Topological Decompositions for 3D Non-manifold Simplicial Shapes

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Abstract

Modeling and understanding complex non-manifold shapes is a key issue in several applications including form-feature identification in CAD/CAE, and shape recognition for Web searching. Geometric shapes are commonly discretized as simplicial 2- or 3-complexes embedded in the 3D Euclidean space. The topological structure of a non-manifold simplicial shape can be analyzed through its decomposition into a collection of components with simpler topology. The granularity of the decomposition depends on the combinatorial complexity of the components. In this paper, we present topological tools for structural analysis of three-dimensional non-manifold shapes. This analysis is based on a topological decomposition at two different levels. We discuss the topological properties of the components at each level, and we present algorithms for computing such decompositions. We investigate the relations among the components, and propose a graph-based representation for such relations.

1 Introduction

Modeling shapes requires representations that integrate geometry, topology, and semantics.. At the geometric level, a shape is described as a collection of elementary cells, such as a cell, or a simplicial complex. A structural representation is a more concise description of the shape in which geometric details are abstracted, and only important features remain. Thus, it is more suitable as a basis for semantic annotation and reasoning. A high-level representation of a shape reflects its overall

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structure, as, for instance, in skeleton-based representations, or its topology, which describes the structure of an object in terms of the connectivity of its parts. An example is provided by the decomposition of solid objects into form features used in CAD/CAM applications.

Most of existing representations are actually for manifold shapes. On the other hand, non-manifold shapes arise in several application contexts, often as the result of an abstraction process applied to manifold ones. This happens, for instance, in the idealization process to which finite element meshes generated from CAD models undergo to meet simulation requirements. Informally, a *manifold* object is a subset of the Euclidean space for which the neighborhood of each internal point is homeomorphic to an open ball. Objects that do not fulfill this property at one or more points are called *non-manifold*.

Here, we focus on structural representations for non-manifold shapes based on topology. We consider a shape modeled as a simplicial complex. Simplicial complexes are commonly used to represent multi-dimensional shapes in a variety of application domains, including finite element analysis, solid modeling, animation, terrain modeling, and visualization of scalar and vector fields, because of their attractive combinatorial properties.

We propose here a two-level topological decomposition for a non-manifold shape. At the lower level, a shape is hierarchically decomposed into basic components, which have a simpler topological structure. Our approach is based on dimensionality and connectivity at non-manifold singularities, since they provide useful information on the structure of a non-manifold shape. The classes of topologically simple shapes can be hierarchically ordered based on their degree of "non-manifoldness". At the upper level, the same shape is decomposed into semantically-rich components. An example is shown in Figure 1. Such composite shapes are the *wire-webs*, which are collections of wires connected together, the *sheets* which are triangulated dangling surfaces, the *2-cycles* which are triangulated surfaces enclosing 3D volumes, and nearly-manifold tetrahedralized parts.

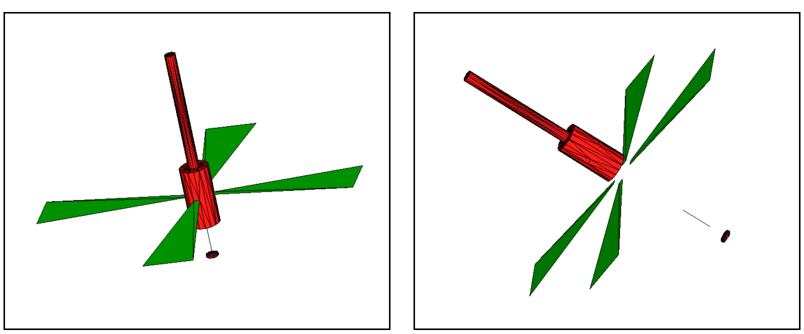


Fig. 1. A fan viewed as a structure of composite shapes: two 2-cycles (the motor and the pendant), four sheets (the blades) and a wire-web (the thread that ties the pendant to the motor)

Based on its topological decomposition, the structure of a non-manifold shape can be described as a hypergraph, that we call the *decomposition graph*, that encodes the connectivity among topological components. The level of structuring reflected in the connectivity of the decomposition graph is determined by the level of structuring within the topological components. The more basic the components are, the more topological details the graph tends to show. As the topological components encompass more structure, the connectivity among them tend to simplify. Thus, the graph reveals a more global structure.

Since all decomposition in our two-level model are unique, they constitute a natural tool for attaching semantic information to the shape description. This is an important step in the direction of semantic-based shape representations and of semantic-oriented tools to acquire, build, transmit, and process shapes with their associated knowledge. Relevant applications of such techniques are in detecting form features from non-manifold simplicial shapes, which are generated from CAD models [8], and shape retrieval and matching for visual data mining over the Web [12].

The remainder of this paper is organized as follows. Section 2 provides some background notions on simplicial complexes and on manifoldness and connectivity. Section 3 reviews related work on decomposition of shapes based on geometry and on topology. Section 4 introduces classes of nearly-manifold topological shapes discretized as simplicial complexes and discusses their properties and relations. In Section 5, we present a hierarchy of decompositions for a shape discretized as a two-dimensional or three-dimensional simplicial complex and an algorithm for the computing such decomposition. In Section 6, we present a semantic-oriented decomposition for a non-manifold shape based on semantic-richer topological shapes, and we discuss an algorithm for computing it based on the lower-level topological decomposition presented in 5. Section 7 presents a graph-based representation for the two-level decompositions, the decomposition graph, and discusses its properties. Finally, in Section 8, we draw some concluding remarks and discuss current and future developments of this work.

2 Background

In this Section, we introduce some background notions on manifold shapes and on simplicial complexes, that we will use throughout the paper (see [1, 24] for more details).

Let $\mathbf{x}$ be a vector and $\|\mathbf{x}\|$ denote the length of $\mathbf{x}$. $S^d = \{\mathbf{x} \in E^{d+1}$ such that $\|\mathbf{x}\| = 1\}$ is called a *unit d-sphere*. $D^d = \{\mathbf{x} \in E^d$ such that $\|\mathbf{x}\| \leq 1\}$ is called the *unit d-disk*. $B^d = \{\mathbf{x} \in E^d$ such that $\|\mathbf{x}\| < 1\}$ is called the *unit d-ball*. A sub-space M of the Euclidean space E^n is called a *d-manifold*, for $d \leq n$, if and only if every point p of M has a neighborhood in M that is homeomorphic to the d-dimensional unit ball B^d. M is called a *d-manifold with boundary* if every point p is homeomorphic to B^d or to the d-dimensional unit ball B^d_+ intersected with a hyperplane. In the following, we only consider connected manifolds with or without boundary embedded in the three-dimensional Euclidean space E^3. We call *non-manifold* any connected subspace which violates the manifold property at at least one point.

A Euclidean *simplex* σ of dimension k is the convex hull of $k+1$ linearly independent points in the n-dimensional Euclidean space E^n, $0 \leq k \leq n$. We simply call a *Euclidean simplex* of dimension k a *k-simplex*. k is called the *dimension* of σ and is denoted $dim(\sigma)$. Any Euclidean p-simplex σ', with $0 \leq p < k$, generated by a set $V_{\sigma'} \subseteq V_\sigma$ of cardinality $p+1 \leq d$, is called a *p-face* of σ. Whenever no ambiguity arises, the dimensionality of σ' can be omitted, and σ' is simply called a *face* of σ. Any face σ' of σ such that $\sigma' \neq \sigma$ is called a *proper face* of σ.

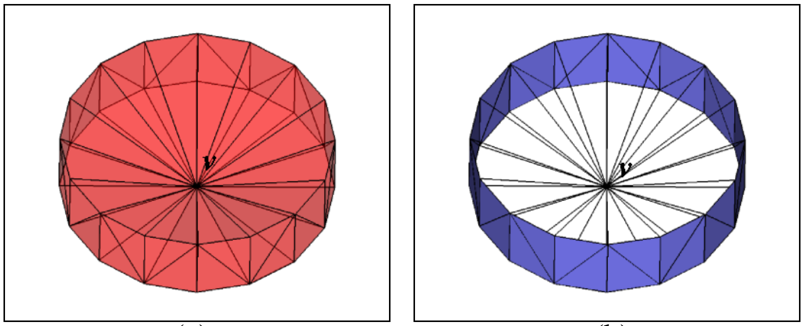
A finite collection Σ of Euclidean simplexes forms a *Euclidean simplicial complex* if and only if (i), for each simplex $\sigma \in \Sigma$, all faces of σ belong to Σ, and (ii), for each pair of simplexes σ and σ', either $\sigma \cap \sigma' = \emptyset$ or $\sigma \cap \sigma'$ is a face of both σ and σ'. If d is the maximum of the dimensions of the simplexes in Σ, we call Σ a *d-dimensional simplicial complex*, or a *simplicial d-complex*. A subset Σ' of Σ is a *sub-complex* if Σ' is a simplicial complex.

The *boundary* of a simplex σ is the set of all faces of σ in Σ, different from σ itself. A simplicial d-complex Σ is said to be *with boundary* if there exists at least one $(d-1)$-simplex that is a face of exactly one d-simplex in Σ. The boundary of Σ is the set of all such $(d-1)$-simplexes. All d-complexes embedded in E^d have boundaries, while d-complexes embedded in $E^n, n > d$ may be without boundary.

The *star* of a simplex σ is the set of simplexes in Σ that have σ as a face, and we denote it as $star(\sigma)$. The *link* of a simplex σ is the set of all the faces of the simplexes in the star of σ which are not incident in σ, and we denote it as $link(\sigma)$. Any simplex σ such that $star(\sigma)$ contains only σ is called a *top* simplex. A d-complex in which all top simplexes are of dimension d is called *regular*, or *of uniform dimension*. For instance, a regular simplicial 2-complex does not contain edges which are not on the boundary of some triangle. A regular simplicial 3-complex does not contain edges or triangles that are not on the boundary of a tetrahedron.

Let us consider a simplicial d-complex Σ. An *h-path* between two $(h+1)$-simplexes in Σ, where $h = 0, 1, ..., d-1$ is a path formed by an alternating sequence of h-simplexes and $(h+1)$-simplexes. A complex Σ is said to be *h-connected* if and only if there exists an h-path joining every pair of $(h+1)$-simplexes in Σ. Any maximal h-connected sub-complex of a d-complex Σ is called an *h-connected* component.

A regular $(d-1)$-connected d-complex in which all $(d-1)$-simplexes are shared by one or two d-simplexes is called a *(combinatorial) d-pseudo-manifold* (possibly with boundary). A simplicial d-pseudo-manifold is a *d-manifold complex* if its carrier in E^n is a d-manifold. It can be easily shown that a d-manifold complex is a d-pseudo-manifold in which the link of every vertex is a triangulation of the $(d-1)$-sphere S^{d-1} or of the $(d-1)$-disk D^{d-1}. Figure 2(a) shows an example of a 3-pseudo-manifold which is not a 3-manifold complex. The solid pinched pie contains a non-manifold vertex v, and its link is not homoemorphic to the sphere or to the disk, as shown in Figure 2(b).



