ABSTRACT

Title of dissertation:	An Investigation on Holomorphic vector bundles and Krichever-Lax matrices over an Algebraic curve
	Taejung Kim, Doctor of Philosophy, 2007
Dissertation directed by:	Professor Niranjan Ramachandran Department of Mathematics

The work by N. Hitchin in 1987 opened a good possibility of describing the cotangent bundle of the moduli space of stable vector bundles over a compact Riemann surface $\mathfrak{R}$ in an explicit way. In [32], he proved that the space can be foliated by a family of certain spaces, i.e., the Jacobi varieties of spectral curves. The main purpose of this dissertation is to make the realization of the Hitchin system in a concrete way in the method initiated by I. M. Krichever in [44] and to give the necessary and sufficient condition for the linearity of flows in a Lax representation in terms of cohomological classes using the similar technique and analysis from the work by P. A. Griffiths in [27].

An Investigation on Holomorphic vector bundles and Krichever-Lax matrices over an Algebraic curve

by

Taejung Kim

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2007

Advisory Committee:
Professor William M. Goldman, Chair/Advisor
Professor Niranjan Ramachandran, Co-Chair/Co-Advisor
Professor Serguei P. Novikov
Professor John J. Millson
Professor P. S. Krishnaprasad

DEDICATION

To my parents

ACKNOWLEDGMENTS

Firstly, I am very indebted to many professors in the department of mathematics in the university of Maryland for their advice, guidance, and teaching in my Ph. D course. I wish to express my sincere gratitude to them. Moreover, I am also very thankful to the department of mathematics in the university of Maryland for its financial support and hospitality during my stay.

Second of all, I should and would like to give my sincere gratefulness to Prof. William Goldman for allowing me to have him as my surrogate father in mathematics. In my mind there is no doubt at all that he has given me his best supports mathematically and financially.

Thirdly, it has been truly my honor and privilege to have an opportunity to have discussions and seminars with Prof. Serguei Novikov in the university of Maryland. His precious insights and valuable suggestions in his seminars consist of a substantial part of this work. I also give my thanks to Krishna Kaipa who participates in the seminar for answering to my questions a number of times and sharing his excellent comprehension with me.

Fourth of all, I would like to acknowledge my profound indebtedness to my advisor, Prof. Niranjan Ramachandran for initiating this wonderful expedition. It is no exaggeration that without his companionship, patience, and wholehearted encouragement during a good and bad time, this journey could have ended in a bitter failure. His contribution is inestimable.

Lastly, but not the least, my great thankfulness goes to those people who have written, published, and distributed manuscriptally and electronically the books, journals, publications, and theses making me educated and enlightened in this beautiful subject. Among the works, I wish to express my sincere thanks to three very influential works [27, 32, 44] on me. At the same time, I would like to mention that any omissions of references, or failure to attribute theorems, reflect only my ignorance, and make an apology, in advance, to those I've inadvertently left out.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

1.1 Motivations and historical perspectives

In some perspective, geometry is another tool for the study of dynamical systems which deal with the portrayal of physical phenomenon in terms of analytic language. Finding a good description of the total hierarchy of dynamics is considered to be very significant and noteworthy to not only the dynamical systems themselves but also geometric problems arising from very different settings. The study of the moduli space of holomorphic vector bundles over an algebraic curve is one sort of such investigation of the totality. However, even though a fair amount of abstract machinery of the research has been developed for the last 70 years, it is well-known that explicit descriptions of such spaces are extremely difficult and not well-understood usually except the trivial case, i.e., the moduli space of rank one vector bundles over a compact Riemann surface (see [25]).

The moduli space of holomorphic line bundles over a compact Riemann surface $\mathfrak{R}$ of genus g is depicted by its Jacobi variety Jac($\mathfrak{R}$), which is a complex torus embedded in a projective space. A divisor D of degree g on $\mathfrak{R}$ corresponds to a holomorphic line bundle L on $\mathfrak{R}$ in one-to-one manner. The Abel-Jacobi map[1] characterizes the isomorphism between the gth symmetric product of $\mathfrak{R}$ and its

[1] See [20, 26, 28, 62].

Jacobi variety:

$$\mathcal{S}^g\mathfrak{R} \cong \mathrm{Jac}(\mathfrak{R}).$$

On the other hand, this classification is not easily extended to the moduli space of higher rank vector bundles, since the moduli space tends to be non-Hausdorff. In 1963, D. Mumford defined a special class of vector bundles to get rid of the non-Hausdorff phenomenon in [51]. The element in the class is called a (semi-)stable vector bundle. Another description of such bundles came from (or rather revived from) the study of A. N. Tyurin in [64, 65, 66]. He studied matrix divisors[2] in order to characterize semi-stable vector bundles and defined parameters to describe them. Moreover, he showed that an open set of $\mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1})$ can parametrize the moduli space of stable vector bundles of rank l over $\mathfrak{R}$.

The analytic aspect of the moduli space is well portrayed in the dynamics of the K-dV hierarchy. The Hamiltonian theory of the K-dV equations started around the late 1960's by Gardner, Greene, Kruskal, and Miura in Princeton. In terms of the moduli space point of views, a significant work, known as the periodic problem of the K-dV hierarchy was investigated by S. Novikov and P. Lax in 1974 [45, 54], simultaneously and independently. After the Novikov's work, he and his students, notably B. Dubrovin and I. Krichever, in Moscow developed a beautiful geometric theory about the K-dV hierarchy. Loosely speaking, a dynamics of a completely integrable system, i.e., Hamiltonian dynamics, is described by foliation of tori or complex vector spaces. In the case when the leaves are tori, we may see the appearance of spectral curves by means of their Jacobi varieties. The geometric

[2] We will explain this concept in Subsection 2.1.2.

theory of the K-dV hierarchy expounds that the solution space of the K-dV hierarchy is parametrized by, i.e., foliated by, the Jacobi varieties of hyper-elliptic curves, which are spectral curves over $\mathbb{P}^1$. See further details in [12, 13, 14, 15].

There are two ways to extend this theory to general cases: The first is motivated by the study of the K-P hierarchy [38, 40, 41]. In this case, the base curve is a projective plane $\mathbb{P}^1$ and the fibers of Hamiltonians, i.e., the leaves of foliation, are the Jacobi varieties of general compact Riemann surfaces. This analysis leads to the famous Novikov's conjecture, which is proved by T. Shiota. See [1, 16, 47, 60] for more detail. The second way to extend the machinery is to change the base curve $\mathbb{P}^1$ to a compact Riemann surface with a positive genus. This direction leads to the theory of the Yang-Mills equations [2] and the Hitchin system [30, 31, 32, 42].

Another facet along this machinery is a representation founded by P. Lax. In [45], P. Lax defined a system of differential equations characterizing an *isospectral* deformation:

$$\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t].$$

For $(l \times l)-$matrices L_t and M_t, the objects invariant under time shift are complex tori associated with the matrix L_t. More precisely, it is the Jacobi variety of a compact Riemann surface $\widehat{\mathfrak{R}}$, which we will call a spectral curve. The eigenvalues of L_t are invariant under the time evolution, yet its eigenvector may vary depending on M_t. The dynamics of flows is governed by M_t and the appearance of a spectral curve is given by the zero locus

$$\widehat{\mathfrak{R}} = \{\det(\mu \cdot \mathrm{id}_{l\times l} - \mathsf{L}_t) = 0\}.$$

Notice that in a Lax representation, we do not have any notion of Hamiltonian dynamics. In order to translate flows induced by a Lax representation into Hamiltonians, we need a symplectic structure on the space where flows stays.

A moduli space is often a symplectic manifold. For example, a Jacobi variety is a symplectic manifold because it is an abelian variety. The realization of the symplectic form on a moduli space has been studied in many instances: W. Goldman proves that the cup product on $\mathrm{H}^1(\pi_1(\mathfrak{R}), \mathfrak{g}_{\mathsf{Ad}})$ is a symplectic form on the variety $\mathrm{Hom}(\pi, G)^-/G$ of representations in [23, 24] where G is a Lie group and $\mathfrak{g}_{\mathsf{Ad}}$ is the associated adjoint representation of G. For another example, S. Wolpert proves that the length and twist parameters construct a symplectic structure on the Teichmüller space, the moduli space of complex structures on a compact Riemann surface $\mathfrak{R}$ in [34, 67, 68, 69]. In order to take an advantage of the symplectic point of view in the study of a moduli space, N. Hitchin studied the cotangent bundle of a moduli space in [32], which has a natural symplectic structure. Comparably, Krichever constructs a symplectic structure associated with Lax matrices in [44]. From this symplectic structure, we will induce Hamiltonians from the flows in a Lax representation. Moreover, it turns out that the cotangent bundles provide much easier and more concrete ways to study the theory than the original moduli spaces alone. Likewise, we will see that the extension of the parameters by A. N. Tyurin on the moduli space to the cotangent bundle by I. Krichever in [44] indeed characterizes the space more definite than Tyurin parameters alone.

The final ingredient in this paper came from the work by P. A. Griffiths. Note that the flows in a Lax representation are not necessarily straight line flows. In order to describe the Hitchin system using a Lax representation, we need a special condition on M. In [27], P. A. Griffiths gave a necessary and sufficient condition where the flows from a Lax representation are straight in the case of spectral curves over $\mathbb{P}^1$. A similar question for the Hitchin system has not been answered yet in the author's knowledge[3]. In this paper, we will investigate this question and give an answer.

It is the author's hope that this investigation would serve a preliminary effort to the big progress in this area as N. Hitchin put it in his seminal 1987 paper very beautifully,

> " Finding some natural, concrete realization of the integrable systems which arise so naturally in this way may lead to an application in the other direction—from algebraic geometry to differential equations. This would be an agreeable outcome, and one consistent with Manin's view of the unity of mathematics."

1.2 Brief descriptions of each chapter

The purpose of Chapter 2 is to give descriptions of the Hitchin system and to present the explicit parameter space constructed by A. N. Tyurin. In Section 2.1, we introduce the definition and properties of the main objects we will investigate, which are called semi-stable bundles over a compact Riemann surface and characterize the

[3] After completing this paper, we made an acquaintance with a paper [21] by Letterio Gatto and Emma Previato.

moduli space of such objects. An exposition about Tyurin parameters in terms of matrix divisors is supplied. In Section 2.2 we describe the cotangent bundle of the moduli space in terms of a symplectic point of views. This naturally brings a spectral curve into our attention.

In Chapter 3, we parametrize the Hitchin system in terms of parameters constructed by I. M. Krichever. In Section 3.1, we explicitly give examples associated with the Hitchin system. Theses examples will also serve as a basis to the further examples in the later sections. In Section 3.2 we establish the relationship between the Hitchin system and the constructed parameter space by Krichever.

Chapter 4 is the principal part of our research. This chapter is devoted to cohomological interpretation of the straightness of flows and allied examples concerning explicit Hamiltonians. First of all, we will give basic facts and preliminaries for the further reading to the reader's convenience in Section 4.1 and Section 4.2. The main part of this chapter is Section 4.3. This section deals with a cohomological theory in a Lax representation. The ambiguity in a Lax representation can be well encoded in cohomology classes. The main results are stated in Theorem 4.2, Theorem 4.3, and Corollary 4.1. They will completely characterize the straightness of flows in terms of cohomology classes. In Section 4.4 we explain a relationship of choices of M in the Lax representation $\frac{d}{dt}\mathsf{L} = [\mathsf{M}, \mathsf{L}]$. In Section 4.5, we calculate explicit Hitchin's Hamiltonians in terms of Hamiltonians given by Krichever. Incidentally, the characterization of the Krichever-Tyurin parameters for classical groups in [32] is established. Those explicit examples are given in Example 4.2, 4.4 and 4.6.

In Appendix, we will explain the theory of commutative rings of differential operators. This is an analytic counter part of the Hitchin system and this part will enhance the historical attribution and understanding in the analytic theory of vector bundles over a compact Riemann surface as well as stimulate further developments along this direction. Section A.1 motivates the proof of Theorem 3.1. In Section A.2, we give the basic property of an (n, m)-curve, which will be used as a basic object in many examples.

Chapter 2

Moduli space of stable vector bundles

2.1 Tyurin parameters and semi-stable vector bundles

2.1.1 Uniquely equipped bundles and semi-stable bundles

Let E be a vector bundle[1] of rank l and degree lg over a compact Riemann surface $\mathfrak{R}$ of genus g. The Riemann-Roch theorem[2] implies

$$\dim \mathrm{H}^0(\mathfrak{R}, \mathrm{E}) - \dim \mathrm{H}^1(\mathfrak{R}, \mathrm{E}) = lg + l(1-g) = l.$$

Clearly, $\dim \mathrm{H}^0(\mathfrak{R}, \mathrm{E}) \geq l$. For a holomorphic line bundle L of degree g over a compact Riemann surface $\mathfrak{R}$ of genus g, we have $\dim \mathrm{H}^0(\mathfrak{R}, \mathrm{L}) \geq 1$. Consequently, such a bundle L has a nonzero section η, and the holomorphic section η generates each fiber L_p where $p \in \mathfrak{R}$ except g points $\gamma_i \in \mathfrak{R}$ associated with the divisor of the line bundle L. Unlike the case of a line bundle, a different phenomenon occurs when we deal with a vector bundle E of rank l and degree lg. In general, a set $\{\eta_1, \ldots, \eta_l\}$ of linearly independent sections of $\mathrm{H}^0(\mathfrak{R}, \mathrm{E})$ does not guarantee to generate even a single fiber E_p by $\mathrm{span}\{\eta_1(p), \ldots, \eta_l(p)\}$ where $p \in \mathfrak{R}$. If there exist a basis $\{\eta_1, \ldots, \eta_l\}$ of $\mathrm{H}^0(\mathfrak{R}, \mathrm{E})$ and a point $p \in \mathfrak{R}$ such that

$$\mathrm{E}_p = \mathrm{span}\{\eta_1(p), \ldots, \eta_l(p)\},$$

[1]Throughout this section, a vector bundle means a holomorphic vector bundle.

[2]See *p.64* in [29].

then we say that $\mathrm{H}^0(\mathfrak{R}, \mathrm{E})$ is an *equipment* of E. In particular, we say that E is a *uniquely equipped* vector bundle if $\dim \mathrm{H}^0(\mathfrak{R}, \mathrm{E}) = l$ (equivalently, $\dim \mathrm{H}^1(\mathfrak{R}, \mathrm{E}) = 0$) and $\mathrm{H}^0(\mathfrak{R}, \mathrm{E})$ is an equipment (*p.250* in [64]).

It is easy to see that an equipment $\{\eta_1, \ldots, \eta_l\}$ generates a fiber E_p for all $p \in \mathfrak{R}$ except lg points γ_i which is the divisor $D = \gamma_1 + \cdots + \gamma_{lg}$ of $\det \mathrm{E}$: If $\dim \operatorname{span}\{\eta_1(p), \ldots, \eta_l(p)\} < l$ over an infinite number of points $p \in \mathfrak{R}$, then $\dim \operatorname{span}\{\eta_1(p), \ldots, \eta_l(p)\} < l$ for all $p \in \mathfrak{R}$, since $\mathfrak{R}$ is a compact Riemann surface. In other words, all the sub-bundles of a uniquely equipped vector bundle E are less ample than E. For example, a *non-special*[3] holomorphic line bundle L of degree g over a compact Riemann surface $\mathfrak{R}$ of genus g is a uniquely equipped bundle. The set of such bundles forms an open set in the Jacobi variety $\mathrm{Jac}(\mathfrak{R})$, which is a moduli space of line bundles.

Definition 2.1. *[51] A holomorphic vector bundle* E *of rank* l *is said to be semi-stable if for all proper sub-bundles* H *of* E *we have*

$$\mathrm{slope}(\mathrm{H}) = \frac{\deg \mathrm{H}}{\operatorname{rank} \mathrm{H}} \le \frac{\deg \mathrm{E}}{\operatorname{rank} \mathrm{E}} = \mathrm{slope}(\mathrm{E}).$$

It is said to be a stable bundle if the strict inequality holds.

Let H be a proper sub-bundle of rank m of a uniquely equipped vector bundle E of rank l and degree lg. Then $\dim \mathrm{H}^0(\mathfrak{R}, \mathrm{H}) \le m$ and $\dim \mathrm{H}^1(\mathfrak{R}, \mathrm{H}) = 0$. By the Riemann-Roch theorem, we have

$$\deg \mathrm{H} \le m - m(1-g) = mg.$$

[3] A holomorphic line bundle L is said to be non-special if $\dim_{\mathbb{C}} \mathrm{H}^1(\mathfrak{R}, \mathrm{L}) = 0$.

Consequently,

$$\mathrm{slope(H)} = \frac{\deg \mathrm{H}}{\mathrm{rank\,H}} \leq \frac{mg}{m} = g = \mathrm{slope(E)}.$$

Hence, we may conclude that a uniquely equipped holomorphic vector bundle E of rank l and degree lg is necessarily a semi-stable vector bundle. Let us look at the converse: From an easy consequence of *Lemma 2.1 (p.16)* in [53], we have

Lemma 2.1. *If* E *is a semi-stable bundle of rank* $l > 1$ *and degree* lg *over* $\mathfrak{R}$ *of genus* g, *then* $\dim_{\mathbb{C}} \mathrm{H}^1(\mathfrak{R}, \mathrm{E}) = 0$.

Proof. Let us remark that the slope of any homomorphic image of a semi-stable bundle E is larger than or equal to the slope of E. This can be proved by the following observation: For a short exact sequence of vector bundles

$$0 \longrightarrow \mathrm{E}_1 \longrightarrow \mathrm{E} \longrightarrow \mathrm{E}_2 \longrightarrow 0\,,$$

we see that $\mathrm{slope(E)} \leq \mathrm{slope(E_2)}$, since

$$\deg \mathrm{E}_1 + \deg \mathrm{E}_2 = \deg \mathrm{E} \;\text{ and }\; \mathrm{rank\,E}_1 + \mathrm{rank\,E}_2 = \mathrm{rank\,E}\,.$$

On the other hands, by the Serre duality[4] we have

$$\mathrm{H}^1(\mathfrak{R}, \mathrm{E}) \cong \mathrm{H}^0(\mathfrak{R}, \mathrm{E}^* \otimes \mathrm{K}).$$

Here K is the canonical bundle over $\mathfrak{R}$ and E^* is the dual bundle of E. If there is a nonzero homomorphism $f : \mathrm{E} \to \mathrm{K}$, i.e., $f \in \mathrm{H}^0(\mathfrak{R}, \mathrm{Hom(E, K)}) \cong \mathrm{H}^0(\mathfrak{R}, \mathrm{E}^* \otimes \mathrm{K})$, then the slope of the image of $f : \mathrm{E} \to \mathrm{K}$ is equal to $2g - 2$, since $\mathrm{slope(K)} = 2g - 2$ and K is a line bundle. Hence, we have

$$\mathrm{slope}(f(\mathrm{E})) = 2g - 2 < 2g \leq \mathrm{slope(E)} = lg.$$

[4] See *p.70* in [29].

This contradicts the remark. $\square$

Now we are going to show that a semi-stable bundle of rank l and degree lg is indeed a uniquely equipped bundle. By the virtue of Lemma 2.1, it suffices to show that $\mathrm{H}^0(\mathfrak{R}, \mathrm{E})$ is an equipment: Suppose there does not exist a point $p \in \mathfrak{R}$ such that $\mathrm{span}\{\eta_1(p), \ldots, \eta_l(p)\} = \mathrm{E}_p$ where $\{\eta_1, \ldots, \eta_l\}$ is a basis of $\mathrm{H}^0(\mathfrak{R}, \mathrm{E})$. Then the linear span of all the elements of $\mathrm{H}^0(\mathfrak{R}, \mathrm{E})$ generates a proper sub-bundle H of E such that $\mathrm{slope}(\mathrm{H}) = \frac{lg}{m} > g = \frac{lg}{l}$, which contradicts semi-stability. Hence, this is an equipment. By summarizing all the results, we have proved

Theorem 2.1. *There is a one-to-one correspondence between semi-stable bundles and uniquely equipped bundles of rank l and degree lg.*

Let us parametrize the set of uniquely equipped bundles of rank l and degree lg by, so-called, *Tyurin parameters*[5]: Let E be a uniquely equipped bundle of rank l and degree lg. Let $D = \gamma_1 + \cdots + \gamma_{lg}$ be the associate divisor of $\det \mathrm{E}$ and take a basis $\{\eta_1, \ldots, \eta_l\}$ of the equipment $\mathrm{H}^0(\mathfrak{R}, \mathrm{E})$. Over the points $\{\gamma_1, \cdots, \gamma_{lg}\}$, we have linear dependence up to constant multiplications

$$0 = \sum_{j=1}^{l} \alpha_{i,j} \eta_j(\gamma_i).$$

The Tyurin parameters associated with a uniquely equipped bundle E up to choosing a basis of $\mathrm{H}^0(\mathfrak{R}, \mathrm{E})$ are given by $\left\{\gamma_i, \{\alpha_{i,j}\}_{j=1}^{l}\right\}_{i=1}^{lg} \in \mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1})$. The diagonal action of $\mathbf{SL}(l, \mathbb{C})$ on the symmetric power of $\mathbb{P}^{l-1}$ induces the action on the space

[5] In the next subsection, we will investigate the specific correspondence between the Tyurin parameters and holomorphic vector bundles in terms of matrix divisors.

of Tyurin parameters

$$\mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1})/\mathbf{SL}(l, \mathbb{C}).$$

In the space of Tyurin parameters, generically γ_i for $i = 1, \ldots, lg$ are distinct and $\{\eta_1(\gamma_i), \ldots, \eta_{l-1}(\gamma_i)\}$ generates an $(l-1)$-dimensional subspace of a fiber E_{γ_i} for $i = 1, \ldots, lg$, i.e,

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{i,j} \eta_j(\gamma_i) \text{ and } \eta_l(\gamma_i) \neq 0. \tag{2.1}$$

In other words, we may find an open set $\mathcal{M}_0$ in $\mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1})$ satisfying the above two conditions and parameterizing stable bundles. Note that the set of stable bundles forms an open set in the set of semi-stable bundles. Later, we will study the following space further

$$\widehat{\mathcal{M}_0} := \mathcal{M}_0/\mathbf{SL}(l, \mathbb{C}) \subset \mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1})/\mathbf{SL}(l, \mathbb{C}).$$

Pictorially, Figure 2.1 shows inclusions of the spaces. Each space[6] is contained in another as an open set in the sense of Zariski.

2.1.2 Tyurin parameters

The theory of relationship between divisors and holomorphic line bundles over a compact Riemann surface $\mathfrak{R}$ has been well developed in many treatises [19, 20, 26, 28]. The main theorem in this scheme is that a divisor D on a compact Riemann surface $\mathfrak{R}$ induces an associated line bundle L_D (*p.132* in [26]). That is, a holomorphic line bundle L can be completely characterized by a divisor D on $\mathfrak{R}$.

[6] For the definition of $\mathcal{M}_0'$, see the below of Equation (3.1).

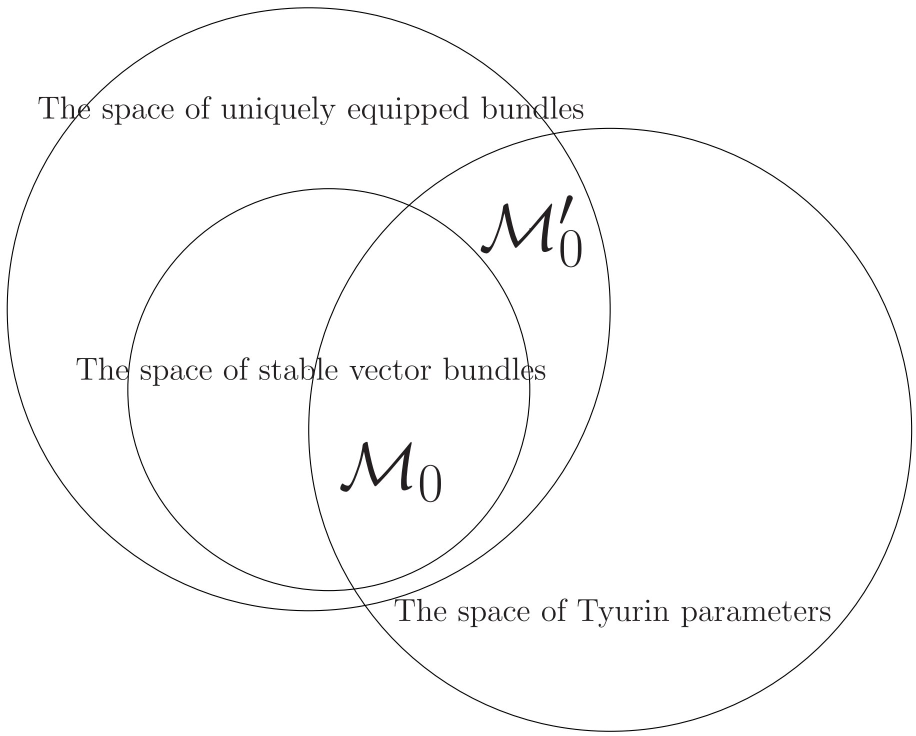

Figure 2.1: The inclusions of the spaces

This comparison of holomorphic line bundles to divisors on $\mathfrak{R}$ is extended to the correspondence of holomorphic vector bundles and matrix divisors on $\mathfrak{R}$ by A. N. Tyurin in [64, 65]. We will elucidate the Tyurin's investigation in [64, 65] informally: Let us remind that a divisor D on $\mathfrak{R}$ is an equivalence class $\left[\{f_i\}\right]$ of sets $\{f_i\}$ of local meromorphic functions associated with an open cover $\{U_i\}_{i\in I}$ of $\mathfrak{R}$ such that $\{f_i\}$ and $\{f_i'\}$ are equivalent, $\{f_i\} \sim \{f_i'\}$, if there are non-vanishing holomorphic functions h_i such that

$$f_i = h_i \cdot f_i' \text{ for } i \in I \text{ where } I \text{ is an index set.}$$

This can be generalized to a matrix divisor E on $\mathfrak{R}$ as follows: A matrix divisor[7] E is an equivalence class $\left[\{\mathsf{E}_i\}\right]$ of sets $\{\mathsf{E}_i\}$ of local matrix-valued meromorphic functions such that $\{\mathsf{E}_i\} \sim \{\mathsf{E}_i'\}$ if there are invertible matrix-valued holomorphic functions A_i such that

$$\mathsf{E}_i = \mathsf{A}_i \cdot \mathsf{E}_i' \text{ for } i \in I.$$

Similar to the case of divisors[8], a matrix divisor $E = \left[\{\mathsf{E}_i\}\right]$ defines an associated holomorphic vector bundle E on $\mathfrak{R}$ by the set $\{\mathsf{G}_{ij}\}$ of transition functions where

$$\mathsf{E}_i \cdot \mathsf{G}_{ij} = \mathsf{E}_j \text{ on } U_i \cap U_j.$$

We will show how the parameter formulated by A. N. Tyurin describes a general matrix divisor $E = \left[\{\mathsf{E}_i\}\right]$: Clearly, a set $\{\det \mathsf{E}_i\}$ defines a divisor on $\mathfrak{R}$.

[7]Note that a geometric meaning of a divisor D as a sum of points on $\mathfrak{R}$ is lost in the definition of a matrix divisor E on $\mathfrak{R}$ in general. However, if the divisor of $\{\det \mathsf{E}_i\}$ consists of distinct points, a matrix divisor E can be geometrically assigned to a sum of points on $\mathfrak{R} \times \mathbb{P}^{l-1}$.

