Relay Placement Approximation Algorithms for k-Connectivity in Wireless Sensor Networks

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Relay Placement Approximation Algorithms for k-Connectivity in Wireless Sensor Networks*

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***Abstract*— Sensors typically use wireless transmitters to communicate with each other. However, sensors may be located in a way that they cannot even form a connected network (e.g, due to failures of some sensors, or loss of battery power). In this paper we consider the problem of adding the smallest number of additional (relay) nodes so that the induced communication graph is k-connected[1]. The problem is NP-hard. We develop algorithms that find close to optimal solutions for both edge and vertex connectivity. For k-connectivity between sensor nodes, we prove the algorithms have an approximation ratio of $O(k^2)$ for vertex connectivity and $O(k)$ for edge connectivity. In addition, our methods extend with the same approximation guarantees to a generalization when the locations of relays are required to avoid certain polygonal regions (obstacles).**

We prove that the algorithms for k-connectivity between sensor and relay nodes have an approximation ratio of $O(k^3)$ for vertex connectivity and $O(k^2)$ for edge connectivity.

I. Introduction

A wireless sensor network is a group of sensor nodes with sensing, processing and communication capabilities, deployed to achieve a certain objective, Akyildiz et al. [1]. Typical applications of sensor networks are habitat monitoring, environmental monitoring, object tracking, etc. Sensor networks may exist in harsh network conditions, thus the network must be designed so that failure of some sensor nodes or some communication links between them does not disrupt the network. We consider the problem of forming a fault-tolerant sensor network topology.

We define fault-tolerance as the existence of multiple internally vertex-disjoint (or edge-disjoint) paths between each pair of terminal nodes. If k vertex (edge) disjoint paths exist between each pair of nodes, the network is said to be k-vertex (edge) connected. A k vertex (edge) connected graph has the property that the failure of any set of $(k-1)$ nodes (edges) cannot disconnect the network. We also consider the problem where fault-tolerance is desired between both terminal and relay nodes. We call this objective as **full k-connectivity**, and the objective of achieving k-connectivity among terminal nodes as **partial k-connectivity**.

[1]We consider both edge and vertex connectivity. We consider k-connectivity between sensor nodes, as well as between sensor and relay nodes.

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Sensor nodes have very limited energy. Thus, they transmit at low power levels, and have a limited transmission range. We assume a fixed transmission range for each sensor node. It may not be feasible to construct even a connected topology among the sensor nodes due to their short transmission range and potential large area deployments. We propose the use of additional relay nodes, whose position we can control, to achieve the desired level of connectivity (number of vertex or edge disjoint paths) among the sensor nodes. The relay nodes are cheaper than sensor nodes as they do not have any sensing capabilities. We assume they have the same communication capabilities as the sensor nodes.

There has been recent work in topology control of sensor networks. Bredin et al. [2] present an $O(k^4)$-approximation algorithm for achieving full k-vertex connectivity using minimum number of relays for nodes distributed in the Euclidean plane. Their algorithms use a similar approach as ours and their analysis for the component that achieves partial k-vertex connectivity gives an approximation ratio of $O(k^3)$. We analyze our algorithms and prove ratios of $O(k^2)$ and $O(k^3)$ for partial and full k-connectivity respectively. This is an $O(k)$ improvement on previous bounds, and we give algorithms for both edge and vertex connectivity, for both partial and full k-connectivity. We provided algorithms for partial k-connectivity in [3], and proved the approximation ratio to be 10 for edge and vertex connectivity for $k = 2$, for nodes in the Euclidean plane. The approximation ratio proved in this paper is consistent with Kashyap et al. [3] for 2-vertex connectivity (see Section III). We provided an analysis of the algorithms for partial 2-connectivity among nodes in higher dimensional metric spaces in [4], and proved the approximation bounds to be $2M$, where M (MST number) is the maximum node degree in a minimum degree MST in the space. The MST number is 5 for the Euclidean plane (Monma and Suri [5]), 13 for the 3-dimensional Euclidean space, and 4 for the rectilinear plane (Robins and Salowe [6]). We prove that the algorithm for k-edge connectivity is an $O(Mk)$-approximation for partial connectivity, and $O(Mk^2)$-approximation for full-connectivity. The approximation ratio proved in this paper is consistent with Kashyap et al. [3], [4] for 2-edge connectivity (see Section IV).