(a) (b)

Fig. 2. (a) A solid pinched pie with a non-manifold vertex v; (b) The link of non-manifold vertex v

3 Related Work

Shape analysis of three-dimensional models is an ongoing active area of research in shape and solid modeling, computer vision, and computer graphics. In Computer Aided Design and Manufacturing (CAD/CAM), objects are described as an organization of meaningful parts, known as form features. In reverse engineering a 3D model, the identification of form features requires an analysis of the shape of the object [30]. In the management of large shape databases, the classification and retrieval of data requires a shape analysis. In molecular biology, analysis of the 3D structure of protein is an important tool to the understanding of phenomena such as protein folding and docking [7].

The major approaches to shape analysis are based on computing the decomposition of a shape into simpler parts. Primitive-based shape decompositions have been developed in computer vision for object recognition. In this case, a shape is described as a collection of volumetric primitives, like generalized cylinders [5], superquadrics [34], or geons [4]. More recent approaches are either *interior-based*, in the sense that they measure properties of the volumes enclosing a shape, or *boundary-based*, in the sense that they consider local properties of the boundary of the shape, such as critical features or curvature (see [40] for a survey). Examples of the former representations are skeletons, like the medial axis [2, 6, 9, 17, 41], or the Reeb graph [3, 22, 42]. Skeletons provide an abstract representation by idealized lines that retain the connectivity of the original shape.

Boundary-based methods provide a decomposition of the boundary of an object into parts, which can be described as a cell complex [23, 27, 28, 31, 32, 33, 46]. All methods try to decompose an object into *meaningful components*, i.e., components which can be perceptually distinguished from the remaining part of the object. Boundary-based methods have been developed in CAD/CAM for extracting the so-called *form features*, like protrusions, depressions or through-holes, which produce a boundary-based decomposition of a 3D object guided by geometric, topological and semantic criteria [10, 19, 29, 26, 36, 38, 39, 43, 45]. Not much work has been done on extracting features from CAD meshes. These latter can be obtained by tessellating the native CAD models, or from a reverse engineering task. Some approaches have been developed for segmenting CAD meshes, which combine curvature with the detection of other measures, like planarity or the size of the dihedral angles.

All above techniques, however, work on manifold shapes. A common approach to represent a non-manifold shape consists of decomposing it into manifold components. Some techniques have been proposed in the literature for decomposing the boundary of regular non-manifold 3D shapes, i.e, for non-manifold shapes which do not contain dangling faces or edges, which are described by their boundary. In [21], the idea of cutting a two-dimensional non-manifold complex into manifold pieces is exploited to develop compression algorithms. In [20], the problem of decomposing r-sets into their manifold parts is discussed. Desaulniers and Stewart [16] propose a representation scheme based on a decomposition of a regular object (called an r-set) into manifold parts. In [37], a decomposition algorithm is presented which minimizes the number of duplications introduced by the decomposition process. In [15] an algorithm for extracting 2-cycles (connected collections of triangles in a mesh bounding a hollow cavity) in a simplicial 3-complex is described with the purpose of computing the topological invariants on a simplicial 3-complex.

Pesco et al. [35] propose a decomposition of a 2D cell complex based on a combinatorial stratification of the complex, inspired by Whitney stratification. The combinatorial stratification is a collection of manifold pieces of different dimensions, the union of which forms the original object. They propose a data structure and a set of operators based on such representation, which allow incremental construction of the object through a sequence of cell attachment, but they do not provide an algorithm for building it from a given (non-decomposed) complex.

In [13], a decomposition of a simplicial complex in arbitatry dimensions is proposed, in which the complex is split into nearly manifold components by cutting it at non-manifold simplexes. This decomposition has been developed as a basis for defining data structures for abstract simplicial complexes not necessarily embeddable in the Euclidean space. It is a good basis for defining both dimension-specific and dimension-independent representations for complex simplicial shapes [14, 25]. The main drawback of this approach is that it produces a too refined shape decomposition, which makes it difficult to use as a basis for reasoning on non-manifold shapes.

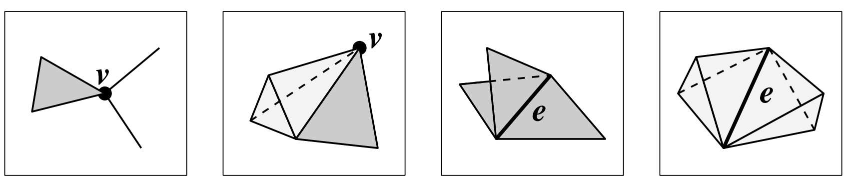
4 Non-Manifold Simplicial Shapes

In this Section, we introduce classes of nearly-manifold simplicial complexes which we will use as the basis for our lower-level shape decompistion . We will restrict to simplicial d-complexes embeddable in the three-dimensional Euclidean space E^3, where $d = 1, 2, 3$.

4.1 Non-manifold Singularities

In this Subsection, we introduce the definition of manifold and non-manifold vertices and edges, which form the connectivity information which characterize the different classes of basic shapes.

A vertex (0-simplex) v in a d-dimensional pseudo-manifold Σ (with $d >= 1$) is a *manifold* vertex if and only if $link(v)$ in Σ is a triangulation of the $(d-1)$-sphere S^{d-1}, or of the $(d$–$1)$-disk B^{d-1}. A vertex is called *non-manifold* otherwise. Figures 3(a) and (b) give two examples of non-manifold vertices. An edge (1-simplex) e in a d-dimensional pseudo-manifold Σ (with $d >= 2$) is a *manifold* edge if and only if $link(e)$ in Σ is a triangulation of the $(d$–$2)$-sphere S^{d-2} or of the $(d$–$2)$-disk B^{d-2}. An edge is called *non-manifold* otherwise. Figures 3(c) and (d) give two examples of non-manifold edges.



(a) (b) (c) (d)

Fig. 3. Examples of non-manifold simplexes: (a) a non-manifold vertex whose link consists of two vertices and one edge; (b) a non-manifold vertex whose link consists of one triangle and one edge; (c) a non-manifold edge whose link consists of three vertices; (d) a non-manifold edge whose link consists of two edges

Informally, manifold vertices and edges are characterized by the following properties:

- In a 1-pseudo-manifold, a vertex v is a manifold vertex if $link(v)$ consists of one or two vertices.
- In a 2-pseudo-manifold, a vertex v is a manifold vertex if $link(v)$ is a 0-connected 1-simplicial complex in which all the vertices, at most with the exception of two, have exactly two incident edges.
- In a 2-pseudo-manifold, an edge e is a manifold edge if $link(e)$ consists of one or two vertices.

- In a 3-pseudo-manifold, a vertex v is a manifold vertex if $link(v)$ is a 1-connected simplicial 2-complex with a domain homemorphic to a 2-sphere or to a disk (1-ball).
- In a 3-pseudo-manifold, an edge e is a manifold edge if $link(e)$ is a 0-connected 1-simplicial complex in which all the vertices, at most with the exception of two, have exactly two incident edges.

Note that a 2-simplex (i.e., a triangle) in a simplicial 3-complex is manifold when its link consists of one or two vertices. Since we restrict our attention to complexes embedded in the 3D Euclidean space, thus all 2-simplexes are manifold.

4.2 Manifold-Connected and Initial Quasi-Manifold Complexes

In this Subsection, we define two new classes of regular complexes. We consider simplicial d-complexes, where $d = 1, 2, 3$.

We consider a regular simplicial d-complex Σ embedded in the 3D Euclidean space, where $d = 1, 2, 3$. In such a complex, a $(d-1)$-simplex σ is a *manifold* simplex if and only if there are at most two d-simplexes in Σ incident in σ. An $(d-1)$-path (i.e., a path formed by alternating d- and $(d-1)$-simplexes) such that every $(d-1)$-simplex in the path is a manifold simplex is called a *manifold* $(d-1)$*-path*. Two d-simplexes in Σ are said to be *manifold-connected* if and only if there exists a manifold $(d-1)$-path connecting them. We define manifold-connected simplicial complexes and initial quasi-manifolds as follows:

- a regular simplicial d-complex Σ is *manifold-connected* if and only if any pair of d-simplexes in Σ are manifold-connected.
- A regular simplicial d-complex Σ is an *initial* d*-quasi-manifold (IQM)* if the star of each vertex in Σ is manifold-connected.

Figure 4(a) shows an example of a manifold-connected 2-complex. It is a 2D duster that is pinched at an edge. Figure 4(b) shows an example of an initial 2-quasi-manifold. Note that all its vertices and edges are manifold (as it is always the case in 2D). The tetrahedralized pinched pie in Figure 2 is an example of an initial 3-quasi-manifold, which has a non-manifold singularity.

4.3 Properties

In this Subsection, we discuss the properties of the classes of complexes we have introduced in the previous subsection and their relations with pseudo-manifold and manifold complexes (defined in the background section). We consider regular simplicial 2- and 3-complexes embedded in the 3D Euclidean space.

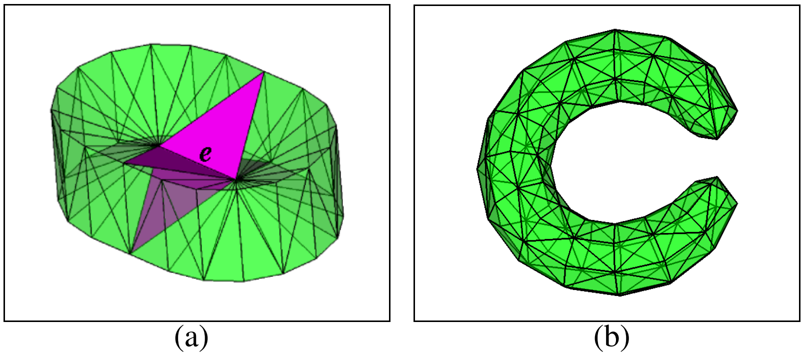


Fig. 4. (a) Example of a 2D manifold-connected complex: a 2D duster that is pinched at an edge. The four triangles incident at the non-manifold edge e are highlighted. (b) Example of a 2D initial quasi-manifold: It is a hollow tube.

We focus first on the two-dimensional case, and analyze the non-manifold singularities contained in the different 2-complexes. Manifold-connected 2-complexes may contain both non-manifold vertices and edges (see Figure 4(a)). On the contrary, a 2-pseudo-manifold cannot contain a non-manifold edge, since all the edges must be shared by one or two triangles, but it may contain non-manifold vertices. Figure 5 shows a hollow torus pinched at one vertex. It has one non-manifold vertex, v, but every pair of triangles on the torus is manifold-connected.

