ABSTRACT

Title of dissertation: SELF-FORCE AND NOISE-KERNEL IN CURVED SPACE-TIME USING QUASI-LOCAL EXPANSION METHODS

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We find a quasi-local expansion for the tail term of the Green's function for a particle with scalar charge moving outside the event horizon of a black hole of mass M. To do that we use a WKB-like ansatz for the mode functions and we solve the resulted differential equation by iteration. We then sum the mode contributions using Plana sum rule. The fact that we find the tail term as an analytic expression is important. We then use our expressions to calculate the self-force exerted upon a particle of scalar charge that has been held at rest from infinite past to some time after which it moves on a general geodesic of the space-time. We perform this computation first for the radial path of a particle released from rest and then generalize the method for a particle launched on a general geodesic.

We then turn to computing the noise kernel. The problem we are primarily concerned with is that of a massless, conformally coupled scalar field in the optical Schwarzschild (the ultrastatic spacetime conformal to the Schwarzschild black hole). In contrast to previous work done on this topic, we keep the two points separate, and

as a result work with non-renormalized Wightman functions. We give an expression in terms of an expansion in coordinate separation and conclude with an outlook.

SELF-FORCE AND NOISE-KERNEL IN CURVED SPACE-TIME USING QUASI-LOCAL EXPANSION METHODS

by

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Dedication

I dedicate this to my father, **Seyed Hossein Eftekharzadeh** and my mother **Maryam Goudasiaei**, They stood by me in my darkest hours and they loved me and protected me above and beyond what was expected from them. They sacrificed whatever convenience that was possible for them so that I achieve my goals in life even when my views were diametrically opposed to theirs. May God bless them and take care of them, the way I could never do.

"If you assume that there is no hope, you guarantee that there will be no hope, if you assume that there is an instinct for freedom, there is a chance to change things, there is a chance for contributing to the making of a better world, that's your choice. "

- Noam Chomsky

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Introduction

1.1 Introduction to Radiation Reaction

1.1.1 Motivation and Background

The subject of effect of radiation emitted from a charged particle on the motion of that particle has long been a topic of interest amongst physicists [1, 3]. DeWitt and Breheme [4] studied the radiation reaction force and motion of particles in the curved space-time. By virtue of the principle of equivalence a particle always moves on a geodesic of space-time, such that a local observer will be unable to distinguish the effect of gravitational field from that of curvature on the motion of particle. However a charged particle is affected by a non-gravitational field and therefore subject to "external" force. It then follows that the charged particle will not move on a geodesic of space-time.

One of the predictions of the General Relativity theory is existence of gravitational radiation emitted by a particle of mass m while moving in curved space-time [6]. This raises new and interesting possibilities, for example now even a neutral particle's motion can be affected by the gravitational radiation reaction. There has been intense activity (for excellent reviews look at [8]) concentrated on finding the effect of the gravitational radiation reaction on the motion of particles in strong gravitational field. The launch of project LISA [7], a space-based interferometric gravitational wave observatory has given these efforts new impetus. Understanding the effect of GRR (Gravitational Radiation Reaction) will enable researchers to create accurate templates for detecting the signal and interpreting the obtained information from the spacecraft.

The initial attempts at calculating the self-force involved using conservation of energy. By computing the energy-momentum tensor of the radiation and obtaining the energy being taken away from the particle through radiation one can find the rate of change in momentum. In flat space-time this approach leads to Abraham-Dirac-Lorentz equation of motion. This approach however has certain limitations as it produces run-away solutions [9] . A general approach was developed later to find the equation of motion of particles in CST (curved space-time). These efforts were culminated in works by Mino, Sasaki and Tanaka [10] and also by Quinn and Wald [11]. Application of the general equation of motion, however, is not trivial and there has been intense activity in this regard. A significant amount of work has been devoted to using various approximation schemes [12, 13] (weak-field, post-newtonian, etc) for solving this problem. A second group of papers have dealt with solving the field equations by mode decomposition and regularization of each mode function [33].The Mode Sum Regularization Prescription has yet to be proven to be generally valid, even though it has been shown that in certain simple cases it provides the correct answer. The gravitational radiation reaction lends itself to one more approach. Since in this case the particle is assumed to carry no charge, the equivalence principle applies. This approach treats the motion of a particle as causing a perturbation in the space-time curvature. It is then possible to separate the effect of the particle on its own path and compute this effect perturbatively.

1.1.2 Motion of Particles in General Space-Time: Electric, Scalar and Gravitational cases

The Abraham-Lorentz-Dirac [3, 2] equation of motion for an electrically charged particle of charge q in flat space-time is often written as:

$$ma^{\mu} = f^{\mu}_{ext} + \frac{2}{3}q^2 \left(\delta^{\mu}_{\nu} + u^{\mu}u_{\nu}\right) \frac{da^{\mu}}{d\tau} \tag{1.1}$$

It has been observed that the equation of motion has run away solutions and remedies have been proposed [9, 14] . This equation of motion was extended to the case of CST [4, 5].

Assuming that metric is denoted by $g_{\alpha\beta}$ and Ricci tensor by R^{α}_{β} the field equation for potential A^{α} is

$$\Box A^{\alpha} - R^{\alpha}_{\beta} A^{\beta} = -j^{\alpha} \tag{1.2}$$

where $\Box$ is the Laplace-Beltrami operator and R^{α}_{β} the Ricci tensor. For j^{α}, the covariant current vector we have:

$$j^{\alpha}(x) = q \int_{-\infty}^{\infty} \frac{dz^{\alpha}(\tau)}{d\tau} \delta^4\Big(x, z(\tau)\Big) \left(-g\right)^{\frac{1}{2}} d\tau \tag{1.3}$$

where $z^{\alpha}(\tau)$ describes the position of moving particle of charge q. The retarded solution to the above equation 1.2 can be written as :

$$A^{\alpha}(x) = q \int G^{\alpha}_{\beta\, R}(x, z(\tau)) u^{\beta} d\tau \tag{1.4}$$

where $G^{\alpha}_{\beta\,R}(x,x')$ is the retarded Green's function , $u^{\beta}=\frac{dz^{\beta}(\tau)}{d\tau}$ the four-velocity and τ the proper time.

The generalized equation of motion in CST is written as

$$ma^{\mu}=f^{\mu}_{ext}+\frac{2}{3}q^{2}\left(\delta^{\mu}_{\nu}+u^{\mu}u_{\nu}\right)\left(\frac{D\,a^{\nu}}{d\tau}+\frac{1}{2}R^{\nu}_{\alpha}u^{\alpha}\right)+q^{2}u_{\alpha}\int_{-\infty}^{\tau^{-}}u^{\beta'}\nabla^{\alpha}G^{\mu}_{\beta'\,R}\Big(z(\tau),z(\tau')\Big)d\tau' \tag{1.5}$$

where $u^{\beta'}=dz(\tau')/d\tau'$ and $\tau^{-}=\tau-\epsilon$ with ϵ being infinitesimal so that the divergence of the Green's function is avoided.

If a particle has scalar charge then the potential is described by the following equation:

$$\left(\Box-\xi R\right)\Phi=-\rho \tag{1.6}$$

where R is the Ricci scalar and ξ identifies the coupling (in this thesis we mostly adopt $\xi=0$ as zero coupling). We also have:

$$\rho(x)=q\int_{-\infty}^{\infty}\delta^{4}\Big(x,z(\tau)\Big)\,(-g)^{\frac{1}{2}}d\tau \tag{1.7}$$

The solution to the scalar equation can be written using the scalar Retarded Green's function as

$$\Phi(x)=q\int G_{R}\Big(x,z(\tau)\Big)d\tau \tag{1.8}$$

The equation of motion for a particle of mass m and scalar charge q has been found as [20]

$$a^{\mu}=\frac{1}{m}\,f^{\mu}_{ext}+\frac{1}{3}q^{2}\left(\delta^{\mu}_{\nu}+u^{\mu}u_{\nu}\right)\left(\frac{D\,a^{\nu}}{d\tau}+\frac{1}{2}R^{\nu}_{\alpha}u^{\alpha}+3\int_{-\infty}^{\tau^{-}}\nabla^{\nu}G_{R}\Big(z(\tau),z(\tau')\Big)d\tau'\right) \tag{1.9}$$

It was noted that since a scalar charge has a potential field of spin zero, unlike the electromagnetic case it can radiate its mass away and so the mass is not constant. The equation governing the variation of mass was found [20] (extended to non-zero coupling in [8]) to be:

$$\frac{dm}{d\tau} = -q^2 \left(\frac{1}{12} (1 - 6\xi) R + u_\mu \int_{-\infty}^{\tau^-} \nabla^\mu G_R\Big(z(\tau), z(\tau')\Big) d\tau' \right) \tag{1.10}$$

As it was already discussed even when the particle is neutral its motion in a curved space-time can cause gravitational radiation to be emitted therefore the effect of the gravitational radiation reaction on particle's motion must be accounted for.

We assume that the particle is moving in a background space-time characterized by the metric $g_{\mu\nu}$. The presence of such particle is going to cause a perturbation in the back-ground space-time, which can be denoted by $\delta g_{\mu\nu}$. The "gravitational potential" produced by a particle of mass m can be then described by the trace-reversed potential $h_{\alpha\beta}$:

$$h_{\alpha\beta} = \delta g_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \left(g^{\mu\nu} \delta g_{\mu\nu} \right) \tag{1.11}$$

the potential satisfies the field equation

$$\Box h^{\alpha\beta} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} h^{\mu\nu} = -4T^{\mu\nu} \tag{1.12}$$

where the Riemann tensor $R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}$ is calculated with respect to the background metric and the harmonic gauge $h^{\alpha\beta}{}_{;\beta} = 0$ is considered. The solution can be formally written as

$$h^{\alpha\beta} = 4m \int G^{\alpha\beta}_{R\mu\nu} u^\mu u^\nu d\tau \tag{1.13}$$

The equation of motion for such particle has been found to be [10, 11]

$$
\begin{aligned}
a^{\mu} = \quad & -2\, m \left(g^{\mu\nu} + u^{\mu}u^{\nu}\right) u^{\lambda}\, u^{\rho} \times \\
& \int_{-\infty}^{\tau-} d\tau' u^{\alpha'} u^{\beta'} \Big[\quad 2\nabla_{\rho} G_{\nu\lambda\alpha'\beta'}(x,x') - \nabla_{\nu} G_{\lambda\rho\alpha'\beta'}(x,x') \\
& \qquad\qquad - \left(g_{\nu\lambda}\nabla_{\rho} - \frac{1}{2} g_{\lambda\rho}\nabla_{\nu} \right) G^{\sigma}_{\sigma\alpha'\beta'}(x,x') \Big]_{x=z(\tau),x'=z(\tau')}
\end{aligned}
\tag{1.14}
$$

f It is important to note that the above equation of motion is not gauge invariant. This is because the equation has been obtained using a specific gauge, namely harmonic gauge.It has been shown [15] that under a gauge transformation a new gauge dependent acceleration appears. Therefore one should be cautioned that the gravitational equation of motion in and out of itself, can not provide answer to physical questions.

1.1.3 The Field and The Green's Function in CST: Massless scalar field with minimal coupling

The single most important ingredient of the above equations is the Retarded Green function. Consider, for example the massless scalar field equation with minimal coupling

$$\Box \Phi = -\rho \tag{1.15}$$

The Green's function is formally a solution of the following equation

$$\Box G(x,x') = -\delta^{4}(x,x') \tag{1.16}$$

By solving this equation it can be shown that in Minkowski FST(flat space-time), there are two linearly independent solutions to this equation, the Retarded and Advanced Green's functions, given as

$$G_R = \theta(t-t')\left(\frac{\delta(t-t'+|\mathbf{x}-\mathbf{x}'|)}{4\pi|\mathbf{x}-\mathbf{x}'|} - \frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{4\pi|\mathbf{x}-\mathbf{x}'|}\right) \tag{1.17}$$

$$G_A = \theta(t'-t)\left(\frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{4\pi|\mathbf{x}-\mathbf{x}'|} - \frac{\delta(t-t'+|\mathbf{x}-\mathbf{x}'|)}{4\pi|\mathbf{x}-\mathbf{x}'|}\right) \tag{1.18}$$

One can see that $G_R = -\theta(t-t')G$ and $G_A = \theta(t'-t)G$ with G (also known as Pauli-Jordan or Schwinger function)

$$G(x-x') = \frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{4\pi|\mathbf{x}-\mathbf{x}'|} - \frac{\delta(t-t'+|\mathbf{x}-\mathbf{x}'|)}{4\pi|\mathbf{x}-\mathbf{x}'|} \tag{1.19}$$

Using the mode decomposition one can separate the positive frequency modes from the negative ones and write $G(x-x')$ as

$$iG(x,x') = G^+(x,x') - G^-(x,x') \tag{1.20}$$

The positive and negative frequency functions are known as Wightman functions and in FST are given by

$$G^+(x,x') = -\frac{1}{4\pi^2}\frac{1}{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2} + \frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)-\delta(t-t'+|\mathbf{x}-\mathbf{x}'|)}{8\pi i|\mathbf{x}-\mathbf{x}'|} \tag{1.21}$$

$$G^-(x,x') = -\frac{1}{4\pi^2}\frac{1}{(t-t')^2-|\mathbf{x}-\mathbf{x}'|^2} - \frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)-\delta(t-t'+|\mathbf{x}-\mathbf{x}'|)}{8\pi i|\mathbf{x}-\mathbf{x}'|} \tag{1.22}$$

From these one can calculate Hadamard's elementary function defined by

$$G^{(1)}(x,x') = G^+(x,x') + G^-(x,x') \tag{1.23}$$

and in FST can be shown to be equal to

$$G^{(1)}(x,x') = \frac{1}{2\pi^2}\frac{1}{|\mathbf{x}-\mathbf{x}'|^2-(t-t')^2} = -\frac{1}{4\pi^2\sigma} \tag{1.24}$$

σ is half of the square of the distance between the two points denoted by x and x' and in FST is equal to $\sigma = \frac{1}{2}(x-x')^2 = \frac{1}{2}\eta_{\alpha\beta}(x^\alpha - x'^\alpha)(x^\beta - x'^\beta)$.

The definition of $\sigma(x,x')$ also known as Synge's World Function [17] will be generalized to the case of curved space time (CST) in the future sections. Another function of interest is Feynman propagator defined as,

$$iG_F = \theta(t-t')G^+(x,x') + \theta(t'-t)G^-(x,x') \tag{1.25}$$

and also of importance is the $\bar{G}$ which is defined as half of the retarded plus half of advanced.

$$\bar{G} = \frac{1}{2}(G_R + G_A) \tag{1.26}$$

It then can be seen that

$$-G_F = \bar{G} + i\,\frac{1}{2}\,G^{(1)} \tag{1.27}$$

or equivalently

$$G^{(1)} \quad = \quad 2\,Im(-G_F) \tag{1.28}$$

There is extensive literature on the Green's function, the above is only for future reference for the purpose of this thesis.

1.1.4 Generalization to Curved Space-Time (CST)

Some of the above notions can be easily generalized to CST. The Green's function for the scalar field will be a solution of the field equation

$$\Box G(x,x') = -\frac{\delta^4(x,x')}{\sqrt{-g}} \tag{1.29}$$

Hadamard [18] ansatz for the Retarded and Advanced Green's function (sometimes called Hadamard Elementary solution) is given by

$$G^{\pm}(x,x') = \frac{u(x,x')}{4\pi}\delta_{\pm}(\sigma) + \frac{v(x,x')}{8\pi}\theta_{\pm}(-\sigma) \tag{1.30}$$

where σ is again, Synge's world function and $\delta_+(\sigma)$ ($\delta_-(\sigma)$) is defined such as it is equal to $\delta(\sigma)$ if x is in the future (past) of a space-like surface on which x' resides and zero otherwise and also similarly $\theta_+(-\sigma)$ $(\theta_-(-\sigma))$ is one if x is in the future (past) of the space-like surface and zero otherwise. Consequently $\delta(\sigma) = \delta_+(\sigma) + \delta_-(\sigma)$ and also $\theta(-\sigma) = \theta_+(-\sigma) + \theta_-(-\sigma)$. It then follows that the Hadamard ansatz for $\bar{G}$ is

$$\bar{G}(x,x') = \frac{u(x,x')}{4\pi^2}\delta(\sigma) + \frac{v(x,x')}{4\pi^2}\theta(\sigma) \tag{1.31}$$

In FST $v(x,x')$ vanishes and therefore the integral part in equations (1.5),(1.9),(1.10) and (1.14) will disappear and in the electrical or scalar case we are back to flat space results and the gravitational result becomes trivial.

The existence of the so called "tail term" $v(x,x')$ makes the acceleration of particle at any point in the space-time dependent on its entire past history.

By substituting the Hadamard ansatz into the differential equation we can find that

[60, 4]

$$u(x, x') = \Delta^{1/2}(x, x') \tag{1.32}$$

where $\Delta(x, x')$ is called Van Vleck-Morrett determinant defined by

$$\Delta(x, x') = -\frac{det[\partial_\alpha \partial_\beta \, \sigma(x, x')]}{\sqrt{g(x)g(x')}} \tag{1.33}$$

It can then be shown that

$$\Box v(x, x') = 0 \tag{1.34}$$

and

$$v(x, x) = -\frac{1}{6}R(x) \tag{1.35}$$

where $R(x)$ is the Ricci scalar.

1.1.5 Self-Force using Quasi-Local expansion

Computation of the self-force for a point particle in orbit around a black hole is a topic of active research today (see [58] and references there in) prompted by the preparation of gravitational wave detectors such as LISA which are capable of detecting gravitational waves emitted when a compact object falls into a supermassive black hole [22, 23].

An exact expression for the self-force in a black hole spacetime has been obtained only for the cases of a scalar or electric charge held at rest in a Schwarzschild spacetime [24, 25, 26, 27] and an electric charge held at rest on the symmetry axis of a Kerr spacetime [28, 29]. Approximate analytical expressions have been obtained using Green's functions for scalar, electric, and gravitational charges in various weak field limits [30, 31, 32]. Most other calculations have involved the use of mode sum techniques [33, 34] in cases of high symmetry such as a static charge [35, 36, 37], radial infall [38, 39], a circular orbit [40, 41, 42, 43], or a slightly elliptical orbit [42].

Since the mode-sum regularization procedure has been developed extensively, aiming at practical calculations, it is important to find independent ways to check its accuracy and reliability. This of course also applies to any other method, numerical or analytical that might be developed in the future. To date the checks on various mode-sum regularization procedures which we are aware of fall into three categories: 1) comparisons with exact analytical results, 2) comparisons with analytical approximations, 3) comparisons of the results of one mode-sum regularization technique with those of another. Most of the comparisons that have been made fall into the third category [45, 43, 42, 41, 39, 44]. Comparisons with exact analytical results are of course the most reliable but also the most limited. They have only been done in the case of static scalar and electric charges held at rest in a Schwarzschild spacetime [35]. Comparison with an analytic approximation in the weak field limit has been made for a static scalar charge at rest in an axisymmetric spacetime [37]. In this paper we present results which can be used as a check in the second category. However, unlike most previous analytical approximations to the self-force, ours is valid in the strong field as well as the weak field limit. Specifically, we consider the case of a particle with scalar charge which is held at rest until a time $t = 0$ and subsequently falls radially towards a Schwarzschild black hole.

1.2 Introduction to Noise Kernel

Consider the simplest form of the semi-classical general relativity theory

$$G_{ab} = \kappa T^q_{ab} \tag{1.36}$$

where G_{ab} is the Einstein tensor and T^q_{ab} is the energy momentum tensor derived from and due to the quantum field. κ depends on the choice of units.

The question of how a quantum field contributes to space-time curvature has been answered by defining T^q_{ab} as the vacuum expectation value of the energy-momentum operator valued tensor of a quantum field. That is $T^q_{ab} = \langle \hat{T}_{ab} \rangle_{VEV}$. Now assume, we are dealing with a "weak" quantum field in the sense that its presence distorts the space-time to the extent that the difference from the classical vacuum can be dealt with perturbatively, with acceptable accuracy. To proceed we assume that the full "quantized" metric is composed of a fully classical part and a fully quantized part that is:

$$\hat{g}_{ab} = g_{ab}\hat{I} + \hat{h}_{ab} \tag{1.37}$$

Consistent with our initial assumptions, we further assume that g_{ab} satisfies the semi-classical field equations and $\hat{h}_{ab}$ satisfies the quantized general relativity field equations, that is

$$G_{ab} = \kappa \langle \hat{T}_{ab} \rangle_{VEV} \tag{1.38}$$

$$\Box \hat{h}_{ab} = \kappa \left(\hat{T}_{ab} - \hat{I} \langle \hat{T}_{ab} \rangle \right) \tag{1.39}$$

We must note that the above equations are ill-defined as it is well known that $\langle \hat{T} \rangle_{VEV}$ diverges and renormalization procedures have been applied [16]. We focus on the second equation and further consider that an operator valued metric has yet to be a well-defined quantity. Nevertheless a formal solution to Eq. (1.39) can be given as

$$\hat{h}_{ab}(x) = \int G_{a\,b\,c'\,d'}(x,x')\hat{t}^{c'\,d'}(x')d^4x' \tag{1.40}$$

where $\hat{t}^{a\,b}(x) = \hat{T}^{ab}(x) - \hat{I}\langle\hat{T}^{ab}(x)\rangle$.

We may ask, what is the closest non-operator valued quantity that could re-produce the quantum effect to the lowest non-vanishing approximation? One could answer this question by considering quantum fluctuations and attempting to re-produce quantum fluctuations within the work frame of stochastic mechanics. To proceed, we note that,

$$\langle\hat{h}_{ab}(x)\rangle_q = \int G_{a\,b\,c'\,d'}(x,x')\langle\hat{t}^{c'\,d'}(x')\rangle_q d^4x' = 0 \tag{1.41}$$

where the $\langle\,\rangle_q$ notation indicates quantum vacuum expectation value of the symmetrized product. At the next order we can look at auto-correlation of metric quantum perturbations. Here we have to be careful since we are dealing with operator valued quantities that may or may not commute with each other when evaluated at different space-time points and multiplied. Bearing this on mind we have,

$$\frac{1}{2}\langle\{\hat{h}_{ab}(x)\hat{h}_{c'd'}(y')\}\rangle_q = \int G_{a\,b\,p\,q}(x,y)\,G_{c'\,d'\,p'\,q'}(x',y')\frac{1}{2}\langle\{\hat{t}^{p\,q}(y)\hat{t}^{p'\,q'}(y')\}\rangle_q d^4y d^4y' \tag{1.42}$$

One can investigate the possibility of replacing the quantum fluctuation operators with stochastic functions in the hope of reproducing the above mentioned expectation values. A complete answer has been given by ref. [65] where the authors have used influence functional methods to show that a stochastic source can reproduce the auto-correlation quantity above. Consequently we introduce the quantum-stochastic tensor ${T_{ab}}^{qs}$ such that,

$$T_{ab}^{qs} \equiv \langle T_{ab} \rangle_q + T_{ab}^s \tag{1.43}$$

where $\langle T_{ab} \rangle_q$ identifies the mean value of the energy momentum tensor and T_{ab}^s is a stochastically fluctuating tensor that identifies the quantum fluctuations around the mean. In semi-classical stochastic gravity the noise kernel extends the role of VEV of energy momentum tensor via the Einstein-Langevin equation [67, 68, 69]:

$$G_{ab}[g] + \Lambda g_{ab} = 8\pi G({T_{ab}}^c + {T_{ab}}^{qs}) \tag{1.44}$$

where G_{ab} is the Einstein tensor Λ, G are the cosmological and Newton constants respectively. We use the superscripts c, s, q to denote classical, stochastic and quantum respectively. As it was noted above, the new term $T_{ab}^s = 2\tau_{ab}$ which is of classical stochastic nature measures the fluctuations of the energy momentum tensor of the quantum field. One defines $\hat{t}_{ab}$ to be,

$$\hat{t}_{ab}(x) \equiv \hat{T}_{ab}(x) - \langle \hat{T}_{ab}(x) \rangle \hat{I} \tag{1.45}$$

which is a tensor operator measuring the deviations from the mean of the stress energy tensor. We are interested in the correlation of these operators at different space-time points. Here we focus on the stress energy (operator-valued) bi-tensor $\hat{t}_{ab}(x)\hat{t}_{c'd'}(y)$ defined at nearby points (x, y). A bi-tensor is a geometric object that has support at two separate spacetime points. In particular, it is a rank two tensor in the tangent space at x (with unprimed indices) and also in the tangent space at y (with primed indices).