[8]The set $\{g_{ij}\}$ of transition functions associated with a divisor $D = \left[\{f_i\}\right]$ where $g_{ij} = f_i^{-1} \cdot f_j$ on $U_i \cap U_j$ defines a holomorphic line bundle L_D.

Let us assume that $\{\det \mathsf{E}_i\}$ is an effective divisor $D = \sum_{k=1}^{N} m_k p_k$ on $\mathfrak{R}$ where m_k is a positive integer. Then the characterization of an equivalence class $E = \left[\{\mathsf{E}_i\}\right]$ is given by a normal form[9] around $p_k \in U_i$

$$\mathsf{E}_{p_k,i} = \begin{pmatrix} z^{d_{1,k}} & 0 & \cdots & \cdots & 0 \\ 0 & z^{d_{2,k}} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & z^{d_{l-1,k}} & 0 \\ 0 & \cdots & \cdots & 0 & z^{d_{l,k}} \end{pmatrix} \begin{pmatrix} 1 & \alpha_{1,2,k,i}(z) & \cdots & \cdots & \alpha_{1,l,k,i}(z) \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & \alpha_{l-1,l,k,i}(z) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Here $\alpha_{r,s,k,i}(z) \in \frac{\mathbb{C}[[z]]}{z^{d_{s,k}-d_{r,k}}\mathbb{C}[[z]]}$ where z is a local coordinate around $p_k \in U_i$ and $\mathbb{C}[[z]]$ is the ring of power series in one variable. Note that $d_{r,k} \leq d_{s,k}$ if $r \leq s$ and $m_k = \sum_{r=1}^{l} d_{r,k}$. For an index j where U_j does not contain p_k, a normal form of E_j is defined by

$$\mathsf{E}_j = \mathrm{id}_{l\times l}\,.$$

Accordingly, if $\{\det \mathsf{E}_i\}$ defines an effective divisor $D = \sum_{k=1}^{N} p_k$ where p_k are distinct, a normal form can get simplified noticeably. The normal form is given by

$$\mathsf{E}_{p_k,i} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_{1,l,k,i} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & 1 & \alpha_{l-1,l,k,i} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \quad \text{where } \alpha_{r,l,k,i} \in \mathbb{C}.$$

Example 2.1. Let us consider a holomorphic vector bundle E of rank 2 on $\mathfrak{R}$ of

[9] *p.253* in [64].

genus g. Over a point γ_k for $k = 1 \ldots, 2g$, we may have

$$\mathsf{E}_{\gamma_k,i} = \begin{pmatrix} 1 & \alpha_{k,i} \\ 0 & z_{k,i} \end{pmatrix} \text{ and } \mathsf{E}_{\gamma_k,j} = \begin{pmatrix} 1 & \alpha_{k,j} \\ 0 & z_{k,j} \end{pmatrix}.$$

Here $z_{k,i}$ is a local coordinate around p_k in U_i. So,

$$\mathsf{G}_{ij} = \mathsf{E}^{-1}_{\gamma_k,i} \cdot \mathsf{E}_{\gamma_k,j} = \begin{pmatrix} 1 & -\frac{\alpha_{k,i}}{z_{k,i}} \\ 0 & \frac{1}{z_{k,i}} \end{pmatrix} \begin{pmatrix} 1 & \alpha_{k,j} \\ 0 & z_{k,j} \end{pmatrix} = \begin{pmatrix} 1 & \alpha_{k,j} - \alpha_{k,i}\frac{z_{k,j}}{z_{k,i}} \\ 0 & \frac{z_{k,j}}{z_{k,i}} \end{pmatrix} \qquad \text{if } \gamma_k \in U_i \cap U_j$$

$$\mathsf{G}_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{if } \gamma_k \notin U_i \cap U_j.$$

Hence, $\{\mathsf{G}_{ij}\}$ defines a holomorphic vector bundle E. Note that $\frac{z_{k,j}}{z_{k,i}}$ is non-zero in $U_i \cap U_j$.

Example 2.2. Let us consider a holomorphic vector bundle E of rank 3 on $\mathfrak{R}$ of genus g. Over a point γ_k for $k = 1 \ldots, 3g$, we may have

$$\mathsf{E}_{\gamma_k,i} = \begin{pmatrix} 1 & 0 & \alpha_{1,k,i} \\ 0 & 1 & \alpha_{2,k,i} \\ 0 & 0 & z_{k,i} \end{pmatrix} \text{ and } \mathsf{E}_{\gamma_k,j} = \begin{pmatrix} 1 & 0 & \alpha_{1,k,j} \\ 0 & 1 & \alpha_{2,k,j} \\ 0 & 0 & z_{k,j} \end{pmatrix}.$$

Hence, on $U_i \cap U_j$ containing p_k we have

$$\mathsf{G}_{ij} = \mathsf{E}^{-1}_{\gamma_k,i} \cdot \mathsf{E}_{\gamma_k,j} = \begin{pmatrix} 1 & 0 & -\frac{\alpha_{1,k,i}}{z_{k,i}} \\ 0 & 1 & -\frac{\alpha_{2,k,i}}{z_{k,i}} \\ 0 & 0 & \frac{1}{z_{k,i}} \end{pmatrix} \begin{pmatrix} 1 & 0 & \alpha_{1,k,j} \\ 0 & 1 & \alpha_{2,k,j} \\ 0 & 0 & z_{k,j} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \alpha_{1,k,j} - \alpha_{1,k,i}\frac{z_{k,j}}{z_{k,i}} \\ 0 & 1 & \alpha_{2,k,j} - \alpha_{2,k,i}\frac{z_{k,j}}{z_{k,i}} \\ 0 & 0 & \frac{z_{k,j}}{z_{k,i}} \end{pmatrix}.$$

Otherwise, G_{ij} is given by the (3×3)-identity matrix.

From these examples, it is not hard to see that for $\gamma_k \in U_i \cap U_j$

$$\mathsf{G}_{ij} = \mathsf{E}^{-1}_{\gamma_k,i} \cdot \mathsf{E}_{\gamma_k,j} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_{1,l,k,j} - \alpha_{1,l,k,i}\frac{z_{k,j}}{z_{k,i}} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & 1 & \alpha_{l-1,l,k,j} - \alpha_{l-1,l,k,i}\frac{z_{k,j}}{z_{k,i}} \\ 0 & \cdots & \cdots & 0 & \frac{z_{k,j}}{z_{k,i}} \end{pmatrix}.$$

Hence, over γ_k it defines a point

$$[\frac{z_{k,j}}{z_{k,i}}, \alpha_{1,l,k,j} - \alpha_{1,l,k,i}\frac{z_{k,j}}{z_{k,i}}, \ldots, \alpha_{l-1,l,k,j} - \alpha_{l-1,l,k,i}\frac{z_{k,j}}{z_{k,i}}] \in \mathbb{P}^{l-1}.$$

Note that this turns out to be the Tyurin parameters in (2.1). This indeed implies that a generic point[10] $(\gamma, \alpha) \in \mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1})$ in the space of Tyurin parameters defines an effective[11] vector bundle $\mathrm{E}_{\gamma,\alpha}$ up to the diagonal action of $\mathbf{SL}(l, \mathbb{C})$.

[10] All γ_k are distinct.

[11] The divisor of the associated determinant vector bundle $\det \mathrm{E}$ is effective.

2.2 Hitchin systems and symplectic geometry

2.2.1 Hamiltonian dynamics

We will briefly give basic definitions in Hamiltonian dynamics. For more detail, we refer to [10] and [48]. Let M be a symplectic manifold with a symplectic form ω. A *Hamiltonian vector field* X_H associated to the symplectic form ω and a smooth function H on M is defined by

$$dH = \iota(X_H)\omega.$$

We will call H a *Hamiltonian or Hamiltonian function.* The *Poisson bracket* $\{\,,\,\}$ (*p.108* in [10]) associated to the symplectic form is defined by

$$\{H, G\} = X_H \cdot G.$$

Two functions H, G are said to be *Poisson commutative* if

$$\{H, G\} = 0.$$

Note that the maximal number of linearly independent Hamiltonians on a symplectic manifold M of dimension $2n$ is n. Accordingly, we say that a symplectic manifold M of dimension $2n$ is a *completely integrable system* if it has n linearly independent Hamiltonians $H_1, \ldots, H_n$ generically, i.e.,

$$dH_1 \wedge \cdots \wedge dH_n \neq 0 \text{ generically.}$$

If M is a completely integrable system, we may define a map

$$\mathbf{H} : M^{2n} \to \mathbb{C}^n \text{ by } \mathbf{H}(m) = (H_1(m), \ldots, H_n(m)).$$

This is a special case of a momentum map (*p.133* in [10]) in symplectic geometry. Indeed, it is a momentum map for the action of an abelian group, i.e., a complex torus. The primary dynamical system to study in this paper is presented as follows:

Definition 2.2. *(*p.96 *in [32]) A dynamical system is said to be an algebraically completely integrable system if*

1 it is a completely integrable system

2 a generic fiber of $\mathbf{H}$ *is an (Zariski) open set of an abelian variety*

3 each Hamiltonian flow of X_{H_i} *is linear on a generic fiber.*

2.2.2 Spectral curve and Hitchin system

In [32], N. Hitchin studied a moduli space of stable vector bundles in terms of symplectic geometry in an infinite dimensional setting. He proves that the cotangent bundle of a moduli space is an algebraically completely integrable system, so-called a Hitchin system. In this section, we will describe this gauge theoretic and symplectic interpretation of the moduli space in detail.

Definition 2.3. *Let* E *be a smooth complex vector bundle of rank* l *over a compact Riemann surface* $\mathfrak{R}$. *A* holomorphic structure *on* E *is a differential operator* d''_A *satisfying the Leibniz rule*

$$d''_A(fs) = \overline{\partial} f \otimes s + f d''_A s \text{ where}$$

$s \in \mathcal{A}^0(\mathfrak{R}, \mathrm{E})$ *and* $f \in C^\infty(\mathfrak{R})$. *Here* $\mathcal{A}^0(\mathfrak{R}, \mathrm{E})$ *is the set of smooth sections on* E.

Since a compact Riemann surface $\mathfrak{R}$ is a 1-dimensional complex manifold, the integrability condition $d''_A \circ d''_A = 0$ is trivially satisfied. That is, we can find l linearly independent local solutions of $d''_A s = 0$. Thus, each holomorphic structure gives rise to a holomorphic vector bundle E over $\mathfrak{R}$ up to a conjugation action of a gauge group $\mathcal{G}$, which consists of smooth maps $g : \mathfrak{R} \to \mathbf{GL}(l, \mathbb{C})$. Let us denote the space of all holomorphic structures on E by $\mathfrak{A}$. This is an affine infinite dimensional space. The quotient space of the space $\mathfrak{A}$ by the action of gauge group $\mathcal{G}$ tends to be non-Hausdorff. In order to overcome this drawback, we can take an open set where the quotient space becomes a manifold. An open set $\mathfrak{A}^s$ in $\mathfrak{A}$ consisting of stable vector bundles is indeed such a set[12]. In fact, the quotient space becomes a projective variety when the degree and the rank of E are coprime, which we will assume throughout this section.

Since $\mathfrak{A}$ is an affine space, we may define a cotangent bundle $\mathrm{T}^* \mathfrak{A}$. Moreover, it is not hard to see that it has a natural symplectic form $\omega_{(A,\Phi)}$ defined by

$$\omega_{(A,\Phi)}((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_{\mathfrak{R}} \mathrm{Tr}(\dot{A}_1 \dot{\Phi}_2 - \dot{A}_2 \dot{\Phi}_1) \text{ where } (A, \Phi) \in \mathrm{T}^*_A \mathfrak{A}.$$

A momentum map $\mu : \mathrm{T}^* \mathfrak{A}^s \to \mathsf{Lie}(\mathcal{G})^*$ induced by the action of gauge group $\mathcal{G}$ is given by

$$\mu(A, \Phi) = d''_A \Phi.$$

The zero locus $\mu^{-1}(0)$ of the momentum map consists of holomorphic fields Φ with respect to the holomorphic structure d''_A. It is called a *Higgs field*. The Marsden-

[12] See Section 2.1 for the definition and properties of a stable vector bundle.

Weinstein quotient[13]

$$\mathrm{T}^*\mathcal{N} := \mathrm{T}^*(\mathfrak{A}^s/\mathcal{G}) \cong \mu^{-1}(0)/\mathcal{G} := (\mathrm{T}^*\mathfrak{A}^s)/\!\!/\mathcal{G}$$

is the space which N. Hitchin investigated in [32]. In this description, a Higgs field $\Phi_{[A]}(z)$ for $z \in \mathfrak{R}$ and $[A] \in \mathcal{N}$ is a cotangent vector in $\mathrm{T}^*_{[A]}\mathcal{N}$ and holomorphic on $\mathfrak{R}$. The main thesis in [32] can be put in the following way:

Theorem 2.2. *[32] Let $\mathcal{N}$ be a moduli space of stable holomorphic vector bundles of rank l over a compact Riemann surface $\mathfrak{R}$ of genus > 1. Then the cotangent bundle $\mathrm{T}^*\mathcal{N}$ is an algebraically completely integrable system. The statement is also true for the cotangent bundle of a moduli space of stable holomorphic principal G-bundles over a compact Riemann surface $\mathfrak{R}$ of genus > 1 where G is a semi-simple complex Lie group..*

The main part of the proof builds up on an observation that a generic fiber of $\mathbf{H}$ is an open set of the Jacobi variety of a spectral curve, which we will construct later. Let $\mathrm{K}_{\mathfrak{R}}$ be the canonical bundle of $\mathfrak{R}$. The Hitchin map $\mathbf{H}$ is defined by invariant polynomials

$$\mathbf{H} : \mathrm{T}^*\mathcal{N} \to \bigoplus_{i=1}^{k} \mathrm{H}^0(\mathfrak{R}, \mathrm{K}_{\mathfrak{R}}^{d_i}) \text{ where } \mathbf{H}(\Phi_{[A]}(z)) = \Big(h_1(\Phi_{[A]}(z)), \ldots, h_k(\Phi_{[A]}(z))\Big).$$

The invariant polynomials $h_1, \ldots, h_k$ are the coefficients of the characteristic polynomial of a Higgs field and the constituents of invariant polynomials are Hamiltonians with respect to the canonical symplectic form on $\mathrm{T}^*\mathcal{N}$ and N functions H_i, the constituents of invariant polynomials, on $\mathrm{T}^*\mathcal{N}$ are Poisson commutative

[13] This is also called a *symplectic quotient*. See *p.141* in [10].

where $N = \dim_{\mathbb{C}} \mathcal{N}$. Thus, $\mathrm{T}^* \mathcal{N}$ is a completely integrable system (*Proposition 4.1*, *Proposition 4.5* in [32]). Consider the following diagram

$$\begin{array}{ccc} \lambda_z \in \mathrm{K}_{\mathfrak{R}} & \xrightarrow{\pi^*} & \pi^*\lambda \in \pi^*\mathrm{K}_{\mathfrak{R}} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ z \in \mathfrak{R} & \xleftarrow{\pi} & \lambda_z \in \mathrm{K}_{\mathfrak{R}}\,. \end{array} \tag{2.2}$$

A spectral curve $\widehat{\mathfrak{R}}$ associated with a Higgs field Φ is the zero locus of a section $\pi^* \det(\lambda_z \cdot I_{l\times l} - \Phi_{[A]}(z)) \in (\pi^* \mathrm{K}_{\mathfrak{R}})^l$

$$\widehat{\mathfrak{R}} = \{\lambda_z \in \mathrm{K}_{\mathfrak{R}} \mid \pi^* \det(\lambda_z \cdot I_{l\times l} - \Phi_{[A]}(z)) = 0\}.$$

2.2.3 Parametrization for $G = \mathbf{GL}(l, \mathbb{C})$

Fixing a value of the Hitchin map $\mathbf{H}$ is equivalent to fixing a spectral curve $\widehat{\mathfrak{R}}_{\boldsymbol{\phi}}$ associated to a fiber $\mathbf{H}^{-1}(\boldsymbol{\phi})$ where $\boldsymbol{\phi} = \mathbf{H}\big(\Phi_{[A]}(z)\big)$. A spectral curve associated with a generic fiber of $\mathbf{H}$ is smooth and is a ramified l-sheeted covering space of $\mathfrak{R}$. In particular, the genus is $l^2(g-1)+1$ and the ramification index is $2(l^2-l)(g-1)$.

Since each point of $\widehat{\mathfrak{R}}$ is an eigenvalue of a Higgs field $\Phi_{[A]}$, a holomorphic vector bundle $\mathrm{E}_{[A]}$ induced from a holomorphic structure d''_A to $\widehat{\mathfrak{R}}$ defines an eigenspace line bundle $\mathbf{Ker}(\lambda_z \cdot I_{l\times l} - \Phi_{[A]}(z))$. Conversely, a holomorphic line bundle L on $\widehat{\mathfrak{R}}$ gives rise to a holomorphic vector bundle E of rank l on $\mathfrak{R}$ by the direct image sheaf $\pi_* L$. In this way, each point of a fiber $\mathbf{H}^{-1}(\boldsymbol{\phi})$ can be parametrized by an open set of $\mathrm{Jac}(\widehat{\mathfrak{R}}_{\boldsymbol{\phi}})$.

The last information we need to identify $\mathrm{T}^* \mathcal{N}$ with an algebraically completely integrable system is the linearity of the Hamiltonian flow of a Hamiltonian vector field X_{H_i}: The main observation is that $\mathrm{T}^* \mathcal{N}$ is contained in a larger symplectic manifold (with singularities) where an open set of $\mathrm{Jac}(\widehat{\mathfrak{R}}_{\boldsymbol{\phi}})$ is naturally compactified[14]. That is, the momentum map can be constructed on the cotangent bundle of the space $\mathfrak{A}$ of all the holomorphic structures instead of the space $\mathfrak{A}^s$ of stable holomorphic structures

$$\widetilde{\mu} : \mathrm{T}^* \mathfrak{A} \to \mathsf{Lie}(\mathcal{G})^*.$$

The construction is natural and

$$\mathrm{T}^* \mathcal{N} = \mu^{-1}(0)/\mathcal{G} \subseteq \widetilde{\mu}^{-1}(0)/\mathcal{G}.$$

Hence, the Hitchin map $\mathbf{H}$ can be naturally extended to $\widetilde{\mu}^{-1}(0)/\mathcal{G}$ and a generic fiber of $\mathbf{H}$ on $\widetilde{\mu}^{-1}(0)/\mathcal{G}$ is in fact the whole of $\mathrm{Jac}(\widehat{\mathfrak{R}}_{\boldsymbol{\phi}})$. Consequently each Hamiltonian vector field X_{H_i} is extended to the whole Jacobi variety $\mathrm{Jac}(\widehat{\mathfrak{R}}_{\boldsymbol{\phi}})$. Therefore, the flow of X_{H_i} must be linear. Let us summarize the results for classical groups in [32]

Theorem 2.3. *[32] Let* $\boldsymbol{\phi} = \mathbf{H}(\Phi_{[A]})$.

1 *For* $G = \mathbf{SL}(l, \mathbb{C})$, *a spectral curve* $\widehat{\mathfrak{R}}_{\boldsymbol{\phi}}$ *associated with a generic fiber is smooth and is an* l*-sheeted covering space of* $\mathfrak{R}$. *A generic fiber* $\mathbf{H}^{-1}(\boldsymbol{\phi})$ *is an open set of* $\mathrm{Jac}(\widehat{\mathfrak{R}}_{\boldsymbol{\phi}})$.

2 *For* $G = \mathbf{SP}(l, \mathbb{C})$, *a generic spectral curve* $\widehat{\mathfrak{R}}_{\boldsymbol{\phi}}$ *is smooth and is a* $2l$*-sheeted covering space of* $\mathfrak{R}$. *A generic fiber* $\mathbf{H}^{-1}(\boldsymbol{\phi})$ *is an open set of* $\mathrm{Prym}(\widehat{\mathfrak{R}}_{\boldsymbol{\phi}})$ *associated with a 2-sheeted covering* $\pi : \widehat{\mathfrak{R}}_{\boldsymbol{\phi}} \to \widehat{\mathfrak{R}}_{\boldsymbol{\phi}}/\sigma$ *ramified at* $4l(g-1)$ *points where* σ *is a natural involution induced by a symplectic form on* E.

[14] See also [31].

3 For $G = \mathbf{SO}(2l, \mathbb{C})$, a generic spectral curve $\widehat{\mathfrak{R}}_{\boldsymbol{\phi}}$ is singular with $2l(g-1)$ ordinary double singularities and is a $2l$-sheeted covering space of $\mathfrak{R}$. A generic fiber $\mathbf{H}^{-1}(\phi)$ is an open set of $\mathrm{Prym}(\widetilde{\widehat{\mathfrak{R}}_{\boldsymbol{\phi}}})$ *associated with a* unramified 2-*sheeted covering $\pi : \widetilde{\widehat{\mathfrak{R}}_{\boldsymbol{\phi}}} \to \widetilde{\widehat{\mathfrak{R}}_{\boldsymbol{\phi}}}/\sigma$ where σ is a natural involution induced by a non-degenerated symmetric bilinear form on* E *and $\widetilde{\widehat{\mathfrak{R}}_{\boldsymbol{\phi}}}$ is the de-singularization of $\widehat{\mathfrak{R}}_{\boldsymbol{\phi}}$*

4 For $G = \mathbf{SO}(2l+1, \mathbb{C})$, a generic spectral curve $\widehat{\mathfrak{R}}_{\boldsymbol{\phi}}$ is smooth and is a $2l+1$-*sheeted covering space of $\mathfrak{R}$. A generic fiber $\mathbf{H}^{-1}(\boldsymbol{\phi})$ is an open set of* $\mathrm{Prym}(\widehat{\mathfrak{R}}_{\boldsymbol{\phi}})$ *associated with a* 2-*sheeted covering $\pi : \widehat{\mathfrak{R}}_{\boldsymbol{\phi}} \to \widehat{\mathfrak{R}}_{\boldsymbol{\phi}}/\sigma$ ramified at* $2(2l+1)(g-1)$ *points.*

In Section 4.5, we will give explicit examples of $\mathrm{T}^* \mathcal{N}$ for various classical groups by applying the techniques[15] in the proof of Theorem 2.3. As by-product, we may see that the description of $\mathrm{T}^* \mathcal{N}$ associated with various classical groups by the Krichever-Tyurin parameters is considerably easier than that of $\mathcal{N}$ for the classical groups by the Tyurin parameters.

[15] See [32] for detail.

Chapter 3

Lax equations on algebraic curves

An explicit parametrization of the cotangent bundle of a moduli space of stable vector bundles over a compact Riemann surface is given in [44]. We will give an exposition about it and provide detailed examples[1] allied with this exposition in terms of the Krichever-Tyurin parameters[2].

3.1 Parametrization of the cotangent bundle of a moduli space of stable vector bundles

Let $\mathrm{E}_{\gamma,\alpha}$ be a holomorphic vector bundle of rank l on a compact Riemann surface $\mathfrak{R}$ of genus g associated with Tyurin parameters $(\gamma, \alpha) = \left\{\gamma_j, \boldsymbol{\alpha}_j\right\}_{j=1}^{lg} \in \mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1})$. A global section $\boldsymbol{\zeta}_{\gamma,\alpha}(z)$ of $\mathrm{E}_{\gamma,\alpha}$ can be written as a vector-valued meromorphic function on $\mathfrak{R}$: Let $\boldsymbol{\alpha}_j = (\alpha_{1,j}, \ldots, \alpha_{l-1,j}, 1) \in \mathbb{C}^l$ for $j = 1, \ldots, lg$. Then

$$\boldsymbol{\zeta}_{\gamma,\alpha}(z) = \frac{c_j \boldsymbol{\alpha}_j}{z - z(\gamma_j)} + O(1) \text{ where } c_j \in \mathbb{C}. \tag{3.1}$$

From the Riemann-Roch theorem and the given constraint (3.1), we have

$$\dim_{\mathbb{C}} \mathrm{H}^0(\mathfrak{R}, \mathrm{E}_{\gamma,\alpha}) \geq l(lg - g + 1) - lg(l-1) = l.$$

[1] These will also serve as backgrounds for examples in Section 4.5.

[2] See Lemma 3.1.

Such vector bundles $\mathrm{E}_{\gamma,\alpha}$ with mutually distinct γ_j for $j = 1, \dots, lg$ and satisfying $\dim_{\mathbb{C}} \mathrm{H}^0(\mathfrak{R}, \mathrm{E}_{\gamma,\alpha}) = l$ form an open set $\mathcal{M}_0'$ of $\mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1})$. Note that $\mathcal{M}_0$ is an open set of $\mathcal{M}_0'$.

Definition 3.1. *A Krichever-Lax matrix*[3] *associated to Tyurin parameters* (γ, α) *and a canonical divisor* K *of a compact Riemann surface* $\mathfrak{R}$ *of genus* g *is a matrix-valued meromorphic function* $\mathsf{L}(p; \gamma, \alpha)$ *with simple poles at* γ_i *and poles at* K *satisfying the following conditions: There exist* $\boldsymbol{\beta}_j \in \mathbb{C}^l$ *and* $\kappa_j \in \mathbb{C}$ *for* $j = 1, \dots, lg$ *such that a local expression in a neighborhood of* γ_j *is given by*

$$\mathsf{L}(p; \gamma, \alpha) = \frac{\mathsf{L}_{j,-1}(\gamma, \alpha)}{z(p) - z(\gamma_j)} + \mathsf{L}_{j,0}(\gamma, \alpha) + O\big((z(p) - z(\gamma_j))\big) \text{ for } j = 1, \dots, lg$$

with the following two constraints

1. $\mathsf{L}_{j,-1}(\gamma, \alpha) = \boldsymbol{\beta}_j^T \cdot \boldsymbol{\alpha}_j$, *i.e., of rank* 1 *and it is traceless*

$$\mathrm{Tr}\, \mathsf{L}_{j,-1} = \boldsymbol{\alpha}_j \cdot \boldsymbol{\beta}_j^T = 0.$$

2. $\boldsymbol{\alpha}_j$ *is a left eigenvector of* $\mathsf{L}_{j,0}$

$$\boldsymbol{\alpha}_j \mathsf{L}_{j,0}(\gamma, \alpha) = \kappa_j \boldsymbol{\alpha}_j.$$

Note that we may find that a holomorphic differential ω *such that the associated divisor* K *with* ω *does not intersect with* $\{\gamma_j\}_{j=1}^{lg}$ *and a local coordinate* z_j *at* γ_j *is given by* $dz_j = \omega$ *in the neighborhood of* γ_j.

Let us denote the set of Krichever-Lax matrices associated to Tyurin parameters (γ, α) *and a canonical divisor* K *by* $\mathcal{L}^K_{\gamma,\alpha}$.

[3] For simplicity, we will also call it a Lax matrix as in [44].