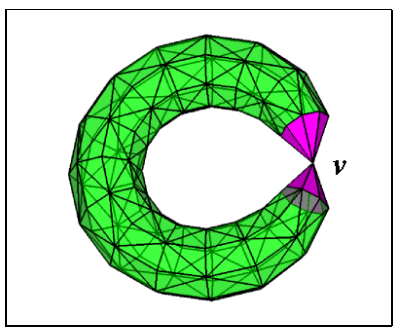


Fig. 5. Example of a 2-pseudo-manifold: It is a hollow torus pinched at one vertex. The neighborhood of the non-manifold vertex v is highlighted

Initial 2-quasi-manifolds and manifold 2-complexes cannot contain non-manifold singularities. Since the star of any vertex v in an initial 2-quasi-manifold is manifold-connected, the link of v must consist of a single manifold-connected 1-complex and thus it must be a graph in which each vertex has one or two incident edges. An initial 2-quasi-manifold cannot contain any non-manifold edge for the same reason.

Thus, we have the following relations among the various complexes in the 2D case:

- An initial 2-quasi-manifold is the same as a manifold 2-complex, since the condition on the star of any vertex in an initial 2-quasi-manifold is translates directly on the condition of the link being homeomorphic to the sphere or to the disk.
- An initial 2-quasi-manifold is a 2-pseudo-manifold (as a consequence of the above), but the reverse is not true because 2-pseudo-manifolds may contain non-

manifold vertices. Figure 5 shows an example of a 2-pseudo-manifold that is not an initial quasi-manifold because of the non-manifold vertex v.

- A 2-pseudo-manifold is a manifold-connected 2-complex, since it is 1-connected, and every edge is shared by at most two triangles, but the reverse is not true (see Figure 4(a)), which shows an example of a manifold-connected complex that is not a 2-pseudo-manifold because there are four triangles incident at the non-manifold edge at the center.

In summary, among all 2-complexes, the classes of manifold (M_2), initial 2-quasi-manifolds (IQM_2), 2-pseud-omanifolds ($PSDM_2$), and manifold-connected 2-complexes (MC_2) are related as follows:

$$M_2 = IQM_2 \subset PSDM_2 \subset MC_2$$

We focus here on the three-dimensional case. In this case, except for manifold complexes, all other 3-complexes, namely manifold-connected complexes, pseudo-manifolds and initial quasi-manifolds can contain both non-manifold vertices and non-manifold edges. We discuss here the relations among the different classes of complexes in the three-dimensional case.

Property 1 *Manifold simplicial* 3*-complexes are initial* 3*-quasi-manifolds.*

Proof: The link of any vertex v in a manifold 3-complex is a triangulation of the sphere or of the 2-disk, thus it is a manifold 2-complex, since all its edges are manifold. This implies that the star of vertex v is a manifold 3-complex, since it is bounded by $link(v)$.

The reverse is not true, that is an initial 3-quasi-manifold is not necessarily a manifold 3-complex. The solid pinched pie in Figure 2 is an example of an initial 3-quasi-manifold, which has a non-manifold singularity at vertex v.

Property 2 *Initial* 3*-quasi-manifolds are* 3*-pseudo-manifolds.*

Proof: We need to prove that an initial 3-quasi manifold is 2-connected. This is a consequence of the fact that an initial 3-quasi-manifold cannot contain a non-manifold edge.

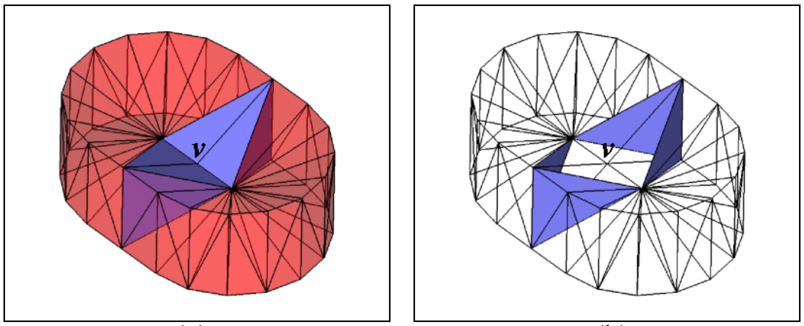
The reverse is not true since there exist 3-pseudo-manifolds which are not initial 3-quasi-manifolds, as shown in the example of Figure 6(a). The non-manifold vertex (shown in Figure 6(b)) between two non-manifold edges has a 0-connected link, which is not manifold-connected.

A 3-pseudo-manifold, that is not an initial quasi-manifold, may contain two types of non-manifold vertices that violate the initial quasi-manifold properties:

- vertices whose link consists of more than one connected component

- vertices whose links are 0-connected

In the latter case, the 3-pseudo-manifold contains non-manifold edges. This case is shown in the example of Figure 6. The first case is shown in the example of Figure 8(a). Figure 8(b) shows the star of the non-manifold vertex, which is connected only through the vertex itself (and thus the link is disconnected)



(a) (b)

Fig. 6. (a) A solid pinched duster with two non-manifold edges. The four tetrahedra incident at the non-manifold vertex v at the center of the pinched duster are highlighted; (b) The link of the non-manifold vertex v.

Unlike in the two-dimensional case, a manifold-connected 3-complex is a 3-pseudo-manifold and viceversa. The motivation is that a 2-connected complex embedded in the three-dimensional Euclidean space must be a 3-pseudo-manifold, since a triangle has to be in this case on the boundary of one or two tetrahedra.

In summary, among simplicial 3-complexes, the classes of manifold (M_3), initial quasi-manifolds (IQM_3), pseudo-manifolds ($PSDM_3$), and manifold-connected 3-complexes (MC_3) are related as follows:

$$M_3 \subset IQM_3 \subset PSDM_3 = MC_3$$

4.4 Relations between 3-complexes and their boundary complexes

In this Subsection, we discuss the relations between classes of regular simplicial 3-complexes embedded in E^3 and their boundary complexes.

Let Σ_3 be a regular simplicial 3-complex. The boundary of Σ_3, denoted by $\partial\Sigma_3$, is a regular simplicial 2-complex. It is well-known that the boundary of a manifold 3-complex with boundary is a manifold 2-complex [24].

If Σ_3 is an initial 3-quasi-manifold, but not a 3-manifold, then Σ_3 must contain non-manifold vertices whose stars are manifold-connected. Thus, $\partial\Sigma_3$ is not an initial 2-quasi-manifold (that means a 2-manifold in the two-dimensional case).

If the link of each non-manifold vertex is manifold, then $\partial\Sigma_3$ is in the class of

$PSDM_2 - IQM_2$, and thus has non-manifold vertices but no non-manifold edge. Otherwise, $\partial\Sigma_3$ is in the class of $MC_2 - PSDM_2$ due to the presence of non-manifold edges, which cause the links of non-manifold vertices in Σ_3 to be non-manifold.

The tetrahedralized pinched pie in Figure 2 is an example of an initial 3-quasi-manifold whose boundary is a 2-pseudo-manifold. In Figure 7 the solid duster with one non-manifold edge has a boundary that is manifold-connected but it is not a 2-pseudo-manifold (see Figure 4(a) which shows the boundary of the solid duster).

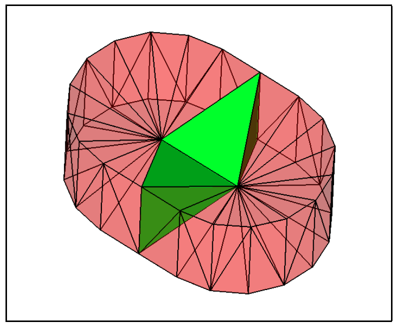
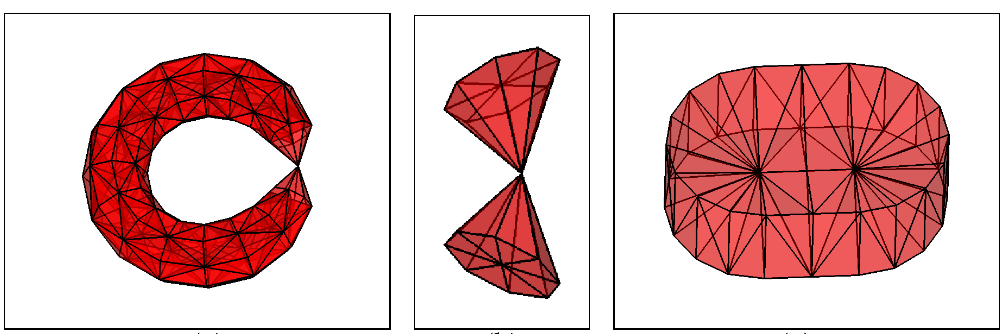


Fig. 7. A solid pinched duster with one non-manifold edge. The tetrahedra incident at the non-manifold edge are highlighted.

Suppose Σ_3 is a 3-pseudo-manifold, but not an initial 3-quasi-manifold. Then, Σ_3 has non-manifold vertices whose links are not manifold-connected. If the links of all non-manifold vertices of Σ_3 consist of disjoint manifold components, then $\partial\Sigma_3$ is a 2-pseudo-manifold (but not a manifold), since it does not contain any non-manifold edge. Figure 8(a) shows a solid torus pinched at one side, which is an example of 3-pseudo-manifold which is not an initial quasi-manifold. The link of its non-manifold vertex consists of two manifold disks of triangles shown in Figure 8(b), and thus $\partial\Sigma_3$ is in the class of $PSDM_2 - IQM_2$. If the non-manifold vertices of Σ_3 have 0-connected links, then $\partial\Sigma_3$ is in the class of $MC_2 - PSDM_2$. In Figure 8(c), the solid pinched duster with two non-manifold edges has a boundary in $MC_2 - PSDM_2$.



(a) (b) (c)

Fig. 8. (a) A solid torus pinched at one vertex; (b) the star of the non-manifold vertex of the pinched torus; (c) a solid pinched duster with two non-manifold edges

5 A Hierarchical Decomposition

In this Section, we discuss a hierarchical decomposition of a simplicial complex based on the basic simplicial shapes introduced in Section 4. The decomposition is hierarchical since it based on the degree of connectivity and on the presence of different non-manifold singularities in the resulting component. There are classes of complexes such as the manifold complexes which do not contain any manifold singularity, others like the 2-pseudo-manifolds that may contain only non-manifold vertices and also complexes like the 3-pseudo-manifolds, the initial quasi-manifold 3-complexes and the manifold connected 2-complexes which may contain both non-manifold vertices and edges.

