me positive or negative about the project.

[9] Problematizing is an act that seeks not to judge something as wrong or right, but to critically examine something by taking the perspective that just because something is a certain way does not mean it should be that way.

warrant, and a truth requirement (Scheffler, 1965). A difficulty with this formal definition is that it requires we have some objective access to truth in order to judge whether someone knows rather than believes. Scheffler and many other philosophers have taken up this question of whether we can know an objective truth; but abandoning this truth leads us in the direction of fallibilism.

In mathematics, objective truth has acquired a seemingly unassailable quality, because of the importance of proof and deduction in establishing certainty. Unlike in empirical sciences, which experience revolutions in thought (Kuhn, 1996), once something is proven in mathematics, it is arguably proven forever. The consequence of this perspective is that, while mathematical truth grows, it does not change (Grabiner, 1998; Kitcher, 1984; Lakatos, 1967). Certainty and Truth are blended. Though many, including those philosophers already mentioned, have questioned this assumption, its effects are far reaching. Knowledge in mathematics in particular is perceived as stable and unchanging (Boaler, 2002a). In schools at all levels, this perspective both feeds and is fed by the consistent, broad use of tests and grades as acceptable and important measures of content knowledge. The measure of one's knowledge then can be likened to an objective trait or characteristic. Knowledge as objective trait extends beyond mathematics as well. In teaching, a substantial amount of research on teachers' content knowledge is based on more objective assessments, such as mathematical courses they have taken as well as the grades they have received.

The importance of assessing teachers' content knowledge is based on the taken-as-shared assumption that a teacher with stronger or more content knowledge will be more effective in helping students to learn. However, in examining the connection

between this fixed, unchanging content knowledge of mathematics (as measured by courses and grades) and teacher effectiveness (often as measured by student testing scores), research does not bear out the assumption that content knowledge and effectiveness are linked (Wilson, Floden & Ferrini-Mundy, 2001). In fact, after a certain number of content courses, effectiveness and content knowledge may be negatively correlated. One strong argument for why the link between content knowledge and teacher effectiveness is not borne out is that there is far more than content knowledge that teachers must draw on in order to teach effectively. What other knowledge is crucial for mathematics teaching?

Prompting a dramatic shift in the landscape of teacher knowledge, Lee Shulman (1987) explored what could constitute a knowledge-base for the professionalization of teaching. He considered sources, content, and character, for this knowledge- base, and he described the minimum categories that could constitute such a knowledge base. Among these categories were content knowledge, general pedagogical knowledge, knowledge of learner, and, of course, his famous addition, pedagogical content knowledge. He described pedagogical content knowledge as:

> the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learner, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understandings of the content specialist from that of the pedagogue (1987, p. 8).

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Regarding the sources of the teaching knowledge-base, I want to draw particular attention to his focus on *scholarship in the content disciplines*, because of its relative importance to both content knowledge and pedagogical content knowledge[10]. His view is broader than a typical perspective of content. While, of course, he highlights the importance of the accumulated knowledge, he also adds that it is necessary to consider historical and philosophical perspectives on the nature of knowledge in the discipline (Shulman, 1987, p. 9). How often in the teaching of mathematics do we also teach its history or philosophy? How often in teacher education? Perhaps, though, with a certain and true mathematics, how ideas have developed lose importance. Yet a teacher also must have understandings not only of the structure of subject matter, but also of its conceptual organization, and in particular, the principles of inquiry that help address the questions of what are important ideas and skills, as well as how new ideas are added by those who produce the knowledge in the discipline (ibid). These understandings are vitally important as these shape what is communicated to students about what is significant in the discipline in question. Shulman further emphasizes that a teacher will convey messages about how "truth" is determined in the field, as well as values and attitudes that he says "markedly influence student understanding" (1987, p. 9). Mathematicians such as R. Hersh (1979) and W. Thurston (1998), philosophers like P. Ernest (1991), and mathematics educators such as A. Schoenfeld (1988) have been making similar arguments regarding the teaching of mathematics in particular. Such a broad focus on content knowledge for teaching, even when content knowledge is only one of the many areas on which teachers draw, is crucial because teaching first requires understanding, as

[10] He actually identified four key sources of knowledge: scholarship in content disciplines, the materials and settings of the educational process, research on schooling, and the wisdom of practice.

well as a critical perspective of the ideas to be taught (Shulman, 1987). Unfortunately, critically examining mathematics, taking ideas apart and putting them back together, exploring how ideas connect and build, and why they are important, are not common in the mathematics education of teachers (Ball & Bass, 2000; Cuoco, 2001), particularly at the elementary level. Thus Shulman's work pushes us to broaden our concepts of knowledge for teaching, as well as our understanding of content knowledge.

Cochran-Smith and Lytle (1999) further push our thinking of knowledge for teaching through exploring three broad traditions of knowledge for teaching. They describe knowledge-*for*-practice, knowledge-*in*-practice, and knowledge-*of*-practice. Knowledge-*for*-practice is generally equated with traditional formal knowledge (p. 254); knowledge-*in*-practice, though, is far more spontaneous. It emphasizes knowledge that is drawn on in the act of teaching; it is rooted in the artistry of practice, including inquiries in practice, accounts of practice, and decisions in practice (p. 262). Knowledge-*of*-practice embodies the stance that knowledge generation and its use are inherently problematic, are open to discussion, and do not exist separately from the knower. Relevant issues herein include what it means to generate knowledge, who generates it, what counts as knowledge, how is knowledge used, and other similar questions (p. 272)—ideas that seem to recall my first activity in Philosophy of Mathematics.

The implications of considering knowledge with these lenses are quite dramatic. As knowledge-for-practice is closely related to formal subject matter knowledge, it is from this perspective that *knowing more* leads to more effective teaching practice (ibid). Teaching then involves the application of received knowledge that was produced by scholars in the discipline. This is the typical knowledge considered in the studies of

teacher effectiveness. Knowledge-in-practice is described as practical rather than formal, suggesting a dichotomy of knowledge that one draws on in teaching. With this lens, teaching is about thinking and acting wisely in ever-changing situations. Expertise and knowledge must come primarily from working in the discipline, because it requires analyzing and understanding not only one's actions but also the source of these actions through reflection and inquiry. Knowledge-of-practice does not then create a "trichotomy", but rather opposes the dichotomy; it draws together the knower and the known in a type of dance across time. The authors state, "Implicit in the idea of knowledge-of-practice is the assumption that, through inquiry, teachers across the professional life span—from very new to very experienced—make problematic their own knowledge and practice as well as the knowledge and practice of others, and thus stand in a different relationship to knowledge" (Cochran- Smith & Lytle, 1999, p. 273). This critical perspective positions teachers as agents of change with reference to knowledge and knowing. Even students are participants in this adventure as knowledge is socially constructed. There is a blurring of teaching and learning, which the authors implicate in why teachers who develop knowledge-of-practice also engage their students in a similar knowledge development (ibid, p. 281). As teaching and learning move towards each other, the act of problematizing one's knowledge and knowing in a content area, seems to assume an aura of pedagogigical content knowledge. While knowledge-*for*-practice has not been linked to effectiveness, aligning with knowledge-*of*-practice seems to allow for the importance of such knowledge to teaching.

Echoing this idea of removing the false dichotomy of formal and practical knowledge, Deborah Ball (2000; see also Ball & Bass, 2000) advocated blurring the

boundaries of content and pedagogy. Paralleling Shulman, Ball highlighted in particular the importance of subject matter understanding as crucial to teaching; yet she argued leaving the integration of this content and pedagogy to the teacher has been ineffective and has promoted a rather fragmented practice. Since teachers' content knowledge education often has focused on acquisition of more knowledge, developing knowledge of content useful for teaching has suffered. She argues, "It is not just what mathematics teachers know, but how they know it and what they are able to mobilize mathematically in the course of teaching" (Ball, 2000, p. 243). To understand what content knowledge is needed, we must first examine teaching practice to determine the demands on their content knowledge. In particular a teacher needs to "deconstruct [her] own knowledge into a less polished and final form, where critical components are accessible and visible" (ibid, p. 245), as well as be able to listen flexibly to students, make sense of where a student is in understandings and where errors might be, and appreciate and make sense of unconventional approaches. Consequently, expert content knowledge[11] is perhaps inadequate for teaching, because of its highly compressed nature. If this were not the case, mathematics professors would certainly be the best teachers. She closes by calling for more research in how teachers can be prepared to know content flexibly enough to be able to work with a wide range of students.

These examinations of different types of knowledge that teachers draw on and how such knowledge is acquired have made significant contributions to our understanding of developing effective teachers. More to the point, they have enlarged our understandings of what we need to attend to in the education of teachers to best prepare

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[11] This description does not feel quite appropriate, because I feel as though she is specifically talking about the products of mathematics, not expert ways of coming to know mathematics.

them to be mathematics teaching professionals. Certainly in talk of interweaving content and pedagogy, or of pedagogical content knowledge, or of knowledge-in-practice, there is a substantive shift to consider how knowledge develops in addition to knowledge itself, but there is only a reflection of the teacher herself in this; knowledge has not completely shed its trait-like feel. Only knowledge-of-practice (Cochran-Smith & Lytle, 1999) explicitly characterizes knowledge as personal when the authors speak of bringing knowledge and the knower together. What might happen if we couple Shulman's enlarged concept of content knowledge and Ball's emphasis on how mathematics is known with the perspective that knowledge is an integral part of who the knower/teacher is? It is with this perspective that I propose another important question to consider: Is there little link between content knowledge and effectiveness because our assumptions about what constitutes content knowledge (for students as well as teacher) are somehow too restricted or otherwise inadequate?

From such a stance of content knowledge, what comes clearly across is that what one *knows* of mathematics is far more than the formal knowledge they have acquired (Ball & Bass, 2000; Pepin, 1999). Teachers have received and internalized messages from their first introduction to mathematics in school about how mathematics is done, what it is about, who finds or creates this knowledge, the importance of their personal questions, and their own role in the doing of mathematics as students and later as teachers, to name a few. The situated perspective of learning (Brown, Collins & Duguid, 1989) helps us to understand that learners are enculturated into the school content disciplines, which may look very different from the professional disciplines they are meant to reflect. Hence teachers do not just learn and draw on formal mathematical

knowledge; they have over time developed a full relationship with the content that has been socialized throughout their experiences around the content. This relationship is dynamic in that it continuously shapes and is shaped by teachers' experiences as learners[12] of mathematics. I suggest that it is not just teachers' reified content knowledge, but also this relationship as a way of knowing mathematics more broadly that impacts their goals, planning, and classroom decision-making. As this perspective has only recently come to the fore, the influence of this relationship remains largely unexplored by teachers themselves as well as researchers, and consequently our awareness of its influence in teaching is not the product of critical consideration. We need to make sense of this relationship that teachers have with mathematics, and its dynamic nature, if we are then to understand its influence in teaching.

What Is the School Mathematics Tradition into which Teachers Are Enculturated?

To begin to make sense of this dynamic relationship, it is particularly important to consider the environment in which it develops. Through sociological and historical research Dan Lortie (1973) and Larry Cuban (1993) have demonstrated the consistency of schooling over the last century, as well as its resiliency to efforts to change it. Lortie (1973) proposed several interrelated theories that account for this resiliency, including the apprenticeship of observation; however, Cuban (1993) documented that, in spite of this lack of gross change, there has been a consistent presence of grassroots progressive education in a small percentage of classrooms over the last century. As part of this school tradition, mathematics education mirrors what has occurred in education more broadly.

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[12] Teachers are not just math learners when they are officially math students. They continue to be learners of mathematics throughout their teaching, as is evidenced by teachers regularly saying that they know the material better after teaching it.

Math education has been fairly stable, perhaps more stable than most areas of education given the perceived absolute[13] nature of mathematics (Hersh, 1979; Lakoff & Nunez, 2000; Ernest, 1991). Though, too, in mathematics there have been progressive efforts, such as in the *New Math* movement and the reform movement of the last 20 years prompted by NCTM's *Principles and Standards of School Mathematics* (2001). Of course, the very need for reform-focused movements and mathematics teacher organizations suggests the tenacity of the traditional school practice. Efforts to change this tradition have resulted in a type of verbal violence, aptly called the *Math Wars* (Schoen, Fey, Hirsch & Coxford, 1999).

Researchers have documented the long history of stable mathematics education practices in this country. Citing Cobb, Wood, Yackel, and McNeal (1992), Gregg labeled this practice the *school mathematics tradition*. In this tradition, there is a standard flow to the mathematics class that involves checking homework, presenting new material in lecture format, and finally practice; ability is seen as the most likely reason for success in mathematics, and success is determined by performance on straightforward tests, which Gregg (1995) argued are anything but straightforward. In this classroom, the teacher and the textbook are the authorities, with teacher talk dominating (ibid; see also Cuban, 1993, for a general account in education). The teacher is the primary if not the sole validator of mathematics (Cobb et al, 1992). Drill and practice with factual recall questions dominate (Fey, 1979; Gregg, 1995; Resnick & Ford, 1981), demonstrating the emphasis on mathematics as a rule-following process (Cobb et al, 1992). Retaining formal mathematics is of utmost importance.

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[13] Formal mathematics represent absolute Truth.

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The nickname *drill and kill* given to the dominant practice in this tradition suggests the unintended negative consequences that accompany this tradition. What students often take away from this practice is that mathematics is a disconnected discipline, that form is privileged over reasoning, that math problems should be solved quickly, and that memorization is crucial to success (Schoenfeld, 1998). This practice has led to the alienation of many learners, particularly those from non-dominant cultures, classes, and, of course, gender (Boaler, William & Zevenbergen, 2000; Burton, 2004). This seems reasonable given the overarching male-dominated, Western view of mathematics that seems to prevail and is not broadly problematized—that mathematics is about Truth and that its entities are real. Lakoff and Núñez (2000) have termed this dominant perspective the *Romance of Mathematics*, which includes the following:

- Mathematics is an objective feature of the universe; mathematical objects are real; mathematical truth is universal, absolute, and certain.

- What human beings believe about mathematics, therefore, has no effect on what mathematics really is. Mathematics would be the same even if there were no human beings, or beings of any sort. Though mathematics is abstract and disembodied, it is real.

- Mathematicians are the ultimate scientists, discovering absolute truths not just about the physical universe but about any possible universe.

- Since logic itself can be formalized as mathematical logic, mathematics characterizes the very nature or rationality. (pp. 339-340).

The idea that mathematics could be something other than this romance is foreign to many people. Yet, in her work with 70 research mathematicians, Leone Burton (2004)

demonstrated that despite common perspectives of mathematical knowledge, how mathematicians engage in research and how new knowledge develops are not in line with this romance. Other examinations of mathematics suggest this as well, particularly in mathematicians accounts of their own work (Thurston, 1998; Muir, 1996), in philosophical literature that prioritizes the act of engaging in mathematics rather than the knowledge that results (Ernest, 1991; Hersh, 1979; Kitcher, 1984; Lakatos, 1967); and in cultural and historical perspectives of mathematics (Bishop, 1989; Grabiner, 1998; Wilder, 1981). Whether in efforts to fend off the negative consequences of the school mathematics tradition, or in an effort to bring school mathematics in closer line with the discipline itself as characterized by the above literature, mathematics education researchers have worked to enlarge our perceptions of mathematics content knowledge that pay particular attention not just to what is known, but to how it is known in school mathematics. This parallels the direction in which knowledge-of-teaching has pushed teacher education.

The editors of *Adding It Up*, developed a five-part description of what they term *mathematical proficiency* (Kilpatrick, Swafford & Findell, 2001); these describe components necessary to learning mathematics successfully. The strands include conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. It is in part hard to argue with such a description. After all, who would not want students to understand why a mathematical idea is important or the different ways in which it is useful (p. 118). Who would not want students to know how to use procedures appropriately and to perform them skillfully (p. 121). As important as delineating these strands is the discussion of their nature. The editors stress that the

strands must develop in an interwoven nature, rather than in isolation. After all, having strong conceptual understanding can facilitate procedural fluency, but also having procedural fluency can allow students to get past some of the workings of a problem to get at the mathematical issues within (Kilpatrick et al, 2001). Moreover, proficiency is neither just present or absent, as it is acquired over time; nor is it measured in terms of some sort of absolute quality. Rather it must be considered in relation to where students are in their mathematics education, as well as their differential access to high-quality mathematics teaching (ibid). Despite these assertions, the editors continually refer to proficiency as knowledge that can be "acquired." They do this in spite of their attempt to situate the knower in relation to this process through the strand of productive disposition; this disposition includes beliefs in one's own ability to do mathematics and to have the motivation to engage in it (ibid).

Others have looked to the discipline of mathematics to make sense of what is important content knowledge for students to develop. Cuoco, Goldenberg, and Mark (1996) have gone as far as to argue for a school mathematics curricula based not on the results of mathematical activity, but on that activity itself, shifting more attention from mathematical knowledge to knowing. They describe general habits of mind—which are actually common processes of sense-making in many disciplines—and mathematical habits of mind, as well as some area-specific ways of thinking in both geometry and algebra. This perspective of content knowledge or knowing, along with Kilpatrick et al's (2001), demonstrate the growing awareness of the interplay between knowledge and practice. Boaler's (2002a; see also Boaler et al, 2000) work with two very different schools, one traditional and one reform, supported this perspective. She demonstrated

how particular pedagogical approaches shaped the forms of knowledge that students developed and the practices that defined their knowledge, as well as in what ways the knowledge was accessible. This prompted her to argue that mathematics is not just about knowing, but about doing (2002b).

Just as enlarging our perspective of what teachers need to know has been powerful in aiding our understanding of teaching, these enlarged perspectives of content knowledge help to provide a better understanding of what to look for in trying to make sense of a teacher's (and of course a students') content knowledge; they also help us consider how to shape teachers' content experience. However, certain questions are not adequately accounted for in these works, such as, How does a learner develop a productive disposition? How do habits of mind grow? Engagement, certainly! Mustn't this involve or rely on one's motivation to do math, or one's willingness to stick with challenging problems? I would argue that the reason practice is so important to what we know mathematically, is because it is in engaging with and puzzling about math that we make and internalize meanings, that we get caught up, or turned off, that, in short, we are affected by it. We need to look at what math means to those who do it and how they are affected by it. As we do not acquire knowledge but rather are shaped and affected by it, how do we bring the learner into our understandings of knowledge and knowing?

Bringing the Knower and Knowledge Together in Knowing

Boaler (2002a, 2002b) has argued for enlarging our view of knowledge in particular by considering what variables we take into account and what frameworks we employ. She has used her research over the last 10 years as a means to consider these alternatives. Her perspective has developed into an argument for looking at the

disciplinary relationship (ibid, 2002a). She considered this disciplinary relationship to be the interaction of the practice-knowledge-identity triad (see following figure). This
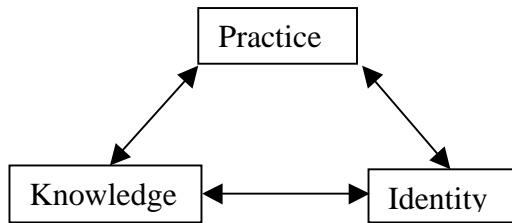
decision was motivated by her research observation that when there is a conflict between the practice and one's identity, involvement is dramatically impacted (ibid), just as happened to me in college. Since making these observations, Boaler has called for investigating the specificities of these relationships and how agency is involved in them. She also has argued that researchers have spent too much time focused on knowledge categories, such as those of Kilpatrick et al (2001), and inadequate time focused on the activity of knowing. This parallels Cuoco et al's (1996) perspective that mathematics educators should shift our focus more from formal mathematics knowledge toward the activity of doing mathematics. We also have spent inadequate time looking at how we are affected by mathematical engagement, in particular what we internalize about the activity as well as the personal feelings, reactions, and connections that arise in the engagement. It would be invaluable to understand how and why these perspectives are brought to bare on new opportunities for engagement.

Recognizing that frameworks of knowledge and knowing do not consider how the knower is affected and transformed by their activity and knowledge development, Handa (2006) has developed a holistic theory of knowing in mathematics. He draws on Martin Buber's image of the *I-Thou* relationship to characterize the interactive process of meaning making between a person(s) and content. Central to Handa's notion of relationship is the idea of mutual affect and co-evolution. He describes two primary

modes of knowing in mathematics, a *savoir* way of knowing that focuses on knowing things, facts, theorems, etc., as well as a *connaître* way of knowing mathematics that highlights a more intimate, personal knowing that embodies this notion of the *I-Thou* relationship; it is similar to how we speak of knowing a person or place well. In order to differentiate the roles of the two modes of knowing and to demonstrate their importance to teaching, Handa described how these operate in the aesthetic field of music, a discipline where holistic modes of knowing are not foreign. A savoir way of knowing involves technical knowledge and mastery. While essential for both a musician and a teacher of music, if this was all the musician or teacher had, she would hardly be considered a musician. A musician also must come to know a piece of music as an "other" in order to be deeply affected and moved by it; she does not just come to know notes, but rather she experiences them through moods, emotions, and intimacy (ibid). It is through the interaction of musician and piece of music that this piece is more than the technical marks on a page. This is the connaître way of knowing.

While Handa notes that in the United States the savoir mode of knowing[14] mathematics is emphasized, which is consistent with the emphasis of the school mathematics tradition, he asserts that it is in developing both modes of knowing, which can support or impede each other, that mathematics meaning making occurs, just as it does in music (ibid). While emphasizing the importance of both ways of knowing, Handa primarily explores the connaître way of knowing specifically because it stands as antithetical to the mathematical experiences of most mathematics learners in the school

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[14] To avoid being overly repetitive, I will use interchangeably expressions such as savoir knowing, savoir modes of knowing, coming to know in the savoir sense, and savoir processes. These all refer to the same process of learning propositional-type knowledge. I will use similar phrasing for connaître ways of knowing.

mathematics tradition. He depicts, through teachers' experiences, what such knowing can look like. To do so, he employs Buber's duality of *will* and *grace*. Will is perhaps better termed *a willingness*; one invests herself in mathematics through a willingness to actively bring-forth-of-self. Grace is passive and can be thought of as an undergoing or as being-affected-by. Thus they are both processes of experience. It is in the intertwining of these that the connaître way of knowing develops: the joining of "the active bringing in of 'will' . . . and the receptive undergoing of 'grace'"—action and reception (p. 40).

Will manifests in a variety of ways—a willingness to struggle, to experience frustration, to put oneself in a vulnerable position by taking on or standing before a challenging problem, to stay with a problem, or to employ purposeful repetition. The image of will-to-overcome (e.g. through brute force or conquering) is not productive here in that it does not allow room for grace; it is not focused on experience, but rather is focused on the end or completion. The receptive nature of grace also is evidenced in a multitude of ways. Affective responses[15], aesthetic reactions, passion, and arriving at understanding are all manifestations of grace. However, the passivity of learning as reception is not relevant to the image of grace; this is *taking-in* rather than *being-affected-by*. Markers of the joining of will and grace are rare, but interestingly are what we would hope all students of mathematics experience. When one is intrinsically motivated to pursue mathematics, when one seeks out and accepts mathematical challenge, when one is simply curious for curiosity's sake or just to understand a question of their own, these are signals of the joining of will and grace that convey a connaître way of knowing and "being in relationship with mathematics" (Handa, 2006, p. 31).

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[15] Affect includes feelings such as love, fear, and excitement as well as the cognitive emotions central to the act of meaning making.

Though the specifics of the evolution of these ways of knowing are not explored, the connaître way of knowing, which is further described as a "becoming," is imbued with a sense of history and direction; such *knowing* is an ongoing process that develops through interaction that may have no real beginning or end. Though this evolution is understood, Handa chooses to focus on the specifics of actual mathematics engagement to understand how relationship manifests through engagement. Thus his framework does not help to analyze the evolution of knowing, or how previous relationships and experiences are brought to bear in new experiences. Consequently, while ideally suited to explore how individuals know mathematics and are affected by it in the doing, it was inadequate to capture all aspects of relationship as I understood it and as came through the data. My data suggests that the historical perspectives that one brings and how one interacts with community members also are critical. Using the term *identity* to describe these characteristics captures the integral nature of these elements to the person. It suggests an evolving entity that is ever-present from interaction to interaction. My hunt for another framework that both fit with my focus on ways of knowing, and could help to make sense of how previous content experiences and interactions within the socio-mathematical context become integral to who we are, brought me to Wenger's broad theory of knowing in organizations (1998). Identity is a central feature of this theory, which filled the area of relationships unexplored by Handa.

In his book *Communities of Practice*, Etienne Wenger (1998) describes his broad socio-historical theory of learning in organizations. While it is unclear how institutions of education fit into this description, the focus on practice and learning certainly facilitates drawing on the work to critically explore the educational setting. Learning is

characterized by the interaction of practice, such as the school mathematics tradition, and the formation of identity in community settings. These are also processes, not static entities. Wenger uses five different viewpoints from which to examine identity, one that I will draw on is identity as *negotiated experience* within the practice. Practice consists of dual processes that interact to support or inhibit each other, in much the same way that the savoir and connaître modes of knowing can interact. These processes are participation[16] and reification[17]. On a large scale, the interaction and balance of participation and reification contribute to the formation of identity (which in turn impacts practice—they are interactively constructed). Identity from this perspective not only relates to how we see ourselves, but it involves the roles we take in the community, the practices and approaches we employ, the perspectives that we hold of the content, and the communities of which we are a part, all of which impact how we engage with the content and how we engage with those with whom we work. A central characteristic of identity is that it is a constant negotiation between the self, the practice, and the community.

While Wenger uses the term *participation* as one aspect of practice, it is conceived of more specifically as a local mix of participation and non-participation. As sources of identity, the mix of participation and non-participation experienced by members of a community shapes, and is shaped by, how members locate themselves in the community, what members will value or disregard, what members strive or know to make sense of or ignore, with whom they seek to connect, how they focus their energies

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[16] "Describes the social experience of living in the world in terms of memberships in social communities and active involvement in social enterprises" (p. 55).

[17] The process of giving form to our experience by producing objects that congeal this experience into "thingness"; this thingness or this form then becomes a focus for the negotiation of meaning (pp. 58-59).

and/or how they attempt to direct their learning (ibid, pp. 167-168). Some trajectories facilitate increased participation, such as peripheral participation; some inhibit participation, such as marginal participation; and surprisingly too, some trajectories achieve escape velocity and propel members out of the community of practice altogether.

Another relevant perspective of identity is that of the *learning trajectory*, which as a "becoming," provides a similar sense of history and direction as the ways of knowing already discussed, but explored in far greater detail (Wenger, 1998). The use of the word trajectory is meant to imply the constant re-negotiation of identity that occurs throughout our lives. Wenger describes it as "continuous motion—one that has a momentum of its own in addition to a field of influences. It has a coherence through time that connect the past, present and future" (p. 154). The direction of these trajectories, which are a joint expression of past learning, present engagement, and future goals, is integral to the negotiation of practice, and thus to the meaning making in which a member will engage in her *negotiated experience* previously highlighted. Drawing on these perspectives of identity, and marrying them with Handa's work, my relationship framework facilitates exploring not only teachers' ways of knowing mathematics in their mathematical interaction, but also how these relationships are integral to and evolve with the person as new or alternative situations in mathematics learning are encountered.

I find combining these two frameworks to make sense of my exploration to be valuable and informative. They share similar holistic perspectives of knowing and knowledge; they share a sense of history in this process of meaning making, and they both are imbued with a sense of relationship, Handa with the content, and Wenger with the others in the communities in which we work. Finally, both avoid dichotomies in their

theories, but rather focus on the importance of dualities. Even so, the scale and focus of these theories are quite different. In particular, Handa focuses narrowly on individual interaction with mathematics to highlight what a connaître way of knowing mathematics could look like. Wenger on the other hand focuses broadly on organizations to make sense of knowing as the community practice and identity interaction[18]. Thus, they each provide an important way of understanding knowing, and interestingly fill the more nebulous aspects of the other's theory without treading on or contradicting each other; each, however, is in itself insufficient to make sense of my questions or data. I developing this merged framework, I have specifically taken on Boaler's charge of taking into account the frameworks we employ. In this paper, as I discuss *relationships with mathematics*, I am speaking of an evolving mathematical knowing that develops through the interaction of an individual meaning making process—à la will and grace—and identity formation, particularly in terms of negotiated experience and learning trajectories. As the interactive construction of these identities and ways of knowing, relationships with mathematics have the potential to develop over the course of a lifetime of mathematical engagement.

Research Questions

Drawing on this combined Handa-Wenger framework of relationships with mathematics, I am able to reconsider the questions I asked previously about the varied engagement I encountered around the parabola investigation. As I consider these

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[18] To use Wenger's construct of identity I must be examining a community of practice, and I have not yet argued that the school mathematics tradition represents such a community. I ask the reader to engage in a mathematical activity. I ask you to assume this for now and see where this will take us. In Appendix A I present an argument for this assumption.

questions in light of the program more broadly, and connect them to the framework just discussed, they ultimately boil down to four inter-related questions:

1. **How do identities mediate how teachers engage in and value their program content experiences?**

This question is related to the understandings that can be developed through using Wenger's (1998) perspectives of identity and practice. The purpose of this questions is to develop a sense of the trajectories that a teacher may draw on in interpreting her experiences, as well as to try to make sense of how the experiences she may have negotiated influence her engagement and responses to it.

2. **How do personal ways of knowing mathematics mediate how teachers engage in and/or value their program content experiences?**

The purpose herein is to explore Handa's (2006) theory of coming to know as related to these experiences. This question will guide explorations into the connection between how a teacher comes to know mathematics and how she thus interprets her experiences as a mathematics learner.

3. **Do relationships with mathematics evolve over a variety of content experiences?**

There always has been an underlying assumption guiding this work once I began to experience shifts in my own relationship with mathematics: that the relationships with mathematics are in fact dynamic and continue to evolve over time. One way to think about how program experiences are valued is to consider whether they facilitate continued evolution of this relationship with mathematics.

4. **How do trajectories of relationships with mathematics influence, and how are they influenced by, the present relationship with mathematics as the interaction of ongoing identity formation and personal ways of coming to know in the context of these program experiences?**

Because the purpose is not to judge whether a program facilitates evolution, but if it does, why does it, and when it does not, why does it not, this question is grounded in a hypothesis. A program likely will not, by its independent nature, be able to facilitate or inhibit shifts in relationships with mathematics. Rather, shifts in these relationships will likely be a product of how program experiences are valued in light of incoming and desired relationships with mathematics. This may have a circular feel, but this is not accidental. In fact, this circularity reflects the assumed nature of relationships with mathematics—that experiences shape relationships, which then shape experiences, which shape relationships, and so on.

These questions together support the initial exploring and teasing apart of this relationships with mathematics. In so doing, this work addresses Boaler's call in future research to investigate the specifics of this relationship (2002a)

Teachers Stories and Their Mathematical Relationships

As I seek to highlight and understand the bonds between teachers' mathematical experiences and their relationships with the discipline, I am going to focus on developing rich descriptions of these phenomena through stories of two middle school mathematics teachers' learning experiences (Carter, 1993). Recall that unlike secondary teachers, who are often mathematics majors at university, middle school mathematics teachers are typically elementary certified, taking only two or three mathematics courses, mostly intended narrowly for future elementary school teachers. Despite this, middle school teachers increasingly are being held accountable for teaching Algebra I/Data Analysis and Geometry for reasons that include increased access and opportunity for students (ITQ Proposal, 2006). In the program with which I am working, many of the teachers already teach middle school, though all but one special education teacher are elementary certified. Regarding their mathematical backgrounds, some have taken little beyond Algebra II, while a few have gone beyond Calculus I. Thus these teachers have received little enculturation into the practices and perspectives of the mathematics discipline.

In focusing on such a group of middle school teachers who are working to further their mathematical knowledge, I hope to shed light on the interactive negotiation of experiences and relationships in a context where this understanding could offer critical, timely insight into the mathematics education of teachers. In doing so, I will work to highlight plausible explanations of why things are as they are, without judging whether they should be that way. The narrative, descriptive structure of story allows me to develop real, personal, and complex pictures of the teachers with whom I worked and their relationships with mathematics (Carter, 1993; Chase, 2005). Even with my goal of

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developing complex images, any presentation must be an oversimplification (Carter, 1993; Chase, 2005). Like using a flashlight to explore a room, this exploration will focus on a few key issues, which will in turn cast some light on other issues, but some things will remain hidden from view. Even so, these stories provide an entrée into making sense of what is at stake in the interaction of experiences and relationships, and into recognizing the variety of issues that must be considered in both further research and in design of content experiences for teachers.

In order to tell these stories and explore the interplay of experiences and relationships, I will highlight three sub-stories for each teacher. I will draw on each teacher's own words to construct and discuss each of their histories as a mathematics learner and teacher. In addition to introducing these two teachers and providing an opportunity for initial exploration of their relationships, these histories then will serve as a reference point from which to critically consider each of the next two sub-stories. Following these histories, I will present the teachers' accounts of how they experienced the various types of program experiences in this master's program. As in the previous sub-story, respecting and understanding these teachers' subjective meanings of their experiences is an important goal (Chase, 2005). I will use these to draw connections between their histories and their current experiences. It will also provide an opportunity for me then to engage in a case comparison. For this comparison, it is important to understand that this comparison is undertaken not because the comparison itself is intrinsically valuable, but because the comparison helps to emphasize through contrast the different ways in which these teachers experienced the program and the opportunities within it. The final focus of this section is on these teachers' actual mathematical

engagement. I will describe and analyze two of each teacher's inquiry experiences. These will be used to explore the alternate components of relationships with mathematics: ways of knowing mathematics (Handa, 2006) and mathematical identities (Wenger, 1998). Drawing on the development and discussion of these two stories, I will then in section 3 be able to discuss what I have learned more broadly and use this to think about implications in terms of mathematics teacher education and future research. Using story and teachers own words in this inquiry is principled on other grounds as well. Employing story (a) allows a direct connection to the overarching framework of this inquiry, which is about journey and process; (b) it reflects the assertion that people, women in particular, live storied lives (Carter, 1993); and (c) it maintains the perspective that teachers are the sources and directors of their growth (Richardson & Placier, 2001).

Two Teachers' Stories

The two middle school teachers on whom I will focus my storytelling efforts were ultimately chosen over a one-and-a-half year process of interaction with the teachers in the program. I initiated data collection on all members of the program cohort who signed the consent forms (all but one teacher in the program). Following 8 months of data collection, as the time for the first interview approached, I examined the data collected to that point to select six teachers to interview in order to gain access to their subjective valuations and meanings. The data I examined included mathematical autobiographies written at the start of the program, two belief surveys given 10 months apart, journal responses written over the course of the two completed inquiry experiences, classroom conversations, e-mail communications, and my own intuitive understandings developed over the long-term interaction with this group of teachers. I used this data to select six

teachers who had been relatively consistent in reflective efforts across the experiences, and who represented a wide-range of experiences as learners of mathematics. All six teachers that I invited agreed to be interviewed up to four times over the following year. The importance of the multiple data sources and the consistency of reflective participation were essential to being able to develop holistic representations of these teachers (Yin, 2003)

I interviewed[19] all six teachers twice, the first interview focusing on their experiences as learners and teachers of mathematics, and the second focused on their experiences in the program. From these six, I selected two focus teachers, on the one hand, whose preparation appeared rather typical of elementary-certified middle school teachers, and on the other hand, whose stories seemed to offer critical windows into understanding learners' content experiences, through the evolution in each story. Their transformations provided unique access into the interaction of experiences and relationships. Though I selected two focus teachers, I actually interviewed four teachers in a third interview. This interview was designed to explore in more detail the previous discussions, to probe issues that had emerged as important for each teacher through previous data collected, and to discuss experiences that had occurred between the second and third interviews. I had opted to interview four of the teachers in case something new emerged from the other cases that seemed to conflict with the broader findings of my two primary cases. While no obvious conflicts emerged, the relationships of these other two teachers were distinguished more by their relative consistency across their histories, their experiences, and their inquiry, rather than by transformations. I had one final interaction

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[19] All interviews were semi-structured in nature, audio-taped and transcribed (Glesne & Peshkin, 1992)

with the two teachers with whom I worked closely. I provided to each of them the portions of the data and analysis that referred specifically to them in order to provide them with an opportunity to verify my presentation and to respond to my interpretations, providing another avenue for their voices to impact this inquiry.