The two constraints imply that a Lax matrix can be thought as a *Higgs field* $\mathsf{L}(p;\gamma,\alpha)\otimes\omega$, i.e., a global section of $\mathrm{End}(\mathrm{E}_{\gamma,\alpha})\otimes\mathrm{K}$: In a neighborhood of γ_j, the first and the second condition respectively imply

$$\begin{aligned}\boldsymbol{\zeta}_{\gamma,\alpha}(z)\mathsf{L}_{j,-1}(\gamma,\alpha) &= O(1)\\ \boldsymbol{\zeta}_{\gamma,\alpha}(z)\mathsf{L}_{j,0}(\gamma,\alpha) &= \kappa_j\boldsymbol{\zeta}_{\gamma,\alpha}(z).\end{aligned} \tag{3.2}$$

Since we are assuming the divisor K of ω does not intersect with $\{\gamma_j\}_{j=1}^{lg}$, we may conclude that $\mathsf{L}(p;\gamma,\alpha)\otimes\omega$ is a global section of $\mathrm{End}(\mathrm{E}_{\gamma,\alpha})\otimes\mathrm{K}$. The dimension of the space of Lax matrices associated to Tyurin parameters (γ,α) is

$$\dim_{\mathbb{C}}\mathcal{L}^K_{\gamma,\alpha} = l^2(2g-2+lg-g+1) - lg\cdot l\cdot(l-1) - lg - lg(l-1) = l^2(g-1).$$

The first term is from the Riemann-Roch theorem, the second term from the condition on rank 1, the third term from the traceless condition, and the fourth term is from the condition on left eigenvectors. Consequently,

$$\dim_{\mathbb{C}}\mathcal{L}^K = l^2(2g-1) \text{ where } \mathcal{L}^K = \bigcup_{(\gamma,\alpha)\in\mathcal{M}'_0}\mathcal{L}^K_{\gamma,\alpha}.$$

In fact, the following lemma shows that $(\alpha,\beta,\gamma,\kappa)$ can be served as coordinates[4] of $\mathcal{L}^K$.

Lemma 3.1. *(*p.236 *in [44]) Let $(\gamma,\alpha)\in\mathcal{M}'_0$. There is a bijection map*

$$\mathsf{L}\mapsto\left\{\boldsymbol{\alpha}_j,\boldsymbol{\beta}_j,\gamma_j,\kappa_j\right\}_{j=1}^{lg}$$

between $\mathcal{L}^K$ and a subset $\mathcal{V}$ of $\mathcal{S}^{lg}(\mathbb{P}^{l-1}\times\mathbb{C}^l\times\mathfrak{R}\times\mathbb{C})$ defined by

[4]We will call them the Krichever-Tyurin parameters.

$$\boldsymbol{\alpha}_j \cdot \boldsymbol{\beta}_j^T = 0 \textit{ for } j = 1, \ldots, lg \textit{ and}$$
$$\sum_{p \in \mathfrak{R}} \operatorname{res}(\mathsf{L} \otimes \omega) = \sum_{j=1}^{lg} \boldsymbol{\beta}_j^T \cdot \boldsymbol{\alpha}_j = \mathsf{O}_{l \times l}. \tag{3.3}$$

Here $\mathsf{O}_{l\times l}$ *is the* $(l \times l)$*-zero matrix.*

Because of the dimension differences, the space $\mathcal{L}^K$ whose dimension is $l^2(2g-1)$ cannot be identified with a cotangent bundle $\mathrm{T}^* \mathcal{M}_0$ whose dimension is $2l^2 g$. However, we may see that $\mathcal{L}^K/\mathbf{SL}(l, \mathbb{C})$ can be identified with

$$\mathrm{T}^* \widehat{\mathcal{M}_0} = \mathrm{T}^* \mathcal{M}_0/\mathbf{SL}(l, \mathbb{C}) \text{ where } \dim_{\mathbb{C}} \mathrm{T}^* \widehat{\mathcal{M}_0} = 2(l^2(g-1)+1).$$

First note that there is a conjugation action, which is not necessarily free, of $W \in \mathbf{SL}(l, \mathbb{C})$ on the set $\mathcal{L}^K$ of Lax matrices $\mathsf{L}(p; \gamma, \alpha)$,

$$\mathsf{L} \mapsto W^{-1} \cdot \mathsf{L} \cdot W.$$

This action defines an action of $W \in \mathbf{SL}(l, \mathbb{C})$ on the space of the Krichever-Tyurin parameters equivariantly:

$$W \cdot \left\{ \boldsymbol{\alpha}_j, \boldsymbol{\beta}_j, \gamma_j, \kappa_j \right\}_{j=1}^{lg} = \left\{ \boldsymbol{\alpha}_j \cdot W, \boldsymbol{\beta}_j \cdot (W^{-1})^T, \gamma_j, \kappa_j \right\}_{j=1}^{lg}.$$

We may find an open set $\mathcal{V}'$ of $\mathcal{V}$ such that any $l+1$ vectors $\{\boldsymbol{\alpha}_{s_1}, \ldots, \boldsymbol{\alpha}_{s_{l+1}}\}$ of lg vectors in $\left\{\boldsymbol{\alpha}_j\right\}_{j=1}^{lg}$ generate an l-dimensional vector space and those $l+1$ vectors satisfy the relation

$$\boldsymbol{\alpha}_{s_{l+1}} = \sum_{j=1}^{l} c_{s_j} \boldsymbol{\alpha}_{s_j} \text{ where } c_{s_j} \neq 0.$$

$\mathcal{V}'/\mathbf{SL}(l, \mathbb{C})$ has an atlas whose charts are indexed by $\{s_1, \ldots, s_{l+1}\}$. For example, let us assume

$$\{s_1, \ldots, s_{l+1}\} = \{1, \ldots, l+1\}.$$

There is an $W \in \mathbf{SL}(l,\mathbb{C})$ such that $\mathbf{e}_j = \boldsymbol{\alpha}_j W$ where $\mathbf{e}_j$ is a basis vector with coordinates $e^i_j = \delta^i_j$ for $j = 1,\dots,l$. Since $\boldsymbol{\alpha}_{l+1} \in \mathbb{P}^{l-1}$, there is $0 \neq c \in \mathbb{C}$ such that

$$c\mathbf{e}_0 = \boldsymbol{\alpha}_{l+1} W \text{ where } \mathbf{e}_0 = \sum_{j=1}^{l} \mathbf{e}_j.$$

For $j = 1,\dots,lg$ let

$$\mathbf{A}_j = \boldsymbol{\alpha}_j W \text{ and } \mathbf{B}_j = \boldsymbol{\beta}_j (W^{-1})^T.$$

This $\left\{\mathbf{A}_j, \mathbf{B}_j, \gamma_j, \kappa_j\right\}_{j=1}^{lg}$ is a coordinate of an $\mathbf{SL}(l,\mathbb{C})$-orbit in the chart indexed by $\{1,\dots,l+1\}$. Notice that $\mathbf{B}_1,\dots,\mathbf{B}_{l+1}$ can be explicitly found from the data $\left\{\mathbf{A}_j, \mathbf{B}_j\right\}_{j=l+2}^{lg}$ using the two conditions (3.3) in Lemma 3.1

$$B^i_{l+1} = -\sum_{j=l+2}^{lg} B^i_j A^i_j$$

$$B^i_k + B^i_{l+1} = -\sum_{j=l+2}^{lg} B^i_j A^k_j \text{ where } k = 1,\dots,l.$$

Here $\mathbf{B}_j = (B^1_j,\dots,B^l_j)$ and $\mathbf{A}_j = (A^1_j,\dots,A^l_j)$. Consequently, we have

$$\dim_{\mathbb{C}} \mathcal{L}^K/\mathbf{SL}(l,\mathbb{C}) = (2l-2)(lg-l-1) + 2lg = 2(l^2(g-1)+1).$$

Example 3.1. Let us parametrize an open set of the cotangent bundle of moduli space of vector bundles of rank 2 over a hyper-elliptic curve of genus 2 defined by

$$y^2 = \prod_{i=1}^{5}(x - c_i) \text{ where } c_i \in \mathbb{C}^1 \text{ and } c_i \text{ are distinct.}$$

Homogeneous coordinates are given by

$$\mathbf{A}_1 = [1,0], \mathbf{A}_2 = [0,1], \mathbf{A}_3 = [1,1] \text{ and } \mathbf{A}_4 = [1,a] \text{ where } a \neq 1.$$

From conditions (3.3), we have

$$\mathbf{B}_1 = (0, ab-b), \mathbf{B}_2 = (a^2b - ab, 0), \mathbf{B}_3 = (ab, -ab) \text{ and } \mathbf{B}_4 = (-ab, b).$$

$\{\mathbf{A}_j, \mathbf{B}_j\}_{j=1}^4$ is a 2-parameter family. Accordingly, $\left\{\mathbf{A}_j, \mathbf{B}_j, \gamma_j, \kappa_j\right\}_{j=1}^4$ is a 10-parameter family by taking 4 distinct points $\gamma_j = (x_j, y_j)$ and the associated eigenvalues κ_j in Definition 3.1 for $j = 1, \ldots, 4$. Note that we are assuming $(\alpha, \gamma) \in \mathcal{M}_0'$. The assumption can be retrieved by the second condition in Definition 3.1: We have 4 vector equations for $j = 1, \ldots, 4$

$$\kappa_j \mathbf{A}_j = \mathbf{A}_j \mathsf{L}_0 + \mathbf{A}_j \mathsf{L}_1 x_j + \sum_{1 \le k \ne j \le 4} \mathbf{A}_j \mathbf{B}_k^T \mathbf{A}_k \frac{y_j + y_k}{x_j - x_k}.$$

Consequently, we are given 8 equations with 8 unknowns. In general, for a hyperelliptic curve, we have $l^2 g$ equations with $l^2 g$ unknowns

$$\kappa_j \mathbf{A}_j = \sum_{i=0}^{g-1} \mathbf{A}_j \mathsf{L}_i x_j^i + \sum_{1 \le k \ne j \le lg} \mathbf{A}_j \mathbf{B}_k^T \mathbf{A}_k \frac{y_j + y_k}{x_j - x_k} \text{ for } j = 1, \ldots, lg.$$

The 4 vector equations become

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} = \mathsf{L}_0 + \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \mathsf{L}_1 + T_1$$

$$\begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}^{-1} \begin{pmatrix} \kappa_3 & \kappa_3 \\ \kappa_4 & a\kappa_4 \end{pmatrix} = \mathsf{L}_0 + \begin{pmatrix} \frac{ax_3 - x_4}{a-1} & \frac{ax_3 - ax_4}{a-1} \\ \frac{x_4 - x_3}{a-1} & \frac{ax_4 - x_3}{a-1} \end{pmatrix} \mathsf{L}_1 + T_2.$$

Here T_1 and T_2 are (2×2)-matrices depending on $\left\{\mathbf{A}_j, \mathbf{B}_j, \gamma_j, \kappa_j\right\}_{j=1}^4$. Thus, L_0 and L_1 are completely determined if and only if

$$\det \begin{pmatrix} x_1 - \frac{ax_3 - x_4}{a-1} & -\frac{ax_3 - ax_4}{a-1} \\ -\frac{x_4 - x_3}{a-1} & x_2 - \frac{ax_4 - x_3}{a-1} \end{pmatrix} \ne 0.$$

This is the condition for $\{\mathbf{A}_j, \gamma_j\}_{j=1}^4 \in \mathcal{M}_0'$.

Example 3.2. Let us consider an open set of the cotangent bundle of moduli space of vector bundles of rank 3 over a hyper-elliptic curve of genus 2. Homogeneous coordinates are given by

$$\mathbf{A}_1 = [1,0,0], \quad \mathbf{A}_2 = [0,1,0], \quad \mathbf{A}_3 = [0,0,1]$$

$$\mathbf{A}_4 = [1,1,1], \quad \mathbf{A}_5 = [1,a_1,a_2], \quad \mathbf{A}_6 = [1,a_3,a_4].$$

From conditions (3.3), we have

$$\mathbf{B}_1 = \big(0, -b_1 - b_3 - b_5, -b_2 - b_4 - b_6\big),$$

$$\mathbf{B}_2 = \big(b_1 + b_2 + a_1(a_1b_3 + a_2b_4) + a_3(a_3b_5 + a_4b_6), 0, -b_2 - a_1b_4 - a_3b_6\big),$$

$$\mathbf{B}_3 = \big(b_1 + b_2 + a_2(a_1b_3 + a_2b_4) + a_4(a_3b_5 + a_4b_6), -b_1 - a_2b_3 - a_4b_5, 0\big),$$

$$\mathbf{B}_4 = \big(-b_1 - b_2, b_1, b_2\big), \mathbf{B}_5 = \big(-a_1b_3 - a_2b_4, b_3, b_4\big), \mathbf{B}_6 = \big(-a_3b_5 - a_4b_6, b_5, b_6\big).$$

Note that we also have

$$\begin{aligned} b_1 + a_1b_3 + a_3b_5 &= 0 \\ b_2 + a_2b_4 + a_4b_6 &= 0. \end{aligned}$$

$\{\mathbf{A}_j, \mathbf{B}_j\}_{j=1}^{6}$ is an 8-parameter family. Accordingly, $\Big\{\mathbf{A}_j, \mathbf{B}_j, \gamma_j, \kappa_j\Big\}_{j=1}^{6}$ is a 20-parameter family by taking 6 distinct points $\gamma_j = (x_j, y_j)$ and the associated eigenvalues κ_j in Definition 3.1 for $j = 1, \ldots, 6$. Let us retrieve the condition for $(\alpha, \gamma) \in \mathcal{M}_0'$. By the second condition in Definition 3.1, we have 6 vector equations for $j = 1, \dots, 6$

$$\kappa_j \mathbf{A}_j = \mathbf{A}_j \mathsf{L}_0 + \mathbf{A}_j \mathsf{L}_1 x_j + \sum_{1 \le k \ne j \le 6} \mathbf{A}_j \mathbf{B}_k^T \mathbf{A}_k \frac{y_j + y_k}{x_j - x_k}.$$

Consequently, we are given 18 equations with 18 unknowns. The 6 vector equations become

$$\begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix} = \mathsf{L}_0 + \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \mathsf{L}_1 + T_1$$

$$\begin{pmatrix} \kappa_4 & \kappa_4 & \kappa_4 \\ \kappa_5 & a_1\kappa_5 & a_2\kappa_5 \\ \kappa_6 & a_3\kappa_6 & a_4\kappa_6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a_1 & a_2 \\ 1 & a_3 & a_4 \end{pmatrix} \mathsf{L}_0 + \begin{pmatrix} x_4 & x_4 & x_4 \\ x_5 & a_1x_5 & a_2x_5 \\ x_6 & a_3x_6 & a_4x_6 \end{pmatrix} \mathsf{L}_1 + T_2.$$

Here, T_1 and T_2 are (3×3)-matrices depending on $\left\{\mathbf{A}_j, \mathbf{B}_j, \gamma_j, \kappa_j\right\}_{j=1}^6$. Hence, L_0 and L_1 are completely determined by the non-degeneracy conditions

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & a_1 & a_2 \\ 1 & a_3 & a_4 \end{pmatrix} \ne 0 \text{ and}$$

$$\det\left(\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 1 & a_1 & a_2 \\ 1 & a_3 & a_4 \end{pmatrix}^{-1} \begin{pmatrix} x_4 & x_4 & x_4 \\ x_5 & a_1x_5 & a_2x_5 \\ x_6 & a_3x_6 & a_4x_6 \end{pmatrix} \right) \ne 0.$$

Example 3.3. Let us parametrize an open set of a cotangent bundle of moduli space of vector bundles of rank 2 over a $(4,3)$-curve[5] $\{R(y,x)=0\}$ of genus 3 defined by

$$R(y,x) = y^4+x^3+c_{3,0}y^3+c_{2,1}y^2x+c_{1,2}yx^2+c_{2,0}y^2+c_{1,1}yx+c_{0,2}x^2+c_{1,0}y+c_{0,1}x+c_{0,0}.$$

As in Example 3.1, we may have

$$\mathbf{A}_1 = [1,0], \mathbf{A}_2 = [0,1], \mathbf{A}_3 = [1,1],$$

$$\mathbf{A}_4 = [1,a_1], \mathbf{A}_5 = [1,a_2], \mathbf{A}_6 = [1,a_3].$$

From conditions (3.3), we have

$$\mathbf{B}_1 = (0, -\sum_{j=1}^{3} b_j - \sum_{j=1}^{3} a_j b_j), \quad \mathbf{B}_2 = (\sum_{j=1}^{3} a_j b_j + \sum_{j=1}^{3} a_j^2 b_j, 0),$$

$$\mathbf{B}_3 = (-\sum_{j=1}^{3} a_j b_j, \sum_{j=1}^{3} a_j b_j), \qquad \mathbf{B}_4 = (-a_1 b_1, b_1),$$

$$\mathbf{B}_5 = (-a_2 b_2, b_2), \qquad \mathbf{B}_6 = (-a_3 b_3, b_3).$$

So, $\{\mathbf{A}_j, \mathbf{B}_j\}_{j=1}^{6}$ is a 6-parameter family. Accordingly, $\left\{\mathbf{A}_j, \mathbf{B}_j, \gamma_j, \kappa_j\right\}_{j=1}^{6}$ is an 18-parameter family where $\gamma_j = (x_j, y_j) \in \mathfrak{R}$ and κ_j is the associated eigenvalues for $j = 1, \ldots, 4$. The condition $(\alpha, \gamma) \in \mathcal{M}_0'$ is given as follows: A meromorphic differential on $\mathfrak{R}$ with residues $\mathbf{B}_j^T \mathbf{A}_j$ at γ_j has the form

$$\mathsf{L}\frac{dx}{R_y} = \left(\mathsf{L}_0 + \mathsf{L}_1 x + \mathsf{L}_2 y + \sum_{k=1}^{6} \mathbf{B}_k^T \mathbf{A}_k \frac{y + R_y(x_k, y_k) - y_k}{x - x_k}\right)\frac{dx}{R_y} \text{ where } R_y = \frac{\partial R(x,y)}{\partial y}.$$

From the second condition in Definition 3.1, there are 6 vector equations for $j = 1, \ldots, 6$

$$\kappa_j \mathbf{A}_j = \mathbf{A}_j \mathsf{L}_0 + \mathbf{A}_j \mathsf{L}_1 x_j + \mathbf{A}_j \mathsf{L}_2 y_j + \sum_{1 \le k \ne j \le 6} \mathbf{A}_j \mathbf{B}_k^T \mathbf{A}_k \frac{y_j + R_y(x_k, y_k) - y_k}{x_j - x_k}.$$

Consequently, theses are 12 equations with 12 unknowns. The 6 vector equations become

[5]See Appendix A.2 for basic facts about an (n,m)-curve.

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} = \mathsf{L}_0 + \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \mathsf{L}_1 + \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \mathsf{L}_2 + T_1$$

$$\begin{pmatrix} \kappa_3 & \kappa_3 \\ \kappa_4 & a_1\kappa_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & a_1 \end{pmatrix} \mathsf{L}_0 + \begin{pmatrix} x_3 & x_3 \\ x_4 & a_1x_4 \end{pmatrix} \mathsf{L}_1 + \begin{pmatrix} y_3 & y_3 \\ y_4 & a_1y_4 \end{pmatrix} \mathsf{L}_2 + T_2$$

$$\begin{pmatrix} \kappa_5 & a_2\kappa_5 \\ \kappa_6 & a_3\kappa_6 \end{pmatrix} = \begin{pmatrix} 1 & a_2 \\ 1 & a_3 \end{pmatrix} \mathsf{L}_0 + \begin{pmatrix} x_5 & a_2x_3 \\ x_6 & a_3x_6 \end{pmatrix} \mathsf{L}_1 + \begin{pmatrix} y_5 & a_2y_5 \\ y_6 & a_3y_6 \end{pmatrix} \mathsf{L}_2 + T_3.$$

Here T_1, T_2, T_3 are (2×2)-matrices depending on $\left\{\mathbf{A}_j, \mathbf{B}_j, \gamma_j, \kappa_j\right\}_{j=1}^{6}$. Hence, $\mathsf{L}_0, \mathsf{L}_1, \mathsf{L}_2$ are completely determined if and only if

$$\det \begin{pmatrix} 1 & 0 & x_1 & 0 & y_1 & 0 \\ 0 & 1 & 0 & x_2 & 0 & y_2 \\ 1 & 1 & x_3 & x_3 & y_3 & y_3 \\ 1 & a_1 & x_4 & a_1x_4 & y_4 & a_1y_4 \\ 1 & a_2 & x_5 & a_2x_5 & y_5 & a_2y_5 \\ 1 & a_3 & x_6 & a_3x_6 & y_6 & a_3y_6 \end{pmatrix} \neq 0.$$

This condition is for $\{\mathbf{A}_j, \gamma_j\}_{j=1}^{6} \in \mathcal{M}_0'$.

We investigate a correspondence between Lax matrices and inverse algebraic spectral data[6]. This section explains how the Hitchin's abstract theory can be translated to the machinery expounded by Krichever. Throughout this section, we assume that $(\gamma, \alpha) \in \mathcal{M}_0$.

3.2 Krichever-Lax matrices and spectral curves

Let $\mathsf{L}(p; \gamma, \alpha)$ be an $(l \times l)$-Lax-matrix on $\mathfrak{R}$ associated with Tyurin parameters (γ, α) and a canonical divisor K where $\gamma = \gamma_1 + \cdots + \gamma_{lg}$. Take a characteristic polynomial

$$R(\mu, p) = \det\Big(\mu \cdot I_{l\times l} - \mathsf{L}(p; \gamma, \alpha)\Big) = 0.$$

The zero locus $\{R(\mu, p) = 0\}$ defines an algebraic curve. We denote the smooth model of this algebraic curve by $\widehat{\mathfrak{R}}$ and call it a *spectral curve* associated with a Lax matrix $\mathsf{L}(p; \gamma, \alpha)$. The coefficients $h_d(p; \mathsf{L})$ of

$$R(\mu, p) = \mu^l + \sum_{d=1}^{l} h_d(p; \mathsf{L})\mu^{l-d}$$

are a priori meromorphic functions on $\mathfrak{R}$ on the neighborhoods U_j of γ_j by definition. It is not hard to prove the next lemma[7]:

Lemma 3.2. *(*p.234 *in [44]) Let* L *have a simple pole at* γ_i *for* $i = 1, \ldots, lg$. *Then* L *satisfies the two constraints in Definition 3.1 if and only if in the neighborhood* U_j *of* γ_j *with a local coordinate* z, L *can be expressed as*

$$\mathsf{L}(z; \gamma, \alpha) = \Phi(z; \gamma, \alpha) \cdot \mathsf{L}\widetilde{(z; \gamma, \alpha)} \cdot \Phi^{-1}(z; \gamma, \alpha)$$

[6] Appendix A.1 provides further premises of this section.

[7] See *p.234* in [44] for the proof.

where $\Phi(z;\gamma,\alpha)$ *and* $\mathsf{L}\widetilde{(z;\gamma,\alpha)}$ *are holomorphic.*

From Lemma 3.2, the eigenvalues of L are holomorphic in U_j and all the $h_d(p;\mathsf{L})$ are holomorphic in U_j. Note that $h_d(p;\mathsf{L})$ has poles of order d at p_i where $K=\sum_{i=1}^{2g-2} p_i$. We denote the space of such meromorphic functions on $\mathfrak{R}$ by $\mathcal{H}^K$. This is just

$$\mathcal{H}^K \cong \bigoplus_{d=1}^{l} \mathrm{H}^0(\mathfrak{R}, \mathrm{K}^d).$$

From the Riemann-Roch theorem,

$$\begin{aligned}\dim_{\mathbb{C}} \mathcal{H}^K &= \dim_{\mathbb{C}} \bigoplus_{d=1}^{l} \mathrm{H}^0(\mathfrak{R}, \mathrm{K}^d)\\ &= g + \sum_{d=2}^{l} (d(2g-2)-(g-1))\\ &= l^2(g-1)+1.\end{aligned}$$

We have a map

$$\mathbf{H} : \mathcal{L}^K \to \mathcal{H}^K \text{ by } \mathbf{H}(\mathsf{L}) = \Big(h_1(p;\mathsf{L}),\ldots,h_l(p;\mathsf{L})\Big).$$

Since it is invariant under the conjugation action of $\mathbf{SL}(l,\mathbb{C})$, the map $\mathbf{H}$ can descend to the quotient space

$$\mathbf{H} : \mathcal{L}^K/\mathbf{SL}(l,\mathbb{C}) \to \mathcal{H}^K \text{ by } \mathbf{H}([\mathsf{L}]) = \Big(h_1(p;\mathsf{L}),\ldots,h_l(p;\mathsf{L})\Big). \tag{3.4}$$

This map is what Hitchin investigated in [32]. By the parameters of the images and fibers of $\mathbf{H}$, we can foliate the space $\mathcal{L}^K/\mathbf{SL}(l,\mathbb{C})$. We summarize the contents of *pp.241–243* in [44] as follows:

Theorem 3.1. *Let* $[\mathsf{L}] \in \mathcal{L}^K/\mathbf{SL}(l,\mathbb{C})$ *be an* $\mathbf{SL}(l,\mathbb{C})$*-orbit of* L *in* $\mathcal{L}^K$*. Then there is a one-to-one correspondence*

$$[\mathsf{L}] \longleftrightarrow \Big((h_1,\ldots,h_l),[\widehat{D}]\Big) = \Big(\widehat{\mathfrak{R}},[\widehat{D}]\Big).$$

$[\widehat{D}]$ *is an equivalence class of an effective divisor of degree* $\widehat{g}+l-1$ *on* $\widehat{\mathfrak{R}}$ *where* $\widehat{g}$ *is the genus of* $\widehat{\mathfrak{R}}$*.*

Theorem 3.1 implies that the fibers of $\mathbf{H}$ can be parametrized by points of $\mathrm{Jac}(\widehat{\mathfrak{R}})$: In order to describe it accurately, we need to pick l points $\widehat{q}_1,\ldots,\widehat{q}_{l-1},\widehat{q}_l$ of $\widehat{\mathfrak{R}}$. The $l-1$ points among l points characterize the translation of degree $\widehat{g}+l-1$ to a point in the space $\mathcal{S}^{\widehat{g}}\widehat{\mathfrak{R}}$ of $\widehat{g}$th symmetric power of $\widehat{\mathfrak{R}}$, and the remaining point plays role of a base point in the Abel map from $\mathcal{S}^{\widehat{g}}\widehat{\mathfrak{R}}$ to $\mathrm{Jac}(\widehat{\mathfrak{R}})$.

Proof. The ramification index ν of an l-sheeted covering $\pi:\widehat{\mathfrak{R}}\to\mathfrak{R}$ is equal to the degree of the divisor of zeros of $\partial_\mu R(\mu,p)$, which is the same as the degree of the divisor of poles of $\partial_\mu R(\mu,p)$ on $\widehat{\mathfrak{R}}$:

$$\nu = \deg(\partial_\mu R)_0 = \deg(\partial_\mu R)_\infty = (2g-2)(l-1)l.$$

From the Riemann-Hurwitz formula[8], the genus of $\widehat{\mathfrak{R}}$ is given by

$$\widehat{g} = l^2(g-1)+1.$$

Each point $\widehat{p}=(p,\mu)\in\widehat{\mathfrak{R}}$ represents an eigenvalue. Accordingly, a generic point $\widehat{p}=(p,\mu)\in\widehat{\mathfrak{R}}$ has a unique normalized eigenvector

$$\boldsymbol{\psi}(p,\mu)\mathsf{L}(p) = \mu(p)\boldsymbol{\psi}(p,\mu) \text{ where } \boldsymbol{\psi}=(\psi_1,\ldots,\psi_l) \text{ with } \sum_{i=1}^{l}\psi_i(\widehat{p})=1.$$

[8]See *p.140* in [20].