The noise kernel $N_{abc'd'}$ bi-tensor is defined as

$$4N_{abc'd'}(x,y) \equiv \frac{1}{2}\langle\{\hat{t}_{ab}(x), \hat{t}_{c'd'}(y)\}\rangle \tag{1.46}$$

where again $\{\}$ means taking the symmetric product.

The noise kernel defines a real classical Gaussian stochastic symmetric tensor field τ_{ab} which is characterized to lowest order by the following relations:

$$\langle\tau_{ab}\rangle_\tau = 0 \quad \langle\tau_{ab}(x)\tau_{c'd'}(y)\rangle_\tau = N_{abc'd'}(x,y) \tag{1.47}$$

where $\langle\ \rangle_\tau$ means taking a statistical average with respect to the noise distribution τ (for simplicity we don't consider higher order correlations). With these new definitions the new stochastic source will generate the same auto-correlation quantities. More specifically considering Einstein Langevin equation one can see that the auto-correlation with respect to quantum vacuum expectation value is equivalent to the one evaluated with respect to stochastic source that is

$$\langle h^s_{ab}(x)h^s_{c'd'}(x')\rangle_\tau = \int G_{a\,b\,p\,q}(x,y)\,G_{c'\,d'\,p'\,q'}(x',y')N^{p\,q\,p'\,q'}(y,y')d^4yd^4y' \tag{1.48}$$

Stochastic gravity contains information about the correlation of fields (and the related phase information) which is absent in semiclassical gravity. This feature moves stochastic gravity closer than semiclassical gravity to quantum gravity in that the correlation in quantum fields and geometry fully present in quantum gravity is partially retained in stochastic gravity, and the background geometry has a way to sense the correlation of the quantum fields through the noise term in the Einstein-Langevin equation, which shows up as metric fluctuations. Further more, the noise kernel can be used to investigate the limits of semi-classical gravity [70, 71, 75].

It can be shown that the noise kernel at the coincidence limit diverges and if the coincidence limit is to be taken, the noise kernel has to be regularized. The regularization of the noise kernel, via covariant point-splitting has been carried out [82]. It is commonly believed that structures in our universe originate from quantum fluctuations of the inflaton field being amplified by the inflationary expansion in the very early universe. Calculating metric fluctuations in the inflationary cosmological models can reveal the characteristics of fields that cause inflation. For example, assuming one takes the following form for the lagrangian for the inflaton field,

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_a\phi\partial^a\phi + \frac{1}{2}m^2\phi^2 \tag{1.49}$$

one can consider a flat Friedmann-Robertson-Walker model and calculate the stochastic average of the two point correlation of the metric function. Comparison of these results and the gravitational fluctuations derived from the small values of the Cosmic Microwave Background anisotropies detected by COBE [79] and WMAP [81], allows one to impose a severe bound on the mass of the inflaton field. This bound can be estimated to be of the order of $(m/m_P) \sim 10^{-6}$ [80], where m_P is the Planck's mass. Applications of the noise kernel in cosmology have been discussed in [72, 73]. Another important application of the noise kernel is in calculating the metric fluctuations near event horizon of black holes. Calculations such as these involve computing the effects of back reaction on space-time and due to complexity involved, complete results are not yet available. However, under some reasonable approximations some significant new results have been obtained. In particular, using the noise kernel, It has been shown [74] that the magnitude of fluctuations near black hole event horizon could grow with time to the extent that semi-classical approximations could break down long before quantum gravity related regime arrives (For more details on these applications see [77] and references there in.)

In chapter 4, we present general relations that give the Noise Kernel for a scalar field conformally coupled to the optical Schwarzschild space-time and introduce a method by which to calculate the elements.

1.3 Overview

In what follows we explain a method to calculate the $v(x, x')$ via Hadamard-WKB ansatz. The method is dependent on a specific coordinate system and it has been carried out for the Schwarzschild metric. We then use this method to calculate the self-force for a particle that has been released from rest after it was held at rest from infinite past. In the following chapter, this method is generalized to the case where the particle is launched with a certain initial velocity on a general geodesic. The expressions are used to calculate the self-force for a particle put on a circular orbit with Keplerian frequency.

Chapter 2

The Retarded Green's Function via Hadamard-WKB Quasi-Local Expansion

2.1 Elements

In this section we briefly introduce the elements of *Hadamard elementary solution* to the following differential equation.

$$g^{\mu\nu}\nabla_\mu\nabla_\nu G(x,x') = 0 \tag{2.1}$$

We only mention these items for completeness and future reference since there are extensive reviews that have already dealt with these issues at length.

2.1.1 Synge's world function

Synge's world function is defined as one half of the geodesic distance between two points. Let $z^\alpha(\lambda)$ describe a geodesic curve connecting two points x and x', that is we have

$$\lambda = 0 \quad z^\alpha(0) = x^\alpha \tag{2.2}$$

$$\lambda = \tau \quad z^\alpha(\tau) = x'^\alpha \tag{2.3}$$

and we have

$$\frac{d^2 z^\alpha}{d\tau^2} = \Gamma^\alpha_{\beta\gamma}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda} \tag{2.4}$$

The Synge's world function is defined as

$$\sigma(x, x') = \frac{1}{2}\int_0^\tau g_{\mu\nu}\frac{dz^\mu(\lambda)}{d\lambda}\frac{dz^\nu(\lambda)}{d\lambda}d\lambda \tag{2.5}$$

The world function is an example of a bi-scalar, a function that maps coordinates of two points in the manifold into a scalar. A derivative of this bi-scalar can be obtained at either of two points. The notations $D\sigma/dx^\alpha \equiv \sigma_{;\alpha} = \sigma_\alpha$ and similarly $D\sigma/dx'^\alpha \equiv \sigma_{;\alpha'} = \sigma_{\alpha'}$ are normally adopted.It can be shown that

$$\sigma = \frac{1}{2}\sigma_\alpha\sigma^\alpha = \frac{1}{2}\sigma_{\alpha'}\sigma^{\alpha'} \tag{2.6}$$

2.1.2 The Van Vleck-Morrett determinant

Another relevant quantity is the Van Vleck - Morrett determinant defined as

$$\Delta(x, x') = \frac{det(\sigma_{\alpha\beta'})}{\sqrt{-g(x)}\sqrt{-g(x')}} \tag{2.7}$$

It can be shown that the above bi-scalar has the following relationship with σ

$$\frac{1}{\Delta}\left(\Delta\sigma^\alpha\right)_{;\alpha} = 4 \tag{2.8}$$

2.1.3 Hadamard elementary solution

Hadamard, in his classic work [64] presented the following ansatz for the Green's function:

$$G(x, x') = \frac{u(x, x')}{\sigma} + v(x, x')\log(\sigma) + w(x, x') \tag{2.9}$$

The solution can be substituted back into the differential equation (2.1) to show that

$$\text{if } \sigma \neq 0 \quad \Box v(x, x') = 0 \quad u_{;\alpha}\sigma^\alpha = \left(2 - \frac{1}{2}\Box\sigma\right)u; \tag{2.10}$$

$$\text{if } \sigma = 0 \quad 2v_\alpha \sigma^\alpha + (\Box\sigma - 2)v = \Box u. \tag{2.11}$$

where $\Box$ is Laplace-Beltrami operator equal to $g^{\alpha\beta}\nabla_\alpha\nabla_\beta$. After substitution into Eq. (2.1), it can be shown that

$$u(x, x') = \Delta^{1/2}(x, x') \tag{2.12}$$

The bi-scalar $v(x, x')$ is called the tail term of the Green's function.

2.2 WKB solution for Euclidean Green function, relationship to the tail term.

We consider a certain space-time with the following line element.

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2 \tag{2.13}$$

where $f(r)$ and $h(r)$ are arbitrary functions of the radial coordinate and $d\Omega^2$ is the metric of a 2-sphere.

2.2.1 The tail term

Now consider the Euclidean Green function. For a massless scalar field with minimal coupling it is given as:

$$\Box G_E(x, x') = -\frac{\delta(x, x')}{\sqrt{g_E}} \tag{2.14}$$

It has the following relationship with the Feynman propagator, G_F.

$$G_E = iG_F \tag{2.15}$$

We make the following analytic continuation

$$-(t-t')^2 \rightarrow -(t-t')^2 + i\epsilon \tag{2.16}$$

and since $ImG_F(x,x') = -\frac{1}{2}G^{(1)}(x,x')$ (with $G^{(1)}(x,x')$ being the Hadamard elementary solution), we find

$$ReG_E = \frac{1}{2}G^{(1)} = \frac{1}{8\pi^2}\left(\frac{u(x,x')}{\sigma} + \frac{1}{2}v(x,x')\log\left(\sigma(x,x')\right) + w(x,x')\right) \tag{2.17}$$

Now the crucial point is that if the separation of point x from point x' is such that, the contribution of time separation is an order or more larger than the contribution of spatial separation that is if

$$\sigma = -f(r)(t-t')^2 + O[(x-x')^3] \tag{2.18}$$

$v(x,x')$ would be proportional the factor of $\log\left[(\tau-\tau')^2\right]$ in expression of G_E. This is essentially how calculation of the tail term is carried out here. In the following we first find a solution for G_E and then introduce a method for collecting all the terms that together constitute the one that is logarithmically divergent when points are brought together then the coefficient of $\log(\tau-\tau')$ is the tail term of the Green's function denoted usually by $v(x,x')$.

Hence to find the tail term, we should find a suitable expression for G_E, one that a term containing $\log(\tau-\tau')$ can be extracted from. To calculate G_E we can use the symmetry of space-time exhibited by the line element (2.13) to write a potential solution as

$$G_E(x,x') \sim \int d\omega \quad \cos\omega(\tau-\tau') \;\; p_1(r)p_2(r') \;\; Y(\gamma) \tag{2.19}$$

where γ is the angle between three dimensional vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{x}'}$. We substitute this ansatz back into the differential equation (2.14) and we find that,

$$G_E(x, x') = \frac{1}{4\pi^2}\int_0^\infty d\omega \cos[\omega(\tau - \tau')] \sum_{l=0}^{\infty} C_{l\omega}(2l+1)P_l(\cos\gamma)p_{\omega l}(r_<)q_{\omega l}(r_>) \tag{2.20}$$

where $P_l(x)$'s are Legendre polynomials and $r_>$ ($r_<$) is the larger (smaller) of r and r'. $C_{\omega l}$'s are to be determined such that the right hand side of (2.14)gives the correct result when integrated over a domain including $r = r'$.

For $r \neq r'$ substituting this ansatz int the (2.14) will yield the following differential equation for both $p_{\omega l}(r)$ and $q_{\omega l}(r)$ (collectively denoted by $S_{\omega l}(r)$):

$$\frac{1}{h(r)}\frac{d^2 S_{\omega l}(r)}{dr^2} + \left(\frac{2}{rh(r)} + \frac{f'(r)}{2f(r)h(r)} - \frac{h'(r)}{2h(r)^2}\right)\frac{dS_{\omega l}(r)}{dr} - \left(\frac{\omega^2}{f(r)} + \frac{l(l+1)}{r^2}\right)S_{\omega l}(r) = 0 \tag{2.21}$$

To satisfy the inhomogeneous differential equation (2.14), the mode functions $p_{\omega l}(r)$ and $q_{\omega l}(r)$ have to satisfy the following Wronskian condition,

$$C_{\omega l}\Big(p_{\omega l}(r)q'_{\omega l}(r) - q_{\omega l}(r)p'_{\omega l}(r)\Big) = -\frac{1}{r^2}\left(\frac{h}{f}\right)^{1/2} \tag{2.22}$$

(we drop the functional dependence of p q,f and h from now on as it is obvious that they all only depend on radial coordinate) First to simplify future relations we define:

$$\varrho = \frac{f}{h} \tag{2.23}$$

Then we proceed by expressing the mode functions in terms of

$$\begin{aligned} p_{\omega l} &= \frac{1}{(2r^2 W)^{1/2}} e^{\int \sqrt{\varrho}\, W\, dr} \\ q_{\omega l} &= \frac{1}{(2r^2 W)^{1/2}} e^{-\int \sqrt{\varrho}\, W\, dr} \end{aligned} \tag{2.24}$$

with

$$2\,W^2 = 2\,\Omega^2 + \frac{\varrho'}{r} + \varrho\,\frac{W''}{W} + \varrho'\frac{W'}{2W} - \frac{3}{2}\varrho\left(\frac{W'}{W}\right)^2 \tag{2.25}$$

and

$$\Omega^2 = \omega^2 + l(l+1)\,\frac{f}{r^2} \tag{2.26}$$

One can solve (2.25) iteratively. To that end, first we take the square root of both sides of (2.25) and use a first order approximation to write it as,

$$W = \Omega + \frac{1}{2\Omega}\left(\frac{\varrho'}{r} + \varrho\,\frac{W''}{W} + \varrho'\frac{W'}{2W} - \frac{3}{2}\varrho\left(\frac{W'}{W}\right)^2\right) \tag{2.27}$$

For reasons that will become clear shortly, we let the consecutive orders of iteration to be spaced from each other by 2. Hence the equation that takes us from a lower order of iteration, (henceforth called WKB order) to the next is given simply by

$$W_{n+2} = \Omega + \frac{1}{2\Omega}\left(\frac{\varrho'}{r} + \varrho\,\frac{W_n''}{W_n} + \varrho'\frac{W_n'}{2W_n} - \frac{3}{2}\varrho\left(\frac{W_n'}{W_n}\right)^2\right) \tag{2.28}$$

To lowest order, one has $W_0 = \Omega$ and higher orders can be obtained by applying Eq.(2.28).

Substituting (2.24) into (2.22) yields $C_{\omega l} = 1$ and as a result,

$$G_E(x,x')|_{r=r'} = \frac{1}{4\pi^2}\int_0^\infty d\omega \cos[\omega(\tau-\tau')] \sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\frac{1}{2r^2 W} \tag{2.29}$$

2.3 Calculating $v(x,x')$ as a quasi-local expansion in coordinate separation via WKB-Hadamard method

Our goal is to use the results of the previous section to find $v(x,x')$ as an expansion in separation of coordinates that is an expansion in powers of $(r-r')$, $(t-t')$, $(\theta-\theta')$ and $(\phi-\phi')$. However it is already clear that the angular dependence is entirely through $\cos\gamma$ and therefore for an expansion in terms of angular coordinates, a first step would be to expand in powers of $\cos\gamma - 1$, therefore our objective is to find $v(x,x')$ as

$$v(x,x') = \sum_{i,j,k} v_{i,j,k}(t-t')^{2i}(\cos\gamma - 1)^j(r-r')^k \tag{2.30}$$

That the expansion has to only involve even orders of $(t-t')$ can be seen from the fact that for every value for separation we have to have $v(x,x') = v(x',x)$. Furthermore from symmetry of the line element (2.13), it is clear that the coefficients can only depend on r. To find the tail term $v(x,x')$ one first uses the iterative procedure detailed in the previous section to calculate $1/W$ to the desired WKB order. The result would be of the following general form,

$$\frac{1}{W} = \sum_{m,p} a_{mp}\frac{(l\,(l+1))^m}{\Omega^p} \tag{2.31}$$

The Eq. (2.29) gives G_E only when $r = r'$. To find the expansion in radial separation, one has to expand Eq. 2.20. Without loss of generality one can assume $r' < r$ and expand $p_{\omega l}(r')$ around r. Using the Eq. (2.21), one can write,

$$p''_{\omega l}(r) = D^{(0)}p_{\omega l}(r) + D^{(1)}p'_{\omega l}(r) \tag{2.32}$$

with

$$D^{(0)} = \left(\frac{1}{\varrho}\right)\left(\omega^2 + \frac{l(l+1)}{r^2}f\right) \tag{2.33}$$

$$D^{(1)} = -\left(\frac{2}{r} + \frac{\varrho'}{\varrho}\right) \tag{2.34}$$

To find higher order derivatives of $p_{\omega l}(r)$ we can use the same procedure, specifically we can use the following relations

$$p_{\omega l}^{(n+2)}(r) = D^{(n)} p_{\omega l}(r) + D^{(n+1)} p'_{\omega l}(r) \tag{2.35}$$

$$D^{(n+2)} = \frac{dD^{(n)}}{dr} + D^{(0)} D^{(n)} + D^{(n+1)} \tag{2.36}$$

$$D^{(n+3)} = \frac{dD^{(n+1)}}{dr} + D^{(1)} D^{(n)} \tag{2.37}$$

From the above recursion relations, one can calculate the contribution of the coefficient of $(r - r')^k$ in the expansion of $v(x, x')$, Let b_k be that coefficient, then it can be shown that,

$$b_k = \left(\frac{1}{2r^2\, k!}\right)\left(\frac{rD^{(k)} - D^{(k-1)}}{r} W^{-1} + D^{(k-1)}\left(\frac{1}{2}\frac{dW^{-1}}{dr} + \varrho^{-1/2}\right)\right) \tag{2.38}$$

The next step is to expand the Legendre Polynomials in powers of $\cos\gamma - 1$ (these two steps are actually independent from each other and one can deal with them in any order that one chooses). $P_l(x)$ is usually expressed as a polynomial in x. However one can always expand them in $x - 1$. To be able to proceed with the computation one needs the general form of this expansion, that is, if we write:

$$P_l(\cos\gamma) = \sum_j c_{j,l} \, (\cos\gamma - 1)^j \tag{2.39}$$

we need the general dependence of $c_{j,l}$ on j and l. This dependence can be obtained from the following recursion relation,

$$c_{j,l} = \frac{l(l+1) - (j-1)(j-2)}{2\,(j-1)^2}\, c_{j-1,l} \tag{2.40}$$

with $c_{1,l} = 1$ for all values of l. Now to find the coefficient of the temporal separation, we substitute all of these coefficients into (2.20) that is

$$G_E = \frac{1}{4\pi^2}\int_0^\infty d\omega \cos[\omega(\tau-\tau')] \sum_{l=0}^{\infty}(2l+1)\; c_{j\,,\,l}\; b_k\; (\cos\gamma - 1)^j (r-r')^k \tag{2.41}$$

The dependence of b_k on ω and $l(l+1)$ is solely through the dependence of W^{-1} and $\frac{dW^{-1}}{dr}$ on Ω, furthermore the dependence of $c_{j,l}$ on l is entirely through $l(l+1)$, from these considerations and also from Eq. (2.31), it is clear that the final mix would only include some odd power of $1/\omega$ and some power of $l(l+1)$ multiplied by $2l+1$. The next step after calculating $c_{j,l}b_k$ is to expand the result in powers of $l(l+1)$ and collect the coefficients, the final expression would be of the following form,

$$G_E = \frac{1}{4\pi^2}\int_0^\infty d\omega \cos[\omega(\tau-\tau')]\; q_{ab}\,\frac{1}{\omega^{2b+1}}\sum_{l=0}^{\infty}(2l+1)\; l^a\,(l+1)^a\;\; (\cos\gamma-1)^j(r-r')^k \tag{2.42}$$

Now we have to perform the sum over l values. This can be done using Plana sum rule, which states,

$$\sum_{n=N}^{\infty} F(n) = \frac{1}{2}F(N) + \int_N^\infty + i\int \frac{dt}{e^{2\pi t}-1}\Big(F(N+it) - F(N-it)\Big) \tag{2.43}$$

The final step of this procedure is to extract the coefficient of temporal expansion, this can be done using the following.

Note that:

$$\int \cos\omega(\tau-\tau')\frac{1}{\omega^{2n+1}} = \frac{(-1)^{(n+1)}}{(2n)!}(\tau-\tau')^{2n}\log(\tau-\tau') \tag{2.44}$$

If the points are split only in the time direction then one finds that using a second order WKB expansion gives v to zeroth order in $(t-t')$, a fourth order one gives v to second order in $(t-t')$, and so forth. For Schwarzschild space-time the lowest non-vanishing order is $(t-t')^4$ which requires use of a sixth order WKB approximation. Finally collecting all coefficients will yield the desired expansion as

$$v(x, x') = \sum_{i,j,k} v_{ijk}(t-t')^{2i} \, (\cos\gamma - 1)^j \, (r-r')^k \tag{2.45}$$

In order to get the complete expansion, one must further expand the angular part in $(\theta - \theta')$ and $(\phi - \phi')$. The result of such expression can be formally expressed as

$$v(x, x') = \sum_{i,j,k} v_{ikmn}(t-t')^{2i} \, (r-r')^k \, (\theta-\theta')^m \, (\phi-\phi')^n \tag{2.46}$$

2.4 Verification, properties of $v(x, x')$ and concluding remarks

Since the calculations are fairly complicated, verification is in order. One should check the results against established properties of $v(x, x')$. The two equations (1.34) and (1.35) being the obvious ones. The important point to bear in mind is that, in this method, all calculations are done so that they are correct to a certain order. It might be worth noting that once that an expression for a quantity has been obtained and found to be correct to a certain order, say N, manipulating it will change its correctness order in ways that are not always straightforward.

The relevant case here is of course $v(x, x')$. If $v(x, x')$ is correct to the N^{th} order then $\Box v(x, x')$ will be correct to an expansion order of $N-2$. It then follows that if we perform a calculation of N^{th} WKB order, we should expect $\Box v(x, x')$ to vanish up to and including series order of $N-4$.

Here it would be helpful to have an expression for the coefficients of expansion of $\Box v$ from coefficient of expansion of v. A calculation of $\Box v$ in the form of Eq. (2.46) yields:

$$
\begin{aligned}
\Box v(x,x') = \quad & \sum_{i,j,k,m,n} \Bigg(-\frac{1}{f}(2i+1)(2i+2)v_{i+1,k,m,n} \\
+ \quad & \frac{1}{h}\left((k+1)(k+2)v_{i,k+2,m,n} + 2(k+1)\frac{dv_{i,k+1,m,n}}{dr} + \frac{d^2v_{i,k,m,n}}{dr^2}\right) \\
+ \quad & \left(\frac{2}{rh} + \frac{1}{(2fh)}f' - \frac{h'}{2h^2}\right)\left((k+1)v_{i,k+1,m,n} + \frac{dv_{i,k,m,n}}{dr}\right) \\
+ \quad & \frac{1}{r^2}\left((m+1)(m+2)v_{i,k,m+2,n} + 2(m+1)\frac{dv_{i,k,m+1,n}}{d\theta} + \frac{d^2v_{i,k,m,n}}{d\theta^2}\right) \\
+ \quad & \frac{\cot\theta}{r^2}\left((m+1)v_{i,k,m+1,n} + \frac{dv_{i,k,m,n}}{d\theta}\right) \\
+ \quad & \frac{1}{r^2\sin^2\theta}(n+1)(n+2)v_{i,k,m,n+2}\Bigg) (t-t')^{2i}(r-r')^k(\theta-\theta')^m(\phi-\phi')^n
\end{aligned}
\tag{2.47}
$$

To verify vanishing of this expression to order $N-4$, one will limit the computation to all terms such that $2i+k+m+n \leq N-4$.