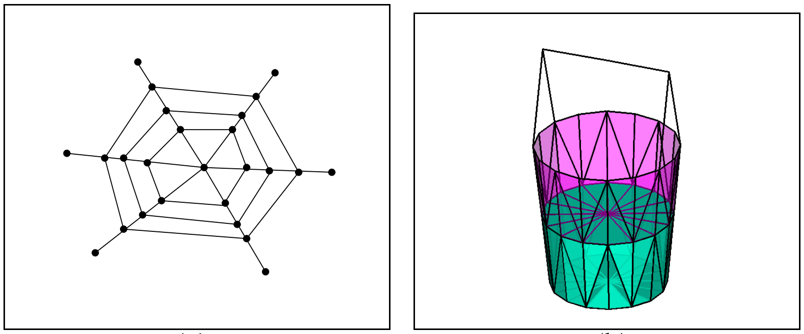
We are interested in the decomposition of non-manifold and non-regular shapes, discretized as simplicial 3-complexes embedded in the three-dimensional Euclidean space. We call the hierarchical decomposition the *basic decomposition*, since it is composed of basic topological shapes, and we denote it as $\mathcal{D}$. We call any complex arising from the hierarchical decomposition of a simplicial 3-complex Σ a *component* of the decomposition.

The first level of the basic decomposition, that we call the *01-decomposition*, consists of decomposing Σ into maximal 0-connected components composed only of top 1-simplexes (that we call *wire-edges*) and into maximal 1-connected components. We call the 0-connected one-dimensional components *wire-webs*. An example of a wire-web is shown in Figure 9(a). The remaining components in the 01-decomposition are not necessarily of uniform dimension, but may contain top simplexes which are triangles, or tetrahedra. Figure 9(b) shows a complex that describes a bucket half-filled with water. In the 01-decomposition, the handle of the bucket (i.e., the wire-web) is detached from the empty part of the bucket (which is formed by triangles alone) and the water-filled part (which is formed by tetrahedra).

The components in a 01-decomposition will share non-manifold vertices. The wire-webs are maintained through all levels of the basic decomposition. Note that such components are not necessarily manifold-connected, since a vertex in a wire-web can have several incident edges.

The second level in the basic decomposition $\mathcal{D}$ is a decomposition of the 1-connected components of the 01-decomposition into two-dimensional and three-dimensional regular components which are 1- and 2-manifold connected, respectively. We call such decomposition an *MC-decomposition*.

To define an MC-decomposition, we consider first a decomposition of the 1-connected components in the 01-decomposition into 2-connected 3-dimensional components and 1-connected 2-dimensional components. The 3-dimensional components are maximal 3-manifold-connected complexes composed of tetrahedra. Recall that in the 3D case are the same as 3-pseudo-manifolds, since 2-manifold-connectivity is

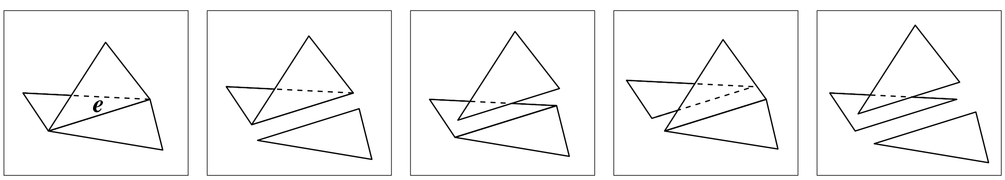


(a) (b)

Fig. 9. (a) Example of a wire-web; (b) A bucket with handle described by a wire-web, the empty upper half of the bucket is formed by triangles, and the water-filled part is formed by tetrahedra.

the same as 2-connectivity. The 2-dimensional components are composed only of triangles. The 3-components and the 2-components will share non-manifold edges and vertices.

The 1-connected 2-components can be further decomposed into manifold-connected components, since 1-connectivity does not imply manifold-connectivity in the two-dimensional case. Unfortunately, while a decomposition into 3-connected components is unique, a 1-connected 2-complex admits several possible decompositions into manifold-connected components. For instance, the 1-connected complex in Figure 10(a) can be decomposed into manifold-components in four different ways, as shown in Figures 10(b)-(e).



(a) (b) (c) (d) (e)

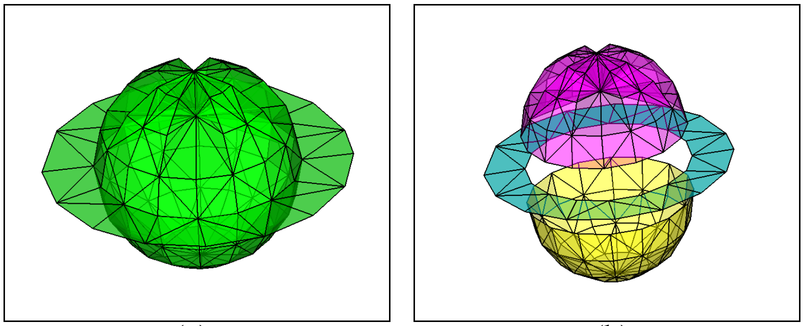
Fig. 10. An example of decompositions of 1-connected simplicial complex that remove the non-manifoldness at a non-manifold edge: (a) original 1-connected complex (b)-(e) decompositions that remove non-manifoldness at e.

Given a 1-connected 2-complex Σ', the MC-decomposition for Σ' consists of manifold-connected components Γ_i satisfying the following conditions:

- Any two components Γ_i and Γ_j can share only one or more non-manifold edge.
- If a component Γ_i contains a non-manifold edge e, all the triangles in $star(e) \cap \Gamma_i$ must be connected within Γ_i through a manifold-connected path which does not include edge e.

The above conditions translate the fact that we do not want to break the complex at manifold edges, but, while keeping this constraint, we want the most refined decomposition at the non-manifold edges into components, which are manifold-connected. In the example of Figure 10(a), we will just break the given 1-connected complex into three triangles as shown in Figure 10(e). This actually will lead to the most refined decomposition into manifold-connected components of a 1-connected simplicial 2-complex. And thus, this decomposition is unique.

Figure 11 shows another example of an MC-decomposition: a hollow ball that is pinched at the top and has a circular wing. The MC-decomposition partitions the complex into three manifold-connected components.



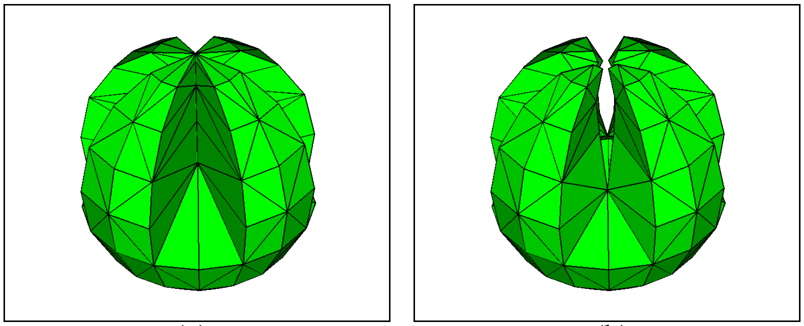
(a) (b)

Fig. 11. An example of an MC-decomposition: (a) A hollow ball that is pinched at the top and has a circular wing; (b) Its MC-decomposition into three manifold-connected components

The third level in the basic decomposition $\mathcal{D}$ consists of a decomposition of the original complex into wire-webs and into pseudo-manifold components, and it is called a *PM-decomposition*. Since a manifold-connected 3-complex is a 3-pseudo-manifold, then 3-dimensional components in the PM-decomposition are the same as the three-dimensional ones in the MC-decomposition. The 2-pseudo-manifold components are obtained from those manifold-connected 2-components in the MC-decomposition, which are not pseudo-manifolds, by duplicating all non-manifold edges.

The fourth level in the basic decomposition $\mathcal{D}$ consists of a decomposition of the original complex into wire-webs and into initial quasi-manifold components, and it is called an *IQM-decomposition*. Note that the two-dimensional components are manifold, while the three-dimensional ones may contain non-manifold vertices. The initial 2-quasi-manifolds are obtained from the 2-pseudo-manifolds in the PM-decomposition, by duplicating the non-manifold vertices. The initial 3-quasi-manifolds are obtained from the 3-pseudo-manifolds by duplicating the non-manifold edges and those non-manifold vertices whose stars are not manifold-connected. Note that the non-manifold vertices which can be present in an initial 3-quasi-manifold are only those which have a manifold-connected star.

Figure 12(a) shows a 2D manifold-connected component of the MC-decomposition. This component is a ball that is pinched on four sides, thus it contains non-manifold edges. Figure 12(b) shows an initial 2-quasi-manifold obtained after applying the PM-decomposition and the IQM-decomposition successively on the manifold-connected component.



(a) (b)

Fig. 12. An example of applying the PM-decomposition and the IQM-decomposition successively on a manifold-connected component of the MC-decomposition: (a) The original object is a hollow ball pinched at four sides; (b) the initial 2-quasi-manifold obtained after duplicating all the non-manifold edges and non-manifold vertices

Note that all decompositions are unique and, thus, the basic decomposition $\mathcal{D}$ is unique as well.

5.1 Computing the basic decomposition

In this Subsection, we present an algorithm for computing the basic decomposition of a simplicial 3-complex Σ.

The 01-decomposition is computed in two steps:

(1) the wire-webs are computed by extracting all the top 1-simplexes and computing the connected components of such set of simplexes;
(2) the other components (the 1-connected ones) are computed by traversing the hypergraph formed by the triangles and the edges of the complex obtained from Σ by eliminating all wire-webs

To compute the MC-decomposition, we observe that in the MC-decomposition of a simplicial complex Σ embedded E^3, any pair of manifold simplexes belonging to the same h-dimensional manifold-connected component (for $h = 2, 3$) must be connected through a manifold $(h-1)$-path. This means that every such component can be traversed by following the manifold $(h–1)$-paths connecting the h-simplexes in the component. The algorithm for computing the MC-decomposition performs the following steps for each 1-connected component Σ' in the 01-decomposition:

(1) Identify all the non-manifold edges and vertices in Σ' to ensure that these simplexes will not be visited in the traversal.
(2) For each unvisited h-dimensional top simplex $\sigma \in \Sigma$ (for $h = 2, 3$), find all other unvisited h-dimensional top simplexes that are reachable from σ by

alternately visiting manifold $(h-1)$-faces and their incident top h-simplexes. All such visited top h-simplexes belong to the same h-dimensional manifold-connected component.

(3) Mark each non-manifold singularity, that is shared by more than one manifold-connected component, as a joint.

The PM-decomposition is computed from the MC-decomposition by removing non-manifold edges from 2-manifold-connected components, which are not 2-pseudo-manifolds. The strategy for removing such edges is the following: for each non-manifold edge e in a manifold-connected 2-component, make one copy of e for each connected component in the link of e. The uniqueness of the PM-decomposition follows directly from the uniqueness of the MC-decomposition.

The IQM-decomposition is computed from the PM-decomposition in two steps:

(1) For the 2-components of the PM-decomposition, which contain non-manifold vertices, we remove non-manifold vertices through duplication. Each such vertex is duplicated in each disjoint component in its link.
(2) For the 3-components of the PM-decomposition, we duplicate first all non-manifold edges by inserting one copy for each connected component in their links. Then, we check whether the non-manifold vertices have a manifold-connected star. We remove the non-manifold vertices which do not satisfy such condition by duplicating them for each disjoint component in their links.