Amelia: The Girl Who Would Be a Teacher

> *Mathematical learning experiences prior to becoming a teacher.*

Amelia always enjoyed mathematics growing up; in fact, she liked school in general. "*I was a big nerd. I just liked being there. I liked learning. I never had a problem with it. I was usually one of the more advanced kids for a long time*" (Interview 1, July, 2006). In her mathematics classes, she felt that everyone had similar backgrounds and that she consistently learned as quickly as others in her class; she suggested she learned the mathematics with a facility that enabled her to explain content to her peers if needed (Interview 3, June, 2007). With her peers as reference, Amelia recognized that she was learning and understanding what she should. Her confidence was strong.

Her interest in math also had a family root; her mother was, and still is, a high school mathematics teacher. She has been a strong influence for Amelia, who described her family in general as very close. Amelia took advantage of that resource by working with her mother on mathematics regularly. However, this relationship was not the typical one of "let me get my mom to help me with my homework." It was much more involved; often her mother was actually her teacher, though only at home. Amelia described a math course in which the work she did with her mother at home had a significant impact.

> I was always in the top group. But then in Eighth grade, we had a lot of
>
> introductory algebra. I would have my mom teach me at home, so that I

would go back to class and I would almost be annoyingly smart, like I was

teaching the teacher. . . . [The teacher] would actually start to ask me

questions because she wanted to compare it to what my mom and I had

done. Which was cool, because she embraced it, instead of getting

annoyed with me. And I got along pretty well with her (Interview 1, July,

2006).

Math class was a place where Amelia was able to demonstrate her knowledge and

expertise, while often her learning of material took place at home with her mother in a

one-on-one situation.

Perhaps reflecting her mom's influence, Amelia just presumed she would be a

math teacher; it was the natural step in her eyes, so much so that she "played" school:

I would always go to my mom's school when I was off in elementary

school, and I would stand on the countertops and teach the kids that were

in there on their break, of course I was doing whatever I was doing in fifth

grade, and they were humoring me. I would do that all the time. So I kind

of knew that I wanted to (Interview 1, July, 2006).

Thus her interactions with her mother not only helped her learn mathematics, but these

also provided the space to consider and act out its role in her future. From an early age,

Amelia seemed to like being in the role of instructor, demonstrating what she knew,

similar to how, in Pre-Algebra, she liked to demonstrate in her classroom what she had

learned with her mother. Looking back though, Amelia acknowledges that what she knew

of math at that time was primarily procedures: "*I never learned meaning behind anything,*

but I always thought that I was very good at remembering my directions, you know?" (Interview 2, September, 2006).

As is all too common in learning mathematics, Amelia eventually had difficulties, though hers came later than for many. She cited Pre-Calculus in her junior year in our first two interviews as the place where she began to have trouble with learning her mathematics, which is not an uncommon location for difficulty.

> Actually, if I am truly being honest, in my junior year of high school, it's
> probably when it started to lose me, cause I can't remember Pre-Calculus.
> . . . I can remember so many things, and I am very good with my
> imagination and memory. . . I can barely remember what the classroom
> looked like and then what they taught me. I can't remember one thing that
> I learned in that classroom. 'Cause it must have been like—memorize,
> memorize, get it out by the test and be done! I don't even know what I got
> in that class. . . . and I think that that was the point when I like, stopped
> feeling so confident. . . . Because for so long I was doing fine. I did really
> well in Algebra. I did really well in Geometry because I could come home
> to my mom, and she would teach it to me. I don't know that it. . . but it's
> still the same thing, at no point did I feel like I really learned the concept.
> It was all procedures (Interview 2, September, 2006).

This type of sudden change or loss of "ability" to do mathematics is a typical description given by those who have math anxiety (Tobias, 1993). While Amelia reiterated the procedural focus to her learning, she credited her mother with her avoiding difficulty up until then. However, it appears that the challenges of Pre-Calculus were such that her

mom could not fully fill the void. Of course, because Amelia could not recall the class, what the nature of those difficulties was we do not know. Even so, there was a large shift in her confidence. She suggested that this coincided with a shift in the nature of the classroom, which changed from being a place where she had all the necessary knowledge to a place where she actually had to learn the material for the first time. "*I think it was really just actually not knowing the material, what I was being taught for the first time, and the need to learn how to learn it*" (Interview 1, July, 2006). Amelia opted not to take mathematics her senior year (leaving a gap in her experience) in part because of her lack of confidence; she decided to try again in college with Calculus. Having been unable to locate the off-campus course until the third session, when she attended the third class, she felt so far behind, she dropped the course immediately and did not take another math course in college except those required for elementary certification. This limited mathematics in college is quite typical of elementary-certified mathematics teachers.

In the span of a few years, Amelia transitioned from a student who loved math, was in the high group always earning strong marks, and worked closely with her mother, to a student who had little confidence and steered away from mathematics courses. The effect was so strong, in fact, that she altered her long-time professional plan. She decided instead that she would teach English and reading (Interview 1, July, 2006). The effects are further evident in the alteration of her perception of herself as a mathematics learner. She talked about not knowing if she really ever was great at math, questioning earlier success (Interview 1, July, 2006), though she did assert that she had been confident when she was younger (Interviews 2, 3). She also seemed to be aware of where her understandings were limited. For example, she felt she had not developed number sense,

at least prior to the program (Interview 2, September, 2006). Even now she says, *"I don't think of myself as a very savvy math person. I just like it"* (Interview 1, July, 2006). She reiterated this in Interview 2 when she said, *"I don't have a lot of confidence when it comes to doing it, but I obviously like it enough that I am teaching it by choice. I was given the choice."* So, if she is teaching it by choice, what then led her to return to wanting to teach mathematics?

Mathematical experiences after becoming a teacher.

In her first year of teaching, Amelia taught middle school reading and English and agreed to teach math only as a favor to the principal. It was in this year, having to work with the mathematics of middle school, that she readily found her early appreciation of mathematics once again. It was simply as if her love of mathematics that she had had in her middle school experience reappeared when she was in the middle school setting working with that content. She requested to have her teaching focus switched to only mathematics the following year. She asserted, *"Math is always a good experience. I think I'd be bored to tears doing anything else. . . . Math is just more fun, you know? More rewarding"* (Interview 1, July, 2006). After asking her to explain more what she liked so much and found so fun, she said:

> I like coming to an answer, and [working to get the answer]. I like figuring
>
> it out, like a puzzle. . . I like working it out and having it work. You know,
>
> there is a process, and I like keeping it very neat, and just having it done, I
>
> don't know. I think I just like that there is an answer (Interview 1, July,
>
> 2006).

In later discussing her rapid transition from discomfort with, to love of, mathematics, she reiterated enjoying its structure, its logical reasoning, and its answers (Interview 3, June, 2007). This prompted a discussion around another pastime that she really enjoyed:

Amelia: *I like to go fishing for crabs or whatever.*

Eden: *What do you like about that?*

Amelia: *There is a result, I either catch the fish or I don't, and if I catch a large one—*

Eden: *There is that competition [laughing].*

Amelia: *Yes! I am very competitive. There are sports, you know, growing up, and I liked it cause of the competitiveness. . . . But, I don't know. I like cleaning. I like the end result, sometimes I like not only the satisfaction that I get out of it, but like the satisfaction that other people might get out of it, like—I don't know.*

Eden: *Somehow I don't know if cleaning gets other people excited –*

Amelia: *It does! If you are doing it for somebody else—*

Eden: *Oh, okay.*

Amelia: *Not like I am running a cleaning service.*

Eden: *You could come and clean my house! [laughing]*

Amelia: *Like for my family or for my boyfriend, like you do things like that that kind of will—and I really do like doing those things...*

Eden: *In what ways, those activities that you described, in what*

> *ways do they give you satisfaction in a way that math does, and in what way are they different?*

Amelia: *I think the same is having a result. And maybe not specific to math, but teaching, the helping other people, like seeing their reaction, like seeing whether or not it works for them or not, you know? The ways that they are different—Actually, no. The way that they would be the same also would be like my own little internal reward that I get out of it, like I have internal reward when they get it in math, and I have a little internal reward when they appreciate, you know, what you are doing for them, or the fish that you caught for them, or for dinner. How they are different, besides the obvious [laughing]. I guess, I mean I don't know. They are all so similar in the sense that—I mean, I go out with my dad and my dad has the knowledge to get the fish, like he is very experienced. I'm not so much, and I am always learning. You know, and I always like that. So that's another comparison. It's laid back, I guess. That's something I can do for leisure, whereas math supposedly—I mean, I don't sit around and do math* (ibid).*

Amelia has enjoyed structure and results in her life more broadly, and math seemed to fit well into this world. In addition, the interest in helping people and in working closely with an expert, in particular an expert parent, also were prominent in her discourse.

Amelia has now been teaching sixth grade math in sixth and seventh grades for eight years and will begin her ninth year in 2007 along with seventh grade mathematics. She talked during the third interview, almost with surprise, about the fact that she was still learning math and that she was not bored with sixth grade mathematics. She actually expressed some discomfort at being asked to teach the on-grade level seventh grade course during the 2007-2008 school year, because she felt she still was trying to learn and master teaching the sixth grade mathematics curriculum. In particular, Amelia has enjoyed teaching at- or below-grade level students, with whom she feels she has worked particularly well.

> I don't really want to have the GT kids, cause I feel that my strength is the
>
> on-grade level or the below-grade level kids. I enjoy the GT level kids,
>
> …[but] they were just on it, and they wouldn't need me. (Interview 1,
>
> July, 2006).

Having had math difficulties herself, she has been able to sympathize with similar students. In her descriptions of teaching, she conveyed that she wants to help her students to achieve in spite of difficulties and will do what she sees as necessary:

> I try and figure out a different way to do things, 'cause I mean, I used to
>
> be, and I think I still am, very lecture and practice, traditional classroom. I
>
> really tried to change that in the past couple years, and that's my goal. I'm
>
> getting better at that, and they do like that. . . . I'm even seeing changes. . .
>
> we just got our scores back, and we made the biggest jump this year, like
>
> one of the top schools, sixth grade math made a huge jump. So that's the
>
> kind of stuff that I get pumped about. I showed the kids in class and like,

> "look at the graph," and that means a lot. So I do like the on-grade level
>
> (Interview 1, July, 2006).

Amelia self-described as a primarily traditional teacher, though she is trying to alter that for her students. Her excitement in working with these students was palpable, particularly when their performance improved dramatically. Gains in performance seem to hold significant meaning for her, and she wanted this to be meaningful for her students as well. Her desire to work with at- or below-level students seemed to be a combination of her ability to connect with these students and that she could effect a large difference in their performance. When talking about establishing rapport with her students in Interview 1, she suggested her organization and strict nature were important, because her students did not get much structure at home. She felt they appreciate having structure and having a comfortable place in which they could learn. She has been active and invested in the school, and the students have seen this. She described extra time investments in her students to help improve performance, reflecting her mother's influence, and what she wants for them in terms of their mathematics education:

> I definitely just want their confidence level to be [high]. You know, like
>
> when I did after-school programs, or a lot of times I just have them stay
>
> after school if they want to for extra help, and it ends up being like its own
>
> little program cause I get a lot of kids that will do it. You just watch them
>
> in class the next day and it's kind of like how I felt when I had my mom
>
> helping me at home. I could go in the next day and raise my hand for all
>
> the homework questions, and you know, you felt comfortable knowing
>
> what was going on. And I love that, like when my little kids do that. And

> they'll even tell you, "You know I worked so hard, Ms. Clef." You know? I love that. So honestly, as long as I see them becoming more confident, and not being so scared of math, that's all I want. And you know, and if that means that one kid knows how to multiply by the end of the year, and another kid knows how to write an incredible essay about their mathematics, that's fine. But as long as everybody kind of grows a little bit [pause]. So I just want them to like it, not be scared of it. I don't like recommending them, like when parents want to push them a little further even if they are really with it, I don't like forcing them because I don't want them to ever not like math. I don't want them to get overly frustrated. A little teeny tiny frustration is good, but not overkill, cause then they don't like it. (Interview 1, July, 2006)

She has not just been finding ways to help students perform, she has identified herself in their activity. She also has consistently looked for other ways to have her classroom be a place of confidence, rather than a place where students struggle to learn. This includes some individualization of expectations. She spoke with enthusiasm about a game she developed that facilitated one's girl's engagement and motivation:

> This one girl that I have typically can't answer the most simple questions. She was sitting there having a great time, and I was listening to her the most; she was one of those people that she 'got it.' And she was answering questions and her confidence level was going up, and she had so much fun in the game (Interview 3, June, 2007).

She explained that by facilitating high levels of confidence in her students through such activities, they were less likely to be scared of math. She is against pushing students forward, because that creates the possibility that they will not like math.

While Amelia described herself as a traditional teacher and reinforced this by noting that she has found it frustrating when students use new algorithms (e.g. lattice method of multiplication) or invent their own when a well-developed, common one is available, her rhetoric suggests that she has not been entirely comfortable with this aspect of herself. She spoke of trying to be open to listening to students present alternative perspectives, trying to make sense of it herself, and then conferring with a colleague "just to make sure" prior to including this alternative perspective in her teaching; though she would only do this if she felt the alternative approach would not confuse students (Interview 1, July, 2006). She has expressed a similar tension between her desire for her students to be able to understand the math and her procedural emphasis (which she has worked to change):

> I think learning mathematics to me in the beginning was simply like, do you know how to do this procedure? Whereas now, I am like "Do you really understand [this]?," "Do you have number sense?", "Do you understand where this is coming from?," "Are you able to kind of give me a reason as to why you did what you did?," or "how that works out" or whatever. . . 'Cause math changed for me, like before it was it was procedural, and it still is cause it's hard to be procedural and then change over, but I feel that I learned procedure—but you know, I'm changing. . . . Things are making more sense to me. I tell the kids all the time. . . . I

understand fractions and decimals, and all these other things so much more now because I am teaching them, you know, which is why I try and have them talk to each other more and explain themselves more, and maybe teach each other. Because I feel that's how you kind of learn it, you know? So that to me has changed a lot, like I am trying to go away from procedure now. And now I feel like I am asking "why?" more often (Interview 2, September, 2006).

There is also a seeming paradox in how she has viewed herself versus how she perceives her students. We spoke briefly about William Thurston's (1998) assertion that a mathematician helps "people" to understand mathematics, and I wondered how this compared to how she viewed her work as a teacher. Agreeing, she said, *"As a teacher, I think that we should be helping the kids to understand all the pieces"* (Interview 2, September, 2006). So then I was curious if, given Thurston's comment, she would consider herself or her students mathematicians. She described herself as, *"an aspiring mathematician [laughing]. 'Cause I don't feel like, and I don't know that I want to feel this, but I don't feel like I have the conceptual understanding about most of the things I'm gonna be teaching. And I am trying to get there"* (Interview 2, September, 2006). However, there was a subtle shift when she began to speak about her students. She said:

I feel like when you say "mathematicians," I almost feel like it's a mastered thing. I don't think that you have to know every algorithm in the world; I think that you just have to be able to learn it, get comfortable with it, and understand like the connections that it has. So yeah, I think that my kids are mathematicians, you know? 'Cause some of them are better than I am because they explain really

> [well]. I think it's more about having an understanding of math than it is about being able to do math (ibid).

The tension in her perspective of herself as a math learner versus that of her students was apparent in more general speech. While she acknowledged she saw herself as not "a very savvy math person," she spoke of her frustration at the misconception that "I am not mathematical," referring to her students' perceptions of themselves as math students. She further stated that her pet peeve is students asserting they cannot do a problem. "Well, yes, [they] can," she said, implying that she does see *them* as mathematically capable.

Amelia indirectly credits her students with how *why* became important to her. She realized her procedural focus did not allow her to answer their questions. She was in part curious to understand why, but she also had come to believe her difficulties with remembering procedures and having to look them up each time she taught them (e.g. how to find new coordinates of a figure under reflections or rotations) were because she did not have a conceptual understanding of the procedures. Not knowing why inhibited her teaching. While she could temporarily relearn procedures, she felt unable to explain why those procedures were the correct procedures (such as why you would multiply a given coordinate by negative one in a rotation). Shifting this perspective was one strong influence in her desire to pursue this master's program: she wanted to get better at teaching. She talked about pushing herself by setting high expectations for herself (and her students), but also hoped it would improve her confidence (Interview 1, July, 2006). She reiterated this in our second interview: "*I am putting money into a program that I want to make me better. You know, like I want to do it. I want to have that confidence, but I don't feel like I am confident.*"

Reflective analysis of Amelia's history.

If we begin by looking broadly at Amelia's interests, two issues readily emerge. She enjoys activities that offer concrete results, as well as working toward those results. In addition, she gravitates towards activities that involve rather centrally a close relationship of assistance, like teaching or learning to fish from her expert father. Of course there is no teaching in cleaning, but she enjoys the reaction and appreciation of those she helps. Mathematics has offered similar rewards. She values that it yields an answer, and that it is quite structured. This emphasis on form and answers suggests that Amelia was enculturated quite strongly into the school mathematics community of practice, and developed the savoir way of knowing mathematics emblematic of this enculturation. The typical benefits of this emphasis on form and answer include providing immediate feedback and allowing for a more objective, tangible form of evaluation (Tobias, 1993), just as Amelia had suggested with fishing. However, what would happen in school when the right answer is elusive? Seemingly more important than the mathematics itself, however, was Amelia's love of explaining. Is this reflective of the importance of a close relationship of assistance where she herself was able to take on the expert role? In working so closely with her mother on mathematics, and having her mother teach her material prior to encountering it in school, a close, purposeful relationship developed that continues still. I often wonder, though, why this did not lead to boredom in her mathematics classes—if anything it seemed to promote participation through knowledge demonstration. Why did she form this type of relationship less often with her teachers in her classrooms? Was it just the lack of the one-on-one dynamic?

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When I think of her playing school as an elementary student, or using her algebra class as a place to demonstrate what she had learned with her mother, I am reminded of Vivian Paley's practice of giving the students in her stories titles that capture their classroom roles, such as *the boy who would be a helicopter (1991)*, *the girl who would be a unicorn* (1999) and *the boy who would be a truck* (1999*)*. As I think of Amelia's early mathematics activities, I regularly return to the title *the girl who would be a teacher*. In Paley's stories some of these children-with-titles wrapped up the entire classroom community in taking on this identity, and providing an important service to the classroom in the process. Speaking to the teacher who had the boy who would be a truck in her classroom, Paley said,

> The moment you told me about [Timmy] I wanted to meet him. Here is a
>
> child who has spent an entire year living one story and bringing all of you
>
> into it, he provided a source of continuity for the entire classroom and for
>
> you as well. What does it mean? One schoolroom recognizes the fact that
>
> a classmate whishes to pretend he is a truck and, it seems to me, the
>
> heavens must rejoice (1999, p. 81).

These identities that young children developed, whether briefly for a story or for the whole year, provided the children with a sense of stability and also provided the means to consider and work through issues of concern (e.g. such as how to deal with anxiety when strangers come into the classroom as in the case of the truck boy, or perhaps in Amelia's case, how to walk into her mother's classroom full of high school students); moreover, these stories enabled children to try on roles. Noticing that students tended to take the same role repeatedly, Paley asked a child why. He simply responded, "It's on their mind.

Is it a good thing?" To which she responded, "Yeah, 'cause then you know who someone is." (ibid, p. 108) Did playing teacher help provide stability and comfort to Amelia and those around her? Did it allow her to try on her future role? Did it help her deal with anxiety or help others in her class deal with theirs? Mathematics is after all an historically anxiety-ridden school practice.

Working with her mother at home was integral to Amelia being able to take on this role of teacher. With her mother's assistance, she was able to work with the community to negotiate a practice with which she felt quite comfortable and in which she could fully participate as a member of the classroom community. She actively participated through knowledge demonstration, at times perhaps encroaching on the teacher's role; she herself recognized that her algebra teacher effectively could have interpreted Amelia's behavior as a challenge of sorts, though the teacher did not, just as Paley and the truck boy's teacher did not. There is no indication that her peers did anything but accept this role, just as occurred in Paley's classrooms. The understanding and acceptance by the communities of these roles, marked by other students riding on the boy's truck or Amelia's teacher actively asking her about her work with her mother, facilitated interaction that might otherwise have been difficult or anxiety-laden.

While there is evidence in Paley's stories that her students would eventually abandon these roles, Amelia's desire to be a teacher perhaps inhibited giving up this role—it had many purposes. In this case though, the ongoing nature of this participation had problematic mathematical consequences. Amelia did not learn to use the classroom as a place to learn mathematics, which she suggested when she talked about needing to learn how to learn in Pre-Calculus. The classroom always had been a place in which she

felt comfortable, but it also had been a place where she walked in ready to explain, using her new background knowledge. Was this classroom comfort dependent on her being able to maintain this role and thus was dependent on her mother? What would have made her uncomfortable? After all, regarding her early schooling, she did not share any discomfort with me. Or was it simply that her mathematics learning trajectory was dominated by the goal of becoming a teacher, which was consistent with her orientation toward results? She provided a hint when she indicated that having background knowledge before going into class helped her feel comfortable because she knew what was going on—it prevented anxiety (Interview 3, June, 2007), and that she in turn helped her students develop this same knowledge to facilitate confidence and prevent anxiety (Interview 1, July, 2006). Comfort was strongly related to background knowledge.

When the practice of her mathematics classroom changed, however, and the classroom became a place where Amelia had to focus primarily on learning; she could not be the girl who would be a teacher, and she could not walk in knowing what was going on. Her sense of control diminished, prompting anxiety (Tobias, 1993). As this occurred, her identity of participation shifted to marginal participation in the classroom community (Wenger, 1998). This shift profoundly impacted both her willingness to engage and her affective response to classroom practice, leading her in the direction of non-participation. She ultimately lost confidence in her ability to learn mathematics. While this may seem unusually sudden, such anxiety and perceived failure in math often seem to appear suddenly as it did with me in college, because difficulty is often masked by good grades (Tobias, 1993). Just as was its effect on me, this difficulty caused Amelia to question whether she had ever been good at math, a doubt that is still with her. One

perceived failure cast a shadow on all previous success, as is common with math anxiety (ibid). In this situation, a focus on form and answers has negative consequences. This emphasis "may result in panic when the right answer is not close at hand" (Tobias, 1993, p. 67). Amelia did in fact drop mathematics after her first year of trouble, further reifying her non-participation. Though she tried again briefly in college, her calculus experience seemed rather to confirm a learning trajectory that ejected Amelia from the mathematics practice.

Another reason for her dropping mathematics seems possible. Amelia was confronted with a practice where her role of teacher could no longer be viable, just as I found I could no longer assume the role of the girl who would be an astrophysicist. As she could not demonstrate her knowledge, her past and future goals combined to imply that math then was something that she could no longer teach. She changed the context of her trajectory, though not the teaching focus itself. One question that I have not been able to shed light on is Why, in her junior year, Amelia's mother was no longer able to help Amelia acquire sufficient knowledge to continue to demonstrate her knowledge. Why did the answers stay out of reach? Her mother conceivably had the knowledge, since she is still a source of mathematical assistance for Amelia. One explanation that I have considered is that given the form and answer focus of her learning, Amelia had developed strong procedural understanding (Skemp, 1976). However, Skemp noted that while procedural knowledge has its advantages, it requires developing much more specific knowledge; in fact, for each process, specific knowledge must be learned, and thus the amount of material that a learner of mathematics must hold onto balloons and becomes vastly unwieldy. Pre-Calculus would be a logical candidate for such difficulty, because in

a traditional curriculum, it is often a mish-mash of disconnected presentations like conic sections, rational functions, limits, trigonometry, etc., so often the topics do not even build on each other. Of course, one can only conjecture, because Amelia herself has almost no memory of what occurred in that class to lead to the difficulty.

When Amelia came back to teaching mathematics, if only as a favor, her discourse suggests that her strong participatory identity re-emerged, because once again the classroom practice was one that involved her knowledge demonstration and a teaching relationship with the others in the classroom. The girl who would be a mathematics teacher was a mathematics teacher! She was excited about the mathematics, she loved it, and she enjoyed the structure, not just of the math but that she was able to create in her own classroom. This renewed practice and identity, however, had to co-exist and negotiate with the other sense of herself as one who had *faked mathematics* years before, to use a description of how those with mathematics anxiety come to perceive themselves (Tobias, 1993). Her confidence in more advanced mathematics was so shaken and was such a strong part of her experience that it has led her to be leery of teaching even seventh grade mathematics, though her troubles technically began much later. Today, she would prefer to master what is in the sixth grade curriculum, which she has taught for eight years already; there she is comfortable, and she knows the practice is one in which she can participate as she wants and she can continue to learn at a steady pace.

Interestingly, it was through teaching, specifically not being able to answer her students' *why* questions, that Amelia began to question whether her savoir way of knowing mathematics was adequate. She wanted to develop more of relational understanding (Skemp, 1976) of mathematics for pedagogical purposes. Again, we see

the teacher emphasis dominate and shape her activities as a learner as she initiated a mathematical shift for pedagogical purposes. I actually wonder if her difficulties in Pre-calculus helped her to hear the questions that her students would later ask. Amelia suggested that she is a better teacher because of the challenges she has faced. On one level this seems to contrast startlingly with Handa's (2006) description of how in music teaching, having a more intimate relationship with music is essential to being able to reach students and help them develop a similarly rich relationship. In mathematics, however, it is difficulty in mathematics that enables teachers to connect with and help their students learn. Not surprisingly, these seem to reflect the commonly perceived aesthetic essence of music and objective essence of mathematics.

In examining Amelia's history as a student through her time as a teacher, her willingness to engage in mathematics learning came in no small part from her drive for the answer, her relationships with her mother, and her to desire to teach and help others learn. Her affective responses similarly seemed to develop in response to the answer (or lack thereof in the case of anxiety), or the demonstration of knowledge, not in the coming to understand the mathematics as is typical of an experience of grace that can promote a willingness to engage (Handa, 2006). Because it has not been the process itself that has motivated her, will and grace have not developed in a manner conducive to their interactive construction, and thus a savoir way of knowing mathematics has been privileged. A connaître process of knowing mathematics has had little opportunity to develop as is quite typical of students and teachers of the school mathematics tradition (Handa, 2006).

Beth: The Girl Who Would Ask Why?

Beth is a middle school mathematics teacher. She has been teaching for 15 years, only the last 5 of which have been in mathematics, and is the head of her department. She is working on her second master's degree; her first was in gifted education. Beth's own mathematical experiences throughout her life have shaped who she has come to be as a doer of mathematics, (and from her perspective, who she is as a mathematics teacher), and this identity has (and continues to) evolved in conjunction with her ongoing mathematical experiences.

Beth's experiences as a learner of mathematics prior to teaching.

From elementary school through college, Beth's experiences can at best be described as difficult, if not disheartening and anxiety-laden, though they were punctuated by a few important positive experiences. When I asked her about her early experiences, she initially said she did not remember anything in particular, but then began recounting negative experiences. The first, to which she then added she would never forget, did not take place in school, but rather in working with her father.

> We were at the kitchen table doing homework, and it somehow turned into a lesson on binary numbers that he was going to teach me. And I didn't get it at all. It was very frustrating to me, and he would get frustrated that I wasn't understanding what he was explaining to me. But he never made me feel stupid about it, so it was okay. But I was 8; I just couldn't get it and he couldn't understand why I wasn't getting it (Interview 1, July, 2006).

While it does not appear that she felt inadequate at this point, she expressed significant frustration at not understanding that which her father expected her to understand (though binary numbers are not typical mathematics for an 8-year old)—a point that was communicated by her father's own frustration at her not "getting it." This seemed to be exacerbated by her perspective of her parents as people who *get math*, so, while she is part of a family that *gets math*, math never seemed to resonate with Beth. Rather, when asked about her perspectives of herself as a learner then, she simply said, she came to expect that math was something she could not get (Interview 1, July, 2006).

What was this math that she could not "get"? In elementary school, Beth was not able to memorize her multiplication tables. In Algebra II, she was never able to memorize the Unit Circle, leading in part to her repeating the class (at her parents' insistence). She also had failed and repeated Pre-Algebra. She clearly did not enjoy algebra; the very thought of symbols invoked a panic response. In considering why symbols gave her such difficulty, she expressed thoughts like, "*I think of Moses with this thing that just came down from heaven [laughing]. 'Cause that's what I thought [symbols were]. Where did this come from? It is a foreign language*" (Interview 1, July, 2006). She tried to make further sense of this particular challenge in our second interview: "*I think since I never had anything connected to them, they just didn't make any sense to me. I didn't know where they came from; I didn't know why they existed, what I was supposed to do with them. There was no making sense of anything*." Echoing what many consider to be typical results of participation in the school mathematics tradition, the math that Beth could not get was disconnected, static, and handed down from on high (Burton, 2004; Schoenfeld, 1988; Ernest, 1991): "*If it was given, you memorized it [pause]; that was it;*

it didn't change. . . . Math is math; it just was. The only thing that existed was what my teacher told me, or what was in the book. That's where it ended—until I got the next book" (Interview 1, July, 2006). Authority and agency were entirely external.

The power of these difficulties with memorization and symbols was profound, as Beth began to question her own intelligence: *"I thought, 'I am so stupid,' because I could not remember, I mean, you know, 'beat yourself up'[20], and if I was smart, I would remember this stuff and I would be able to do it"* (Interview 2, September, 2006). Moreover, rather than isolating this experience, math infringed on other content areas and shaped how she assessed them; her understanding of mathematics as disconnected and having one correct expected answer became a litmus test of sorts. She eventually steered away from the science she had loved (particularly Astronomy and Chemistry; Biology was fine), because it was too "mathy" for her and she did not feel she could do it. She even paralleled an English course that she hated to mathematics: *"I wrote my paper and the professor said that I thought about it wrong. The right answer was such and such. And I thought, 'god, that's like math!' That was my experience with math"* (Interview 2, September, 2006).

Of course, her perspectives were not just shaped by how the mathematics was presented, and whether or not she got it, but also by the student-teacher interactions. In particular, she observed that her algebra teachers did not really interact with students, and that they would mainly put problems up on the board from the textbook, which the students were required to sit and do during class. Complicating matters, she noted that this was contrary to her way of making sense of things.

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[20] Leone Burton (2004) has suggested that traditional school mathematics actually inflicts a type of violence on its participants, as is echoed in this image of violence.

'Cause I was the kind of kid—not the 'sit in the back', but the 'why do I
need to know this? When am I going to use this? Who cares?' That was
definitely me with a lot of stuff. . . . I want connections. . . . And I ended
up taking Consumer Math my senior year, 'cause I had it. I was done.
(Interview 2, September, 2006).

Rather than valuing these (natural) questions, her teachers conveyed to Beth that she was

a discipline problem for asking these questions; math was not about *why*. Reflecting a

willingness to push back that she still seems to possess, Beth acknowledged that she then

did become a bit of a discipline problem by acting out instead of asking questions. It is

worth highlighting that, despite these negative experiences and her own admission of

being *done* with math, she still took Consumer Mathematics in her senior year, a typically

low-track terminating math course, rather than no math. What is particularly interesting

about this experience is that she described it as "*easy*," but then added "*it was boring*,"

suggesting that success was not what piqued her interest (Interview 1, July, 2006). This

course certainly did not add to her enjoyment of mathematics, which she avoided almost

entirely in college. She took only the math required for an Elementary Education

certification—a common situation for elementary certified teachers, as well as teachers in

the cohort.

As I previously indicated, Beth's early mathematics experiences were punctuated

by moments of positive teachers and interesting content. Engaging with Geometry

provides particularly fond memories: "*I remember building a lot of stuff with straws and

all that. I thought proofs were really cool. I loved that logical sequence, and everything

had a reason. The theorems and postulates and all that*" (Interview 1, July, 2006). When

talking with her about algebra in the context of geometry, she said that it was fine, because she had something to which she could anchor the algebra: "*I'm very visual anyway and the geometry, obviously, was visual.* $a^2 + b^2 = c^2$ *meant something on the triangle, and I could attach it*" (Interview 2, September, 2006). Beth further added that math done in the context of science, at least for a while, was meaningful and consequently made sense[21]. Being visual, she felt as long as she could draw a picture, she was all right. Despite this strong experience and success, Geometry did not alter Beth's view of math more broadly. Rather, Geometry was separate and distinct—almost *not math*. It made sense, she could play and build, follow her own paths to results like proofs and it connected to familiar ideas.

In addition to Geometry, Beth is fond of talking about her second experience in Algebra II/Trigonometry in spite of her familiar difficulty with symbols and memorizing. Here, however, it was the teacher, not the content, that made the difference:

> Then my Algebra II/Trig teacher the second time, she let me kind of cheat. Part of my thing with the symbols was that I can't remember them. I get them confused; I can't remember which step is which. I flip things; I transpose some. And she let me keep a unit circle on my desk, so as long as I could refer to that, I was fine. I could do all the other stuff but couldn't remember the formulas (Interview 1, July, 2006).

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[21] Recall that her aversion to mathematics was eventually so strong that it overcame her love of science.

While she called this "kind of" cheating, reflecting her understanding of what mathematics at this point was about, how pivotal this experience was to her perspectives of herself came out in the following interview:

> She let me keep a sheet on my desk with a unit circle with all the cosines and sines and all the relationships, and just being allowed to have that reference made me think I was smart, and that I could do it. Even though I couldn't remember everything. Like I previously thought, "I am so stupid," because I couldn't remember. And that changed the way I thought of myself as a learner (Interview 2, September, 2006).

While she seemingly did not have the connections necessary to make sense of the unit circle, as long as she had a reference and did not need to remember them, she performed well. Beth also remarked that this teacher let another student who had trouble with computation use the calculator. In seeing this teacher allow students to use what they needed to be successful, Beth began to feel that math difficulty was not a reflection of who she was. She was capable. Moreover, she added, "*the Algebra II teacher was accommodating. . . If I asked a question, she did not shoot me down or think I was being a smart aleck.*" Whether or not this teacher was able to answer her questions, that she valued Beth's questions as legitimate reinforced her feelings of value. Just as Geometry did not alter Beth's image of math, experiences with this teacher did not seem to either. Rather it impacted her image of herself. This helps to clarify why she was still "done" with math after this course; math and symbols in general still invoked panic. They were non-sensical, and math was disconnected from anything familiar.

Beth's mathematical experiences after becoming a teacher.

Of course, Beth is a middle school mathematics teacher, so something obviously happened to push her in that direction. As one might expect from her K-12 experiences, she was not a math teacher initially or even for much of her years of teaching (something I did not know before asking her to participate). She has taught in both elementary and middle school, and has been a reading, a social studies, and a science teacher. After receiving her master's degree in gifted education, Beth agreed to work at a middle school with a fellow teacher on the stipulation that she would teach all the science in the gifted program, and he would teach all the mathematics [a pre-algebra level seventh grade class]. A few months into the year, this fellow teacher felt he really could not handle all his math classes so Beth had to take on one of his classes and he took one of her science classes. Given that she herself had failed Pre-Algebra, her reaction to the material feels familiar:

> And I looked at the [material] and he sat down with me, and it was literally like head in my hands. I couldn't handle it. I called in Julie Barron[22], and she came and sat down with me. . . . She teaches a class [for the district teachers], and it's similar kinds of stuff. So I felt kind of comfortable with it, but as long as I could ask my teammate what I was doing [pause]. And I taught that for three years. And then this past year, I started liking it again (Interview 1, July, 2006).

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[22] She is also a participant in the program

While Beth was forced to teach mathematics that first year, it was in its continued teaching that Beth actually began to enjoy the mathematics. When asked about why she was able to find some enjoyment, she said:

> The kids came up with such interesting questions regarding the math and how it's used, that it made me think about the math differently. I had to change my own thinking from "math is a set of steps and procedures we must memorize" to "math is a way to solve 'problems' that occur in the real world." My students forced that change in me[23] because I had to, as their teacher, come up with ways to help them make sense of what they were doing. It took other people (children!) who could see connections that I had never been exposed to, to change my thinking. Math became fun again because it was more like solving a puzzle than what I had been used to in my life as a student, and as a teacher. Once I started thinking like my students, I started becoming curious again, which led me to research things like fractals and the use of math in art. It was my students who asked questions that led me to find things about math that were enjoyable. And, my kids weren't afraid of it. They were willing to (most of them!) work through things and not give up, whereas I was exposed to people who either got it, or didn't. I never actually saw anyone work through a math problem the way my gifted students did. Maybe it was a combination of their intelligence and their age (9-10) that allowed them to be so flexible in their thinking, which in turn forced me to be more flexible in

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[23] She initially asserted the role of her students in this transformation in our first interview.