One important property that $v_{i,j,k}$'s and consequently $v_{i,k,m,n}$'s possess is that there is a certain recursion relation among them. This can be seen from the following considerations. Let us assume we have an expansion of $v(x,x')$ around $r=r'$. For brevity we assume that every time we write $v(r,r')$ we implicitly mean $v(t,r,\theta,\phi;t',r',\theta',\phi')$

$$v(r,r') = \sum v_n(r)(r'-r)^n \tag{2.48}$$

we also have

$$v(r',r) = \sum v_n(r')(r-r')^n \tag{2.49}$$

Now we use $v(r,r') = v(r',r)$ to equate the left sides and therefore

$$\sum v_n(r)(r'-r)^n = \sum v_n(r')(r-r')^n \tag{2.50}$$

we take the k^{th} derivative of both sides with respect to r.

$$\frac{d^k}{dr^k}\, v(r,r') = \sum_n \frac{d^k}{dr^k}\left(v_n(r) \ \ (r'-r)^n\right) = \sum_n v_n(r') \ \ \frac{d^k}{dr^k}(r-r')^n \tag{2.51}$$

and then use:

$$\frac{d^k}{dr^k}\,(fg) = \sum_{l=0}^{k} \frac{k!}{l!\ (k-l)!} f^{(l)} g^{(k-l)} \tag{2.52}$$

to write:

$$\sum_{n=l}\sum_{l=0}^{k}\left(\frac{k!}{l!\ (k-l)!}\right)\ \frac{d^{k-l}v_n(r)}{dr^{k-l}}\ \frac{n!}{(n-l)!}\ (-1)^l\ (r'-r)^{n-l} = \sum_{n=k} v_n(r')\frac{n!}{(n-k)!}\ \ (r-r')^{n-k} \tag{2.53}$$

This can be written as

$$\sum_{n=0}\sum_{l=0}^{k}\left(\frac{k!}{l!\ (k-l)!}\right)\ \frac{d^{k-l}v_{n+l}(r)}{dr^{k-l}}\ \frac{(n+l)!}{n!}\ (-1)^l\ (r'-r)^{n} = \sum_{n=0} v_{n+k}(r')\frac{(n+k)!}{n!}\ \ (r-r')^{n} \tag{2.54}$$

We put $r = r'$, all terms vanish except $n = 0$. We have

$$\sum_{l=0}^{k} \frac{k!}{(k-l)!}\ \frac{d^{k-l}v_l(r)}{dr^{k-l}}\ (-1)^l = k!v_k(r) \tag{2.55}$$

we separate the last term on the left side.

$$\sum_{l=0}^{k-1} \frac{k!}{(k-l)!}\ \frac{d^{k-l}v_l(r)}{dr^{k-l}}\ (-1)^l + k!(-1)^k v_k(r) = k!v_k(r) \tag{2.56}$$

and set $k = 2m+1$,this yields

$$\sum_{l=0}^{2m} \frac{(2m+1)!}{(2m+1-l)!}\ \frac{d^{2m+1-l}v_l(r)}{dr^{2m+1-l}}\ (-1)^l - (2m+1)!v_{2m+1} = (2m+1)!v_{2m+1} \tag{2.57}$$

From this we get the following recursion relation for odd numbered coefficients:

$$v_{2m+1} = \frac{1}{2}\sum_{l=0}^{2m} \frac{(-1)^l}{(2m+1-l)!}\frac{d^{2m+1-l}}{dr^{2m+1-l}}v_l(r) \tag{2.58}$$

Some special cases are:

$$m = 0 \qquad v_1 = \frac{1}{2}\frac{dv_0}{dr} \tag{2.59}$$

$$m = 1 \qquad v_3 = \frac{1}{2}\left(\frac{1}{3!}\frac{d^3v_0}{dr^3} - \frac{1}{2}\frac{d^2v_1}{dr^2} + \frac{dv_2}{dr}\right) \tag{2.60}$$

$$m = 2 \qquad v_5 = \frac{1}{2}\left(\frac{1}{5!}\frac{d^5v_0}{dr^5} - \frac{1}{4!}\frac{d^4v_1}{dr^4} + \frac{1}{3!}\frac{d^3v_2}{dr^3} - \frac{1}{2}\frac{d^2v_3}{dr^2} + \frac{dv_4}{dr}\right) \tag{2.61}$$

The even ordered coefficients however are not related to each other and must be found using the WKB method.

The WKB-Hadamard method yields analytical expressions valid in the region that points denoted by x and x' are close and can be applied in a variety of investigations. One pertinent question is the radius of convergence of the series expansion in Eq. (2.45) or Eq. (2.46). A general expression for the series coefficients is not available, therefore a rigorous answer is not possible. Even though, one can still ask for an approximate measure. An upper limit to the radius of convergence can be obtained by noting that the Hadamard ansatz will break down as soon as the points are too far to be described adequately within the domain of validity of Riemann Normal Coordinates. The reason is usually attributed to caustics since it can be shown that for points that are connected by more than one geodesics, Van Vleck determinant diverges, making Hadamard solution invalid [4]. One method to investigate this issue is to look at all points that are residing on a certain path (preferably a radial line or a circle) and investigate the growth of the ratio of a higher order term to a lower one, say,$v_{i,j,k+1}(r+d)/v_{i,j,k}(r)$ for various values of r. One can use the rate at which this ratio grows to estimate a measure of convergence radius. It can be seen, then, that one might be able to formulate certain reasonable criteria in order to ascertain the convergence of the series and estimate its radius of convergence. The issue is still an open problem and the work on the radius of convergence is underway. We hope to report on this topic in near future.

In the following chapters we use Eq. (2.45) and Eq. (2.46) to calculate the effect of radiation reaction and find a quasi-local series expansion of the self-force.

Chapter 3

Calculation of self-force using quasi-local expansion of the Green's function

3.1 Self-force for a particle in motion after being held at rest from infinite past.

In this chapter we review a method to calculate the self-force for a particle that was held at rest from $t = -\infty$ to $t = 0$ at which it begins to move on a geodesic of space-time. In doing so we are working in an approximation that ignores the effect of the self-force on the motion. In other terms we assume that even with the presence of the self-force the particle is still moving on the same geodesic. This assumption can be given more validity if we assume that there is an external force that cancels the self-force and change the problem to one that attempts to find this external force.

3.2 Introduction

An exact expression for the self-force in a black hole space-time has been obtained only for the cases of a scalar or electric charge held at rest in a Schwarzschild space-time [24, 25, 26, 27] and an electric charge held at rest on the symmetry axis of a Kerr space-time [28, 29]. Approximate analytical expressions have been obtained using Green's functions for scalar, electric, and gravitational charges in various weak field limits [30, 31, 32]. Most other calculations have involved the use of mode sum techniques [33, 34] in cases of high symmetry such as a static charge [35, 36, 37], radial infall [38, 39], a circular orbit [40, 41, 42, 43], or a slightly elliptical orbit [42].

Since the mode-sum regularization procedure has been developed extensively, aiming at practical calculations, it is important to find independent ways to check its accuracy and reliability. This of course also applies to any other method, numerical or analytical that might be developed in the future. To date the checks on various mode-sum regularization procedures which we are aware of fall into three categories: 1) comparisons with exact analytical results, 2) comparisons with analytical approximations, 3) comparisons of the results of one mode-sum regularization technique with those of another. Most of the comparisons that have been made fall into the third category [45, 43, 42, 41, 39, 44]. Comparisons with exact analytical results are of course the most reliable but also the most limited. They have only been done in the case of static scalar and electric charges held at rest in a Schwarzschild spacetime [35]. Comparison with an analytic approximation in the weak field limit has been made for a static scalar charge at rest in an axisymmetric spacetime [37]. In this chapter we present results which can be used as a check in the second category. However, unlike most previous analytical approximations to the self-force, ours is valid in the strong field as well as the weak field limit. Specifically, we consider the case of a particle with scalar charge which is held at rest until a time $t = 0$ and subsequently falls radially towards a Schwarzschild black hole.

In the following section we detail a method which is based on the quasi-local expansion of the tail term of the Green's function which we have calculated **in the last chapter**.

3.3 Review of the method

Consider a massless scalar field with a source term consisting of a point particle of scalar charge q. The wave equation that was given in (1.15) is

$$\Box\Phi = -\rho \tag{3.1}$$

where

$$\rho(x) \;=\; q\int_{-\infty}^{\infty}\frac{\delta^4(\,x, z(\tau)\,)}{\sqrt{-g}}d\tau \tag{3.2}$$

The self-force is given formally by

$$f_\mu(\tau) = q\left[\nabla_\mu\Phi(x)\right]_{x=z(\tau)}\;. \tag{3.3}$$

This expression is divergent and must be regularized. Quinn [20] has shown that the regularized expression for the self-force can be split into a local term plus a finite integral over the gradient of the retarded Green's function. The latter is often called the 'tail term'.

The Hadamard expansion for the retarded Green's function [20, 52, 49, 50] was given in (1.30) as

$$G_R(x,x') \;=\; \theta(x,x')\left\{\frac{u(x,x')}{4\pi}\delta[\sigma(x,x')] - \frac{v(x,x')}{8\pi}\theta[-\sigma(x,x')]\right\} \tag{3.4}$$

Here $\theta[-\sigma(x,x')]$ is defined to be zero if the point x' is inside the light cone of the point x and zero otherwise, while $\theta(x,x')$ is defined to be one if the point x resides in the future of a space-like hypersurface involving the point x' and zero otherwise. The quantity $\sigma(x,x')$ is equal to one-half the square of the proper distance between x and x' along the shortest geodesic connecting them. The function $v(x,x')$ contributes to the tail part of the self-force [20]. It obeys the equation

$$\Box_x v(x,x') = 0 \tag{3.5}$$

and is symmetric under the exchange of the two points, i.e. $v(x,x') = v(x',x)$. In a general spacetime [48, 60]

$$v(x,x) = -\frac{1}{6}R(x) \; . \tag{3.6}$$

The tail term for the self-force obtained from Eqs. (3.5) and (3.1) can be written in the form (see, e.g., [54])

$$\begin{aligned} (f_\mu(\tau))_{\text{tail}} &= -\frac{q^2}{8\pi}\int_{\tau_0}^{\tau}\left(\frac{\partial}{\partial x^\mu}v[x,z(\tau')]\right)_{x=z(\tau)} d\tau' \\ &\quad + q^2\int_{-\infty}^{\tau_0}\left(\frac{\partial}{\partial x^\mu}G_R[x,z(\tau')]\right)_{x=z(\tau)} d\tau' \; . \end{aligned} \tag{3.7}$$

It is necessary that τ_0 be chosen so that the Hadamard-WKB expansion for v converges throughout the region of integration of the first integral in Eq. (3.7). From a WKB expansion of order $(2N)$ (detailed in the previous chapter) one can obtain an expansion for $v(x,x')$ that includes terms up to order $(x-x')^{2N-2}$. For the metric

$$ds^2 = -(1-2M/r)dt^2 + dr^2/(1-2M/r) + r^2 d\Omega^2 \; , \tag{3.8}$$

the expansion for v is of the form

$$v(x,x') = \sum_{i,j,k=0} v_{ijk}(r)(t-t')^{2i}(\cos\gamma - 1)^j (r-r')^k \tag{3.9}$$

with

$$\cos\gamma \equiv \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \ . \tag{3.10}$$

To calculate the self-force we need to calculate the two terms in (3.7).

3.4 The case of a particle released from rest

3.4.1 Preliminaries

Consider the following problem. A particle has been held at rest from infinite past till the time that it is released from rest in the Schwarzschild spacetime described by

$$ds^2 = -(1-2M/r)dt^2 + dr^2/(1-2M/r) + r^2 d\Omega^2 \ , \tag{3.11}$$

the particle is going to fall on a radial path toward the center of black hole. The geodesic equations of motion can be written as [56]

$$\tau = -\sqrt{E}\int_{r'}^{r} \frac{d\bar{r}}{(1-Ef(\bar{r})-\frac{J^2 f(\bar{r})}{\bar{r}^2})^{1/2}} \tag{3.12}$$

$$t-t' = -\int_{r'}^{r} \frac{d\bar{r}}{f(\bar{r})(1-Ef(\bar{r})-\frac{J^2 f(\bar{r})}{\bar{r}^2})^{1/2}} \tag{3.13}$$

$$\phi-\phi' = -J\int_{r'}^{r} \frac{d\bar{r}}{\bar{r}^2(1-Ef(\bar{r})-\frac{J^2 f(\bar{r})}{\bar{r}^2})^{1/2}} \tag{3.14}$$

where $f(r) \equiv 1-2M/r$. Note that $J=0$ yields a radial geodesic and $E=0$ a null one.

For radial fall from rest, these equations of motion can be solved in detail and its path can be described by [6]

$$\frac{t}{2M} = \left(\frac{r_0}{2M}-1\right)^{1/2}\left(\left(\frac{r_0}{4M}+1\right)\eta+\frac{r_0}{4M}\sin\eta\right)+\ln\left|\frac{(\frac{r_0}{2M}-1)^{1/2}+\tan(\eta/2)}{(\frac{r_0}{2M}-1)^{1/2}-\tan(\eta/2)}\right| \tag{3.15}$$

where

$$\eta = \cos^{-1}\left(\frac{2r}{r_0}-1\right) \tag{3.16}$$

In terms of r and r_0 it can be written as

$$\begin{aligned}\frac{t}{2M} &= \left(\frac{r_0}{2M}-1\right)^{1/2}\left(\left(\frac{r_0}{4M}+1\right)\cos^{-1}\left(\frac{2r}{r_0}-1\right)+\frac{r_0}{2M}\sqrt{\frac{r}{r_0}\left(1-\frac{r}{r_0}\right)}\right)\\ &+ \ln\left|\frac{(\frac{r_0}{2M}-1)^{1/2}+\sqrt{\frac{r_0}{r}-1}}{(\frac{r_0}{2M}-1)^{1/2}-\sqrt{\frac{r_0}{r}-1}}\right|\end{aligned} \tag{3.17}$$

Assuming a photon that arrives at $x=(t,r)$ from $x_0=(t_R,r_0)$ we have

$$t_R = t-(r'-r)-2M\ln\frac{r'-2M}{r-2M}\ . \tag{3.18}$$

3.4.2 Review of the method

To calculate the self-force we need to calculate the potential which is formally given as

$$\Phi(x) = -\frac{q^2}{8\pi}\int_{\tau_0}^{\tau}\left(v[x,z(\tau')]\right)_{x=z(\tau)}d\tau' + q^2\int_{-\infty}^{\tau_0}\left(G_R[x,z(\tau')]\right)_{x=z(\tau)}d\tau'\ . \tag{3.19}$$

To that end, first consider the problem of computing the field Φ_{static} at the point y due to an eternally static charge q at the position $r=r_0,\theta=\theta_0,\phi=\phi_0$. One can use Eq. (3.1), to write the field due to the static charge as

$$\Phi_{static}(y) = q \int_{-\infty}^{\infty} G_R[t, r; t'(\tau'), r_0] d\tau' \ . \tag{3.20}$$

On the other hand, The solution to this problem in a Schwarzschild spacetime has been given by Wiseman [26]. For the above locations of the charge and field point, with a metric of the form (3.11), and while taking into account a difference in sign conventions, it can be written as

$$\Phi_{static}(y) = \frac{1}{4\pi} q \sqrt{1 - \frac{2M}{r_0}} \frac{1}{r_0 - r} \ . \tag{3.21}$$

Here and for the rest of this section we suppress the dependence of various quantities on the angles θ_0 and ϕ_0. Dividing the region of integration for Eq. (3.20) in the same way as was done in Eq. (3.7) and using the Hadamard expansion (3.4) one can write this latter equation as

$$\begin{aligned} \Phi_{static}(y) &= \frac{q}{4\pi} \sqrt{1 - \frac{2M}{r_0}} \int_0^t u[t, r; t', r_0] \, \delta[\sigma(t, r; t', r_0)] \, dt' \\ &- \frac{q}{8\pi} \sqrt{1 - \frac{2M}{r_0}} \int_0^{t_R} v[t, r; t', r_0] \, dt' \ + q \int_{-\infty}^{0} G_R[t, r; t'(\tau'), r_0] \, d\tau' \end{aligned} \tag{3.22}$$

In the first two terms the integration variable has been changed from τ' to t'. The two theta functions in Eq. (3.4) result in an upper limit for the second integral which is equal to the retarded time t_R which is given in Eq. (3.18). The time t is taken to be the time that it would take a particle to fall from r_0 to r assuming that it starts at rest. Then the third term on the right in Eq. (3.57) is the same, except for the gradient and a factor of q, as the second term on the right in Eq. (3.7). The value of this term can be obtained by computing the other three terms in the equation. The term on the left is given in Eq. (3.56) and the second term on the right can be computed using the Hadamard-WKB expansion.

To calculate the first term on the right in Eq. (3.57), we note that the argument of the delta function vanishes on the light cone of the point y. Since the charge is static

$$
\begin{aligned}
\sigma(t,r;t',r_0) &= \sigma(t,r;t_R,r_0) + \left(\frac{\partial}{\partial t'}\sigma(t,r;t',r_0)\right)|_{t'=t_R}\,(t'-t_R) + ... \\
&= \sigma_{t_R}(t'-t_R) + ...\,, \qquad (3.23)
\end{aligned}
$$

where the shorthand notation $\sigma_\mu \equiv \sigma_{;\mu}$ has been used. Then

$$
\delta(\sigma(t,r;t_R,r_0)) = \delta[\sigma_{t_R}(t'-t_R)] = \frac{\delta(t'-t_R)}{|\sigma_{t_R}|}\,. \qquad (3.24)
$$

Next one must calculate $u(y,y_R)$ with y and $y_R \equiv (t_R, r_0, \theta_0, \phi_0)$ connected by a null radial geodesic. By substituting the Hadamard expansion into the equation satisfied by the Green's function it is possible to show that in general [47, 48, 60]

$$
\begin{aligned}
u(x,x') &= \Delta(x,x') \qquad (3.25) \\
\Delta(x,x') &= -\frac{\det(-\sigma_{;\mu\nu'})}{\sqrt{-g(x)}\sqrt{-g(x')}}
\end{aligned}
$$

Thus what remains is to calculate σ_{t_R} and $\sigma_{;\mu\nu'}$ for the two points $x = y$ and $x' = y_R$. Although there may be some simple way to reason out the answer, as shown in Appendix A, it can be obtained by solving the geodesic equations and integrating the result to obtain the proper distance along the geodesic. The result is

$$
\sigma_{t_R}(t,r;t_R,r_0) = r_0 - r
$$

$$u(t,r;t_R,r_0) \quad = \quad 1\ . \tag{3.26}$$

Substituting Eq. (3.26) into Eq. (3.57) and computing the integral one finds that

$$\int_{-\infty}^{0} G_R[t,r;t'(\tau'),r_0]\,d\tau' = \frac{1}{8\pi}\sqrt{1-\frac{2M}{r_0}}\int_0^{t_R} v[t,r;t',r_0]dt' \tag{3.27}$$

With the definitions

$$\Phi_s(y) \quad = \quad \frac{q}{8\pi}\sqrt{1-\frac{2M}{r_0}}\int_0^{t_R} v[t,r;t',r_0] \quad dt' \tag{3.28}$$

$$\Phi_f(y) \quad = \quad \frac{q}{8\pi}\int_0^{\tau} v[t,r;t'(\tau'),r'(\tau')]\,d\tau' \tag{3.29}$$

Eq. (3.7) becomes

$$f_\mu(\tau) = q\left[\frac{\partial}{\partial y^\mu}(\Phi_s(y)-\Phi_f(y))\right]\ . \tag{3.30}$$

Here the fact that $v(x,x) = 0$ in Schwarzschild spacetime has been used to interchange the order of integration and differentiation in the Φ_f term. Note that this derivation only works for the time and radial components of the self-force. Because of spherical symmetry, the angular components of the self-force for a radial trajectory are zero. Finally the subscript "tail" has been dropped because for a geodesic trajectory the local part of the self-force is zero in a Schwarzschild spacetime [20].

As a result of Eq. (3.30), the problem of calculating the self-force reduces to calculating Φ_s and Φ_f. This is an exact result. We now calculate the right hand side of Eq. (3.30) using the Hadamard-WKB expansion for $v(x,x')$ whose form is given in Eq. (3.9). For radial geodesics, $\cos\gamma = 1$, so only the coefficients $v_{i0k}(r)$

contribute. The result for Φ_s is

$$\Phi_s(y) = \frac{q}{8\pi}\sqrt{1-\frac{2M}{r_0}}\sum_{i,k=0}^{\infty}\left(\frac{1}{2i+1}\right)v_{i0k}(r)\left[t^{2i+1}-(t-t_R)^{2i+1}\right](r-r_0)^k \tag{3.31}$$

To calculate Φ_f one can use the geodesic equations (3.4.1) to convert the integral (3.62) to an integral over the radial coordinate r. One can further solve the geodesic equations to obtain the trajectory $t(r)$. After substituting the Hadamard-WKB expansion for v, the integral (3.62) can be computed numerically.

An alternative is to expand all relevant quantities in both Φ_s and Φ_f in Taylor series about r_0. This allows one to compute the integrals analytically order by order. The derivation is given in more detail in the next section. We find

$$f_t(\tau) = q^2\frac{(5\,r_0-12M)\,\sqrt{1-\frac{2M}{r_0}}}{5600\,\pi\,r_0^6}(r-r_o)^3+O[(r-r_0)^4] \tag{3.32}$$

$$f_{r*}(\tau) = -q^2\,\frac{3\sqrt{M}(r_0-2M)}{11200\sqrt{2}\pi{r_0}^6}(r-r_0)^{5/2}+O[(r-r_0)^{7/2} \tag{3.33}$$

with r^* the Regge-Wheeler coordinate defined by

$$r^* \equiv r + 2M\log\left(\frac{r-2M}{2M}\right)\ . \tag{3.34}$$

It turns out that each subsequent order of the WKB expansion adds another term to the series. Using a 16th order WKB expansion we have results for a total of six terms in the expansions for both f_t and f_{r*}. The coefficients of these terms are displayed in Appendix B.

3.4.3 Computing the self-force: Details

Φ_s has been calculated in (3.92). However calculation of Φ_f is not entirely straightforward. In terms of the geodesic equations (3.4.1) Φ_f can be written as:

$$\Phi_f = -\frac{q}{8\pi}\sum v_{ik00}(r)\int_{r_0}^{r_1}\frac{dr'}{\sqrt{E^2-(1-\frac{2M}{r'})}}\;(t'(r')-t)^{2i}\;(r-r')^k \quad (3.35)$$

With $t'(r')$ given in (3.17).

3.4.3.1 Numerical evaluation of the integrals in Φ_f

The self force contribution coming from the portion of path,where the charge is falling can be calculated as

$$f_{falling} = -\frac{q^2}{8\pi}\Bigg[\sqrt{\frac{r_0}{2M}}\sum_{i,k}\left(\frac{dv_{ik00}}{dr}+(k+1)v_{i,k+1,0,0}\right)\times \int_{r_0}^{r_1} dr'\sqrt{\frac{\frac{r'}{r_0}}{1-\frac{r'}{r_0}}}\,(t(r')-t)^{2i}\,(r-r')^k\;\Bigg]_{r=r_1\;t=t(r_1)} \quad (3.36)$$

To go further, we need to calculate the integral. We change the variables so that all the variables are dimensionless. We adopt

$$x=\frac{r'}{2M}\;,\;a=\frac{r_0}{2M}\;,\;b=\frac{r_1}{2M}\;,\;u=\frac{r}{2M}\;,\;v=\frac{t}{2M}\;,\;y(x)=\frac{t(r')}{2M} \quad (3.37)$$

Now one can write the self force as

$$f_{falling} = -\frac{q^2}{8\pi}\sqrt{a}\sum_{i,k}\;I(i,k)\;\left[\frac{dv_{ik00}}{dr}+(k+1)v_{i,k+1,0,0}\right]_{r=r_1\;t=t(r_1)} \quad (3.38)$$

where we have

$$I(i,k)=(2M)^{(2i+k+1)}\int_a^b dx\;\left(\frac{a}{x}-1\right)^{-1/2}(y(x)-v)^{2i}\,(u-x)^k \quad (3.39)$$

and

$$y(x) = \sqrt{a-1}\left(\left(1+\frac{a}{2}\right)\cos^{-1}\left((2x/a)-1\right)+a\sqrt{\frac{x}{a}-\frac{x^2}{a^2}}\right)+\ln\left|\frac{\sqrt{a-1}+\sqrt{\frac{a}{x}-1}}{\sqrt{a-1}-\sqrt{\frac{a}{x}-1}}\right| \quad (3.40)$$

The dimensionless part of $I(i,k)$ can be evaluated numerically.