Let $\widehat{D}$ be a divisor of poles of $\boldsymbol{\psi}(\widehat{p})$. Note that $\widehat{p}'$ is a pole of $\boldsymbol{\psi}(\widehat{p})$ if and only if $\widehat{p}'$ is a pole of all the components $\psi_i(\widehat{p})$ for $i = 1, \ldots, l$ by definition. Suppose that all the ramification points have the ramification index $l-1$, i.e., every branching points are totally ramified. Let $F(p) = \det^l \Psi$ where

$$\Psi(p) = \Big(\boldsymbol{\psi}(\widehat{p}_1), \ldots, \boldsymbol{\psi}(\widehat{p}_l)\Big)_{l\times l} \text{ where } \widehat{p}_1, \ldots, \widehat{p}_l \in \pi^{-1}(p).$$

Of course, $\Psi(p)$ is well-defined up to permutation, but its determinant $\det\Psi$ is a well-defined meromorphic function on $\mathfrak{R}$. We may see that

$$l \cdot \deg \widehat{D} = \deg(F)_\infty = \deg(F)_0.$$

The zeros of F are at $\gamma_1 + \cdots + \gamma_{lg}$ and totally ramified $(2g-2)l$ points on $\mathfrak{R}$ counting multiplicities. The degree of zeros of F is equal to

$$\begin{aligned}
\deg(F)_0 &= l \cdot lg + l \cdot (\frac{1}{l} + \cdots + \frac{l-1}{l}) \cdot (2g-2)l \\
&= l \cdot lg + l \cdot \frac{l(l-1)}{2l} \cdot (2g-2)l \\
&= l^2 g + l^2(l-1) \cdot (g-1) \\
&= l \cdot \deg \widehat{D}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\deg \widehat{D} &= lg + l(l-1)(g-1) \\
&= l^2(g-1) + l \\
&= \widehat{g} + l - 1.
\end{aligned}$$

The conjugation action of $W \in \mathbf{SL}(l, \mathbb{C})$ on L is carried on $\boldsymbol{\psi}$ as

$$\boldsymbol{\psi} \mapsto \boldsymbol{\psi'} = \big(\sum_{i,j}^{l} w_{ji}\psi_i\big)^{-1} \boldsymbol{\psi} W \text{ where } W = \big(w_{ji}\big)_{l\times l}.$$

Note that $\sum_{i=1}^{l} \psi_i'(\widehat{p}) = 1$ generically. The divisor of poles of a meromorphic function $\sum_{i,j}^{l} w_{ji}\psi_i(\widehat{p})$ on $\widehat{\mathfrak{R}}$ is $\widehat{D}$. So, the divisor $\widehat{D}'$ of poles of $\boldsymbol{\psi'}$ is the divisor of zeros of $\sum_{i,j}^{l} w_{ji}\psi_i$. Consequently, $\widehat{D}$ is linearly equivalent to $\widehat{D}'$. Hence, $[\mathsf{L}] \in \mathcal{L}^K/\mathbf{SL}(l,\mathbb{C})$ defines algebraic spectral data $\Big((h_1,\ldots,h_l),[\widehat{D}]\Big) = \Big(\widehat{\mathfrak{R}},[\widehat{D}]\Big)$.

Suppose that we are given $\Big((h_1,\ldots,h_l),[\widehat{D}]\Big)$. The spectral curve $\widehat{\mathfrak{R}}$ is given by

$$0 = R(\mu,p) = \mu^l + \sum_{i=1}^{l} h_i(p)\mu^{l-i}.$$

For a generic point $q \in \mathfrak{R}$, we have l pre-images $\widehat{q}_1,\ldots,\widehat{q}_l$ on $\widehat{\mathfrak{R}}$. For a divisor $\widehat{D} \in [\widehat{D}]$, since $\deg \widehat{D} = \widehat{g} + l - 1$, the Riemann-Roch theorem implies that there are l linearly independent meromorphic functions $\psi_1(\widehat{p}),\ldots,\psi_l(\widehat{p})$ having poles at $\widehat{D}$ on $\widehat{\mathfrak{R}}$ with normalization $\psi_i(\widehat{q}_j) = \delta_i^j$. Let $\boldsymbol{\psi}(\widehat{p}) = (\psi_1(\widehat{p}),\ldots,\psi_l(\widehat{p}))$ and

$$\Psi(p) = \Big(\boldsymbol{\psi}(\widehat{p}_1),\ldots,\boldsymbol{\psi}(\widehat{p}_l)\Big)_{l\times l} \text{ where } \widehat{p}_1,\ldots,\widehat{p}_l \in \pi^{-1}(p).$$

$\Psi(p)$ is well-defined up to permutation. The ambiguity can be removed by the conjugation action of $\mathbf{SL}(l,\mathbb{C})$. A matrix-valued function $\Psi'(p)$ associated with a different base point $q' \in \mathfrak{R}$ is conjugate to

$$\Psi(p) = W \cdot \Psi'(p) \cdot W^{-1} \text{ where } W \in \mathbf{SL}(l,\mathbb{C}).$$

Let $\mu_1(p),\ldots,\mu_l(p)$ be the roots of $R(\mu,p)$ and $\mathsf{K}(p) = \operatorname{diag}\Big(\mu_1(p),\ldots\mu_l(p)\Big)$ be the associated $(l \times l)$-diagonal matrix. Define

$$\mathsf{L}(p) = \Psi(p) \cdot \mathsf{K}(p) \cdot \Psi^{-1}(p).$$

Since $h_d \in \mathrm{H}^0(\mathfrak{R},\mathrm{K}^d)$, $\Psi(p)$ and $\mathsf{K}(p)$ are holomorphic except possibly at a multiple of the canonical divisor K. So, Lemma 3.2 implies that L is a Lax matrix associated with K and (γ, α). Moreover, choosing another equivalent divisor $\widehat{D}' \in [\widehat{D}]$ is characterized by the multiplication of a meromorphic function $f(\widehat{p})$ to $\Psi(\widehat{p})$. Hence, an equivalent divisor gives the same Lax matrix, but a different normalization gives a Lax matrix conjugate to the original Lax matrix. Therefore, the algebraic data $\Big((h_1, \ldots, h_l), [\widehat{D}]\Big)$ define an $\mathbf{SL}(l, \mathbb{C})$-orbit $[\mathsf{L}]$ in the space of Lax matrices. For more information, see Appendix A.1. $\square$

Chapter 4

A cohomological interpretation of straight line flows and examples

The theory of straight line flows on the Jacobi variety of a spectral curve over a projective plane $\mathbb{P}^1$ is well-known around 1960s from the investigation of the K-dV equation. The classical notion of a spectral curve over $\mathbb{P}^1$ can be constructed as follows: Consider the following diagram[1]

$$\begin{array}{ccc} \lambda_\xi \in \mathcal{O}_{\mathbb{P}^1}(n) & \xrightarrow{\pi^*} & \pi^*\lambda \in \pi^*\mathcal{O}_{\mathbb{P}^1}(n) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ \xi \in \mathbb{P}^1 & \xleftarrow{\pi} & \lambda_\xi \in \mathcal{O}_{\mathbb{P}^1}(n). \end{array} \tag{4.1}$$

If $\mathsf{A}(\xi) = \sum_{i=1}^{n} A_i \cdot \xi^i \in \mathrm{H}^0(\mathbb{P}^1, \mathrm{End}\,\mathrm{E})$ and A_i are constant $(l \times l)-$matrices, then a spectral curve is the zero locus

$$\mathfrak{R} = \{\pi^* \det(\lambda_\xi \cdot I_{l\times l} - \mathsf{A}(\xi)) = 0\}.$$

Note that $\pi^* \det(\lambda_\xi \cdot I_{l\times l} - \mathsf{A}(\xi))$ is a section of a line bundle $(\pi^*\mathcal{O}_{\mathbb{P}^1}(n))^l$ over $\mathcal{O}_{\mathbb{P}^1}(n)$ where $\mathcal{O}_{\mathbb{P}^1}(n) = \bigotimes^n \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\mathbb{P}^1}(1)$ is the sheaf of a hyperplane bundle H over $\mathbb{P}^1$. In [27], P. Griffiths gave a cohomological criterion about the straightness of flows in a Lax representation:

$$\frac{d}{dt}\mathsf{A}(\xi, t) = [\mathsf{B}(\xi, t), \mathsf{A}(\xi, t)].$$

Here ξ is a rational parameter on $\mathbb{P}^1$. The dynamics of $\mathsf{A}(\xi, t)$ in $\mathfrak{gl}(l, \mathbb{C})[\xi, \xi^{-1}]$ is characterized in a cohomological class induced by the singularities of $\mathsf{B}(\xi, t)$. It is

[1] Compare it with diagram (2.2) in Subsection 2.2.2.

clear that if $\mathsf{B} \in \mathfrak{gl}(l,\mathbb{C})[\xi,\xi^{-1}]$, then $[\mathsf{B},\mathsf{A}]$ is tangent to $\mathfrak{gl}(l,\mathbb{C})[\xi,\xi^{-1}]$. In [27], the author describes a straight line flow in terms of a cohomology class in the sheaf cohomology group for a skyscraper sheaf $\mathbb{C}_D$ (see Subsection 4.1.1 for the definition and properties of a skyscraper sheaf): Let $\pi : \mathfrak{R} \to \mathbb{P}^1$ and $D = \pi^{-1}(0)+\pi^{-1}(\infty)$. In *p.1475* of [27], it was shown that the flows are straight in $\mathrm{Pic}^d(\mathfrak{R})$ where $d = \deg D$ if and only if

$$\frac{d}{dt}\rho(\mathsf{B}) \equiv 0 \text{ modulo } \operatorname{span}\{\mathrm{H}^0(\mathfrak{R}, L_D), \rho(\mathsf{B})\} \text{ where } \rho(\mathsf{B}) \in \mathrm{H}^0(\mathfrak{R}, \mathbb{C}_D).$$

Here L_D is the sheaf of the associated line bundle with the divisor D on $\mathfrak{R}$. For the definition of a residue section $\rho(\mathsf{B})$ associated with B, see Definition 4.2. Moreover, we may formulate Hamiltonians corresponding to the linear flows explicitly. Indeed, in *p.429* of [58] the corresponding Hamiltonians to the linear flows are explicitly identified as

$$H(\mathsf{A}) = \operatorname{res}_{\xi=0} \xi^{-m} h(\mathsf{A}).$$

Here $h(\mathsf{A})$ is an invariant polynomial on $\mathfrak{gl}(l,\mathbb{C})[\xi,\xi^{-1}]$ and it depends on B. Note that another characterization of the linear flows are also possible. In *p.1476* of [27], the author characterizes the linear flows associated with B_t as linear functions[2]. They are given by

$$F(\mathsf{B}_t,\omega) = t\sum_{p_i \in D} \operatorname{res}_{p_i}\left(\rho_i(\mathsf{B}_t)\omega\right) \text{ where } \omega \in \mathrm{H}^0(\mathfrak{R}, \mathrm{K}).$$

On the other hand, when we deal with the Hitchin system the framework becomes a little bit different. The principal cohomology groups are changed in

[2]Note that the Hamiltonian is a constant function on the linear flow.

order to fit the scheme of Hitchin system. Nevertheless, the similar cohomological interpretation of a straight line flow in the classical case can be still valid in the case of Hitchin system. It is possible because the fundamental underlying concepts in [27] are applicable to the case of Hitchin system, namely, an eigenvector mapping (see *p.1456* in [27]) and the fact that the sum of all residues of a meromorphic 1-from is zero on a compact Riemann surface. However, the machinery used in [27] can not be applied directly. A small modification is needed. This is what we will analyze in this section. As a byproduct, we will reprove several theorems in [44] in this setting.

A spectral curve $\widehat{\mathfrak{R}}$ appeared in what Hitchin investigated is constructed an l-sheeted covering space of a compact Riemann surface $\mathfrak{R}$ of genus at least greater than one. Unlike the spectral curves over a projective plane $\mathbb{P}^1$, a spectral curve $\widehat{\mathfrak{R}}$ over $\mathfrak{R}$ is defined as the zero locus in the canonical bundle $\mathrm{K}_{\mathfrak{R}}$ of a compact Riemann surface $\mathfrak{R}$. The divisor $D = \pi^{-1}(0) + \pi^{-1}(\infty)$ in [27] is replaced with a lifting divisor $\pi'^{-1}(nK)$ of a canonical divisor K on a compact Riemann surface $\mathfrak{R}$ where $\pi' : \widehat{\mathfrak{R}} \to \mathfrak{R}$ and n is a positive integer. The necessary and sufficient condition that $[\mathsf{M}, \mathsf{L}]$ is tangent to the space $\mathcal{L}^K$ of Lax matrices should be investigated. Moreover, we need to characterize the space where M should belong. The characterization of linear flows in the Hitchin system is given in Corollary 4.1 and an explicit formula of the Hitchin's Hamiltonians is expressed in Section 4.4. Moreover, explicit calculations of the Hitchin's Hamiltonians in terms of examples are also given in Section 4.5. Furthermore, these explicit calculations of Hamiltonians help us to characterize the conditions[3] on the Krichever-Tyurin parameters for classical groups described in

[3]See Example 4.2, 4.4 and 4.6.

[32].

Let us begin with this chapter by giving general definitions and preliminaries for reader's convenience. For more detailed theory of sheaf cohomology groups, we refer to [20, 26, 28].

4.1 Basic facts in sheaf theory

4.1.1 Skyscraper sheaf

Let p be a point of a compact Riemann surface $\mathfrak{R}$ and U be an open set of $\mathfrak{R}$. Define a sheaf on $\mathfrak{R}$ by

$$\mathbb{C}_p(U) := \begin{cases} \mathbb{C} & \text{if } p \in U \\ \\ 0 & \text{if } p \notin U \end{cases}$$

This is called a *skyscraper sheaf* $\mathbb{C}_p$ on $\mathfrak{R}$ associated with a point p. In particular, we have

$$\mathrm{H}^0(\mathfrak{R}, \mathbb{C}_p) \cong \mathbb{C}^1$$

$$\mathrm{H}^1(\mathfrak{R}, \mathbb{C}_p) \cong 0.$$

Let $D = \sum_{k=1}^{d} m_k p_k$ be a divisor of a compact Riemann surface $\mathfrak{R}$. A *skyscraper sheaf* $\mathbb{C}_D$ on $\mathfrak{R}$ associated with a divisor D is defined as

$$\mathbb{C}_D = \bigoplus_{k=1}^{d} \big(\underbrace{\mathbb{C}_{p_k} + \cdots + \mathbb{C}_{p_k}}_{m_k}\big).$$

For instance, we have

$$\mathrm{H}^0(\mathfrak{R}, \mathbb{C}_D) \cong \mathbb{C}^{\sum_{k=1}^{d} m_k}$$

$$\mathrm{H}^1(\mathfrak{R}, \mathbb{C}_D) \cong 0.$$

A meromorphic function $\lambda(p)$ on $\mathfrak{R}$ can define a global section of a skyscraper sheaf associated with the divisor D as follows: In the neighborhood of p_k, $\lambda(p)$ can have the tail of a Laurent expansion

$$\lambda(z_k) = \frac{c_{k,m_k}}{z_k^{m_k}} + \cdots + \frac{c_{k,1}}{z_k}.$$

Consequently, a vector $\big((c_{1,1}, \ldots, c_{1,m_1}), \ldots, (c_{d,1}, \ldots, c_{d,m_d})\big)$ can be regarded as an element of a vector space $\mathbb{C}^{\sum_{k=1}^{d} m_k} \cong \mathrm{H}^0(\mathfrak{R}, \mathbb{C}_D)$. We will call this element of $\mathrm{H}^0(\mathfrak{R}, \mathbb{C}_D)$ associated to $\lambda(p)$ the *Laurent tail* of $\lambda(p)$ at D.

4.1.2 Definition of a connecting homomorphism

Let $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ be sheaves on a Riemann surface $\mathfrak{R}$. A short exact sequence

$$0 \to \mathcal{F} \xrightarrow{i} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \to 0$$

induces a long exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(\mathfrak{R}, \mathcal{F}) &\xrightarrow{i_*^0} \mathrm{H}^0(\mathfrak{R}, \mathcal{G}) \xrightarrow{\pi_*^0} \mathrm{H}^0(\mathfrak{R}, \mathcal{H}) \xrightarrow{\delta_1} \\ \mathrm{H}^1(\mathfrak{R}, \mathcal{F}) &\xrightarrow{i_*^1} \mathrm{H}^1(\mathfrak{R}, \mathcal{G}) \xrightarrow{\pi_*^1} \mathrm{H}^1(\mathfrak{R}, \mathcal{H}) \xrightarrow{\delta_2} \\ \mathrm{H}^2(\mathfrak{R}, \mathcal{F}) &\xrightarrow{i_*^2} \mathrm{H}^2(\mathfrak{R}, \mathcal{G}) \xrightarrow{\pi_*^2} \cdots . \end{aligned}$$

Let $\{U_i\}$ be a Leray covering[4] of $\mathfrak{R}$. Suppose that $\{h_i\} \in \mathrm{H}^0(\mathfrak{R}, \mathcal{H})$. There exists $g_i^\alpha \in \mathcal{G}(U_i)$ such that

$$\pi(g_i^\alpha) = h_i.$$

[4] An open covering $\{U_i\}_{i \in I}$ of $\mathfrak{R}$ such that $\mathrm{H}^1(U_i, \mathcal{F}) = 0$ for every index $i \in I$ is called a Leray covering. Throughout this paper, by an open cover of a compact Riemann surface $\mathfrak{R}$ we mean a Leray cover without further indication.

$\{h_i\} \in \mathrm{H}^0(\mathfrak{R}, \mathcal{H})$ implies that $h_i - h_j \equiv 0$ on $U_i \cap U_j$. Since $\pi(g_i^\alpha - g_j^\alpha) \equiv 0$ on $U_i \cap U_j$, there exists $\{f_{ij}^\alpha\} \in C^1(\mathfrak{R}, \mathcal{F})$ such that

$$i(f_{ij}^\alpha) = g_i^\alpha - g_j^\alpha.$$

The injectivity of i implies that $f_{ij}^\alpha + f_{ji}^\alpha + f_{ki}^\alpha = 0$. Define

$$\delta_1(\{h_i\}) = \{f_{ij}^\alpha\} \in \mathcal{Z}^1(\mathfrak{R}, \mathcal{F}).$$

Now suppose that we choose a different $g_i^\beta \in \mathcal{G}(U_i)$ such that

$$\pi(g_i^\beta) = h_i.$$

Then we have $\{f_{ij}^\beta\} \in C^1(M, \mathcal{F})$ such that $i(f_{ij}^\beta) = g_i^\beta - g_j^\beta$. So, $\delta_1(\{h_i\}) = \{f_{ij}^\beta\} \in \mathcal{Z}^1(\mathfrak{R}, \mathcal{F})$. Consequently, δ_1 is not well-defined in $\mathcal{Z}^1(\mathfrak{R}, \mathcal{F})$. On the other hand, δ_1 is indeed well-defined in $\mathrm{H}^1(\mathfrak{R}, \mathcal{F})$:

$$\begin{aligned} i(f_{ij}^\alpha) - i(f_{ij}^\beta) &= g_i^\alpha - g_j^\alpha - (g_i^\beta - g_j^\beta) \\ &= (g_i^\alpha - g_i^\beta) - (g_j^\alpha - g_j^\beta). \end{aligned}$$

Since $\pi((g_i^\alpha - g_i^\beta)) = h_i - h_i \equiv 0$, we can find $\{f_i\} \in C^1(\mathfrak{R}, \mathcal{F})$ such that

$$i(f_i) = g_i^\alpha - g_i^\beta.$$

Combining this with the injectivity of i, we get

$$f_{ij}^\alpha - f_{ij}^\beta = f_i - f_j.$$

Hence,

$$\{f_{ij}^\alpha\} = \{f_{ij}^\beta\} \in \mathrm{H}^1(\mathfrak{R}, \mathcal{F}).$$

4.1.3 Euler sequence

Consider the *Euler sequence* over $\mathbb{P}^{l-1}$ (*p.409* in [26]):

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{l-1}} \longrightarrow \mathbb{C}^l \otimes \mathcal{O}_{\mathbb{P}^{l-1}}(1) \longrightarrow \Theta_{\mathbb{P}^{l-1}} \longrightarrow 0. \tag{4.2}$$

Here $\mathcal{O}_{\mathbb{P}^{l-1}}(1)$ is the sheaf of a hyperplane bundle H over $\mathbb{P}^{l-1}$ and $\Theta_{\mathbb{P}^{l-1}}$ is the sheaf of a *holomorphic* tangent bundle $\mathrm{T}\,\mathbb{P}^{l-1}$: The dual bundle H^* of a hyperplane bundle H over $\mathbb{P}^{l-1}$ is called a universal bundle. This universal bundle H^* is a canonical realization of a point $[\mathbf{v}]$ in $\mathbb{P}^{l-1}$ as a line in the product space $\mathbb{P}^{l-1} \times \mathbb{C}^l$ (*p.145* in [26]). The hyperplane bundle H can be regarded as the set of linear functionals on a line $[\mathbf{v}]$ in $\mathbb{C}^l$. This construction gives a short exact sequence of sheaves over $\mathbb{P}^{l-1}$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{l-1}}(1)^* \longrightarrow \mathbb{C}^l \longrightarrow \mathbb{C}^l/\mathcal{O}_{\mathbb{P}^{l-1}}(1)^* \longrightarrow 0: \tag{4.3}$$

Let U be an open set of $\mathbb{P}^{l-1}$ and $([\mathbf{w}], \mathbf{v}) \in \mathbb{C}^l(U)$ be a local section. A local section in $\mathcal{O}_{\mathbb{P}^{l-1}}(1)(U)$ is written as $([\mathbf{w}], \sigma|_{[\mathbf{w}]})$ where $\sigma|_{[\mathbf{w}]}$ is a linear functional restricted to the line $[\mathbf{w}]$ in $\mathbb{C}^l$. Define

$$\mathbf{v} : \mathcal{O}_{\mathbb{P}^{l-1}}(1)(U) \to \Theta_{\mathbb{P}^{l-1}}(U) \text{ by } \mathbf{v}(\sigma|_{[\mathbf{w}]}) = \big(\mathbf{v} - \sigma|_{[\mathbf{w}]}(\mathbf{v}) \cdot (\sigma|_{[\mathbf{w}]})^*\big).$$

From this, we may conclude that

$$(\mathbb{P}^{l-1} \times \mathbb{C}^l)/\,\mathrm{H}^* \cong \mathrm{Hom}(\mathrm{H}, \mathrm{T}\,\mathbb{P}^{l-1}).$$

Thus,

$$\big((\mathbb{P}^{l-1} \times \mathbb{C}^l)/\,\mathrm{H}^*\big) \otimes \mathrm{H} \cong \mathrm{Hom}(\mathrm{H}, \mathrm{T}\,\mathbb{P}^{l-1}) \otimes \mathrm{H} \cong \mathrm{T}\,\mathbb{P}^{l-1}.$$

Consequently, in a tangent space $\mathrm{T}_{[\mathbf{w}]}\,\mathbb{P}^{l-1}$, we have $\sigma|_{[\mathbf{w}]} \otimes (\sigma|_{[\mathbf{w}]})^* = 0$, i.e.,

$$\sum_{i=1}^{l} X_i \frac{\partial}{\partial X_i} = 0.$$

Here $X_1, \ldots, X_l$ are homogeneous coordinates of $\mathbb{P}^{l-1}$. After tensoring $\mathcal{O}_{\mathbb{P}^{l-1}}(1)$ to (4.3), the Euler sequence is obtained:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{l-1}} \longrightarrow \mathbb{C}^l \otimes \mathcal{O}_{\mathbb{P}^{l-1}}(1) \longrightarrow \Theta_{\mathbb{P}^{l-1}} \longrightarrow 0.$$

In other words, over a point $[\mathbf{w}] \in \mathbb{P}^{l-1}$ we have

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{O}_{\mathbb{P}^{l-1},[\mathbf{w}]} & \longrightarrow & \bigoplus^l \mathcal{O}_{\mathbb{P}^{l-1}}(1)_{[\mathbf{w}]} & \longrightarrow & \Theta_{\mathbb{P}^{l-1},[\mathbf{w}]} & \longrightarrow 0 \\ & c & \longmapsto & c \cdot (X_1|_{[\mathbf{w}]}, \ldots, X_l|_{[\mathbf{w}]}) & & & \\ & & & (\sigma_1|_{[\mathbf{w}]}, \ldots, \sigma_l|_{[\mathbf{w}]}) & \longmapsto & \sum_{i=1}^{l} \sigma_i|_{[\mathbf{w}]} \cdot \frac{\partial}{\partial X_i}. & \end{array} \tag{4.4}$$

4.2 Short exact sequences of vector bundles over an algebraic curve and eigenvector mappings on a spectral curve

Let $\widehat{\mathfrak{R}}$ be a spectral curve associated with a Lax matrix $\mathsf{L}(p;\gamma,\alpha)$:

$$\widehat{\mathfrak{R}} = \{\det\big(\mu \cdot I_{l\times l} - \mathsf{L}(p;\gamma,\alpha)\big) = 0\} \text{ where } p \in \mathfrak{R}.$$

Each point $(\mu, p) := \widehat{p} \in \widehat{\mathfrak{R}}$ is an eigenvalue of $\mathsf{L}(p;\gamma,\alpha)$. From the proof of Theorem 3.1, it is easy to see the following lemma.