The uniqueness of the IQM-decomposition follows from the uniqueness of the PM-decomposition.

5.2 Our Implementation

We have implemented the basic hierarchical decomposition by using a compact dimension-independent data structure, we have developed, *Incidence Simplicial (IS) data structure* to encode the given complex [11]. The IS data structure is an incidence-based representation for simplicial complexes in arbitrary dimensions, that encodes all the simplexes in the complex and some relations among such simplexes. It has been developed specifically as a representation for finite element meshes, and for this reason it encodes all simplexes explicitly and uniquely. Compared with the incidence graph [18] commonly used to encode general simplicial complexes, the IS data structure has the same representation power, but it is much more compact.

For each p-dimensional simplex σ, the IS encodes all the $(p-1)$-simplexes on the boundary of σ, and one $(p+1)$-simplex for each connected component of the link of σ. The latter, that we call the partial co-boundary relations, are illustrated in Figures 13(a)-(c) for simplexes of dimensions 0, 1 and 2 in a simplicial 3-complex. For the example in Figure 13(a), the IS data structure encodes the two tetrahedra that are incident at triangle t. In Figure 13(b), there are four triangles incident at edge e, but two of them belong to the same component (i.e., the tetrahedron) that is incident at e. The IS data structure encodes one triangle for each connected component in the link of e. Figure 13(c) shows a vertex whose link consists of three components. The IS encodes one edge for each component of the linkof v. Notice that the complete co-boundary relation of a p-simplex σ consists of all the $(p+1)$-simplexes that are incidents at σ. The difference between the incidence graph and the IS data structure is that the latter encode only partial co-boundary relations, while the former encode the corresponding complete relations.

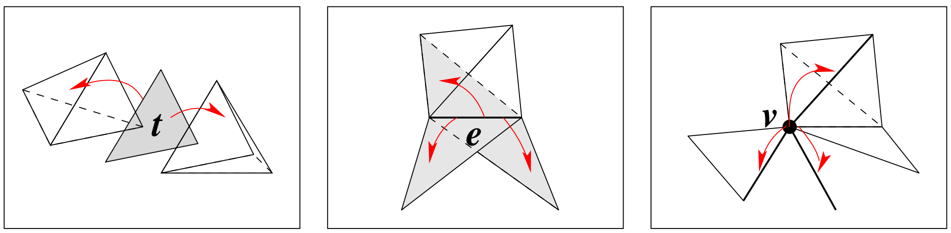


Fig. 13. An example of the topological partial co-boundary relations encoded by the IS for simplexes in a simplicial 3-complex: (a) The star of triangle t; (b) the star of edge e; (c) the star of vertex v

In what follows, we discuss the computation of the hierarchical decomposition of a simplicial complex Σ represented in an IS data structure.

The 01-decomposition is computed by examining the co-boundary relations of all the vertices of Σ. For each vertex v, that is connected to some wire-edges, one copy of v is created. All the wire-edges incident at v are transferred to the copy of v.

In the computation of the MC-decomposition, the non-manifold edges in the simplicial complex can be identified from their partial co-boundary relations. An edge is manifold if its partial co-boundary consists of either just one triangle, or two triangles that are top simplexes. Non-manifold vertices are detected as extreme vertices of non-manifold edges, or as vertices whose partial co-boundary relations consists of more than one edge. The p-manifold-connected components are traversed by exploring the boundary relations of the p-simplexes and the top simplexes in the partial co-boundary relations of the $(p-1)$-simplexes. The non-manifold edges and vertices are duplicated accordingly.

The PM-decomposition and the IQM-decomposition are computed by duplicating the appropriate non-manifold edges and vertices within each manifold-connected component. These operations involve only the star of the non-manifold simplex involved.

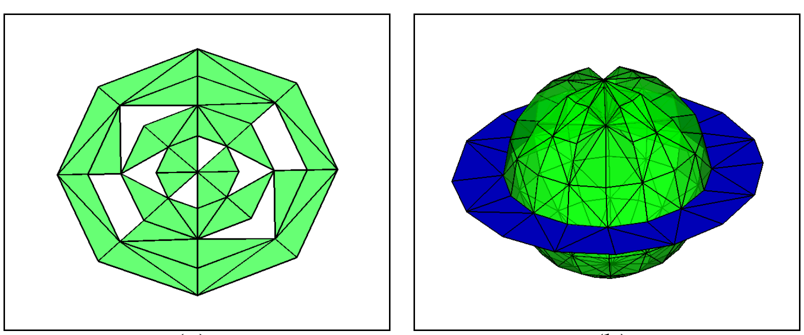
By using the IS data structure, the computation of the basic hierarchical decomposition has a time complexity that is linear in terms of the number of simplexes in the simplicial complex Σ.

6 A Semantic-Oriented Decomposition

In this Section, we discuss a decomposition of a non-manifold shape into three composite topological shapes which are of interest because of their richer semantics in several application scenarios. When we consider a simplicial 2-complex representing a non-manifold shape, we want to identify the parts of the shape which are 1-dimensional (the wire-webs we introduced before), parts that form sheets, that is 2-dimensional parts forming dangling surfaces, and parts that can enclose volumes. These latter will be connected set of triangles forming a closed surface. If we have a representation of the shape with solid parts (filled with tetrahedra) and, thus, a simplicial 3-complex, we need to distinguish between triangulated closed and open surfaces. This will produce a semantic-oriented decomposition which is computed based on the MC-decomposition, and is intended as a basis for analyzing the structure of a shape and for reasoning on shapes.

6.1 Components of a semantic-oriented decomposition

We consider the following composite shapes as constituent parts of a semantic-oriented decomposition of a shape, namely, the *wire-web*, introduced in Section 5, the *sheet* and the *2-cycle*. A *sheet* is a manifold-connected simplicial 2-complex with boundary. Figure 14(a) shows an example of a sheet. A simplicial 2-complex Σ_2 in E^3 induces a partition of E^3. A *2-cycle* C of Σ_2 is any 1-connected component of Σ_2 without boundary, such that the three-dimensional region in space enclosed by C is connected, that is, every two points in the region can be joined by a curve which does not intersect any simplex of Σ_2. Intuitively, a 2-cycle is a minimal sub-complex that completely encloses a three-dimensional region. Figure 5 shows a 2D torus pinched at one side, which is an example of a 2-cycle.



(a) (b)

Fig. 14. Examples of (a) a sheet and (b) a decomposition of a 2-complex into a 2-cycle (the hollow pinched ball) and a sheet (the circular wing).

Let us consider a simplicial 3-complex Σ and its MC-decomposition. We start from an MC-decomposition to define a semantic-oriented one. Wire-webs are already part of the MC-decomposition. We characterize sheets and 2-cycles in terms of components of the MC-decomposition. This is also the way we construct the semantic-oriented decomposition.

Each component in the MC-decomposition is a manifold-connected 2-complex with or without boundary, a wire-web or a 3-pseudo-manifold. Let Σ' be a manifold-connected 2-complex with boundary. The boundary of Σ' consists of either edges that are boundary edges in Σ, or non-manifold edges. If Σ' consists of an edge that is a boundary edge in Σ_2, then Σ' is a sheet in Σ.

For the sake of simplicity, let us consider first the MC-decomposition of a simplicial 2-complex Σ_2. A 2-cycle in Σ_2 is a manifold-connected component without boundary and with empty interior, or it is composed of a collections of manifold-connected 2-components with boundary, or is generated by collection of components which are enclosed in another component.

Note that each manifold-connected component in the MC-decomposition may belong to at most two 2-cycles. If a 2-cycle is formed by a collection of manifold-connected components with boundary, the boundary of manifold-connected components which form a 2-cycle must be formed only by non-manifold edges in the original complex Σ_2 (it cannot contain any boundary edge of Σ_2). Thus, the intersection of the manifold-connected 2-components forming a 2-cycle must be at a chain of non-manifold edges.

Figure 14(b) shows a hollow pinched ball with a circular wing. The cycle of non-manifold edges and non-manifold vertices connect the circular wing and the pinched ball. As a result, the non-manifold edges partition the 2-cycle into upper and lower parts, each of which is a manifold-connected component with boundary. The figure illustrates the decomposition of a 2-complex into a 2-cycle (the hollow pinched ball) and a sheet (the circular wing).

It is known that the triangles and the edges of a 2-cycle can be assigned a consistent orientation in such a way that, given one triangle and a specific orientation, all triangles on the same 2-cycle can be found incrementally by matching orientations [15]. There is a unique way to assign orientations consistently to all the 2-cycles in a simplicial 2-complex Σ_2 so that, if a triangle appears in two 2-cycles, its orientations in the two 2-cycles are opposite. Moreover, at a non-manifold edge e, two triangles t_1 and t_2, that belong to the same 2-cycle, induce two opposite orientations on e. In this way, t_1 and t_2 are successors of each other when they are sorted based on the orientation they induce around e [44]. Thus, given an MC-decomposition D of a simplicial 2-complex, we compute the 2-cycles from the components of D based on orientation.

We consider now a simplicial 3-complex and its MC-decomposition, and we address the properties of 2-cycles based on such decomposition.

We know that the boundary of a 3-pseudo-manifold component in the MC-decomposition of a 3-complex is a 2-cycle or a collection of 2-cycles. This is the consequence of the fact that the boundary of a 3-pseudo-manifold is a collection of manifold-connected 2-complexes without boundary.

Now let us consider how to identify all the 2-cycles in a simplicial 3-complex. In the MC-decomposition of a simplicial 3-complex Σ_3, the 3-pseudo-manifold components can be viewed as collection of 2-cycles that are tetrahedralized. Let Σ_2 be the 2-complex obtained from Σ_3 by removing all the tetrahedra forming any 3-pseudo-manifold component, but not the triangles bounding the component. The 2-cycles in Σ_2 are the boundaries of the 3-pseudo-manifolds plus the actual 2-cycles in Σ_3. Thus, the 2-cycles in a simplicial 3-complex Σ_3 can be found by computing the 2-cycles in the associated 2-complex Σ_2 followed by filtering those 2-cycles that bound 3-pseudo-manifolds in the MC-decomposition of Σ_3.

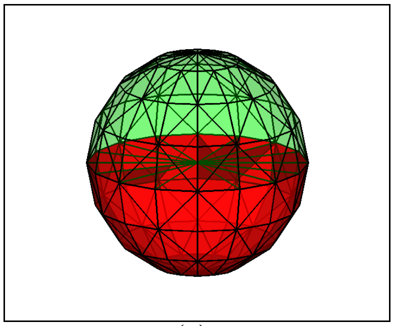
The example in Figure 15(a) shows a 3-complex Σ with an empty upper hemisphere and a tetrahedralized lower hemisphere. The 2-complex Σ' associated with Σ is computed by replacing the solid lower hemisphere with its boundary (shown in Figure 15(b)). The MC-decomposition of Σ' consists of three components: empty upper and lower hemispheres and a flat circular disc, as shown in Figure 15(c). The circular disc in the middle and the empty lower hemisphere belong to the boundary of the solid part of Σ. There are two 2-cycles in Σ', namely, the one formed by the upper empty hemisphere and the circular disc, and the other by the lower empty hemisphere and the circular disc. The latter 2-cycle is tetrahedralized in Σ.