However there is an obstacle in the way of numerical integration. The integrand diverges at $x = a$ and all numerical integration schemes involve evaluating the integrand at the initial point. A way out of this is to notice that

$$\frac{dy}{dx} = \sqrt{a-1}\sqrt{\frac{x}{a-x}}\left(\frac{x}{x-1}\right) \quad (3.41)$$

Therefore the above integration can be written as:

$$I(i,k) = \frac{1}{\sqrt{a-1}}\int_a^b dx\ \left(\frac{x-1}{x}\right)\left(\frac{1}{2i+1}\right)\left(\frac{d}{dx}\left(y(x)-y(b)\right)^{2i+1}\right)(b-x)^k \quad (3.42)$$

Now integration by part, can be used to write the integral as:

$$\begin{aligned} I(i,k) = \quad & \left(\frac{1}{(2i+1)\sqrt{a-1}}\right)\times \\ & \left(\left[\frac{x-1}{x}\left(y(x)-y(b)\right)^{2i+1}(b-x)^k\right]_a^b - \right. \\ & \left.\int_a^b dx\ \Big(y(x)-y(b)\Big)^{2i+1}\frac{d}{dx}\left[\frac{x-1}{x}(b-x)^k\right]\right) \end{aligned} \quad (3.43)$$

which in turn is equal to:

$$\begin{aligned} I(i,k) = \quad & \left(\frac{1}{(2i+1)\sqrt{a-1}}\right)\Bigg\{\ \left(\frac{a-1}{a}\right)y(b)^{2i+1}(b-a)^k + \\ & \int_a^b dx\ \frac{1}{x^2}\Big(y(x)-y(b)\Big)^{2i+1}(b-x)^{k-1}\Big(kx^2-(k-1)x-b\Big)\Bigg\} \end{aligned} \quad (3.44)$$

The last term in the above expression is finite in all points in the range $[a,b]$ and the integral can be evaluated numerically without a problem.

3.4.3.2 Analytical evaluation of the self-force: The power series expansion.

Since we are using essentially a power series expansion for $v(x,x')$ one can see that the integrals do not have to be accurate to all orders. There is an error in calculation nevertheless which comes from an approximate series expansion of $v(x,x')$ and the evaluation of the integrals only need to be correct to the same order that we have expanded that function to. Since this is the main approach we have taken elsewhere [57], we now present the full calculation of the self-force using *this* method. In this section an expansion for the self-force in powers of $r - r_0$ will be derived. The coefficients of the expansion depend on the mass M of the black hole, the radius r_0 at which the particle begins falling, the radius r which is its present location, and the charge q.

To begin, consider Φ_s in Eq. (3.28). Its contribution to the self-force is given in Eq. (3.30). From Eq. (3.18) it is clear that $\partial_t = \partial_{t_R}$. Making use of the fact that $v(t,r;t_1,r_1)$ is a function of $(t-t_1)^2$ one finds that

$$\begin{aligned}\frac{\partial \Phi_s}{\partial t} &= \frac{q}{8\pi}\sqrt{1-\frac{2M}{r_0}}\left[v(t,r;t_R,r_0) - \int_0^{t_R} \frac{\partial}{\partial t_1} v(t,r;t_1,r_0)dt_1\right] \\ &= \frac{q}{8\pi}\sqrt{1-\frac{2M}{r_0}}\, v(t,r;0,r_0)\ . \qquad (3.45)\end{aligned}$$

Here as in section 3.4.2 we suppress the dependence of v on θ_0 and ϕ_0.

Next consider Φ_f in Eq. (3.62). The geodesic equations (3.4.1) can be used to change the integration variable from the proper time to the coordinate time. In this case $J = 0$ as the geodesic is radial. Since it starts from rest at r_0 it can be seen from the geodesic equations that $E = (1 - 2M/r_0)^{-1}$. The result is

$$\Phi_f = \frac{q}{8\pi} \frac{1}{\sqrt{1 - \frac{2M}{r_0}}} \int_0^t v(t, r; t_1, r_1) \left(1 - \frac{2M}{r_1}\right) dt_1 \ . \tag{3.46}$$

Noting that in a Schwarzschild spacetime $v(x, x) = 0$ and $v(x, x_1)$ is a function of $(t - t_1)^2$ due to the time translation and time reversal invariance of the metric (3.11), one finds

$$\frac{\partial \Phi_f}{\partial t} = -\frac{q}{8\pi} \frac{1}{\sqrt{1 - \frac{2M}{r_0}}} \int_0^t \left(\frac{\partial v(t, r; t_1, r_1)}{\partial t_1}\right) \left(1 - \frac{2M}{r_1}\right) dt_1 \ . \tag{3.47}$$

Since the particle is freely falling, $r_1 = r_1(t_1)$ and one can write

$$\frac{\partial v}{\partial t_1} \left(1 - \frac{2M}{r_1}\right) = \frac{d}{dt_1} \left[v \left(1 - \frac{2M}{r_1}\right)\right] - \frac{dr_1}{dt_1} \frac{\partial}{\partial r_1} \left[v \left(1 - \frac{2M}{r_1}\right)\right] \tag{3.48}$$

with the result that

$$\begin{aligned} \frac{\partial \Phi_f}{\partial t} = & \frac{q}{8\pi} \frac{1}{\sqrt{1 - \frac{2M}{r_0}}} \left\{ v[t, r; 0, r_0] \left(1 - \frac{2M}{r_0}\right) \right. \\ & \left. + \int_{r_0}^r \frac{\partial}{\partial r_1} \left[v(t, r; t_1, r_1) \left(1 - \frac{2M}{r_1}\right)\right] dr_1 \right\} \ . \end{aligned} \tag{3.49}$$

Thus

$$f_t = -\frac{q^2}{8\pi} \frac{1}{\sqrt{1 - \frac{2M}{r_0}}} \int_{r_0}^r \frac{\partial}{\partial r_1} \left[v(t, r; t_1, r_1) \left(1 - \frac{2M}{r_1}\right)\right] dr_1 \ . \tag{3.50}$$

The computation of f_{r^*} is straightforward. Taking the derivative of Eqs. (3.92) and (3.62) and then using the geodesic equations (3.4.1) and Eq. (3.34) one finds

$$\begin{aligned} f_{r^*} = & \frac{q^2}{8\pi} \sqrt{1 - \frac{2M}{r_0}} \left(1 - \frac{2M}{r}\right) \sum_{i,k=0}^{\infty} \left(\frac{k}{2i+1}\right) v_{i0k}(r) \left[t^{2i+1} - (t - t_R)^{2i+1}\right] (r - r_0)^{k-1} \\ & + \frac{q^2}{8\pi} \left(1 - \frac{2M}{r}\right) \sqrt{\frac{r_0}{2M}} \int_{r_0}^r \frac{\partial v(t, r; t_1, r_1)}{\partial r} \sqrt{\frac{r_1}{r_0 - r_1}} dr_1 \ . \end{aligned} \tag{3.51}$$

The next step is to expand t in powers of $r_0 - r$ and t_1 in powers of $r_0 - r_1$. This is done using the geodesic equation (3.13). Then making the change of variables

$$\begin{aligned} s &= \sqrt{\frac{r_0 - r}{r_0}} \\ s_1 &= \sqrt{\frac{r_0 - r_1}{r_0}} \\ x_0 &= \sqrt{\frac{2M}{r_0}} \\ f_0 &= \sqrt{1 - \frac{2M}{r_0}} \end{aligned} \tag{3.52}$$

gives

$$t = \frac{r_0}{x_0}\left[\frac{2s}{f_0} + \left(\frac{2}{3} - y_0^2\right)\left(\frac{s}{f_0}\right)^3 + \ \ldots \ .\right] \tag{3.53}$$

Substituting this and the corresponding expression for t_1 into Eqs. (3.50) and (3.51), expanding in powers of s and s_1 and computing the integrals give

$$f_t = q^2 \sum_{n=3}^{\infty} a_{2n} s^{2n} \tag{3.54}$$

$$f_{r*} = q^2 \sum_{n=2}^{\infty} b_{2n+1} s^{2n+1} \ . \tag{3.55}$$

In Appendix B, we have given the results that we have obtained using the highest WKB order result for $v(x, x')$ which is available to us.

3.5 The case of a particle launched from rest with an initial velocity on a general geodesic

3.5.1 Preliminaries

We begin by considering the scalar charge being held at rest at $r = r_0, \theta = \frac{\pi}{2}, \phi = 0$ from $t = -\infty$ to $t = 0$. At $t = 0$ the charge is given an initial velocity in a general direction and it continues to go along a geodesic for a duration of proper time equal to τ and coordinate time equal to t until it reaches the point with spacial coordinates r, θ and ϕ.

The general framework for calculating the self-force has been laid out in the last section. Here we generalize the arguments given there to apply them to the case of a general geodesic. It has been already noted that the self-force comes from two parts. The first part is the contribution of the particle trajectory from infinite past to the moment at which it is released and the second part is the contribution due to particle's motion after that. However since the first part is identical between the proposed trajectory here and the trajectory of an eternally static particle and since the potential for such a particle (that is the result of the full integration of the Green's function from $t = -\infty$ to $t = +\infty$) is found by Wiseman [26], we can use that result to calculate the first part. In the last section the particle trajectory was in the radial direction so we only needed to know the potential for the points that reside on the same radial direction that the charge's location is (that is $\theta = \theta'$ and $\phi = \phi'$). Here we need to know the potential everywhere.

3.5.2 Generalizing the Method to apply to a general Geodesic

The Wiseman potential for a stationary scalar charge at $x = (r_0, \theta = \frac{\pi}{2}, \phi = 0)$ is:

$$\Phi_{static}(r, \eta; r_0) = -\frac{1}{4\pi} q \sqrt{1 - \frac{2M}{r_0}} \frac{1}{\sqrt{\Delta r^2 - 2\eta r_0(r_0 - 2M) - 2\eta \Delta r(r_0 - M) + M^2\eta^2}} . \tag{3.56}$$

where $\eta = \cos\gamma - 1$ and $\cos\gamma = \cos\theta\cos\theta' + \cos(\phi - \phi')\,\sin\theta\,\sin\theta'$

One can formally obtain the same result by using Hadamard ansatz as:

$$
\begin{aligned}
\Phi_{static}(y) = \quad & \frac{q}{4\pi}\sqrt{1-\frac{2M}{r_0}}\int_0^t u(t,\mathbf{x};t',\mathbf{x}')\,\delta[\sigma(x,x')]\,dt' \\
- \quad & \frac{q}{8\pi}\sqrt{1-\frac{2M}{r_0}}\int_0^{t_R} v(t,\mathbf{x};t',\mathbf{x}')\,dt' \\
+ \quad & q\int_{-\infty}^{0} G_R[t,\mathbf{x};t'(\tau'),\mathbf{x}'(\tau')]\,d\tau'
\end{aligned}
\tag{3.57}
$$

It was then shown in the last section, that for the first term we have:

$$
\int_0^t u(t,\mathbf{x};t',\mathbf{x}')\,\delta[\sigma(x,x')]\,dt' = \frac{u(x,x')}{\sigma_t}\,|_{\sigma=0}
\tag{3.58}
$$

It can be shown to the desired order that the right side of (3.58) is identically equal to the Wiseman result (For details look at Appendix A). The fact that the tail term of the Green's function does not contribute could be in and out of itself interesting. It then, follows that:

$$
\int_{-\infty}^{0} G_R[t,\mathbf{x};t'(\tau'),\mathbf{x}'(\tau')]\,d\tau' = \frac{1}{8\pi}\sqrt{1-\frac{2M}{r_0}}\int_0^{t_R} v(t,\mathbf{x};t',\mathbf{x}')\,dt'
\tag{3.59}
$$

The result of this calculation is that, one of the results of the last section, Eq.(3.27), along with Eqs (3.28) and (3.62) are basically valid for any general geodesic path that the scalar charge is moving on after its release. For reference we repeat that result with new notation which will be explained shortly.

$$
f_a^{\ tail} = q\,\partial_a\,(\Phi_s - \Phi_f) \tag{3.60}
$$

$$
\Phi_s = \frac{q}{8\pi}\int_0^{t_R} v[t,\mathbf{x};t',r_0,0,0]\sqrt{1-\frac{2M}{r_0}}\;dt' \tag{3.61}
$$

$$
\Phi_f = \frac{q}{8\pi}\int_0^{\tau} v[t,\mathbf{x};t'(\tau'),\mathbf{x}'(\tau')]\;d\tau' \tag{3.62}
$$

3.6 Calculation of f_t, the temporal part of the self-force

In the section for radial trajectory, arguments were given to simplify the calculation of f_t here we generalize those arguments. For time derivative of the Φ_s part we have the following:

$$\partial_t \int_0^{t_R} v(t, \mathbf{x}; t', \mathbf{x_0}) \sqrt{1 - \frac{2M}{r_0}} \; dt' = \sqrt{1 - \frac{2M}{r_0}} \, v(t, \mathbf{x}; 0, \mathbf{x_0}) \tag{3.63}$$

To move further we note that the $d\tau'$ as the integral measure in the last equation is a complete differential, that is, we have substituted the r' as $r'(t')$ and also other coordinates. Therefore one can write

$$d\tau' = \frac{d\tau'}{dt'} dt' \tag{3.64}$$

From here one can follow the rest of argument

$$\begin{aligned} \partial_t \Phi_f &= \frac{q}{8\pi} \int_0^{\tau} \partial_t v[t, \mathbf{x}; t', \mathbf{x}'(t')] \; \frac{d\tau'}{dt'} dt' \\ &= -\frac{q}{8\pi} \int_0^{\tau} \frac{d\tau'}{dt'} \partial_{t'} v[t, \mathbf{x}; t', \mathbf{x}'] \mid_{\mathbf{x}'=\mathbf{x}'(t')} \; dt' \end{aligned} \tag{3.65}$$

From the geodesic equations, Eqs. (3.4.1) we one can see that,

$$\frac{d\tau}{dt} = \sqrt{E} f(r) \tag{3.66}$$

We have the following:

$$\frac{d\tau'}{dt'} \, \partial_{t'} v[t, \mathbf{x}; t', \mathbf{x}'] = \partial_{t'} \left(\frac{d\tau'}{dt'} \, v[t, \mathbf{x}; t', \mathbf{x}'] \right) - v[t, \mathbf{x}; t', \mathbf{x}'] \, \partial_{t'} \left(\frac{d\tau'}{dt'} \right) \tag{3.67}$$

The derivative $\partial_{t'}$ is with respect to explicit time dependence not the dependence through $x'(t')$. From (3.66), we notice that $\frac{d\tau'}{dt'}$ does not explicitly depend on t' the coordinate time along the geodesic path. Hence the second term in the recent equation vanishes. For any function (say, $V(t', x')$) whose t' dependence is both explicit and implicit (e.g. through dependence of $\mathbf{x}'$ on t') we can write

$$\frac{dV(t',\mathbf{x}')}{dt'} = \partial_{t'} V(t',\mathbf{x}') + \sum_{i=1}^{3} \frac{dx'_i}{dt'} \partial_{x'_i} V(t',\mathbf{x}') \tag{3.68}$$

Therefor we have

$$\frac{d\tau'}{dt'}\, \partial_{t'} v[t,\mathbf{x};t',\mathbf{x}'] = \frac{d}{dt'} \left(\frac{d\tau'}{dt'}\, v[t,\mathbf{x};t',\mathbf{x}'] \right) - \sum_{i=1}^{3} \frac{dx'_i}{dt'} \partial_{x'_i} \left(\frac{d\tau'}{dt'}\, v[t,\mathbf{x};t',\mathbf{x}'] \right) \tag{3.69}$$

From above we can deduce,

$$\partial_t \Phi_f = \frac{q}{8\pi} \left[\frac{d\tau'}{dt'}\, v[t,\mathbf{x};t',\mathbf{x}'] \right]_{t'=0;x'^i=x^i_0} + \frac{q}{8\pi} \sum_{i=1}^{3} \int_{x^i_0}^{x^i} dx'_i \;\; \partial_{x'_i} \left(\frac{d\tau'}{dt'}\, v[t,\mathbf{x};t',\mathbf{x}'] \right) \tag{3.70}$$

Putting all the pieces together, we can write:

$$\begin{aligned} f_t = \quad & \frac{q^2}{8\pi} \left(\sqrt{1 - \frac{2M}{r_0}} - \frac{d\tau'}{dt'} \|_{t'=0,x'^i=x^i_0} \right) \; v(t,\mathbf{x};0,\mathbf{x_0}) \\ - \quad & \frac{q^2}{8\pi} \sum_{i=1}^{3} \int_0^t dt' \frac{dx'^i}{dt'} \;\; \partial_{x'_i} \left(\frac{d\tau'}{dt'}\, v[t,\mathbf{x};t',\mathbf{x}'] \right) \end{aligned} \tag{3.71}$$

One point to clarify is that in the above equation, second term, we first take the derivative ∂_{x^i}, then substitute the path equations and *then* take the integration. At the end, depending on the path equation one can write the final result as an expansion in powers of t , $(r - r_0)$ or $(\phi - \phi_0)$.

For the general case, expanding the self-force in terms of the time series has this advantage that it can be easily applied to all geodesics. Henceforth our basic goal is to write the self-force as a series expansion in terms of the coordinate time. To that end first we need to find the parametric path equation of a particle following a geodesic path in Schwarzschild space-time. Without loss of generality we presume that the geodesic is in the equatorial plane, and therefore the only coordinates that change with time are $r(t)$ and $\phi(t)$. One can then easily write the equations of motion for such a particle in terms of t as follows:

$$\ddot{r} = \frac{1}{f(r)}\left(\frac{3M}{r^2}\dot{r}^2 + rf(r)\,\dot{\phi}^2 - \frac{M}{r^2}f^2(r)\right) \tag{3.72}$$

$$\ddot{\phi} = -\frac{2\dot{r}\dot{\phi}}{r}\left(1 - \frac{M}{rf(r)}\right) \tag{3.73}$$

where $\dot{r}$,$\ddot{r}$,$\dot{\phi}$ and $\ddot{\phi}$ are equivalent to $\frac{dr}{dt}$, $\frac{d^2r}{dt^2}$,$\frac{d\phi}{dt}$ and $\frac{d^2\phi}{dt^2}$ and $f(r) = 1 - 2M/r$. This is a non-linear system of coupled differential equations. To solve this we assume that $r(t)$ and $\phi(t)$ can be written as time series:

$$r(t) = \sum_{n=0}^{N} a_n t^n \tag{3.74}$$

$$\phi(t) = \sum_{n=0}^{N} b_n t^n \tag{3.75}$$

Then we substitute the two series into (3.72) and (3.73) and solve for the coefficients a_n's and b_n's. The final result for the most general equatorial geodesic in Schwarzschild spacetime will be determined in terms of a_1 and b_1, these are of course the initial velocities given to the particle at the time of lunch. To find the temporal component of the self-force in for a general geodesic, we then need to compute the two parts in (3.71) in a way that is convenient for calculation as a time series. To that end we use the symmetries of the space-time and the Eq. (3.66) to write the integrand in the second part of the equation (3.71) as follows

$$\frac{dr'}{dt'}\partial_{r'}\left(\frac{d\tau'}{dt'}v(t,\mathbf{x};t',\mathbf{x}')\right) + \frac{d\phi'}{dt'}\partial_{\phi'}\left(\frac{d\tau'}{dt'}v(t,\mathbf{x};t',\mathbf{x}')\right) =$$
$$\sqrt{E}\left(\dot{r'}\frac{df(r')}{dr'}v(t,\mathbf{x};t',\mathbf{x}') + f(r')\left(\dot{r'}\frac{dv(t,\mathbf{x};t',\mathbf{x}')}{dr'} + \dot{\phi'}\frac{dv(t,\mathbf{x};t',\mathbf{x}')}{d\phi'}\right)\right) \tag{3.76}$$

where, $\dot{r'}$ and $\dot{\phi'}$ are defined as $\frac{dr'}{dt'}$ and $\frac{d\phi'}{dt'}$, respectively. Using the result of [46], we can write :

$$v(t,\mathbf{x};t',\mathbf{x}') = \sum v_{i,k,m,n}(t-t')^{2i}(r-r')^k(\theta-\theta')^m(\phi-\phi')^n \tag{3.77}$$

Then the integrand can be written as:

$$\begin{aligned} &\frac{dr'}{dt'}\partial_{r'}\left(\frac{d\tau'}{dt'}v(t,\mathbf{x};t',\mathbf{x}')\right) + \frac{d\phi'}{dt'}\partial_{\phi'}\left(\frac{d\tau'}{dt'}v(t,\mathbf{x};t',\mathbf{x}')\right) = \\ &\sqrt{E}\sum(t-t')^{2i}(r-r')^k(\phi-\phi')^n\Big(\dot{r'}\frac{df(r')}{dr'}v_{i,k,0,n} - \dot{r'}f(r')(k+1)v_{i,k+1,0,n} - \\ &\qquad \dot{\phi'}f(r')(n+1)v_{i,k,0,n+1}\Big) \end{aligned} \tag{3.78}$$

After substitution of $\mathbf{x} = \{r(t), \phi(t)\}$ and $\mathbf{x}' = \{r'(t'), \phi'(t')\}$ and expansion one can evaluate the integral and find the result as a power series expansion in time. If the particle has non-zero initial velocity, for the three lowest non-vanishing terms, we can write f_t as follows:

$$f_t = f_{t_4}t^4 + f_{t_5}t^5 + f_{t_6}t^6 + O[t^7] \tag{3.79}$$

The fourth order term can be written as:

$$\begin{aligned} f_{t_4} = \frac{3\,q^2M^2}{17920\,\sqrt{f_0}\,\pi\,{r_0}^8}\Big(& 5\,{f_0}^4\,(1-Q) - 3\,{f_0}^2\,(2-Q)\,{a_1}^2 + a_1^4 \\ + & 9\,{f_0}^3\,(2-Q)\,{r_0}^2\,{b_1}^2 - 6\,f_0\,{r_0}^2\,{a_1}^2\,{b_1}^2 + 3\,{f_0}^2\,{r_0}^4\,{b_1}^4\Big) \end{aligned} \tag{3.80}$$

where

$$Q = \sqrt{\left(1 - \frac{a_1^2 + r_0^2 f_0 b_1^2}{f_0^2}\right)} \tag{3.81}$$

$$f_0 = 1 - \frac{2M}{r_0} \tag{3.82}$$

The fifth order term can be given as:

$$
\begin{aligned}
f_{t_5} = \quad & \frac{3\, f_0{}^{\frac{3}{2}}\, q^2 M^2\, a_1}{89600\, \pi\, r_0{}^{11}} \times \\
& \Big(\quad 106\, Q\, M^2 + \Big(305\, f_0\ (1-Q) - 93\, Q\Big) M r_0 - 20 \Big(5\, f_0\ (1-Q) - Q\Big)\, r_0{}^2 \\
& \quad + \ \frac{2\, r_0\, a_1{}^2}{f_0}\ (37\, M - 12\, r_0)\ (3Q-5) + \frac{5\, r_0\, a_1{}^4}{f_0^3}\, (13\, M - 4\, r_0) \\
& \quad - \ 6\, r_0{}^3\, b_1{}^2\, (45\, M - 16\, r_0)\ (3Q-5) - \frac{10\, r_0{}^3\, a_1{}^2\, b_1{}^2}{f_0^2}\ (47\, M - 16\, r_0) \\
& \quad + \ \frac{15\, r_0{}^5\, b_1{}^4}{f_0}\ (21\, M - 8\, r_0)\Big)
\end{aligned}
\tag{3.83}
$$

In the special case of a particle released from rest the above terms vanish and what remains is of sixth order which for a general case can be written as:

$$
\begin{aligned}
f_{t_6} = \quad & \frac{M^2 f_0{}^{\frac{7}{2}}}{537600\ \pi\, r_0{}^{12}} \times \\
& \Big(\quad 3\, M\, Q\ (53\, M - 20\, r_0) + 40\ (1-Q)\ \Big(3\, M^2 - 23\, M\, r_0 + 7\, r_0{}^2\Big) \\
& \quad - \frac{2\, a_1{}^2}{f_0^2} \Big(\quad 21\, M^2\ (-1035 + 1396\, Q) - 2\, M\ (-6675 + 9427\, Q)\ r_0 \\
& \qquad\qquad + \ 10\ (-195 + 292\, Q)\ r_0{}^2\Big) \\
& \quad + \frac{3\, a_1{}^4}{f_0{}^4} \Big(\quad M^2\ (-17820 + 12521\, Q) - 20\, (-560 + 397\, Q)\ M\, r_0 \\
& \qquad\qquad + \ 8\ (-210 + 151\, Q)\ r_0{}^2\Big) \\
& \quad + 2\, \frac{r_0{}^2\, b_1{}^2}{f_0} \Big(\quad 3\, M^2\ (-870 + 767\, Q) + M\ (615 - 1109\, Q)\ r_0 \\
& \qquad\qquad + \ 5\ (15 + 26\, Q)\ r_0{}^2\Big) \\
& \quad - \frac{6\, r_0{}^2\, a_1{}^2\, b_1{}^2}{f_0{}^3} \Big(\quad 3\, M^2\ (-12990 + 9067\, Q) + 2\ (13395 - 9431\, Q)\, M\ r_0 \\
& \qquad\qquad + \ 2\ (-2235 + 1591\, Q)\ r_0{}^2\Big) \\
& \quad + \frac{9\, r_0{}^4\, b_1{}^4}{f_0{}^2} \Big(\quad M^2\ (-2160 + 1189\, Q) - 4\ (-445 + 257\, Q)\, M\, r_0 \\
& \qquad\qquad + \ 72\ (3\, Q - 5)\ r_0{}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{6\,{r_0}^2\,{a_1}^4\,{b_1}^2}{f_0^5}\left(M^2\,\left(13980-44\,Q\right)+\left(-9305+13\,Q\right)M\,\,r_0+1495\,{r_0}^2\right)\\
&+\frac{18\,\,{r_0}^4\,{a_1}^2\,{b_1}^4}{{f_0}^4}\,\left(M^2\,\left(4185-11\,Q\right)+4\,\left(-775+Q\right)M\,r_0+560\,{r_0}^2\right)\\
&-\frac{2\,{a_1}^6}{f_0^6}\left(21\,M^2\,\left(-235+Q\right)+\left(2990-6\,Q\right)M\,r_0-430\,{r_0}^2\right)\\
&+\frac{2\,{r_0}^7\,{b_1}^6}{{f_0}^2}\left(3\,M\,\left(-335+3\,Q\right)+370\,r_0\right)\Bigg)
\end{aligned}
\tag{3.84}
$$

For the case of particle released from rest, that is for $a_1 = 0$,$b_1 = 0$, the sixth order result reduces to

$$
f_t \;=\; \left(\frac{{f_0}^{\frac{7}{2}}\,M^3\,\left(53\,M-20\,r_0\right)}{179200\,\pi\,{r_0}^{12}}\right)t^6 \tag{3.85}
$$

which after converting the t-dependence to r-dependence, via geodesic equation, will agree with the result presented in Eqs (3.32).