Hao et al. [7] consider the problem of placing the minimum number of backbone nodes (relays) among a set of candidate locations such that each sensor node has paths to at least two backbone nodes, and the backbone nodes have at least two vertex-disjoint paths between them. They provide an approximation algorithm having an $O(D \log n)$ approximation ratio, where D depends on the diameter of the network and n is the number of sensor nodes in the network. Liu et al. [8] consider the problem of placing relays in a network of sensor nodes so that the network is 2-connected. They provide a $(6+\epsilon)$-approximation algorithm for connectivity and two approximation algorithms for 2-connectivity with ratios $(24 + \epsilon)$ and $(6/T + 12 + \epsilon)$, where T is the ratio of relays needed for connectivity to the number of sensor nodes. Their problem is different from ours as they want the set of relays to be a dominating set among the sensor nodes, i.e., each sensor node should be directly connected to at least one relay node.

The nodes on which a fault tolerant topology is desired are referred to as terminal nodes in the rest of the paper. The problem of constructing a connected network on terminal nodes using a minimum number of relay nodes has been considered in Lin et al. [9], Măndoiu and Zelikovsky [10] and Chen et al. [11]. Lin et al. [9] showed the problem to be NP-Hard and proposed an approximation algorithm for constructing a tree using relay nodes. They showed algorithm to be a 5-approximation. The algorithm restricts the placement of relay nodes on lines joining pairs of terminal nodes. It then assigns a weight function to each pair of terminal nodes according to the number of relay nodes needed to connect them directly. They find a minimum spanning tree (MST) on this graph. Proofs of 4-approximation ratio for the algorithm are provided by Măndoiu and Zelikovsky [10] and Chen et al. [11], and the bound is proved to be tight. Măndoiu and Zelikovsky [10] prove the approximation ratio to be $M-1$ for nodes distributed in higher dimension metric spaces. Chen et al. [11] also provide a 3-approximation algorithm for the problem. Cheng et al. [12] provide a 2.5-approximation randomized algorithm for placement of relay nodes to connect a given set of terminal nodes.

We consider the problem of providing k-(edge, vertex) connectivity for $k \geq 2$ among terminal nodes using minimum number of relay nodes. The contributions of this paper are as follows: (1) we prove the algorithms of Kashyap et al. [3] to be $O(k^2)$-approximation with respect to the optimal for achieving partial k-vertex connectivity among terminals distributed in the Euclidean plane; (2) we prove the algorithms of Kashyap et al. [4] to be $O(Mk)$-approximation with respect to the optimal for achieving partial k-edge connectivity among terminals distributed in a metric space of MST number M ($M = 5$ for Euclidean plane); (3) we extend our algorithms to the generalization where the relays cannot be placed in certain polygonal regions and show the same approximation ratios hold for this generalization as well; (4) we provide an analysis of the algorithm of Bredin et al. [2] for full k-vertex connectivity, and provide $O(k^3)$ bounds; (5) we extend our partial k-edge connectivity algorithm to provide an $O(Mk^2)$-approximation full k-edge connectivity algorithm for terminals distributed in a metric space of MST number M.

The paper is organized as follows: Section II gives the network model and problem statement. Section III describes the partial k-vertex connectivity approximation algorithm, and gives the proof of its approximation ratio. Section IV describes the algorithm for achieving partial k-edge connectivity and gives the proof of its approximation ratio. Section V extends the algorithms to work with the same approximation ratio for the generalization where relays cannot be placed in certain polygonal regions of the network. Section VI analyzes the algorithm of Bredin et al. [2] for full k-vertex connectivity, to give an improved approximation ratio. It also describes the full k-edge connectivity approximation algorithm. Section VII concludes the paper.