Lemma 4.1. *For a Lax matrix $\mathsf{L}(p;\gamma,\alpha)$, there exists a unique eigenspace complex line bundle* L *of $\mathsf{L}(p;\gamma,\alpha)$ on $\widehat{\mathfrak{R}}$ which is a sub-bundle of a trivial bundle $\mathbb{C}^l$ on $\widehat{\mathfrak{R}}$.*

Definition 4.1. *We shall call* (4.5)

$$\overline{\boldsymbol{\Psi}}_t(\gamma(t),\alpha(t)) : \widehat{\mathfrak{R}} \to \mathbb{P}^{l-1} \tag{4.5}$$

an eigenvector mapping *associated to a Lax representation*

$$\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t]. \tag{4.6}$$

In other word, letting $\overline{\boldsymbol{\psi}}_t(\widehat{p}; \gamma(t), \alpha(t)) = \mathbb{C} \cdot \boldsymbol{\psi}_t(\widehat{p}; \gamma(t), \alpha(t))$, we have

$$\boldsymbol{\psi}_t(\widehat{p}; \gamma(t), \alpha(t))\mathsf{L}_t(p; \gamma(t), \alpha(t)) = \mu(\widehat{p}) \cdot \boldsymbol{\psi}_t(\widehat{p}; \gamma(t), \alpha(t)).$$

A vector-valued meromorphic function $\boldsymbol{\psi}_t(\widehat{p}; \gamma(t), \alpha(t))$ on $\widehat{\mathfrak{R}}$ defines a vector-valued (and multi-valued) meromorphic function $\pi_*\boldsymbol{\psi}_t$ on $\mathfrak{R}$ where $\pi : \widehat{\mathfrak{R}} \to \mathfrak{R}$ and the degree of each component of a vector $\boldsymbol{\psi}_t$ is $\widehat{g}+l-1$. Moreover, the multi-valued function $\pi_*\boldsymbol{\psi}_t$ has poles at $\gamma(t) = \gamma_1(t) + \cdots + \gamma_{lg}(t)$, and it is written as (3.1) associated with a Tyurin parameter $(\gamma(t), \alpha(t))$ (see also Equation (3.2)). The eigenvalue $\mu(\widehat{p})$ can be regarded as a *multi-valued* meromorphic function on $\mathfrak{R}$ with poles at the canonical divisor K of $\mathfrak{R}$. Let

$$\mathrm{L}_t = \overline{\boldsymbol{\psi}}_t^*\big(\mathcal{O}_{\mathbb{P}^{l-1}}(1)\big) \in \mathrm{Pic}^{\widehat{g}+l-1}(\widehat{\mathfrak{R}}).$$

Note that the degree of L_t is $\widehat{g}+l-1$ by Theorem 3.1 and L_t is a line bundle associated with an equivalence class $[\widehat{D}_t]$ of divisors in Theorem 3.1. Let $\mathcal{L}^K_{\widehat{\mathfrak{R}}}/\mathbf{SL}(l, \mathbb{C}) \subset \mathcal{L}^K/\mathbf{SL}(l, \mathbb{C})$ be the pre-images of the Hitchin map (3.4) associated to a spectral curve $\widehat{\mathfrak{R}}$. The eigenvector mapping $\overline{\boldsymbol{\psi}}_t$ induces

$$\varphi_{\widehat{\mathfrak{R}}} : \mathcal{L}^K_{\widehat{\mathfrak{R}}}/\mathbf{SL}(l, \mathbb{C}) \to \mathrm{Pic}^{\widehat{g}+l-1}(\widehat{\mathfrak{R}}) \text{ by } \varphi_{\widehat{\mathfrak{R}}}([\mathsf{L}_t]) = \overline{\boldsymbol{\psi}}_t^*\big(\mathcal{O}_{\mathbb{P}^{l-1}}(1)\big). \tag{4.7}$$

We will also call it an *eigenvector mapping* associated to a spectral curve $\widehat{\mathfrak{R}}$. Since the tangent space of $\mathrm{Pic}^{\widehat{g}+l-1}(\widehat{\mathfrak{R}})$ is isomorphic to $\mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}})$, we have

$$\frac{d}{dt}\,\mathrm{L}_t\,|_{t=0} \in \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}).$$

Pulling back the Euler sequence (4.2) on $\widehat{\mathfrak{R}}$ by $\overline{\boldsymbol{\psi}}_t(\widehat{p}; \gamma(t), \alpha(t))$ induces the following short exact sequence on $\widehat{\mathfrak{R}}$:

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathfrak{R}}} \longrightarrow \mathbb{C}^l \otimes \mathrm{L}_t \longrightarrow \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}} \longrightarrow 0. \tag{4.8}$$

From short exact sequence (4.8), we have a long exact sequence:

$$\cdots \longrightarrow \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}^l \otimes \mathrm{L}_t) \longrightarrow \mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}}) \xrightarrow{\ \delta\ } \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) \longrightarrow \cdots . \tag{4.9}$$

Let $\{g_{t,i}(\widehat{p})\}$ be the set of local trivializations of a line bundle L_t associated to an open cover $\{U_i\}$ of $\widehat{\mathfrak{R}}$ and denote a transition function $g_{t,ij}(\widehat{p})$ by the restriction of $g_{t,i}(\widehat{p})^{-1} \cdot g_{t,j}(\widehat{p})$ to $U_i \cap U_j$. Since $\overline{\boldsymbol{\psi}}_t(\widehat{p}; \gamma(t), \alpha(t)) = \mathbb{C} \cdot \boldsymbol{\psi}_t(\widehat{p}; \gamma(t), \alpha(t))$, any global section[5] $\mathbf{s}_t$ of $\mathbb{C}^l \otimes \mathrm{L}_t \cong \bigoplus^l \mathrm{L}_t$ can be given by $\{\rho_{t,i}^{-1}(\widehat{p}) \cdot \boldsymbol{\psi}_{t,i}\}$ where $\boldsymbol{\psi}_{t,i}$ is the restriction of $\boldsymbol{\psi}_t(\widehat{p}; \gamma(t), \alpha(t))$ to an open set U_i and

$$\rho_{t,i}(\widehat{p})^{-1} \cdot \rho_{t,j}(\widehat{p}) = g_{t,ij}(\widehat{p}) \text{ on } U_i \cap U_j.$$

Here $\{\rho_{t,i}(\widehat{p})\}$ is the set of local non-vanishing holomorphic functions.

Lemma 4.2. *A time-derivative* $\frac{d}{dt}\mathbf{s}_t$ *can be regarded as an element of* $\mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}})$.

Proof. Note that a line bundle L_t has a global section. Let $\mathbf{s}_t := \{\rho_{t,i}^{-1}(\widehat{p}) \cdot \boldsymbol{\psi}_{t,i}\}$ be

[5]Consequently, without loss of generality we may also regard $\boldsymbol{\psi}_t$ as a global section of $\mathbb{C}^l \otimes \mathrm{L}_t$.

a global section of $\mathbb{C}^l \otimes \mathrm{L}_t$. Accordingly,

$$\begin{aligned}\frac{d}{dt}\boldsymbol{\psi}_{t,i} &= \frac{d}{dt}(\rho_{t,i} \cdot \mathbf{s}_{t,i}) \\ &= (\frac{d}{dt}\rho_{t,i}) \cdot \mathbf{s}_{t,i} + \rho_{t,i} \cdot (\frac{d}{dt}\mathbf{s}_{t,i}).\end{aligned}$$

Here $\mathbf{s}_{t,i}$ is the restriction of $\mathbf{s}_t$ to U_i. Thus,

$$\rho_{t,i}^{-1} \cdot \frac{d}{dt}\boldsymbol{\psi}_{t,i} = \rho_{t,i}^{-1} \cdot (\frac{d}{dt}\rho_{t,i}) \cdot \mathbf{s}_{t,i} + (\frac{d}{dt}\mathbf{s}_{t,i}).$$

From Sequence (4.4), we see that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\widehat{\mathfrak{R}}} & \xrightarrow{\mathbf{s}_t} & \mathbb{C}^l \otimes \mathrm{L}_t & \longrightarrow & \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}} & \longrightarrow & 0. \\ & & \{\xi_i\} & \longrightarrow & \{\xi_i \cdot \mathbf{s}_{t,i}\} & & & & \end{array}$$

Consequently, $\{\rho_{t,i}^{-1} \cdot \frac{d}{dt}\boldsymbol{\psi}_{t,i}\} = [\frac{d}{dt}\mathbf{s}_t]$ defines an element of $\mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}^l \otimes \mathrm{L}_t / \mathcal{O}_{\widehat{\mathfrak{R}}})$. Since

$$\mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}^l \otimes \mathrm{L}_t / \mathcal{O}_{\widehat{\mathfrak{R}}}) \cong \mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}}),$$

we can see that $\{\rho_{t,i}^{-1} \cdot \frac{d}{dt}\boldsymbol{\psi}_{t,i}\}$ defines an element of $\mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}})$. □

The mapping $\varphi_{\widehat{\mathfrak{R}}} : \mathcal{L}_{\widehat{\mathfrak{R}}}^K/\mathbf{SL}(l, \mathbb{C}) \to \mathrm{Pic}^{\widehat{g}+l-1}(\widehat{\mathfrak{R}})$ induces a mapping between tangent spaces

$$\mathrm{T}\,\varphi_{\widehat{\mathfrak{R}}} : \mathrm{T}_{[\mathsf{L}]}\,\mathcal{L}_{\widehat{\mathfrak{R}}}^K/\mathbf{SL}(l, \mathbb{C}) \to \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) \text{ where } [\mathsf{L}] \in \mathcal{L}_{\widehat{\mathfrak{R}}}^K/\mathbf{SL}(l, \mathbb{C}).$$

In other word,

$$\mathrm{T}\,\varphi_{\widehat{\mathfrak{R}}}\Big(\frac{d}{dt}[\mathsf{L}_t(p; \gamma(t), \alpha(t))]|_{t=0}\Big) \in \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}).$$

We can observe the following result:

Theorem 4.1. *Let* $\mathbf{s}_t := \{\rho_{t,i}^{-1}(\widehat{p}) \cdot \boldsymbol{\psi}_{t,i}\}$ *be a global section of* $\mathbb{C}^l \otimes \mathrm{L}_t$ *and* $[\frac{d}{dt}\mathbf{s}_t]$ *in Lemma 4.2 be regarded as an element of* $\mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}})$. *Then we may have*

$$\mathrm{T}\,\varphi_{\widehat{\mathfrak{R}}}\Big(\frac{d}{dt}[\mathsf{L}_t]|_{t=0}\Big) = \delta([\frac{d}{dt}\mathbf{s}_t|_{t=0}]).$$

Moreover, it is independent of a section $\mathbf{s}_t$ *we chose. Hence, we may write*

$$\mathrm{T}\,\varphi_{\widehat{\mathfrak{R}}}\big(\frac{d}{dt}[\mathsf{L}_t]|_{t=0}\big) = \delta(\frac{d}{dt}\boldsymbol{\psi}_t|_{t=0}).$$

Proof. The infinitesimal change $\mathrm{T}\,\varphi_{\widehat{\mathfrak{R}}}\big(\frac{d}{dt}[\mathsf{L}_t]|_{t=0}\big)$ of line bundles is characterized by $\frac{d}{dt}\log g_{ij}(t)$ where $\{g_{ij}(t)\}$ is the set of transition functions of a line bundle L_t over $\widehat{\mathfrak{R}}$ associated with an open cover $\{U_i\}$. From Section 4.1.2, the connecting homomorphism δ of long exact sequence (4.9) is given by

$$\delta([\frac{d}{dt}\mathbf{s}_t]) = \{\rho_{t,j}^{-1}\cdot(\frac{d}{dt}\rho_{t,j}) - \rho_{t,i}^{-1}\cdot(\frac{d}{dt}\rho_{t,i})\} \in \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}).$$

Since $\rho_{t,i}^{-1}\cdot\rho_{t,j} = g_{ij}(t)$ on $U_i \cap U_j$, we have

$$\begin{aligned}\rho_{t,j}^{-1}\cdot(\frac{d}{dt}\rho_{t,j}) - \rho_{t,i}^{-1}\cdot(\frac{d}{dt}\rho_{t,i}) &= \frac{d}{dt}\log\rho_{t,j} - \frac{d}{dt}\log\rho_{t,i}\\ &= \frac{d}{dt}\log\rho_{t,i}^{-1}\rho_{t,j}\\ &= \frac{d}{dt}\log g_{ij}(t).\end{aligned}$$

So,

$$\mathrm{T}\,\varphi_{\widehat{\mathfrak{R}}}\big(\frac{d}{dt}\mathsf{L}_t|_{t=0}\big) = \delta([\frac{d}{dt}\mathbf{s}_t|_{t=0}]).$$

Also notice that the explicit expression

$$\delta([\frac{d}{dt}\mathbf{s}_t]) = \{\rho_{t,j}^{-1}\cdot(\frac{d}{dt}\rho_{t,j}) - \rho_{t,i}^{-1}\cdot(\frac{d}{dt}\rho_{t,i})\} \in \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}})$$

implies that it only depends on $\boldsymbol{\psi}_t$ and is independent of choosing $\mathbf{s}_t$. □

4.3 Cohomological interpretation of residues

The dynamics of a Lax representation $\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t]$ on $\mathcal{L}^K$ is invariant under the addition of a polynomial $\mathsf{P}(\mathsf{L}_t)$ of L_t or an element Q_t commuting with L_t, since

$$\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t] = [\mathsf{M}_t + \mathsf{P}(\mathsf{L}_t), \mathsf{L}_t] = [\mathsf{M}_t + \mathsf{Q}_t, \mathsf{L}_t]. \tag{4.10}$$

Thus, the dependence of flows on M might be indicated by an equivalence object associated with M. We will characterize it in terms of a cohomological class associated with M.

In fact, what we are interested in is a flow in the quotient space $\mathcal{L}^K/\mathbf{SL}(l, \mathbb{C})$. Note that for $W \in \mathbf{SL}(l, \mathbb{C})$, we have

$$\begin{aligned} \frac{d}{dt}(W^{-1}\mathsf{L}W) &= [\mathsf{M}, W^{-1}\mathsf{L}W] = \mathsf{M}(W^{-1}\mathsf{L}W) - (W^{-1}\mathsf{L}W)\mathsf{M} \\ &= W^{-1}(W\mathsf{M}W^{-1}\mathsf{L} - \mathsf{L}W\mathsf{M}W^{-1})W \\ &= W^{-1}[W\mathsf{M}W^{-1}, \mathsf{L}]W. \end{aligned} \tag{4.11}$$

Thus, if $\frac{d}{dt}\mathsf{L} = [\mathsf{M}, \mathsf{L}]$, then

$$\frac{d}{dt}\mathsf{L} = [W\mathsf{M}W^{-1}, \mathsf{L}].$$

So, the characteristic class of M should be invariant under the change of gauges. We will show the gauge-invariance of the associated cohomology class of M_t in Lemma 4.5.

First, we describe the condition on isospectral deformations, that is, the condition that the flow of a Lax representation stays in a leaf $\mathcal{L}^K_{\widehat{\mathfrak{R}}}$ in the foliation of the Hitchin map.

Lemma 4.3. *If the flow of a vector field $[\mathsf{M}_t, \mathsf{L}_t]$ is tangent to $\mathcal{L}^K$, then $[\mathsf{M}_t, \mathsf{L}_t]$ has poles only at the canonical divisor K of $\mathfrak{R}$ other than $\gamma(t) = \gamma_1(t) + \cdots + \gamma_{lg}(t)$.*

Proof. Suppose that a vector field $[\mathsf{M}_t, \mathsf{L}_t]$ on the space of matrix-valued meromorphic functions on $\mathfrak{R}$ is tangent to $\mathcal{L}^K$. Then the flow L_t stays in $\mathcal{L}^K$. So, we can write

$$\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t].$$

From Definition 3.1, it is easy to see that $\frac{d}{dt}\mathsf{L}_t$ has a double pole possibly at $\gamma_j(t)$ for $j = 1, \ldots, lg$ and a simple pole at p_i where $K = \sum_{i=1}^{2g-2} p_i$. Thus, we have the desired result. $\square$

Suppose that $[\mathsf{M}_t, \mathsf{L}_t]$ is tangent to $\mathcal{L}^K_{\widehat{\mathfrak{R}}}$. From eigenvector mapping (4.7), we have

$$\boldsymbol{\psi}_t(\widehat{p}; \gamma(t), \alpha(t))\mathsf{L}_t(p; \gamma(t), \alpha(t)) = \mu(\widehat{p}) \cdot \boldsymbol{\psi}_t(\widehat{p}; \gamma(t), \alpha(t)).$$

After differentiating $\boldsymbol{\psi}_t\mathsf{L}_t = \mu \cdot \boldsymbol{\psi}_t$ with respect to t, we have

$$(\frac{d}{dt}\boldsymbol{\psi}_t)\mathsf{L}_t + \boldsymbol{\psi}_t(\frac{d}{dt}\mathsf{L}_t) = \mu \cdot (\frac{d}{dt}\boldsymbol{\psi}_t).$$

Note that $\mu(\widehat{p})$ does not depend on t, i.e., it is isospectral. It only depends on $\widehat{p} \in \widehat{\mathfrak{R}}$. From $\boldsymbol{\psi}_t\mathsf{L}_t = \mu \cdot \boldsymbol{\psi}_t$ and $(\frac{d}{dt}\boldsymbol{\psi}_t)\mathsf{L}_t + \boldsymbol{\psi}_t[\mathsf{M}_t, \mathsf{L}_t] = \mu \cdot (\frac{d}{dt}\boldsymbol{\psi}_t)$, we have

$$\big(\boldsymbol{\psi}_t\mathsf{M}_t + (\frac{d}{dt}\boldsymbol{\psi}_t)\big)\mathsf{L}_t = \mu \cdot \big(\boldsymbol{\psi}_t\mathsf{M}_t + (\frac{d}{dt}\boldsymbol{\psi}_t)\big).$$

Since the eigenspace of $\mathsf{L}_t(p)$ associated with the eigenvalue μ is 1-dimensional generically, we find a meromorphic function $\lambda_t(\widehat{p}; \gamma(t), \alpha(t))$ such that

$$\boldsymbol{\psi}_t\mathsf{M}_t + (\frac{d}{dt}\boldsymbol{\psi}_t) = \lambda_t\boldsymbol{\psi}_t. \tag{4.12}$$

This meromorphic function λ_t certainly depends on M_t and $\boldsymbol{\psi}_t$. However, the Laurent tails of λ_t at poles only depend on M_t: For another $\boldsymbol{\psi}'_t = \varrho_t \cdot \boldsymbol{\psi}_t$ associated with a line bundle L_t where $\varrho_t(\widehat{p})$ is a local non-vanishing holomorphic function on $\widehat{\mathfrak{R}}$, λ_t is transformed to

$$\lambda_t + \varrho_t^{-1}\frac{d}{dt}\varrho_t.$$

Thus the Laurent tails are well-defined quantities associated to M_t only. Hence, the meromorphic functions λ_t in Equation (4.12) can be regarded as a global section of a skyscraper sheaf $\mathbb{C}_{\pi^{-1}(nK)}$ for some positive integer n where $\pi : \widehat{\mathfrak{R}} \to \mathfrak{R}$ and K is a canonical divisor of $\mathfrak{R}$. In this notation we make a definition:

Definition 4.2. *A* residue section $\rho(\mathsf{M}_t) \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}_{\pi^{-1}(nK)})$ *associated to* M_t *is defined to be the* Laurent tail $\{\lambda_{t,i}\}$ *of* λ_t *in Equation* (4.12) *at* $\pi^{-1}(nK)$ *where* $K = \sum_{i=1}^{2g-2} p_i$.

We may observe the following lemma[6]:

Lemma 4.4. *Suppose that* $\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t]$. *Then in the neighborhood of* γ_j, M *can be written, up to commuting elements with* L_t, *as*

$$\mathsf{M} = \frac{\mathsf{M}_{j,-1}}{z - z(\gamma_j)} + \mathsf{M}_{j,0} + O(z - z(\gamma_j)) \text{ for } j = 1, \ldots, lg. \tag{4.13}$$

In particular, the $(l \times l)$*-matrix* $\mathsf{M}_{j,-1}$ *is given by* $\mathbf{v}_j^T \cdot \boldsymbol{\alpha}_j$ *and*

$$\boldsymbol{\alpha}_j \mathsf{M}_{j,0} = \kappa_j \boldsymbol{\alpha}_j - \frac{d}{dt}\boldsymbol{\alpha}_j + \mathbf{w}_j \text{ where } \mathbf{v}_j, \mathbf{w}_j \in \mathbb{C}^l.$$

Proof. Let us remind Equation (4.10)

$$\frac{d}{dt}\mathsf{L} = [\mathsf{M}, \mathsf{L}] = [\mathsf{M} + \mathsf{P}(\mathsf{L}), \mathsf{L}] = [\mathsf{M} + \mathsf{Q}, \mathsf{L}] \text{ where } \mathsf{P}(x) \in \mathbb{C}[x].$$

[6] In [44], the condition for M is given in the beginning and it is the starting point of the paper. This corollary confirms the definition of [44]. See *p.233 and Lemma 2.3* in [44] for detail.

Accordingly, M can be specified up to commuting elements with L. So, it suffices to prove that $\mathsf{M}_{j,-1}$ is of rank 1. We know that $\pi_*\boldsymbol{\psi}_t$ is a multi-valued function on $\mathfrak{R}$ where $\pi : \widehat{\mathfrak{R}} \to \mathfrak{R}$. Taking a branch of it, from Equation (3.2) we may write $\pi_*\boldsymbol{\psi}_t$ in the neighborhood of γ_j for $j = 1, \dots, lg$ as

$$\pi_*\boldsymbol{\psi}_t(z) = \frac{c_j \boldsymbol{\alpha}_j(t)}{z - z(\gamma_j(t))} + \boldsymbol{\psi}_{j,0}(t) + O\big((z - z(\gamma_j))\big). \tag{4.14}$$

Here $c_j \in \mathbb{C}$ and $\boldsymbol{\alpha}_j(t) = (\alpha_{1,j}(t), \dots, \alpha_{l-1,j}(t), 1) \in \mathbb{C}^l$. Consequently,

$$\frac{d}{dt}\pi_*\boldsymbol{\psi}_t = \frac{c_j \cdot \boldsymbol{\alpha}_j(t) \cdot \frac{d}{dt} z(\gamma_j(t))}{\big(z - z(\gamma_j(t))\big)^2} + \frac{c_j \cdot \frac{d}{dt}\boldsymbol{\alpha}_j(t)}{z - z(\gamma_j(t))} + O(1). \tag{4.15}$$

Equation (4.12) implies that the possible poles which λ_t can have are at the poles of $\boldsymbol{\psi}_t$ and the pre-images $\pi^{-1}(nK)$ of the canonical divisor K, which are the poles of the global meromorphic function $\mu(\widehat{p})$. Consequently, after taking a branch of a multi-valued function $\pi_*\lambda_t$ on $\mathfrak{R}$ around γ_j, we have

$$\pi_*\lambda_t = \frac{\lambda_{j,-1}(t)}{z - z(\gamma_j(t))} + \lambda_{j,0}(t) + O\big((z - z(\gamma_j(t)))\big). \tag{4.16}$$

Thus, from Equation (4.15) and (4.12) we see that

$$\begin{gathered} c_j \boldsymbol{\alpha}_j(t) \mathsf{M}_{j,-1} + c_j \frac{d}{dt} z(\gamma_j(t)) \boldsymbol{\alpha}_j(t) = c_j \lambda_{j,-1}(t) \boldsymbol{\alpha}_j(t) \\ \boldsymbol{\psi}_{j,0}(t) \mathsf{M}_{j,-1} + c_j \boldsymbol{\alpha}_j(t) \mathsf{M}_{j,0} + c_j \frac{d}{dt}\boldsymbol{\alpha}_j(t) = c_j \lambda_{j,0}(t) \boldsymbol{\alpha}_j(t) + \lambda_{j,-1}(t) \boldsymbol{\psi}_{j,0}(t). \end{gathered} \tag{4.17}$$

From Equation (4.17), we conclude that there is a vector $\mathbf{v}_j(t) \in \mathbb{C}^l$ such that

$$\mathsf{M}_{j,-1}(t) = \mathbf{v}_j^T(t) \cdot \boldsymbol{\alpha}_j(t) \text{ where } \boldsymbol{\alpha}_j(t) \cdot \mathbf{v}_j^T(t) = -\frac{d}{dt} z(\gamma_j(t)) + \lambda_{j,-1}(t)$$

and letting $\kappa_j(t) = \lambda_{j,0}(t) - \frac{1}{c_j}\boldsymbol{\psi}_{j,0}(t) \cdot \mathbf{v}^T$,

$$\boldsymbol{\alpha}_j(t)\mathsf{M}_{j,0}(t) = \kappa_j(t)\boldsymbol{\alpha}_j(t) - \frac{d}{dt}\boldsymbol{\alpha}_j(t) + \lambda_{j,-1}(t)\boldsymbol{\psi}_{j,0}(t).$$

□

Generically, the poles of $\boldsymbol{\psi}_t$ are simple and $\deg \boldsymbol{\psi}_t = \widehat{g} + l - 1$. So in the neighborhood of a pole $\widehat{\gamma}_j(t)$ of $\boldsymbol{\psi}_t$ we may write $\boldsymbol{\psi}_t$ as

$$\boldsymbol{\psi}_t(\widehat{z}) = \frac{\mathbf{c}_j(t)}{\widehat{z} - \widehat{z}(\widehat{\gamma}_j(t))} + O(1) \text{ where } \mathbf{c}_j(t) \in \mathbb{C}^l.$$

Consequently,

$$\frac{d}{dt}\boldsymbol{\psi}_t = \frac{\mathbf{c}_j(t) \cdot \frac{d}{dt}\widehat{z}(\widehat{\gamma}_j(t))}{\left(\widehat{z} - \widehat{z}(\gamma_j(t))\right)^2} + \frac{\frac{d}{dt}\mathbf{c}_j(t)}{\widehat{z} - \widehat{z}(\gamma_j(t))} + O(1). \tag{4.18}$$

The next theorem indicates how the behavior of the poles of the global meromorphic function λ_t on $\widehat{\mathfrak{R}}$ governs the dynamics of Lax representation.

Theorem 4.2. *Suppose that* $\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t]$. *Then there is* $\lambda_t \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n_{\mathfrak{R}})$ *for some positive integer* n *such that* $\boldsymbol{\psi}_t(\widehat{p};\gamma,\alpha)\mathsf{M}_t - \lambda_t\boldsymbol{\psi}_t(\widehat{p};\gamma,\alpha)$ *defines a global section of* $\mathbb{C}^l \otimes \mathrm{L}_t$ *if and only if the flows are constant, i.e.,*

$$\frac{d}{dt}\mathsf{L}_t = 0.$$

Proof. Suppose that there is $\lambda_t \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n_{\mathfrak{R}})$ for some positive integer n such that $\boldsymbol{\psi}_t(\widehat{p};\gamma,\alpha)\mathsf{M}_t - \lambda_t\boldsymbol{\psi}_t(\widehat{p};\gamma,\alpha)$ defines a global section of $\mathbb{C}^l \otimes \mathrm{L}_t$. Accordingly, there is a global meromorphic function ξ_t on $\widehat{\mathfrak{R}}$ such that

$$\boldsymbol{\psi}_t\mathsf{M}_t - \lambda_t\boldsymbol{\psi}_t = \xi_t \cdot \boldsymbol{\psi}_t.$$

Of course, the only possible poles of ξ_t are at $\pi^{-1}(nK)$, since $\lambda_t \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n_{\mathfrak{R}})$. This implies that M_t preserves the eigenspaces of L_t. Thus, M_t and L_t commute. From $\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t]$, we conclude that

$$\frac{d}{dt}\mathsf{L}_t = 0.$$

Suppose that $\frac{d}{dt}\mathsf{L}_t = 0$. Since $[\mathsf{M}_t, \mathsf{L}_t] = 0$, the M_t preserves the eigenspaces of L_t. What this amounts is that there is a global meromorphic function $\varsigma_t(\widehat{p})$ on $\widehat{\mathfrak{R}}$ such that

$$\boldsymbol{\psi}_t \mathsf{M}_t = \varsigma_t \boldsymbol{\psi}_t.$$

Notice that ς_t only has poles possibly at $\pi^{-1}(nK)$ where n is a positive integer and K is the canonical divisor of $\mathfrak{R}$, since M preserves the eigenspace of L and $\boldsymbol{\psi}_t \mathsf{L}_t = \mu \cdot \boldsymbol{\psi}_t$ where $\mu(\widehat{p})$ takes poles only at $\pi^{-1}(K)$. Thus,

$$\boldsymbol{\psi}_t(\widehat{p}; \gamma, \alpha)\mathsf{M}_t - \lambda_t \boldsymbol{\psi}_t(\widehat{p}; \gamma, \alpha) = (\varsigma_t - \lambda_t)\boldsymbol{\psi}_t$$

defines a global section of $\mathbb{C}^l \otimes \mathrm{L}_t$. Moreover, since $\frac{d}{dt}\mathsf{L}_t = 0$ implies that $\frac{d}{dt}\widehat{z}(\widehat{\gamma}_j(t)) = 0$ for $j = 1, \ldots, \widehat{g} + l - 1$, we see that $\frac{d}{dt}\boldsymbol{\psi}_t$ has only first order poles at γ_j from Equations (4.18). Since

$$(\varsigma_t - \lambda_t)\boldsymbol{\psi}_t = -\frac{d}{dt}\boldsymbol{\psi}_t,$$

we conclude that $\lambda_t \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n_{\mathfrak{R}})$. □

Theorem 4.2 exhibits how the dynamics on $\mathcal{L}^K_{\widehat{\mathfrak{R}}}$ of a Lax representation $\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t]$ is related with M_t in terms of the residue section $\rho(\mathsf{M}_t) = \{\lambda_{t,i}\} \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}_{\pi^{-1}(nK)})$. When the flow is constant, i.e., it is fixed, then $\rho(\mathsf{M}_t) = \{\lambda_{t,i}\}$ defines a global section in $\mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n_{\mathfrak{R}})$. In other word, if L_t and M_t commutes, then M_t defines an endomorphism of $\mathrm{E}_{\gamma(t),\alpha(t)}$. In Corollary 4.1, we will give a necessary and sufficient condition for $\{\lambda_{t,i}\}$ when the flow of L_t is linear, which is the second simplest case next to the constant flows.