3.7 Calculation of f_i, The spacial components of the self-force.

For calculating the spacial coordinates of the self-force, we go back to calculating Φ_s. Incidentally this part is straightforward and can be written as:

$$
\Phi_s = \frac{q}{8\pi}\sqrt{1-\frac{2M}{r_0}}\sum_{i,j,k}\left(\frac{1}{2i+1}\right)v_{i,j,k}\left(t^{2i+1}-(t-t_R)^{2i+1}\right)(\cos\gamma-1)^j\,(r-r_0)^k \tag{3.86}
$$

Therefore, self-force can be written rather trivially as:

$$
\begin{aligned}
f_i \;&=\; \frac{\partial}{\partial x^i}\left(\frac{q^2}{8\pi}\sqrt{1-\frac{2M}{r_0}}\sum_{i,j,k}\left(\frac{1}{2i+1}\right)v_{i,j,k}\left(t^{2i+1}-(t-t_R)^{2i+1}\right)(\cos\gamma-1)^j\,(r-r_0)^k\right)\\
&-\;\frac{q^2}{8\pi}\int_0^\tau d\tau'\partial_{x^i}v(t,\mathbf{x};t'(\tau'),\mathbf{x}'(\tau'))
\end{aligned}
\tag{3.87}
$$

We further redefine the two terms of (3.87) as follows

$$
\begin{aligned}
f_{R_i} &\equiv \frac{\partial}{\partial x^i}\left(\sqrt{1-\frac{2M}{r_0}}\sum_{i,j,k}\left(\frac{1}{2i+1}\right)v_{i,j,k}\left(t^{2i+1}-(t-t_R)^{2i+1}\right)(\cos\gamma-1)^j(r-r_0)^k\right)
\end{aligned} \tag{3.88}
$$

$$
f_{P_i} \equiv \int_0^\tau d\tau'\partial_{x^i}v(t,\mathbf{x};t'(\tau'),\mathbf{x}'(\tau')) \tag{3.89}
$$

$$
f_i = \frac{q^2}{8\pi}\left(f_{R_i}-f_{P_i}\right) \tag{3.90}
$$

As for the first part, first we re-expand the angular part in powers of $(\theta-\theta')$ and $(\phi-\phi')$ and again without loss of generality assume that the geodesic is in the equatorial plane. We break the first part as follows

$$
\Phi_{s_1} \equiv \sum_{i,j,k}\left(\frac{1}{2i+1}\right)v_{i,k,0,n}t^{2i+1}\left(\phi-\phi'\right)^n\left(r-r'\right)^k \tag{3.91}
$$

$$
\Phi_{s_2} \equiv \sum_{i,j,k}\left(\frac{1}{2i+1}\right)v_{i,k,0,n}(\Delta t_R)^{2i+1}\left(\phi-\phi'\right)^n\left(r-r'\right)^k \tag{3.92}
$$

$$
f_{R_i} = \left(\sqrt{1-\frac{2M}{r_0}}\right)\frac{\partial}{\partial x^i}\left(\Phi_{s_1}-\Phi_{s_2}\right) \tag{3.93}
$$

t_R must be expanded as indicated in the Appendix A, in Eq. (A.8). The difference between t which characterizes the current coordinate time at which the particle is at r,θ and ϕ and t_R for which a point with coordinate r',θ' and ϕ' resides on the past light cone of the former point, depends on r,$\Delta r=(r-r')$, θ and $(\theta-\theta')$ and also $(\phi-\phi')$. The dependence can be extracted from the expansion given in Eq. (A.8). After substituting the expansion and combining everything we have for Φ_{s_2} (3.92):

$$
\Phi_{s_2} = \sum_{k,m,n}P_{k,m,n}(r,\theta)\left(\theta-\theta'\right)^m\left(\phi-\phi'\right)^n\left(r-r'\right)^k \tag{3.94}
$$

After taking the derivative and substituting the path equations, The final result would be an expansion in the powers of t. For a general geodesic with general a_1

and b_1, the result of this computation can be written as

$$f_{r*} = fr_4 t^4 + fr_5 t^5 + O[t^6] \tag{3.95}$$

where we have:

$$fr_4 = \frac{3\, {f_0}^{\frac{5}{2}}\, M^2\, a_1}{17920\, \pi\, {r_0}^8} \left(3\,Q - 4 + \frac{{a_1}^2 - 3\,{r_0}^2\, f_0\, {b_1}^2}{{f_0}^2}\, (4 - Q) + \frac{4\, {a_1}^3}{{f_0}^3} - \frac{12\,{r_0}^2\, a_1\, {b_1}^2}{{f_0}^2} \right) \tag{3.96}$$

$$\begin{aligned} fr_5 = & \quad -\frac{3\, {f_0}^{\frac{7}{2}}\, M^2}{89600\, \pi\, {r_0}^{10}} \Bigg(M + 4\, (1 - Q)\, (16\, M - 5\, r_0) \\ & + \ 2\frac{a_1^2}{f_0^2}\Big(M\, (-185 + 144\, Q) + 12\, (5 - 4\, Q)\, r_0\Big) - \frac{40 M a_1^3}{f_0^3} \\ & + \ \frac{a_1^4}{f_0^4}\Big(M\, (305 - 88\, Q) - 4\, (25 - 7\, Q)\, r_0\Big) + \frac{40 a_1^5}{f_0^5}(11M - 4r_0) \\ & + \ \frac{r_0^2 b_1^2}{f_0}\Big(M(290 - 171Q) - (100 - 63Q) r_0\Big) + \frac{60\, M\, {r_0}^2\, a_1\, {b_1}^2}{{f_0}^2} \\ & - \ \frac{3 r_0^2 a_1^2 b_1^2}{f_0^3}\Big(M\, (390 - 111\, Q) - (140 - 39\, Q)\, r_0\Big) - \frac{20\, {r_0}^2\, {a_1}^3\, {b_1}^2}{{f_0}^4}\, (73\, M - 26\, r_0) \\ & + \ \frac{3\, {r_0}^4\, {b_1}^4}{{f_0}^2}\Big(M\, (45 - 11\, Q) - 5\, (4 - Q)\, r_0\Big) - \frac{60\, {r_0}^5\, a_1\, {b_1}^4}{{f_0}^2} \Bigg) \end{aligned} \tag{3.97}$$

After putting, $Q = 1$,$a_1 = 0$ and $b_1 = 0$ for the case of release from rest, we have $fr_4 = 0$ and the above result reduces to

$$f_{r^*} = -\frac{3 f_0^{7/2}\, q^2\, M^3}{89600 \pi r_0^{10}} t^5 + O[t^6] \tag{3.98}$$

which again after using geodesic equations to convert t-dependence to r-dependence we arrive at the result presented in Eq. (3.33).

The azimuthal component of the self-force can be calculated along the same lines and the result for the lowest non-vanishing order is:

$$f_\phi = \frac{-9\, {f_0}^{\frac{5}{2}}\, M^2\, b_1}{17920\, \pi\, {r_0}^6} \Big(3Q - 4 - \frac{a_1^2 - r_0^2 b_1^2}{f_0^2}(Q-4) + \frac{4a_1^3}{f_0^3} + \frac{4\, {r_0}^2\, a_1\, {b_1}^2}{{f_0}^2} \Big) t^4 + O[t^5] \tag{3.99}$$

In practice, one must expand the relevant quantities only to the necessary order. Since, expanding some quantities to a higher order than is needed will produce undue computational cost in terms of memory and time, which sometimes can render the computation impractical.

3.8 A Specific example: Particle launched on a Keplerian circular orbit

Here we deal with an specific example. The particle is held at $r = r_0$ from $t = -\infty$ to $t = 0$ at which time, the particle is launched with Keplerian velocity on a Keplerian circular orbit. The path is,

$$z^\alpha(t) = \{r_0, \frac{\pi}{2}, \sqrt{\frac{M}{r_0}}\frac{t}{r_0}, t\} \tag{3.100}$$

$$\frac{dt}{d\tau} = \sqrt{\frac{r_0}{r_0 - 3M}} \tag{3.101}$$

$$\frac{d\phi}{d\tau} = \frac{1}{r_0}\sqrt{\frac{M}{r_0 - 3M}} \tag{3.102}$$

For comparison we note that these formulae are equivalent to a geodesic with

$$a_1 = 0 \tag{3.103}$$

$$b_1 = \sqrt{\frac{M}{r_0^3}} \tag{3.104}$$

With the Keplerian circular orbit, one can see that the calculations will be particularly simplified. For example f_t can be readily calculated to all orders for which the quasi-local expansion of $v(x, x')$ as noted in (3.77) is available . It, then can be seen that for f_t we have:

$$
\begin{aligned}
f_t^{(kc)} = \quad & \frac{q^2}{8\,\pi}\sqrt{\frac{r_0 - 3\,M}{r_0}} \times \\
& \sum_{i,n} \left(\frac{M}{r_0^3}\right)^{\frac{n}{2}} \left(\left(1 - \sqrt{\frac{r_0 - 2M}{r_0 - 3M}}\right) v_{i,0,0,n} - \Omega\, t \left(\frac{n+1}{2i+n+1}\right) v_{i,0,0,n+1}\right) t^{2i+n}
\end{aligned}
\tag{3.105}
$$

This formula can be compared with the general formula applied to Keplerian orbit using (3.103). The equivalence provides a further check on that result.

Calculation of the spacial components has two parts. The part denoted by f_R involves t_R and even with the simplification afforded by using the circular Keplerian orbit, providing a general formula is neither useful nor straightforward. The second part, f_P, that is the path dependent part can be calculated using a general formula similar to (3.105). At any rate the overall result of the calculation of $f_{r^\star}$ for the lowest non-vanishing order is

$$
\begin{aligned}
f_{r^\star} = \frac{-3\, f_0\, M^2\, q^2}{89600\,\pi\, {r_0}^{12}} \Bigg(& \quad (185\,\omega_0 - 53\,\beta_0)\ M^3 - 18\ (5\,\omega_0 + 3\,\beta_0)\ M^2\, r_0 \\
& - \quad 9\ (5\,\omega_0 - 9\,\beta_0)\ M\, {r_0}^2 + 20\ (\omega_0 - \beta_0)\ {r_0}^3 \Bigg) t^5 + O[t^6]
\end{aligned}
\tag{3.106}
$$

$$
\beta_0 \;=\; \sqrt{1 - \frac{2M}{r_0}} \tag{3.107}
$$

$$
\omega_0 \;=\; \sqrt{1 - \frac{3M}{r_0}} \tag{3.108}
$$

and for f_ϕ

$$f_\phi = \frac{-9\, f_0\, M^2}{17920\, \pi\, {r_0}^8} \sqrt{\frac{M}{r_0}} \Big(4\, \beta_0\, \left(M - r_0 \right) - \omega_0 \left(5\, M - 3\, r_0 \right) \Big) t^4 + O[t^6] \qquad (3.109)$$

Higher order results are given in Appendix C.

Chapter 4

The noise kernel of a massless, conformally coupled scalar field in the optical Schwarzschild space-time.

4.1 Introduction and Background

In chapter 1, we introduced the noise kernel and showed how it presents the next step from semi-classical gravity on the way to quantum gravity. Here we bring the relationships that yield the noise kernel and proceed by calculating its trace from two related approaches.

The general form of the noise kernel for a general space-time and coupling has been obtained [66]. For massless scalar field and conformal coupling, the complete form of the noise kernel functional $N_{abc'd'}[G]$ is

$$N_{abc'd'}[G] = \tilde{N}_{abc'd'}[G] + g_{ab}\tilde{N}_{c'd'}[G] + g_{c'd'}\tilde{N}'_{ab}[G] + g_{ab}g_{c'd'}\tilde{N}[G] \tag{4.1}$$

with [1]

$$\begin{aligned} 72\tilde{N}_{abc'd'}[G] &= 4\left(G_{;c'b}\,G_{;d'a} + G_{;c'a}\,G_{;d'b}\right) + G_{;c'd'}\,G_{;ab} + G\,G_{;abc'd'} \\ &\quad -2\left(G_{;b}\,G_{;c'ad'} + G_{;a}\,G_{;c'bd'} + G_{;d'}\,G_{;abc'} + G_{;c'}\,G_{;abd'}\right) \end{aligned}$$

[1] Notice that these equations have two slight but crucial differences with the equations of [66]. The sign of the last term of the equation for $N_{abc'd'}$ and also the sign of the term $GG^{p'}_{a,b,p'}$ have been corrected.

$$+2\,(G_{;a}\,G_{;b}\,R_{c'd'} + G_{;c'}\,G_{;d'}\,R_{ab})$$

$$-\,(G_{;ab}\,R_{c'd'} + G_{;c'd'}\,R_{ab})\,G + \frac{1}{2}R_{c'd'}\,R_{ab}G^2 \qquad (4.2)$$

$$\begin{aligned} 288\tilde{N}'_{ab}\,[G] \;=\;& 8\left(-G_{;p'b}\,G_{;}{}^{p'}{}_{a} + G_{;b}\,G_{;p'a}{}^{p'} + G_{;a}\,G_{;p'b}{}^{p'}\right) \\ & 4\left(G_{;}{}^{p'}\,G_{;abp'} - G_{;p'}{}^{p'}\,G_{;ab} - G\,G_{;abp'}{}^{p'}\right) \\ & -2\,R'\,\left(2\,G_{;a}\,G_{;b} - G\,G_{;ab}\right) \\ & -2\,\left(G_{;p'}\,G_{;}{}^{p'} - 2\,G\,G_{;p'}{}^{p'}\right)\,R_{ab} - R'\,R_{ab}G^2 \end{aligned} \qquad (4.3)$$

$$\begin{aligned} 288\tilde{N}\,[G] \;=\;& 2\,G_{;p'q}\,G_{;}{}^{p'q} + 4\,\left(G_{;p'}{}^{p'}\,G_{;q}{}^{q} + G\,G_{;p}{}^{p}{}_{q'}{}^{q'}\right) \\ & -4\,\left(G_{;p}\,G_{;q'}{}^{pq'} + G_{;}{}^{p'}\,G_{;q}{}^{q}{}_{p'}\right) \\ & +R\,G_{;p'}\,G_{;}{}^{p'} + R'\,G_{;p}\,G^{;p} \\ & -2\,\left(R\,G_{;p'}{}^{p'} + R'\,G_{;p}{}^{p}\right)G + \frac{1}{2}R\,R'G^2 \end{aligned} \qquad (4.4)$$

In ref. [82] the Gaussian approximation for the Green's function has been computed and used to calculate a re-normalized expression for the noise kernel.

However we believe that the noise kernel is a distribution and physical observable quantities are obtained by integrating suitable functions with the noise kernel. As such, the noise kernel with points kept separated contains additional information over and beyond its coincidence limit. On the other hand, the noise kernel in and out of itself can be used as a device to study fluctuations of the stress-energy tensor. Studying correlation of these fluctuations at two different points can yield new and important information, hence, calculating the noise kernel at two separated points is useful and hitherto unexplored. Consequently, in the following sections we take a different approach as we assume that the points at which the noise kernel is to be evaluated are not brought together. This assumption frees us from the requirement of renormalization and allows us to consider other forms of Green functions.

For the rest of this chapter, we calculate the Wightman Green functions and verify that they indeed satisfy the field equations. We then demonstrate how Wightman and other functions with the same form can be used for calculating the noise kernel bi-tensor.

4.2 Wightman function in optical Schwarzschild space-time and calculation of the noise kernel

4.3 Preliminaries

We are considering the following general metric

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \tag{4.5}$$

Any metric that has $g_{tt} = -1$ in Minkowskian or $g_{\tau\tau} = 1$ in Euclidean space-time is called an optical metric. Furthermore if the metric is time independent we have an ultra-static metric. One can go from Minkowskian to Euclidean space by using

$$\tau = -i\, t \tag{4.6}$$

for non-zero temperature we define:

$$\kappa = \frac{T}{2\pi} \tag{4.7}$$

For a conformally coupled massless scalar field we have:

$$(\Box - \xi R)G^+(x, x') = (\nabla^\mu \nabla_\mu - \frac{1}{6})G^+(x, x') = 0 \tag{4.8}$$

The optical Schwarzschild metric is defined as

$$ds^2 = -dt^2 + \frac{1}{\left(1-\frac{2M}{r}\right)^2}dr^2 + \frac{r^2}{1-\frac{2M}{r}}d\Omega^2 \tag{4.9}$$

which is conformally related to normal Schwarzschild metric. Synge's world function then is equal to

$$\sigma(x,x') = \frac{1}{2}\left(-(t-t')^2 + \mathrm{r}^2\right) \tag{4.10}$$

where

$$\mathrm{r} = \sqrt{2\,^{(3)}\sigma} \tag{4.11}$$

Note that r (with bold roman font) denotes the square root of three dimensional Synge's world function while r (with normal italic font) denotes the radial coordinate. and $^{(3)}\sigma$ is the three dimensional world function.

4.4 Wightman function by Analytical Extension of Hadamard Green function : Gaussian Approximation

The Gaussian approximation for Hadamard Green function of a massless scalar field, conformally coupled to an Euclidean optical Schwarzschild spacetime has been calculated as follows [78]:

$$G_{Gauss}(\tau,\mathbf{x},0,\mathbf{x}') = \frac{1}{8\pi^2}\frac{\kappa\Delta^{1/2}}{\mathrm{r}}\frac{\sinh\kappa\mathrm{r}}{\cosh\kappa\mathrm{r}-\cos\kappa\tau} \tag{4.12}$$

This can be expanded to yield [82]:

$$G_{\text{Gauss}} = \frac{\Delta^{\frac{1}{2}}}{8\,\pi^2\,\sigma} + \frac{\Delta^{\frac{1}{2}}}{8\,\pi^2}\left\{\frac{\kappa^2}{6} + \frac{\kappa^4}{180}\left(2\,\Delta\tau^2 - \sigma\right)\right.$$

$$+\frac{\kappa^6}{3780}\left(4\,\Delta\tau^4 - 6\,\Delta\tau^2\,\sigma + \sigma^2\right)\Bigg\} + O\left(\sigma^{\frac{5}{2}}, \delta\tau^5\right) \quad (4.13)$$

To arrive at the Wightman function we apply the following prescription

$$t - t' \to t - t' + i\epsilon \quad (4.14)$$

and use the principal value theorem as

$$\frac{1}{x + i\epsilon} = \text{P.V.}\frac{1}{x} - \pi i \delta(x) \quad (4.15)$$

Since except for the first term in (4.106) the rest are regular, it follows that we only need to apply the prescription to the first term. Working in ultra-static Euclidian metric we write

$$\sigma = \frac{1}{2}\left(\Delta\tau^2 + \mathrm{r}^2\right) \quad (4.16)$$

where $\Delta\tau = i\Delta t$ and $\mathrm{r} = (2\,{}^{(3)}\sigma)^{1/2}$ the three dimensional σ. Carefully applying the prescription

$$\Delta t \to \Delta t - i\epsilon \quad (4.17)$$

this means,

$$\sigma = \frac{1}{2}\left(\Delta\tau^2 + \mathrm{r}^2\right) \to \frac{1}{2}\left(-(\Delta t - i\epsilon)^2 + r^2\right) = \frac{1}{2}\left(-\Delta t^2 + \mathrm{r}^2 + 2i\,\Delta t\,\epsilon + \epsilon^2\right) \quad (4.18)$$

We have:

$$\frac{1}{-\Delta t^2 + \mathrm{r}^2 + 2i\,\Delta t\,\epsilon + \epsilon^2} = \frac{1}{2\mathrm{r}}\left(\frac{1}{-\Delta t + \mathrm{r} + i\epsilon} + \frac{1}{+\Delta t + \mathrm{r} - i\epsilon}\right) \quad (4.19)$$

That is:

$$\frac{1}{-\left(\Delta t - i\epsilon\right)^2 + \mathrm{r}^2} = \frac{1}{2\mathrm{r}}\left(\frac{1}{-\Delta t + \mathrm{r} + i\epsilon} + \frac{1}{+\Delta t + \mathrm{r} - i\epsilon}\right) \quad (4.20)$$

$$
\begin{aligned}
&= \frac{1}{2\mathrm{r}}\left(\frac{1}{-\Delta t+\mathrm{r}} - i\pi\delta(\mathrm{r}-\Delta t) + \frac{1}{\Delta t+\mathrm{r}} + i\pi\delta(\mathrm{r}+\Delta t)\right) \\
&= \frac{1}{-\Delta t^2+\mathrm{r}^2} - i\pi\left(\frac{\delta(-\Delta t+\mathrm{r})-\delta\left(\Delta t+\mathrm{r}\right)}{2\mathrm{r}}\right)
\end{aligned} \tag{4.21}
$$

We then conclude that

$$
G^+ = G_{Gauss} - \frac{i\Delta^{1/2}}{8\pi}\left(\frac{\delta(-\Delta t+\mathrm{r})-\delta\left(\Delta t+\mathrm{r}\right)}{2\mathrm{r}}\right) \tag{4.22}
$$

4.5 Verification

We have to show that

$$
\left(\nabla^\mu\nabla_\mu - \frac{R}{6}\right)G^+(x,x') = 0 \tag{4.23}
$$

First we start with the imaginary part of $G^+(x,x')$, that is we try to show that

$$
\left(\nabla^\mu\nabla_\mu - \frac{R}{6}\right)\left(\frac{i\Delta^{1/2}}{8\pi}\left(\frac{\delta(-\Delta t+\mathrm{r})-\delta\left(\Delta t+\mathrm{r}\right)}{2\mathrm{r}}\right)\right) = 0 \tag{4.24}
$$