6.2 *Computing a semantic-oriented decomposition*

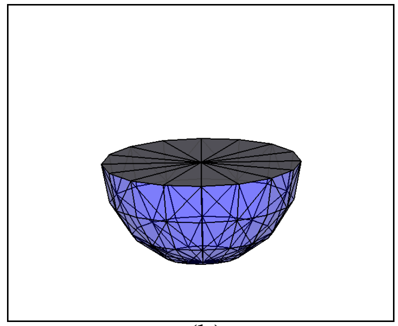
In this Subsection we present a topological decomposition for a complex at a level of richer semantics, with the wire-webs, sheets, the 2-cycles and the 3D pseudo-manifolds as components. We call this decomposition a *semantics-oriented decomposition*.

Consider a 3-simplical complex Σ embedded in E^3. The building blocks of the semantics-oriented decomposition are found from the MC-decomposition of Σ. Let D be the MC-decomposition of Σ. Wire-webs are collections of 0-connected one-dimensional top-simplexes and are not elaborated. The sheets in Σ are identified as the 2-components in D that do not belong to any 2-cycle in Σ. In what follows, we elaborate on the computation of the 2-cycles in Σ based on its MC-decomposition D.

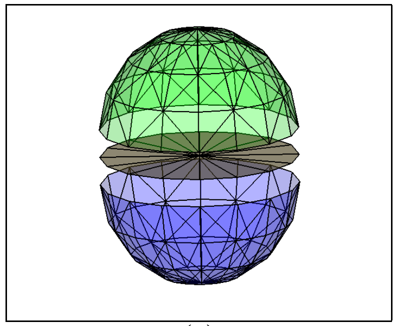
In the computation of the 2-cycles, if Σ is a 3-complex, it is reduced to a 2-complex first. This is how the 2-complex associated with Σ (defined in the previous section)



(a)



(b)



(c)

Fig. 15. Example of 2-cycles in a 3-complex: (a) a complex Σ with empty upper hemisphere and solid lower hemisphere; (b) the boundary of the lower hemisphere; (c) the MC-decomposition D of the reduced 2-complex Σ' of Σ consists of three components.

is computed: for each 3-pseudo-manifold C, its boundary ∂C is found by simply traversing all the triangles that belong to exactly one tetrahedron in C. The 2-complex Σ' associated with Σ is formed by replacing the 3-pseudo-manifold components by their boundaries, i.e., $\Sigma' = \Sigma - \bigcup C \cup \bigcup(\partial C)$.

The 2-cycles of Σ' are then extracted on the basis of the MC-decomposition D' of Σ'. The manifold-connected 2-components without boundaries are 2-cycles. Thus, we need to compute only those cycles which are formed by joining manifold-connected components with boundary. The basic idea here is like fixing a 3D jigsaw puzzle: the manifold-connected 2-components form the patches which will have to be pasted together through chains of non-manifold edges. Each patch may be shared by two 2-cycles. The algorithm for computing the 2-cycles performs the following basic steps:

(1) For each manifold-connected 2-component with boundary C in the MC-decomposition D', we create a duplicate C'. We then introduce a consistent orientation for every triangle of C, and an opposite orientation to all the triangles of C'. The orientation of the triangles of C induces an orientation to the boundary of C.
(2) Starting with any manifold-connected 2-component with boundary, we perform a depth-first search over all other components, and sew together the components whose oriented boundaries partially match each other.

Figure 16 illustrates the steps in the computation of 2-cycles. The object shown in Figure 16(a) consists of two boxes sharing a common face (for clarity, we do not show the faces subdivided into triangles and thus the simplicial complex). The MC-decomposition consists of three components, shown in Figure 16(b), with the boundary labeled. The components are duplicated, and orientations are assigned to each component and its duplicate as shown in Figure 16(c). Based on the orientation of the boundaries, pieces belonging to the same 2-cycle are matched, see Figure 16(d).

7 The Decomposition Graph

In this Section, we introduce a graph-based representation for topological decompositions of non-manifold two- and three-dimensional simplicial complexes. Such representation captures the complexity of the connectivity among the components and supports the extraction of interesting global topological features of the decomposed complex.

Consider a simplicial complex Σ with vertices V and embedded in E^3. Let $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ be the set of components in a decomposition $\mathcal{D}$ on Σ. If $\mathcal{D}$ is the MC-decomposition, a component C_i can be a wire-web, a manifold-connected 2- or 3-dimensional component. If $\mathcal{D}$ is the semantic-oriented decomposition, then a component C_i can be a wire-web, a sheet, a 2-cycle or a 3-pseudo-manifold.

The basic elements that describe the connections among components in a topological decompositions are the vertices. The simplest way to define the relations among the components in $\mathcal{C}$ would be consider all non-manifold vertices which are common to such components (disregarding those that are inside the wire-webs). Thus, the relations among the components would be described as a hypergraph in which the nodes are the components in $\mathcal{C}$ and the hyperarcs correspond to the non-manifold vertices. A hyperarc will thus connect all components sharing the corresponding non-manifold vertex. This graph-based representation is unambiguous, since any simplex in a simplicial complex is a set of vertices, but it does not convey information about the global connectivity among two ore more components.

Let $S_{ij}, i, j = 1, \cdots, k$ be the non-empty set of simplexes such that σ belongs to S_{ij} if and only if σ is shared by components C_i and C_j, and σ is not a proper face to some other simplex that is also common to C_i and C_j. That is, $S_{ij} = \{\sigma \in S_{ij} \mid \sigma \in C_i \cap C_j, \text{ and } \nexists\, \sigma' \in C_i \cap C_j \text{ where } \sigma \subset \sigma'\}$.

The set $\mathcal{S} = \{S_{ij} \in S \mid i, j = 1, \cdots, k \text{ and } S_{ij} \neq \emptyset\}$ consists of the simplexes shared between all pairs of components C_i and C_j in a given decomposition $\mathcal{D}$, and the simplexes in each S_{ij} are of maximal dimension, in the sense that they are not on the boundary of some other simplexes common to components C_i and C_j.

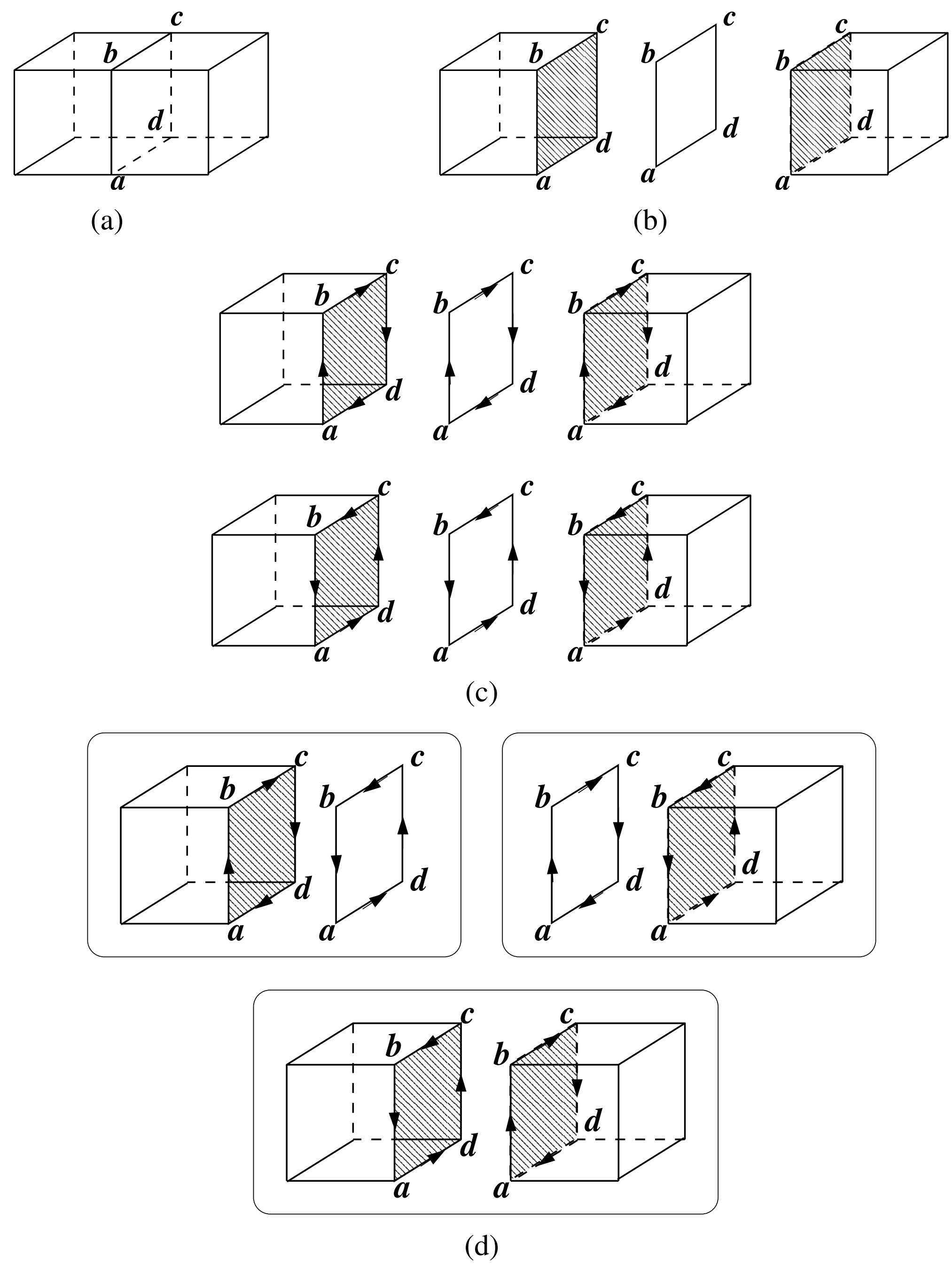


Fig. 16. Example to illustrate the computation of 2-cycles: (a) A 2D simplicial complex Σ (actually the triangles splitting the faces are not shown for the sake of clarity); (b) components of the MC-decomposition of Σ; (c) orientations assigned to each component and its duplication; (d) resulting 2-cycles

Let S denote the collection of all elements in the sets S_{ij}. We consider the partition of the simplexes in set S into a collection of sets $\mathcal{H}$ induced by $\mathcal{S}$. In this way, every H_k in $\mathcal{H}$ is formed by the collection of simplexes in S which belong to the same sets S_{ij} and every H_k is maximal. We denote with I_k the set of indexes of the components in $\mathcal{C}$ connected by the set of simplexes in H_k. These are the components generating the sets S_{ij} of which H_k is the intersection.