> my thinking. I hope this makes sense. It became fun for me because math was no longer something that made no sense, and that I couldn't remember. It was something problematic and beautiful. I know that sounds strange, but it really did seem beautiful to me to watch my kids love it, to learn that there is so much more "out there" in the world of mathematics than facts and algorithms. Seeing how closely mathematics is tied to the world made it enjoyable (Interview 3, April, 2007).

With this evolving perspective, Beth sought out district math courses—just as she had sought out peer support when she began teaching pre-algebra—to improve her comfort with the material she would be teaching. These courses helped spark a reconsideration of what made math math.

> I guess [the work we did was] problems, but they didn't look like math problems. They were situations. And you used the manipulatives to describe the situation and figure out the situation, so it didn't seem like math. It was definitely problem solving. But there was no computation, there wasn't a formula to gather, an algorithm—it was totally open (Interview 1, July, 2006).

In this setting, learning departed so strikingly from her previous experiences that she had a hard time describing it as math. She had the opportunity for the first time to explore with manipulatives and to develop understanding. In particular, she recalled learning the table method of multiplication and multiplication as repeated addition, both of which facilitated her understanding of a process that had always challenged her (Interview 1,

July, 2006). How would a master's program in middle school mathematics education shape these shifting perspectives?

In the spring of 2006[24], Beth made a pivotal decision: she applied for and took a position as a department head at a new middle school that involved teaching only mathematics and obviously working with the mathematics teachers in her department. This seemed an unusual move for someone who had such troubling mathematics experiences. However, Beth was able to clearly articulate why she did it: not because of the mathematics, directly:

> **Eden:** *For somebody who not that long ago felt very uncomfortable with math, why did you feel comfortable taking on this full-time math position where you are head of this department?*
>
> **Beth:** *…It was less the math, and more the leadership skills. That's it. I knew that I knew the strategies to reach the students, and that my leadership ability would eventually get my department on board with me and like the way we know math should be taught.*
>
> **Eden:** *And how's that coming?*
>
> **Beth:** *Well, slow. But I knew it was going to be slow, and there are bumps in the road. I had a couple people on board and some others were like, "Who is this person who thinks that they can come in and suddenly teach," but I knew it was*

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[24] This was her first year in this master's program.

> *going to take time. I knew that I was going to have to push*
>
> *back* (Interview 3, April, 2007).

While the choice stemmed from leadership, mathematics is not far from her mind. She has strong convictions about what mathematics is useful in promoting understanding. She also had developed a strong desire to influence how others taught in order to help students understand mathematics. Moreover, she felt she was getting a better idea of what mathematics is about and what leads to poor performance:

> I am starting to think that the most likely reason a student does not understand mathematics is because they haven't "seen" the big picture, or the underlying concept. For example, a quadratic equation is the product of two binomials. If students do not understand the area model of multiplication, "the product of two binomials" looks like magic to them! It always did to me! How on earth can you factor a quadratic without having an understanding of the area model, either? There are steps you can follow, yes, but they are easily forgotten. Understanding is so much more than memorizing.
>
> To be good at math, a person needs to understand that math is a way of viewing the world. I can look out the window of my classroom and see trees blowing in the wind. The amount of bend in the trees is a result of how hard the wind is blowing. The direction the tree bends is a result of the direction the wind is blowing. These factors can be expressed mathematically. I can sit and observe the trees and find out the wind speed and direction, and make a table of my observations. I can then look for

patterns in my data. Once I find a pattern, I can make a generalization and most likely, make some sort of equation for this, or a symbolic representation. I then can communicate my findings and someone would call that mathematics. I can go on and on about the different ways I can view the trees mathematically—the patterns in the branches and how they become more numerous the higher you go, the patterns in the leaves, the symmetry in the structures, the area and space each takes up in the ground… All of this is "math" and all of it is a way to view the trees outside my window. I could paint them or write about them too—other ways of viewing the world. All three (painting, writing, mathematics) are ways to represent the trees. This is part of the reason I have always loved doing art projects in my math classes and with my math classes. Art seems more accessible, for whatever reason, and it is a "safe" way to start talking about mathematics; plus it is problem-based, so it might make more sense to kids (E-mail follow-up to Interview 3: April 30, 2007)

It in fact seems difficult now to separate pedagogy from views of mathematics, recalling mathematician Reuben Hersh's (1979) argument that the discussion should not be about what is the best way to teach mathematics, but rather, *what is mathematics about*? This was further evident in her response to my question about what she wanted for her students:

Oh, the first thing is that I want them to feel like they can do it. Just—I don't want to use the word confidence, but just with a feeling of being able to solve problems almost. Like, "if I'm faced with a problem,

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I'll be able to figure it out one way or another." That's the first thing that I want them to feel, that "I can do it."

The second thing is that it's okay to make mistakes and be wrong 'cause I am often wrong. Well, they ask me questions and I say, "I don't know." And that's okay not to know, but they have to know how to find the answers. So you come upon a problem and you can't figure it out, but you also need to be able to figure it out. I think the third thing would be that it's all connected. There is some place that you are going to find a connection to this particular thing. If you are having a problem with it, and you can't figure it out, there will be a connection somewhere. And that's—it's kind of like, learning how to learn. I think with math, once I figured out that it was all patterns, and it's like the way the world works, like I know that now, and I want them to know it, but I don't want to tell them that. I want them to figure it out. And they do, a lot of them already have. They understand that there is going to be a rhyme and reason to things and I can use math to kind of explain that (Interview 1, July, 2006).

In trying to find ways to impact teaching, the mathematics is also never far from her mind. Near the end of this year, Beth actually scheduled a "play date" with another teacher to work through some content in Algebra so that they could explore it as sense-making; unfortunately, this date fell through due to other commitments. Even so, Beth is now working to organize monthly department meetings next year that will focus on exploring and making sense of the math they will teach. While Beth is quite confident in her approaches and what she is trying do to, some of her colleagues (as she alluded to in

talking about influencing the teachers in her department) are less than confident, and Beth has experienced some professional undermining as a consequence of her allegiances.

> I'm really ticked off about this: another math resource teacher in the county tutors one of my Algebra students and told the kid's mom that I'm teaching the "wrong" way, and that the child's grades are inflated because he gets A's on unit tests. Imagine that! Students, actually being successful! The tutor is recommending that the kid repeat Algebra and no one has actually asked me about it, except my assistant principal (who loves the way I teach by the way). I know what I'm doing is right. I know I'm teaching well, and the kids "get it." This guy, the tutor, is a traditional "plug and chug" type teacher who probably doesn't understand why I'm doing this or what I'm doing because it's not in the text book. My kids actually understand what the zeros are in a quadratic - they know how to find them and what they mean, and they didn't get to that understanding by memorizing anything. They investigated quadratic functions and came to their own conclusions. This guy is undermining me with my own students and doesn't have the sense to even ask me about my teaching methods! ARGHH! (E-mail communication, May 18, 2007)

Her conviction tells her she is doing the right thing, but as she moves against the grain of both school mathematics as well as perceptions of mathematics, she has (and will likely continue to) hit a variety of roadblocks. Even so, she is trying to find ways to reach out to her colleagues in order to start conversations about teaching through working together making sense of math.

Reflective analysis of case history.

For a brief summary: We have seen a transition in Beth's willingness to engage in mathematics, as well as in her responses to, and perspectives of, mathematics. While she actively tried to engage early on even when she did not understand, by asking *why*, she could not measure up to the external bar of mathematical capacity or ability. This prompted significant math anxiety and the associated negative self-talk (Tobias, 1993), which was reinforced in that her parents and her peers seemed to understand the math through absorption. Moreover, her personal questions were not just devalued, but she was labeled a discipline problem when asking them. These questions had no place in the school classroom (something she pushed against in various ways). School mathematics did not make sense to her, and she eventually gave up. Through what turned out to be a fortuitous circumstance, Beth had to reengage. Through working with colleagues and students, she eventually recovered and valued her own mathematical questions, and learned to see math as connected and intriguing, something she tries to instill in her students.

These changing attitudes and perspectives represent the history of her learning, and thus are part of her identity in the practice of school mathematics. This identity was shaped over time through the three-part interaction between Beth, the mathematics, and those she worked with (most often her teachers) around two often competing goals: Beth's goal to understand the *why*s, of and connections in, mathematics, and the goal of the school mathematics practice for her to acquire mathematical knowledge and perform successfully on assessments of this knowledge. While these goals do not necessarily need

to be in conflict, the importance of the teacher's role in shaping what was learned, and thus identity, comes into sharp focus when there is goal conflict.

There is the sense that in Beth's general math experiences, when the school's goals did not coincide with hers, she attempted to negotiate the practice to make it livable with her goals; she asked why? Unfortunately many teachers, algebra teachers in particular, devalued these questions, labeling her a discipline problem, prompting the further development of a negative self image. As she did not understand, little productive algebra knowledge developed, but a negative affective response to a subject had, above all anxiety. On the flip side, Geometry, with its visual components, dependence on logic and reasoning in proof, and its connection to familiar objects, was such that Beth had sufficient access to what she needed (including the physical constructions) to make sense of the material. While she did at one point mention that her geometry teacher was one who did interact positively with students, she rarely talks about this teacher's role in her understanding of geometry. In Algebra II/Trigonometry, a course that did not seem accessible to her in terms of understanding, Beth attributes her success in the second class almost entirely to the teacher. In valuing her questions and allowing her to use what she needed, this teacher promoted engagement in which Beth had earlier been less willing to partake. Recalling one of the two lenses through which to consider identity,

> as **trajectories [of learning],** our identities incorporate the past and the future in the very process of negotiating the present…. They provide a context in which to determine what… actually becomes significant learning. A sense of trajectory gives us ways of sorting out what matters

and what does not, what contributes to our identity and what remains

marginal (Wenger, p. 155; emphasis added).

As we consider Beth's past algebra learning, and her goal of understanding, the relative

significance of the second Algebra II teacher becomes clear: positive identity as a learner

implied at least that the goal of understanding in mathematics was possible.

In terms of her participation, Beth developed over her mathematical experiences

an identity of marginal non-participation,[25] and her role of disciplinary problem was

reified by her teachers condemnations and her subsequent behavior; her participation

became increasingly restricted as she advanced through her courses, and eventually

carried her out of the practice entirely. It seemed as though rather than sacrifice her goal

of understanding, she sacrificed mathematics. As this process took place, this marginal

non-participation seemed to involve, on the one hand, a sharp decrease in the will to

engage in school mathematics, and, on the other hand, strong affective reaction, such as

when she talks of the panic that she would experience when having to participate

according to the established rules of the practice and when she acknowledged the

developing negative self-image. These emotional responses call to mind images of grace,

yet a grace that is counter-productive to engagement (Handa, 2006). Will and grace seem

mutually reinforcing but in a direction of non-participation, rather than connaître

knowing.

Beth's experiences in Geometry in particular seem to complicate this

understanding given that these were contrary to her typical participation in school

mathematics. However, it seems likely that Beth viewed Geometry as a very different

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[25] Wenger uses particpation to characterize mixes of participation and non-participation; however, I have attached marginality to non-participation to highlight that marginality is a type of participation where non-participation dominates (Wenger, 1998).

practice than the rest of mathematics, and thus the identity of participation that she formed and drew on in that regard was largely distinct from her identity of non-participation in school mathematics more broadly. This would help explain why, when Geometry was such a powerful and enjoyable experience for her, she does not associate it with changing her image of, or identity in, mathematics writ large. Because Algebra II/Trig was part of the school mathematics practice, that experience is associated with directly impacting how she experienced herself, if not mathematics (she was still done). This evolving learner identity becomes quite important for Beth later.

Both Wenger (1998) and Handa (2006) talk of learning as *becoming*, implying that what we learn changes us and becomes a part of us. While this is not a typical way to view learning, the effect that learning can have on who we are became evident for Beth years later. When forced to teach Pre-Algebra (a course that she had failed), her identity of marginal non-participation that had remained a quiet part of her was immediately resurrected. Old affective responses promptly kicked in, panic in particular, and accompanied the view that she would not be able to handle it. This demonstrates that her learner identity as shaped by her experiences was quite alive and well, even as a teacher. Of course, at this early stage, Beth's new role as the mathematics authority-figure in no way facilitated an alternative identity. Her role as a teacher was enough to evoke a certain will to participate in order to help her students; she could not walk away. As in her previous mathematics practice, the role of those she worked with proved critical. With her partner and another knowledgeable teacher, she worked through (or in spite of) both her uneasiness and the mathematics itself. While this was certainly important, it was

active interaction and negotiation of practice with her students that proved pivotal in Beth's continued engagement in school mathematics.

When Beth began to teach middle school mathematics, she was working with students in the gifted and talented program, students who often are highly motivated and involved in their academic work. In my own experience, these are also the students who have many enrichment opportunities in mathematics, so that they seemed to think about math in ways that departed from Beth's school experience is not surprising. Beth recognized that witnessing and participating with these students (and following their lead in regards to mathematics as sense-making) slowly helped her move from a more marginal non-participation to an identity of peripheral[26] participation, where she increasingly engaged in no small part due to her desire to help students answer their questions. That this trajectory was leading toward full participation is visible as her own curiosity was uncovered, first for her students, but then for herself. This served to motivate increased participation by looking for additional avenues through which to learn and explore mathematics. Curiosity is a type of affective, aesthetic response to mathematics, in particular a *motivational aesthetic response* that attracts "mathematicians to certain problems and even to certain fields of mathematics" (Sinclair, p. 264). As aesthetic response, curiosity can be considered a form of grace, but as motivation propels will, Handa (2006) considers curiosity a signal of the joining of will and grace. Thus perhaps this is the beginning of a connaître way of knowing mathematics. I have often wondered if Beth had been involved with students who still did have such natural

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[26] As with marginality, peripherality is a mix of participation and non-participation, but in this case, participation dominates, guiding her to full participation.

curiosity and desire to make sense of mathematics, would this aesthetic have developed as readily (though it did take a few years)?

Before moving forward, I think it worth highlighting that since school mathematics practice is negotiated, as a teacher Beth certainly had the option or opportunity to devalue her students questions and desire to make sense of mathematics; however, given her own anger and subtle rejection of authority when her own questions were devalued, as well as her wish as a learner that it had made sense, contributed to the negotiation of the sense-making environment that led to the awakening of her mathematical questions. Just as her Algebra II/ Trig teacher facilitated particular identity development through her valuing of Beth, it was in working with others around content, both students and professional developers, that seemed to facilitate new perspectives of mathematics and renewed engagement. However, as we will see in her program, these new perspectives did not simply replace the old ones. Rather they lived together in Beth, and she had to constantly negotiate both the previous identity and what was newly learned, supporting Wenger's argument that one's identity is a complex embodiment of one's learning history, present and future.

I would like to close with an examination of two of Beth's previous comments:

1. *It became fun for me because math was no longer something that made no sense. It was something problematic and beautiful.*

2. *All of this is "math" and all of it is a way to view the trees outside my window. I could paint them or write about them too—other ways of viewing the world.*

As perspectives that in part constitute her identity, what is most striking about these for me is the invoking of will and grace. The word *problematic* suggests activity, because a problem is one that requires thinking and action to solve or make sense of; what is *beautiful* then supplies the affect to motivate engagement with what is problematic, and perhaps is also the reward. The second comment, too, creates an active image of representing the world through mathematics. But the tool itself—the math, paint or pen—is aesthetic. These are aesthetic modes of actively representing a world that Beth originally saw as completely disconnected from mathematics. Does such a seeming joining of will and grace emerge in her mathematical activity in her master's program, or does being back in the position of mathematics learner recall vividly other historical, and far less productive, perspectives?

Contrasting Experiences in Program Courses

In this subsection, I will highlight Amelia's and Beth's responses to their program experiences. In contrasting their responses, certain themes and issues emerge as salient to consider. My data collection for this discussion focused on the second through sixth courses; the cohort has just completed their seventh course and the second year. The primary sources of data for program experiences are the three interviews (particularly the latter two), as well as journal responses, informal inquiry write-ups, mathematical autobiographies and experiences I am aware of as liaison between the cohort and the university. The design of this program explicitly considered literature on effective professional development[27] (ITQ Proposal, 2006). One aspect of this is that it provides extensive content experiences in addition to the pedagogy courses, both of which were desired by the school district,. The foci of the content are three of the four content areas recommended by CBMS (2001), Algebra, Data Analysis, and Geometry, which are explored though three different modes of experience. For each content area, one integrated course interweaves content with pedagogy (Ball & Bass, 2000), which is then followed up directly with the formal content courses. In addition, in the second through sixth courses, one inquiry strand (Cuoco, 2001; Burton, 2004) was interwoven into and related specifically to the coursework. The following section contrasts and discusses Amelia's and Beth's responses to these different program experiences, including a brief discussion of the inquiry situations that will not be discussed in the inquiry section.

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[27] For example, strong content focus grounded in what teachers teach, cohort design, and long duration (Garet, Porter, Desimone, Birman, & Yoon, 2001; Little, 1993).

Why Did These Teachers Pursue This Program?

The teachers in this program have come for a variety of reasons. Some are involved for reasons of learning mathematics, some to acquire the required master's degree, some to have access to higher pay, and some to improve their teaching. Amelia wanted to improve her teaching and attended the interest meeting to see if this program would fit the bill. In her personal autobiography that she wrote upon starting the program, she described her reaction to this meeting:

> The night of the interest meeting for this [district-university] partnership program was very scary for me. I left thinking that I was really going to struggle with the mathematics aspect of this program, and that everyone else knew more than I did. The worst part is that the demonstration was of **children** performing tasks that I didn't seem to understand (July 2005; emphasis in original).

Her historical anxiety still was present after several years of teaching and still reflected that reified belief of not being mathematically capable (Tobias, 1993). Given her reaction, it might seem surprising that she opted to get involved, however, she wanted to push herself. Moreover, this particular program offered two distinct advantages that were particularly attractive and seemed to add to her comfort:

> I set really high expectations for myself. . . . Like just kind of making the decision to go for my master's, you know, and this program, just deciding at this point that this is what I want to do, this is what I want to focus on, I want to get better at it. . . . I liked the idea of being a cohort, where I could keep that comfort zone of being with the same group of people. I liked the

idea that my county was involved with the college, 'cause I felt like that was another support system, and it would be geared toward what I needed to do in the county. So it just seemed like a good fit (Interview 1, July, 2006).

While Amelia was trying to improve her teaching through learning in an environment that she felt would be both supportive of her efforts and within her *comfort zone*, Beth decided to pursue the program from primarily personal reasons, which is consistent with a comment in her mathematics autobiography: "I am beginning to see mathematics as a very personal thing" (July 2005). Learning about this program coincided with her desire to pursue math. *"I remember saying to my husband, 'Maybe I could go to [the local] college and take Pre-Calculus and Calculus, maybe I can do that at night.' And it just so happened that I read about this program in our bulletin at the same time. . . . I felt that I was ready to learn more"* (Interview 1, July, 2006). She wrote in her incoming autobiography that:

> I am at a point in my life where I have the confidence to try higher-level math, and I have a great desire to learn more. Please understand that, for me, Algebra is "higher-level." I still have a "panic" reaction and it takes me a while to figure things out, but I do figure them out. I know I might have trouble with some of it, but I also know that with discussion and reflection, I can find the relationship and attach them to my prior learning (July 2005).

While she presumed that the program would help her students, her focus was really "*all about me*" (Interview 1, July, 2006). This sentiment also was expressed in our second interview.

While both Amelia and Beth expected program difficulties upon entering, these expectations seemed more of an obstacle for Amelia (an obstacle that was seemingly unique to her, as somehow she already knew that everyone else knew more) and more par for the course for Beth. Noting this contrast, as well as their different sources of motivation (to learn math versus wanting to improve teaching and pushing self), I wondered about Wenger's (1998) suggestion that both the past and the future help us interpret the importance of the present. I was particularly curious if or how these different goals would impact either how they engaged or how they interpreted their engagement in the integrated, content, and pedagogy courses, as well as the inquiry experiences.

How Did They See Their Experiences in the Different Types of Courses?

Integrated courses.

The integrated content-pedagogy courses each focused on specific content areas of middle school mathematics. These preceded the content courses because the program designers believed that this sequence would provide a more gentle and accessible entrée to each content area than jumping right into the more abstract content course. While Amelia came to the program in large part to improve her pedagogy, she has tended to find these courses somewhat frustrating, because she did not feel she was able to get out of them what she had hoped. "*I kind of wish that I had more content knowledge before these courses, and not intertwined in it. Because I don't feel that I am getting much of the teacher aspect of it, like how to teach it, because I am really trying to learn it right now*"

(Interview 1, July, 2006). This desire to have more knowledge going into a course is actually very typical of those with mathematics anxiety; it is one expression of a phenomenon referred to as the "dropped stitch" problem, which refers to beliefs that somewhere along the way, the person lost something or did not have something that they would later need to do mathematics (Tobias, 1993). Though she expresses that it would have been more appropriate to have the content courses prior to the integrated ones so that she could get more pedagogy, she simultaneously did appear to find the mathematics relevant to her work as a teacher:

> I start thinking about things, like especially [during Integrated Algebra], when I was teaching my algebra class. Some of the things we've talked about in the summer program, I'd be like, "oh!" like, I'd raise a question [to my students], "Oh do you really think that's true?" "'cause, you would not believe what I've just learned." Like, I would tell them what I was learning—stuff like that. So that was kind of neat (Interview 1, July, 2006).

The integrated geometry course also provided significant frustrations regarding background knowledge and her desire for structure, but in this case she did not feel able to take either the mathematics or the pedagogy discussed into her own classroom:

> I felt, again, I felt as though it's hard to talk about a lot of the things that we were doing without having gone through the content first. So to me, that's very backwards. I really don't feel like I gained much of anything out of it. Anything I feel like I gained out of it came from my individual [inquiry] assignment and a couple of the things that other people taught. I

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feel like, especially with geometry, which I am starting to learn more and more, there are just so many pieces of prior knowledge, especially at a high school level. . . . I felt like there wasn't much structure either. I felt like there was a lot of wasted time. I don't mind doing some of those bigger projects sometimes, but I felt like none of them got finished; there was no closure on anything. We started the cube activity, there was never any closure on that, I still had my cubes until one of the last classes, you know. And we just turned them in. The dilation [activity] I was getting excited about it, especially since it was around that time that I was doing proportions in [my grade six math] class, and similarity and things of that nature. And I wanted some closure to that, too, 'cause I wanted to be able to do it with my kids, 'cause I thought that was kind of neat. But then I ended up not doing it, 'cause I didn't feel like I was secure enough with doing it to understand like where I would want my kids to go (Interview 3, June, 2007).

The integrated data analysis felt somewhat different for her because we explored some of the mathematics before we ever discussed pedagogical issues. In addition, she felt better able to keep abreast of the material, perhaps because the teaching style coincided more closely with her preferred method of learning: "*[I] don't know if it's the best way for me [to learn], but it's what I'm used to, which seems to be my comfort zone. So start in small groups, and try and practice, and understand where it comes from. But I really need a lot of practice before it gets harder. I like learning the computation of it*" (Interview 1, July, 2006).

The only inquiry investigation in an integrated course that we did as a group was an investigation around simulating results of a gender discrimination study in integrated data analysis; it was based on an NCTM Navigations activity. The context of the simulation was around exploring if a particular hiring result was atypical enough to label it gender discrimination. Personally, I found this to be a very challenging investigation to work with, because, of all the directions I had foreseen that we might go in, this group did not come close to any of them until the last day of work. Even then it took them two full sessions to move past discussions of appropriate methods of simulation. It was very hard to try to make sense of how they were interpreting the problem. While looking back I feel there were three to four larger difficulties around this work, I firmly believe one of the major obstacles was the value-laden nature of the context itself. While I did not have a strong positive interpretation, Amelia enjoyed this investigation.

> I liked the [discrimination investigation] a lot. . . . because I wanted to do it. Two, because probability has not been my strength in the past; that's the unit we never got to until last year. So that was kind of cool to do that and get me a little more into it. Like I used that stuff a lot when I taught probability last year. Yeah. And we did it well. And I had fun with it. I have all this stuff now. So I really liked that one. I also like the whole question, like some people kind of took it almost like in theory—like philosophy, like sexist versus [pause]. And then again, I just kind of got frustrated when it gets to like how we are proving something, or not proving, and we are trying to support our answers with things that I didn't know how to do. But I liked the card one, that challenged me a little bit,

but not so much that I was blown out of the water or anything (Interview

2, September, 2006).

However she then added that even with this investigation she had to manage the open-

endedness and still had wanted closure:

I like the whole hypothetical thing, you know the "what-ifs" and the

constant questioning, although at some point I kind of feel like it has to

kind of end. . . . So that's why with the investigations, sometimes it will be

hard for me to like, get a grasp of it, because in my head I would like there

to be some sort of ending—some sort of answer. And so, I liked the

discussions, but I think I am more satisfied when there is an actual answer.

. . . But I still like to be right [laughing], and think that there is an answer,

I don't know, so—I guess I wasn't—I don't like it to be too open-ended, I

would like there to be some sort of culmination, or at least be comfortable

with the fact that there isn't, which I don't think could happen too often

(Interview 2, September, 2006).

During Integrated Geometry, the investigation was completely of an individual nature

and will be discussed in more detail following these more general program experiences.

Beth, too, had moments of frustration with the integrated courses, but given her

desire to pursue higher-level mathematics, these were not generally prevalent, and were

basically tied to her previous learning experiences:

I remember sitting in that [integrated] algebra class we had here, just

really wanting to slam my head on the table, 'cause a lot of people knew

that stuff and there was a lot of calling out. Matt, god love him, with his

calculator, pushing his calculator, which—god, its another thing I don't know how to do! People who get it don't understand people who don't get it (Interview 1, July, 2006).

While originally she wished that she had had the content courses first, she ultimately changed her mind, and overall quite enjoyed learning through these experiences.

The [integrated] algebra one felt different. But I think it was a function of the teachers, 'cause you could tell that in the algebra course, they were really trying to teach us how to teach algebra. Even though a lot of people felt like they didn't know how, I think learning about the open-ended stuff, how to break problems down—that was the pedagogy part for me. And looking back on it, I don't think it would have helped any if we had had the algebra content course first. I really don't. And I know that people go back and forth, and people feel differently. And I was one of those people that wanted the content course first, I'm like, "I don't know this stuff already, how am I supposed to learn how to teach it?" But I think it allowed me to be more open-minded about how to teach something (Interview 2, September, 2006).

Integrated Geometry (which took place during the final set of interviews) offered more opportunity for Beth to learn mathematics. While one might attribute it to her long-standing love of the subject, the instructor's interaction with them played heavily in these experiences. *"I mean, she has already figured us out and where we are, which is awesome. So that's another thing that makes me feel confident, is that my teacher knows me. So she knows what to ask me to get me thinking. That makes me feel great"*

(Interview 3, April, 2007). Given that I knew that at different times several people had been frustrated with this teacher's running of the course, I asked Beth about her interpretation of others' feeling regarding the course. She echoed some of the concerns that had been raised by Amelia, but also talked about why she herself did not find this troubling. She suggested, *"that she's not disjointed, but kind of like, she doesn't stick to her syllabus, like her plan. But I keep wanting to say, 'It's not about the plan. It's about where we're going.' You know? So what if she has 10 things planned and we only get through three. So what?"* (ibid).

The Integrated Data Analysis course, however, was the course Beth disliked intensely. In particular, she did not feel as though she was learning, since the level of the material was not challenging for at least half of the members the class (though half found the pace just grand).

> **Beth:** I hated it, but I felt confident [with] the [content] part of the statistics.
>
> **Eden:** Okay, what could you feel confident about?
>
> **Beth:** I knew the answers (Interview 3, April, 2007).

More to the point though, she had serious conflict with the teacher, how he taught and how he spoke about his own students: *"The stats course—that just was different, I don't feel like, if that was supposed to be pedagogy and methods. I would never teach the way that class was taught. I was kind of like, 'If this is what the teaching is supposed to be, then I need to find a different life'"* (Interview 2, September, 2006).

Regarding the gender-discrimination investigation, Beth was not at all enticed by the subject matter. She could not get into this one, evidenced by the fact that she did not

write a single journal response that semester until we moved from the investigation onto discussions of the use of intuition in mathematics. It did not interest her, and she felt it should have ended well before it did. Moreover, she clearly did not want to pursue the same type of things many of her classmates were pushing on, such as wanting to find a way to simulate the results in an authentic way. *"And for the discrimination one, I was like. 'whatever.' I didn't care if it was discrimination or not, I thought it had been exhausted. And that is why it was like, 'I think it's over,' I was ready to move on"* (Interview 2, September, 2006)[28]. One might just presume that the subject matter did not interest her, which perhaps is in part true; however, when later we see her paired research project, she found a way into the material that was exciting for her. While she really thought the investigation had ended, I asked her about why the focus was so much in the direction of authentically simulating the results. Her response echoed my feelings about the context, and linked to Amelia's comment about theory as well. *"Well, the situation I think, too, was something that we all thought would really happen. And so I think some of the more rule-minded people, since it really could happen, wanted to make sure we simulated the way it really would happen"* (Interview 2, September, 2006).

   Content courses.

   Each content course directly followed its associated integrated course. These three content courses (Algebra, Statistics, and Geometry) have each been taught by the same instructor. My data collection focused on the algebra and statistics courses. It was in the algebra course that we engaged in our first inquiry investigation with the parabola, and in

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[28] That Beth felt comfortable conveying to me her feelings around something that I had designed suggests that she was being open with her perspectices, rather than telling me what she thought I would like to hear.

the statistics course, these teachers for the first time independently designed and investigated their own inquiry questions in small groups.

Overall, Amelia also has found these content courses troubling. In part, she thought that she would do better than she did, even though she expected she would have difficulties. She feels she is not learning enough content, which is invoking her lack of confidence:

> I thought I was going to do better than I am. Like, trying out for the
>
> program and everything and applying, I really thought that I was going to
>
> do better, and I don't know what I was anticipating. I feel like I just
>
> thought that I was going to learn the content more. And I think that I have
>
> been exposed to it, so it's not like I haven't been exposed to it, and I have
>
> had great teachers, I just think that it definitely went faster than what I was
>
> ready for—which kind of knocked my confidence down (Interview 2,
>
> September, 2006).

These feelings seem to be reinforced by her midterms and finals on which she felt she was barely getting by (though she has received B's in both classes). She believes the course instructor has been very good and has helped her as much as possible, but the courses are just too fast-paced for her, which was a disappointment. In terms of the specific courses, she discussed her difficulties with Algebra, her favorite math strand:

> I tell you, with [the instructor], she is really good about starting off slow,
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> but she has no choice but to up it up. One, because we've got to get places,
>
> but two because everyone else is ready for it. So it starts slow, but it has to

get really fast, and it's really hard to go with that pace (Interview 1, July,

2006).

There is this sense that somehow everyone else was more advanced. Statistics though, a topic with which she had far less comfort, had a more confidence-blowing effect. Amelia expressed frustration similar to what Beth had described following her high school mathematics. Amelia said, "*Yeah, like last semester was way over my head. Once we were done with last semester, I was done*" (Interview 2, September, 2006). In each of these descriptions there is a common theme of almost being left behind, and that the student has no recourse. As a typical feeling for the mathematically anxious, it often leads to a desire to abandon the subject matter (Tobias, 1993). There was a sense of giving up, though Amelia still remained in the program. This coincides with Amelia's assertion that she never quits anything (Interview 1, July, 2006), reflecting a strong will, but a will from where?

In general she highlights two primary explanations for her difficulty: her own background knowledge and the backgrounds of the other students in the class, recalling the dropped stitch (Tobias, 1993). Recalling her desire to effectively know material before walking into class, she asserted, *"I don't feel like I have enough background knowledge going into anything right now"* (Interview 2, September, 2006). This feeling was perhaps reinforced when she looked around the classroom at her peers whom she felt knew more than she did. Together, these prevented the classroom from being a comfortable place to learn.

I really wish there was a way to have the class be more consistent in

terms of the people in it, with their background. Like, we have people in our

class that range from a first grade or kindergarten teacher to actually writing curriculum. That is really hard, I mean I know as teachers, you are supposed to be able to differentiate, and we are supposed to be able to handle it as students, too, but I don't see how. It just has not worked so far, like it is really hard to try and stay on top of things (Interview 1, July, 2006).

I don't necessarily judge it on the grade levels that we teach but I'm obviously better with things like fractions and decimals and things like that at a sixth grade level than I am with things they are doing in Algebra or Geometry or Pre-Algebra. Plus I feel like a lot of people have better knowledge right now of how to do stuff than I do (Interview 2, September, 2006).

While in the first and second interviews, she stressed her difficulties with the classroom; in our third interview, as she described her ideal learning environment, she offered some insight into why she is uncomfortable when others have this different background knowledge:

A group of people that I would want to learn with are those kind of people like that have respect, that have that knowledge, but at the same time are open to my questions, and will explain it to me, and will be patient. I kind of feel like an open environment, where people aren't just trying to show you how they do it. You know what I mean? Like when I am trying to learn, there is a difference. I don't want to hear about—I want it to be

about the learning. I don't want it to be about a competition (Interview 3,

June, 2007).

Amelia does not feel comfortable when learning seems related to competition with her peers, which is exacerbated by the different levels of knowledge in the group to whom she compares herself. This does not promote an open environment for her where people are able to learn from each other. One might question her desire for an open learning community without competition, given her own desire to demonstrate her knowledge in class, but what if we use this new perspective to look back again at her history. How better to remove the learning competition than to have your mother teach you the material ahead of time—a security blanket rather than a chance to show off? How better to not have to compare yourself to the group than to not have to learn what they are learning? Going at their pace then is peaceful. How better to negotiate a practice that is related to your goals and to be able to help your peers learn than to come in to class with the knowledge already in hand? It seems likely that in having her mother teach her the mathematics before coming into the classroom, Amelia effectively was trying to counteract the typical negative consequences of the school mathematics tradition, as well as the associated mathematics anxiety. However, this tactic ceased to work.

The first inquiry experience in which these teachers engaged, the parabola investigation that was detailed earlier in this dissertation, occurred during the algebra content course. Amelia expressed difficulty with this investigation, and in interviews would avoid directly speaking about it, though she did mention it in comparison to other investigations. She contrasted her enjoyment of the discrimination one to her parabola experience (Interview 2, September, 2006). The primary reactions I have received from

Amelia around this investigation were through her journals. Her final journal response is highlighted below.

The investigations project has been very frustrating for me. I don't really feel like I am coming up with anything new. I know that there are people I can ask to teach me what they could come with, but I feel like that would defeat the purpose of the project. I decided to "focus" on two items.

Knowing that the idea of these set of points form a parabola was a hot topic for our class, I asked my mom for help using the graphing calculator to try to develop an equation for this possible parabola. I only had three points that I felt confident about: (1, 1), (-3, 1), (5, -1). I entered those points into a table. I used the calculator to find the quadratic regression. We found what would be an equation IF IT IS ACTUALLY A PARABOLA. The equation was $(y = -\frac{1}{8}x^2 + \frac{1}{4}x + \frac{7}{8})$. My mom then showed me the vertex form. Once I manipulated the problem, it was the same as the quadratic equation. Even though I was able to work through the "algebra," I still wasn't sure how it was "justifying" anything. I'm not even sure that I understand what the equations mean. I also don't know how to use this information to find other points.

One topic I felt more comfortable addressing was questioning what would happen if the line were vertical… Basically, I imagined turning my original picture to make the line vertical. It just makes sense that the

points can't go toward the line and be the same distance from both the line

and the given point (Points Journal, October 11, 2005)

Here she described her frustration at not developing new understandings. While she said she did not want to go to peers, in the next paragraph, she discussed asking her mother to help her attempt to connect the set of points to a parabola, the task that the class had negotiated for themselves. In the paragraph following this description, which is significantly shorter, she suggested that she actually did not feel comfortable with the parabola proof task. Why did she do it then? That which she did feel more comfortable investigating is given only two sentences, whereas the parabola work was given 11.