4.5.1 The Imaginary part

We define,

$$
D \equiv \frac{\delta(-\Delta t+\mathrm{r})-\delta\left(\Delta t+\mathrm{r}\right)}{2\mathrm{r}} \tag{4.25}
$$

we then have

$$
\left(\Box - \frac{R}{6}\right)8\pi ImG^+ = D\left(\nabla^\mu\nabla_\mu - \frac{R}{6}\right)\Delta^{1/2} + 2\nabla_\mu D\ \nabla^\mu\Delta^{1/2} + \Delta^{1/2}\Box D \tag{4.26}
$$

To evaluate the right hand side of the above equation, let us calculate the effect of two relevant operators on $\frac{\delta(\mathrm{r}\pm\Delta t)}{\mathrm{r}}$. First let us calculate the effect of operating $\Box$,

$$
\begin{aligned}
\Box\frac{\delta(\mathrm{r}\pm\Delta t)}{\mathrm{r}} &= \left(-\frac{\partial^2}{\partial t^2} + \nabla_\mu\nabla^\mu\right)\frac{\delta(\mathrm{r}\pm\Delta t)}{\mathrm{r}} \\
&= \mathrm{r}^{-1}\left(-\frac{\partial^2}{\partial t^2} + \nabla_\mu\ \nabla^\mu\right)\delta(\mathrm{r}\pm\Delta t) + 2\nabla_\mu\mathrm{r}^{-1}\ \nabla^\mu\delta(\mathrm{r}\pm\Delta t) + \delta(\mathrm{r}\pm\Delta t)\Box\mathrm{r}^{-1}
\end{aligned} \tag{4.27}
$$

In an optical ultrastatic metric, r and $\Delta^{1/2}$ are independent of time, hence,

$$\Box \frac{\delta(\mathrm{r} \pm t)}{\mathrm{r}} = -\frac{1}{\mathrm{r}} \Big[\Big(1 - \nabla_i \mathrm{r}\ \nabla^i \mathrm{r}\Big) \delta''(\mathrm{r} \pm t) + \frac{1}{\mathrm{r}^2} \Big(\mathrm{r} \nabla_i \nabla^i \mathrm{r} - 2 \nabla_i \mathrm{r}\ \nabla^i \mathrm{r}\Big) \left(\mathrm{r}\delta'(\mathrm{r} \pm t) - \delta(\mathrm{r} \pm t)\right) \Big] \tag{4.28}$$

where we have adopted the notation that $\delta'(u) = \frac{d\delta(u)}{du}$ and $\delta''(u) = \frac{d^2\delta(u)}{du^2}$ and Δt is redefined as t with $t' = 0$. We then calculate $\left(\Delta^{1/2}\right)_{,i} \left(\mathrm{r}^{-1}\delta(\mathrm{r} \pm t)\right)^{,i}$

$$\left(\Delta^{1/2}\right)_{,i} \left(\frac{\delta(\mathrm{r} \pm t)}{\mathrm{r}}\right)^{,i} = \frac{1}{\mathrm{r}^2} \left(\Delta^{1/2}\right)^{,i} \nabla_i \mathrm{r} \Big(\mathrm{r}\delta'(\mathrm{r} \pm t) - \delta(\mathrm{r} \pm t)\Big) \tag{4.29}$$

Using the above relationships we can show that the last two terms of (4.26) are equal to

$$\begin{aligned} & 2\nabla_\mu \left(\frac{\delta(\mathrm{r} \pm t)}{\mathrm{r}}\right) \nabla^\mu \Delta^{1/2} + \Delta^{1/2} \Box \left(\frac{\delta(\mathrm{r} \pm t)}{\mathrm{r}}\right) \\ = & \left(\frac{1}{\mathrm{r}^2}\, 2 \left(\Delta^{1/2}\right)_{,i} \nabla^i \mathrm{r} + \frac{1}{\mathrm{r}^2} \Delta^{1/2} (\mathrm{r} \nabla_i \nabla^i \mathrm{r} - 2 \nabla_i \mathrm{r}\ \nabla^i \mathrm{r})\right) (\mathrm{r}\ \delta'(\mathrm{r} \pm t) - \delta(\mathrm{r} \pm t)) \end{aligned} \tag{4.30}$$

We now use the properties of $\Delta^{1/2}$, r and σ:

$$\mathrm{r} = (2\ {}^3\sigma)^{1/2} \tag{4.31}$$

$$\nabla_i \mathrm{r} = \frac{{}^3\sigma_i}{\mathrm{r}} \tag{4.32}$$

$$\nabla^2 \mathrm{r} = \frac{{}^3\sigma_i^i - 1}{\mathrm{r}} \tag{4.33}$$

$$\nabla_i \mathrm{r}\ \nabla^i \mathrm{r} = \frac{{}^3\sigma_i\ {}^3\sigma^i}{2\ {}^3\sigma} = 1 \tag{4.34}$$

$$2 \left(\Delta^{1/2}\right)_{,i} \nabla^i \mathrm{r} = \left(\frac{2}{\mathrm{r}} - \nabla^2 \mathrm{r}\right) \Delta^{1/2} \tag{4.35}$$

where $\nabla^2 = \nabla^i \nabla_i$. Because of Eq. (4.34) and Eq. (4.35), the equation Eq. (4.30) vanishes. Furthermore, the first term of Eq. (4.28) is identically zero. From these considerations one can see that the last two terms of Eq. (4.26) vanish. As a result we have,

$$\left(\Box - \frac{R}{6}\right) 8\pi\ Im\ G^{+} = D\left(\nabla^{\mu}\nabla_{\mu} - \frac{R}{6}\right)\Delta^{1/2} \tag{4.36}$$

To proceed further, we use the sixth order expansion of $\Delta^{1/2}$ given by

$$\begin{aligned} \Delta^{1/2} &= 1 + \Delta^{(2)}_{pq}\sigma^p\sigma^q + \Delta^{(3)}_{pqr}\sigma^p\sigma^q\sigma^r + \Delta^{(4)}_{pqrs}\sigma^p\sigma^q\sigma^r\sigma^s + \Delta^{(5)}_{pqrst}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t \\ &\quad + \Delta^{(6)}_{pqrstu}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u \end{aligned} \tag{4.37}$$

where

$$\Delta^{(2)}{}_{ab} = \frac{R_{ab}}{12} \tag{4.38}$$

$$\Delta^{(3)}{}_{abc} \doteq -\frac{R_{ab;c}}{24} \tag{4.39}$$

$$\Delta^{(4)}{}_{abcd} \doteq \frac{1}{1440}\left(18\, R_{ab;cd} + 5\, R_{ab}\, R_{cd} + 4\, R_{paqb}\, R_{c}{}^{q}{}_{d}{}^{p}\right) \tag{4.40}$$

$$\Delta^{(5)}{}_{abcde} \doteq -\frac{1}{1440}\left(4\, R_{ab;cde} + 5\, R_{ab;c}\, R_{de} + 4\, R_{paqb;c}\, R_{d}{}^{q}{}_{e}{}^{p}\right) \tag{4.41}$$

$$\begin{aligned} \Delta^{(6)}{}_{abcdef} &\doteq \frac{1}{362880}\left(315\, R_{ab;c}\, R_{de;f} + 180\, R_{ab;cdef} + 378\, R_{ab;cd}\, R_{ef}\right. \\ &\quad 35\, R_{ab}\, R_{cd}\, R_{ef} + 84\, R_{ab}\, R_{pcqd}\, R_{e}{}^{q}{}_{f}{}^{p} - 270\, R_{paqb;c}\, R_{d}{}^{q}{}_{e}{}^{p}{}_{;f} \\ &\quad \left. -288\, R_{paqb;cd}\, R_{e}{}^{q}{}_{f}{}^{p} + 64\, R_{paqb}\, R_{rdc}{}^{q}\, R_{e}{}^{r}{}_{f}{}^{p}\right) \end{aligned} \tag{4.42}$$

as shown in [76] where $\doteq$ is defined to mean equal after symmetrization.

Consequently we have

$$\left(\nabla_{\mu}\nabla^{\mu} - \frac{R}{6}\right)\Delta^{1/2} = Q_0 + Q_p\sigma^p + Q_{pq}\sigma^p\sigma^q + \cdots \tag{4.43}$$

where the right hand side is to be determined. To do so, we substitute (4.37) into (4.43) and consider that

$$\sigma_{ab} = g_{ab} + \frac{1}{3}\left(R_{acdb} + R_{adcb}\right)\sigma^c\sigma^d + \cdots \tag{4.44}$$

For the zeroth order term on the right side of Eq. (4.43) we find:

$$Q_0 = 2\Delta^{(2)}_{pq} g^{pq} - \frac{1}{6}R = \frac{R}{6} - \frac{R}{6} = 0 \tag{4.45}$$

For the first order we find,

$$Q_a = \Delta^{(2)}_{pq;a} - 3\Delta^{(3)}_{pqa} \tag{4.46}$$

Using Eqs. (4.38) and (4.39) for the values of the coefficients of expansion of $\Delta^{1/2}$, we have:

$$Q_a = R_a{}^b{}_{;b} - \frac{1}{2}R_{;a} = G_a{}^b{}_{;b} = 0 \tag{4.47}$$

Since Einstein tensor is divergenceless by virtue of the Bianchi identities. This proves that $\Delta^{1/2}$ expanded to the third order satisfies the conformally coupled field equation. For the second order term on the right hand side of Eq. (4.43) we have:

$$\begin{aligned} Q_{ab} = \left(\frac{1}{360}\right)\Big(& 9\,R_{;ab} + 9\,R_{ab;p}{}^{p} - 42\,R_{bp;a}{}^{p} + 18\,R_{bp;}{}^{p}{}_{a} + 30\,R_{a}{}^{p}\,R_{bp} \\ - & 36\,R_{pq}R_{a}{}^{p}{}_{b}{}^{q} + 4\,R_{a}{}^{upq}\,R_{bupq} + 4\,R_{apqu}\,R_{b}{}^{qpu}\Big) \end{aligned} \tag{4.48}$$

This expression does not always vanish. However, one can show that for the Optical Schwarzschild metric, it vanishes identically, consistent with [78, 76].[2]

4.5.2 The Real Part

The real part of the Green's function is G_{gauss} and we now attempt to find out to what order it satisfies the field equation, that is, we try to verify

$$\left(\nabla_\mu \nabla^\mu - \frac{1}{6}R\right) G_{gauss} = 0 \tag{4.49}$$

[2]The author thanks Dr. Albert Roura for pointing out a previous error in the calculation of (4.48)

We remind ourselves that

$$G_{gauss} = \frac{u}{\mathrm{r}} \frac{\kappa \sinh(\kappa\,\mathrm{r})}{\cosh(\kappa\,\mathrm{r}) - \cos(\kappa\,\tau)} \tag{4.50}$$

Due to the fact that the metric is ultra-static, the field equation can be written as a spatial and a temporal part, (with $\tau = -it$).

$$\nabla_\mu \nabla^\mu G_{gauss} = \left(\frac{\partial^2}{\partial \tau^2} + \nabla_i \nabla^i \right) G_{gauss} \tag{4.51}$$

and after applying the differential operator and using (4.34) and (4.35)the right hand side contains two main terms, one is a factor of $\nabla^2 \mathrm{r} = \nabla_i \nabla^i \mathrm{r}$ and the other a factor of $\nabla^2 u$, (where $u \equiv \Delta^{1/2}$).

A rigorous calculation has to include using Eq. (4.34) and Eq. (4.35). The result of such a calculation is

$$\nabla_\mu \nabla^\mu G_{gauss} = C_u \nabla^2 u \tag{4.52}$$

$$C_u = \frac{1}{\mathrm{r}} \frac{\kappa \sinh(\kappa\,\mathrm{r})}{\cosh(\kappa\,\mathrm{r}) - \cos(\kappa\,\tau)} \tag{4.53}$$

(All other terms cancel each other in the process of applying the above identities)

Therefore

$$\left(\Box - \frac{1}{6} R \right) G_{gauss} = C_u \left(\nabla^2 - \frac{1}{6} R \right) \Delta^{1/2} \tag{4.54}$$

Once the divergent parts of C_u is factored out, the rest is at least of second order, or higher, showing that the original expression, if expanded to fourth order, satisfies the field equation.

We have already demonstrated that $(\Box - R/6)\, \Delta^{1/2}$ is of second order.

4.6 Comparing and contrasting various formulations for Green functions

Calculating the Green's functions in curved space-times is usually at the heart of many inquiries involving quantum field processes. Here we go over some of the known results [66] and compare them with our result obtained above.

In all these expressions, the Hadamard ansatz for the Green's function is assumed as follows:

$$G(x,x') = \frac{1}{(4\pi)^2}\left(\frac{u(x,x')}{\sigma} + v(x,x')log(\sigma) + w(x,x')\right) \tag{4.55}$$

where, σ is Synge's world function and $u(x,x')$, $v(x,x')$ and $w(x,x')$ are functions to be determined such that the Hadamard function satisfies the following equation:

$$(\Box - \xi R)\, G(x,x') = 0 \tag{4.56}$$

where ξ is called the conformal factor and we have $\xi = 0$ for minimal coupling or $\xi = \frac{1}{6}$ for conformal coupling. One can easily substitute the Hadamard ansatz into this differential equation and find relationships amongst $u(x,x')$, $v(x,x')$ and $w(x,x')$ [4]. It is worth mentioning that $w(x,x')$ can not be determined from the field equation since it depends on specific boundary conditions. After substituting the Hadamard ansatz, and using the properties of the van Vleck determinant we find that

$$u(x,x') \quad = \quad \Delta^{1/2} \tag{4.57}$$

$$(\Box - \xi R)\, v(x,x') \quad = \quad = 0 \tag{4.58}$$

From the above equations, (provided that $\sigma \neq 0$):

$$(\Box - \xi R)\Delta^{1/2} = (2 - \Box\sigma)v - 2v_{;\alpha}\sigma^{\alpha} \tag{4.59}$$

Phillips and Hu [82] obtained the following expressions for $v(x, x')$ for a massless conformally coupled scalar field,

$$v(x,x') = v_0(x,x') + \sigma v_1(x,x') + \sigma^2 v_2(x,x') + O(\sigma^3) \tag{4.60}$$

$$v_0(x,y) = v_0^{(0)} + \sigma^p v_{0p}^{(1)} + \sigma^p\sigma^q v_{0pq}^{(2)} + \sigma^p\sigma^q\sigma^r v_{0pqr}^{(3)} + \sigma^p\sigma^q\sigma^r\sigma^s v_{0pqrs}^{(4)} \tag{4.61}$$

$$v_1(x,y) \approx v_1^{(0)} + \sigma^p v_{1p}^{(1)} + \sigma^p\sigma^q v_{1pq}^{(2)} \tag{4.62}$$

$$v_{0ab}^{(2)} = \left(R_{;ab} - 3\,R_{ab;p}{}^{p} + 4\,R_{pa}\,R_{b}{}^{p} - 2\,R_{pq}\,R_{a}{}^{q}{}_{b}{}^{p} + 2\,R_{pqra}\,R_{b}{}^{rpq}\right)/360 \tag{4.63}$$

$$\begin{aligned} v_{0abc}^{(3)} = {} & (5\,R_{;abc} - 14\,R_{pa;bc}{}^{p} - 7\,R_{ab;pc}{}^{p} + 5\,R_{ab;p}{}^{p}{}_{c} + 8\,R_{ab;cp}{}^{p} \\ & +10\,R_{pa;b}\,R_{c}{}^{p} + 15\,R_{ab;p}\,R_{c}{}^{p} - 10\,R_{paqb;c}\,R^{pq} \\ & -24\,R_{pq;a}\,R_{b}{}^{q}{}_{c}{}^{p} + 4\,R_{pa;q}\,R_{b}{}^{q}{}_{c}{}^{p} - 8\,R_{paqr;b}\,R_{c}{}^{pqr} \\ & +2\,R_{paqb;r}\,R_{c}{}^{pqr} - 2\,R_{paqr;b}\,R_{c}{}^{qpr} + 2\,R_{paqb;r}\,R_{c}{}^{qpr} \\ & +2\,R_{paqr;b}\,R_{c}{}^{rpq})/1440 \end{aligned} \tag{4.64}$$

The second result we want to compare with is due to Décanini and Folacci [83]. Their results, when specialized to the conformal coupling massless case yield:

$$v(x,x') = \sum_{n=0}^{\infty} v_n(x,x')\sigma^n \tag{4.65}$$

$$\begin{aligned} v_0(x,x') = {} & v_0 - v_{0\ a}\sigma^{;a} + \frac{1}{2!}v_{0\ ab}\sigma^{;a}\sigma^{;b} - \frac{1}{3!}v_{0\ abc}\sigma^{;a}\sigma^{;b}\sigma^{;c} \\ & + \frac{1}{4!}v_{0\ abcd}\sigma^{;a}\sigma^{;b}\sigma^{;c}\sigma^{;d} + O\left(\sigma^{5/2}\right) \end{aligned} \tag{4.66}$$

$$v_1(x,x') = v_1 - v_{1\ a}\sigma^{;a} + \frac{1}{2!}v_{1\ ab}\sigma^{;a}\sigma^{;b} + O\left(\sigma^{3/2}\right) \tag{4.67}$$

$$v_2(x,x') = v_2 + O\left(\sigma^{1/2}\right) \tag{4.68}$$

$$v_0 = 0 \tag{4.69}$$

$$v_{0\ a} = 0 \tag{4.70}$$

$$v_{0\ ab} = -(1/120)\,\Box R_{ab} + (1/360)\ R_{;ab}$$

$$+(1/90)\,R^{\rho}{}_{a}R_{\rho b} - (1/180)\,R^{\rho\sigma}R_{\rho a\sigma b} - (1/180)\,R^{\rho\sigma\tau}{}_{a}R_{\rho\sigma\tau b} \tag{4.71}$$

$$v_{0\ abc} = (1/240)\,R_{;(abc)} - (1/80)\,(\Box R_{(ab)})_{;c)} + (1/8)\,m^2\,R_{(ab;c)}$$

$$+(1/240)\,RR_{(ab;c)} + (1/30)\,R^{\rho}{}_{(a}R_{|\rho|b;c)}$$

$$-(1/120)\,R^{\rho}{}_{\sigma}R^{\sigma}{}_{(a|\rho|b;c)} - (1/120)\,R^{\rho}{}_{\sigma;(a}R^{\sigma}{}_{b|\rho|c)} - (1/60)\,R^{\rho\sigma\tau}{}_{(a}R_{|\rho\sigma\tau|b;c)} \tag{4.72}$$

$$v_1 = (1/720)\,\Box R - (1/720)\,R_{\rho\sigma}R^{\rho\sigma} + (1/720)\,R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} \tag{4.73}$$

$$v_{1\ a} = (1/1440)\,(\Box R)_{;a} - (1/720)\,R_{\rho\sigma}R^{\rho\sigma}{}_{;a} + (1/720)\,R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa}{}_{;a} \tag{4.74}$$

where a notation like $R^{\rho}{}_{(a}R_{|\rho|b;c)}$is defined as being symmetrized with respect to indices a, b and c excluding the index ρ.

The third result is due to Anderson and Hu [84], also detailed in this thesis in chapter 2. For a massless scalar field in Schwarzschild space-time, using specific Schwarzschild coordinates, It expresses $v(x,x')$ as a coordinate expansion as follows:

$$v(x,x') = \sum v_{i,j,k}(t-t')^{2i}(r-r')^k(cos\gamma - 1)^j \tag{4.75}$$

where γ is the angle between the three dimensional vectors $\mathbf{x}$ and $\mathbf{x}'$. The relevant part of their result shows that the coefficients of terms of collective order of three and lower vanish. The last result we quote is the result related to Gaussian approximation which has been obtained above and with renormalization in [76]. At zero temperature the non-renormalized result is given as

$$G_{gauss,T=0} = \frac{\Delta^{1/2}}{\sigma} \tag{4.76}$$

Using specific coordinates we have computed that the first three results agree on the fact that for the coefficient of the logarithmically divergent term, (i.e. $v(x, x')$) in the conformally coupled case, both in Schwarzschild and optical Schwarzschild for all the terms of third order and lower in coordinate separation vanish. That is, if we write:

$$v(x, x') = v_0(x) + v_a(x)\sigma^a + v_{ab}(x)\sigma^a\sigma^b + v_{abc}(x)\sigma^a\sigma^b\sigma^c + O(\sigma^a\sigma^b\sigma^c\sigma^d) \tag{4.77}$$

The first three terms vanish for Schwarzschild and optical Schwarzschild, for conformally coupled massless fields. Now we state that this result is in agreement with the zero-temperature result we obtained using Gaussian approximation for the following reason.

Using Eq.(4.59) one can see that the order up to which the right hand side of the Eq.(4.43) is correct, depends on the order up to which $v(x, x')$ has been calculated. Specifically one can see that if v is written as the coincidence limit expansion (i.e. as in Eq.(4.77)), correct up to a certain order, the right hand side of (4.43) would be correct to the same order. Therefore it can be seen, that if $v(x, x')$ is vanishing up to a certain order, the right hand side of the Eq.(4.43) should vanish to the same order. We have computed and shown that the right hand side of Eq.(4.43) is vanishing for all terms up to and including the third order. Therefore for our result to agree with the three mentioned published results above, one has to make sure that all of their expressions for $v(x,x')$ vanish when computed for their respective metrics up to and including the third order in expansion. Such calculation will also yield a consistency check amongst these same results. Indeed we have calculated zeroth, first , second and third order terms in the expansions derived in Phillips and Hu[66] and also in Décanini and Folacci [83] and found that, all of them are vanishing. Anderson and Hu [84] have already shown the vanishing of $v(x,x')$ expansion coefficients up to the third order in their paper.