It is not hard to see that a residue section $\rho(\mathsf{M}_t)$ of M_t is gauge-invariant:

Lemma 4.5. *For $W \in \mathbf{SL}(l, \mathbb{C})$, we have*

$$\rho(\mathsf{M}_t) = \rho(W^{-1} \cdot \mathsf{M}_t \cdot W).$$

Proof. Let $\boldsymbol{\psi}_t \mathsf{L}_t = \mu \cdot \boldsymbol{\psi}_t$ and $\boldsymbol{\psi}_t \mathsf{M}_t + \frac{d}{dt}\boldsymbol{\psi}_t = \lambda_t \cdot \boldsymbol{\psi}_t$. Since

$$\begin{aligned}(W\boldsymbol{\psi}_t W^{-1})\mathsf{L}_t &= W \cdot \boldsymbol{\psi}_t (W^{-1}\mathsf{L}_t W) \cdot W^{-1} \\ &= W \cdot \mu\boldsymbol{\psi}_t \cdot W^{-1} \\ &= \mu \cdot (W\boldsymbol{\psi}_t W^{-1}),\end{aligned}$$

we have

$$(W\boldsymbol{\psi}_t W^{-1})\mathsf{M}_t + \frac{d}{dt}(W\boldsymbol{\psi}_t W^{-1}) = \lambda_t \cdot W\boldsymbol{\psi}_t W^{-1}.$$

Accordingly,

$$\begin{aligned}\boldsymbol{\psi}_t(W^{-1}\mathsf{M}_t W) + \frac{d}{dt}\boldsymbol{\psi}_t &= W^{-1} \cdot \big((W\boldsymbol{\psi}_t W^{-1})\mathsf{M}_t + \frac{d}{dt}(W\boldsymbol{\psi}_t W^{-1})\big)W \\ &= W^{-1} \cdot (\lambda_t \cdot W\boldsymbol{\psi}_t W^{-1}) \cdot W \\ &= \lambda_t \boldsymbol{\psi}_t.\end{aligned}$$

□

For a positive integer n, consider a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathfrak{R}}} \longrightarrow \mathcal{O}_{\widehat{\mathfrak{R}}} \otimes \pi^* \mathrm{K}^n \stackrel{\jmath}{\longrightarrow} \mathbb{C}_{\pi^{-1}(nK)} \longrightarrow 0. \tag{4.19}$$

This induces a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) \longrightarrow \mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n) \xrightarrow{\jmath} \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}_{\pi^{-1}(nK)}) \\ \xrightarrow{\partial} \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) \longrightarrow \mathrm{H}^1(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n) \longrightarrow \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathbb{C}_{\pi^{-1}(nK)}). \end{aligned} \tag{4.20}$$

Note that

$$\dim_{\mathbb{C}} \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) = 1, \quad \dim_{\mathbb{C}} \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}_{\pi^{-1}(nK)}) = ln(2g-2)$$

$$\dim_{\mathbb{C}} \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) = \widehat{g}, \quad \dim_{\mathbb{C}} \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathbb{C}_{\pi^{-1}(nK)}) = 0.$$

The time dependence of the residue section $\rho(\mathsf{M}_t) = \{\lambda_{t,i}\}$ associated to M_t can be characterized by the following theorem:

Theorem 4.3. *If* $[\mathsf{M}_t, \mathsf{L}_t]$ *is tangent to* $\mathcal{L}^K_{\widehat{\mathfrak{R}}}$*, then*

$$\frac{d}{dt}\,\mathrm{L}_t = \partial\rho(\mathsf{M}_t) \in \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) \cong \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathrm{K}_{\widehat{\mathfrak{R}}}).$$

Proof. In the notation of the proof of Lemma 4.2, we let

$$\varpi_1 = \{\rho_{t,i}^{-1} \cdot \frac{d}{dt}\boldsymbol{\psi}_t\} \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}}).$$

Similarly, we may let

$$\varpi_2 = \{\rho_{t,i}^{-1} \cdot \lambda_{t,i} \cdot \boldsymbol{\psi}_t\} = \{\rho_{t,i}^{-1} \cdot (\boldsymbol{\psi}_t \mathsf{M}_t + \frac{d}{dt}\boldsymbol{\psi}_t)\} \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}^l \otimes \mathrm{L}_t \otimes \pi^* \mathrm{K}^n).$$

Since $\mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}^l \otimes \mathrm{L}_t \otimes \pi^* \mathrm{K}^n / \mathcal{O}_{\widehat{\mathfrak{R}}}) \cong \mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}} \otimes \pi^* \mathrm{K}^n)$, ϖ_2 may induce an element in $\mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}} \otimes \pi^* \mathrm{K}^n)$. Let us denote this element by $\tau(\varpi_2)$. Now we let

$$\varpi_3 = \{\lambda_{t,i}\} \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}_{\pi^{-1}(nK)}).$$

Since $\rho_{t,i}$ is a non-vanishing local holomorphic function, from Section 4.1.2 and the short exact sequence (4.19)

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathfrak{R}}} \longrightarrow \mathcal{O}_{\widehat{\mathfrak{R}}} \otimes \pi^* \mathrm{K}^n \xrightarrow{\jmath} \mathbb{C}_{\pi^{-1}(nK)} \longrightarrow 0,$$
$$\rho_{t,i}^{-1} \cdot \lambda_{t,i} \xrightarrow{\quad\jmath\quad} \lambda_{t,i}$$

it is clear that

$$\partial\{\lambda_{t,i}\} = \{\frac{d}{dt} \log g_{t,ij}\}.$$

From Theorem 4.1, we also have

$$\delta\{\rho_{t,i}^{-1} \cdot \frac{d}{dt}\boldsymbol{\psi}_t\} = \{\rho_{t,j}^{-1} \cdot \frac{d}{dt}\rho_{t,j} - \rho_{t,i}^{-1} \cdot \frac{d}{dt}\rho_{t,i}\} = \{\frac{d}{dt} \log g_{t,ij}\}.$$

Hence,

$$\frac{d}{dt}\,\mathrm{L}_t = \partial\rho(\mathsf{M}_t).$$

Cohomologically, this is just chasing the following diagram:

$$\begin{array}{ccccccc}
\longrightarrow & \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) & \longrightarrow & \mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n) & \xrightarrow{\jmath} & \varpi_3 \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}_{\pi^{-1}(nK)}) & \xrightarrow{\partial} \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle\sigma} & \\
& \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}^l \otimes \mathrm{L}_t) & \longrightarrow & \varpi_2 \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}^l \otimes \mathrm{L}_t \otimes \pi^* \mathrm{K}^n) & \xrightarrow{\jmath} & \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}^l \otimes \mathrm{L}_t \otimes \mathbb{C}_{\pi^{-1}(nK)}) & \\
& \downarrow & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & \\
& \varpi_1 \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}}) & \xrightarrow{\imath} & \mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}} \otimes \pi^* \mathrm{K}^n) & \xrightarrow{\jmath} & \mathrm{H}^0(\widehat{\mathfrak{R}}, \overline{\boldsymbol{\psi}}_t^* \Theta_{\mathbb{P}^{l-1}} \otimes \mathbb{C}_{\pi^{-1}(nK)}) & \\
& \downarrow{\scriptstyle\delta} & & & & & \\
\xrightarrow{\partial} & \mathrm{H}^1(\widehat{\mathfrak{R}}, \mathcal{O}_{\widehat{\mathfrak{R}}}) & & & & &
\end{array}$$

Since $\tau \circ \jmath = \jmath \circ \tau$ and $\imath(\varpi_1) = \tau(\varpi_2)$, we have

$$\tau \circ \jmath(\varpi_2) = \jmath \circ \tau(\varpi_2) = \jmath \circ \imath(\varpi_1) = 0.$$

Hence, there is $\varpi_3 \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \mathbb{C}_{\pi^{-1}(nK)})$ such that

$$\sigma(\varpi_3) = \jmath(\varpi_2).$$

From the chasing the diagram, we see

$$\partial(\varpi_3) = \delta(\varpi_1).$$

□

Note that Theorem 4.3 confirms Theorem 4.2 again and this cohomological proof of Theorem 4.3 again shows that the gauge-invariance of the residue section, which was verified in Lemma 4.5. Moreover, we can deduce from Theorem 4.3 that the flow on the quotient $\mathcal{L}^K_{\widehat{\mathfrak{R}}}$ of a Lax representation is described by the Laurent tails of λ_t at $\pi^{-1}(nK)$ of $\widehat{\mathfrak{R}}$. A corollary we can have from Theorem 4.3 is as follows:

Corollary 4.1. L_t *is linear on* $\mathrm{Pic}^{\widehat{g}+l-1}(\widehat{\mathfrak{R}})$ *if and only if*

$$\frac{d}{dt}\rho(\mathsf{M}_t) \equiv 0 \text{ modulo } \operatorname{span}\{\jmath\big(\mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n)\big), \rho(\mathsf{M}_t)\}.$$

Proof. Clearly, we can observe that the flow L_t is straight if

1 $\frac{d^2}{dt^2}\mathrm{L}_t = 0$ or

2 $\frac{d^2}{dt^2}\mathrm{L}_t = c \cdot \frac{d}{dt}\mathrm{L}_t$ where $c \neq 0$.

By Theorem 4.3, $\frac{d^2}{dt^2}\mathrm{L}_t = 0$ if and only if $\frac{d}{dt}\rho(\mathsf{M}_t) \equiv 0$ modulo $\jmath\big(\mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n)\big)$. And $\frac{d^2}{dt^2}\mathrm{L}_t = c \cdot \frac{d}{dt}\mathrm{L}_t$ if and only if $\frac{d}{dt}\rho(\mathsf{M}_t) \equiv 0$ modulo $\rho(\mathsf{M}_t)$. This proves the claim. □

4.4 A characterization of flows in terms of M

The dynamics of a Lax representation $\frac{d}{dt}\mathsf{L}_t = [\mathsf{M}_t, \mathsf{L}_t]$ is completely described by M_t up to addition of a polynomial $\mathsf{P}(\mathsf{L}_t)$ or commuting element Q_t with L_t. As in the case of meromorphic functions on a compact Riemann surface, a matrix-valued meromorphic function on a compact Riemann surface is determined by the behavior of its poles. Consequently, the characterization of poles of M_t determines the dynamics of the Lax representation. From Lemma 4.3, we may see that at the poles of M_t other than lg points γ_j, the poles of $[\mathsf{M}_t, \mathsf{L}_t]$ are no greater than the poles of L_t. This is one restriction for defining tangent flows and it turns out to be the only one.

The existence of a meromorphic (matrix-valued) function on a compact Riemann surface is manifested by the Riemann-Roch theorem. Accordingly, we may not have M for generally prescribed poles D. What this means is that we need special ansatz to have the existence of M. In [44], Krichever defines special ansatz which guarantees the existence of M. That is, M exits if M has a special form in Equation (4.13) at lg points: Let us denote the space of all M having representations in Equation (4.13) at lg points by $\mathcal{N}^D$. For a given $(\gamma, \alpha) \in \mathcal{M}_0'$, we have

$$\dim_{\mathbb{C}} \mathcal{N}^D_{\gamma,\alpha} = l^2(d + lg - g + 1) - lg(l^2 - l) = l^2(d+1) \text{ where } d = \deg D.$$

From this we conclude that we do have M for any prescribe poles D as long as it obeys Equation (4.13). Thus any flow from the Lax representation comes from M satisfying Lemma 4.3 and Equation (4.13).

The description of straight line flows in terms of M_t will be given as follows:

Let $K = \sum_{i=1}^{2g-2} p_i$ be a canonical divisor of $\mathfrak{R}$ where all p_i are distinct. Consider M_t satisfying Equation (4.13) around γ_j for $j = 1, \ldots, lg$ and locally given by

$$\mathsf{M}_t(w_i) = w_i^{-m_i} \mathsf{L}_t^{n_i} \text{ around } p_i.$$

Here w_i is a local coordinate around p_i. From Lemma 4.3, we see that $[\mathsf{M}_t, \mathsf{L}_t]$ is tangent to $\mathcal{L}_{\widehat{\mathfrak{R}}}^K$. By Equation (4.14), we may see that

$$\boldsymbol{\psi}_t \mathsf{M}_t = \zeta_{t,i}(\widehat{p}) \cdot \boldsymbol{\psi}_t \text{ locally.}$$

Note that the set $\{\zeta_{t,i}\}$ of local meromorphic functions has poles only at the pre-images $\pi^{-1}(K)$ of the canonical divisor K on $\mathfrak{R}$ and they are invariant under time shift, since $\zeta_{t,i}(\widehat{p}) = \widehat{w}_i(\widehat{p})^{m_i} \mu(\widehat{p})^{n_i}$ in the neighborhoods of $\pi^{-1}(p_i)$ where $\pi : \widehat{\mathfrak{R}} \to \mathfrak{R}$ and $\widehat{w}_i$ is the lifting of w_i. From Equation (4.12), we have

$$(\lambda_t - \zeta_{t,i}) \cdot \boldsymbol{\psi}_t = \frac{d}{dt} \boldsymbol{\psi}_t \text{ around } p_i.$$

What this says is that the poles of λ_t at $\pi^{-1}(p_i)$ are also isospectral, since $\frac{d}{dt}\boldsymbol{\psi}_t$ does not have poles at $\pi^{-1}(p_i)$. Consequently, Theorem 4.3 confirms the linearity of this flow, since

$$\frac{d}{dt} \mathsf{L}_t = \partial \rho(\mathsf{M}_t) = \text{ constant.}$$

We may see that adding an element in $\mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n)$ to $\rho(\mathsf{M}_t)$ is equivalent to adding an element commuting with L to M in the Lax representation $\frac{d}{dt}\mathsf{L} = [\mathsf{M}, \mathsf{L}]$: Consider a time-dependent matrix $\mathsf{Q}_t(p)$ such that $[\mathsf{Q}_t, \mathsf{L}_t] = 0$ where $p \in \mathfrak{R}$. Since Q_t and L_t commute with each other, Q_t preserves the eigenspaces of L_t. Accordingly, there is a global meromorphic function $\vartheta_t(\widehat{p})$ on $\widehat{\mathfrak{R}}$ such that

$$\boldsymbol{\psi}_t \mathsf{Q}_t = \vartheta_t(\widehat{p}) \cdot \boldsymbol{\psi}_t. \tag{4.21}$$

Moreover, since $\boldsymbol{\psi}_t \mathsf{L}_t = \mu(\widehat{p}) \cdot \boldsymbol{\psi}_t$ and the poles of $\mu(\widehat{p})$ are at $\pi^{-1}(nK)$, we see that the poles of ϑ_t are only at $\pi^{-1}(nK)$. Thus we conclude that

$$\vartheta_t \in \mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n).$$

Note that ϑ_t is not necessarily isospectral unless Q_t is of form $\mathsf{P}(\mathsf{L}_t)$ where P is a polynomial. Combining Equation (4.12) with Equation (4.21), we have

$$\boldsymbol{\psi}_t(\mathsf{M}_t + \mathsf{Q}_t) + \frac{d}{dt}\boldsymbol{\psi}_t = (\lambda_t + \vartheta_t) \cdot \boldsymbol{\psi}_t.$$

Consequently, we see that

$$\rho(\mathsf{M}_t + \mathsf{Q}_t) \equiv \rho(\mathsf{M}_t) \text{ modulo } \mathrm{H}^0(\widehat{\mathfrak{R}}, \pi^* \mathrm{K}^n).$$

After normalizing by $\mathsf{M}(p_0) = 0$, we denote this straight line flow by

$$\mathsf{a} = (p_i, n_i, m_i).$$

Note that m_i can be a negative integer. The underlying machinery of this observation is that the sum of residues is zero. More precisely, what this implies is that the behavior of $\widehat{g} + l - 1$ poles is translated into the behavior of the lifting divisor in $\pi^{-1}(nK)$. The linearity of the dynamics of $\widehat{g} + l - 1$ poles is portrayed by the linearity of the dynamics of the lifting divisor in $\pi^{-1}(nK)$.

It is not hard to see that theses flows commute with each other (*Theorem 2.1* in [44]). Moreover, by constructing a symplectic structure on $\mathcal{L}^K/\mathbf{SL}(l, \mathbb{C})$, Krichever calculates Hamiltonians. The Hamiltonian of the flow associated with $\mathsf{a} = (p_i, n_i, m_i)$ is given by

$$H_{\mathsf{a}}(\mathsf{L}) = -\frac{1}{n_i} \operatorname{res}_{p_i} \operatorname{Tr}(w^{-m_i} \mathsf{L}^{n_i}) dz \text{ for } \mathsf{a} = (p_i, n_i, m_i) \text{ where}$$

w_i is a local coordinate around p_i. See *p.248* in [44] for more detailed investigation.

We will give examples of Hitchin's Hamiltonians in Section 4.5.

4.5 Hamiltonians in terms of Krichever-Lax matrices

In the Hitchin's investigation [32], the dynamics of Hamiltonians on the cotangent bundle of the moduli space of stable vector bundles on a compact Riemann surface is characterized by straight line flows. Indeed, it is a basic distinction between algebraically completely integrable systems and completely integrable systems. The essence of this characterization in [32] comes from the existence of a larger symplectic manifold containing the cotangent bundle where each fiber, an open set of the Jacobi variety of a spectral curve, is naturally compactified. The extension of Hamiltonian vector fields to the larger symplectic manifold is equivalent to the straightness of the associated Hamiltonian flows, since each fiber is a complex torus. See *p.101* in [32] or Section 2.2.2.

In the space of Lax matrices, we have not defined a symplectic structure nor a Poisson structure. Because of this reason, we do not have any Hamiltonian dynamics yet. The starting point of [44] is to define the dynamics of system on the space of Lax matrices in terms of what is called a Lax equation:

$$\partial_{t_{\mathsf{a}}} \mathsf{L} = [\mathsf{M}_{\mathsf{a}}, \mathsf{L}]. \tag{4.22}$$

Note that M_{a} is a function of L. The matrix M_{a} characterizes the dynamics of flows in the space of Lax matrices. Krichever gives the condition of M_{a} when the flows of the Lax equation become straight (*Theorem 2.1, Theorem 2.2* in [44]).

Moreover, he constructs a symplectic structure on the space of Lax matrices and shows that the straight line flows coming from the Lax equation indeed are Hamiltonian flows. That is, they define Hamiltonians associated with the symplectic structure.

Theorem 4.4. *(p.248 in [44]) Let $K = \sum_{i=1}^{2g-2} p_i$ and $\partial_{t_{\mathsf{a}}}$ be the linear vector fields corresponding to the Lax equation (4.22). Then the Hamiltonian H_{a} corresponding to $\partial_{t_{\mathsf{a}}}$ is given by*

$$H_{\mathsf{a}}(\mathsf{L}) = -\frac{1}{n+1}\operatorname{res}_{p_i}\operatorname{Tr}(w^{-m}\mathsf{L}^{n+1}dz) \text{ for } \mathsf{a} = (p_i, n, m) \text{ where}$$

w is a local coordinate in the neighborhood of $p_i \in \mathfrak{R}$.

In this section, we illustrate how Hamiltonians in the Hitchin system can be expressed in terms of Lax matrices. Moreover, we will also provide the conditions on the Krichever-Tyurin parameters for the Hitchin systems allied with classical groups in [32].

Example 4.1. Let us follow Example 3.1 where $\mathfrak{R}$ is given by

$$y^2 = \prod_{i=1}^{5}(x - c_i) \text{ with } c_i \neq 0.$$

A Higgs field is given by

$$\begin{aligned}
\mathsf{L} \otimes \omega &= \Big(\mathsf{L}_0 + \mathsf{L}_1 x + \sum_{k=1}^{4} \mathbf{B}_k^T \mathbf{A}_k \frac{y+y_k}{x-x_k}\Big)\frac{dx}{2y} \\
&= \begin{pmatrix} l_{11}^0 + l_{11}^1 x + ab\frac{y+y_3}{x-x_3} - ab\frac{y+y_4}{x-x_4} & l_{12}^0 + l_{12}^1 x + (a^2b - ab)\frac{y+y_2}{x-x_2} + ab\frac{y+y_3}{x-x_3} - a^2b\frac{y+y_4}{x-x_4} \\ l_{21}^0 + l_{21}^1 x + (ab - b)\frac{y+y_1}{x-x_1} - ab\frac{y+y_3}{x-x_3} + b\frac{y+y_4}{x-x_4} & l_{22}^0 + l_{22}^1 x - ab\frac{y+y_3}{x-x_3} + ab\frac{y+y_4}{x-x_4} \end{pmatrix}\frac{dx}{2y} \\
&= \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\frac{dx}{2y} \text{ where } \mathsf{L}_0 = \big(l_{ij}^0\big)_{2\times 2}, \mathsf{L}_1 = \big(l_{ij}^1\big)_{2\times 2} \text{ and } \omega = \frac{dx}{2y}.
\end{aligned}$$

The condition for $\Big\{\mathbf{A}_j, \mathbf{B}_j\Big\}_{j=1}^{4}$ was given in Example 3.1. The Lax matrix $\mathsf{L}(p)$ has poles at (x_j, y_j) for $j = 1, \ldots, 4$ with residue $\mathbf{B}_j^T \mathbf{A}_j$. We have the components of the

Hitchin map (3.4) as

$$h_1(p;\mathsf{L}) = \mathrm{Tr}\,\mathsf{L}(p) \text{ and } h_2(p;\mathsf{L}) = \det \mathsf{L}(p) \text{ where } p \in \mathfrak{R}.$$

We may let w be a local coordinate at $p_1 = (\infty, \infty)$ where $x = \frac{1}{w^2}$. Note that the canonical divisor K of $\mathfrak{R}$ is given by

$$(\frac{dx}{y})_0 = 2p_1.$$

From this,

$$\begin{aligned} h_1((x,y);\mathsf{L}) &= (l_{11}^0 + l_{22}^0) + (l_{11}^1 + l_{22}^1)x \\ &= \mathrm{res}_{p_1} \mathrm{Tr}(x^{\frac{3}{2}}\mathsf{L}\frac{dx}{y}) + \Big(\mathrm{res}_{p_1} \mathrm{Tr}(x^{\frac{1}{2}}\mathsf{L}\frac{dx}{y})\Big)x. \end{aligned}$$

Accordingly, after identifying $\mathcal{H}^K$ with $\mathbb{C}^5$ the Hamiltonians coming from the Hitchin map

$$\mathbf{H} : \mathcal{L}^K/\mathbf{SL}(l,\mathbb{C}) \to \mathbb{C}^5 \text{ by } \mathbf{H}(\mathsf{L}) = (H_1(\mathsf{L}), \dots, H_5(\mathsf{L}))$$

are given by

$$\begin{aligned} H_1(\mathsf{L}) &: = \mathrm{res}_{p_1} \mathrm{Tr}(x^{\frac{3}{2}}\mathsf{L}\frac{dx}{y}) \\ H_2(\mathsf{L}) &: = \mathrm{res}_{p_1} \mathrm{Tr}(x^{\frac{1}{2}}\mathsf{L}\frac{dx}{y}). \end{aligned}$$

By the second condition in Definition 3.1, we have

$$\begin{aligned} L_{12}(\gamma_1) &= l_{12}^0 + l_{12}^1 x_1 + (a^2b - ab)\frac{y_1+y_2}{x_1-x_2} + ab\frac{y_1+y_3}{x_1-x_3} - a^2b\frac{y_1+y_4}{x_1-x_4} = 0 \\ L_{21}(\gamma_2) &= l_{21}^0 + l_{21}^1 x_2 + (ab - b)\frac{y_2+y_1}{x_2-x_1} - ab\frac{y_2+y_3}{x_2-x_3} + b\frac{y_2+y_4}{x_2-x_4} = 0. \end{aligned}$$

Consequently, $L_{12}(p)\cdot L_{21}(p)$ is holomorphic at $\gamma_1 = (x_1, y_1)$ and $\gamma_2 = (x_2, y_2)$. Thus, $h_2(p;\mathsf{L}) = \det \mathsf{L}(p)$ is holomorphic at $\gamma_1 = (x_1, y_1)$ and $\gamma_2 = (x_2, y_2)$. Let

$$\mathsf{L}(p) = \begin{pmatrix} L_{11}(p) & L_{12}(p) \\ L_{21}(p) & L_{22}(p) \end{pmatrix} := \begin{pmatrix} J_{11}(p) + ab\frac{y+y_3}{x-x_3} & J_{12}(p) + ab\frac{y+y_3}{x-x_3} \\ J_{21}(p) - ab\frac{y+y_3}{x-x_3} & J_{22}(p) - ab\frac{y+y_3}{x-x_3} \end{pmatrix}$$

The second condition in Definition 3.1 implies

$$J_{11}(\gamma_3) + J_{21}(\gamma_3) = J_{12}(\gamma_3) + J_{22}(\gamma_3) \text{ and } J_{ij}(\gamma_3) \in \mathbb{C}.$$

Consequently,

$$\begin{aligned}\det \mathsf{L}(\gamma_3) &= J_{11}(\gamma_3)J_{22}(\gamma_3) - J_{12}(\gamma_3)J_{21}(\gamma_3) \\ &\quad + ab\frac{y_3 + y_3}{x_3 - x_3}\big(J_{11}(\gamma_3) + J_{21}(\gamma_3) - J_{12}(\gamma_3) - J_{22}(\gamma_3)\big)\end{aligned}$$

is a complex number. Thus, $h_2(p; \mathsf{L}) = \det \mathsf{L}(p)$ is holomorphic at $\gamma_3 = (x_3, y_3)$. In the similar way, we can prove that it is also holomorphic at γ_4. The fact that $\det \mathsf{L}(p)$ is holomorphic at γ_i for $i = 1, \ldots, 4$ implies that

$$\begin{aligned}\det \mathsf{L} &= \frac{1}{2}\Big(\big(\operatorname{Tr}\mathsf{L}\big)^2 - \operatorname{Tr}\mathsf{L}^2\Big) \\ &= \frac{1}{2}\Big(\big(\operatorname{Tr}\mathsf{L}\big)^2 - \operatorname{Tr}\big(\mathsf{L}_0 + \mathsf{L}_1 x\big)^2 \\ &\qquad - 2\operatorname{Tr}\big((\mathsf{L}_0 + \mathsf{L}_1 x)(\sum_{k=1}^{4}\mathbf{B}_k^T\mathbf{A}_k\frac{y + y_k}{x - x_k})\big) + \operatorname{Tr}\big(\sum_{k=1}^{4}\mathbf{B}_k^T\mathbf{A}_k\frac{y + y_k}{x - x_k}\big)^2\Big) \\ &= \frac{1}{2}\Big(\big(\operatorname{Tr}\mathsf{L}\big)^2 - \operatorname{Tr}\big(\mathsf{L}_0 + \mathsf{L}_1 x\big)^2\Big) \\ &= \frac{1}{2}\Big(\big(\operatorname{res}_{p_1}\operatorname{Tr}(x^{\frac{3}{2}}\mathsf{L}\frac{dx}{y}) + \operatorname{res}_{p_1}\operatorname{Tr}(x^{\frac{1}{2}}\mathsf{L}\frac{dx}{y})x\big)^2 \\ &\qquad - \operatorname{res}_{p_1}\operatorname{Tr}(x^{\frac{3}{2}}\mathsf{L}^2\frac{dx}{y}) - \big(\operatorname{res}_{p_1}\operatorname{Tr}(x^{\frac{1}{2}}\mathsf{L}^2\frac{dx}{y})\big)x - \big(\operatorname{res}_{p_1}\operatorname{Tr}(x^{-\frac{1}{2}}\mathsf{L}^2\frac{dx}{y})\big)x^2\Big).\end{aligned}$$

Thus, we may write Hamiltonians in the Hitchin's map explicitly in terms of the

Hamiltonians constructed by Krichever as follows:

$$\begin{aligned}
H_1(\mathsf{L}) &= \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{3}{2}}\mathsf{L}\frac{dx}{y}) \\
H_2(\mathsf{L}) &= \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{1}{2}}\mathsf{L}\frac{dx}{y}) \\
H_3(\mathsf{L}) &= \frac{1}{2}\Big(\big(\operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{3}{2}}\mathsf{L}\frac{dx}{y})\big)^2 - \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{3}{2}}\mathsf{L}^2\frac{dx}{y})\Big) \\
H_4(\mathsf{L}) &= \frac{1}{2}\Big(2 \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{3}{2}}\mathsf{L}) \cdot \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{1}{2}}\mathsf{L}\frac{dx}{y}) - \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{1}{2}}\mathsf{L}^2\frac{dx}{y})\Big) \\
H_5(\mathsf{L}) &= \frac{1}{2}\Big(\big(\operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{1}{2}}\mathsf{L}\frac{dx}{y})\big)^2 - \operatorname{res}_{p_1} \operatorname{Tr}(x^{-\frac{1}{2}}\mathsf{L}^2\frac{dx}{y})\Big).
\end{aligned}$$

The Hitchin's Hamiltonians H_i are in the ring generated by Krichever's Hamiltonians

$$\Big\{\big(\operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{m}{2}}\mathsf{L}^n\frac{dx}{y})\big)^r \mid n, r \in \mathbb{N} \text{ and } m \in \mathbb{Z}\Big\}.$$

We will investigate Example 4.1 more closely in the next example for classical groups in Theorem 2.3.