While Amelia has struggled to acquire the content she had hoped to, Beth has felt that she has been learning a substantial amount in these courses, and she stresses that being able to pursue things her own way really is helpful, a mark of the instructor whom she very much enjoys. This enjoyment did not prevent her from experiencing quite a bit of anxiety as she had in her previous learning experiences. It is interesting that stronger anxiety and poor self-efficacy surface particularly when she is not able to shape the environment to her needs:

> Well just thinking about it now, I do, I have to talk myself out of it. I have
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> to take a deep breath, I'll skip problems, I'll go into something that I know
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> and start on something that I know and then go back, because I think that
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> anxiety is still ingrained in me, that even if I do know it, the testing
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> situation freaks me out. So it was mostly the testing situation, I mean, I
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> panicked for that midterm. It was awful. And it was like totally irrational.
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> Like, I know I am going to do fine, I know this stuff. But it's just the

pressure. 'Cause I can't do it at my own pace, and I can't talk to anybody about it. . . . In the algebra course, and some of the statistics, I did get that feeling of, "There is no way I'm going to get through this." At least now I know a little bit better than I did in high school. Mostly, the courses themselves are [better]. The open-ended stuff that you did helped, and knowing that, it's almost like I have permission to do it my way, and so I know I am going to have trouble remembering stuff, so I know I have to work harder at some things than I do at other things. So I guess my anxiety level has gone down. And I also know that if I don't get it the first time, it's not the end of the world because I know there is another way to understand it. I just have to figure it out for myself (Interview 2, September, 2006).

There is an ever-present negotiation between Beth's anxiety-laden past and her sense of efficacy in shaping her own learning. Tobias (1993) describes this self-talk that people engage in as they encounter difficult situations—it is not whether or not they encounter difficulties, but rather how they talk themselves through it. Those who are in the midst of mathematical anxiety engage in negative self-talk, whereas those who are overcoming their anxiety like Beth (or perhaps never had it), the talk is positive. Beth has developed a confidence that has not been previously present in her mathematics, a confidence in her ability to work through difficulties, and this did not seem to come from any coursework, but rather from her own interests and determination, combined with maturity.

In her discussions around the specific courses, we see her working to resolve these tensions. She not only has developed more positive feelings towards algebra, but

she has a perspective on two things that have facilitated this growth—the historical perspective of the subject and the instructor.

> I'm learning how to tolerate algebra because I have learned, through all the classes, that it was a guy, like somebody was sitting down looking at things, and decided[29] to use symbols. And now we all have to use them (Interview 2, September, 2006).

> I felt like [the instructor] didn't start with the math part of it. She started with situations where we had to apply mathematical thinking, which I think is different from memorizing symbols and formulas. There's—I now think that there really is something as mathematical thinking (Interview 3, April, 2007).

While she was learning this about symbols in algebra and beyond, two semesters after Algebra, she still had her anxiety reaction: "*Here's another thing I don't get: Why are there so many symbols for 'mean.' Can't there just be one? [The instructor] started doing that and I was like, 'God, I'm never going to be able to remember all this*" (Interview 1, July, 2006). Though she experienced these tensions and knee-jerk reactions, this did not impede her from learning in the course. In fact, to distinguish the type of courses that she did not like, she specifically invoked courses that were too easy. She then made a large distinction between feeling overwhelmed and not liking a course (Interview 3, April, 2007).

Further contrasting how these teachers experienced the content courses, Beth quite enjoyed the parabola investigation. Though she felt pulled to move in the direction

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[29] This comment contrasts with her early invocation of Moses when discussing symbols.

of the colleague she was working with (Points Journal, December 12, 2005), she instead opted to pursue something she had "seen" in the location of points: isosceles triangles.

> Well, I thought I was on to something and had a definite pattern going with the triangles, but some didn't fit proportionally. It's really hard because none of the points so far lands on anything "even"—I mean like a definite intersection—so it's really hard to figure out the proportions and relationships that I can see happening. . . . I'd like to work backwards, maybe using an existing parabola and measuring angles, etc., and try to match that to the points you gave in class. But, I really don't care if it's a parabola or not - is that bad? I just want to see the relationships formed by the different triangles and then try to assign a formula or something. (Points Journal, October 5, 2005).

There is a distinct move on Beth's part to disregard the task the class developed and to work on what she had found interesting and motivating. While it had not yet worked out, she was trying to find alternative ways to see if the set of points could be built with isosceles triangles. Engaging in the work motivated a shift in her perceptions of what doing mathematics entailed and how it related to other content areas:

> I think it was stuff with the inquiry [that challenged my perception of math]. The stuff that not everything has been done yet…. And that throughout history, it had been that way. You know, people used to think a certain way, and then somebody would discover something and it would change. But I never realized that until I got into the program. I knew about it in science. . . . But nobody ever talks about how math developed. And

> so, you know, they talk about how art developed. How writing has
> developed throughout history, how science has. . . . I think if math were
> taught in a more connected way—I think a lot of the math that was
> discovered influenced science, and vice versa; I think it influenced art.
> People learned about perspective. They learned how to draw a different
> way. I mean if math was taught more like that, it might be more
> comprehensible to more people (Interview 2, September, 2006).

Somehow math was becoming more like other subject areas for Beth, and inquiry was reinforcing this progression. This changing perspective echoes perspectives that have been developing over the last 50 years. The mathematicians Philip Davis and Reuben Hersh (1981) stated:

> Mathematics does have a subject matter, and its statements are
> meaningful. The meaning, however, is to be found in the shared
> understanding of human being, not in an external nonhuman reality. In this
> respect, mathematics is similar to an ideology, a religion, or an art form; it
> deals with human meanings, and is intelligible only within the context of
> culture. In other words, mathematics is a humanistic study. It is one of the
> humanities (p. 410).

Pedagogy courses.

Only one such course took place during my data collection, a course on mathematical misconceptions. In general this course was perceived as very difficult, but important. Another student in the program told the incoming cohort of teachers that while

it was the hardest class she ever took, it was the best class she had ever had (program information meeting, January 2007). The course instructor responded to the group's difficulty by significantly pairing down readings and finding ways to share the reading load about a third of the way through the course. Amelia though loved this course in particular because of how useful it was to her pedagogy. In addition, as compared to the others in the course, she did not feel her content knowledge inhibited her ability to learn:

> I really liked [the misconceptions] class. I typically go against the norm I guess [laughing]. I really loved [that] class. I still feel like—no, for the most part I think I liked her class, I feel that I got stuff out of it, like I was using stuff from her class. I can't think of specific examples right this second, but I was constantly—even though I knew some stuff. . . I felt like I was constantly referring back to her class and thinking, "Oh, let's not think of this." And I think it's great, I mean, I think that the reason I liked it so much, too, is because even though some people have a lot of math content knowledge, teaching it is a much different thing. That's why I don't feel like I am so off track, 'cause even though my content might not be at the same level as somebody else's, I still feel like teaching it is much different (Interview 3, June, 2007).

In fact, it seemed that Amelia did not feel that more content knowledge implied that someone would be a better teacher, which resonated with research that has questioned the link between content knowledge and teaching practice. Her enjoyment of what has universally in the program been acknowledged as the hardest course, helps us to understand that Amelia is not disturbed by engaging in difficult work, thus her anxiety

with mathematics is not an issue of the amount of effort it requires, but more of how the nature of that work affects her.

Beth found this misconceptions course troubling, but is aware of why. It was not the course or the instructor. She was frustrated because she was unable to focus or put the time in because of personal family issues that were going on at the time:

> I hated [that class] when I couldn't focus, because of stuff that was going on in my personal life, because—I knew I was not doing as well as a could have been, and it wasn't because of the content of the math. It was because I couldn't pay attention to it. But I thought as a teacher that is very relevant. Because there are going to be times when my students are going to be tuned out, not because of me, not because the math is hard, but just because (Interview 3, April, 2007).

While she was not able to enjoy this class much, she still found value in reflecting on the experience. She used the experience as another way to try to understand how students experience school and mathematics.

In a way similar to discrimination, there was significant trouble around investigation this semester of Zeno's Paradox. While most seemed to enjoy reading the story of the slow tortoise convincing Achilles that if Achilles gave the tortoise a head-start Achilles could not win, they initially did not even see what mathematics there was to discuss in the story. The story begins with the Tortoise challenging Achilles to a race with the caveat that he have a head start. He then proceeds to convince Achilles he cannot win. The argument part of the story follows[30]:

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[30] Source: http://www.mathacademy.com/pr/prime/articles/zeno_tort/

Tortoise: "Suppose, that you give me a 10-meter head start. Would you say that you could cover that 10 meters between us very quickly?"

Achilles: "Very quickly,"

Tortoise: "And in that time, how far should I have gone, do you think?"

Achilles: "Perhaps a meter – no more,"

Tortoise: "Very well, so now there is a meter between us. And you would catch up that distance very quickly?"

Achilles: "Very quickly indeed!"

Tortoise: "And yet, in that time I shall have gone a little way farther, so that now you must catch that distance up, yes?"

Achilles: "Ye-es,"

Tortoise: "And while you are doing so, I shall have gone a little way farther, so that you must then catch up the new distance."

Achilles said nothing.

"And so you see, in each moment you must be catching up the distance between us, and yet I – at the same time – will be adding a new distance, however small, for you to catch up again."

Achilles: "Indeed, it must be so,"

Tortoise: "And so you can never catch up,"

Achilles: "You are right, as always,"—and conceded the race.

Many of the teachers came to class with ready-made arguments for themselves as to why Achilles would win, and thus it simply did not seem worthwhile to be investigating this situation. They simply suggested that if you consider the 1 meter the tortoise ran during the time Achilles covered the 10 meters, and then that same time elapsed again, then Achilles would have gone 20 meters total, and the tortoise would only have gone 2, hence Achilles must have passed the tortoise[31]. I, of course, wanted to use the story to get at the tortoise's actual mathematical argument, which despite many arguments to the contrary, does ideally represent an accurate breakdown of the continuous race into a sequence of infinite events. However, it rests on an assumption that is a misconception: this infinite sequence of events must take an infinite amount of time. While I thought this was a fascinating doorway into our mathematical perspectives and unquestioned assumptions, it was extremely challenging to get them to even attend to the tortoise's argument, given that they knew what would happen in the real world. In fact, some teachers simply accepted that what would happen mathematically (the tortoise would win) was different from what would happen in the real world (Achilles would win), and they were fairly satisfied with this. In her project write-up of a related activity, another teacher, Susannah, recalled this investigation and stated, "It may seem frustrating that there are two clear answers to the situation, but that is what I have decided. Mathematically, yes, the Tortoise would win. If they actually raced however, Achilles would win" (Interval Project Write-up, January 2007). I eventually had to abandon the context and create three related activities (one on intervals, one on fractals, and one on decimal expansion in

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[31] This is my summary of some teachers' primary argument.

different bases) with which these teachers could engage that had solely mathematical context.

Themes that Emerge Through Contrasting the Cases

If we were to compare these teachers based on their "objective" histories, in some ways they are very similar (both elementary-certified middle school mathematics teachers, and both have experienced periods of great frustration in mathematics learning, but both, after teaching, came to love mathematics), and in some ways they are very different (most of Amelia's mathematics learning was actually very positive and she did well; whereas Beth struggled the whole way, and Amelia went through Pre-Calculus where as Beth effectively stopped after Algebra II). The more mathematically proficient and comfortable would appear to be Amelia. However, such a comparison offers us little to work with as we try to understand how they are actually experiencing their program coursework. In fact, their reactions to the program experiences provide a startling contrast regarding how their learning trajectories have shaped their engagement in the mathematics and their participation in the practice. A variety of areas in which this seems evident include how they respond to the content, what helps them with learning the content, the role that pedagogy plays in these experiences, the role the social practice plays for each of them, and their own sense of efficacy and agency.

As they react to the content, there seems to be an ongoing identity negotiation that reflects their histories. Beth reacts quite anxiously to the content courses in particular, but then she draws on a variety of coping mechanisms that elicit her confidence, particularly when she can pursue the work in her own way. This enables her to develop understanding of the mathematics. The power of these coping mechanisms for people who have

experienced math anxiety is both simple and profound: "If we can talk ourselves into feeling comfortable and secure, we may let in a good idea" (Tobias, 1993, p. 69). Amelia tends to do well and feel at ease when the courses start off slow. However, when content activities push past that comfort zone, she feels the courses move too fast, are thus over her head, and she feels unable to learn the material; she engages in self-defeating self-talk, which often leads to freezing, if not quitting (ibid). Without confidence, she feels increasingly less able to succeed. The reactions to inquiry specifically also echo these identities. Since in inquiry Beth is able to pursue understanding in her own way, this activity facilitates understanding and resonates with her as meaningful; however, for Amelia, where the end-result is not a focus and background knowledge is not necessarily established, her discomfort with inquiry is obvious. It is interesting that up to this point, the one group inquiry activity Amelia has enjoyed, gender discrimination, had the most external structure, was the most result-oriented, and did not unfold rapidly.

These highlight those things that make Amelia feel comfortable in her content experiences: when their pedagogical connection is clear, when there is structure within which to work, where closure is important (making it pedagogically useful), and when it moves at a slow pace to allow her significant practice. In particular, she believes that if she had more content knowledge coming into the practice, and if her peers were in similar places as she was, that learning would be much more accessible. This certainly contrasts with Beth, who wants to be pushed and challenged in mathematics, because if it does not interest her, she is far less willing to invest in the activity. Those elements that she feels most help her to make progress, aside from being able to pursue it her own way, are working on math thinking prior to formalization, learning how these ideas developed

historically so that she understands why and in what ways they were important, and working with people who bring different knowledge and backgrounds so they can work together. In contrasting this to Amelia, while she wants mathematics to be easier for her, in general she is not against being challenged. In fact, she loved the most difficult course in the program. She just does not want that same type of challenge in mathematics where she has significant anxiety and adheres to the dropped stitch theory: That she has trouble getting it because she missed something along the way—in her case, more background knowledge. So she looks for pedagogical ways to make the mathematics more accessible—ways that are really distinct from the math. Meanwhile, it is not that those things that Beth wants don't have pedagogical importance, but that her choices tend to have a distinctly mathematical relevance.

This perhaps sheds light on the position pedagogy holds for them in these experiences. Because Beth feels as though she is learning mathematics, she is learning how to think mathematically, and she has the room to do it in a way that is relevant; she perceives that she is in fact getting pedagogy. She feels she is learning how to teach through learning how to explore mathematics and break down the content. Thus mathematics and pedagogy are intertwined. For Amelia, they are distinct. She feels as though she is having a hard time getting at the pedagogy because she has to invest so much in learning the content. This potentially creates a quandary, though, because the value of what she is learning is connected to how Amelia can draw on it in her classroom.

It is not just primarily through pedagogical judgments that Amelia's math experience is shaped. The community is quite powerful in shaping that to which Amelia attends. In the inquiry experiences that were not individual, it was clear that she follows

the community negotiation of what the task is, regardless of with what she feels comfortable. Moreover, she uses the community as a reference point from which to judge her own learning and activities. When she does not compare favorably or has trouble with the community task, lack of confidence emerges. This helps to understand why she would prefer to be working with a group of peers that she felt were similar to her; it reduces competition, making her comfortable. What is interesting though is she suggests she wants to work in an open environment without competition to get questions answered, just as Beth does, but that may be difficult to achieve when there is constant comparison of self to others. I suppose it could be less of an issue if they were all just like her. It would certainly resemble more closely the one-on-one situation in which she enjoyed learning mathematics from her mother/teacher.

Beth, too, has felt pulled by the community-negotiated practice regarding how to work and on what to work. However, if given the room, she will choose to prioritize her own interests if these are not parallel to the community (or co-worker) focus, while Amelia prioritizes the group. Though on one level this may sound contradictory, Beth also wants the teacher and her peers to be involved in her sense-making process and she in theirs—valuing, interacting, and learning from each other. In order to make such activity worthwhile and productive, group members need to bring different knowledge and skills to the table.

Considering these issues, these teachers portray contrasting identities regarding agency and self-efficacy. Beth certainly feels she can serve her own learning best when she is able to shape the work and the path she follows, when she is able to make connections and work with others as a community. When she cannot do these things, as

in tests, and agency is removed, her previous difficulties with mathematics tend to emerge. If she is able to direct her own learning, she is fully confident that she will make progress and learn. Agency breeds self-efficacy. Amelia, however, looks to external circumstances to facilitate learning. Thus in some senses she has little agency in the learning process, which is coherent with a notion of mathematics as answer, rather than procedure, focused. However, she does exhibit some agency in selecting the external circumstances. In our third interview, she said she often shuts down in class prior to when she might otherwise, because she knows she can just go home and ask her mother to help her. Where these teachers have and want to have agency are quite different, and as such, one exhibits a high degree of self-efficacy, whereas the other does not, as she constantly looks externally to effect learning, supporting the notion that she did not learn to use the classroom as a place to learn.

The issues above suggest interesting implications regarding different ways of knowing mathematics, and in terms of will and grace in particular. There is a very strong sense from Beth that her internal motivation to engage is particularly important; moreover, it appears as though she looks for and accepts challenging mathematics. Both of these are hallmarks of the joining of will and grace that characterize a connaître way of knowing mathematics (Handa, 2006). In Amelia, too, there seems to be a type of joining, but it does not seem to be located in the mathematics itself. Her will seems to be located in the pedagogical purposes or the community practice, whereas grace seems to require a pedagogical warrant or depends on external circumstances. This reflects her perspective that she needs pedagogical support to make up for her mathematical holes. Thus math for math's sake does not seem to enter her realm. A savoir way of knowing continues to be

cultivated, even though this has felt unsatisfactory to her in teaching. This way of knowing would seem to inhibit what she is able to learn mathematically, through constraining her work space. It will be interesting to see if these ways of knowing are in fact present in their mathematical inquiries.

Exploring Inquiry Opportunities

This group of teachers had additional opportunities to engage in open-ended mathematical questions that they themselves designed either individually or in pairs, in addition to those that were previously described. My purpose for choosing to explore more in depth those projects that were of these teachers' own design is, given that these are their own questions, their interest in, and motivation for, engagement is likely high, thereby supporting their willingness to seriously engage. Difficulties that arise then are likely of a relationship origin, not something inherent in their interest in the topic itself. The first set of investigations discussed will focus on the first opportunity these teachers had to design their own project. It was focused in statistics and they generally worked in pairs. Amelia worked with Heidi and Robin on trying to understand why an activity in experimental probability was carried out in a particular way; meanwhile Beth and Julie explored the relationship between Pascal's Triangle and probability. I will use these investigations to explore different ways in which these teachers come to know mathematics, and relate this to the process of knowing in terms of the interaction of will and grace. The second set of investigations I discuss focuses around the final large scale mathematical inquiry experience in this master's program. This was an individual research project in Integrated Geometry. As with the project in statistics, these teachers were able to choose any topic in geometry about which they wanted to develop their understanding. This set of investigations will be used to highlight how various aspects of identity continually shape engagement with inquiry experiences in mathematics. Thus between the two sets of projects, both aspects of relationships with mathematics will be explored and discussed.

In order to discuss these projects, I need to highlight some additional literature related to mathematical inquiry that I will employ as I work to make sense of Beth's and Amelia's mathematical thinking. The three pieces of literature I will discuss all emerge in large part from interactions the authors had with research mathematicians around their own mathematical research, and in how they come to make sense of what they are investigating. The first piece is related to the active processes one engages in during inquiry, the second is related to the role of intuition in inquiry, and the third relates to how aesthetic responses comes into play for mathematicians when engaged in inquiry activities. In looking at mathematical activity, the first piece connects to the willingness to engage; whereas the second and third pieces, which speak more to responses to activity, connect to the experiences of grace.

Mental Habits and Actions in Approaching Problem Situations

Cuoco, Goldenberg, and Mark (1996)[32] have argued to develop curricula not around mathematical results, but rather around what they term *habits of mind* or mental approaches that facilitate student thinking about mathematics, just as might occur in the discipline with mathematicians. While they clearly state they do not expect students to be able to understand the same topics with which mathematicians concern themselves, they assert that students can approach their work in similar ways. Their reason for focusing on mathematics learning in this way is:

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[32] Levasseur and Cuoco have a more recent article that highlights particular habits of mind that should be focused on in schools. That work did not resonate as fully with me as this one. That work condensed the habits and approaches into eight. In speaking with Al Cuoco, he noted that he himself uses the 1996 framework for his work, not the more recent.

Much more important than specific mathematical results are the habits of mind used by people who create those results. We envision a curriculum that elevates the methods by which mathematics are created and the techniques used by researchers to a status equal to that enjoyed by the results of that research. . . . A habits of mind curriculum is devoted to giving students a genuine research experience (1996, pp. 375-376).

As such, this recalls the earlier knowledge discussion where *knowing* is conceived of as *doing*. The general habits of mind described include students being pattern sniffers, experimenters, describers, tinkerers, inventors, visualizers, conjecturers, and guessers (ibid). In addition to describing these general habits of mind, which they suggest should be emphasized in all areas of study, the authors describe other general mathematical approaches that have proved useful historically, as well as subject specific algebraic and geometric ways of thinking. While I agree that these are vitally important, for examining the inquiry work of teachers, I will focus exclusively on the general habits of mind. I do this in no small part because, as these are not singularly mathematical, these are likely more accessible to teachers than the other approaches, which may require significant enculturation into the discipline of mathematics, an enculturation of which many elementary-certified teachers have not had the advantage.

For my own purposes, I group pattern sniffing, experimenting, and tinkering under the heading *playing*. Cuoco et al (1996) use this word specifically under experimenting stating, "when faced with a mathematical problem, a student should immediately start playing with it, using strategies that have proved successful in the past" (p. 378). While they may be talking specifically about the recording of, or being skeptical

of results, or about experimental tactics like keeping all but one variable fixed, central aspects of tinkering (e.g. taking apart and putting back together, leaving things out, or reassembling in a different way) also invoke images of play such as models, puzzles, and blocks; pattern sniffing, which the authors liken to criminal detection or analyzing historical events, also brings to mind images of play, likely because of the delight that draws us to them. It is not accidental that many of the games kids and adults play, (e.g. Battleship, Sudoku, Gin Rummy, Pat-a-cake, and Say, say my playmate) focus on finding and/or using patterns. Grouping these under one heading will be useful for me later.

Guessing and conjecturing also are strongly linked; guessing can be used as an early form of conjecturing. Conjecturing should be based on evidence, such as patterns, but also from previous experiences and understanding of what is taking place, such as comprehending the algorithm used to produce pictures or results (ibid). Conjecturing often involves making a stronger generalization from what has been learned. Guesses tend to be a bit less secure and do not necessarily involve increasing strength. Guessing and checking can facilitate making sense of a problem, and using a guess to work backwards can contribute to developing both new insights and new approaches.

The words visualizing, describing, and inventing reflect their intent. While there are three primary types of visualizing, in breaking these down, further examples of visualization include, visualizing *data* as with a table, *relationships* as on a plane or in space (both of which relate to different representations), *processes* like a machine, *change* such as what happens as the foci of an ellipse move apart, as well as *mental calculations* (ibid). Description becomes central in mathematics particularly if one wants to communicate what is learned. Such common types of description include detailing

precise steps, inventing notation, and crafting arguments to convince others. Less commonly, types of description in mathematics that the authors emphasize result in producing a record of work, such as writing down thoughts, results, conjectures, arguments, questions, and opinions to name a few (ibid). How often in picking up a piece of mathematical writing do we see opinions, questions or thought processes? This type of description facilitates collaboration, asking questions and providing feedback. The distinction between these two types of description can provide important information about one's mathematical perspectives (Sinclair, 2004). The final habit of mind, invention, may not feel common in mathematics either, though under *describing* the invention of notation is mentioned. The authors note inventions are not arbitrary but are based on reasoning, akin to the rules of a game. A way to invent may be to take something known and to change one feature and examine the result (sounding similar to tinkering), or to look for analogous situations or similar structures. Algorithms, explanations of how things work, and axioms are all mathematical inventions.

Examining if and how students make use of habits of mind as they engage in inquiry can facilitate understanding of how they see mathematics and how they position themselves in the doing of mathematics; moreover, it could help to isolate why someone is encountering difficulty with developing understanding. As the inquiry projects in which these teachers are engaged are focused on providing a type of research experience for them in order to create new mathematical understandings (in their own eyes, rather than in the discipline), considering their work in terms of habits of mind feels appropriate. That said, these teachers were not expressly encouraged or taught to use these particular habits.

Essential Intuition Develops From Knowledge and *Experience*

Leone Burton (2004) has actively examined mathematicians' use of intuition in their own research work. While a few do not believe it exists, sixty-five of seventy of the research mathematicians with whom she worked believed in its importance in mathematical thinking, though many preferred to use words such as *instinct* or *insight* presumably because this did not invoke notions of the feminine (ibid). Wilder (1981) asserted that while intuition was central to the creative process of mathematics, it depended on the growth of knowledge, suggesting an ongoing cycle. The research mathematicians with whom Burton worked echoed this perspective saying in particular that intuition depended on the development of both knowledge and experience.

These mathematicians asserted a variety of roles intuition plays, noting that it is at work all the time. They described intuition as seeing connections, as reflected in metaphors and analogies, or as the necessary impetus to start working on something (p. 76-77). Many likened it as well to having an *aha!* experience or simply an enhanced understanding. One primary concern that mathematicians seemed to have with intuition (for those that subscribed to it) is that it is not always reliable (Burton, 2004). This is why it must be checked on the one hand, and on the other, why it is important to understand that it grows and changes with knowledge and experience. The more experience one has with a system or a situation, the more reliable will be one's intuition around these. Davis and Hersh (1981) added that another reason for the difficult acceptance of intuition in mathematics is that intuition is antithetical to an absolutist perspective of mathematics; however, given that a large number of Burton's mathematicians held absolutist perspectives and touted the importance of intuition in their own work suggests that if

there is a problem between an infallible mathematics and intuition, the problem resides in the formal discipline, rather than in the private work of mathematicians.

Aesthetic Responses Shape Mathematical Inquiry

Similarly to Leone Burton's work with intuition (and aesthetics), Natalie Sinclair (2004) has described the role that aesthetic response plays in mathematician's research. While this may seem unusual to someone outside mathematics who does not often associate the discipline with aesthetics, Sinclair noted that mathematicians have touted the role that aesthetic plays, noting in particular that mathematician Henri Poincaré felt it actually was aesthetic rather than logic that distinguished the mathematical mind[33] (ibid). As thinking about the role habits of mind plays in teachers' inquiry experiences, examining the role of aesthetic can help to understand again how they view mathematics, what mathematics is about, and how they go about making sense of what they are learning. Sinclair writes:

> A student's aesthetic capacity is not simply equivalent to her ability to identify formal qualities such as economy, unexpectedness, or inevitability in mathematical entities. Rather her aesthetic capacity relates to her sensibility in combining information and imagination when making purposeful decisions regarding meaning and pleasure. This is the use of the term aesthetic drawn from interpretations such as Dewey (1934) (2004, p. 262).

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[33] Davis and Hersh (1981) cited in addition to Poincaré several others, including themselves, who find mathematics beautiful, and on some levels equate it with an art form.

This description seems to link cognition and intuition, and similarly connects sense-making and affect. She reiterates this as she describes aesthetic response as "an active, lived experience geared toward meaning and pleasure" (ibid, p. 269), which sounds strikingly like a intertwining of will and grace.

There are three broad categories of aesthetic responses to mathematics, the evaluative, the generative, and the motivational (ibid). The evaluative aesthetic draws attention to mathematical entities such as theorems or proofs, and is at play when considering or judging beauty, elegance, and significance (p. 264); an ugly result is determined to be so because of this aesthetic. The obvious focus on the evaluative is on products of work. There is another less obvious area in which the evaluative aesthetic plays a strong part in mathematics—how one chooses to write up or describe one's work. It is not uncommon in mathematical presentations to remove all connections to thought processes simply in order to develop the logic of a formal argument or proof (Thurston, 1998). This certainly reflects an aesthetic where the beauty is in argument or result alone, which should thus be compelling on its own, and it invokes the common or typical modes of description in Cuoco et al (1996). Of course such aesthetic appreciation is learned from participation in the discipline. Graduate students of mathematics often do not appreciate this type of description of mathematical results, but rather prefer more *messy*, less elegant, accounts of mathematics (Sinclair, 2004), which also are relevant to the describing habit of mind (Cuoco et al, 1996): "The messy formulations better encapsulated the students' personal history with the problem as well as its genealogy and that the students wanted to remember the struggle more than the neat end product" (Sinclair, 2004, p. 268). This suggests first, that different goals promote different

aesthetics, particularly in terms of description, and second, that their place in the work, or their relationship to it, is critical. What students find compelling is their thinking process, not the argument[34]. This suggests that for many the meaning is in the act of sense-making, or the relationship with the math, rather than in the products (Davis & Hersh, 1981).

While the evaluative might focus on products, the generative aesthetic response is integral to process. It is "responsible for generating new ideas and insights that could not be derived by logical steps alone" (Sinclair, 2004, p. 264). This generative response is strongly connected to intuition, as described in Burton (2004), and spawns new questions and directions, as well as inventive methods or approaches to answering questions, by leading mathematicians down certain paths. To follow where feelings and intuition lead can be a risky venture, particularly when the disciplinary value of work rests on logical arguments and reasoning. Consequently to make use of this aesthetic response, mathematicians must trust that feelings and intuition are worth listening to (which is typical per Burton), and they must believe that feelings work in conjunction with reasoning and effort (p. 271). In addition to intuition, Sinclair identifies two other tools that mathematicians use to trigger their generative aesthetic response: developing intimacy with the mathematics and playing, which links directly to habits of mind. Sinclair actually draws on all forms of playing-experimenting, tinkering and pattern sniffing—when she noted, "The phase of playing is aesthetic as the mathematician is

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[34] I wonder if appreciation for the argument is something that emerges naturally over time subsequent to appreciation of the thought processes. The mathematician already has engaged in the thought processes, so a new logical argument actually can represent greater understanding, because they have new ways to think about and argue about the result. However, for someone who has not gone through all the process work, it is difficult to appreciate the argument for its inherent value, since they do not have the experience of the writer on which to draw. Perhaps arguments actually become inherently valued when experience is there to support it.

framing an area of exploration, qualitatively trying to fit things together, and seeking patterns that connect or integrate" (p. 272). Meanwhile, intimacy invokes images of relationships, suggesting both a deeper knowing, and affective involvement. Sinclair argues that mathematicians' reliance on intuition, playing, and intimacy to evoke the generative aesthetic suggests it is believed and accepted that mathematical behavior is often not goal-directed, and that frustration is integral to the process (ibid).

The final aesthetic response certainly is related closely to the other two: the motivational aesthetic. Motivation is crucial to any type of engagement, and this feeling is often driven by an aesthetic response. Simplicity order, sense of fit, visual appeal, surprise, and paradox often are sources of motivation[35], (p. 275), as they stimulate curiosity, desire to repair, or yearning to understand. When these are not present and the motivation is thus not inherently mathematical, this creates potential obstacles to involvement or allows attention to get "snagged"[36] by something non-mathematical, which would then impede mathematical progress. This motivation may not rest entirely in the mathematics, but could relate as well to the community in which the mathematics is done, such as the desire to help others makes sense of mathematics, to work on an important problem in the field, or to seek appreciation (ibid). Thus the motivational aesthetic must draw on enculturation in particular communities of practice, as well as on the identity of the one doing the work. It has been observed that the motivational aesthetic does not seem to become a driving force until graduate school (ibid), which is when students for the first time are truly able to involve themselves in their own research

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[35] Many of these sources seem to fit well with the evaluative or generative aesthetic. This does not seem accidental. Rather the evaluative and generative can trigger the motivational.

[36] I am borrowing this term from Daniel Chazan.

(source of motivation relates to generative work) and in answering questions that might be important to the discipline (source of motivation in part tied to communities of practice). What role will these habits of mind, intuition, and aesthetic responses have in Beth's and Amelia's inquiry experiences?

Pair Inquiry in Statistics: The Interaction of the Active and the Passive
Probability and Pascal's Triangle: Intertwining Intuition, Aesthetics, and Habits

This exploration occurred in the summer of 2006 during Statistics. It was this group of teachers' third opportunity to engage in inquiry, but their first chance to design the initial questions and work in pairs or triplets. To do so, each group was to use a specific problem posing heuristic, involving identifying attributes, engaging in *what-if-nots*[37], and generating questions from these to define their investigation (Brown & Walter, 2005). Beth and Julie worked together on this investigation with the goal of making sense of the connections between probability and Pascal's triangle. The pair provided a joint summary as well as their individual project journals, which demonstrated their different ways of thinking about and approaching the questions.

As the process began, they worked together on listing attributes, such as each row begins and ends with a one. Julie had worked extensively with Pascal's triangle, and began citing attributes with which Beth was not comfortable, such as "the sum of each row is a power of 2" (Beth, Probability Project Journal, p. 2), which then became a question for Beth, as did what would happen if the leading number was not a one. While

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[37] A *what-if-not* is basically a process of looking at the attributes of an object, such as the leading 1s of Pascal's triangle, and then proposing changing those—What if not a leading one. This leads to questions like What if leading with 2s? It is a tinkering process.

Julie began to focus on probability, Beth's questions related primarily to Pascal's Triangle (see Figure 3).

1. *How/why did he "do" this? and What was he doing mathematically to get these results?*

2. *Can formulas be written for the patterns?*

3. *What would happen if we graphed any of the patterns?*

4. *Is there a formula you can write for the nth row?*

5. *What are some practical applications beyond probability (beyond coin tosses)?*

6. *Do the mini-triangles relate to each other?* (Beth, Probability Journal, p. 2-3)

```
                        1
                    1       1
                1       2       1
            1       3       3       1
        1       4       6       4       1
    1       5      10      10       5       1
  1     6      15      20      15       6       1
1     7      21      35      35      21      7       1
...
```
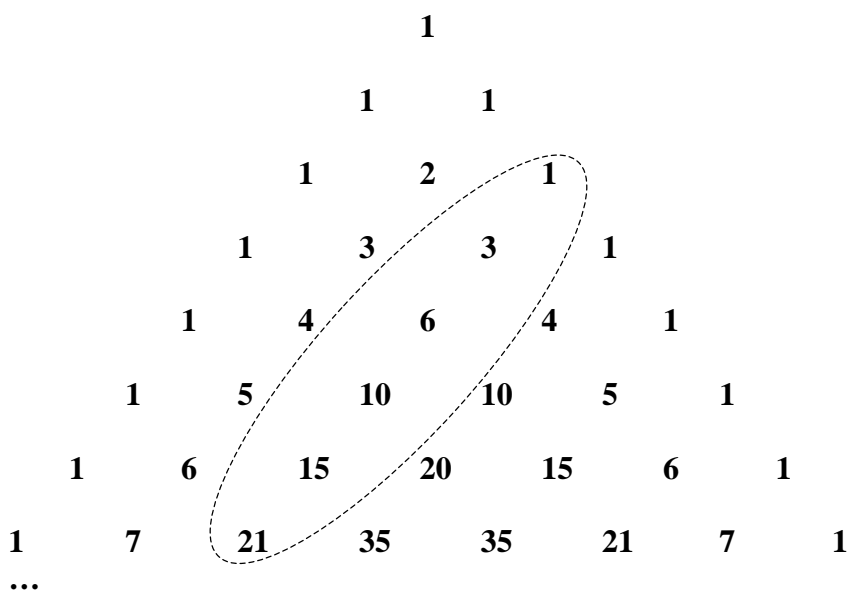
Figure 3: The top eight rows of Pascal's Triangle.[38]

As she began her investigation, Beth represented her data in *x-y* tables of values where the *x-* coordinate was the position in the row of the term, and the *y*-coordinate was the actual value of the term. The rows that she chose actually were the diagonal rows (see

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[38] To find the numbers in any space in the triangle, you start the row with a 1 and then simply add the two numbers directly above the space to get the value in the new line.

oval above), rather than the more typical horizontal rows, which she had familiarized herself with in her initial construction of the triangle (Member Checking, July, 2007).