4.7 Calculation of the trace of the noise kernel

For the massless scalar conformal case, the trace of the noise kernel reduces to

$$N\left[G\right] = \frac{1}{144}\left\{RR'G^2 - 6G\left(R\Box' + R'\Box\right)G + 18\left(\left(\Box G\right)\left(\Box' G\right) + G\Box'\Box G\right)\right\} \quad (4.78)$$

This equation can be written in terms of $\left(\Box - (R/6)\right)G$ and $\left(\Box' - (R'/6)\right)G$. To do that, we add and subtract $RG/6$ and also $R'G/6$ to terms containing $\Box G$ and $\Box' G$ respectively. The result can be shown to be equal to:

$$N(G) = \frac{1}{8}\left(G\left(\Box' - \frac{R'}{6}\right)\left(\Box - \frac{R}{6}\right)G + \left(\Box G - \frac{RG}{6}\right)\left(\Box' G - \frac{R'G}{6}\right)\right) \quad (4.79)$$

From this expression it becomes clear that, the vanishing of the noise kernel depends explicitly on G satisfying the field equation. We already know that the Gaussian approximation does not satisfy the field equation, in fact we have:

$$\begin{aligned}\left(\Box - \frac{R}{6}\right)G_{Wightman} &= \left(\Box - \frac{R}{6}\right)(C+iD)\,\Delta^{1/2} = (C+iD)\left(\Box - \frac{R}{6}\right)\Delta^{1/2} \\ &= (C+iD)\left(Q_{ab}\sigma^a\sigma^b + O(\sigma^a\sigma^b\sigma^c)\right) \quad (4.80)\end{aligned}$$

$$C = \frac{1}{8\pi^2 \mathrm{r}} \frac{\kappa \sinh(\kappa \mathrm{r})}{\cosh(\kappa \mathrm{r}) - \cos(\kappa \tau)} \tag{4.81}$$

$$D = \frac{\delta(\mathrm{r} - \Delta t) - \delta(\mathrm{r} + \Delta t)}{16\pi \mathrm{r}} \tag{4.82}$$

To compute the trace, we make the following definitions:

$$P(x, x') = C + iD \tag{4.83}$$

$$Q(x, x') = \left(\Box - \frac{R}{6}\right) \Delta^{1/2} \tag{4.84}$$

$$Q'(x, x') = \left(\Box' - \frac{R'}{6}\right) \Delta^{1/2} \tag{4.85}$$

Incidentally it can be inferred that

$$Q'(x, x') = Q_{ab}(r')\sigma^{a'}\sigma^{b'} + \cdots \tag{4.86}$$

Therefore We have

$$\begin{aligned} \left(\Box' - \frac{R'}{6}\right)\left(\Box - \frac{R}{6}\right) G &= \left(\Box' - \frac{R'}{6}\right)(P(x, x')Q(x, x')) \\ &= P(x, x')\left(\Box' - \frac{R'}{6}\right) Q(x, x') + Q(x, x')\Box' P(x, x') \\ &\quad + 2\nabla_{\alpha'} P(x, x') \cdot \nabla^{\alpha'} Q(x, x') \end{aligned} \tag{4.87}$$

With $P(x, x')$ being in the form given by Eq.(4.83), we have already shown that

$$2\nabla_{\alpha'} P(x, x') \cdot \nabla^{\alpha'} \Delta^{1/2}(x, x') + \Delta^{1/2} \Box' P(x, x') = 0 \tag{4.88}$$

This results in

$$\Box' P(x, x') = -\nabla_{\alpha'} P(x, x') \cdot \nabla^{\alpha'} \ln \Delta(x, x') \tag{4.89}$$

Combining the last two results yields

$$\left(\Box' - \frac{R'}{6}\right)\left(\Box - \frac{R}{6}\right) G = \qquad P(x, x')\left(\Box' - \frac{R'}{6}\right) Q(x, x')$$

$$+ \quad \nabla_{\alpha'} P(x,x') \cdot \left(2\nabla^{\alpha'} Q(x,x') - Q(x,x')\nabla^{\alpha'} \ln \Delta(x,x')\right)$$

(4.90)

Combining all of the above, we have:

$$\begin{aligned} 8\,N(G) = \quad & P^2(x,x')Q(x,x')Q'(x,x') + \Delta^{1/2}(x,x')P^2(x,x')\left(\Box' - \frac{R'}{6}\right)Q(x,x') \\ + \quad & \Delta^{1/2}(x,x')P(x,x')\nabla_{\alpha'}P(x,x') \cdot \left(2\nabla^{\alpha'}Q(x,x') - Q(x,x')\nabla^{\alpha'} \ln \Delta(x,x')\right) \end{aligned} \tag{4.91}$$

The reason for trying to find a relationship using $P(x,x')$ and $Q(x,x')$ is because we want to find a formula that can be used readily to calculate the noise kernel by using what we have already computed in the previous sections. Since the dependence in $P(x,x')$ is through the followings:

$$\Delta t = t - t' \tag{4.92}$$

$$\mathrm{r} = \sqrt{2\ {}^{(3)}\sigma} \tag{4.93}$$

we can rewrite the trace of the noise kernel in terms of P and its derivatives,

$$N(G) = P(\mathrm{r},\Delta t)\left(P(\mathrm{r},\Delta t)T_1(x,x') + \frac{1}{\mathrm{r}}\frac{\partial P(\mathrm{r},\Delta t)}{\partial \mathrm{r}}T_2(x,x')\right) \tag{4.94}$$

$$T_1(x,x') = \frac{1}{8}\left(Q(x,x')Q'(x,x') + \Delta^{1/2}\left(\Box' - \frac{R'}{6}\right)Q(x,x')\right) \tag{4.95}$$

$$T_2(x,x') = \frac{1}{4}\left(\nabla_{i'}\ {}^{(3)}\sigma\right)\left(\Delta^{1/2}\nabla^{i'}Q(x,x') - Q(x,x')\nabla^{i'}\Delta^{1/2}\right) \tag{4.96}$$

A calculation of $T_1(x,x')$ and $T_2(x,x')$ up to second order yields the following:

$$\begin{aligned} T_1(x,x') = \; & \frac{3\,M^2\,(r-2\,M)^2}{28\,r^{10}}\left((r-r')^2 + (\phi-\phi')^2 + 7\,r\,(r-2\,M)\,(\theta-\theta')^2\right) \\ & + O[(x-x')^3] \end{aligned} \tag{4.97}$$

$$T_2(x,x') = O[(x-x')^3] \tag{4.98}$$

4.8 The noise kernel in the optical Schwarzschild space-time

To calculate the noise kernel we notice two crucial points. First, the Green's function is of the following general form:

$$G(x,x') = P(x,x')U(x,x') \tag{4.99}$$

and second, the dependence of $P(x,x')$ on the coordinates of the two point is of the form $P(\mathrm{r},\Delta t)$. one can see that,

$$\begin{aligned} P_{;a} &= \delta_{4,a}\frac{\partial P(\mathrm{r},\Delta t)}{\partial \Delta t} + \delta_{i,a}\left(\frac{\sigma^i}{\mathrm{r}}\right)\frac{\partial P(\mathrm{r},\Delta t)}{\partial \mathrm{r}} & (4.100)\\ P_{;c'} &= -\delta_{4,c'}\frac{\partial P(\mathrm{r},\Delta t)}{\partial \Delta t} + \delta_{i,c'}\left(\frac{\sigma^i}{\mathrm{r}}\right)\frac{\partial P(\mathrm{r},\Delta t)}{\partial \mathrm{r}} & (4.101)\end{aligned}$$

We can continue this operation to express higher covariant derivatives of $P(x,x')$ in terms of regular partial derivatives of $P(\mathrm{r},\Delta t)$. Then we can substitute all of these covariant derivatives into general formulae Eqs.(4.1) and subsequent equations for the noise kernel. The result of this procedure is that the noise kernel functional dependence on P , U and σ is clearly split in such a way that all the different components of the noise kernel are determined via covariant derivatives of $U = \Delta^{1/2}$ and σ with normal partial derivatives of $P(\mathrm{r},\Delta t)$ carried along as factors. This expression for the noise kernel is detailed in the following.(Here u and U both denote $\Delta^{1/2}$

We write the noise kernel as the following combination:

$$N_{abc'd'} = \sum_{n=0}^{38} A_n(P)\; B^{(n)}_{abc'd'}(U,\sigma) \tag{4.102}$$

Quoting B factors is a non-trivial task. For example this is the first one.

$$
\begin{aligned}
B^{(1)}_{abc'd'}(U,\sigma) = \quad & \frac{U_{;c'b}\,U_{;d'a}}{18} + \frac{U_{;c'a}\,U_{;d'b}}{18} + \frac{U_{;c'd'}\,U_{;ab}}{72} - \frac{U_{;b}\,U_{;c'ad'}}{36} \\
- \quad & \frac{U_{;a}\,U_{;c'bd'}}{36} - \frac{U_{;d'}\,U_{;abc'}}{36} - \frac{U_{;c'}\,U_{;abd'}}{36} + \frac{U\,U_{;abc'd'}}{72} \\
- \quad & \frac{R'\,U_{;a}\,U_{;b}\,g_{c'd'}}{72} + \frac{R'\,U\,U_{;ab}\,g_{c'd'}}{144} - \frac{U_{;p'}{}^{p'}\,U_{;ab}\,g_{c'd'}}{72} \\
+ \quad & \frac{U_{;p'}\,U_{;ab}{}^{p'}\,g_{c'd'}}{72} - \frac{U\,U_{;abp'}{}^{p'}\,g_{c'd'}}{72} - \frac{U_{;p'b}\,U_{;q'a}\,g_{c'd'}\,g^{p'q'}}{36} \\
+ \quad & \frac{U_{;b}\,U_{;p'aq'}\,g_{c'd'}\,g^{p'q'}}{36} + \frac{U_{;a}\,U_{;p'bq'}\,g_{c'd'}\,g^{p'q'}}{36} - \frac{R\,U_{;c'}\,U_{;d'}\,g_{ab}}{72} \\
+ \quad & \frac{R\,U\,U_{;c'd'}\,g_{ab}}{144} - \frac{U_{;c'p}\,U_{;d'}{}^{p}\,g_{ab}}{36} - \frac{U_{;c'd'}\,U_{;p}{}^{p}\,g_{ab}}{72} \\
+ \quad & \frac{U_{;p}\,U_{;c'd'}{}^{p}\,g_{ab}}{72} + \frac{U_{;d'}\,U_{;c'p}{}^{p}\,g_{ab}}{36} + \frac{U_{;c'}\,U_{;d'p}{}^{p}\,g_{ab}}{36} - \frac{U\,U_{;c'd'p}{}^{p}\,g_{ab}}{72} \\
+ \quad & \frac{R\,R'\,U^2\,g_{c'd'}\,g_{ab}}{576} + \frac{R\,U_{;p'}\,U_{;}{}^{p'}\,g_{c'd'}\,g_{ab}}{288} + \frac{R'\,U_{;p}\,U^{;p}\,g_{c'd'}\,g_{ab}}{288} \\
- \quad & \frac{R\,U\,U_{;p'}{}^{p'}\,g_{c'd'}\,g_{ab}}{144} + \frac{U_{;p'p}\,U_{;}{}^{p'p}\,g_{c'd'}\,g_{ab}}{144} - \frac{R'\,U\,U_{;p}{}^{p}\,g_{c'd'}\,g_{ab}}{144} \\
+ \quad & \frac{U_{;p'}{}^{p'}\,U_{;p}{}^{p}\,g_{c'd'}\,g_{ab}}{72} - \frac{U_{;p}\,U_{;p'}{}^{pp'}\,g_{c'd'}\,g_{ab}}{72} - \frac{U_{;}{}^{p'}\,U_{;p}{}^{p}{}_{p'}\,g_{c'd'}\,g_{ab}}{72} \\
+ \quad & \frac{U\,U_{;p}{}^{p}{}_{p'}{}^{p'}\,g_{c'd'}\,g_{ab}}{72} + \frac{U_{;a}\,U_{;b}\,R_{c'd'}}{36} - \frac{U\,U_{;ab}\,R_{c'd'}}{72} \\
- \quad & \frac{R\,U^2\,g_{ab}\,R_{c'd'}}{288} - \frac{U_{;p}\,U^{;p}\,g_{ab}\,R_{c'd'}}{144} + \frac{U\,U_{;p}{}^{p}\,g_{ab}\,R_{c'd'}}{72} \\
+ \quad & \frac{U_{;c'}\,U_{;d'}\,R_{ab}}{36} - \frac{U\,U_{;c'd'}\,R_{ab}}{72} - \frac{R'\,U^2\,g_{c'd'}\,R_{ab}}{288} - \frac{U_{;p'}\,U_{;}{}^{p'}\,g_{c'd'}\,R_{ab}}{144} \\
+ \quad & \frac{U\,U_{;p'}{}^{p'}\,g_{c'd'}\,R_{ab}}{72}
\end{aligned}
\tag{4.103}
$$

There are 44 terms in this expression. One can continue quoting other B's. However the form of the A factors can be determined from the following considerations. A_1 is $P(\mathrm{r}, \Delta t)$ the rest are of the following form,

$$
A_N = \frac{\partial^m \partial^n P}{\partial \mathrm{r}^m \partial \Delta t^n} \frac{\partial^p \partial^q P}{\partial \mathrm{r}^p \partial \Delta t^q} \tag{4.104}
$$

where

$$0 \le m, n, p, q \le 3 \quad \text{and} \quad m + n + p + q = 4 \tag{4.105}$$

They can be found in Table 4.1.

One immediate application of this form is the expression of the noise kernel for hot flat space-time [82]. To reproduce the Eq. (4.3) in [82] via our method it suffices to write the renormalized Gaussian approximation expression for P as

$$P(\mathrm{r}, \Delta t) = \frac{1}{120960\,\pi^2}\kappa^2 \left(2520 - 42\,\kappa^2 \left(\mathrm{r}^2 - 3\,\tau^2\right) + \kappa^4 \left(\mathrm{r}^4 - 10\,\mathrm{r}^2\,\tau^2 + 5\,\tau^4\right)\right) \tag{4.106}$$

This expression can be further simplified by noting that in this case r is simply the Euclidean expression. In flat space-time we further have the following simplifications

$$U = 1\ ,\ R_{ab} = 0\ ,\ R_{abcd} = 0\ ,\ g_{ab} = \eta_{ab} \tag{4.107}$$

Using our expression quoted (partially) here in Eqs. (4.102) and (4.103) and in Table (4.1) , we can evaluate the coincidence limit of the noise kernel, the answer is:

$$\begin{aligned} N_{abc'd'} = \quad & \frac{41\,\pi^4\,T^8\,\delta^4{}_a\,\delta^4{}_b\,\delta^4{}_c\,\delta^4{}_d}{85050} - \frac{13\,\pi^4\,T^8\,\delta^4{}_c\,\delta^4{}_d\,g_{ab}}{204120} - \frac{29\,\pi^4\,T^8\,\delta^4{}_b\,\delta^4{}_d\,g_{ac}}{510300} \\ - \; & \frac{29\,\pi^4\,T^8\,\delta^4{}_b\,\delta^4{}_c\,g_{ad}}{510300} - \frac{29\,\pi^4\,T^8\,\delta^4{}_a\,\delta^4{}_d\,g_{bc}}{510300} + \frac{\pi^4\,T^8\,g_{ad}\,g_{bc}}{85050} \\ - \; & \frac{29\,\pi^4\,T^8\,\delta^4{}_a\,\delta^4{}_c\,g_{bd}}{510300} + \frac{\pi^4\,T^8\,g_{ac}\,g_{bd}}{85050} - \frac{13\,\pi^4\,T^8\,\delta^4{}_a\,\delta^4{}_b\,g_{cd}}{204120} + \frac{41\,\pi^4\,T^8\,g_{ab}\,g_{cd}}{4082400} \end{aligned} \tag{4.108}$$

This expression is in agreement with Eq. (4.3) in [82], except for a factor of 2. That the correct expression has to be twice bigger can be seen from the following

$$A_1(P) = P(\mathrm{r},\tau)^2 \qquad A_2(P) = P^{(1,0)}(\mathrm{r},\tau)\,P^{(3,0)}(\mathrm{r},\tau)$$

$$A_3(P) = P^{(1,0)}(\mathrm{r},\tau)\,P^{(2,1)}(\mathrm{r},\tau) \qquad A_4(P) = P^{(1,0)}(\mathrm{r},\tau)\,P^{(1,2)}(\mathrm{r},\tau)$$

$$A_5(P) = P^{(0,3)}(\mathrm{r},\tau)\,P^{(1,0)}(\mathrm{r},\tau) \qquad A_6(P) = P^{(0,1)}(\mathrm{r},\tau)\,P^{(3,0)}(\mathrm{r},\tau)$$

$$A_7(P) = P^{(0,1)}(\mathrm{r},\tau)\,P^{(2,1)}(\mathrm{r},\tau) \qquad A_8(P) = P^{(0,1)}(\mathrm{r},\tau)\,P^{(1,2)}(\mathrm{r},\tau)$$

$$A_9(P) = P^{(0,1)}(\mathrm{r},\tau)\,P^{(0,3)}(\mathrm{r},\tau) \qquad A_{10}(P) = P^{(2,0)}(\mathrm{r},\tau)^2$$

$$A_{11}(P) = P^{(1,1)}(\mathrm{r},\tau)\,P^{(2,0)}(\mathrm{r},\tau) \qquad A_{12}(P) = P^{(0,2)}(\mathrm{r},\tau)\,P^{(2,0)}(\mathrm{r},\tau)$$

$$A_{13}(P) = P^{(1,1)}(\mathrm{r},\tau)^2 \qquad A_{14}(P) = P^{(0,2)}(\mathrm{r},\tau)\,P^{(1,1)}(\mathrm{r},\tau)$$

$$A_{15}(P) = P^{(0,2)}(\mathrm{r},\tau)^2 \qquad A_{16}(P) = P^{(1,0)}(\mathrm{r},\tau)^2$$

$$A_{17}(P) = P^{(0,1)}(\mathrm{r},\tau)\,P^{(1,0)}(\mathrm{r},\tau) \qquad A_{18}(P) = P^{(0,1)}(\mathrm{r},\tau)^2$$

$$A_{19}(P) = P^{(1,0)}(\mathrm{r},\tau)\,P^{(2,0)}(\mathrm{r},\tau) \qquad A_{20}(P) = P^{(1,0)}(\mathrm{r},\tau)\,P^{(1,1)}(\mathrm{r},\tau)$$

$$A_{21}(P) = P^{(0,2)}(\mathrm{r},\tau)\,P^{(1,0)}(\mathrm{r},\tau) \qquad A_{22}(P) = P^{(0,1)}(\mathrm{r},\tau)\,P^{(2,0)}(\mathrm{r},\tau)$$

$$A_{23}(P) = P^{(0,1)}(\mathrm{r},\tau)\,P^{(1,1)}(\mathrm{r},\tau) \qquad A_{24}(P) = P^{(0,1)}(\mathrm{r},\tau)\,P^{(0,2)}(\mathrm{r},\tau)$$

$$A_{25}(P) = P(\mathrm{r},\tau)\,P^{(1,0)}(\mathrm{r},\tau) \qquad A_{26}(P) = P(\mathrm{r},\tau)\,P^{(0,1)}(\mathrm{r},\tau)$$

$$A_{27}(P) = P(\mathrm{r},\tau)\,P^{(2,0)}(\mathrm{r},\tau) \qquad A_{28}(P) = P(\mathrm{r},\tau)\,P^{(1,1)}(\mathrm{r},\tau)$$

$$A_{29}(P) = P(\mathrm{r},\tau)\,P^{(0,2)}(\mathrm{r},\tau) \qquad A_{30}(P) = P(\mathrm{r},\tau)\,P^{(3,0)}(\mathrm{r},\tau)$$

$$A_{31}(P) = P(\mathrm{r},\tau)\,P^{(2,1)}(\mathrm{r},\tau) \qquad A_{32}(P) = P(\mathrm{r},\tau)\,P^{(1,2)}(\mathrm{r},\tau)$$

$$A_{33}(P) = P(\mathrm{r},\tau)\,P^{(0,3)}(\mathrm{r},\tau) \qquad A_{34}(P) = P(\mathrm{r},\tau)\,P^{(4,0)}(\mathrm{r},\tau)$$

$$A_{35}(P) = P(\mathrm{r},\tau)\,P^{(3,1)}(\mathrm{r},\tau) \qquad A_{36}(P) = P(\mathrm{r},\tau)\,P^{(2,2)}(\mathrm{r},\tau)$$

$$A_{37}(P) = P(\mathrm{r},\tau)\,P^{(1,3)}(\mathrm{r},\tau) \qquad A_{38}(P) = P(\mathrm{r},\tau)\,P^{(0,4)}(\mathrm{r},\tau)$$

Table 4.1: A Factors

considerations. After $\kappa = 2\pi T$ is substituted, The Gaussian function will be(from [82] Eq. 3.1)

$$G^{Gauss}_{ren}(\tau,\sigma) = \Delta^{1/2}\left(\frac{T^2}{12} - \frac{\pi^2\,\sigma\,T^4}{90} + \frac{2\,\pi^4\,\sigma^2\,T^6}{945} + \frac{\pi^2\,T^4\,\tau^2}{45} - \frac{4\,\pi^4\,\sigma\,T^6\,\tau^2}{315} + \frac{8\,\pi^4\,T^6\,\tau^4}{945}\right) \tag{4.109}$$

If G is any function Q times $U = \Delta^{1/2}$ we have :

$$G_{;ab} = Q_{;ab}U + Q_{;a}U_{;b} + Q_{;b}U_{;a} + U_{;ab}Q \tag{4.110}$$

Since $U = 1$, which is true when space-time is flat,

$$G_{;ab} = Q_{;ab} \tag{4.111}$$

Let us apply this to the renormalized Gaussian function, followed by taking the coincidence limit $(\sigma \to 0, \tau \to 0)$.

$$G_{;ab} = \frac{2\,\pi^2\,T^4\,\tau_a\tau_b}{45} + \frac{4\,\pi^4\,T^6\,(\sigma_a)\,(\sigma_b)}{945} - \frac{\pi^2\,T^4\,(\sigma_{ab})}{90} \tag{4.112}$$

since $\tau \to \tau - \tau'$ it can be seen that $\tau_a = \delta^4_a$ and just for completeness $\tau_{c'} = -\delta^4_{c'}$ considering that $[\sigma_a] = 0$ we have:

$$[G^{ren}_{ab}] = -\frac{\pi^2\,T^4\,(\sigma_{ab})}{90} + \frac{2\,\pi^2\,T^4\,\tau_a\tau_b}{45} \tag{4.113}$$

which is Eq.(4.2b) in the same ref. [82]. The more important part is that if we accept (4.2b) and (4.2c) of ref. [82], we should be able to find out (4.3), specially the first term. The N_{ab}, $N_{c'd'}$ and N terms in the basic equations of noise kernel given by ref. [66] do not contribute. Only the first line of Eq. (3.25 a) contributes to the first term of (4.3) of [82]. That line is

$$(4(G_{;c'b}G_{d'a} + G_{c'a}G_{d'b}) + G_{c'd'}G_{ab} + GG_{abc'd'}) \tag{4.114}$$

All that we have to do to find the first line of (4.3) is to substitute (4.2a, 4.2b and 4.2c) into the above line. The result is

$$4\left(\left(\frac{2}{45}\right)\times\left(\frac{2}{45}\right)\delta_a^4\delta_b^4\delta_c^4\delta_d^4+\frac{2}{45}\times\left(\frac{2}{45}\right)\delta_a^4\delta_b^4\delta_c^4\delta_d^4\right)$$
$$+\quad\left(\frac{2}{45}\right)\times\left(\frac{2}{45}\right)\delta_a^4\delta_b^4\delta_c^4\delta_d^4+\frac{64}{315\times 12}\delta_a^4\delta_b^4\delta_c^4\delta_d^4 \tag{4.115}$$

The final result of this calculation is

$$\frac{41}{85050}\delta_a^4\delta_b^4\delta_c^4\delta_d^4 \tag{4.116}$$

Therefore the first term can be easily calculated and shown to agree with the elaborated result, particularly that would confirm the factor of 2.

A second verification can be obtained by calculating the Trace of $B^{(1)}_{abc'd'}(U,\sigma)$ and expanding the result in separation of points (i.e. expanding it in $(r-r')$, $(\theta-\theta')$ and $(\phi-\phi')$). One can compare this calculation with the indirect result calculated in (4.94). A thorough calculation has to overcome several challenges in terms of computing. At the time of this writing we have computed the contribution of the first term that is Eq.(4.103) to the total trace of the noise kernel expressed as Eq.(4.102) and up to the second order we have confirmed that this contribution matches Eq.(4.97). However a complete verification of our results requires calculation of the trace of all 38 terms in Eq. (4.102).

Our ultimate aim is to calculate the noise kernel as a quasi-local expansion in coordinate separations up to a certain order Ω that is, we want to express the noise kernel as sum of terms in the following form

$$B_{abc'd'}(x,x') = \sum_{q=0}^{4} \left(\frac{1}{\mathrm{r}}\right)^{q} \sum_{i,j,k=0}^{i+j+k\leq\Omega} B^{(q)}_{abc'd'\ \{ijk\}} \Delta r^{i} \Delta\theta^{j} \Delta\phi^{k} \tag{4.117}$$

where $\Delta r = (r - r'), \Delta\theta = (\theta - \theta')$ and $\Delta\phi = (\phi - \phi')$ and the coefficients of expansion, namely, $B^{(q)}_{abc'd'\ \{ijk\}}$ for a massless scalar field conformally coupled to an optical Schwarzschild metric, depend on r and θ. (the inclusion of braces in the above, that is, the notation, $\{ijk\}$, imparts no further property and is only to distinguish expansion related indices from tensor related indices).The factors of these terms will depend on the specific form of $P(\mathrm{r}, \tau)$ functions. For example for That the power of square root of three dimensional Synge's world function can not be more than 4 can be easily seen from Eq. (4.100), since from Eq. (4.1) and equations that follow it, we know that the final expression will include up to four covariant derivative of $P(x, x')$ and for each one there will be one $1/\mathbf{r}$. The steps that are required to arrive at this recent equation, starting from Eq. (4.102) are cumbersome but more or less straightforward. One obviously has to substitute for the relevant quantities and expand to the desired order. The expansion for $U = \Delta^{1/2}$, ${}^{(3)}\sigma$ and their derivatives are obtained in [85].