Example 4.2. Let us characterize a subspace of the space of the Krichever-Tyurin parameters coming from symplectic vector bundles in Example 4.1. From the Hitchin's investigation in [32], the characteristic polynomial of a Higgs field associated with a symplectic vector bundle must be of form

$$R(\mu, p) = \mu^{2l} + h_2(p)\mu^{2l-2} + \cdots + h_{2l-2}(p)\mu^2 + h_{2l}(p),$$

since a symplectic form makes eigenvalues be a pair $\mu, -\mu$. Accordingly, we have $h_1((x,y);\mathsf{L}) = 0$ in Example 4.1. Thus, the condition on a Higgs field is given by

$$\operatorname{Tr}\mathsf{L}_0 = \operatorname{Tr}\mathsf{L}_1 = 0.$$

On the other hand, the characterization of a Higgs field associated with a holomorphic vector bundle of rank 2 with a non-degenerate symmetric bi-linear form is given as follows: In this case, the symmetric form also provides a pair $\mu, -\mu$ of eigenvalues. However, the characteristic polynomial of a Higgs field has another condition in addition to having a pair of eigenvalues. The condition is that $h_2((x,y);\mathsf{L}) = \det(\mathsf{L})$ is the square of a polynomial $\mathrm{Pfaff}(\mathsf{L})$, which is called the *Pfaffian*. Consequently, from the following equation

$$\begin{aligned}\det \mathsf{L} &= \frac{1}{2}\Big(\big(\operatorname{Tr}\mathsf{L}\big)^2 - \operatorname{Tr}\left(\mathsf{L}_0 + \mathsf{L}_1 x\right)^2\Big)\\ &= -\frac{1}{2}\Big(\operatorname{Tr}\left(\mathsf{L}_0 + \mathsf{L}_1 x\right)^2\Big)\\ &= \Big(\mathrm{Pfaff}(\mathsf{L})\Big)^2,\end{aligned}$$

we have a relation

$$\Big(\operatorname{Tr}\left(\mathsf{L}_0\mathsf{L}_1\right)\Big)^2 = \operatorname{Tr}\mathsf{L}_0^2 \cdot \operatorname{Tr}\mathsf{L}_1^2$$

along with $h_1((x,y);\mathsf{L}) = 0$.

Example 4.3. Let us follow Example 3.2 where the moduli space of vector bundles of rank 3 over a hyper-elliptic curve was considered. Let $p_1 = (\infty,\infty)$. Similarly to Example 4.1, $h_1(p;\mathsf{L}), h_2(p;\mathsf{L})$, and $h_3(p;\mathsf{L})$ are holomorphic at γ_i for $i = 1,\ldots,6$ where

$$\mathbf{H} : \mathcal{L}^K/\mathbf{SL}(l,\mathbb{C}) \to \mathcal{H}^K \text{ by } \mathbf{H}(\mathsf{L}) = (h_1(p;\mathsf{L}), h_2(p;\mathsf{L}), h_3(p;\mathsf{L})).$$

Identifying $\mathcal{H}^K$ with $\mathbb{C}^{10}$, Hamiltonians in the Hitchin map are given by

$$\mathbf{H} : \mathcal{L}^K/\mathbf{SL}(l,\mathbb{C}) \to \mathbb{C}^{10} \text{ by } \mathbf{H}(\mathsf{L}) = (H_1(\mathsf{L}),\ldots,H_{10}(\mathsf{L})).$$

Note that

$$\dim_{\mathbb{C}} \mathrm{H}^0(\mathfrak{R}, \mathrm{K}_{\mathfrak{R}}) = 2$$

$$\dim_{\mathbb{C}} \mathrm{H}^0(\mathfrak{R}, \mathrm{K}_{\mathfrak{R}}^2) = 3$$

$$\dim_{\mathbb{C}} \mathrm{H}^0(\mathfrak{R}, \mathrm{K}_{\mathfrak{R}}^3) = 5.$$

It is easy to see that

$$\begin{aligned} h_1((x,y);\mathsf{L}) &= (l_{11}^0 + l_{22}^0 + l_{33}^0) + (l_{11}^1 + l_{22}^1 + l_{33}^1)x \\ &= \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{3}{2}}\mathsf{L}\frac{dx}{y}) + \Big(\operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{1}{2}}\mathsf{L}\frac{dx}{y})\Big)x. \end{aligned}$$

Thus,

$$\begin{aligned} H_1(\mathsf{L}) &:= \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{3}{2}}\mathsf{L}\frac{dx}{y}) \\ H_2(\mathsf{L}) &:= \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{1}{2}}\mathsf{L}\frac{dx}{y}). \end{aligned}$$

The fact that $h_2(p;\mathsf{L})$ is holomorphic at γ_i for $i = 1, \dots, 4$ implies that

$$\begin{aligned} h_2((x,y);\mathsf{L}) =& \frac{1}{2}\Big(\big(\operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{3}{2}}\mathsf{L}\frac{dx}{y}) + \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{1}{2}}\mathsf{L}\frac{dx}{y})x\big)^2 \\ &- \operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{3}{2}}\mathsf{L}^2\frac{dx}{y}) - \big(\operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{1}{2}}\mathsf{L}^2\frac{dx}{y})\big)x - \big(\operatorname{res}_{p_1} \operatorname{Tr}(x^{-\frac{1}{2}}\mathsf{L}^2\frac{dx}{y})\big)x^2\Big). \end{aligned}$$

Letting $\operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{m}{2}}\mathsf{L}^n\frac{dx}{y})$ by $H_{\{m,n\}}(\mathsf{L})$ in order to simplify the notations, we have

$$\begin{aligned} h_3((x,y);\mathsf{L}) &= \det \mathsf{L} = \frac{1}{6}\big(\operatorname{Tr}\mathsf{L}\big)^3 - \frac{1}{2}\big(\operatorname{Tr}\mathsf{L}^2\big)\operatorname{Tr}\mathsf{L} + \frac{1}{3}\operatorname{Tr}\mathsf{L}^3 \\ &= \frac{1}{6}\big(H_{\{3,1\}}(\mathsf{L}) + H_{\{1,1\}}(\mathsf{L})x\big)^3 \\ &\quad + \frac{1}{2}\big(H_{\{3,2\}}(\mathsf{L}) + H_{\{1,2\}}(\mathsf{L})x + H_{\{-1,2\}}(\mathsf{L})x^2\big)\big(H_{\{3,1\}}(\mathsf{L}) + H_{\{1,1\}}(\mathsf{L})x\big) \\ &\quad + \frac{1}{3}\big(H_{\{3,3\}}(\mathsf{L}) + H_{\{1,3\}}(\mathsf{L})x + H_{\{-1,3\}}(\mathsf{L})x^2 + H_{\{-3,3\}}(\mathsf{L})x^3\big). \end{aligned}$$

Let us remind that we may take a basis of $\mathrm{H}^0(\mathfrak{R}, \mathrm{K}_{\mathfrak{R}}^3)$ as

$$\Big\{x^i\frac{dx\otimes dx\otimes dx}{y^3}, y\frac{dx\otimes dx\otimes dx}{y^3} \mid i = 0,1,2,3\Big\}.$$

The Hitchin's Hamiltonians H_i are in the ring generated by Krichever's Hamiltonians

$$\Big\{\big(\operatorname{res}_{p_1} \operatorname{Tr}(x^{\frac{m}{2}}\mathsf{L}^n\frac{dx}{y})\big)^r \mid n, r \in \mathbb{N} \text{ and } m \in \mathbb{Z}\Big\}.$$

Explicitly, they are given as follows:

$$
\begin{aligned}
H_1(\mathsf{L}) &= H_{\{3,1\}}(\mathsf{L})\\
H_2(\mathsf{L}) &= H_{\{1,1\}}(\mathsf{L})\\
H_3(\mathsf{L}) &= \frac{1}{2}\Big(H_{\{3,1\}}(\mathsf{L})^2 - H_{\{3,2\}}(\mathsf{L})\Big)\\
H_4(\mathsf{L}) &= \frac{1}{2}\Big(2H_{\{3,1\}}(\mathsf{L})H_{\{1,1\}}(\mathsf{L}) - H_{\{1,2\}}(\mathsf{L})\Big)\\
H_5(\mathsf{L}) &= \frac{1}{2}\Big(H_{\{1,1\}}(\mathsf{L})^2 - H_{\{-1,2\}}(\mathsf{L})\Big)\\
H_6(\mathsf{L}) &= \frac{1}{6}H_{\{3,1\}}(\mathsf{L})^3 + \frac{1}{2}H_{\{3,1\}}(\mathsf{L})H_{\{3,2\}}(\mathsf{L}) + \frac{1}{3}H_{\{3,3\}}(\mathsf{L})\\
H_7(\mathsf{L}) &= \frac{1}{2}\big(H_{\{3,1\}}(\mathsf{L})^2H_{\{1,1\}}(\mathsf{L}) + H_{\{1,2\}}(\mathsf{L})H_{\{3,1\}}(\mathsf{L}) + H_{\{3,2\}}(\mathsf{L})H_{\{1,1\}}(\mathsf{L})\big) + \frac{1}{3}H_{\{1,3\}}(\mathsf{L})\\
H_8(\mathsf{L}) &= \frac{1}{2}\big(H_{\{3,1\}}(\mathsf{L})H_{\{1,1\}}(\mathsf{L})^2 + H_{\{-1,2\}}(\mathsf{L})H_{\{3,1\}}(\mathsf{L}) + H_{\{1,2\}}(\mathsf{L})H_{\{1,1\}}(\mathsf{L})\big) + \frac{1}{3}H_{\{-1,3\}}(\mathsf{L})\\
H_9(\mathsf{L}) &= 0\\
H_{10}(\mathsf{L}) &= \frac{1}{6}H_{\{1,1\}}(\mathsf{L})^3 + \frac{1}{2}H_{\{-1,2\}}(\mathsf{L})H_{\{1,1\}}(\mathsf{L}) + \frac{1}{3}H_{\{-3,3\}}(\mathsf{L}).
\end{aligned}
$$

Example 4.4. Let us describe a Higgs field associated with a holomorphic vector bundle of rank 3 with a non-degenerate symmetric bi-linear form among the Higgs fields in Example 4.3. In this case, the characteristic polynomial of a Higgs field is given by

$$R(\mu, p) = \mu(\mu^2 + h_2(p)).$$

Consequently, the condition is given by

$$h_1((x, y); \mathsf{L}) = h_3((x, y); \mathsf{L}) = 0.$$

Example 4.5. Following Example 3.3, consider vector bundles of rank 2 over a $(4, 3)$-curve $\mathfrak{R} = \{R(x, y) = 0\}$ of genus 3 defined by

$$R(x, y) = y^4 + x^3 + c_{3,0}y^3 + c_{2,1}y^2x + c_{1,2}yx^2 + c_{2,0}y^2 + c_{1,1}yx + c_{0,2}x^2 + c_{1,0}y + c_{0,1}x + c_{0,0}.$$

This curve is a 4-sheeted covering over $\mathbb{C}$

$$\pi : \mathfrak{R} \to \mathbb{C} \text{ by } \pi(x, y) = x.$$

We first characterize the ramification points of π. For a generic $(4, 3)$-curve, it is ramified over 4 points with multiplicity 4. On the other hands, the possible places of the zeros of $\frac{dx}{R_y}$ must be at the ramification points where $\deg \frac{dx}{R_y} = 4$: When a ramification point is over a complex number, we may find a local coordinate z such that

$$x = z^4 \text{ and } y = t_1 z \text{ where } t_1 \in \mathbb{C}.$$

Accordingly, $\frac{dx}{R_y(z^4, t_1 z)} = \frac{4z^3 dz}{t_2 z^3} \neq 0$. Thus, the holomorphic differential does not take zeros at the ramification points over complex numbers. Hence, we conclude that it must be ramified over the infinity where a local coordinate w is given by

$$x = \frac{1}{w^4} \text{ and } R_y(\frac{1}{w^4}, t_1 \frac{1}{w^3}) = t_2 \frac{1}{w^9} \text{ where } t_1, t_2 \in \mathbb{C}.$$

Therefore, it is ramified over 3 points in $\mathbb{C}$ and the infinity ∞ with multiplicity 4 and the canonical divisor K of $\frac{dx}{R_y}$ is $4 \cdot (\infty, \infty)$. The Hitchin map is given by $\mathbf{H} : \mathcal{L}^K/\mathbf{SL}(l, \mathbb{C}) \to \mathcal{H}^K \cong \mathbb{C}^9$ by

$$\mathbf{H}(\mathsf{L}) = (h_1(p; \mathsf{L}), h_2(p; \mathsf{L})) = (H_1(\mathsf{L}), \ldots, H_9(\mathsf{L})).$$

Here

$$\dim_{\mathbb{C}} \mathrm{H}^0(\mathfrak{R}, \mathrm{K}_{\mathfrak{R}}) = 3 \text{ and } \dim_{\mathbb{C}} \mathrm{H}^0(\mathfrak{R}, \mathrm{K}_{\mathfrak{R}}^2) = 6.$$

Let $p_1 = (\infty, \infty)$. A Higgs field is given by

$$\mathsf{L}\frac{dx}{R_y} = \Big(\mathsf{L}_0 + \mathsf{L}_1 x + \mathsf{L}_2 \frac{y}{t_1} + \sum_{k=1}^{6} \mathbf{B}_k^T \mathbf{A}_k \frac{y + R_y(x_k, y_k) - y_k}{x - x_k}\Big) \frac{dx}{R_y} \text{ where } R_y = \frac{\partial R(x, y)}{\partial y}.$$

We see that

$$h_1((x,y);\mathsf{L}) = (l^0_{11}+l^0_{22}) + (l^1_{11}+l^1_{22})x + (l^2_{11}+l^2_{22})\frac{y}{t_1}$$

$$= \mathrm{res}_{p_1}\,\mathrm{Tr}(w^{-5}\mathsf{L}\frac{t_2dx}{R_y}) + \Big(\mathrm{res}_{p_1}\,\mathrm{Tr}(w^{-1}\mathsf{L}\frac{t_2dx}{R_y})\Big)x + \Big(\mathrm{res}_{p_1}\,\mathrm{Tr}(w^{-2}\mathsf{L}\frac{t_2dx}{R_y})\Big)\frac{y}{t_1}.$$

Thus,

$$H_1(\mathsf{L}) := \mathrm{res}_{p_1}\,\mathrm{Tr}(w^{-5}\mathsf{L}\frac{t_2dx}{R_y})$$

$$H_2(\mathsf{L}) := \mathrm{res}_{p_1}\,\mathrm{Tr}(w^{-1}\mathsf{L}\frac{t_2dx}{R_y})$$

$$H_3(\mathsf{L}) := \mathrm{res}_{p_1}\,\mathrm{Tr}(w^{-2}\mathsf{L}\frac{t_2dx}{R_y}).$$

Let us denote $\mathrm{res}_{p_1}\,\mathrm{Tr}(w^{-m}\mathsf{L}^n\frac{t_2dx}{R_y})$ by $H_{\{m,n\}}(\mathsf{L})$ in order to simplify the notations.

The fact that $h_2(p;\mathsf{L})$ is holomorphic at γ_i for $i=1,\dots,4$ implies that

$$\begin{aligned} h_2((x,y);\mathsf{L}) = \frac{1}{2}\Big(&\big(H_{\{5,1\}}(\mathsf{L}) + H_{\{1,1\}}(\mathsf{L})x + H_{\{2,1\}}(\mathsf{L})\frac{y}{t_1}\big)^2 \\ &- H_{\{5,2\}}(\mathsf{L}) - H_{\{2,2\}}(\mathsf{L})\frac{y}{t_1} - H_{\{-1,2\}}(\mathsf{L})(\frac{y}{t_1})^2 \\ &- H_{\{1,2\}}(\mathsf{L})x - H_{\{-2,2\}}(\mathsf{L})\frac{xy}{t_1} - H_{\{-3,2\}}(\mathsf{L})x^2\Big). \end{aligned}$$

The Hamiltonians induced from the Hitchin's map are given explicitly in terms of the Hamiltonians constructed by Krichever as follows:

$$H_1(\mathsf{L}) = H_{\{5,1\}}(\mathsf{L}),\quad H_2(\mathsf{L}) = H_{\{1,1\}}(\mathsf{L}),\quad H_3(\mathsf{L}) = H_{\{2,1\}}(\mathsf{L})$$

$$H_4(\mathsf{L}) = \frac{1}{2}\Big(H_{\{5,1\}}(\mathsf{L})^2 - H_{\{5,2\}}(\mathsf{L})\Big),\quad H_5(\mathsf{L}) = \frac{1}{2}\Big(2H_{\{5,1\}}(\mathsf{L})H_{\{2,1\}}(\mathsf{L}) - H_{\{2,2\}}(\mathsf{L})\Big)$$

$$H_6(\mathsf{L}) = \frac{1}{2}\Big(H_{\{2,1\}}(\mathsf{L})^2 - H_{\{-1,2\}}(\mathsf{L})\Big), H_7(\mathsf{L}) = \frac{1}{2}\Big(2H_{\{1,1\}}(\mathsf{L})H_{\{1,1\}}(\mathsf{L}) - H_{\{1,2\}}(\mathsf{L})\Big)$$

$$H_8(\mathsf{L}) = \frac{1}{2}\Big(2H_{\{2,1\}}(\mathsf{L})H_{\{1,1\}}(\mathsf{L}) - H_{\{-2,2\}}(\mathsf{L})\Big), H_9(\mathsf{L}) = \frac{1}{2}\Big(H_{\{1,1\}}(\mathsf{L})^2 - H_{\{-3,2\}}(\mathsf{L})\Big).$$

Example 4.6. Following Example 4.5, consider a holomorphic vector bundles of rank 2 over a (4,3)-curve $\mathfrak{R} = \{R(x,y)=0\}$ of genus 3 with a symplectic form. The characterization of such bundles is exactly same as in the case of Example 4.2.

That is, the condition is simply to put

$$h_1((x, y); \mathsf{L}) = 0.$$

Furthermore, analogously to Example 4.2, it is not hard to see that the condition on a Higgs field associated with a holomorphic vector bundle of rank 2 with a non-degenerate symmetric bi-linear form becomes

$$\left(\operatorname{Tr} \left(\mathsf{L}_0 \mathsf{L}_1 \right) \right)^2 = \operatorname{Tr} \mathsf{L}_0^2 \cdot \operatorname{Tr} \mathsf{L}_1^2$$
$$\left(\operatorname{Tr} \left(\mathsf{L}_0 \mathsf{L}_2 \right) \right)^2 = \operatorname{Tr} \mathsf{L}_0^2 \cdot \operatorname{Tr} \mathsf{L}_2^2$$
$$\left(\operatorname{Tr} \left(\mathsf{L}_1 \mathsf{L}_2 \right) \right)^2 = \operatorname{Tr} \mathsf{L}_1^2 \cdot \operatorname{Tr} \mathsf{L}_2^2$$

in addition to $h_1((x, y); \mathsf{L}) = 0$.

Appendix A

This part is an exposition of papers by I. Krichever [35, 37] which construct a correspondence effectively between commutative rings of ordinary differential operators in one variable and certain geometric data. This effective and constructive algorithm is worth reviewing for our future investigation. For a general categorical approach using abstract machinery, let us note [49, 52]:

A.1 Commutative rings of ordinary differential operators in one variable

Consider two monic differential operators of one variable:

$$L_1 = \sum_{i=0}^{n} u_i(x) \frac{d^i}{dx^i} \text{ and } L_2 = \sum_{i=0}^{m} v_i(x) \frac{d^i}{dx^i}$$

satisfying a commutative relation $[L_1, L_2] = 0$. The subring $\mathcal{R}(L_1, L_2)$ generated by L_1 and L_2 is a commutative subring of the ring $\mathcal{D}$ of differential operators in one variable x. The Burchnall-Chaundy lemma in [9] states that there is a polynomial relation

$$R(L_1, L_2) = 0.$$

Let us mention that the commutative relation $[L_1, L_2] = 0$ is a system of differential equations. We will call this a system of *Novikov equations*.

A.1.1 When m and n are coprime

The direct problem asks us to construct a Riemann surface with a distinguished point P_0 and g points $\gamma_1, \ldots, \gamma_g$ on it from a given subring $\mathcal{R}(L_1, L_2)$.

For $L_1\psi(x,k) = k^n\psi(x,k)$, there is a unique normalized formal solution which is called a Baker-Akhiezer function

$$\psi(x,k,x_0) = \exp\big(k(x-x_0)\big)\big(\sum_{s=0}^{\infty}\xi_s(x)k^{-s}\big) \text{ with } \psi(x_0,k,x_0) = 1.$$

It is meromorphic in $k \in \mathbb{C}$. Notice that for $\lambda = k^n$, we have an n-dimensional solution space of $L_1\psi = \lambda\psi$ given by

$$\psi_i(x,\lambda,x_0) = \psi(x,\epsilon_i k,x_0) \text{ with } \epsilon_i^n = 1.$$

Since $[L_1, L_2] = 0$, we can solve a simultaneous eigenvalue problem

$$\begin{cases} L_1\psi(x,k,x_0) & = \lambda\psi(x,k,x_0) \\ L_2\psi(x,k,x_0) & = \mu\psi(x,k,x_0). \end{cases}$$

From the $(n \times n)$-matrix representation $L_2(\lambda)$ on the n-dimensional space $\mathcal{L}(\lambda)$ of solutions of $L_1\psi = \lambda\psi$, we have

$$R(\lambda,\mu) = \det(\mu \cdot I_{n\times n} - L_2(\lambda)).$$

Now the key observation is

$$\begin{cases} \lambda(k) = k^n \\ \mu(k) = k^m + \sum_{i=-m+1}^{\infty} a_i k^{-i} \end{cases}$$

From this observation and $(n,m)=1$, it is obvious that for all $\lambda(k)=\lambda(\epsilon_i k)$ but a finite number of λ, we have n distinct μ's given by

$$\mu_i = \mu(\epsilon_i k).$$

Moreover, by the same reasoning, we have exactly one point $P_0 = (\lambda,\mu) = (\infty,\infty)$ as $k \to \infty$. This proves that $\mathfrak{R} = \{R(\lambda,\mu)=0\}$ is an irreducible algebraic curve, not necessarily smooth, with one point P_0 at the infinity. We will call it the associated Burchnall-Chaundy Riemann surface. It is not hard to see that

$$R(\lambda,\mu) = \mu^n + \lambda^m + \sum_{i,j} c_{i,j}\mu^i\lambda^j \text{ where } 0 \le i \le n-1, 0 \le j \le m-1, im+jn < nm.$$

For a generic pair (L_1, L_2), the associated Burchnall-Chaundy Riemann surface $\mathfrak{R}$ is smooth and the genus is given by

$$g = \frac{(n-1)(m+1)}{2} - (n-1) = \frac{(n-1)(m-1)}{2}.$$

This irreducible algebraic curve is called an (n,m)-*curve* in literature, i.e., [9, 18]. One example is a hyper-elliptic curve, which is a $(2, 2g+1)$-curve. Furthermore, since we constructed a one-dimensional space of solutions of the simultaneous eigenvalue problem over a point $P = (\lambda,\mu)$ in the (n,m)-curve, we have a unique function $\psi(x,P,x_0) = \psi(P)$ on $\mathfrak{R}$ by the normalization condition $\psi(x_0,P,x_0)=1$.

Lemma A.1. *The number of poles of $\psi(P)$ is given by $(n-1)(m-1)/2$ for a generic pair (L_1, L_2).*

Proof. Let $\mathfrak{R}$ be the constructed smooth (n,m)-curve with genus $g = (n-1)(m-1)/2$ with P_0 and consider the n-sheeted covering map $\lambda : \mathfrak{R} \to \mathbb{P}^1$. Letting $Q_i =$

(λ, μ_i) for $i = 0, \dots, n-1$ be the inverse images of λ, we can define a meromorphic function

$$F(\lambda, x_0) = \det(\partial_x^i \psi(x, Q_j, x_0)) \text{ on } \mathbb{P}^1.$$

Since the zeros of F are the ramification points except the infinity $\lambda = \infty$, the number of zeros of F is m with multiplicity

$$\frac{1}{n} \cdot \frac{n(n-1)}{2} = \frac{n-1}{2}.$$

The poles of F consist of poles of $\psi(P)$ and the infinity $\lambda = \infty$ with multiplicity of $(n-1)/2$ from $\psi(x, k, x_0) = \exp\big(k(x - x_0)\big)(1 + \sum_{s=1}^{\infty} \xi_s(x) k^{-s})$. Since the number of zeros of F is the same as the number of poles of F, we have

$$N + \frac{n-1}{2} = m \cdot \frac{n-1}{2},$$

where N is the number of poles of $\psi(P)$ on $\mathfrak{R}$. Consequently,

$$N = \frac{(n-1)(m-1)}{2} = g.$$

□

Let us consider the *inverse problem*: Suppose we are given an *algebraic spectral data* $(\mathfrak{R}, P_0, \gamma_1, \dots, \gamma_g)$. Here, $\mathfrak{R}$ is a compact Riemann surface with one distinguished point P_0 and g points $\gamma_1, \dots, \gamma_g$ in general position on $\mathfrak{R}$. By the Riemann-Roch theorem, we have a unique Baker-Akhiezer function ψ having poles at those points and a local expression at P_0 is given by

$$\psi(x, k, x_0) = \exp\big(k(x - x_0)\big)(\sum_{s=0}^{\infty} \xi_s(x) k^{-s}) \text{ with } \psi(x_0, k, x_0) = 1.$$

k^{-1} is a local parameter at P_0. Let $\mathcal{A}(\mathfrak{R}, P_0)$ be the set of meromorphic functions regular except (possibly) at P_0. If $f \in \mathcal{A}(\mathfrak{R}, P_0)$, then locally

$$f = k^n + \sum_{i=-n+1}^{\infty} a_i(x)k^{-i}.$$

Formally we can construct a unique monic differential operator L_f in one variable x such that

$$(L_f - f)\psi = O(k^{-1})\exp(k(x - x_0)).$$

The function $(L_f - f)\psi$ is also defined on $\mathfrak{R}$ and satisfies all the properties of the Baker-Akhiezer function. Consequently $(L_f - f)\psi \equiv 0$ on $\mathfrak{R}$. Therefore, the Baker-Akhiezer function defines an isomorphism from $\mathcal{A}(\mathfrak{R}, P_0)$ to a commutative subring of the ring $\mathcal{D}$ of differential operators

$$\mathcal{A}(\mathfrak{R}, P_0) \overset{\psi}{\hookrightarrow} \mathcal{D} \text{ by } f \mapsto L_f.$$

We need to show that the image of $\mathcal{A}(\mathfrak{R}, P_0)$ contains a pair of mutually coprime two monic operators. Let n be the minimal positive number where there exists a function $f \in \mathcal{A}(\mathfrak{R}, P_0)$ with $\mathsf{ord}_{P_0}(f) = -n$. If $g > 0$, then $n > 1$ and there exists another function $g \in \mathcal{A}(\mathfrak{R}, P_0)$ with $\mathsf{ord}_{P_0}(g) = -an-1$ where a is a positive integer. Hence, L_f and L_g are a mutually coprime pair. Notice that the coprime pair does not necessarily generate $\mathcal{A}(\mathfrak{R}, P_0)$.