Table 1: Entries of the 2ⁿᵈ and 3ʳᵈ diagonal rows by position[39]

x	y		x	y
1	1		1	1
2	2		2	3
3	3		3	6
4	4		4	10
5	5		5	15
6	6		6	21
…	…		…	…

(Beth, Probability Project Journal, p. 3). In graphing these, she recognized that row one was the line $y = 1$, and that the second diagonal row was a linear function $y = x$; the graph of the third row produced a curve that went up increasingly quickly and resembled a curve that she had encountered in her algebra courses and inquiry. She then returned to the tables to examine the first and second differences (see Table 2) in order to test her

Table 2: Exploration of 1ˢᵗ and 2ⁿᵈ differences of the 3ʳᵈ diagonal row

x	y	*1ˢᵗ diff.*	*2ⁿᵈ diff.*
1	1		
2	3	2	
3	6	3	1
4	10	4	1
5	15	5	1
6	21	6	1

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[39] The journal was handwritten so all tables, graphs, etc., are my reproductions of the work that was in her journal.
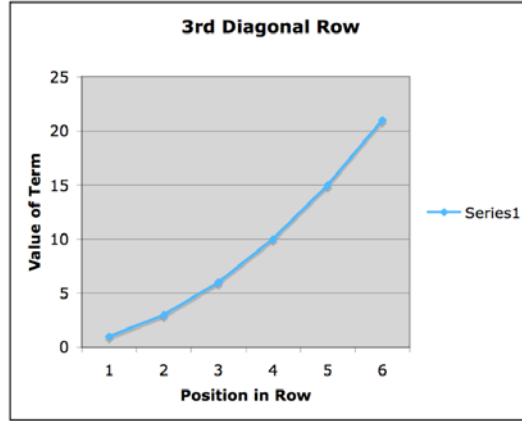
Figure 4: Graph of 3rd Diagonal Row

intuitive guess. From the constant second difference, she realized that the points

generated half of a parabola (see Figure 4) with the equation $y = .5x^2 + .5x + 0$. She

continued sketching the graphs of additional diagonal rows, and all graphs resembled in

shape the third row half-parabola (see Figure 5).
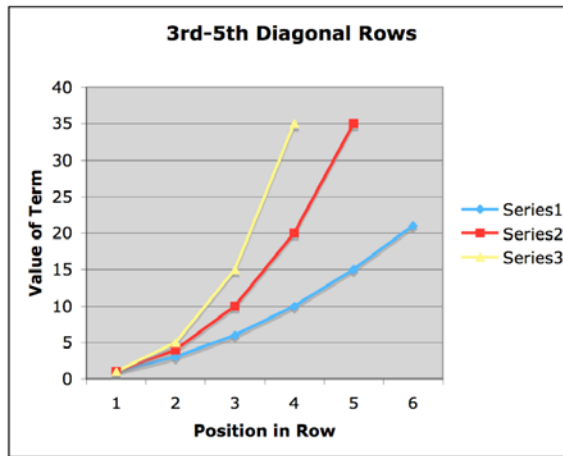
Figure 5: Graph of 3rd-5th diagonal rows

This similarity prompted her to examine the fourth row for first and second differences.

When she did not get another constant second difference (a quadratic), she guessed that

she would get a cubic function. She noted that she could do a cubic regression on the

calculator, but if she did, the result is not included in the project journal. She continued

displaying and exploring many other data charts, which together prompted the development of a conjecture that the power (highest exponent) of the resulting function increased by one degree[40] with each progressive row. She wrote, "Are all these exponential[41] functions of growth? They all take the same shape" (ibid, p. 3b). That all the graphs except the first and second took the same shape seemed important to Beth; it was a pattern.

Following up this part of the exploration, Beth opted to graphically represent the horizontal rows of Pascal's triangle, producing a familiar-looking graph (see Figure 6).
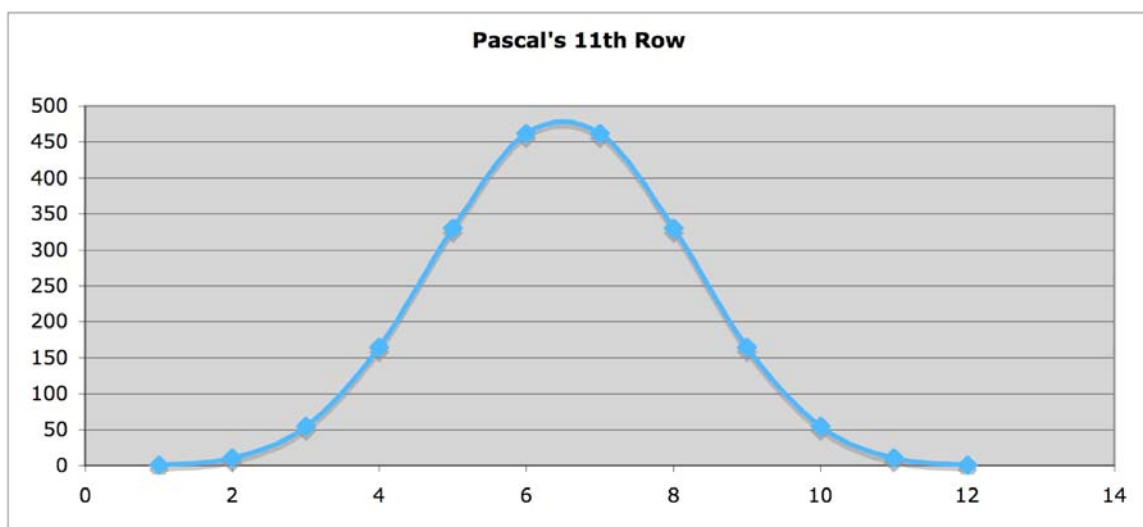
Figure 6: Graph of the 11th horizontal row

Thinking about the connection of this triangle to probability, she made a generative observation: "They almost look like bell curves. Wouldn't it be interesting if those two things were connected—the shape of the bell curve, the proportion of data in each standard deviation, and the ratio of those numbers to each other??????" (ibid, p. 4a). Her

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[40] A linear function has degree 1, a quadratic has degree 2, a cubic has degree 3, etc.

[41] While she used the word *exponential*, in noting that the 3rd row yields a quadratic and the 4th row yields a cubic, and that the 5th row is degree 4, etc., indicates her conjecture relates to power functions. This appears a problem of vocabulary or oversight, not math.

curiosity piqued, Beth called on what she knew about the bell curve—the data percentage breakdown by standard deviation (see Figure 7)—and applied this breakdown to the number distribution of Pascal's triangle by row. Her division of the 7th row is reproduced in Figure 8.
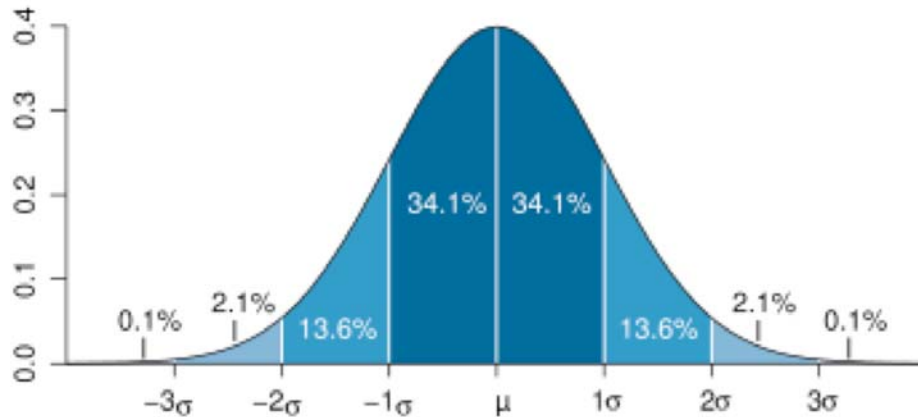
Source: http://en.wikipedia.org/wiki/Standard_deviation
Figure 7: Standard deviation divisions in normal curve
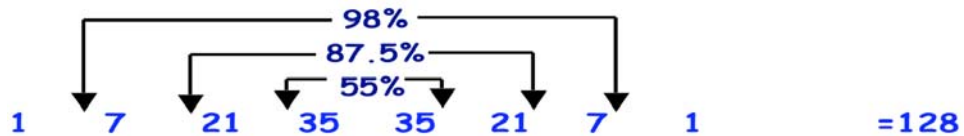
Figure 8: Division of 7th horizontal row into percentages

While not exact, she wondered if the distribution would become more precise. So she

Figure 9: Division of 10th horizontal row into percentages

Repeated the process with row 10 (see Figure 9), which seems more closely connected to the standard deviation breakdowns to which she was accustomed. Though she had not found an exact match, Beth felt as though she had found a direct link between Pascal's triangle and probability. It was in carrying out this process that she also recognized that the sums of each of the horizontal rows seemed to be double the sum of the previous row. After this recognition, she went back and wrote each row as a power of 2 (see Figure 10), and made an *aha*! discovery—"Now I see the powers of 2 [that Julie had been mentioning as an attribute from the beginning]!" (ibid, p. 4b).

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & =1 & & =2^{0} \\
 & & & & 1 & & 1 & & =2 & & =2^{1} \\
 & & & 1 & & 2 & & 1 & & =4 & & =2^{2} \\
 & & 1 & & 3 & & 3 & & 1 & & =8 & & =2^{3} \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & =16 & & =2^{4} \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & & =32 & & =2^{5} \\
1 & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & =64 & & =2^{6}
\end{array}
$$

Figure 10: Pascal's Triangle with sums written as powers of 2

With this new understanding, she decided to link back to the way that Julie[42] was pursuing probability; she began looking at permutations of boy-girl births, while Julie had been looking at head-tails flipping. Figure 11 represents how she connected these ideas:

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[42] Julie's work focused on making probability trees for coin flips, and connecting the probabilities of different combinations of events to the elements of Pascal's Triangle.

$$2^3 = 2*2*2 = 8 \qquad 1/8 \text{ ways to have all girls}$$
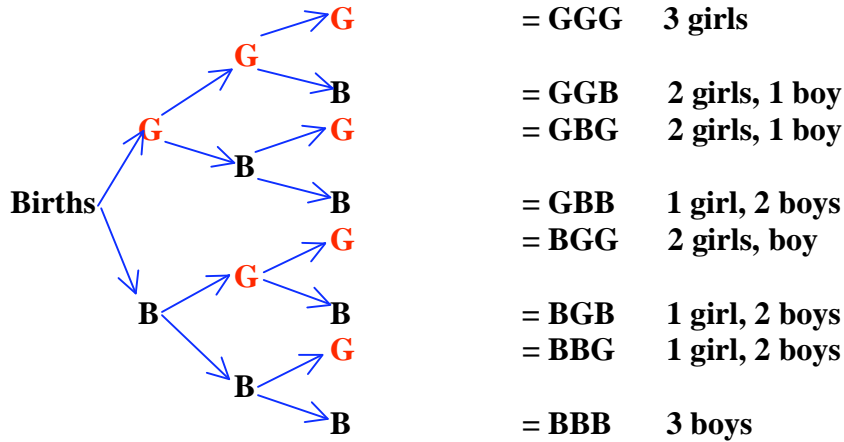
1 way to have 3 girls
3 ways to have 2G, 1B
3 ways to have 2B, 1G
+1 way to have 3 boys
$2^3 = 2*2*2 = 8$ 1/8 ways to have all girls
(ibid, p. 5).
Note : **1, 3, 3, 1** is the third row of Pascal's triangle

Figure 11: Combinations and permutations of boy/girl births.

As Beth moved from the eight ordered permutations to the four general combinations, she realized how the row of the triangle provided the probability of each combination. Using this understanding, she looked at the 5th row of Pascal, which summed to 16, and then guessed that this meant that 1/16 of the combinations[43] of four children would be all girls. Beth then conjectured a generalization, suggesting that the exponent of the power of 2 corresponds to the number of items in each sample set grouping (e.g. in the example above the exponent 3 represents the 3 children) and that the individual numbers in each row correspond to the number of ways to get particular combinations.[44] Her journal

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[43] Combinations refer to an unordered set of elements. In terms of combinations, the birth sequence GBG is no different from BGG or GGB. Permutations depend on order.

[44] She also tinkered with Pascal in other ways. She looked at what happened with leading 1s, 2s, and 3s. After observing what occurred with leading 2s, she drew on her understanding of number

writing around this investigation closes with the following reflection: "I think it's really cool that I followed my own thinking and now all the connections make more sense to me. I wonder if Mendel saw these numbers in his plant research?" (ibid, p. 5b). This suggests that what Sinclair learned about grad students in mathematics finding the process meaningful can relate to mathematics education grad students as well.

After sharing their work with each other, Julie and Beth were able to see the connections in each other's work. In their joint summary, Beth wrote: "We enjoyed being able to bounce ideas off each other and were intrigued by the different discoveries each of us was making." Their summary also included each of their "most exciting" discoveries from the different directions they took. In particular, Beth noted:

> My most exciting discovery became a conjecture that I would like to explore further. While experimentally graphing each row of the triangle, it seemed to me that the visual shape each row took was similar to the normal curve. In my notes, you will see where I added all the values in particular rows and then divided the "data" into standard deviations around the middle section of data. I did not consider this the mean, because as I am typing this, I now wonder if the middle value in each row is the arithmetic mean of the row (or could this be geometric, or harmonic?). I noticed that the more values I used, the closer the percentages came to fitting the Empirical Rule for normal curves. I found this very exciting because it fit with some of what we had been talking

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to conjecture that leading 3s would be different but leading 4s would behave similarly. This conjecture was incorrect, but she made sense of why.

about in class with regard to binomial experiments" (Project Summary,

July 25, 2006).

Coming to know in the Pascal inquiry project.

What readily characterizes the mathematical inquiry work in which Beth is

engaged is not the presence of habits of mind, intuition or aesthetic, though they are of

course present, but rather how their interaction invites interpreting and making sense of

her work.

As soon as Beth and Julie decided on the problem, there was immediately the

question of how to proceed: Did they work in the same way or did they each proceed in

their own manner? Given that Julie wanted to focus on probability and Beth, with her

love of patterns, wanted to focus on the triangle, they opted to go separate ways, though

they agreed to come together in class and talk about their work. If aesthetic responses are

motivational and thus guide the direction of work, then it is logical that Beth would

choose to focus in a direction related to something that she found aesthetically pleasing:

patterns. Julie seemed to have more experience with Pascal's triangle, which may have

influenced her desire to go in the other direction.

Once the decision was made in which directions to move, Beth did set to work

playing with the elements of the triangle, and in particular, looking for patterns. She did

this through visualizing data in tables and then in graphs of diagonal rows. In the

graphing of these, the shape of the graph of row 3 connected to an entity she had worked

with previously—or at least to half of it—though it was only an approximate replica of

the curve. Such a connection has an intuitive feel because she was not actively looking

for analogous entities. This connection shaped further investigation. She immediately

began looking at the first and second differences of row 3 to test her guess that the shape was indeed a parabola. The confirmation of her guess motivated her to continue this process with further rows. She began looking at first and second differences, but discovered with later rows these were not constant. Rather than abandon her work, she seemed motivated to find out what these rows represented, suggesting a gut feeling that there was a pattern to find.

Drawing on the fact that row 2 was linear and row three was quadratic, she guessed that perhaps this 4th row was cubic (which has constant 3rd difference). While again it is unclear how she confirmed this, the pattern she found seemed to be developing into a new conjecture about the rows representing power functions that she tested empirically through graphing. She was, however, quite surprised when row 10 turned out to be linear. In this case however both her intuition (which was based on her knowledge and experience) and the likely involvement of a violation of the evaluative aesthetic, she was motivated not to explain why it was linear, but rather to repair the misleading graph. She felt there was an explanation for why it appeared linear when she was sure it was not. It was in working on this reparation that she recognized a problem with her graphing tool; it could not effectively handle the large numbers in row 10. Results not fitting the expected pattern played a very different role with intuition in the next part of her work.

As Beth began looking for patterns in the horizontal rows, this visual representation sparked an intuitive[45] connection to the bell curve. Her excitement at such a direct connection was obvious as she included this feeling in her journal. Echoing the authors of habits of mind, this intuitive connection led to the invention of a unique, but

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[45] This connection is intuitive in as much as it just came to her, but it also depended on her knowledge of the Bell Curve. She was not trying to find something to which to link the shape of the graph of the horizontal rows. This was very motivating to her.

not arbitrary, way of exploring this connection, using the percentage breakdowns by standard deviation. If her graphs and the bell curve were connected, she felt the one should apply to the other. What stands out as interesting herein is that, as with her diagonal row 10 above, the breakdown did not fit her expectations or hopes well. However in this case, there was no indication that she was remotely upset by this lack of fit. Rather it felt plausible to her. Why was one previous counter-example problematic, but not one in this case? The previous case had even had other linear functions in it, so it was not a unique result. Her suggestion that the fit with the normal curve might improve as she goes down demonstrates her intuitive understandings, again based on both knowledge and experience, of how larger numbers of trials behave when probability is concerned. This directly parallels the relationship of experimental and theoretical probability, though Beth never once mentioned this idea. Why would anyone without this understanding expect the percentage breakdowns to change (let alone use this approach)? As she watched the fit improve in successive rows, Beth's motivation to engage in further exploration grew as noted in her summary. It was exciting (affect).

I wanted to highlight this connection of habits, aesthetic, and intuition in one other place in her exploration that I did not previously highlight but in a footnote, as it seemed somewhat isolated from the other work. She played with Pascal's Triangle in one other way: she experimented or tinkered with leading digits. Traditionally Pascal's Triangle leads with a one, but the process of summing the numbers above to get the numbers below can take place with any leading digit. She clearly expressed her *aha*! recognition at seeing the rows double with leading ones. She proceeded to examine leading 2s and found the same pattern. At this point, though, rather than conjecturing that

rows always double, she in fact guessed that leading 3s would act differently, but leading 4s would not. Why would one expect or look for a seemingly pleasing pattern (rows doubling) to break? One very plausible explanation is intuitive understandings of how individual numbers behave. 1, 2, and 4 can all be written as integer powers of 2; 3 cannot. Thinking another way, in particular 2 and 3 are prime numbers and as such have very little relationship to each other (except sequence). Someone with a weak or simply superficial understanding of number as quantity would unlikely make such a conjecture. This conjecture has to be based on understanding that these numbers behave differently. And again, this was simply a guess that occurred in the process of experimenting with the triangle. In checking, Beth discovered her guess was incorrect, and that the rows still doubled. One way to think about this is that her intuition was faulty. Rather, I think her intuition is sound, but her number intuition was drawn on more readily than the developing Pascal intuition. Her desire to understand motivated her to check her guess, which was wrong; surprise then motivated her to seek a new explanation, prompting an exploration of the prime factorization of the rows. While the sums of rows with leading 2s and 1s could be written as powers of 2 (2^n), the sums of rows with leading 3s could be written as 3 times a power of 2 ($3*2^n$). In pursuing other intuition and then reconciling the mismatch, she perhaps developed even greater understanding. Not only did she now know that the rows double each time, but she understood the role of the leading digit.

As we look at Beth's mathematical work from these perspectives, two things become quite clear: she both trusts and responds to her feelings and her intuition in the doing of mathematics, which Sinclair stressed was central to making use of the generative aesthetic response, to the development of ideas and of paths to follow. Moreover, Beth

uses habits of mind in an interweaving way—they lead into and fall back onto each other—both to promote and as an active response to her affective, intuitive, and aesthetic responses. Her actions and responses each cyclically propelled and shaped the process of knowing. When Cuoco et al (1996) discuss the importance of developing habits of mind, they suggest these "give students the tools they will need in order to use, understand, and even make mathematics that does not yet exist" (p. 376). However, in examining Beth's work, these tools are not just those listed as habits of mind; habits of mind are the active tools that facilitate coming to know. Just as Sinclair (2004) argued playing is a tool to develop the generative aesthetic response, intuitive and aesthetic responses are tools to facilitating habit of mind. In the motivational aesthetic, Sinclair highlights the mathematician's belief that he will be able to resolve tension and work through frustration. Beth has more than once stated that while she knows she sometimes panics, she believes that encountering frustration in doing the work that she does is not inherently problematic; she feels that she will eventually be able to make sense of things, one way or another, as long as she keeps trying. Thus somewhere amidst the interaction of thee active and passive tools is another unspoken too: confidence. This is an affective response, but for the mathematician it reassures that activity will be productive and worthwhile. Thus developing confidence that activity will be productive seems critical to the process of connaître knowing.

To Drop or Not to Drop? –If Not In Habits and Aesthetics, Where Is the Math?

Though in general a pair investigation, Amelia worked with Robin and Heidi, both of whom happened to be interviewees for this research. The source of their

investigation was a sixth math activity with which their own students had had some difficulty. This group had not felt confident answering students' questions. They wrote:

> The curriculum guide suggests a lab that explores probability and area. Students draw a small square inside of larger square on dot paper. They are supposed to drop 50 beans onto the paper from about 8 inches above. They record the ratio of the number of beans that fall in the small square to those that fall within the larger square. The directions specifically say, "Do not count those that land outside the squares." Students have often asked what they should do with the beans that fall outside the squares. Why don't they count? Shouldn't they count? We questioned what the difference would be, if any, if we re-dropped the ones that fell outside the larger square so that they would be included in the ratio (Joint Project Summary, July, 2006).

With their written journals, they included a photocopy of the textbook activity, large copies of the squares drawn on grid paper, and a data record of the various experiments they conducted.

As the assignment suggested, the group technically worked with the heuristic of Brown and Walter (2005) to develop an area of investigation, but as they already had a specific question they wanted to investigate, this part of the process does not appear to have guided much of their work. Their closest question related to a what-if-not is: "*Is the data valid if you do or do not count those items that land outside of the figure*" (Joint "Brainstorming" Page). They also asked questions such as: "*What are the givens in the experiment? What are the variables in the experiment? What is random when you have a*

randomly thrown dart? How does skill affect the relationship between theoretical and experimental probability?" (Ibid). The trio then made particular decisions about how to proceed with the bean dropping experiment, such as deciding that they would use elbow macaroni, not beans, and that they would drop them from a height of 8 inches.

The experiment involved dropping the beans on a board like Figure 12 where the
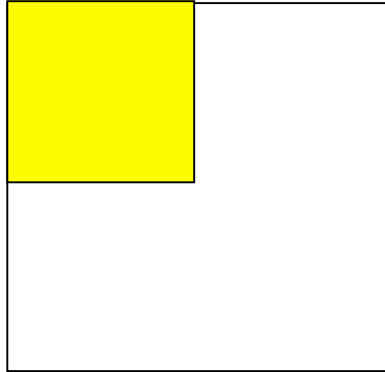
Figure 12: Bean/Macaroni Drop Board

yellow represents the shaded portion. This shaded square was one quarter of the whole, thus the theoretical probability of randomly landing in the shaded region is 25%. They conducted their trials on their own time at home. Amelia kept track of her experiments in a table.

Table 3: Partial replication of data record of experiment

Trial	# in shaded	# in whole	P(shade/whole)		Redo Shaded	Redo Whole	P(shade/whole)	
1	1	16	1/16	6%	2	25	2/25	8%
2	3	18	3/18	16%	3	25	3/25	12%
3	2	12	2/12	17%	3	25	3/25	12%
4	3	15	3/15	20%	5	25	5/25	20%
5 …	2	12	2/12	17%	5	25	5/25	20%

When her trials yielded an experimental probability of near 20%[46], Amelia observed:

> The theoretical probability was 25%. When I look at my data of the ratio
> of those that fell in the shaded out of the whole, I think the probabilities
> are low compared to 25%. Does this mean that all the beans should be
> included in this experiment? So then I looked at my ratios when I included
> the beans that feel outside the square with re-do's. The percentages
> improved a little bit but most were still lower than I expected (25%) (To
> Drop? Project Journal, p. 1).

As the group reconvened in the next class, they began by comparing their results, which led to a discussion of the differences in their method of trials. Two of them had dropped the macaroni on a hard table and another dropped their macaroni onto a carpet. This prompted wondering about the role of variables in the experiment. Amelia commented that following this discussion they were stuck, but then eventually they decided to combine their respective data. The other members had recorded 63/239 or 26%, and 83/284 or 29%. Their combined total of 186/768 or 24% was very close to the theoretical probability. Amelia then wrote, "*What does all of this mean? It seemed to reinforce for me the idea that the more trials there are, the closer the experimental will be to the theoretical probability*" (To Drop? Project Journal, p. 2). They did the same with their data that involved re-dropping the macaroni that landed outside the squares, which yielded an experimental probability of 21%. This seemed to them reasonably close to the theoretical probability, though not as close. The following description from Amelia's journal highlights her perspectives up to this point:

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[46] To find this probability, she summed the total number of beans that landed in the shaded area and then divided by the total number of beans dropped in the larger square.

> We weren't really sure what to do next and I don't really feel like anything
> got answered either. I still don't understand why the activity in the text
> book said not to count the ones that feel outside the whole figure. Why
> wouldn't they be dropped again? Is it a convenience thing? If you are
> comparing what fell in the shaded area to the whole area, then I can
> understand not including another factor such as what fell outside, but does
> re-dropping the ones that fell outside give a better representation? What's
> "better"? (p. 2)

This prompted another discussion of variables and a decision to modify the experiment to focus on what role variables might play. Choosing this time to not re-drop the macaroni since they were not sure how to make sense of the results, the trio decided that one person would drop the macaroni from 2 inches, one from 8 inches, and one from 18 inches. Amelia included a full chart of 20 trials from 8 inches (see Table 4).

Table 4: Partial replication of data record for height experiment

Trial	Shaded	Whole		
1	2	10	2/10	20%
2	3	17	3/17	18%
3	2	9	3/9	22%
4	4	8	4/8	50%
5 ...	7	14	7/14	50%

She noted that the theoretical probability was still 25%, and that the experimental probability of her trials was 27%, which is quite close. Amelia did not include more in

her journal; however, the journals of her colleagues can be used to highlight the state of their thinking at completion. Robin, who did the experiment from 2 inches, wrote:

> Why was this data so close to the data from 8 inches above? Does this mean that height is not a factor? How can that be? . . . As you can see, the questions continue. As **we** worked, **we** asked more and more questions. **We** didn't really come to any conclusion. **I** guess that is a lot of the point to this investigation (Robin, To Drop? Project Journal, p. 5; emphasis added).

She switches pronouns to distinguish her group from herself, suggesting that she perceived Amelia and Heidi felt similarly. Heidi's work also echoed this same feeling of not having answered the questions, and of not understanding why the height data was not much different than in their first experiment, though she did seem to feel that variables played a significant role. Based on the closings of her partners, I presume that Amelia did not come to any more firm conclusions regarding her questions at this time, or if she did, she did not share them. That this is a reasonable assumption is found in my second interview with Amelia where we talked some about this investigation. She said:

> I liked the macaroni one, obviously, 'cause then I used that and I kept talking about that, and I kind of brought that back to people I work with, but again, I kind of reach a point where I am like, "Okay, now what?" you know? And then like if I do a new one, is my question really going to [be answered]?. . . We were talking about the fact that, "Well let's get a dartboard." You know, like let's get a dartboard and try it that way, or whatever. And I feel like we could have probably done a lot of these

things, but then I think it kind of went back to simply the wording of the

question—which is important to the understanding too. But it was at least

interesting to me and it had meaning to me.

From this quotation, it is clear she enjoyed the investigation in that it was relevant and

meaningful to her, yet there is some sense that questions didn't get answered, and that

maybe they should have gone further. She simultaneously suggested though that the

answer might go back to "simply the wording of the question," seemingly implying that

the answer may not have been in the work they did. She does not indicate she drew on

her understanding of probability to make this statement, though to a certain extent, all

probability questions rely on the wording of the questions.

Coming to know in To Drop or Not to Drop*? and finding the math.*

As I begin to talk about this mathematical work, I want to reiterate that I only

have access to what Amelia (and the others) wrote in their journals and said in their

interviews. It is quite likely that much more went on here, but in their choices of how to

describe their activity, they may well have hidden a great deal of that from view, which is

typical of mathematical writing (Thurston, 1998). Sinclair (2004) argued these choices

reflect the evaluative aesthetic of those engaged in the work, and in particular, what they

value. When I argue below that Amelia's does not make purposeful use of habits of mind,

intuition, or aesthetic response in a connected way, maybe it is simply because those

types of connections did not feel important for the three teachers to write down, though I

might expect at least one of them to give more indication if this was the case. It is

important to stress also that highlighting this disconnect in her approaches to coming to

know is not done to criticize Amelia. I engage in highlighting the disconnect to make

sense of how and why inquiry might be problematic for her and her peers, not that she is somehow deficient (Carter, 1993). After I highlight these issues in coming to know, I will then discuss some relevant mathematical issues that Amelia seems to be confronting. While on the one hand, these issues at least temporarily seem to inhibit her progress, they parallel important issues in the historical development of probability, and perhaps are important to her own mathematical development.

In terms of habits of mind, I will just focus on playing as this relates closely to aesthetic. As was the nature of their task, Amelia, Heidi, and Robin collected different types of data to see what they could learn about whether to re-drop the beans. They compared their results, were skeptical of them, and, not feeling that issues were resolved, they made new choices, such as combining their data. Episodes of playing may be considered present in other ways as well. While the trio followed the given directions in the textbook, they tinkered with the experiment in ways they hoped would shed light on their question, such as considering variables, height being the main one. While these habits were technically present, the role of playing is also important to consider. In mathematical inquiry playing is used to generate thinking and ideas, suggest conjectures, or raise questions for further investigation (Sinclair, 2004; Cuoco et al, 1996). Yet, Amelia and her peers seem to have been expecting/hoping the actual results of experimenting to provide ready answers. This is evident in the previously quoted comment "I don't feel like anything got answered. . . . Why wouldn't they be dropped again? Is it a convenience thing?" First she used the expression "got answered," as opposed to an expression like "we were not able to answer…"; This suggests a

perspective that external results have an intrinsic power; moreover, it places the mathematics in the results rather than in the process of developing understanding.

Each time their question does not get answered by the data, Amelia conveyed frustration and seemed to ask, "Okay, what should we do now?" They combined their results for this reason; they began to explore variables like height for this reason. Another way of stating this is that they did not make decisions based on what they learned from their actions, but rather decide what to do because they did not learn what they wanted from their activities.

Regarding intuition, which was present in Amelia's work, its emergence did not enable Amelia to capitalize on it, nor did it propel the investigation by facilitating the making of connections or feelings of understanding. When Amelia examined her results from her first round of trials, her experimental probabilities were both lower than the theoretical. This evoked consternation on her part, because it did not feel right, suggesting an intuitive expectation. Rather than consider why this might make sense mathematically, she immediately went to see if her other results were more "accurate." She certainly was not drawing on an intuitive understanding that, with experimental probability, it is not uncommon to get results that vary, sometimes widely, from the theoretical—that is why it is theoretical. Wide variance would be expected when the number of trials is low. Of course someone might suggest that Amelia did not understand this aspect of the relationship between experimental and theoretical probability. However, she certainly has the propositional knowledge. After combining their results she said, "It seemed to reinforce for me the idea that the more trials there are, the closer the experimental will be to the theoretical" (To Drop? Project Journal, p. 2). She had the

knowledge that the experimental did not have to reflect the theoretical. Perhaps she simply lacked the experience requirement of intuition to be able to draw on the knowledge in another situation. Her metaphor of reinforcing suggests another plausible perspective. Amelia's playing might serve as a confirmatory act rather than a generative one, as a symptom of the theoretical absence of empiricism in mathematics (Lakatos, 1967). Amelia again later demonstrated having intuitive understanding when she offered a plausible answer to the very question she and her peers had asked. She reasoned that if the task was to compare the areas of shaded to whole, then those pieces outside the square were irrelevant. This is exactly one basic understanding that allows you to disregard the macaroni outside the square. Yet why does she only suggest it when nothing else has worked? Moreover, why does this not satisfy her, as it clearly does not because her next statement searches for another explanation? The lack of drawing on intuition to make sense of this problem helps us to understand both the lack of generative aesthetic and the disconnected feel of any habits of mind. It further suggests a lack of trust in one's feelings or intuition, which Sinclair (2004) argued was central to mathematical activity.

Intuition figured prominently in this investigation in another way. I believe Amelia and her peers had intuitively answered their question regarding why the directions said not to count the beans that fell outside well before they carried out a single trial. Unfortunately, in not recognizing this, they began collecting data on a quite different question: Would re-dropping the ones that fell outside provide more accurate results (linking back to Amelia's notion that the answer was in the wording of the problem)? Their experiment did not consist of trying to find ways to count the macaroni

that fell outside, or even what it might mean to do that. Most likely, as Amelia said later, they "knew" this did not make sense given the context of the problem. This problem was made more challenging because they were trying to answer one question while effectively collecting data on another. Perhaps this occurred in part because of the pedagogical motivation to the problem. They were looking for an answer to be able to address students' questions, but also to decide if they should alter the activity. The effect of the external motivation for finding an answer rather than allowing the generative or motivational aesthetic response to drive the situation conceivably further focused these teachers on *the* answer.

While their processes of sense making may have been disconnected, this trio was actively engaged in their work. Were they doing math? An easy answer to this would be "no." As they looked for a mathematical argument regarding whether or not to drop the beans, they were distracted by what many might call inconsequential matters. In general, a mathematician would disregard *What is random*, *What are the variables*, and *How is skill involved*. The problem was, they were looking for an answer where an objective mathematical answer did not exist. They were trying to make a practical decision regarding whether or not to drop the beans, yet this is not a mathematical decision; various decisions could be justified. I can certainly come up with a variety of arguments for either answer. A pedagogical argument for not re-dropping the bean could be that it will take too much time. A statistical argument from study design to re-drop them relates to the notion that larger sample sizes are more powerful. Drawing on the fact that both experiments have the same theoretical probability and will exhibit the same type of varied experimental probability, one could argue it simply does not matter.

Interestingly, their feeling that probability could make the decision is actually reflective of historical challenges to the development of probability. Davis and Hersh (1986) argued that it is exactly this decision-making thinking, made famous by Luca Aioli, a fifteenth century Venetian mathematician, that delayed the development of probability until the seventeenth century; probability is not about decision-making after all. These two mathematicians distinguished three levels of probability theory: pure probability theory [PPT] with its axioms and theorems; applied probability theory [APT], which serves to connect PPT with real world contexts; and applied probability at the bottom line [APBL], which involves using PPT and APT in the consideration of practical decisions, as Amelia and her colleagues were trying to do (ibid). APBL potentially inhibited the development of probability because of how the thinking in APBL required a focus on real world issues and policy. Certainly probability is not about decision-making, yet for centuries, notions of randomness, on which probability theory depends, were employed specifically for decision-making—of course this randomness involved throwing irregular dice or carving up carrion (ibid). It was not until our current notion of randomness was theorized that Probability could develop (ibid). Thus Amelia et al's struggles with using probability for decision-making and with what randomness is reflect important historical issues. Just as randomness needed to transcend its everyday applications, probability itself needed to be separated from the everyday practices to develop; it needed to be idealized (ibid).

Is this practice of idealization that distinguishes mathematics? Philip Kitcher (1984) argued that arithmetic was an idealizing theory as well. Its truth did not lay in what we as humans actually were able to do, but rather what an idealized being would be

able to do ideally. Thus while probability and arithmetic and arguably much of mathematics are idealizing theories, Amelia, Robin, and Heidi are tied into their real situation and exploring the realities of the experiment (height, surface texture, skill). Davis and Hersh go one step beyond and in fact claim that APBL is not even mathematics. Do I agree? Would I think that their question is not a mathematical one?

I would agree that they are no longer considered mathematical, because the discipline has dealt with these issues and discarded them. As naturally arising problems, they were once mathematical in that they reflect epistemological issues in the development of mathematics. Would some suggest that conceptualizing zero was not mathematical work? It too took centuries to develop. While Davis and Hersh argue that such issues may not be mathematical, if they were of issue for the discipline and for the development of mathematics, I would argue they are mathematical, though certainly of a different flavor. After all quantity and measurement are still important mathematical notions (Bishop, 1989), yet number required separation from quantity in order to give rise to negative and imaginary numbers, and geometry is no longer a synonym for measurement of the earth, though this is what its name means. Thus these three teachers are getting bogged down in a historical issue in probability and in variables that most mathematicians would not even consider. Shouldn't those involved in the doing of mathematics encounter the issues that lead to important mathematical decisions, even separation from related ideas, rather than just suspend their disbelief and accept? Given that the problems encountered herein reflect both historical and epistemological issues in mathematics, perhaps wrapping these types of considerations into mathematical study would be worthwhile, particularly for teachers who may run up against these issues or

whose students may run up against these issues. Isn't this similar to the case that Beth finally realized that symbols were a human invention and the difference that made in her understanding of mathematics?