We consider a graph-based representation of the decomposition, that we call the *decomposition graph*, in which the nodes represent the components and the hyperarcs connect two or more components, as defined below. We consider the isolated vertices, the 0-connected and 1-connected components formed by the simplexes in each set H_k. Such components group the simplexes in H_k into hyperarcs. We can classify hyperarcs as follows (based on the connectivity):

- 0-hyperarcs consist only of one vertex
- 1-hyperarcs are graphs of edges (they are hyperarcs of type edge)
- 2-hyperarcs are manifold-connected collections of triangles

The sets H_k in $\mathcal{H}$ are thus a way of grouping the hyperarcs. We call such sets *macro-hyperarcs*.

The decomposition graph, when applied to the 01-decomposition, consists only of 0-hyperarcs, while, when applied to the MC-decomposition, it consists of 0- and 1-hyperarcs. 2-hyperarcs will be present only when we apply such representation to the semantic-oriented decomposition. A 2-hyperarc can be common only to two 2-cycles or to one 2-cycle and a 3-psueso-manifold. A 2-hyperarc cannot be shared by two or more 3-pseudo-manifolds.

The decomposition graphs for a PM-decomposition or for an IQM-decomposition are the same as the graph for the MC-decomposition from which they are obtained. Each component in the PM- or IQM-decomposition is an expansion of an MC-component obtained by duplicating edges and/or vertices. The non-manifold singularities in a component in both PM- or IQM-decompositions are represented as self-loops, the hyperacs describing the connections of a component with other components are the same as in the MC-decomposition and are described by 0- and 1-hyperarcs.

We illustrate the decomposition graph through some examples. Consider the semantics-oriented decomposition of the object in Figure 17(a). The object consists of two 2-cycles, C_1 and C_2. The connectivity between these two 2-cycles is shown in Figures 17(b) and (c). C_1 and C_2 share vertex v and the four edges e_1 to e_4. Thus there are two hyperarcs, a 1-hyperarc defined by the connected component $\{e_1, \cdots, e_4\}$ shown in Figure 17(b), a 0-hyperarc defined by the standalone vertex shown in Figure 17(c). The macro-hyperarc connecting them corresponds in this case to set $S_{\{1,2\}}$ Figure 17(d) shows the hypergraph with the macro-hyperarc connecting components C_1 and C_2. The components C_1 and C_2 are represented by circles. The macro-hyperarc is labeled as $S_{\{1,2\}}$. The two joints within $S_{\{1,2\}}$ are shown in the dash-lined bubbles. Note that the number of hyperarcs between the two components is related to the number 1-cycles in the object. In this case, the two components form a 1-cycle in the object.

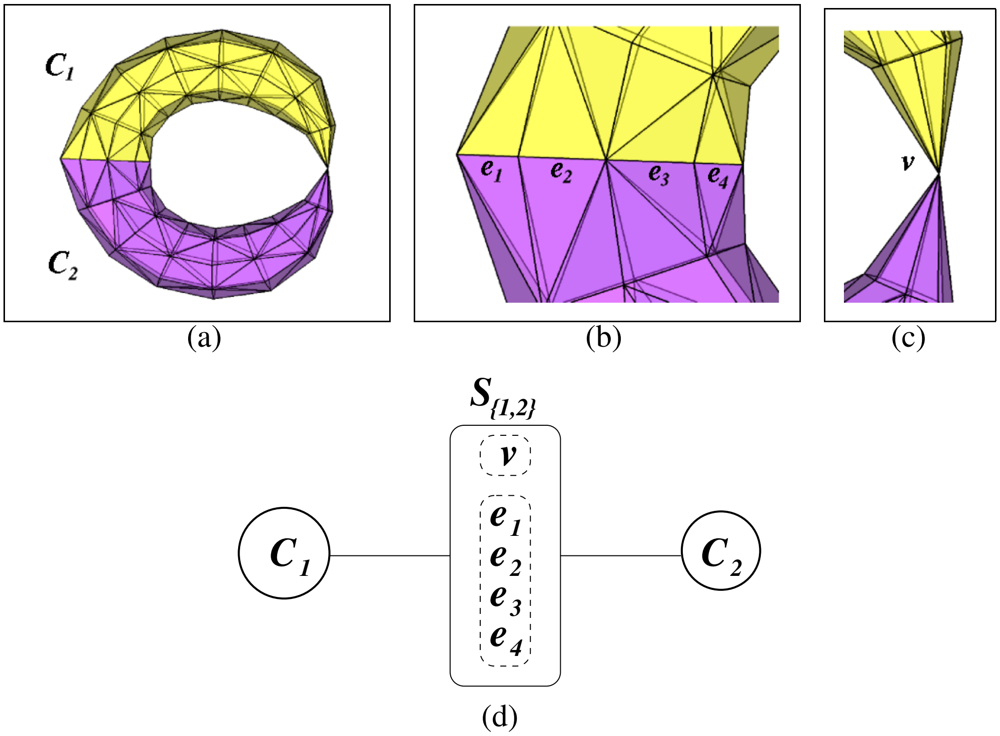


Fig. 17. An example showing the multiple connectivity between two 2-cycles: (a) the semantics-oriented decomposition consists of two 2-cycles C_1 and C_2; (b) connection through a chain of four edges e_1 to e_4; (c) connection at vertex v; (d) the hypergraph showing the macro-hyperarc with two hyperarcs (shown in dash-lined bubbles) connecting C_1 and C_2

Consider the example of Figure 18(a) which consists of two boxes sharing a common triangulated surface. In the semantics-oriented decomposition of this object, the two boxes are components C_1 and C_2. The set S_{12} consists of just one element which is the set of triangles $t_1, \cdots, t_4$ on the common surface shown in Figure 18(b). The connectivity is described by one 2-hyperarc corresponding to the triangles in $\{t_1, \cdots, t_4\}$.

Consider the example of the object in Figure 19(a). The semantics-oriented decomposition of this object consists of 2-cycles C_1 and C_2, which are the boxes, and sheet C_3, which is the surface. There are two macro-hyperarcs, each consisting of a hyperarc, here: the set $\{t_1, \cdots, t_4\}$ composed by the four triangles defining the common surface, and the set $\{e\}$.

8 Concluding Remarks

In this paper, we have presented topological tools for the structural analysis of three-dimensional non-manifold shapes. Our analysis is based on a topological decomposition of the simplicial complex discretizing the shape at two different lev-

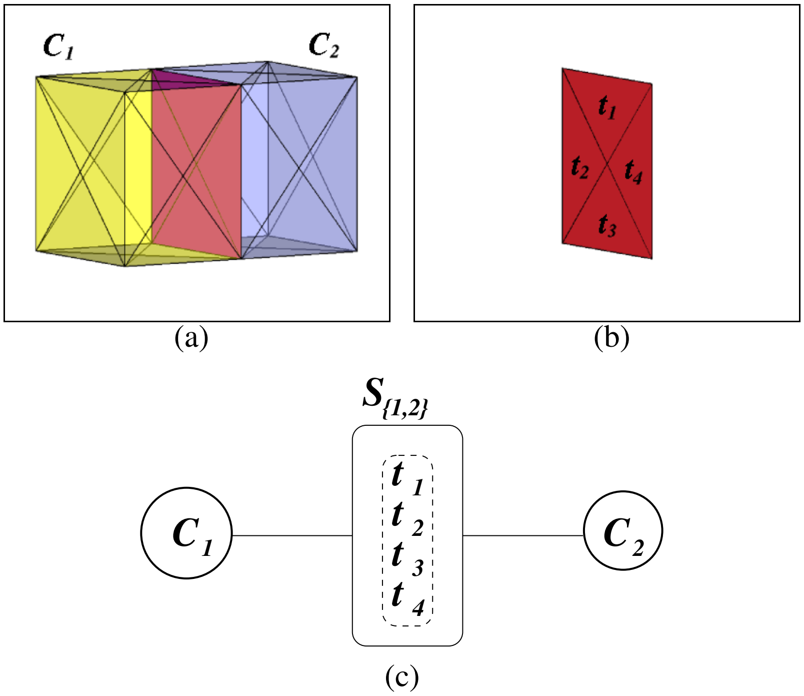


Fig. 18. An example showing a common surface between two 2-cycles: (a) the semantics-oriented decomposition consists of two 2-cycles, sharing a surface; (b) the common surface formed by triangles $t_1, \cdots, t_4$ shared by the two boxes; (c) the hypergraph showing hyperarc connecting C_1 and C_2

els. The first level in the decomposition, that we called the *basic decomposition*, consists of a hierarchy of topological decompositions based on simpler simplicial shapes, such as 1-connected, manifold-connected, pseudo-manifold and initial quasi-manifold components. The second level is a semantic-oriented decomposition into more complex shapes oriented to semantic annotation and reasoning. The semantic-oriented decomposition is built on top of the basic one. We have discussed the topological properties of the different components at each level, and we have described algorithms for computing such decompositions. We have investigated the relations among the components, and proposed a graph-based representation for such relations which can be used for describing both the basic hierarchy of decompositions and the semantic-oriented one.

The two-level decomposition is relevant in two applications, that we are considering in our research. The first one is in shape matching and retrieval. The uniqueness property of both the basic and semantic-oriented levels makes them very useful in such application. The semantic-oriented decomposition could be combined with with descriptions of the manifold parts, thus forming the basis for a two-level shape recognition process. Moreover, topological invariants can be easily computed from our semantic-based decomposition which will act as an effective topological shape signature.