The inverse problem deals with a broader class of Riemann surfaces than the considered direct problem. That is, two coprime generic operators can only construct an (n, m)-curve among all Riemann surfaces, but in the inverse problem we considered a general compact Riemann surface as a part of algebraic spectral data $(\mathfrak{R}, P_0, \gamma_1, \ldots, \gamma_g)$. In fact, there is a considerably more general version of the inverse problem than the direct problem. That is, the inverse problem associated with the following algebraic spectral data

$$(\widetilde{\mathfrak{R}}, P_1, \ldots, P_l, \gamma_1, \ldots, \gamma_{g+l-1+d}, E_1, \ldots, E_d)$$

Here, $\widetilde{\mathfrak{R}}$ is the unique smooth model of a singular Riemann surface $\mathfrak{R}$ with singularities $E_1, \ldots, E_d$. This inverse construction can characterize all the commutative subrings containing a coprime pair of differential operators in $\mathcal{D}$, which is stated in [35].

A.1.2 When m and n are not coprime

We will retain the same notation in this section as in the previous section. The *rank* of a ring $\mathcal{R}(L_1, L_2)$ generated by L_1 and L_2 is the minimal number l such that

$$\mu(k) = \mu(\epsilon^l k) \text{ where } \epsilon = \exp(\frac{2\pi i}{n}).$$

This is equivalent to the existence of $\widetilde{\mu}$ such that

$$\mu(k) = \widetilde{\mu}(k^l).$$

Certainly, l is a common divisor of n and m, not necessarily the greatest common divisor. In the previous section, we considered the case $l = 1$. Notice that it is possible that $l = 1$ even if $(m, n) \neq 1$. In this section we will consider only when $(m, n) = l > 1$ in the direct problem.

Let us consider the Burchnall-Chaundy Riemann surface associated with a higher rank. It is when $\mu(k) = \widetilde{\mu}(k^l)$ for $l > 1$: In this case,

$$R(\lambda, \mu) = \prod_{j=1}^{n} (\mu - \mu(\epsilon_i k)) = \prod_{j=1}^{n'} (\mu - \widetilde{\mu}(\widetilde{\epsilon_i k}))^l = (\widetilde{R}(\lambda, \mu))^l,$$

where $(\epsilon_i k)^n = (\widetilde{\epsilon_i k})^{n'} = \lambda$ for $n'l = n$. This associated Burchnall-Chaundy Riemann surface $\mathfrak{R} = \{\widetilde{R}(\lambda, \mu) = 0\}$ is an irreducible algebraic (n', m')-curve with genus $g = (n'-1)(m'-1)/2$ where $m'l = m$, $(n', m') = 1$ and compactified with one point at the infinity, since

$$\begin{cases} \lambda(\widetilde{k}) = \widetilde{k}^{n'} \\ \mu(\widetilde{k}) = \widetilde{k}^{m'} + \sum_{i=-m'+1}^{\infty} a_i \widetilde{k}^{-i}. \end{cases}$$

From the simultaneous eigenvalue problem, we have an l-dimensional space of eigenvectors over each point $P = (\lambda, \mu) \in \mathfrak{R}$ with a basis of normalized Baker-Akhiezer functions

$$\psi_j(x, P, x_0) = \sum_{s=0}^{n-1} \chi_j^s(P, x_0) c_s(x, \lambda, x_0) \text{ with}$$

$$\partial_x^i \psi_j(x, P, x_0)|_{x=x_0} = \delta_{ij} \text{ where } 0 \leq i, j \leq l-1.$$

Here $c_i(x, \lambda, x_0)$ for $i = 0, \ldots, n-1$ is a normalized basis of the n-dimensional space of solutions of $L_1\psi = \lambda(k)\psi$ with

$$\partial_x^j c_i(x, \lambda, x_0)|_{x=x_0} = \delta_{ij} \text{ where } 0 \leq i, j \leq n-1.$$

Lemma A.2. *The number of poles of $\psi(P)$ is lg for a generic pair (L_1, L_2).*

Proof. Let $\mathfrak{R}$ be the constructed smooth (n', m')-curve with genus $g = (n'-1)(m'-1)/2$ with P_0 and consider the n'-sheeted covering map $\lambda : \mathfrak{R} \to \mathbb{P}^1$. Letting $Q_i =$

(λ, μ_i) for $i = 0, \ldots, n'-1$ be the inverse images of λ, we can define a meromorphic function

$$F(\lambda, x_0) = \det \left(\partial_x^i \psi(x, Q_j, x_0) \right)_{n' \times n'}.$$

Similar to the proof of Lemma A.1, we may conclude the desired result using the constructed meromorphic function. □

Note that the poles of $\psi_i(P)$ only depend on the base point x_0. Of course, the zeros of $\psi(P)$ will move as x varies. For a generic pair (L_1, L_2), the poles $\gamma_1(x_0), \ldots, \gamma_{lg}(x_0)$ are simple and there are constants $\alpha_{i,j}(x_0)$ such that

$$\alpha_{i,j}(x_0) \operatorname{res}_{\gamma_i(x_0)} \psi_{l-1} = \operatorname{res}_{\gamma_i(x_0)} \psi_j.$$

The set $\left\{ \gamma_i(x_0), \{\alpha_{i,j}(x_0)\}_{j=0}^{l-1} \right\}_{i=1}^{lg} \in \mathcal{S}^{lg}(\mathfrak{R} \times \mathbb{P}^{l-1})$ is a set of *Tyurin parameters.* The residue $\operatorname{res}_{\gamma_i(x_0)} \psi_j$ will vary as x varies. Let

$$\boldsymbol{\alpha}_i(x_0) = (\alpha_{i,0}(x_0), \ldots, \alpha_{i,l-2}(x_0), 1) \text{ and } \beta_i(x) = \operatorname{res}_{\gamma_i(x_0)} \psi_{l-1}.$$

Around a point γ_i with a local variable z_i, the vector-valued Baker-Akhiezer function $\boldsymbol{\psi} = (\psi_0, \ldots, \psi_{l-1})$ can be written as

$$\boldsymbol{\psi}(z_i) = \frac{\beta_i(x)\boldsymbol{\alpha}_i(x_0)}{z_i - z_i(\gamma_i(x_0))} + O(1).$$

Let

$$\Psi(x, P, x_0) = \left(\partial_x^i \psi_j(x, P, x_0) \right)_{l \times l} \text{ where } P \in \mathfrak{R}.$$

Since the poles of ψ do not depend on x, letting $\beta_{i,j}(x) = \operatorname{res}_{\gamma_i(x_0)} \partial_x^j \psi_{l-1}$, Ψ can be written as around a point γ_i with a local variable z_i,

$$\Psi(x, z_i, x_0) = \frac{\boldsymbol{\beta}_i^T(x)\boldsymbol{\alpha}_i(x_0)}{z_i - z_i(\gamma_i(x_0))} + O(1) \text{ where } \boldsymbol{\beta}_i(x) = \big(\beta_{i,0}(x), \ldots, \beta_{i,l-1}(x)\big).$$

From Ψ we can extract the final geometric information, namely, the set $\{\widetilde{u_0}, \widetilde{u_1}, \ldots, \widetilde{u_{l-2}}\}$ of $l-1$ functions from

$$A = (\partial_x \Psi)\Psi^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ k + \widetilde{u_0} & \widetilde{u_1} & \cdots & \widetilde{u_{l-2}} & 0 \end{pmatrix} + O(k^{-1}).$$

In the case of rank $l = 1$, this is

$$A = \partial_x \log \psi = k + O(k^{-1}).$$

When we reconstruct Ψ from the information $\{\widetilde{u_0}, \widetilde{u_1}, \ldots, \widetilde{u_{l-2}}\}$ in the inverse problem, we encounter the ambiguity of the singular part of Ψ. It can be removed by solving the Riemann problem of factoring Ψ into a product of an entire part and a singular part around $k = \infty$:

$$\Psi_0(x, k, x_0) = \big(\sum_{i=0}^{\infty} \Xi_i(x, x_0) k^{-i}\big)\Psi \text{ with } \Xi_0(x, x_0) = \Psi_0(x_0, k, x_0) \equiv \mathrm{id}_{l\times l}\,.$$

Note that $\Psi_0(x, k, x_0)$ is entire in $k \in \mathbb{C}$ and the exponential analogue in a scalar-valued Baker-Akhiezer function $\psi(x, k, x_0) = \exp\big(k(x - x_0)\big)(\sum_{i=0}^{\infty} \xi_i(x) k^{-i})$. Indeed, the vector-valued Baker-Akhiezer function $\boldsymbol{\psi}$ is given by

$$\boldsymbol{\psi} = \big(\sum_{i=0}^{\infty} \boldsymbol{\xi}_i(x, x_0) k^{-i}\big)\Psi_0(x, k, x_0) \text{ with } \boldsymbol{\xi}_0(x, x_0) = (1, 0, \ldots, 0).$$

Moreover,

$$A_0 = (\partial_x \Psi_0)\Psi_0^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ k+u_0 & u_1 & \cdots & u_{l-2} & 0 \end{pmatrix}.$$

In the case of rank $l = 1$, this is

$$A_0 = \partial_x \log \exp(k(x - x_0)) = k.$$

This datum A_0 has a significant geometric meaning for a higher rank: In the case of rank one, a divisor $D = \gamma_1 + \cdots + \gamma_g$ in the inverse problem is enough to construct a line bundle of degree $g + 1$ on $\mathfrak{R}$ by taking $P_0 = (\infty, \infty)$ as another divisor. However, in a higher rank case Tyurin parameters alone are not enough to construct a vector bundle, i.e., a corresponding commutative subring of differential operators. Data combining Tyurin parameters with a set $\{u_0(x), u_1(x), \ldots, u_{l-2}(x)\}$ of "control parameters" are indeed sufficient. These control parameters determine the behavior of a vector bundle of rank l over $\mathfrak{R}$ at the infinity $P_0 = (\infty, \infty)$. That is, Ψ_0 is the transition function in the neighborhood of P_0. The x-dynamics of control parameters in the moduli space of vector bundles will be investigated later.

In the inverse problem, we are given an *algebraic spectral data*

$$\Big(\mathfrak{R}, P_0, \Big\{\gamma_i, \{\alpha_{i,j}\}_{j=0}^{l-1}\Big\}_{i=1}^{lg}, \{u_0(x), u_1(x), \ldots, u_{l-2}(x)\}\Big).$$

From these data, we may construct a unique vector-valued Baker-Akhiezer function $\boldsymbol{\psi} = (\psi_0, \ldots, \psi_{l-1})$ having poles at those points with the conditions $\alpha_{i,j} \operatorname{res}_{\gamma_i} \psi_{l-1} =$

$\operatorname{res}_{\gamma_i} \psi_j$ and a local expression at P_0 is given by

$$\boldsymbol{\psi} = \big(\sum_{i=0}^{\infty} \boldsymbol{\xi}_i(x) k^{-i}\big) \Psi_0(x,k) \text{ with } \boldsymbol{\xi}_0(x) = (1,0,\ldots,0).$$

Let $\mathcal{A}(\mathfrak{R}, P_0)$ be the set of meromorphic functions regular except (possibly) at P_0. Formally we can construct a unique monic differential operator L_f of order ln' in one variable x such that

$$(L_f - f)\boldsymbol{\psi} = O(k^{-1})\Psi_0 \text{ where } \mathsf{ord}_{P_0} f = -n'.$$

Remark that the vector-valued Baker-Akhiezer function $\boldsymbol{\psi}$ is uniquely associated with a given Tyurin parameter by the Riemann-Roch theorem. From this remark we conclude that

$$(L_f - f)\boldsymbol{\psi} \equiv 0.$$

Hence, the vector-valued Baker-Akhiezer function defines an isomorphism from $\mathcal{A}(\mathfrak{R}, P_0)$ to a commutative subring of the ring $\mathcal{D}$ of differential operators

$$\mathcal{A}(\mathfrak{R}, P_0) \overset{\boldsymbol{\psi}}{\cong} \mathcal{A} \subseteq \mathcal{D} \text{ by } f \mapsto L_f.$$

We saw that the commutative subring associated with rank one contains a coprime pair of operators. By the same reasoning, we may conclude that l is the greatest common divisor of the orders of the operators in $\mathcal{A}$. Therefore, we have solved the inverse problem. See Theorem 2.3 in [37] for detailed proofs.

A.2 (n, m)-curve

Let n and m be coprime. An (n, m)-curve [6, 9] is an algebraic curve with a representation for $0 \leq i \leq n-1, 0 \leq j \leq m-1, im + jn < nm$

$$0 = R(x, y) = y^n + x^m + \sum_{i,j} c_{i,j} y^i x^j.$$

For example, a hyper-elliptic curve is a $(2, 2g+1)$-curve: In general, a $(2, 2g+1)$-curve is given by

$$0 = y^2 + x^{2g+1} + \sum_{j=0}^{g} c_j y x^j + \sum_{i=0}^{2g} d_i x^i.$$

A hyper-elliptic curve is a $(2, 2g+1)$-curve with all $c_j = 0$ for $j = 0, \ldots, g$.

Definition A.1. *Let n and m be coprime. The* Weierstrass gap sequence $\mathbf{W}_{n,m}$ *is the set of positive integers which are not representable in the form*

$$an + bm \text{ where } a, b \text{ are non negative integers.}$$

The number of these integers is called the length *of the sequence.*

It is not hard to prove that the length of $\mathbf{W}_{n,m} = \{w_1, \ldots, w_g\}$ is equal to $\frac{(n-1)(m-1)}{2}$ which is the genus of an (n, m)-curve and its maximal element w_g is $2g - 1$ (*p.84* in [6] or *p.561* in [9]). Using the Weierstrass gap sequence, we can find g linearly independent holomorphic differentials on the curve. The g linearly independent holomorphic differentials are given by

$$x^{a_i} y^{b_i} \frac{dx}{R_y} \text{ where } i = 1, \ldots, g$$

where $\{(a_i, b_i)\}_{i=1}^{g}$ is the set of the first g non-gaps, i.e., $a_i n + b_i m \notin \mathbf{W}_{n,m}$ with a_i, b_i non-negative and $R_y(x, y) = \frac{\partial R(x,y)}{\partial y}$.

For instance, the Weierstrass gap sequence $\mathbf{W}_{2,2g+1}$ of a $(2,2g+1)$-curve is

$$\{1,3,5,\ldots,2g-1\}.$$

Thus, g linearly independent holomorphic differentials are given by

$$\frac{dx}{R_y}, x\frac{dx}{R_y}, \ldots, x^{g-1}\frac{dx}{R_y}.$$

In particular, for a hyper-elliptic curve we have $R_y(y,x) = 2y$. Hence,

$$\frac{dx}{2y}, x\frac{dx}{2y}, \ldots, x^{g-1}\frac{dx}{2y}.$$

The Weierstrass gap sequence $\mathbf{W}_{4,3}$ of a $(4,3)$-curve is

$$\{1,2,5\}.$$

Hence, 3 linearly independent holomorphic differentials of a $(4,3)$-curve are given by

$$\frac{dx}{R_y}, x\frac{dx}{R_y}, \text{ and } y\frac{dx}{R_y}.$$

BIBLIOGRAPHY

[1] Enrico Arbarello and Corrado De Concini. Another proof of a conjecture of S. P. Novikov on periods of abelian integrals on Riemann surfaces. *Duke Math. J.*, 54(1):163–178, 1987.

[2] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[3] Michèle Audin. *Spinning tops*, volume 51 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1996. A course on integrable systems.

[4] H. F. Baker. *Abelian functions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995. Abel's theorem and the allied theory of theta functions, Reprint of the 1897 original, With a foreword by Igor Krichever.

[5] Rolf Berndt. *An introduction to symplectic geometry*, volume 26 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated from the 1998 German original by Michael Klucznik.

[6] V. M. Bukhshtaber, D. V. Leĭkin, and V. Z. Ènol′skiĭ. Rational analogues of abelian functions. *Funktsional. Anal. i Prilozhen.*, 33(2):1–15, 95, 1999.

[7] V. M. Bukhshtaber, D. V. Leĭkin, and V. Z. Ènol′skiĭ. σ-functions of (n, s)-curves. *Uspekhi Mat. Nauk*, 54(3(327)):155–156, 1999.

[8] J. L. Burchnall and T. W. Chaundy. Commutative ordinary differential operators. *Pro. London Math. Soc.*, **21**:420–440, 1922.

[9] J. L. Burchnall and T. W. Chaundy. Commutative ordinary differential operators. *Proceedings Royal Soc. London (A)*, 118:557–583, 1928.

[10] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[11] Ron Donagi and Eyal Markman. Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, volume 1620 of *Lecture Notes in Math.*, pages 1–119. Springer, Berlin, 1996.

[12] B. A. Dubrovin and S. P. Novikov. Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg-de Vries equation. *Ž. Èksper. Teoret. Fiz.*, 67(6):2131–2144, 1974.

[13] B. A. Dubrovin and S. P. Novikov. A periodic problem for the Korteweg-de Vries and Sturm-Liouville equations. Their connection with algebraic geometry. *Dokl. Akad. Nauk SSSR*, 219:531–534, 1974.

[14] B. A. Dubrovin. A periodic problem for the Korteweg-de Vries equation in the class of finite band potentials. *Funkcional. Anal. i Priložen.*, 9(3):41–51, 1975.

[15] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov. Nonlinear equations of Korteweg-de Vries type, finite-band linear operators and Abelian varieties. *Uspehi Mat. Nauk*, 31(1(187)):55–136, 1976.

[16] B. A. Dubrovin. Theta-functions and nonlinear equations. *Uspekhi Mat. Nauk*, 36(2(218)):11–80, 1981. With an appendix by I. M. Krichever.

[17] B. A. Dubrovin, I. M. Krichever, and S. P. Novikov. *Topological and algebraic geometry methods in contemporary mathematical physics*, volume 2 of *Classic Reviews in Mathematics and Mathematical Physics*. Cambridge Scientific Publishers, Cambridge, 2004.

[18] J. C. Eilbeck, V. Z. Enolskii, and E. Previato. Varieties of elliptic solitons. *J. Phys. A*, 34(11):2215–2227, 2001. Kowalevski Workshop on Mathematical Methods of Regular Dynamics (Leeds, 2000).

[19] Hershel M. Farkas and Irwin Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1980.

[20] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Translated from the 1977 German original by Bruce Gilligan, Reprint of the 1981 English translation.

[21] Letterio Gatto and Emma Previato. A remark on Griffiths' cohomological interpretation of Lax equations: higher-genus case. *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, 126(3-4):63–70, 1992.

[22] Bert van Geemen and Emma Previato. On the Hitchin system. *Duke Math. J.*, 85(3):659–683, 1996.

[23] William M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984.

[24] William M. Goldman. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. *Invent. Math.*, 85(2):263–302, 1986.

[25] William M. Goldman and Eugene Xia. Rank one Higgs bundles and fundamental groups of Riemann surfaces. *To be appeared.*

[26] Phillip A. Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[27] Phillip A. Griffiths. Linearizing flows and a cohomological interpretation of Lax equations. *Amer. J. Math.*, 107(6):1445–1484 (1986), 1985.

[28] R. C. Gunning. *Lectures on Riemann surfaces.* Princeton Mathematical Notes. Princeton University Press, Princeton, N.J., 1966.

[29] R. C. Gunning. *Lectures on vector bundles over Riemann surfaces.* Princeton Mathematical Notes. Princeton University Press, Princeton, N.J., 1967.

[30] N. J. Hitchin. Monopoles and geodesics. *Comm. Math. Phys.*, 83(4):579–602, 1982.

[31] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[32] N. J. Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54(1):91–114, 1987.

[33] N. J. Hitchin. Gauge theory on Riemann surfaces. In *Lectures on Riemann surfaces (Trieste, 1987)*, pages 99–118. World Sci. Publ., Teaneck, NJ, 1989.

[34] Y. Imayoshi and M. Taniguchi. *An introduction to Teichmüller spaces.* Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors.

[35] I. M. Krichever. Integration of nonlinear equations by the methods of algebraic geometry. *Funkcional. Anal. i Priložen.*, 11(1):15–31, 96, 1977; translation in *Funct. Anal. Appl.* 11:12–26, 1977.

[36] I. M. Krichever. Methods of algebraic geometry in the theory of nonlinear equations. *Uspehi Mat. Nauk*, 32(6(198)):183–208, 287, 1977.

[37] I. M. Krichever. Commutative rings of ordinary linear differential operators. *Funktsional. Anal. i Priložen.*, 12(3):20–31, 96, 1978; translation in *Funct. Anal. Appl.* 12:175–185, 1978.

[38] I. M. Krichever and S. P. Novikov. Holomorphic vector bundles over Riemann surfaces and the Kadomcev-Petviašvili equation. I. *Funktsional. Anal. i Prilozhen.*, 12(4):41–52, 1978.

[39] I. M. Krichever. Rational solutions of the Kadomcev-Petviašvili equation and the integrable systems of N particles on a line. *Funkcional. Anal. i Priložen.*, 12(1):76–78, 1978.

[40] I. M. Krichever and S. P. Novikov. Holomorphic bundles and nonlinear equations: Finite-gap solutions of rank 2. *Dokl. Akad. Nauk SSSR*, 247(1):33–37, 1979.

[41] I. M. Krichever and S. P. Novikov. Holomorphic bundles over algebraic curves, and nonlinear equations. *Uspekhi Mat. Nauk*, 35(6(216)):47–68, 215, 1980.

[42] I. M. Krichever. Elliptic solutions of the Kadomcev-Petviašvili equations, and integrable systems of particles. *Funktsional. Anal. i Prilozhen.*, 14(4):45–54, 95, 1980.

[43] I. M. Krichever. Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations. *Mosc. Math. J.*, 2(4):717–752, 806, 2002. Dedicated to Yuri I. Manin on the occasion of his 65th birthday.

[44] I. M. Krichever. Vector bundles and Lax equations on algebraic curves. *Comm. Math. Phys.*, 229(2):229–269, 2002.

[45] Peter D. Lax. Periodic solutions of the KdV equation. *Comm. Pure Appl. Math.*, 28:141–188, 1975.

[46] Yingchen Li and Motohico Mulase. Hitchin systems and KP equations. *Internat. J. Math.*, 7(2):227–244, 1996.

[47] Giambattista Marini. A geometrical proof of Shiota's theorem on a conjecture of S. P. Novikov. *Compositio Math.*, 111(3):305–322, 1998.

[48] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[49] Motohico Mulase. Category of vector bundles on algebraic curves and infinite-dimensional Grassmannians. *Internat. J. Math.*, 1(3):293–342, 1990.

[50] Motohico Mulase. Algebraic theory of the KP equations. In *Perspectives in mathematical physics*, Conf. Proc. Lecture Notes Math. Phys., III, pages 151–217. Internat. Press, Cambridge, MA, 1994.

[51] D. Mumford. Projective invariants of projective structures and applications. In *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, pages 526–530. Inst. Mittag-Leffler, Djursholm, 1963.

[52] D. Mumford. An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg deVries equation and related nonlinear equation. In *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, pages 115–153, Tokyo, 1978. Kinokuniya Book Store.

[53] M. S. Narasimhan and S. Ramanan. Moduli of vector bundles on a compact Riemann surface. *Ann. of Math. (2)*, 89:14–51, 1969.

[54] S. P. Novikov. A periodic problem for the Korteweg-de Vries equation. I. *Funkcional. Anal. i Priložen.*, 8(3):54–66, 1974.

[55] S. P. Novikov, S. V. Manakov, L. P. Pitaevskiĭ, and V. E. Zakharov. *Theory of solitons*. Contemporary Soviet Mathematics. Consultants Bureau [Plenum], New York, 1984. The inverse scattering method, Translated from the Russian.

[56] Emma Previato. Seventy years of spectral curves: 1923–1993. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, volume 1620 of *Lecture Notes in Math.*, pages 419–481. Springer, Berlin, 1996.

[57] Emma Previato and George Wilson. Vector bundles over curves and solutions of the KP equations. In *Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987)*, volume 49 of *Proc. Sympos. Pure Math.*, pages 553–569. Amer. Math. Soc., Providence, RI, 1989.

[58] A. G. Reyman and M.A. Semenov-Tian-Shansky. Reduction of Hamiltonian systems, affine Lie algebras and Lax equations. II. *Invent. Math.*, 63(3):423–432, 1981.

[59] I. Schur. Uber vertauschbare lineare Differentialausdrücke. *Sitzungsber. Berl. Math. Ges.*, 4:2–8, 1905.

[60] Takahiro Shiota. Characterization of Jacobian varieties in terms of soliton equations. *Invent. Math.*, 83(2):333–382, 1986.

[61] C. L. Siegel. *Topics in complex function theory. Vol. I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Elliptic functions and uniformization theory, Translated from the German by A. Shenitzer and D. Solitar, With a preface by Wilhelm Magnus, Reprint of the 1969 edition, A Wiley-Interscience Publication.

[62] C. L. Siegel. *Topics in complex function theory. Vol. II.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Automorphic functions and abelian integrals, Translated from the German by A. Shenitzer and M. Tretkoff, With a preface by Wilhelm Magnus, Reprint of the 1971 edition, A Wiley-Interscience Publication.

[63] C. L. Siegel. *Topics in complex function theory. Vol. III.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1989. Abelian functions and modular functions of several variables, Translated from the German by E. Gottschling and M. Tretkoff, With a preface by Wilhelm Magnus, Reprint of the 1973 original, A Wiley-Interscience Publication.

[64] A. N. Tyurin. The classification of vector bundles over an algebraic curve of arbitrary genus. *Izv. Akad. Nauk SSSR Ser. Mat.*, 29:657–688, 1965; translation in *Am. Math. Soc., Translat., II. Ser.*, 63:245–279 (1967)

[65] A. N. Tyurin. Classification of n-dimensional vector bundles over an algebraic curve of arbitrary genus. *Izv. Akad. Nauk SSSR Ser. Mat.*, 30:1353–1366, 1966; translation in *Am. Math. Soc., Translat., II. Ser.*, 73:196–211 (1968)

[66] Andre Weil. Généralisation des fonctions abeliennes. *J. Math. Pures Appl.*, 17(9):47–87, 1938.

[67] Scott Wolpert. An elementary formula for the Fenchel-Nielsen twist. *Comment. Math. Helv.*, 56(1):132–135, 1981.

[68] Scott Wolpert. The Fenchel-Nielsen deformation. *Ann. of Math. (2)*, 115(3):501–528, 1982.

[69] Scott Wolpert. On the symplectic geometry of deformations of a hyperbolic surface. *Ann. of Math. (2)*, 117(2):207–234, 1983.