I think the difficulty is mathematics educators don't want students to have to get bogged down. Perhaps this suggests then that there is another mathematical habit of mind, though not as common as the others described, related to idealizing. Though one is not described, there seems to be a related one: **Mathematicians Talk Small and Think Big** (Cuoco et al, 1996). While the analogy is not precise, in the elaboration of this habit, idealized images seem relevant:

> Much of this "thinking big" goes under the name abstraction. Modeling is also used to describe some of it. Once again, getting good at building and applying abstract theories and models comes form immersion in a motley of experiences. . . . However, experience, all by itself, does not do it for most students. They need explicit help in what connections to look for, in how to get started. Unfortunately, (for curriculum developers), sometimes the only way to do this is to apprentice with someone who knows how to play the game (ibid, p. 385).

If idealizing is anything akin to these ideas of abstracting and modeling, to develop this habit would require significant enculturation in the practice of doing mathematics.

Amelia and her colleagues did not make significant progress in inquiry. Certainly some reasons for this include its pedagogical motivation, and the disconnectedness of their activities. However how successful could they be if not given the chance to confront

problematic philosophical and historical issues when they emerge? How helpful could I have been, not being able to recognize that these issues were involved?

Individual Inquiry in Integrated Geometry: The Role of Identity in Inquiry

In this sub-section, I will explore the final large inquiry activity in which these teachers engaged as part of their program work. I will use this as an opportunity to connect when possible to their previous inquiry work to demonstrate whether this work seems typical of these teachers, and I also will draw on other discussions surrounding their identities, including their histories, goals, perspectives, and approaches to mathematics. As these projects are individual, the negotiated experience in the community of practice is not as explicitly relevant, though what these teachers have come to see as the purpose of the inquiry is certainly a reflection of this aspect of identity.

Transformations and Tessellations: The Dominance of the Teacher Identity

Similar to her investigation during statistics, Amelia wanted to do a research project around something that would be useful for her teaching. Because she had never learned it in school, and has had some difficulty teaching it with understanding because all she knew was procedure, Amelia decided to investigate transformations. In particular she said, "I wanted to explore M. C. Escher's work with tessellations to spark the interest in the students. . . . I hope to use [The Stone Lad], as well as many other examples from Escher to motivate the students to experiment with translations, rotations and reflections" (Transformations Project Report, p. 1). She added that, with her understandings that develop with this research, she hoped to create lesson plans that would develop her

students conceptual understanding of "what occurs with transformations" (Transformations Project Report, p. 2).

In her report, Amelia took stock of the prior procedural knowledge that she had regarding transformations, such as "when you reflect a point over the y-axis, you keep the same y-coordinate and multiply the x-coordinate by (-1)" (ibid, p. 1). It bothered her that she always had to refer to her notes to remember how to do the rotation and reflection, though she felt fairly comfortable with what to add and subtract to coordinates based on translations. She highlighted a variety of questions including, What is the difference between a tessellation and a transformation? To answer her questions, she searched through resources where she found the definitions. As she was researching, other questions arose, such as which polygons would tessellate. She continued researching and found that to tessellate, polygon angles that meet must sum to 360 degrees, and that only a triangle, quadrilateral, and hexagon would self-tessellate; in other cases, more than one polygon is necessary.

Drawing from her textbook and a suggestion from another colleague, Amelia engaged in an Escher-type activity. She described how to take a note-card, cut a piece out ,and translate it to create a tessellation: "You take an index card and cut out a random piece from the bottom. Take the piece and tape it to the other side of the index card. It needs to be in the exact same location vertically or horizontally (a translation)" (ibid, p. 2). After creating her own figure and tessellating it, she noted:

> I tried the activity and it worked! It was fun to see how the figure
> continued to fit together seamlessly. I tried to look at the original figure to
> see how it could be interpreted. The one I did looked like a fighting man

or a candle with a flame. I thought this would be an interesting activity for

my students to try" (ibid, p. 3).

In the remainder of her write-up, Amelia described the resulting lesson plan and response to the lesson in detail. It was a three to four day lesson that began with researching Escher on the Internet; she provided a questionnaire to have students comment on what they saw in his art in terms of shapes, figures, and movement. She then planned to have students experiment with creating figures from note cards that would tessellate. Ultimately they would choose at least one to tile. For the third part of the lesson, which she said she created after she researched for this project (though she does not note which aspect of her research led to this), she designed coordinate planes and figures that the students could physically transform, because her curriculum requires the students "identity the ordered pairs of the vertices of both the original figure and the image of the figure" (ibid, pp. 3-4) Her desire to have them physically do this was to "help them to discover what happens to the figure and the coordinates of the vertices so that they develop more of a conceptual understanding" (ibid, p. 5). To facilitate their work, Amelia color-coded the different angles of the figure so that students could easily track which vertices were related to each other. She then described what she wanted students to see in terms of translations—that one coordinate would stay the same, and the other would change depending on the direction of the slide[47].

Amelia provided a personal account of what took place during the lessons with her students. She definitely felt that the Escher work helped motivate her students, particularly as they began to do their own tessellations; their motivation was impressive.

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[47] This procedure only applies to horizontal or vertical translations

She remarked, "They made their own rules for positive and negative numbers since they had not yet learned operations with integers. For example, instead of adding a (-10) to each *x*-value in the ordered pair for a translation, they decided to count 10 spaces left for each vertex of a figure" (ibid, p. 6). She said they also understood where figures would move in terms of quadrants, and they used this knowledge to make sense of the sign of the coordinates. This learning was particularly pleasing to her in that she had sometimes avoided teaching the topic due to her lack of understanding; when she had taught it, she had assumed students really had not learned much. This, however, felt very different for her, and she was grateful that she did not feel as though she had to refer to her notes anymore. In closing her project report, she noted, "During this investigation, I developed a better understanding of the vocabulary (tessellation, transformation, translation, rotation, reflection). I would like to explore more about angle measures of the figures in a tessellation and how that relates to forming shapes that definitely tessellate" (ibid, p. 7).

In our third interview (June, 2007), we talked some about this personal investigation and what she felt she learned.

> **Eden:** *Do you think that in the stuff you did and in watching them, do you see it any differently?*
>
> **Amelia:** *Yeah, like I have a better understanding of how things happen. Like, I didn't understand, I mean even as simple as the reflection, I didn't understand why it was changing. And then now I kind of feel stupid, 'cause when I am talking to kids, I am like, "Think, you know, if we are going over the y, what direction is changing?" They are like, "Oh, that's your*

*x value." **Like**, I mean, I came to those realizations after this, like I kind of feel like the kids sometimes. Like once you start to think about it, some things just make a lot more sense, which I didn't think about before—Grace actually showed me something with the rotations, cause I still feel kind of wishy-washy when it comes to—*

Eden: *So now when you say that you feel wishy-washy on it, is it being able to find the numbers that you feel wishy-washy on, or—?*

Amelia: *Yeah, like, why do the numbers work that way? Why do we flip one and multiply by -1 for **x** or for both, and then multiply by negative 1 for **y** for clockwise—Why does that work?. . . . So like I am trying to "figure it out" a way for them to think about it. So, instead of memorize it.*

Eden: *So you are really satisfied?*

Amelia *As a start, yeah; I am happy as a start. Like I still want to do more, you know maybe make it a little bit easier, and again, do more with rotations to try and understand that little bit better. But I definitely feel better with reflections and translations and whatnot.*

In watching her students work, Amelia was able to develop better understanding of why some of the procedures for determining vertices are what they are. She shared her current questions regarding rotations, of which she does not feel as though she has a strong

understanding, reflecting that she primarily improved her understanding on transformations with which she already was more comfortable, as noted in her introduction and as noted by the course instructor in her feedback. It also comes across clearly in this discussion that her students are never far from her mind when she is trying to make sense of mathematics.

Identity and transformations.

There are several similarities between Amelia's work on this investigation and on her previous work, suggesting that these in fact are more typical of how she negotiated the practice of mathematical inquiry. This investigation had a distinct pedagogical motivation; she wanted to develop her understanding of transformations in order to design a lesson plan that would facilitate her students' learning of them. She did not productively connect habits of mind in her sense-making activity. Certainly one could argue that Amelia engaged in various type of playing while she was creating her figure and tessellating it; however, this activity did not generate further questions, or prompt ideas or thinking. Rather it was used to confirm an externally defined procedure, just as collecting data had confirmed for her that with more trials, experimental probability approaches theoretical probability. Given the lack of playing, the absence of the generative aesthetic response is reasonable. However, aesthetic response is key to her work.

The motivational aesthetic figures in quite prominently. Visual appeal is a primary source of motivation for mathematicians (Sinclair, 2004), and Escher's work certainly held appeal for Amelia. Theoretically, though, this work was about transformations rather than tessellations. That tessellations rather than transformations are

what motivated her can be seen in her one-page summary to the class. Her findings that she conveys all deal with tessellations, none refer directly to transformation, such as "a tessellation does not have any gaps and fills the entire surface" (Class Project Summary). In terms of her lingering questions, the only one that related to transformations is a pedagogical extension of the one that prompted this project: "What are other ways to teach students about what happens with the coordinates of the vertices of a figure compared to the coordinates of the vertices of an image of different transformations?" (ibid). In her class summary, she did not refer to the understandings of transformations that we discussed in the interview, such as what she learned about determining vertices or her lingering questions about rotations. All her findings in this summary related to what was found through external research. This highlights another difference between this work and her work on the macaroni project. In this work, there are no expressions of frustration at not being able to answer questions. This is likely because the external resources she used to learn about transformations and tessellations, held the answers to any questions. In the project she does not engage in any other type of activity geared toward answering questions, though our discussions suggest working with her students may have filled this role. She is developing knowledge in manner that is consistent with her savoir way of knowing, and she is comfortable; she can move at her own pace, for her own purposes, and around content in her sixth-grade curriculum. Amelia's willingness to engage is located more in pedagogical purposes, and aesthetic appeal is what shapes how she chooses to do this. She searches to develop formal knowledge in order to develop a product, not to prompt further thinking. This is certainly goal-oriented behavior facilitated through her savoir way of knowing mathematics. In choosing to use external

resources almost exclusively for a mathematical investigation aimed at pedagogical growth, she reified her mathematical identity as a received knower of mathematics. Belenky, Clinchy, Goldberger and Tarule's (1986) characterize such individuals as, "want[ing] to understand what others think, to know what others know. As they conceive of themselves as capable of receiving ideas but not creating them, they listen to others to find out" (p. 15). This perspective does not conflict with her valuing of savoir modes of knowing. Whether receiving knowledge from her mother, from these external resources, or hoping to receive it from the program, Amelia at no time indicates that she believes she can or even that she should be able to answer her own mathematical questions. She has little sense of agency as a doer of mathematics, other than turning to external resources.

Her Identity as teacher is quite prominent in this investigation beyond its pedagogical motivation. Mathematics, which, through her descriptions, seem to hold a similar interest as the other hobbies in which Amelia participates, appears simply to be the chosen context in which to develop this teacher identity and the expert-novice relationship more broadly; it also suggests why she would use explorations for teaching rather than mathematical purposes. Savoir knowing also fits well within a traditional perspective of expert helping novice acquire knowledge. Interestingly, as I gain more of a sense of the role of Amelia's teacher identity, this adds a new layer of understanding in consideration of why demonstrating her knowledge was so important. Could it be this type of performance that made mathematics meaningful, because she was placed more in the role of explainer rather than learner? Not only did coming with background knowledge help her to feel comfortable, but also it actually served as her sense of purpose

for engaging at all. These both would be consistent with her abandoning teaching math when it was no longer possible to demonstrate her knowledge in the classroom.

Pedagogical considerations seem to be prioritized as well in the lesson. For example, Amelia color-coded the vertices of the angles so students would readily know which angles went together under a transformation. On the one hand, it would be considered purposeful pedagogy, because she removed a possible complicating feature that could slow down her students as they were trying to see the patterns to develop the rules. This move certainly seems to be in line with Amelia's desire to minimize students' frustration, but mathematically speaking, the color-coding focuses attention on the vertices rather than on the transforming process. Thus, while it facilitates procedure-generating, it could potentially inhibit the development of conceptual understanding because students do not have to struggle with certain concepts.

Such a tension between math and pedagogy in carrying out her investigation also reflects previously suggested perspectives that Amelia is more mathematically generous with her students that she is with herself, perhaps because she believes they are mathematicians while she is mathematically anxious. She designed an activity in which her students could play, guess, conjecture, and invent procedures for determining new vertex coordinates under particular transformations. She designed their activity to be generative, while hers was only confirmatory. It is not surprising than that she seemed to really develop understanding when her students were working, not before. Though there was mathematical richness, she also pedagogically constrained the space in which they could work. Their activity was designed for them to invent particular procedures[48], rather

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[48] I am not suggesting this is inappropriate, particularly given the challenges of accountability and curriculum breadth, but it suggests a tension.

than to generate their own ideas about transformations. This expectation on her part stands in contradiction to her own expressed discomfort at feeling as though she was expected to come to particular understandings, a feeling that she felt inhibited her ability to work in open-ended situations (Interview 2, September, 2006).

Despite a possible mathematics/pedagogy tension, it was in working with students that she realized that savoir knowing might be inadequate for pedagogical purposes. Working with students in this project was instrumental as well. In choosing to work in opposition to her typical mode of delivering knowledge first, Amelia gave her students more space than she gave herself; but in making room for their thinking, she seemed to further her own mathematical understandings. In discussing this lesson plan, she noted,

> I thought it worked really well. I tried to do that project before we focused
>
> on any procedural stuff, which was hard for me, 'cause I am still learning
>
> how to do that. . . . And then now I kind of feel stupid, cause when I am
>
> talking to kids, I am like, "Think, you know, if we are going over the y,
>
> what direction is changing?" They are like, "Oh, that's your x value." . . . I
>
> mean I came to those realizations after this [lesson] . . . I kind of feel like
>
> the kids sometimes. (Interview 3, June 2007).

Rather than mathematics knowledge serving pedagogy, it is almost as if her changing pedagogy is vicariously facilitating her own mathematical growth.

This inquiry activity was designed to facilitate Amelia's understanding of a topic in geometry, a purpose that largely was unrealized in the pages of the project itself, as was echoed in many of the course instructor's comments to Amelia in the feedback around this project. That she realized this is only apparent in her closing assertion that she

would like to try to make sense of the angle measures of tessellating polygons. In the interview, more was apparent; however, what she chose to share in her class presentation with her peers, suggests certain perceptions of the understandings that she did develop. I do not feel that she valued as highly learning that did not come from authoritative sources. Had she valued as readily what she learned with her students as through her external research, it is likely that she would have highlighted these in her findings category.

The Unit Circle- Taking on Historical Challenge

As we know from Beth's history, she had significant difficulty with the Unit Circle when she took Algebra II/Trigonometry. She has stated many times that the reason she was able to pass that class was because her teacher allowed her to use a reference unit circle (which she had to make herself) for all homework and tests. When Beth was given the opportunity this past spring to do an individual research project in a topic related to Geometry, she chose to do the Unit Circle, because,

> having [the relationships] on a piece of paper was useful in terms of finishing a high school course, but I want to be able to discover the relationships [of the trig functions] for myself, construct my own understanding and then use that understanding, to attempt higher level mathematics" (Project Summary, p. 1).

The project has a formal summary at the beginning, where she highlighted her ingoing knowledge, as well as her motivation for this project:

> I worked in a center for the highly gifted for three years, and in the last year, had a prodigious math student who needed much more content than I could provide. He was a natural when it came to Algebra and was very interested in learning Calculus. Knowing that he would need the Trigonometry background, I looked in my catalogs and found a workbook and manipulative tool called The Trig Trainer. I gave it to my student and watched him learn the unit circle and its relationships with ease. It made me want to go back and use the workbook myself—it seemed a good way for me to investigate something I've always wanted to know about (Project Summary, p. 1)

She then pointed out that she was investigating math that had already "been established in the mathematics community," but that she wanted to understand it. She has a less formal handwritten summary, includes a few other details, such as "the following pages list my thinking as it develops from one idea to the next—it is not a "straight" path (Unit Circle Project Journal, p. 1). She mentioned making "new leaps of understanding" and that as she answered many questions, she highlighted those in the margins. Finally she describes the current state of her thinking following the project: "I have also come to think of sin (y) and cos (x) as the point where the unit circle intersects the coordinate plane. As 'the point' moves around the circle, its position on the coordinate plane changes—its sin & cosine (y and x respectively) measure change." This idea of motion is something that she did not seem to conceive of as she originally talked about what she "knew" of the unit circle.

On page 3 we begin to see her work and her thinking process. She drew a large

unit circle and labeled everything. She highlights the fact that the triangle that is created

is a right triangle, and used her understanding of this object to draw a connection to the

trigonometric identity that $\cos^2 + \sin^2 = 1$; by changing the 1 to a 1^2, she explicitly

connected this identity to the Pythagorean Theorem $a^2 + b^2 = c^2$, saying that *it applied* to

the identity (Project Work, p. 3). To investigate the relationships, Beth began making

tables keeping track of a variety of data related to the sin and cosine (see Table 5).

Table 5: Sine & cosine relationships in intervals of 1°.

Angle	sin/cos	cos/sin	Y . sine	X cosine	sin+cos
1°	.0174	57.29	.0174	.9998	1.017
2°	.0349	28.64	.0174	.9994	1.034
3°	.0524	19.08	.0174	.9986	1.050
4°	.0699	14.30	.0174	.9976	1.067
5°	.0844	11.43	.0173	.9962	1.083
…					
	tan?				

Even though tangent (sin/cos) and cotangent (cos/sin) are represented, it is clear that she

was just exploring a variety of relationships between the sine and cosine, because at the

bottom of the sin/cos column she wrote: "tan?" and then in the next table she relabeled

the column "tan" (see Table 6). She also removed the cos/sin column and the sin + cos

column from this table. The first table explored angle measures in increments of 1°.

Noting that the changes she was seeing were quite small, she changed her strategy to

examine angles in increments of 5° to see greater change (compare first column of tables

5 and 6). She had written: "very small increments; want to move to larger increments"

next to the transition. At the bottom of these charts, Beth highlighted what she has seen in

Table 6: Sine & cosine relationships in intervals of 5°

Angle	tan	Y sine	X cosine
5°	.087	.08715	.99619
10°		**.17364**	.98480
15°		.25881	.96592
…	…	…	…
75°		.96592	.25881
80°	5.67	.98480	**.17364**
85°		.99619	.08715
90°	error	1	0

the numbers: "I am noticing pairs of opposites—the sine of 10° is the cosine of 80°; the

sine of 25° is the cosine of 65°. . . they (the pairs of angle measures) come to a meeting

place at 45° where tan = 1" (Project Journal, p. 4).

Beth then highlighted that in her table at 90°, where sine = 1 and cosine = 0, the

"slope" is undefined. She then wrote:

*interesting: tan of 90° gives an error in the calculator

tan of 89 = 57.28996

tan of 89.99 = 5729.57789

tan of 89.99999 = 5729577.951 (Project Journal, p. 5)

After demonstrating this tangent explosion, she then worked on a connection she felt was

relevant, highlighting that if cosine is akin to run and that sine is akin to rise, perhaps the

tangent is the slope. She then altered her notation and referred to slope in terms of $\Delta y/\Delta x$

(change in y over change in x) and wondered if the tangent actually was the change in

sine and the change in cosine. She seemed to be trying to work through the subtleties of

the vocabulary and putting what she knows together with what she is learning.

Doing what had been so helpful in her previous work and her own perspective of herself as a visual learner, Beth opted to graph her sine and cosine values. Beth also goes on an unusual exploration that results from an unclear error. Putting the data from her tables into the calculator, she plotted the points. It is not clear if she was expecting anything in particular to result, but she seemed surprised when she got a "parabola!!!"[49] but it sparked curiosity since it connected to previous work. She computed a quadratic regression to get an equation, and then wondered what the coefficients represented. Though this graph was incorrect, it prompted an interesting idea. Beth knew that parabolas have a unique definition based on distance (see parabola investigation). She thought this related well to her pairs of opposite angle measures: "The angle measures with reciprocal sin & cos are equal distances from 45°, and that 45° (interior angle) is the halfway point between 0°and 90°" (Project Journal, p. 6). She then tried to figure out how to write this as an absolute value function[50] such as $|x - \mathbf{45°}| \leq \mathbf{30°}$?? (ibid, p. 7). From this observation, she decided to begin looking for other patterns in the pairs of opposite angles for the tangent function. When she did not see an immediate pattern since the tangent function grows in an unbounded manner, she tried various other possibilities, such as adding the tangents from the pairs of opposites (e.g. tan 65° + tan 25° = $\boldsymbol{x}$). Again no discernable relationship, so she then divided them… again no discernable relationship.

Abandoning this search, the final two pages focused on asking and resolving some outstanding questions. She was disappointed not to be able to use a pattern in her sin cosine data to be able to develop a method for finding the sine and the cosine without

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[49] There was obviously some type of error. . . In plotting this x and y values the result should in fact be the unit circle.

[50] One common conception of absolute value is distance; direction does not matter.

a calculator aside from using the Pythagorean Theorem, and wondered if this was how they were originally calculated. She then asked again why the slope at 90° is undefined, and wondered if it had to do with the nature of the tangent. Trying to work through an argument, she described the tangent in reference to a picture (see Figure 13): "The tangent measurement of the unit circle is the length of the segment made by the rays (radii) in the unit circle forming an interior angle" (Project Journal, pp. 8-9). While she was unable to fully reconcile the slope tangent connection, using this alternative visual process approach, she was able to justify for herself why the tangent of 90° should be undefined. She included the following pictures:

(x,y)

length of orange segment is equivalent to value of tangent function [and value of slope of ray through (x,y)]
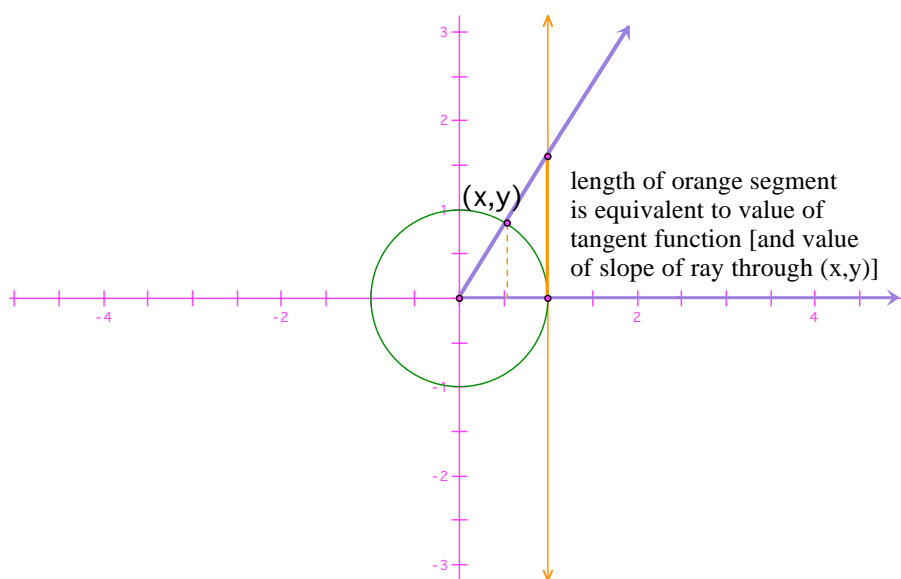
Figure 13: Tangent representation as segment from the *x*-axis to the ray

She wrote, "As the interior angle gets bigger, the tangent function measurement gets larger. You can imagine that at 90° there would be no intersection of the ray that makes the 90° angle [purple ray] and the tangent line passes through 0° [orange line] because they are parallel. I wonder how this works on a sphere?" (p. 9; see Figure 14). It is quite reflective of Beth that the last thing written in her project journal is a question to pursue.

the ray through (x,y) never intersects the orange line to determine a value for the tangent of the angle

(x,y)

Figure 14: Undefined tangent represented by no intersection of ray and line

Identity and the unit circle.

To assert that Beth's activity in this exploration is not an isolated incident, but rather a reflection of her mode of engaging in mathematical sense-making,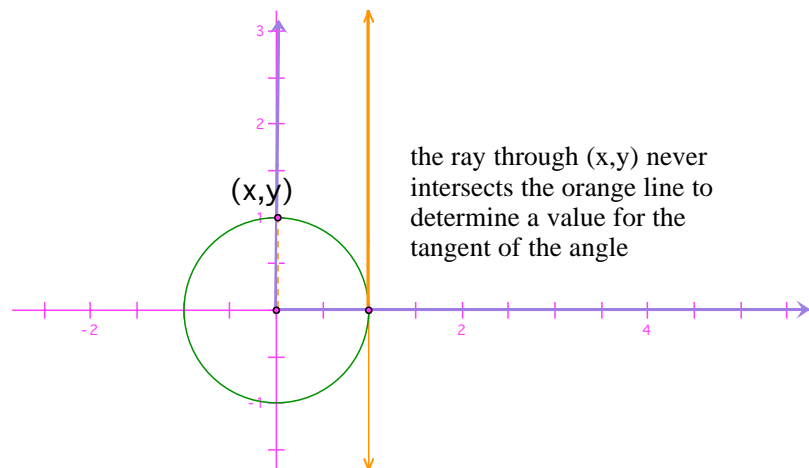 I will first draw connections between this activity and the Pascal's triangle one, before moving on to more analytic features. As in the previous activity, habits of mind were frequently used in an interconnected way facilitating the making of connections, suggesting the role of intuition. Through her pattern sniffing, her sense of how these relationships behaved enabled her to begin to visualize the unit circle in terms of motion and process. She used this visualizing process later as a way to invent and describe a mathematical explanation for why the tangent is undefined at 90°. As playing and intuition are two of the primary modes of invoking the generative aesthetic response, its presence is natural, such as in her shift to conceiving of the angle growth process as similar to absolute value; she even closed the project with a newly generated question. That perhaps her intuition regarding the unit circle is not as strong as in the Pascal case may be hypothesized in that, while she

was surprised [51]to see the parabola graph, she had not developed a sense that this graph was problematic; perhaps all her other parabola work with triangles suggested that this could be a reasonable connection. Finally, just as Beth's previous activity originated in attributes from the what-if-not process (Brown & Walter, 2005), this activity began with a presentation of formal knowledge, from which playing, intuition, and aesthetic facilitated sense-making. This then solidified more former knowledge, such as the tangent in fact represents a relationship between the sine and the cosine, and that angles in the first quadrant equidistant from $45°$[52], had opposite sine and cosine values. Thus knowing and knowledge facilitated each other.

Beth's identity as a learner with her goal of understanding is strongly reflected in this work. At the outset we see her familiar interest in taking on something challenging. She purposefully chose to explore a topic with which she had had significant trouble learning in the past. She did not use her holes to justify her difficulties, but rather uses them as a guide to develop knowledge. This desire for challenge echoes her early dislike of her *easy* consumer mathematics course, as well as her recent desire to be team leader in order to help teachers to modify their teaching of mathematics by helping them to see mathematics as something different, though she knew this would be slow and that she would encounter resistance.

To take on this challenge, however, requires a sense of confidence and understanding of frustration as part of the process. Beth has discussed her confidence that if something does not work out one way, that there will be another way that will make sense. This came out explicitly in this work regarding making sense of why the tangent of

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[51]A common motivational aesthetic response (Sinclair, 2004)

[52] These are complementary angles.

90° is undefined. She worked with it for awhile by trying to connect with slope (a connection that she seemed to still be working on) and rise over run; there was no indication in her journal that she felt she had come to understanding, nor was there any sense of doubt or lack of confidence that she could. She did temporarily stop working on it, but at the end of her project, she returned to this lingering question. Using another approach she was able to develop an explanation for the result. Returning to something you were unable to figure out previously might feel risky, but with confidence that success will eventually be rewarded manages this risk. It is also important to managing mathematics anxiety (Tobias, 1993).

This work further reflects Beth's goal of sense-making. While Beth is goal-oriented in the sense of developing understanding, she is not oriented toward particular results, though she did find important results (as previously suggesting knowing and knowledge interact in supportive ways). When I spoke with her two months after completing this investigation, she was not able to remember all the precise definitions, but she was able to recall the understandings she had developed, and felt that if she looked up the definitions, she would not feel lost in terms of meaning.

> **Beth:** *As soon as I saw the pictures again, if I looked at them [I could make sense of the definitions], like even tonight—Emma was asking me, that her kids wanted to know about the angles when you create the slope, and I said, "Oh, I know it's a sine/cosine relationship, and this is the tangent," but then when she was asking me, I'm like, "I cant remember which one is which."*

Eden: *Aside from remembering those types of specifics, how about relationships? Do you feel—*

Beth: *Oh yeah. I remember those. I remember that when you have the **y** axis and the **x** axis, whatever they are. . . and when the tangent is at—ok, 45º, when you cut it in half, 45º, there was a relationship on either side of that 45º where—okay, so we're at 60º, cosine and sine was the opposite for [pause] 30º. Whatever the absolute value distance away from that 45º line was, the things were opposite. So you just flip them. So that's something.*

Eden: *Okay*

Beth: *…knowing now why a vertical line has a slope of [undefined], that makes sense. Because as the hypotenuse moves up towards the y axis. The numbers get like…and I did this out on paper, the numbers get ridiculously big very quickly. So you cant actually calculate it anymore* (Interview 3, April, 2007).

Her ways of engaging and the connections she made facilitated Beth's retention of understandings, if not the specifics of her work; this lack of specifics bothered her little. Thus for Beth, learning and doing math have a synonymous feel. If learning and doing have a similar feel, could teaching as well? If students have to learn by doing, teaching would not be useful unless somehow teaching were related to doing. Perhaps this is one reason Beth feels that in doing the mathematics and in watching others support this work, she is developing her pedagogy, or her pedagogical reasoning. If I think about my own

work in FM, a course I took to have an opportunity to engage with mathematics and develop my own understandings, yet as a teacher I recognized the value of what I was engaged in, and this recognition necessitated its sharing, seemingly as Beth felt the need to become a team leader to share these approaches as well. Perhaps in regularly separating learning and teaching we are creating a false dichotomy where one does not really exist; perhaps it is in part in separating these that the savoir-connaître dichotomy arises.

Reflecting how she was able to reconnect with the interest and beauty in mathematics, the source of this investigation actually arose from watching a student explore and make sense of these relationships. The students re-awoke her historical curiosity. However, rather than simply accept vicarious understanding [and given her goal of pursuing more advanced mathematics], taking on this challenge was important both to her trajectory and to her sense of herself as one who constructs knowledge. Belenky et al (1986) describe such knowers as ones who recognize

> that the knowledge acquired depends on the whole frame of reference of
>
> the knower as well as on the context in which the evens to be known have
>
> occurred. As knowers, they draw out and listen to all of their own voices-
>
> to the inner voice of intuition as well as to the voice of reason (p. 20).

Others stimulate and support Beth, but she needs to make sense of things herself. Echoing her assertion in our third interview that her ideal learning environment would have people of all different types or levels of mathematical understanding, she is happy to learn from and engage with anyone if they are willing to honor her need to pursue things in her own

way. Yet she also enjoys watching and connecting to their approaches. Doing this her own way does not involve fully disconnecting from others.

Discussion, Directions for Future Research and Lessons Learned

In this discussion section, I will draw on the previous data and analyses to answer the research questions that guided this exploration and discuss potential implications for mathematics teacher education. As I respond to each question, I will highlight themes that emerged as important to consider in shaping mathematics teacher education. I think it is important to stress that these responses and themes are highly interrelated; however, I have chosen to discuss them separately for both convenience and clarity. It would be a mistake, though, to assume that because I discuss the frustration/confidence dichotomy in response to the first question, that it is not also relevant to answering the second or fourth question. I will then draw on these discussions to highlight areas for future research.

Following these responses and a discussion of the implications for future research, I will reflect on what I have learned regarding the importance of inquiry and about structuring such meaningful experiences in mathematics. Much of what I have found relevant to inquiry in this exploration relates to the structure of the experiences that were created. The structure we worked within significantly constrained teachers' experiences with inquiry. Consequently, I can only suggest, that inquiry in itself is unlikely to make a profound difference in how a teacher perceives mathematics. These types of considerations will be important for both future research and attempts to design such experiences for teachers or mathematics students more broadly.

How Do Identities Mediate How Teachers Engage

In and Value Their Program Content Experiences?

There are two main perspectives of identity that I have so far considered in relation to ways of knowing mathematics: *learning trajectories*—"we define who we are by where we have been and where we are going"—and *negotiated experience*—"we define who we are by the ways we experience our selves through participation as well as by the ways we and others reify our selves" (Wenger, p. 149, 1998). While these issues are discussed separately in the literature, the division is false. These are interrelated. Learning trajectories allow us to connect the past and future to the present ebb and flow of participation in negotiated experience. Wenger explained, "Learning events and forms of participation are thus defined by the current engagement they afford, as well as by their location on a trajectory" (p. 155, 1998). With Amelia and Beth as exemplars, these processes that constitute identity formation strongly influence how content is experienced and valued, reflecting the interactive constitution of identity and practice (Wenger, 1998). More specifically, I would further hypothesize that feelings of confidence and frustration in the doing of mathematics, which impact how content is experienced, are further reflections of teachers' identities in the school mathematics practice.

Regarding Beth's learning trajectory, two salient factors emerged in how she experienced content, both of which relate to goals and motivation: first, she wanted to develop understanding particularly through questioning mathematics; and second, her primary objective for coming to the program was her own interest. There was no external goal, beyond her own learning, from which to judge the value of her experiences.

Consequently, the primary gauge of value was whether Beth felt that she was able to learn.

The integrated classes, which focused on the depth of understanding in middle school mathematics, offered Beth an opportunity to return to mathematics she had previously encountered (and sometimes had not enjoyed as in the case of algebra) and probe the meaning within. This was significant in helping her transition from hating algebra, to tolerating it, to choosing to teach it in her school. Questioning mathematics as a method of learning was critical (reflecting her historical emphasis on *why* and the resulting disappointments); thus inquiry also resonated quite strongly with her, particularly when the activity addressed questions of personal value and interest. The one inquiry opportunity she did not value was of constrained focus. Consequently, she barely engaged, which she could choose to do as there was no grade received for these investigations.

These integrated and inquiry experiences helped Beth begin to see mathematics as a connected, sense-making activity. Beth also valued her more formal content experiences as an important tool in developing understanding, though these brought out her historical anxiety, which she managed through positive self-talk (Tobias, 1993), reflecting her more recent confidence. This anxiety was elicited primarily in activities where knowledge acquisition was prioritized and she could not pursue her own routes (e.g. examinations). What seems to have been central in allowing content experiences to be productive was the instructor's valuing of Beth's way of thinking. In particular, starting with concepts eased anxiety and facilitated Beth's development of understanding.

The other primary facilitator was Beth's own confidence that she would eventually figure things out one way or another.

For Amelia, there appear to be three interrelated features that influenced how she engaged in and valued her experiences: first, her historical and ongoing appreciation of the emphasis on form and answer in mathematics; second, her related desire to create a comfortable space in which to work; and third, her goal of developing her pedagogy (both her ability to answer her students' *why*[53] questions and to incorporate less traditional methods into her repertoire). As such, Amelia tended to value experiences in terms of whether they helped her to achieve her teaching goals, whether they were structured to engage preferred ways of coming to know, and whether they allowed her to stay in her learning "comfort zone." The integrated courses were frustrating because she felt she was not getting the pedagogy she thought she should. In contrast, the content experiences were frustrating because she sensed competition while also feeling unable to perform comparably to her peers, destabilizing her mathematical wellbeing.

Amelia's experience in integrated geometry, which seemed to stand in contrast to both her pedagogical goals and her desire for a safe working space, offers a plausible explanation, rooted in the interaction of wanting to know why and her own lack of mathematical agency, for why closure was essential. Many of the open-ended activities in integrated geometry created situations for this group of teachers to explore why things worked as they did. For example, using unifix cubes, each teacher physically *dilated* three-dimensional figures along one, two, and three dimensions in order to make sense of why these different types of growth caused the volume to increase according to linear,

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[53] I do not mean to imply that this is not a mathematical issue; I am stressing that the source of this mathematical question stems from pedagogy.

quadratic, and cubic relationships[54] respectively. While there were class discussions about this growth, the structure was flexible and there was no formal summing up in which the instructor explicitly told the teachers what they had learned. There were teacher presentations and discussions around much of the work, but because Amelia has little agency and relies on external direction, she has not learned independently what to value in activities. Her discomfort with the lack of formal closure seems to relate to her difficulty gauging what mathematics she learned. I do not believe she was unable to learn anything without closure, as her discussion of the matter might suggest.