The second application is in the context of finite element meshes generated from

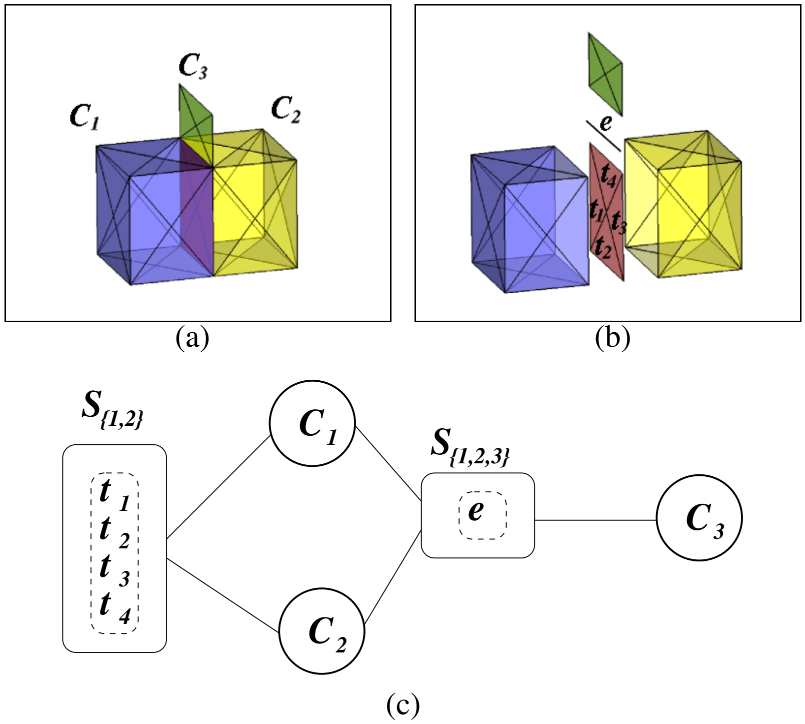


Fig. 19. An example showing the connectivity among two 2-cycles and one sheet: (a) In the semantics-oriented decomposition, two boxes form components C_1 and C_2, and the surface forms component C_3; (b) The two 2-cycles share a common surface formed by triangles $t_1, \cdots, t_4$. The edge e is shared by all the three components of the decomposition; (c) the hypergraph showing the two macro-hyperarcs

CAD models for simulation. In this case, one objective is to detect form features in a non-manifold shape, like protrusions, depressions, handle or through-holes, based on the structure of the single components and on the combinatorial structure of the decomposition. In [8], we have defined a taxonomy for form features in non-manifold shapes and we have proposed a first classification based on the mutual relations among a form feature and its adjacent ones. In our future work, we will apply the decomposition graph which describes the relations among the components in the semantic-oriented decomposition as the basis for form feature identification and classification.

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References

[1] M. Agoston. *Computer Graphics and Geometric Modelling*. Springer, 2005.

[2] N. Amenta, S. Choi, and R. Kolluri. The power crust. In *Proceedings 6th ACM Symposium on Solid Modeling*, pages 249–260. ACM Press, June 2001.

[3] S. Biasotti, B. Falcidieno, and M. Spagnuolo. Extended Reeb Graphs for surface understanding and description. In G. Borgefors and G. Sanniti di Baja, editors, *Proceedings of the* 9^{th} *Discrete Geometry for Computer Imagery Conference*, volume 1953 of *Lecture Notes in Computer Science*, pages 185–197, Uppsala, 2000. Springer Verlag.

[4] I. Biederman. Human image understanding: recent research and a theory. *Computer Vision, Graphics, and Image Processing*, 32:29–73, October 1985.

[5] T. O. Binford. Visual perception by computer. In *Proceedings IEEE Conference on Systems and Control 1971*. IEEE Computer Society, 1971.

[6] G. Borgefors, I. Nystroem, and G. Sanniti di Baja. Computing skeletons in three dimensions. *Pattern Recognition*, 32(3):1225–1236, September 1999.

[7] F. Cazals, F. Chazal, and T. Lewiner. Molecular shape analysis based upon the morse-smale complex and the connolly function. In *Proceedings of the nineteenth annual symposium on Computational geometry*, pages 351–360, New York, USA, 2003. ACM Press.

[8] C. Crovetto, L. De Floriani, and F. Giannini. A first classification of form features in non-manifold shapes. Technical report, University of Genova, Italy, November 2006.

[9] T. Culver, J. Keyser, and D. Manocha. Accurate computation of the medial axis of a polyhedron. In *Proc. of the fifth ACM Sympos. on Solid modeling and applications*, pages 179–190, Ann Arbor, Michigan, United States, 1999. ACM Press.

[10] L. De Floriani. Feature extraction from boundary models of solid objects. *IEEE Pattern Analysis and Machine Intelligence*, 11(9):785–798, August 1989.

[11] L. De Floriani and A. Hui. A compact dimension-independent data structure for non-manifold simplicial complex. Technical report, University of Maryland, College Park, October 2005.

[12] L. De Floriani, A. Hui, and L. Papaleo. Topology-based reasoning on non-manifold shapes. In *First International Symposium on Shapes and Semantics*, Japan, 17 June 2006.

[13] L. De Floriani, M. M. Mesmoudi, F. Morando, and E. Puppo. Non-manifold decompositions in arbitrary dimensions. *CVGIP: Graphical Models*, 65(1/3):2–22, 2003.

[14] L. De Floriani, F. Morando, and E. Puppo. Representation of non-manifold objects in arbitrary dimension through decomposition into nearly manifold parts. In *8th ACM Symposium on Solid Modeling and Applications*, pages 103–112. ACM Press, June 2003.

[15] CJ. A. Delfinado and H. Edelsbrunner. An incremental algorithm for betti numbers of simplicial complexes on the 3-sphere. *Computer Aided Geometric Design*, 12(7):771–784, 1995.

[16] H. Desaulnier and N. Stewart. An extension of manifold boundary representation to R-sets. *ACM Transactions on Graphics*, 11(1):40–60, 1992.

[17] T. K. Dey and W. Zhao. Approximate medial axis as a Voronoi subcomplex. In *Proc. 7th ACM Sympos. Solid Modeling Appl.*, pages 356–366, 2002.

[18] H. Edelsbrunner. *Algorithms in Combinatorial Geometry*. Springer Verlag, Berlin, 1987.

[19] B. Falcidieno and F. Giannini. Automatic recognition and representation of shape based features in a geometric modeling system. *Computer Vision, Graphics and Image Processing*, 48:93–123, 1989.

[20] B. Falcidieno and O. Ratto. Two-manifold cell-decomposition of R-sets. In A. Kilgour and L. Kjelldahl, editors, *Proceedings Computer Graphics Forum*, volume 11, pages 391–404, September 1992.

[21] A. Gueziec, G. Taubin, F. Lazarus, and W. Horn. Converting sets of polygons to manifold surfaces by cutting and stitching. In *Conference abstracts and applications: SIGGRAPH 98*, Computer Graphics, pages 245–245. ACM Press, 1998.

[22] M. Hilaga, Y. Shinagawa, T. Kohmura, and T. L. Kunii. Topology matching for fully automatic similarity estimation of 3D shapes. In *ACM Computer Graphics, (Proc. SIGGRAPH 2001)*, pages 203–212, Los Angeles, CA, August 2001. ACM Press.

[23] D. D. Hoffman and M. Singh. Salience of visual parts. *Cognition*, 63:29–78, 1997.

[24] C. Hoffmann. *Geometric and Solid Modeling: An Introduction*. Morgan Kaufmann, San Francisco, CA, USA.

[25] A. Hui, L Vaczlavik, and L. De Floriani. A decomposition-based representation for 3d simplicial complexes. In *Fourth Eurographics Symposium on Geometry Processing*, pages 101–110, Cagliari (Italy), June 2006. ACM Press.

[26] S. Joshi and T. C. Chang. Graph-based heuristics for recognition of machined features from 3d solid model. *Computer-Aided Design*, 20(2):58–66, 1988.

[27] S. Katz and A. Tal. Hierarchical mesh decomposition using fuzzy clustering and cuts. *ACM. Transactions on Graphics*, 22:954–961, July 2003.

[28] J. J. Koenderink and A. J. van Doorn. The shape of smooth objects and the way contours end. *Perception*, 11:129–137, 1982.

[29] S. H. Lee. A cad-cae integration approach using feature-based multi-resolution and multi-abstraction modelling techniques. *Computer Aided Design*, 37(9):941–955, August 2005.

[30] J-C. Leon and L. Fine. A new approach to the preparation of models for FE analyses. *International Journal of Computer Applications in Technology*,

23(2/3/4):166 – 184, 2005.

[31] A. Mangan and R. Whitaker. Partitioning 3D surface meshes using watershed segmentation. *IEEE Transaction on Visualization and Computer Graphics*, 5(4):308–321, 1999.

[32] M. Mortara, G. Patané, M. Spagnuolo, B. Falcidieno, and J. Rossignac. Blowing bubbles for multi-scale analysis and decomposition of triangle meshes. *Algorithmica*, 38(2):227–248, 2003.

[33] D. L. Page. *Part Decomposition of 3D Surfaces*. PhD thesis, 2003.

[34] P. Pentland. Perceptual organization and representation of natural form. *Artificial Intelligence*, 28:293–331, 1986.

[35] S. Pesco, G. Tavares, and H. Lopes. A stratification approach for modeling two-dimensional cell complexes. *Computers and Graphics*, 28:235–247, 2004.

[36] W. C. Regli and D. S. Nau. Building a general approach to feature recognition of material removal shape element volumes. In *Second ACM Symposium on Solid Modeling Foundations and CAD/CAM Applications*, pages 259–270, Montreal, Canada, May 1993. ACM Press.

[37] J. Rossignac and D. Cardoze. Matchmaker: manifold BReps for non-manifold R-sets. In W. F. Bronsvoort and D. C. Anderson, editors, *Proceedings Fifth Symposium on Solid Modeling and Applications*, pages 31–41. ACM Press, 9–11 June 1999.

[38] H. Sakurai and D. C. Gossard. Recognizing shape features in solid models. *IEEE Computer Graphics & Applications*, 10(5):22–32, 1990.

[39] J. J. Shah. Assessment of features technology. *Computer-Aided Design*, 23(5):331–343, 1991.

[40] A. Shamir. Segmentation and shape extraction of 3d boundary meshes. In *State-of-the-Art Report, Eurographics*, Vienna, September 2006.

[41] E. C. Sherbrooke, N. M. Patrikalakis, and E. Brisson. An algorithm for the medial axis transform of 3D polyhedral solids. *IEEE Transactions on Visualization and Computer Graphics*, 2:44–61, 1996.

[42] Y. Shinagawa and T. L. Kunii. Constructing a Reeb Graph automatically from cross sections. *IEEE Computer Graphics and Applications*, 11(6):44–51, nov 1991.

[43] R. K. Tapadia and M. R. Henderson. Using a feature-based model for automatic determination of assembly handling codes. *Computer & Graphics*, 14(2):251–262, 1990.

[44] K. Weiler. The radial-edge data structure: a topological representation for non-manifold geometric boundary modeling. In J. L. Encarnacao, M. J. Wozny, and H. W. McLaughlin, editors, *Geometric Modeling for CAD Applications: Selected and Expanded Papers from the IFIP WG5.2 Working Conference, Rensselaerville, NY, USA, 12-16 May 1986*, pages 3–36. Elsevier Science Publishers B. V. (North–Holland), Amsterdam, 1988. ISBN: 0444704167.

[45] P. R. Wilson and P. R. Pratt. A taxonomy of features for solid modeling. In M. J. Wozny, H. W. McLaughlin, and J. L. Encarnaao, editors, *Geomet-*

ric Modeling for CAD Applications - Proceedings IFIP WG5.2 Conference, pages 247–277. North Holland, Albany, NY, USA, May 1988.

[46] Y. Zhang, J. Paik, A. Kroschan, and M. A. Abidi. A simple and efficient algorithm for part decompositions of 3-D triangulated models based on curvature analysis. In *Proceedings International Conference on Image Processing*, volume 3, pages 273–276. IEEE Computer Society, 2002.