The task of computation however is non-trivial as it is evident from the large number of terms and expressions that are involved. The real challenge is in finding the proper algorithm such that the entire project can be carried out within limitations of computing processing time and memory. The key to accomplishing this task is to find out which terms contribute to the total sum and take the necessary precautions as to include only the contributing terms and nothing more.

4.9 Summary

In this chapter, we found an expression for the Wightman function for a massless scalar field conformally coupled to optical Schwarzschild metric. We then verified the result, finding the order up to which it will satisfy the field equation. In the course of this investigation, we also checked for the consistency of all of the known published results for the tail-term function $v(x, x')$ and verified that these results are also consistent with ours. Having found the Wightman function we used a generic form written as any bi-scalar denoted by $P(x, x')$ multiplied by $U = \Delta^{1/2}$ with Δ being the square root of van Vleck determinant. We then substituted this generic form, writing the noise kernel as the sum of terms consisting of two factors, one entirely dependent on $P(r, \tau)$ and its derivatives and the other entirely dependent on U and σ and their covariant derivatives. This enabled us to use the already calculated expanded forms of U and σ to finally calculate the general point-separation expanded form of the Noise Kernel. The procedure, in principle, can be carried out to the desired order limited only by computing time and memory resources. Such calculation can be important and useful in investigating:

1. metric fluctuations
2. correlation of stress energy tensor fluctuations in two distinct points
3. certain physical quantities that are obtained by integrating suitable functions over the noise kernel factored in as a tensor with distribution like properties.

Future developments can follow using these expressions [86].

Appendix A
Calculation of $\sigma(x,x')$, t_R and $u(x,x')$ as coordinate expansion in separation of end-point coordinates

Van Vleck determinant, $\Delta(x,x')$ and Syng's world function, $\sigma(x,x')$ have been calculated using the covariant geodesic point separation ([61],[60] and [66]). Concrete calculations however, need to be done in coordinate dependent fashion in one of the convenient coordinate systems. Specially Syng's world function, is often needed and a calculation of this function in a useful coordinate system seems to be relevant. We start by noting that $\sigma(x,x')$ satisfies the following non-linear differential equation:

$$\sigma = \frac{1}{2} g^{\alpha\beta}\sigma_\alpha\sigma_\beta \tag{A.1}$$

We do this for geometries that have the following line element:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{A.2}$$

For the metric (A.2) the diff. eq. can be explicitly written as

$$\sigma = \frac{1}{2}\left(-\frac{1}{f(r)}\left(\frac{\partial\sigma}{\partial t}\right)^2 + f(r)\left(\frac{\partial\sigma}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\sigma}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial\sigma}{\partial\phi}\right)^2\right) \tag{A.3}$$

It can be seen that the angular part can be entirely described in terms of $\cos\gamma = \cos\theta\cos\theta' + \cos(\phi-\phi')\sin\theta\sin\theta'$ In the following manner:

$$g^{\theta\theta}\left(\frac{\partial\sigma}{\partial\theta}\right)^2 + g^{\phi\phi}\left(\frac{\partial\sigma}{\partial\phi}\right)^2 = \frac{1}{r^2}\left(\frac{\partial\sigma}{\partial\cos\gamma}\right)^2(1-\cos^2\gamma) \tag{A.4}$$

Therefore one can describe the angular dependence of $\sigma(x,x')$ entirely in terms of $\cos\gamma$. We decide to describe this dependence in terms of $(\cos\gamma - 1)$ and to make the expressions shorter we define:

$$\eta \equiv \cos\gamma - 1 \tag{A.5}$$

Now since the metric does not depend on time, we can see that $\sigma(x,x')$ must be a function of $\Delta t \equiv (t-t')$ since any time translation should leave it unchanged. we therefore write $\sigma(x,x')$ as a function of η,Δt and $\Delta r \equiv r - r'$ as follows:

$$\sigma(x,x') = \sum_{i,j,k} s_{i,j,k}(r)\Delta t^{2i}\Delta r^k \eta^j \tag{A.6}$$

It has to be in terms of even powers of $(t-t')$ since $\sigma(x,x') = \sigma(x',x)$. In terms of these variables one can write the differential equation as

$$\sigma = \frac{1}{2}\left(-\frac{1}{f(r)}\left(\frac{\partial\sigma}{\partial\Delta t}\right)^2 + f(r)\left(\frac{\partial\sigma}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial\sigma}{\partial\eta}\right)^2(\eta^2+2\eta)\right) \tag{A.7}$$

Now one can substitute the expansion (A.6) into the differential equation (A.7) and attempt to solve for the coefficients, order by order. The procedure succeeds since in the course of calculation, it becomes clear, that all the equations are one-equation-one-unknown algebraic equations. Furthermore, there will be no need to solve any differential equation to find the coefficients. To fourth order we have

$$\begin{aligned}
\sigma^{(4)}(x,x') = \quad & -\frac{1}{2}f\Delta t^2 + \frac{1}{2f}\Delta r^2 - r^2\eta + r\Delta r\eta + \frac{1}{6}r^2\eta^2(1-f) \\
+ \quad & \frac{\Delta t^2\,\Delta r\,f'}{4} + \frac{\Delta r^3\,f'}{4\,f^2} + \frac{r\,\Delta r^2\,\eta\,f'}{12\,f} \\
- \quad & \frac{(r\,\Delta t^2\,\eta\,f\,f')}{12} - \frac{\Delta t^4\,f\,f'^2}{96} + \frac{r\,\Delta r\,\eta^2\,(-2+2\,f+r\,f')}{12} \\
- \quad & \frac{\Delta t^2\,\Delta r^2\,\left(-3\,f'^2+4\,f\,f''\right)}{48\,f} - \frac{\Delta r^4\,\left(-15\,f'^2+8\,f\,f''\right)}{96\,f^3}
\end{aligned}$$

The expansion is particularly useful for calculating t_R. To that end we write t_R as an expansion in powers of Δr and η

$$\Delta t_R = \sum_{j,k} q_{j,k}(r)\eta^j \Delta r^k \tag{A.8}$$

The q coefficients must be calculated in a way that after substitution in the expression for $\sigma(x,x')$, the expression vanishes.

To calculate van Vleck determinant $\Delta(x,x')$ or its square root, $u(x,x')$ we can use the fact that $u(x,x')$ satisfies the following differential equation:

$$(\ln\Delta)_\alpha \sigma^\alpha = 4 - \Box\,\sigma \tag{A.9}$$

It could be worth mentioning that $\Box\,\sigma$ has the same angular dependence that σ has. Indeed It can be shown that $\Box\,\sigma = \sigma^\alpha_\alpha$ can be written as an expansion as

$$\begin{aligned}
\Box\,\sigma &= \sum_{i,j,k} d_{i,j,k}(r)\Delta t^{2i}\Delta r^k \eta^j && \text{(A.10)}\\
d_{i,j,k} =& \quad f(r)\left(\frac{d^2\ s_{i,j,k}}{dr^2} + 2(k+1)\frac{ds_{i,j,k+1}}{dr} + (k+1)(k+2)s_{i,j,k+2}\right)\\
&+ X\left(\frac{ds_{i,j,k}}{dr} + (k+1)s_{i,j,k+1}\right)\\
&- \frac{1}{f(r)}(2i+2)(2i+1)s_{i+1,j,k} - \frac{2\,j^2}{r^2}s_{i,j+1,k} - \frac{j(j-1)}{r^2}s_{i,j,k} && \text{(A.11)}\\
X &= f'(r) + \frac{2}{r}f(r) && \text{(A.12)}
\end{aligned}$$

The left side of the Eq. (A.9) can be written as:

$$\begin{aligned}
(\ln\Delta)_\alpha\sigma^\alpha = \frac{1}{2}\Bigg(& -\frac{1}{f(r)}\left(\frac{\partial\sigma}{\partial\Delta t}\right)\left(\frac{\partial\ln\Delta}{\partial\Delta t}\right) + f(r)\left(\frac{\partial\sigma}{\partial r}\right)\left(\frac{\partial\ln\Delta}{\partial r}\right)\\
& -\frac{1}{r^2}\left(\frac{\partial\sigma}{\partial\eta}\right)\left(\frac{\partial\ln\Delta}{\partial\eta}\right)(\eta^2+2\eta)\Bigg)
\end{aligned}$$

Hence one can write $\ln\Delta$ as an expansion similar to the expansion of σ and solve for the undetermined coefficients of expansion. Δ then can be found by exponentiating

$\ln \Delta$ and expanding to the desired order.

After calculation of $\Delta(x, x')$ and substitution of $t_R - t$ from $A.8$, One can calculate $\Delta(x, x')$ for x and x' separated with a null geodesic. the result is:

$$\Delta(x, x')|_{\sigma=0} = 1 + \frac{2\,\eta^2\,M^2}{5\,r^2} + \frac{2\,\Delta r\,\eta^2\,M^2}{5\,r^3} + O(|x - x'|^3) \tag{A.13}$$

Calculating σ_t^2 at the null separation yields

$$\begin{aligned}
\sigma_t^2|_{\sigma=0} &= \Delta r^2 + 2\,\eta\,(2\,M - r)\,r + 2\,\Delta r\,\eta\,(-M + r) + \eta^2\,M^2 \qquad \text{(A.14)}\\
&+ \frac{2\,\Delta r^2\,\eta^2\,M^2}{5\,r^2} + \frac{4\,\eta^3\,M^2\,(2\,M - r)}{5\,r} + \frac{2\,\Delta r^3\,\eta^2\,M^2}{5\,r^3} + \frac{4\,\Delta r\,\eta^3\,M^3}{5\,r^2} + \frac{2\,\Delta r^4\,\eta^2\,M^2}{5\,r^4}\\
&+ \text{higher order terms}\\
&= \left(\Delta r^2 + 2\,\eta\,(2\,M - r)\,r + 2\,\Delta r\,\eta\,(-M + r) + \eta^2\,M^2\right)\left(1 + \frac{2\,\eta^2\,M^2}{5\,r^2} + \frac{2\,\Delta r\,\eta^2\,M^2}{5\,r^3}\right)
\end{aligned}$$

Hence it can be deduced that:

$$\frac{u(x, x')}{\sigma_t}\,|_{\sigma=0} = \frac{1}{\sqrt{\Delta r^2 - 2\eta r_0(r_0 - 2M) - 2\eta\Delta r(r_0 - M) + M^2\eta^2}} \tag{A.15}$$

Appendix B
Power Series Expansion for the Self-Force

Using a 16^{th} order WKB expansion for $v(x, x')$ we find

$$\begin{aligned}
a_6 &= \frac{-1}{5600\,\pi\,{r_0}^2}\left(f_0 - 6\,{f_0}^3\right) \\
a_8 &= \frac{-1}{3763200\,\pi\,{r_0}^2\,{x_0}^2\,f_0}\left(33 + 1673\,{f_0}^2 - 6505\,{f_0}^4 - 801\,{f_0}^6\right) \\
a_{10} &= \frac{-1}{1862784000\,\pi\,{r_0}^2\,{x_0}^4\,{f_0}^3}\left(1485 + 58135\,{f_0}^2 + 1318420\,{f_0}^4 - 6320024\,{f_0}^6 + 9543963\,{f_0}^8\right. \\
&\quad \left. -8129979\,{f_0}^{10}\right) \\
a_{12} &= \frac{-1}{4262049792000\,\pi\,{r_0}^2\,{x_0}^6\,{f_0}^5}\left(64350 + 17996550\,{f_0}^2 + 260849550\,{f_0}^4 + 3865599526\,{f_0}^6\right. \\
&\quad \left. -23615537141\,{f_0}^8 + 39720954511\,{f_0}^{10} - 8647606803\,{f_0}^{12} - 20916240543\,{f_0}^{14}\right) \\
a_{14} &= \frac{-1}{775693062144000\,\pi\,{r_0}^2\,{x_0}^8\,{f_0}^7}\left(-50450400 + 601200600\,{f_0}^2 + 7645255800\,{f_0}^4\right. \\
&\quad +66115588280\,{f_0}^6 + 793355275637\,{f_0}^8 - 6317820122409\,{f_0}^{10} + 15546245547034\,{f_0}^{12} \\
&\quad \left. -21861205897898\,{f_0}^{14} + 24851959906665\,{f_0}^{16} - 14951493087309\,{f_0}^{18}\right) \\
a_{16} &= \frac{-1}{1054942564515840000\,\pi\,{r_0}^2\,{x_0}^{10}\,{f_0}^9}\,(-59339129850 \\
&\quad +199482633350\,{f_0}^2 + 2284989775250\,{f_0}^4 + 15281113074290\,{f_0}^6 + 99509041653021\,{f_0}^8 \\
&\quad +1135360622213657\,{f_0}^{10} - 11642046270515187\,{f_0}^{12} + 35642217287656961\,{f_0}^{14} \\
&\quad -48909059709636311\,{f_0}^{16} + 9979576112044621\,{f_0}^{18} + 50856188133399573\,{f_0}^{20} \\
&\quad \left. -39894377180673375\,{f_0}^{22}\right) \qquad \text{(B.1)}
\end{aligned}$$

and

$$b_5 = \frac{3\,x_0\,{f_0}^2}{22400\,\pi\,r_0^2}$$

$$b_7 = \frac{1}{940800\,\pi\,x_0 r_0^2}\left(-147 + 1031\,{f_0}^2 - 2004\,{f_0}^4\right)$$

$$b_9 = \frac{1}{1117670400\,\pi\,x_0^3 r_0^2}\left(725274 - 4840427\,{f_0}^2 + 7544064\,{f_0}^4 - 1384911\,{f_0}^6\right)$$

$$b_{11} = \frac{1}{3196537344000\,\pi\,x_0^5\,r_0^2}\left(-5192394350 + 35645887586\,{f_0}^2 - 76724155827\,{f_0}^4 \right.$$
$$\left. +74055537336\,{f_0}^6 - 34933610745\,{f_0}^8\right)$$

$$b_{13} = \frac{1}{27703323648000\,\pi\,x_0^7\,r_0^2}\left(88650418610 - 652839443586\,{f_0}^2 + 1715470441205\,{f_0}^4 \right.$$
$$\left. -2048237904519\,{f_0}^6 + 938994638717\,{f_0}^8 + 27641613573\,{f_0}^{10}\right)$$

$$b_{15} = \frac{1}{197801730846720000\,\pi\,x_0^9\,r_0^2}\left(-1082712168000450 + 8690022729803252\,{f_0}^2 \right.$$
$$-27039642594514215\,{f_0}^4 + 43159798466443548\,{f_0}^6 - 40035939562445204\,{f_0}^8$$
$$\left. +24271024582691064\,{f_0}^{10} - 8499901220531595\,{f_0}^{12}\right)\ . \qquad \text{(B.2)}$$

Appendix C
Self-force Calculation Results for a particle previously held at rest but launched on a circular orbit with Keplerian frequency

Here we bring the results of the calculation of the self-force for Keplerian circular orbit to the highest order available to us. Let $f_t^{(kc)}$ denote the time component of the self-force for Keplerian circular orbit.

To the extent that it is possible and meaningful, We try to express the results in terms of the following dimensionless parameters,

$$\begin{aligned} x_0 &= \frac{r_0}{M} \\ f_0 &= 1 - \frac{2\,M}{r_0} \\ \omega_0 &= \sqrt{1 - \frac{3\,M}{r_0}} \\ \beta_0 &= \sqrt{1 - \frac{2\,M}{r_0}} \end{aligned} \tag{C.1}$$

the following The result of the calculation can be written as

$$f_t^{(kc)} = \sum f_{2n} \left(\frac{t}{M}\right)^{2n} \tag{C.2}$$

and we have :

$$f_4{}^t = \frac{3\,f_0}{17920\,\pi\ \ r_0{}^2\,x_0{}^8}\left(\left(13 + 2\,x_0\ - 5\,x_0{}^2\right)\,\beta_0 + \left(2 -\ 11\,x_0 + 5\,x_0{}^2\right)\,\omega_0\right) \tag{C.3}$$

$$\begin{aligned} f_6{}^t = \frac{-f_0}{107520\,\pi\ \ r_0{}^2\,x_0{}^{12}}\Big(&\beta_0\left(-1302 +\ 688\,x_0 + 522\,x_0{}^2 - 378\,x_0{}^3 +\ 56\,x_0{}^4\right) \\ &\omega_0\left(327 +\ 614\,x_0 - 1063\,x_0{}^2 + 448\,x_0{}^3 - 56\ \ x_0{}^4\right)\Big) \end{aligned} \tag{C.4}$$

$$
\begin{aligned}
f_8{}^t &= \frac{f_0}{141926400\,\pi\ \ r_0{}^2\,x_0{}^{16}} \times \\
&\Big(2\,\beta_0\left(4103055 - 5118465\ \ x_0 + 472505\,x_0{}^2\right. \\
&\qquad \left.+1948602\,x_0{}^3 - 1093758\,x_0{}^4 + 223705\,x_0{}^5 - \ 15750\,x_0{}^6\right) + \\
&3\,\omega_0\left(- 1274130 + \ 417945\,x_0 + 2023852\,x_0{}^2 - \right. \\
&\qquad \left. 2166754\,x_0{}^3 + 881199\,x_0{}^4 - \ 159170\,x_0{}^5 + 10500\,x_0{}^6\right)\Big) \qquad \text{(C.5)}
\end{aligned}
$$

$$
\begin{aligned}
f_{10}{}^t &= \frac{f_0}{6199345152000\,\pi\ \ r_0{}^2\,x_0{}^{20}} \times \\
&\Big(\beta_0\left(1616491337742 - \ 3187571421681\,x_0 + 1879624942785\,x_0{}^2 + \right. \\
&\qquad 278397740030\,x_0{}^3 - 835465289640\,x_0{}^4 + \ 416773889931\,x_0{}^5 - \\
&\qquad \left. 97913745743\,x_0{}^6 + \ 11232774240\,x_0{}^7 - 498960000\,x_0{}^8\right) \ + \\
&3\,\omega_0\left(- 321226132428 + 462825450396\,x_0 + \ 55536384285\,x_0{}^2 - \right. \\
&\qquad 507007548036\,x_0{}^3 + \ 423045891786\,x_0{}^4 - 167783526144\,x_0{}^5 + \\
&\qquad \left. 35671434261\,x_0{}^6 - 3872778080\,x_0{}^7 + 166320000\,x_0{}^8\right)\Big) \qquad \text{(C.6)}
\end{aligned}
$$

The result for f_{r^*} can be presented as :

$$
f_{r^*} = \sum f_n^{r^*}\left(\frac{t}{M}\right)^n \qquad \text{(C.7)}
$$

And we have :

$$
\begin{aligned}
f_5{}^{r^*} &= \frac{3}{89600\,\pi\ \ r_0{}^2\,x_0{}^{10}}\Big(- 5\,\beta_0\ \left(\ 37 - 18\,x_0 - 9\,x_0{}^2 + 4\,x_0{}^3\right) \\
&\qquad +\omega_0\ \left(53 + 54\,x_0 - \ 81\,x_0{}^2 + 20\,x_0{}^3\right)\Big) \qquad \text{(C.8)} \\
f_6{}^{r^*} &= \frac{9\,\beta_0\ (2\,x_0 - 5)}{35840\,\pi\,r_0{}^2\,x_0{}^{12}} \qquad \text{(C.9)}
\end{aligned}
$$

$$
\begin{aligned}
{f_7}^{r^*} &= \frac{1}{1505280\,\pi\ \ {r_0}^2\,{x_0}^{14}} \times \\
&\Big(-14\,\beta_0\,\left(6714 - 7153\,x_0 + 256\,{x_0}^2 + 2025\,{x_0}^3 - 764\,{x_0}^4 + 80\,{x_0}^5\right) \\
&\quad +\omega_0\,\left(21510 - 5281\,x_0 - 26084\,{x_0}^2 + 21465\,{x_0}^3 - 6020\,{x_0}^4 + 560\,{x_0}^5\right)\Big) \\
{f_8}^{r^*} &= \frac{\beta_0\,(72 - 85\,x_0 + 22\,{x_0}^2)}{430080\,\pi\ \ {r_0}^2\,{x_0}^{16}} && \text{(C.10)} \\
{f_9}^{r^*} &= \frac{\omega_0}{1277337600\,\pi\ \ {r_0}^2\,{x_0}^{18}} \times \\
&\Big(103311450 - 124020270\,x_0 - 35018287\,{x_0}^2 + 128131276\,{x_0}^3 \\
&-83133090\,{x_0}^4 + 24674038\,{x_0}^5 - 3498265\,{x_0}^6 + 189000\,{x_0}^7\Big) && \text{(C.11)} \\
{f_{10}}^{r^*} &= \frac{-\beta_0\,(90 - 140\,x_0 + 41\,{x_0}^2)}{6451200\,\pi\ \ {r_0}^2\,{x_0}^{19}} && \text{(C.12)}
\end{aligned}
$$

Similarly we can present the result of calculation of f_ϕ as follows:

$$
f_\phi = \sum f_n^\phi \left(\frac{t}{M}\right)^n \qquad \text{(C.13)}
$$

The non-vanishing coefficients of the above expansion they have been computed so far are,

$$
\begin{aligned}
f_4^\phi &= \frac{9\,f_0}{17920\,\pi\,r_0\,{x_0}^{\frac{15}{2}}}\,\left(\omega_0\,\left(5 - 3\,x_0\right) + 4\,\beta_0\,\left(-1 + x_0\right)\right) && \text{(C.14)} \\
f_5^\phi &= \frac{-9\,\beta_0\,f_0}{8960\,\pi\,r_0\,{x_0}^{\frac{19}{2}}} && \text{(C.15)} \\
f_6^\phi &= \frac{f_0}{107520\,\pi\,r_0\,{x_0}^{\frac{23}{2}}}\,\Big(\omega_0\,\left(975 - 1302\,x_0 + 541\,{x_0}^2 - 70\,{x_0}^3\right) + \\
&\qquad \beta_0\,\left(-882 + 1356\,x_0 - 606\,{x_0}^2 + 84\,{x_0}^3\right)\Big) && \text{(C.16)} \\
f_7^\phi &= \frac{\beta_0\,f_0\,\left(3 - 2\,x_0\right)}{35840\,\pi\,r_0\,{x_0}^{\frac{27}{2}}} && \text{(C.17)}
\end{aligned}
$$

$$
\begin{aligned}
f_8^{\phi} &= \frac{f_0}{141926400\,\pi\,r_0\,{x_0}^{\frac{31}{2}}} \times \\
&\Big(\omega_0\Big(\quad 4383720 - 8983095\,x_0 + 7016566\,{x_0}^2 \\
&\qquad -2603058\,{x_0}^3 + 456081\,{x_0}^4 - 30100\,{x_0}^5 \quad) + \\
&\quad \beta_0\Big(- 4062960 + 8940360\,x_0 - 7329824\,{x_0}^2 \\
&\qquad +2817456\,{x_0}^3 - 507144\,{x_0}^4 + 34400\,{x_0}^5\Big)\Big) \qquad \text{(C.18)}
\end{aligned}
$$

$$
\begin{aligned}
f_{10}^{\phi} &= \frac{f_0}{6199345152000\,\pi\,r_0\,{x_0}^{\frac{39}{2}}} \times \\
&\Big(\quad \omega_0\Big(\quad 652812940458 - 1799095070493\,x_0 + 2046234095640\,{x_0}^2 \\
&\qquad -1242624904078\,{x_0}^3 + 433672385718\,{x_0}^4 - 86576688501\,{x_0}^5 \\
&\qquad +9100557040\,{x_0}^6 - 385560000\,{x_0}^7\Big) \\
&\quad +\beta_0\Big(- 609320914020 + 1759649090850\,x_0 - 2072310397920\,{x_0}^2 \\
&\qquad +1292397157340\,{x_0}^3 - 460467765420\,{x_0}^4 + 93463692930\,{x_0}^5 \\
&\qquad -9966677600\,{x_0}^6 + 428400000\,{x_0}^7\Big)\Big) \qquad \text{(C.19)}
\end{aligned}
$$

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