As regards inquiry, I originally thought it strange that she wanted more closure with a well-defined answer. Recall her comments about inquiry when we spoke of the discrimination investigation. She said:

> I kind of feel like it has to kind of end. . . . So that's why with the investigations, sometimes it will be hard for me . . . because in my head **I would like there to be some sort of ending**. . . . And so, I liked the discussions, but I think I am more satisfied when there is an actual answer. . . . But I still like to be right , **and think that there is an answer,** I don't know, so—I guess I wasn't—I don't like it to be too open-ended… (Interview 2, emphasis added).

Her speech indicates that these endings and answers reside externally to her activity. She has not recognized that she, herself, is able to decide where and how to end inquiry, or that she could say, "I have made sense of this." This suggests why, in her individual inquiry, she opted to use external resources, rather than mathematical activity. With these

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[54] Working on this understanding was particular requested by the district in which these teachers work.

resources, Amelia had direct access to knowing what she "should" learn. The inquiry experiences with pedagogical bases (her macaroni and her transformations investigations) were more highly valued, as was the far more structured, directed activity of gender discrimination. Those inquiry experiences that had neither pedagogical relevance nor external direction (parabola and Zeno's paradox) were more frustrating for her as she was unsure of how to proceed; as with Integrated Geometry, there was no formal closure and minimal structure. Helping to explain why she was unable to make use of intuition in inquiry, Amelia has not been able to surrender to the undergoing of grace in terms of coming to an understanding.

With regards to negotiated experience, the cycles of participation also mediated and responded to content opportunities, as did reified perspectives of self. Beth, who has affirmed her own confidence and ability to work through problems, moved between positions of peripheral and full participation. Because she perceives the practice of doing mathematics as related to her goals of understanding, she actively engaged in a negotiated journey of understanding with other community members. When the practice seemed to deviate either from her interests or from this sense of journey towards understanding, she assumed a more peripheral role temporarily by pursuing her own work, as occurred explicitly in the parabola investigation when she decided not to pursue the negotiated proof goal. This shift occurred implicitly as well in content course as was indicated by her assertion that she felt she had permission to work in her own way, though the material was the same. These timely shifts to peripheral participation seemed critical to Beth's ongoing involvement in the practice; it positioned her to fully re-join the community. By taking on a more peripheral role, she prevented the practice from alienating her and

pushing her to a more marginal position. In order for her to be able to take this step however, to work in her own way, it is likely that knowing how to make use of the generative aesthetic response is essential.

As a member in the same community of practice, Amelia's participation changed as well, though perhaps less purposefully, given that her shifts in participation were less voluntary. Though Amelia's descriptions of what she hoped for in a program seemed to indicate that she too would have preferred to perceive content experiences as a journey, she seemed to only know how to work for the destination—the acquisition of knowledge—rather than experiencing knowing. In each content-related course, Amelia moved over time from a peripheral to a marginal role from which her peers, the instructor, and formal knowledge became an external bar by which she could measure her learning and how she should engage. When she could not meet these external bars, feelings of frustration, lack of confidence, etc., emerged to prevent her full engagement. These feelings at times even caused her to shut down. She hoped to participate fully in an externally defined practice; but when she was not able to meet the full demands of this externally defined practice, her lack of agency promoted marginal participation. The difference in how Beth and Amelia reacted when the practice was moving in a direction they neither appreciated nor felt confident with highlights their own senses of agency in negotiating the practice; moreover, these actions then mediated how they were able to value their experiences. While Beth continued to value the learning experiences, Amelia had an increasingly difficult time doing so or feeling positively about either her learning or herself. This reified her sense of self as not a math-savvy person.

This brings me to an important theme that has emerged regarding the presence of frustration and confidence in mathematical activity. Both of these teachers experienced frustration. In truth, any mathematician has experienced significant frustration with his/her own work (Burton, 2004). It is certainly common among students of mathematics. How Beth and Amelia experienced this frustration was mediated by their trajectories and their participation as members of this practice in predictable ways: "some can see, and in turn experience, only the deception and disappointment [of frustration], while others are able to grasp the necessity or inevitability of it—even sense the potential joy in it—as part of the process of coming to know" (Handa, 2003, p. 26). As Beth was focused on a journey of knowing, her actions and behavior were thus directed, rather than toward achieving specific goals. Consequently episodes of frustration were perceived not just as a part of the process, but as an important part of the process, evidenced in her desire to look for that which challenged her mathematically. Burton (2004), Sinclair (2004) and Handa (2006) have all highlighted how, in the discipline, frustration and other feelings of being stuck, are accepted as part of the mathematical process, at times even revered. It is in part through frustration that we recognize and come to appreciate what is learned; the reward of coming through the frustration is instrumental to the willingness to experience it (Handa, 2006). Hence, frustration is integral to coming to know. What is more, when accepted as part of the learning process, frustration seems not to prompt large drops in confidence; it did not with Beth. In fact, it is confidence in being able to work with and through the frustration that allows frustration to be productive. Because productive frustration requires confidence, these exist together as a duality in the doing of mathematics, facilitating staying with the mathematics until one arrives at understanding.

For Amelia, however, confidence and frustration present a dichotomy. As she is focused on the destination—knowledge rather than knowing—non-goal directed activities were experienced not just as uncomfortable, but also as wasted (Handa, 2003). This conflicts with images of doing mathematics where non-goal directed behavior is essential (Sinclair, 2004; Cuoco et al, 1996). Knowing how to value the end, but unsure of how to value the journey, Amelia experienced frustration as an obstacle. The thinking seems to follow the lines of, *if I could do it, I would not hit snags along the way*. It served as a reminder that the goal had not been achieved. It thus shook her confidence in her ability to achieve the goal, as is common with math anxiety (Tobias, 1993). This suggests why Amelia works to avoid frustration. This is why she wants things to move particularly slowly, why she wants to have people just like herself in her classes, and why she wants to have all the background knowledge ahead of time. As long as she stays within her comfort zone, her frustration-avoiding tools are in place, and she is able to be confident. Unfortunately, avoiding frustration by staying in one's comfort zone is an obstacle in and of itself: "those students who do not return to the scene of their struggle are likely those who do not find success with mathematics" (Handa, 2003, p. 27). This suggests that it is important in education to identify the source and role of one's frustration in the doing of mathematics. In doing so, educators are better positioned to help students work productively with this emotion.

How Do Personal Ways of Knowing Mathematics Mediate How Teachers Engage In and/or Value Their Program Content Experiences?

This frustration/confidence interplay as a result of identity also feels strongly related to how these teachers come to know mathematics. In his later work, Handa (2006)

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argued that frustration is in fact a cognitive emotion (Scheffler, 1991) as it is integral to the process of coming to know—frustration can be an experience of grace. Confidence, also an affective response, facilitates the will to stick with challenging work. Thus the interplay of confidence and frustration in the doing of mathematics speaks to the productivity of the ongoing interaction of will and grace. When these emotions exist in duality, they likely aid the process of coming to know reflective of the joining of will and grace; however, when they exist in dichotomy, frustration inhibits confidence and thus stands in opposition to any joining, helping to keep ways of knowing mathematics traditionally entrenched.

Using ways of knowing to understand how content is experienced adds an additional layer of complexity. Amelia's favoring of savoir modes of knowing fits well with how she views mathematics; it is probably a large part of why she views mathematics in this manner, given that practice not only is shaped by identity but shapes identity itself (Wenger, 1998). If this way of coming to know mathematics is prioritized, then the emphasis must surely be on results. In terms of communities of practice, this reflects the traditional prioritizing of reification over participation in education (ibid). As integrated and inquiry experiences did not readily engage her preferred ways of knowing, it was difficult for Amelia to recognize any knowledge growth she experienced therein. The emphasis on reification further explicates the importance of closure. This instructor-led activity would effectively put what was to be learned from the journey into a form coherent with and accessible to Amelia's preferred way of knowing mathematics.

Amelia's sense of frustration was different in the content courses in part because of this coherence in the preferring of savoir modes of knowing and the perceiving of

math as result-oriented. In those courses, she did not yearn for closure, for structure, or for external direction; the focus on formal mathematics[55] engaged her ways of coming to know. The factors in the content courses, which she felt impeded her learning, were the absence of those external tools that put her in her learning comfort zone. Amelia has a problem herein though. For pedagogical reasons, she has realized that her savoir mode of knowing mathematics has been inadequate to her teaching; however, the tools she values have not facilitated her recognition of, or the making use of, other ways of knowing (e.g. drawing on intuition, seeing frustration as important, and recognizing when she herself has come to understanding).

Beth's way of knowing in mathematics also fits well with what she has come to understand mathematics is about: sense-making. Since beginning to teach mathematics, she has over time been striving to understand mathematics holistically. Though the journey is particularly important to her, it would be a mistake to say that she neither appreciates *the end* nor makes use of a savoir modes of knowing. Rather, her early difficulties seemed to stem, not from the focus on knowing in the savoir sense, but that this did not develop in conjunction with connaître ways of knowing. Through this program, while focused heavily on developing connected understanding, savoir modes of knowing progressed concurrently.

Beth enjoyed her content courses as well because the instructor helped develop the concept before the formality. There was no indication that the specific knowledge was all that mattered. Consequently, Beth was free to try to make sense of things in her own way. The specific math and symbols still caused anxiety, but with more open space in

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[55] This is not to suggest this was the only focus, but a focus that was critical for Amelia.

which to move, she was able to work with them. In fact, while memorization had previously been difficult, with a conceptual focus, memorization was not necessary.

To help demonstrate how connaître ways knowing have encouraged Beth's savoir knowing, I want to draw attention to two previously discussed experiences. First, before integrated geometry, Beth had encountered the unit circle as a mathematical product. Yet she had trouble retaining it. When she took on the challenge[56] of trying to make sense of the unit circle, she actively explored various ways of relating the sine and the cosine. It was in this process that she came to recognize the tangent as the ratio of the sine to the cosine. Certainly she "knew" this fact before, but she had no real sense of it, nor did she retain it. However, through her exploration, this fact became relevant and meaningful; she has since retained the knowledge that tangent is a ratio of sine and cosine. Coming to know through connaître modes facilitated not only understanding how (and why) the tangent behaves, but facilitated her coming to know what the tangent represents definitionally. This was, however, not the end of the process of savoir knowing. Rather, this process then actively re-ignited connaître processes as well. This definitional understanding served as a jumping off point for explorations geared toward developing intimacy around the unit circle, in particular, why the tangent was undefined at 90 degrees. So while in some senses a savoir way of knowing represents coming to a destination, when developed in conjunction with connaître knowing, it also shapes the direction of ongoing processes, just as deciding which road of a crossroads to take would.

Second, engaging in the *what-if-not* process (Brown and Walter, 2005) as Beth did with Pascal's Triangle, is another process of activating one's savoir modes of

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[56] The acts of looking for and taking on mathematical challenges are distinguishing features of one who is engaged in connaître ways of knowing.

knowing. It requires drawing on attributes of a topic in order to prompt questions that are pursued through developing intimacy with the situation. As Beth continued in the program, she did not just allow for the joining of will and grace; she experienced an intertwining of savoir and connaître modes of knowing as well. Handa (2006) suggested that these could in fact support or inhibit each other. Through these two teachers, Amelia and Beth, two of these cases have been exhibited: savoir knowing inhibiting connaître processes, and savoir and connaître knowing supporting each other.

Regarding the mutual support these modes of knowing can provide, I originally thought in terms of directionality: one process bolstering the other. However, I am coming to believe that support between the two must be bi-directional. This is true not just for Beth, but also arguably for all mathematicians. Research mathematicians must strongly value the savoir way of knowing mathematics in order to publish and make an impact; however, as Burton (2004) demonstrated, and Cuoco et al (1996) and Sinclair (2004) have argued, it is in the playing and the generating of ideas and the establishing of intimacy that mathematicians develop the understandings that lead to these formal products of activity. Moreover, as knowledge grows, new reifications serve as objects around which future participation takes place (Wenger, 1998). This draws to mind two other descriptions of mathematical knowledge development. Paul Ernest (1991), who theorized a social constructivist vision of mathematics, describes the interplay of objective and subjective knowledge, and the process by which subjective becomes objective and then objective becomes subjective. Pickering (1995) described mathematical activity and knowing in terms of a *dance of agency*. While the mathematician works, s/he wears alternate hats representing partners in a dance. S/he

conducts her/his own work through personal agency, but in order to communicate what s/he has learned, s/he espouses disciplinary agency, which is shared roughly by all members of the mathematics community. Once the knowledge has been communicated, both the original mathematician, as well as any to whom s/he has communicated, can then invoke personal agency to work with this new piece of knowledge.

For me, the question that remains is whether connaître ways of knowing can inhibit savoir modes. I find this difficult to envision, because in most cases one cannot help but learn things if engaging in a connaître way of knowing. However, drawing on the dualities above, I can conceive of ways this potentially could happen. If I think of the connaître mode as akin to subjective knowing, or as a personal agency, it might be so highly valued that there is no move to connect to objective knowledge or to disciplinary agency. In these circumstances I feel savoir knowing could be impeded. However, this makes me question if it is only disciplinary reifications that count in savoir modes of knowing. This does not feel quite right either. This is one question that for me remains fully open and needs exploring.

As I try to understand ways of coming to know and how they influence engagement, another theme has surfaced as important to productive mathematical relationships: using the generative aesthetic response. As previously described, aesthetic responses indicate an experience of grace in coming to know mathematics, because it represents a way in which the mathematics is affecting the one engaged with it (Handa, 2006). Sinclair (2004) echoed Poincaré in identifying this response as crucial to the mathematician being able to engage creatively and productively in the discipline. Based on what I have learned, this feels not only plausible, but also likely; we have seen that

Beth employs the generative aesthetic response often in her own work, while Amelia rarely draws on it. Somehow though this response feels different from the aesthetic experience that characterizes grace, or from the other two aesthetic responses described by Sinclair. Drawing on the generative aesthetic response seems to actually signify a joining of will and grace in much the same way as looking for and accepting a challenge, or being internally motivated to engage in doing mathematics (Handa, 2006). Given that the generative aesthetic response requires a process of letting go in order to be affected by mathematics (Sinclair, 2004), it thus also requires a sense of agency and courage to surrender to the experience. Below, as I consider the three ways in which mathematicians invoke the generative aesthetic response (ibid), all of which are stated in the active, I hope to highlight the interplay of will and grace.

Playing is an active process, but it is not a goal-oriented process. I posit that perhaps it is, in fact, affect-oriented. Through this process, a mathematician will begin to get a sense of how things work—intuition can develop, motivation can develop, understanding can develop. However, if mathematicians only wanted to engage in goal-directed activity, playing would be risky, if not a waste of time, because it might be unproductive. Playing requires not just taking risks, but also the belief that the activity in itself will be productive regardless of the specific outcome (Sinclair, 2004). Productive playing thus requires both act and feeling, a willingness to experience grace. This explains why, when I drew on habits of mind to examine inquiry, it felt powerful, yet inadequate. It did not provide access to the affective.

Just as playing seems to be an affect-generating activity, listening to intuition can be considered activity-generating affect. By listening to intuition a mathematician might

draw a particular connection or develop a new idea, which would direct the course of future activities. To capitalize on intuition again requires both the action of risk-taking and the feeling of trust. The presence of intuition suggests that the person has been affected by the mathematics on a cognitive level. But listening to intuition also requires a risk: after all intuition can be faulty. Thus again there must be a trust in the productivity of the activity, regardless of the result. There is action based on that affect. Listening to intuition then represents a willingness to surrender to grace and the courage to act on it.

Finally, establishing intimacy invokes the generative aesthetic response in the work of mathematicians (Sinclair, 2004). The very use of the word intimacy elicits an image not of a goal, but of relationship. Just as someone may know a person well enough to have a feeling about what makes him or her tick, one who does mathematics also can come to know a situation, or a problem, or a system well enough to have a sense of how it works, long before there is any way to communicate this understanding to others. A productive relationship, of course, involves both taking risks and letting go in order to allow oneself to come to understand the other. Intimacy then, too, seems to require the interplay of will and grace. I would argue that the reason the generative aesthetic response is perceived as crucial to productive mathematical thought is that it signifies that one is proceeding through a connaître way of knowing mathematics, which then could be a tool to both facilitate and draw on the savoir mode of knowing. This suggests not only why Beth seems to make use of this aesthetic, but also why she is able to make productive use of the different types of content experiences. Amelia, however, avoids taking risks and letting go, and thus can neither make adequate use of this aesthetic, nor engage fully in the connaître way of knowing mathematics for which she seems to yearn.

Do Relationships with Mathematics Evolve

Over a Variety of Content Experiences?

If learning is *becoming* (Handa, 2006; Wenger, 1998*)*, and then learning by its very nature changes who we are by how it affects us, it is perhaps impossible for shifts in relationships not to occur over the course of the content experiences. In particular, this question seems to be more about shifts that are apparent with regards to ways of knowing and identity as teacher and learner of mathematics (which includes perspectives of mathematics and one's role in the doing of math). It is, however, important to reiterate that a program does not independently effect change, any more than teaching independently effects learning. As discussed in the previous two questions, it is in large part how these experiences are mediated through ways of knowing and identities that would effect change. In particular, experiences must be valued or be otherwise powerful to facilitate change. Looking to Amelia and Beth, while their relationships evolved over the course of their program experiences, this program did not initiate these processes. More importantly, the changes may have as likely been facilitated by what was occurring in these teachers' own classrooms as well as by the program. After all, in both cases, the initiation of change was rooted in interaction with their students. They came to the program to learn. They came to change.

Prior to the start of the program, Beth's identity as a mathematics teacher and learner was growing in new directions from working with her students and her county math experiences; these developed her confidence and motivation, precipitating a renewed yearning for alternative ways of engaging in mathematics that contrasted with her old experiences. Beth has stated that this program, and inquiry in particular, helped

her to alter her perspectives of mathematics to the point that mathematics is not only no longer a disconnected, inaccessible discipline, but that it is similar to other disciplines that involve questioning; she even has likened it to a form of aesthetic expression. This program provided the space to experience these early joinings of will and grace, and of connaître and savoir ways of knowing. Her growing agency and sense of self-efficacy help position her to continue to learn from practice regardless of future curriculum changes. This is effectively what program designers hoped to facilitate.

For Amelia, the process might not feel as dramatic, but she has been working to alter her pedagogy and shift her current role in the school mathematics practice. Her recent *transformations* lesson plan—where for the first time she initiated working with concepts prior to developing her students' savoir knowledge—is a dramatic progressive pedagogical change. There did not appear to be concurrent change in her approaches to or experiences of content, though her content knowledge grew. She certainly felt her understanding improved dramatically in the misconceptions course. Keep in mind, she was not looking to change her perspectives of mathematics; she was hoping to add a layer of knowledge to develop her pedagogy. However, it seems as though her relationship with mathematics has inhibited her achievement of this goal by obstructing her recognition of learning, particularly in terms of experiencing grace. Since both teachers seem to have changed along the lines of what they were looking for in this program, one should question whether providing varied content experiences is adequate to assist growth that was not looked for or desired. While there is no indication Beth came purposefully looking for new perspectives of mathematics, since her other perspectives

were incongruous with her identity as a learner and teacher, when another more relevant option presented itself, it resonated with her.

I find it intriguing that shifts occurred where they were personally meaningful or otherwise looked for. Program experiences reinforced both active trajectories. This supports the image of teachers as architects of their own learning, even if only subconsciously. This seems reasonable given that, in areas where changes were not looked for or the teacher was not open to them, evolution would be stifled because experiences would be mediated by previous experience. Of course, this presents a challenge. In teacher education, we do not look to assist those who are ready and willing to progress. Is there a way to support teachers shifting their relationships if they are not actively desirous of change?

How do trajectories of relationships with mathematics influence, and how are they influenced by, the present relationship with mathematics as the interaction of ongoing identity formation and personal ways of coming to know in the context of these program experiences?

What the changes above seem to suggest is that a program that is geared toward facilitating growth in relationships may have difficulty facilitating what is not looked for or welcome. What can this help us to understand about shifts in relationships? The reason that I believe this question is so important is that it relates to how teacher change is understood. Thinking about this question and the teachers I have worked with has prompted me to re-conceptualize[57] my notions of change. In much literature, teacher change is considered in light of some destination, somewhere we would like them to be

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[57] A process which is ongoing

(e.g. Mrs. Oublier). However, given that it was a destination focus in learning mathematics that inhibited Amelia's growth in understanding, isn't it possible this destination perspective inhibits our understandings of shifts in relationships and other change? What if we instead assume a journey frame of reference? Looking at location along a trajectory relative to where once we were makes greater sense[58]. Using incoming relationships as a reference point from which to observe the process of becoming can dramatically alter interpretation of change. This is akin to looking at a student's mathematical growth in terms of how far they have come, rather than where they have not yet been. It also counters the common (though receding) perspective in teacher education research of the deficient teacher (Carter, 1993). The two teachers with whom I worked closely have influenced this process of change in my own perspective, because I feel there would likely be a tendency for readers of this research to interpret Beth's and Amelia's changes as fundamental and superficial respectively (Cuban, 1993). This feels inherently problematic, as it at least devalues the work in which Amelia is engaged. However, if we examine these shifts using these teachers' incoming relationships as a reference point, the interpretation changes quite dramatically, at least for one of the teachers. Recalling that while Amelia and Beth have been in the same community of practice, their interpretations of this practice have been mediated by their relationships to mathematics, leading them to value different things, enjoy different things, and view the roles of their peers and teachers differently. It is such considerations that help to re-examine shifts that have occurred.

When Beth came to the program, her perspectives of the role of the mathematics learner, and her perspectives of mathematics were not truly coherent. As a learner,

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[58] This perhaps gives the false impression of linear path. This is not what is intended.

questioning and understanding were of utmost importance. However, her perspectives of the discipline as externally determined and disconnected did not mesh fully with this. I sometimes wonder what would have happened had the program been about learning facts and knowledge performance. Would she have again abandoned mathematics learning, at least in a formal program? Through the varied experiences that presented math as something other than how Beth had come to know it, something other than what she had seriously disliked, and something that resonated with her perspectives as learner and teacher, her shift actually brought her perspectives into a more coherent whole. Everything was fitting increasingly well together!

Amelia's case presents a rather startling contrast. While Amelia had wanted to learn *why* for pedagogical reasons, her perspectives of herself as a learner and as a teacher, and of mathematics reinforced each other early on. In coming to the program to facilitate a change in her pedagogy, she actually was upsetting a balance. She was pushing towards incoherence, though it was neither through challenging math as about form and answers, nor about challenging her perception of herself as a learner of mathematics. Perhaps it is because these were not all being challenged simultaneously that they became obstacles. Her perspectives could not reinforce each other or support her change. This would seem to impede change; paradoxically, it also would seem to move her, the woman who wanted a mathematics comfort zone, from a point of comfort to a point of discomfort. While Beth was working to resolve her discomfort by developing coherence, Amelia was initiating discomfort and incoherence in order to make changes that were important for her students. Her working to embrace more progressive mathematics pedagogy presented risks to her identity that had relished the expert role of

imparting knowledge and being needed. While I by no means want to diminish what Beth has been working towards, it is important to highlight the difficulty of what Amelia has begun. I have since questioned Cuban's (1993) argument that, while researchers often underestimate a teacher's change, teachers often overestimate their own change as fundamental. In characterizing levels of change, he distinguished first-order change, which is a more superficial alteration often of behavior, and second-order change, which involves more deep shifts in perspective that accompany further shifts in behavior. If we employ an external measure of progressive teacher, Beth is apparently closer to where researchers and reformers would want her to be as a mathematics teacher. In terms of perspectives, Amelia though seemingly had much further to go to begin with. Thus she knows both that she is making changes, and that these changes are quite difficult, as was noted in her discussion of her switch to pursue conceptual first. She is correct. Her change should not be minimized or underestimated. By some standards, she is the one engaging in fundamental change.

Let's reconsider Cohen's (1990) analysis of Mrs. Oublier[59]. The result of his case study was a paradox—a teacher who perceived her changes as substantial, and a researcher who viewed her changes as superficial. Cohen recognized that the teacher's views of learning did change, Mrs. O felt was of great benefit to her students; they now "understood" the math as they learned through exploration of manipulatives and activities. However, the relative roles of teacher and students did not change, nor did Mrs. O's absolutist view of mathematics. The class remained teacher-centered, and the emphasis was still on correct mathematics—student ideas were not explored or explained.

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[59] See Wilson & Goldenberg, 1998, and Tzur, Simon, Heinz & Kinzel, 2001, for similar case analyses of teachers "not achieving" fundamental change.

Consistent with this, the teacher did not feel she had more to learn to effectively embrace reform.

Cohen argued that the teacher did not have the necessary content knowledge, nor an awareness that math could be other than absolute and platonic, to alter her teaching more dramatically. This observation speaks to the importance of developing teachers' relationships as the interaction of ways of knowing mathematics and identity. While I do not technically disagree with much of his assessment of what changed or why, I feel that Cohen's positionality created a deficit perspective of Mrs. Oublier's change. Shouldn't one ask from where she started? Perhaps he was uncomfortable with her not feeling as though she needed to learn more in order to teach reform-oriented mathematics. That is quite possible, but that also may be temporary; it likely had something to do with the professional development she experienced around the new curriculum she was implementing and how her students responded—until something challenged that image, why should she believe anything else? Amelia did not recognize a need to alter her pedagogy until after consistently not being able to answer her students' questions. As occurred with Amelia and Beth, continued interaction with students can help teachers to recognize areas of needed growth in their own relationships with mathematics, though this can take time.

Mrs. Oublier's change is actually reflective of a more common trend in progressive education. While Cuban (1993) noted small consistent progressive change in schooling over the last century, this often became watered down over time or hybridized, just as Mrs. O represented a hybrid of progressive and traditional perspectives, just as Amelia is trying to hybridize progressive and traditional. Could one of the primary

reasons for this hybridization in mathematics, in addition to external pressures, be the incoherence of perspectives[60] of mathematics, of teaching, and of learning? Perhaps this is the result of one's identity being pulled in different directions by different communities of practices. This identity work of negotiating the demands of different communities of practice demonstrates identity also as a nexus of multi-membership (Wenger, 1998)? "She must find an identity that can reconcile the demands of these forms of accountability [in the different practices] to a way of being in the world. The work of reconciliation may be the most significant challenge faced by learners who move from one community of practice to another" (ibid, p. 160). As third of the five perspectives of identity, this nexus of multi-membership provides a way to understand the significant challenge Amelia is undertaking. She is embarking on a pedagogical shift that is in tension with her ways of knowing mathematics and her perspectives of herself as a mathematics learner. How will she reconcile these?

I feel that external measures of change are losing their usefulness, and as I think about reconciliation, I am concerned for Amelia. Amelia seems to be in a vulnerable position. What if she pushes for change, which then becomes watered down because her perspectives of math, teaching, and learning don't congeal, thereby inhibiting her goals? Coherence is certainly not a identifying characteristic of a nexus of multi-membership, but lacking it certainly creates tension and more reconciling work. To further explore this notion of coherent perspectives and extend beyond where Cohen's story of Mrs. O leaves

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[60] Alba Thompson's (1992) meta-analysis demonstrated that it is quite common for beliefs (which are part of one's identity) to be held in isolation, enabling incoherence.

off, I want to highlight another story of change that initially began with the development

of progressive pedagogy: the story of Marty Schnepp[61].

As Schnepp (in press) himself indicated, his teacher training "sought" to make

him a constructivist. This impacted his perspectives of learning, though he still possessed

a platonic, absolutist view of mathematics. At this time, he wanted students to learn

constructively, but he did not hold students' conceptions at the forefront, similar to Mrs.

O and Amelia. As he engaged in the identity work of reconciliation, Marty struggled with

the dilemma of whether he should continue to ask students to construct meaning, while

looking for a *right* answer (p. 10). It was through studying philosophy of mathematics

that Schnepp realized that it was not constructivist learning that required questioning, but

rather he needed to problematize his absolutist view of mathematics. In this process of

identity development, Schnepp developed a social constructivist (Ernest, 1991) view of

mathematics, one that was coherent with his constructivist perspectives on learning.

These mutually reinforcing views promoted both inquiry in teaching and inquiry in

mathematics. While he encourages this perspective in his students, he is also committed

to helping his students develop their formal knowledge, as is required on tests and in the

classes of others (Chazan & Schnepp, 2002). Perhaps he is helping his student to

intertwine their connaître and savoir ways of knowing.

As with Marty, Amelia feels a push to change, but part of her resists that change,

because her mathematics perspectives don't support it. I raised the question of whether a

program geared toward change was likely to facilitate change that was not looked for. I

hypothesize that in order to facilitate and support such change (unless all identities of

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[61] See Gormas, 1998, for a full account of Schnepp's change.

participants already are suited to the desired change), a program would have to find ways to help teachers problematize their perspectives of teaching, learning *and* mathematics, because problematizing could aid the work of reconciliation[62]. It would unlikely be enough to model "good" teaching, and to provide a variety of mathematical opportunities, in no small part because these will be interpreted and experienced differently based on identity and modes of knowing. Considering the stories of Beth and Amelia, of Marty Schnepp and Mrs. Oublier, and other accounts of teacher change, problematizing feels increasingly important to facilitate shifts in teachers' relationships, which would position them to be life-long learners and critically participate in the process of education. This is anything but a new idea. Freire (1998) argued that problematizing is a critical act in all education, in order to enable all learners to direct not just their learning, but participation in society more broadly— problematizing is the *pedagogy of freedom*. Well over two hundred years ago, in a letter to her 10-year old son, John Quincy Adams, Abigail Adams had a similar perspective of critical involvement in the community that hinted at will and grace:

> These are times in which a Genious (sic.) would wish to live. It is not in the still calm of life, or the repose of a pacific station, that great characters are formed. Would Cicero have shone so distinguished an orator (sic.), if he had not been roused, kindled and enflamed by the Tyranny of Cadline, Millo, Verres and Mark Anthony. The Habits of a vigorous mind are formed in contending with difficulties. All History will

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[62] See Davis, 1997 for a story of change that evolved first through problematizing pedagogy and then through problematizing the nature of mathematics.

convince you of this, and that wisdom and penetration are the fruits of experience, not the Lessons of retirement and leisure.

Great necessities call out great virtues. When a mind is raised, and animated by scenes that engage the Heart, then those qualities, which would otherways lay dormant, wake into Life, and form the Character of the Hero and the Statesman.

Considering my own experience, on the one hand I would argue that Geometry and FM were so appealing to me because these experiences resonated with my identity as a learner more broadly and how I preferred to engage in learning. On the other hand, it was still in pursuing the philosophy of mathematics (and I would not nearly have considered myself an absolutist or Platonist) that I was able to reconsider my own mathematical perspectives and direct my identity development and ways of knowing toward increased coherence. This is why the second FM experience was far more dramatic in many ways. My actions were not just what I, as a learner, was trying to do to make sense of things; my actions actually represented to me important mathematics. Though nothing seemed overtly out of place previously, problematizing still made a significant difference for me. These processes are perpetual. I believe this is perhaps why for Beth, thinking about the historical development of mathematics was so important: it helped her to challenge mathematics as handed down from on high, and enabled her to see mathematics as a human creation rather than something out "there" waiting to be discovered—a creation in which she and her students could participate. It was created to help represent and find answers to questions. It is in these senses that mathematics is aesthetic for her.

I feel troubled by Amelia's case, because, without adequately problematizing mathematical perspectives in inquiry in particular (but in other content experiences as well), she was not supported to work to reconcile the practices in which she was being asked to engage with her identity. Her valuing of many of the practices we encouraged was inhibited. In some senses, her time was not productively used. This has helped me to see what roles I need to continue to develop if I am to engage others in inquiry.

While I will not argue for such activities for mathematicians[63], I think it important that we consider whether programs of teacher education should look to help pre-service and in-service teachers develop coherent perspectives of learning, teaching, and mathematics, or at least problematize incoherence through activities that both support and respect identities and ways of knowing as relationships with mathematics. There is significant literature on the importance of program coherence, but perhaps this needs to go one step further. Perhaps programs ought not to just model coherence, but also facilitate and present it as valuable over time.

Directions for Future Research

Relationship-Teaching Link

This study started under the presumption that it was relationships with mathematics rather than the traditionally defined content knowledge, which influenced teaching. The emphasis in this exploration was to try to understand how teachers experience and value program experiences as mediated through their relationships with mathematics, and how program experiences might then further influence these

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[63] I am not saying it's unimportant for mathematicians to problematize their activity; it could facilitate their critical involvement in the discipline rather than indoctrination. This stance is contrary to arguments that teaching history and philosophy of science to students could interfere with their being able to learn to engage properly in the discipline (see Brush, 1974).

relationships. This work suggests that the dynamic relationship with mathematics is critical to the teacher's own learning, and if, as Lortie (1973) suggests, their learning influences teaching, than relationships would be critical to teaching. This presumption feels reasonable based on the work of others (e.g. Cuoco, 2001; Ball & Bass, 2000), as well as on my data collection. It is important to remember that the situation is far more complex than this. Many things actually influence what teachers do in the classroom; this work strongly suggests only that one of those things that teachers would consider, even if only implicitly, is their own relationship to mathematics.

Lortie (1973) also demonstrated that one of the primary motivators and rewards for teachers was in reaching students, even if only one or two. Thus if relationships with mathematics is reflected in what they want for their students, then this suggests that relationships could be a strong factor in how teachers try to reach their students. In their teaching histories with both teachers we had glimpses that these teachers relationships were echoed in what they worked for in the classroom, be it Amelia's after-school programs that develop confidence or Beth's alternative methods that engage her students in sense-making. As they spoke of their students, their own relationships came through quite strongly.

Recall Amelia's discussion of what she wanted for her students. Issues of confidence, frustration, a mathematical comfort zone, and developing background knowledge prior to class all were present:

> I definitely just want their confidence level to be [high]. You
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> know, like when I did after-school programs, or a lot of times I just have
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> them stay after school if they want to for extra help, and it ends up being

like its own little program cause I get a lot of kids that will do it. You just watch them in class the next day and it's kind of like how I felt when I had my mom helping me at home. . . . A little teeny tiny frustration is good, but not overkill, cause then they don't like it.

And if that means that one kid knows how to multiply by the end of the year, and another kid knows how to write an incredible essay about their mathematics, that's fine. But as long as everybody kind of grows a little bit (Interview 1, July, 2006)

Amelia herself connects what she wants for her students to the experiences that shaped who she is as a mathematics learner. However, there is the ever-present paradox in how she sees herself versus her students. While she looks to others to judge her own learning, she talks about valuing her students' learning in ways that are individual. Perhaps this reflects the open environment she wishes she had for herself, though she did not know how to cultivate it.

When Beth talked of her students, her well-defined desires for her students also reflected her own relationship. However, I was not surprised a their interconnected nature. She actually numbered her desires, as if she already had a set list:

Oh, the first thing is that I want them to feel like they can do it. . . . The second thing is that it's okay to make mistakes and be wrong, 'cause I am often wrong. . . . So you come upon a problem and you can't figure it out, but you also need to be able to figure it out. I think the third thing would be that it's all connected. . . . I think with math, once I figured out that it was all patterns, and it's like the way the world works, like I know that

now, and I want them to know it, but I don't want to tell them that. I want

them to figure it out. (Interview 1, July, 2006).

In describing what she wanted for her students, Beth very much described the types of things that she came to perceive about mathematics later in life after she began teaching: confidence to work through problems, mathematics as a connected discipline—connections can be used to facilitate sense making— and that it is not just about getting right answers, but that not knowing is the reason for the journey.

They also drew on their own experiences to explain student difficulty. Amelia discussed issues of lacking background knowledge, while Beth offered students not seeing or having a sense of the big picture in mathematics. Based on this data and my own exploration over the last year two years, future research should focus on the connections between teachers' own ways of knowing mathematics, their identities, and their classroom decision-making. A colleague of mine is working with this same cohort (Beth and Amelia are two of her participants) to examine whether they bring what they have learned in the program into their classrooms. One thing that will be important to consider in this work, for those teachers who draw on program learning in their classrooms, is the degree to which what they draw on reflects their relationship with mathematics and the value attributed to the associated program experiences. For those that do not bring program experiences in, is it because they did not value what was learned or are other factors involved? What seems to be the primary influence on classroom decision-making then other than relationships with mathematics?

Change and Coherence of Perspectives

Another area of future research suggested from the findings of this exploration is to explore teacher growth in light of the coherence of their perspectives regarding teaching, learning, and mathematics. Wenger (1998) argues that identity is in part an ongoing negotiation based on the influences of, not just one but, the various communities of practice in which one participates, as well as on one's trajectory. To the degree that this is possible, this suggests that there is an unconscious need to develop coherence or accept incoherence. Perhaps Thompson (1992) found that beliefs are held in isolation in part because full coherence is not achievable based on the influences of the different practices; even it is was achievable it would likely be temporary as identity negotiation is ongoing. When teachers change, how is coherence involved? If the change fosters greater incoherence, is it followed by a push towards greater coherence (which could happen in either direction) as there was with Marty and Beth? Does change (temporarily) slow when the incoherence becomes too great or when coherence is achieved (however temporarily)?

Growth and Problematizing Perspectives

Related to this question of coherence and reconciliation is the question of the role of problematizing one's own perspectives. Inquiry as we have seen can serve to problematize mathematical perspectives; it also offers a means to facilitate teachers becoming the architects of their own learning and growth; however necessary, for many it is insufficient to this process. Freire (1998) argued that the act of problematizing was

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central to growth and learning, and served as a liberatory act. In his earlier work, he demonstrated in what way this counters traditional education.

> There is no such thing as a neutral educational process. Education either functions as an instrument which is used to facilitate integration of the younger generation into the logic of the present system and bring about conformity to it, or, it becomes the practice of freedom the means by which men and women deal critically and creatively with reality and discover how to participate in the transformations of their world (Freire, 1972, p. 38).

Do teacher education programs or professional development experiences that have a tendency to foster significant growth focus on helping pre-service or in-service teachers problematize their perspectives; do they have the structures in place that support teachers as they engage in this activity? Does growth reflect the areas that were problematized (e.g. teaching versus mathematics versus both)? I have two specific interests herein: I am curious as to the role courses such as Philosophy of Mathematics can play in facilitating growth through problematizing the nature of mathematics and how one comes to know mathematics; keeping in mind that even the value of this course would likely be mediated by relationships with mathematics. In addition, can problematizing the confidence/ frustration dichotomy encourage teachers' personal mathematics risk-taking, as well as how they see the importance of these in their students' learning?

Generative Aesthetic and Ways of Knowing Mathematics

Both my data and others' work with mathematicians (e.g. Sinclair, 2004) suggest the importance of the generative aesthetic to connaître ways of knowing mathematics. But is this broadly true with teachers? Is this a distinguishing feature of those that have more productive or holistic ways of coming to know mathematics? If so, this could help us to understand what is important to focus on in teachers' mathematical engagement if we hope to facilitate connaître ways of knowing mathematics that draw on and support the savoir ways of knowing which currently predominate (Handa, 2006). By encouraging teachers to play —not just through physical manipulatives, but cognitive play as well— to draw on their intuition, and to develop intimacy with problems or situations, can programs facilitate mathematical learning, agency, and risk-taking? As a related question, by demonstrating the importance of and facilitating the drawing on non-goal directed behavior necessary to the generative aesthetic, is a frustration/confidence duality encouraged?

Relationships and Program Experiences

Finally, for the teachers that I have worked with closely, there is a strong indication that teachers can value and engage in program experiences differently based on their identities as learners and teachers of mathematics, as well as their ways of knowing mathematics. However, as this is such a new area of exploration, drawing on newer theories and frameworks, it is critical that more work is done investigating this phenomenon. Amelia and Beth offered useful contrasts, which helped to highlight a variety of issues for consideration; however, in many cases, I don't think things will be

quite so striking due to the complexity of the interaction of relationships and content experiences. It will be important to make better sense of this complexity and the subtleties involved, particularly when/if there seems to be a disconnect between relationships and experiences.

Reflections on What I Learned About Inquiry

The Necessity of Inquiry

From the beginning of my desire to explore relationships with mathematics, I felt sure that inquiry was somehow crucial, though not sufficient, to teachers developing full, productive relationships with mathematics. At that time I did not have notions of identity and ways of knowing, but I felt convinced that this activity was central, and not just in mathematics. I had originally talked about inquiry *in* and *about* mathematics, reflecting the work of Ball & Cohen (1999) who talked about the need of teachers to inquire *in* and *about* teaching practice. Now I feel more comfortable talking about *mathematical inquiry* and *problematizing mathematical relationships*. However, the essence of what I meant has not changed. I still feel that these are necessary activities, even though many teachers had difficulty with or did not value them. It is now about finding ways to help them experience activities like this as meaningful. Inquiry represents a stance that education arising out of teachers' own questions and experiences embraces the basic human right to learn through critically reflective, meaningful experience (Dewey, 1938). This further promotes a *bold, adventurous, self-critical epistemological curiosity* (Freire, 1998, pp. 37-38), which places teachers as the intentional engineers of their own life-long development as educators and doers of mathematics, reflecting the normative-reeducative tradition of teacher education and change (Richardson & Placier, 2001).

Engaging in this exploration with these teachers supported these feelings, as has the literature I have explored. Whether I consider Cuoco et al's (1996) and Sinclair's (2004) press of playing, or Beth's suggestion that mathematics is an aesthetic expression of the world, or Davis ans Hersh's (1981) and Lakoff and Núñez's (2000) argument that mathematics is a human construction similar to other subject areas, or Burton's (2004) argument that in depriving students of inquiry we rob them of what makes mathematics so interesting and exciting, there is something that seems to connect each of these perspectives—that these are natural, human ways of working through issues and developing understanding not only of ourselves, but of our world and our inventions.

It was, however, reading about *storytelling* and *story playing* with pre-school and kindergarten children that helped me to capture the essence of how I felt about inquiry, and its role in life-long mathematics learning. Vivian Paley has used storytelling and story playing constructively in her classroom. In the following passage when she speaks of the importance of these two acts, note the invocation of will and grace, of continuity and connections, of abstraction, of intuition, of journeys to understanding, of being able to find your own way, of community friendship, safety, and hence comfort (themes that have emerged as important in this exploration):

> Eli and Edward, using fantasy play, are able to visualize a concept, the child finds the natural method for concentration and continuity and satisfies the intuitive belief in hidden meanings.

> This is why play feels so good. Discovering and using the essence of any part of ourselves is the most euphoric experience of all. It opens blocked passages and establishes new routes. Any approach to language

and thought that eliminates dramatic play, and its underlying theme of friendship and safety lost and found, ignores the greatest incentive to the creative process.

> Play and its necessary core of storytelling are the primary realities in the preschool and kindergarten, and they may well be the prototypes for imaginative endeavors throughout our lives. For younger students, however, it is not too much to claim that play contains the only set of circumstances understandable from beginning to end (1990, p. 6).

Though the transformation is not perfect, re-read the above paragraph by replacing *fantasy* and *play* with the word *inquiry*. Inquiry offers just this type of opportunity for people engaging in mathematics. Moreover, as Paley talks about the naturalness of fantasy play for children, she connects this to the natural development of walking, and then offers a picture of what could circumvent this process:

> Play, that most ordinary of human functions, as natural as crawling, walking, and running. Without instruction, theses skills flourish. No one is taught to walk—or to act out a fantasy. The patterns and incentives arise from within.

> Children pay no attention to the way they walk; they would stumble and fall if they continually observed themselves moving along. The effect would be worse if they watched everyone else as well (ibid, p. 9).

As I read this last excerpt, my mind immediately turned to Amelia and her concern with how everyone else was doing and how she compared, and how this inhibited her ability to grow and act. Inquiry is a most natural extension of the play we begin as children. It is in mathematical inquiry that we are story players, and through this inquiry that we are and can become mathematical storytellers, even if only to ourselves.

Inquiry is not Inherently Powerful

Keeping in mind both that inquiry, like story playing and telling, is one embodiment of our natural mode of coming to know, and that several of the teachers and some colleagues I worked with have not found inquiry valuable, I do not believe that mathematical inquiry is an independently powerful form of learning—at least for those of us who have been enculturated into the school mathematics tradition. Forgive me if I will still allow for this possibility if mathematical learning is experience- or play-based from the beginning. For those of us enculturated into the school mathematics tradition, the power of inquiry will, I believe, depend on the participant's incoming relationship with mathematics. We cannot look to solve things simply by providing these experiences to teachers (or other students of mathematics); otherwise, the transformations would have likely been far more dramatic.

In order to make such productive use of inquiry, it seems likely that structures need to be in place to help teachers learn to recognize and value their own learning, and to develop their own agency and self-efficacy. Problematizing might be enough, but I doubt it. Time to reflect will be important; as is time to work both independently and with someone who appreciates the difficulty of the task of inquiry, who can recognize the mathematical value of the work, and who then can help the teacher make better sense of

what their work means, to what it connects, and where it could lead; as is time to be stuck. Five to six hours per semester was horribly inadequate, though it was in other ways useful to have the strand carry over multiple courses. It was certainly useful as a research prompt, though this was not to be its primary purpose. In general we had to cease our work just as we were getting comfortable. I frequently wonder if we had had more time to work, if it could have been more valuable for Amelia? It then should be particularly important in early experiences to have the reflection time and discussion time to really build such activities as useful to those who participate. Once developed, shorter experiences over time would likely be more productive, as they were for Beth, but she was in part ready for them (though she also at times expressed the desire to have more time with this kind of work). Teachers did tend to become more comfortable with the activities overall, but I do not have the sense that most teachers came to view it as something they could independently engage in to further their own learning. Only Beth ever talked about it as valuable enough to take to her school and to her teachers.

The Role of Context

I have learned to be careful of context. Typically, real-world context is employed to help foster interest in a problem or situation, as well as to convey mathematics as connected to our human world and experiences. It is also conceived of as a way to make mathematics more accessible and less abstract. These are the perceived benefits. But I have encountered significant drawbacks. In the two investigations I used that had some realistic element to which these teachers could connect—gender discrimination and a foot race—we encountered difficulties rooted in the context. The value-laden aspect of gender discrimination made it difficult for some to agree on how to simulate the experiment.

Though we were simulating non-discrimination, which is more value-neutral, because the simulation was used to compare to a gender discrimination situation, many teachers felt the non-discrimination data was invalid if it did not mimic exactly the discrimination case. These were the rule-minded people to whom Beth had referred. This group of teachers was effectively snagged by something many would not consider mathematically relevant or reasonable. In the Zeno's Paradox investigation, familiarity with races made it quite easy to demonstrate that Achilles would win, if the race ran for a short time. It did not matter to this group that they were not dealing with the "intended" mathematics in the story, echoing a difficulty with manipulatives (Ball, 1992). Real life understanding made it clear what would happen, so the mathematical arguments in the story were not worth considering, though for two days we tried. Of course this does not always happen. I have engaged in the study of Zeno's Paradox twice before, once as a student in the first FM class, and once with a group of high school students—in neither instance did the real-world context pose significant difficulties, perhaps because in both cases the populations were those who had chosen to engage in inquiry activities.

These real-world snags recall the difficulties Amelia and her group had in their macaroni investigation. While trying to locate a mathematical justification for the textbook directions, the trio was distracted by traditionally non-mathematical issues. Of course I already argued that there is a sense in which these issues are mathematical, so when I said to be careful of context, this did not mean to avoid it. Certainly in textbooks it may be a valuable way to develop interest where there might not otherwise be any (though this does not absolve us of responsibility for the boring material to begin with). When work is adequately structured and space is constrained, as is common in various

investigative approaches to mathematics, context may not offer the types of challenges encountered herein. However, choosing contextualized situations for open-ended inquiry situations, might require wrestling with both what and where the mathematics is. This is not an inappropriate activity. In fact, struggling with these issues might provide access for many students to the importance of mathematics as an idealization (Kitcher, 1984; Davis & Hersh, 1986) that, while grounded in, is not tied to our physical world and its limitations. In other words, perhaps this idealizing process reflects a mathematical habit of mind (Cuoco et al, 1996) that is valuable to enculturate. That said, if there is neither the time nor the desire to tackle such issues (for they must naturally arise from time to time), perhaps mathematical contexts are more appropriate.

Anyway, I have often been troubled by the emphasis on context. In my own experience, and in watching teachers like Julie, Susannah and Beth, it is quite clear that a purely mathematical situation can be taken on as one's own, just as readily as real-world contexts can, given the right opportunities to engage with it. Deborah Ball (1992) made a related argument with manipulatives. Mathematics is not inherent to these objects of play, but rather in what is made of working with them. Yet somehow these tools have acquired some intrinsic power in people's minds. Manipulatives though are just one of many tools to support mathematics. Moreover, it is important to mathematics to be able to move beyond them, just as it is important to step away from context, just as number had to move away from quantity. Great care must be taken to ensure that these are appropriately used as one of many tools, lest they become obstacles to continued learning.

This emphasis on manipulatives and even context in theory is to make mathematical ideas more accessible. However this rests on the assumptions (a) that

learning progresses from concrete to abstract, and (b) that what makes something concrete is its physically tangible or real-world nature (Wilensky, 1991). Wilensky (1991) problematized this view of learning, instead suggesting that learning is a process of concretizing the abstract. From this perspective, until a learner works with any idea, problem, situation, manipulative, or even a context, adequately to make it her own, it remains effectively abstract. This is perhaps why we always have to learn how to use manipulatives. Context does not make something more or less abstract or concrete, only one's relationship with the context does. (Is this bringing to mind contrasting images of mathematics and music?). In the following two descriptions, imagine Beth and her work with the unit circle. "When our relationship with an object is poor, our representations of it limited in number, and our modes of interacting with it few, the object becomes inaccessible to us" (1991, ¶ 21). However, "it is only through use and acquaintance in multiple contexts, through coming into relationship with other words/concepts/ /experiences, that the word has meaning for the learner and in our sense becomes concrete for him or her" (ibid, ¶ 21). These descriptions—which recalled the *I-Thou* relationship on which Handa (2006) built his framework—characterize intellectually Beth's experience with the unit circle, as well as with Pascal's Triangle. When abstract, the unit circle was inaccessible; when she concretized it through exploration it was meaningful. As Amelia does not take mathematical risks, she rarely engages in experiences that would concretize the abstract ideas—looking something up in a book would not readily render an idea less abstract, though this is not impossible. Given that any context, real or mathematical, can be abstract, this seems an inconsequential question in theory (though practically it might be critical). Rather a more relevant question in

either case is where there is adequate opportunity for engagement to afford the development of a relationship with the material, allowing the abstract to become concrete.

Closure and Affect

Other smaller issues seem important to consider regarding this type of activity. In addition to an alternative structure and a longer duration, I am not sure formal closure is warranted, despite Amelia's desires. Leaving questions unanswered is important. If learners are dependent on an instructor to provide final closure, that teacher will be less able to make the move to independently pursuing sense-making. This is why being able to talk through findings with others with different knowledge and presenting one's work are so central. Another concern with closure is that if the closure does not address a particular learner's thinking, will that learner feel that their ideas are less important? Regarding these affective responses, I think it important to be far more open about the types of feelings that may be experienced throughout this work and why; perhaps having them read Paley, Cuoco, or Tobias, as well as communicate their own feelings would support this. When you invest heavily in an activity, affect is a given, but as mathematics in particular is often interpreted as objective and affect-less, emotion might seem troubling. Frustration, excitement, passion, anger, and worry are all emotions that may be experienced as part of the activity, and they do not mean something independently about the one doing mathematics. All are legitimate, if not likely. Understanding this could facilitate the productive use of affect and inhibit emotions serving as an obstacle to continued learning. Being open and working with these as they arise may be one way to

help problematize perspectives of mathematics, including the confidence/frustration dichotomy.

Finale

I began this dissertation highlighting some current issues in mathematics and mathematics teacher education, such as progressive reform, NCLB and shifting content. I will close by connecting back to these issues. Much of what I say is grounded in one ardent belief: I am not willing to let go of progressive education, though I believe there are multiple approaches to this, which, by the way, could include things that look very traditional. I want to stress again, that as with ways of knowing, it is not the presence of traditional that is problematic, but rather the supremacy of it.

Because of this dedication I am predisposed to recognize the CBMS recommendations for mathematics teacher education and the PSTM as thoughtful and critically important contributions. With their heavy focus on content, including emphases on integration with pedagogy, activity that fosters and uses habits of mind, and its historical and cultural perspectives of mathematics, these recommendations have the possibility of enabling teachers (pre-service or in-service) to develop critical knowledge and perspectives of the mathematics discipline and school mathematics. However, there is a challenge herein that, while present, is not explored in these recommendations: many of those who would be learning under these recommendations have been enculturated into a very different practice and bring with them their own unique identities. Consequently these experiences will be valued quite differently by each person who participates. This must be so. Considering Beth and Amelia who encountered the same program that considered not only CBMS recommendations, but other literature of

effective professional development, they reacted very differently and valued very different things. Lest I suggest that we should abandon facing this issue because there is no way to possibly vary a program to meet each person's individual needs, it is essential to consider how we can implement such recommendations in ways that facilitate the learning of all teachers and their valuing of the experience.

In this inquiry, I have explored the importance of personal mathematical histories (and to one teacher, history of mathematics as well) to how these two teachers experienced mathematics in their in-service teacher education. If one's history (or goals, or feeling, etc.) can be integral to one's learning, wouldn't it make sense to include explicit opportunities to reflect on and critically consider these as part of the learning process? This would seem particularly important for teachers who must also develop a sense of how these can influence their own students' learning Deborah Ball (1996) had suggested that teachers needed opportunities to unlearn old views. Though I understand this perspective, I am very troubled by the metaphor of *unlearning*. As identity represents a historical process of learning, how can we unlearn a part of who we have become? Rather, providing the opportunity to consider these old views in light of personal histories and how these old views have shaped participation and identity seems more powerful. Not only does this counter the suggestion that somehow there was something deficient in these old views (which after all developed in conjunction with a sanctioned practice), but these serve as valuable learning opportunities in ways that truly cast the teacher as the primary actor in their learning overall, just as they are in their own inquiry. It is the teacher who must negotiate the old and the new; for some, like Beth, this won't be much of a struggle (at least at the time of the program; there was much struggle earlier).

However, as we saw with mathematical frustration, struggle perhaps should be honored as it may ultimately make learning more valuable. The teachers who have difficulty or even who seem to reject negotiating the old and the new, should not be considered deficient or inadequate. They are still negotiating a practice in a way that is true to themselves. We though must continue to support and push for critical reflection.

We can never fully predict all the areas in which a teacher would need help and support, but in structuring significant opportunity to engage in adequate reconsideration/ /reconciliation/problematizing in various contexts, we can promote and support this activity. The difficulty will be in not circumventing the process in the rush to move somewhere else; this type of activity should hold a place of importance at least equal to content and pedagogy if teachers are to embrace this as critical to their development. Perhaps dedicated space for this is important. That said, open-ended inquiry, more structured investigations, history (both personal and disciplinary), philosophy of mathematics, capstone sequences, and simple time for critical and public (supported) reflection can all provide prompts to promote engaging in trying to negotiate and reconcile identity and ways of knowing. Thus providing dedicated space does not necessarily imply revamping recommendations, but thinking more of how to structure them.

Of course there is the complicating tension of NCLB. This is quite a noble idea that probably is at the heart of progressive mathematics reform, and why people like Leone Burton (2004) have called for students and teachers to experience mathematics as a process similar to the process found in the discipline. Mathematics after all has been one of the most alienating of all content areas for students in the K-12 setting.

Unfortunately, it is not with this that NCLB is concerned. Rather, NCLB holds teachers accountable for students passing tests, which generally assess the type of isolated, acquired knowledge that I argued against early in this work. Moreover, the High Quality Teacher (HQT) requirements don't look at how teacher education programs are structured, and do not specify the types of experiences teachers should have. Rather it specifies requirements like *degrees earned* and *subject-area majors*, which is not vastly different from looking at courses and grades to assess effectiveness. I don't see anything in these stipulations that would have made any difference for Beth as a mathematics student. There is nothing in here that would facilitate Amelia's ability to answer her students' questions as a mathematics teacher. Yet NCLB is law, and the idea is useful. Thus there must be work also to reconcile these demands with what teachers are learning and doing. In particular this is one place where epistemological and historical perspectives can be valuable as these could enable teachers to justify what they do in mathematics classes as important mathematics, even if these aren't recognizable to all. It may also help them consider how to move students toward the "objective" knowledge that is tested, as Marty Schnepp did; these do after all reflect the products of engagement in the discipline. Beth has already encountered challenges to her teaching methods. Not adequately prepared to face these challenges, it would not be entirely surprising if she eventually hybridized (Cuban, 1993) in the face of opposition—after all practice is negotiated.

NCLB and HQT requirements also do not seem to make stipulations for teachers to become life-long learners. Yet without such a stance, how can teachers be expected to effectively deal with (and be held accountable to) landscape changes such as those we are

seeing with middle school content? There is thus not only an urgency to educate teachers in the ways recommended by CBMS and with the perspectives of PSTM; there is an urgency (right now particularly for middle school teachers) to assist teachers to problematize the landscape and the discipline, their identities and ways of coming to know mathematics, and to help them learn to ask and answer their own mathematical questions. In this way we can hope to position them to intentionally grow and develop with the ever-changing demands of public school mathematics education and to critically reconcile, coherently or not, all these with their actions as teachers and learners of mathematics.

Appendix A: School Mathematics as a Community of Practice

The consistency of the descriptions of the school mathematics tradition and the degree to which these descriptions fit within the larger context of the schooling tradition support the perspective that school mathematics is a well-developed practice that is resistant to change. However, that the practice is well established does not automatically characterize school mathematics as a community. I want to note at this point, that I have not separated school mathematics learners and teachers as separate communities. I have chosen to consider school mathematics as one community of practice for a variety of reasons, though teachers could also be considered as members of a teaching community of practice, which I am not considering herein. First, what happens in the mathematics classroom and the meaning of these is jointly constructed between students and teachers (Cobb, Wood, Yackel & McNeil, 1992; Gregg, 1995); second, teachers tend to spend much more time engaging with students around mathematics, than with each other, and except for the very rare occasion, their primary influences and motivations come from the classroom, rather than teacher interaction (Lortie, 1973); third, communities of practice inherently make use of generational relationships (Wenger, 1998)—if teachers are separated from students, the experts in school mathematics are separated from the more novice participants, and the notion of different generations negotiating the practice is lost; fourth, as was noted by Lortie (1973) in his designation of the *apprenticeship of observation*, much of what teachers bring to their classrooms and practice was learned historically as learners of mathematics—while their identities certainly continue to evolve, these identities were begun as learners; finally, teachers often continue to be learners of mathematics throughout their profession, whether in professional

development, in higher education or in their own classrooms, and their identities that have formed over the course of their experiences as learners will be brought to bear on (and perhaps modified by) ongoing experiences. The highly formal division between the teachers and other participants troubled me as I made this decision, but separation into different practices was even more so. To portray school mathematics as a community, I will draw on Wenger's (1998) three central aspects of community: mutual engagement, joint enterprise and shared repertoire.

Mutual Engagement

In examining how people engage with each other, Wenger considers (a) formal and informal structures and activities that facilitate working together in the practice (*enabling engagemen*t), (b) the ways participants negotiate and assert their similarities and differences (*diversity and partiality*), and (c) the complex bonds between participants, positive and negative, that arise from ongoing engagement (*mutual relationships*). These characterizations facilitate highlighting common elements and occurrences across school mathematics classrooms. Most universal, the very structure of schooling—with its various courses and classrooms—provides formal structures for engagement to occur. The intended purpose of tracking might also be seen in this light as it is supposed to allow groups of students to participate in ways considered more beneficial for them. Department meetings and faculty rooms are other structures that enable engagement, though primarily between experts, who otherwise rarely have other opportunities to interact in the school setting.

Working to draw out similarities and differences among participants is also very common in school mathematics, though likely rarely viewed as an element of the

practice. Within a classroom, students and teachers negotiate particular roles (which may relate to roles in school more broadly); students sitting with certain peers, and certain students rushing to answer with others rolling their eyes at this are ways of conveying these. Identities do just emerge passively, but rather how students and teachers choose to participate can assert certain roles and characteristics, such as being good at math, such as not caring about the class, such as a student suggesting he knows better than the teacher, being the class clown, etc. Tracking, which does not often occur within a single classroom, but rather across math classrooms, can be seen as a formal means to assert sameness and diversity in terms of mathematics success. Teachers themselves will also assert, though not always publicly, their similarities and differences; they come to know who has similar perspectives or mathematical interests, who takes their work very seriously, who asserts him/herself as an authority figure, who gives students the benefit of the doubt, or who has smooth running classrooms.

This brings us to the idea of mutual relationships. As participants engage, their relationships grow deeper and often stronger, as can be seen in alliances between teachers, between students, or even teachers and students. Gregg (1995) demonstrated how strong yet different relationships developed between a teacher and her high track versus her low-track students. The teacher had significant power over her high track students, primarily through the use of grading; however the low-track students challenged her increasingly as they did not see mathematics as relevant and the teacher was not able to help them to understand its relevance. The students challenged her at every turn not allowing her to be primary director of classroom practice

Joint Enterprise

Ongoing learning and negotiation of the practice are well entrenched in the school mathematics practice, though one might simply assume because teacher-centered classrooms dominate that the enterprise is hardly "joint". Yackel and Cobb highlight how socio-mathematical norms develop in classrooms through a negotiation process, such as around what constitutes a valid response to justification questions; this occurs while students offer certain responses and the teacher, and perhaps other students, accepts, rejects, or pushes on the answer. Gregg (1995) further demonstrated that a similar negotiation occurs around the emphasis on forms and procedures. In both cases, as students are actively involved in negotiating the practice, they are also involved in reproducing it (Cobb, Wood, Yackel & McNeil, 1992)[64]. It is not simply something they learn passively over time. Another area of important negotiation focuses on what is the purpose of doing mathematics. Gregg depicts how teachers at one school invoke common justifications for engaging in mathematical study, such as: it helping to promote reasoning, helping in real-life situations, and helping in later schooling. The teachers shared these with each other, and depending on students goals, identities, etc., these non-mathematical justifications are perceived very differently; many interpret these as saying that studying mathematics is a meaningless pursuit, whereas others might buy in to this argument, thereby reinforcing the arguments for teachers

Wenger also stresses the importance of negotiating the meaning of internal contexts, called *indigenous enterprise*. High-stakes environments, tracking, and teacher-centered instruction are contexts that can be differentially negotiated. Teachers may feel

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[64] This suggests that legitimate peripheral participation may be a better descriptor than apprenticeship of observation.

the pressures of *No Child Left Behind* differently based on their years teaching (such as whether they have tenure), or based on their goals or perceptions of their students[65]. This may affect how they communicate their perspectives on things like High Stakes Testing, which then impact students' interpretations. Tracking is another area in which negotiation occurs, as it is not uncommon for additional meanings to be attributed to these classes. Unfortunately, participation in low track classes commonly come to mean a student is less able, which- as teachers and students come to believe this- could get reinforced by students putting forth less effort because success is out of reach. Other contexts that might take on different meanings in different classrooms, schools, etc, could be the racial or socio-economic contexts.

Finally as part of making sense of the practice, formal and informal ways of holding each other accountable develop. In schools in general and in mathematics, grading is a formal way of holding students accountable for their learning. Students may also find ways of holding teachers accountable (and vice versa), through their behavior and treatment. What are accepted answers or rejected answers or explanations can serve as the basis on which to enact accountability as was seen in Yackel and Cobb when students evaluated the appropriateness of another's solution as *different*. In classrooms where performance is emphasized, a very common form of accountability is actually self-imposed as some students become afraid to respond for fear of looking unintelligent in front of their teacher or peers. Lortie (1973) highlights even another form of accountability that has an unusual manifestation. As it is hard in the structure of schooling to reach all students, teachers often reduce their emphasis to a few students in

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[65] Are these students liable to stay in school and go on to college? Do they want them to perform or to make sense of math?

order that they do not feel a sense of failure in not reaching all students; such action can reinforce and is reinforced by the perspective that ability is central to achievement.

Shared Repertoire

A final aspect of community that Wenger emphasizes is the shared stories, symbols, routines, heuristcs, etc., that participants can call on and use. In the previous description of the practice of school mathematics, elements of the shared repertoire were highlighted, such as the "drill and practice" routine and the architecture of the typical class. Some elements of the repertoire of school mathematics are not inherent to this practice, but rather come from the practice of mathematics itself, such as the symbols and the formal mathematical knowledge that are used and/or learned, depending on level. There are also common taken-as-shared shared perspectives, such as success equating with ability, the central role of the textbook, the perspective that math is unchanging, that form and procedures are fixed and self-evident, or that it is fine to forget material as long as you remember it for the test. The previously mentioned justifications teachers use to try to get students to buy into the study of math are common and could be considered part of the repertoire. That students and teachers walk into classrooms knowing to a certain extent what to expect reflects this shared repertoire. Perhaps this is one reason that it can be so difficult when particular classes deviate from the norm- participants have less of a repertoire from which to draw on to inform participation.

My goal in describing school mathematics as a community of practice was to justify using Wenger's (1998) construct of identity in examining the mathematical identities of the teachers in this cohort, as well as how this identity shapes and responds to current participation.

Appendix B: Additional Discussion of Methods

Summary Table of Data Collection

Semester/Course	Inquiry Activities and Formal Data Collection
Summer I 2005- June Trends in Mathematics Education Pedagogy	Beliefs Surveys*
Summer II 2005- July Integrated Algebra	Mathematical Autobiographies*
Fall 2005 Algebra Content	Inquiry Activity: Parabolas & Proof ✓ Class Transcripts ✓ Class Summaries ✓ Journals and E-mails regarding inquiry
Spring 2006 Integrated Data Analysis	Inquiry Activity: Gender Discrimination & Intuition ✓ Class Transcripts ✓ Class Summaries ✓ Journals and E-mails regarding inquiry Beliefs Surveys*
Summer 2006- July Statistics Content	Inquiry Activity: Paired Investigation & The Art of Problem Posing ✓ Inquiry Journals Interview 1- July
Fall 2006 Misconceptions Pedagogy	Inquiry Activity: Zeno's Paradox (and revamped activities) & Basic Metaphor of Infinity ✓ Class Transcripts ✓ Journals and E-mails regarding inquiry ✓ Project write-ups Interview 2- September/August
Spring 2007 Integrated Geometry	Inquiry Activity: Individual Investigation ✓ Formal Project Write-up Beliefs Surveys* Interview 3- April-June

* This data was collected in the context of the program and not in the context of my research in particular.

Reflections on Coming to Know in the Research Process

I purposefully collected various forms of data. This was in part a necessity as the phenomenon I was investigating, relationships with mathematics, did not primarily live in one source. I needed access to perspectives, as well as actual engagement. It was also important because I wanted to focus on individuals who on the one hand were representative, in terms of mathematical preparation, of middle school mathematics teachers, but also who would offer some valuable, yet distinct, insight into the dynamic nature of relationships with mathematics. These multiple data sources I knew were important for an in depth understanding, and I also knew that this could be valuable for *triangulation*. Thus my choice was also practical.

However, I have had increasing difficulty with this concept of triangulation. The very notion, that different data sources confirm each other, while useful, is an obstacle if looking at dynamic change. Over time data sources may actually contradict each other, or different data may simply provide access to different perspectives. If Wenger and Thompson are correct, that beliefs are held in isolation and that, as a nexus, identity is complex and at times inconsistent, then consistency is not a reliable measure of the data. Inconsistency is not inherently problematic. In fact it could offer true insight. For example, Amelia had inconsistent views of the mathematics learner. However, there was an internal consistency to them in that she viewed her own learning and that of her students differently. When I first ran across this difference, I was deeply concerned that the data was not reliable. It was in fact the long duration of the project with multiple interviews that allowed me to see this as a tension that was very real for her.

While I am uncomfortable with the process of triangulation, there is a metaphor that I strongly prefer, if one must be used: the metaphor of *crystallization*:

> The central imagery is the crystal, which combines symmetry and substance with an infinite variety of shapes, substances, transmutations, multidimensionalities, and angels of approach. Crystals grow, change, and are altered, but they are not amorphous. Crystals are prisms that reflect externalities and refract within themselves, creating different colors, patterns, and arrays casting off in different directions. What we see depends on our angle of repose—not triangulation but rather crystallization. In creative analytical process (CAP) texts, we have moved from plane geometry to light theory, where light can be both waves and particles (Richardson & St. Pierre, 2005, p. 963).

Crystallizations are dynamic and somewhat time sensitive. This seems relevant given that depending on which lens I am viewing from, I see something quite different, though generally related. For example, when I look at Amelia's mathematical work in terms of her identity, I have seen marginal non-participation, yet when I have looked from the ways of knowing perspective, I see a valuing of savoir ways of knowing; mathematical and epistemological views all give different windows of relationships. It is in working through these differences that crystallization seems to happen… that I gain some still partial but complex understanding. While objective validity does not feel viable, through member-checking I was able to ensure that I offered these teachers more voice than simply recounting their stories, and that I can work to represent them in way

that does not feel alien to or a misinterpretation of who they are, even though different

then how they might represent themselves

My methods of analysis further reflect a creative process. As with mathematics, I

struggled with what it would mean to come to know in this work. This was not as

problematic for me in the beginning as I examined the data to determine which six I

would interview; it became more problematic as I designed further interviews, which,

while they each had their own particular theme, required previous data analysis in order

to probe previous comments, themes, and some mathematical work as well. It was

particularly difficult as I engaged in looking at the teachers over time. I *knew* I was

supposed to engage in coding, but that process did not seem to make much sense to me. I

tried to find ways to develop complex understandings from the data, but coding was not

helpful yet. At a certain point I realized that in addition to the multiple data sources, I had

yet to examine myself. I had neglected to see myself as an instrument. In this case,

because I had many long-term interactions with these teachers, I had begun to have a

sense of them; *Je les ai connais*[66]! So I began then with what came most naturally. I told

their stories orally. I simply began to speak with others (professors and peers) who these

teachers were as I understood them to date. Though informal, I was playing with story.

This was invaluable. As these colleagues recounted to me what they had heard from my

stories, issues emerged that I had not recognized; I realized I had highlighted issues that

did not feel salient; and I had left things out that felt central. From here I began exploring

themes across interviews through developing data matrices (which I did not recognize as

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[66] *I knew them.* Using the conjugation of the connaître form of to know.

a valid analytic method initially). From here I began a more defined storytelling process, which continued to push my understanding and helped me design the third interview.

Of course amidst this, I simultaneously struggled with coding interviews, class transcripts, and journals. My codes were developed from the literature I was using to try to make better sense of ways of knowing and identity. In particular codes were rooted in Cuoco et al (1996), Handa (2006), Sinclair (2004), Burton (2004), and issues of my own concern, such as those in the Philosophy of Mathematics. This work on the one had was very important in that it was through this work that I was able to fine tune some understandings. For example, it was through coding that I came to understand it was not the presence of aesthetic that distinguished these two teachers from each other, as I had originally insinuated in their stories, but that it was the type of aesthetic response they made use of that characterized their different ways of coming to know. That said, I had trouble with coding. I had learned about the importance of coding in my qualitative courses, but it never felt adequately meaningful, though it certainly helped me familiarize myself with the data. Perhaps there was a problem with my coding, but I felt I was missing Beth's and Amelia's heart and voice. I struggled with ultimately letting go of this method as central. I understand that coding can help make the work rigorous, but it took awhile to accept that this was not the only way. I feel that what this represented was working to balance savoir and connaître ways of knowing in this research.

Again though, not quite sure what to do, I did the natural thing: writing (Richardson & St. Pierre, 2005). This would help me draw on all the previous work I had done. It turned out writing as inquiry and analysis was the most productive process for me. Again I was a storyteller, but writing was different. It was meditation in action. I

worked to integrate my intuitive understandings with emergent themes, my frameworks—I allowed the data to effect me. As such I made connections that had previously seemed elusive. As this helped me enter into a relationship with the data, I developed a more holistic sense of these two teachers; moreover I was able to reflect more deeply about my identity as a researcher, teacher and doer of mathematics in this work. It was through writing that I was able to recognize the intuition in Amelia's mathematical work. I developed a sense of the difference between having it and drawing on it. It was through writing that I came to see the process of drawing on the generative aesthetic response as akin to the process of the joining of one's willingness to engage in mathematics and their undergoing of grace in response to the mathematics. It was in writing that I was able to let go of inquiry as an inherently powerful form of learning.

I have struggled with this research process. Struggle though is invaluable, and helps us to value what we learn. Making use of the processes I had been taught was useful, but it was in finding my own way that was instrumental to my development of understanding.

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