II. Network Model and Problem Statement

We model the network as a graph $G = (V, E)$, where V is the set of sensor nodes, which we call terminal nodes, and E is the set of links between them. We assume each node has a limited transmission range, which we normalize to one. It is assumed that a node can connect to all nodes within its transmission range. A link $e = (x, y)$ belongs to E if nodes x and y are within unit distance of each other. The links can be either omnidirectional RF, directional RF or Free Space Optical (without obscuration).

We assume we have relay nodes that are identical to the terminal nodes in terms of their transmission range and type of links. We assume we have control over the location of relay nodes. Thus, we place the relay nodes in the network so that the desired level of connectivity is achieved. We consider two objectives: one of achieving the desired connectivity between terminals, and the other of achieving it for both terminals and relays. The problems can be stated as follows:

Partial k-connectivity: Given a graph $G = (V, E)$, find the minimum number of relay nodes (denoted by set R) needed (and their locations) such that the set of nodes V is k-edge (vertex) connected ($k \geq 2$) in the resulting graph $G' = (V + R, E'), E \subseteq E'$. The objective is to construct a graph such that $\forall\, u, v \in V$, $\lambda(u, v) \geq k$; where $\lambda(u, v)$ is the number of edge-disjoint (or internally vertex-disjoint) paths between u and v in G'.

Full k-connectivity: Given a graph $G = (V, E)$, find the minimum number of relay nodes (denoted by set R) needed (and their locations) such that the set of nodes $V + R$ is k-edge (vertex) connected ($k \geq 2$) in the resulting graph $G' = (V + R, E'), E \subseteq E'$.

III. Algorithm for k-vertex connectivity

We use the algorithm of Kashyap et al. [3] for achieving k-vertex connectivity among terminal nodes using relays. To connect two terminal nodes outside each other's transmission range, the relay nodes are placed on the straight line connecting the two nodes. The algorithm proceeds by forming a complete graph G_c on the terminal nodes. Equation 1 gives the weight function used for the edges, where $|e|$ is the length of an edge. The weight represents the number of relay nodes required to form an edge. We do not allow the relay nodes to have edges other than the ones required to form the edge they are placed on. Then we compute an approximate minimum cost spanning k-vertex connected subgraph (G'_c) of G_c.

$$c_e = \lceil |e| \rceil - 1 \tag{1}$$

The problem of finding the minimum cost spanning k-vertex connected subgraph of a graph is NP-Hard (Gary and Johnson [13]. Thus, we use the 2-approximation algorithm of Khuller and Raghavachari [14] for $k = 2$, and the k-approximation algorithm of Kortsarz and Nutov [15] for $k > 2$. The algorithm takes $O(k^2 n^3 m)$ time, where n is the number of terminals and m is the number of edges in the graph (which is $n(n-1)$ for a complete graph, as in our case). For $k \leq 7$, we can use the improved approximation algorithms proposed by Auletta et al. [16] and Dinitz and Nutov [17]. It is worth noting that the weight function of Equation 1 is not a metric as it does not satisfy triangle inequality. In the resulting k-vertex connected subgraph, the relay nodes are placed to form the edges (of length greater than one) of the subgraph. We later prove that this algorithm has an approximation ratio of $O(k^2)$. The solution is then improved by removing some relays. The relays are allowed to form edges with all nodes in their transmission range and sequentially removed if k-vertex connectivity is preserved. We call this step the *sequential removal step*. Algorithm 1 describes the algorithm.

Algorithm 1 Relay placement for k-vertex connectivity

```
1: Construct a complete $G_c = (V, E_c)$ by adding an edge
   between each pair of vertices of graph $G$.
2: Weight the edges of the graph as follows. $|e|$ represents
   the length of edge $e$.
       $c_e = \lceil |e| \rceil - 1$
3: Compute an approximate minimum cost spanning $k$-vertex
   connected subgraph from this graph $G_c$. Let the resulting
   graph be $G'_c$.
4: Place relay nodes (number equal to the weight of the edge)
   on the edges in $G'_c$ with link costs greater than zero.
5: For all pairs of nodes (including the relay nodes) in $G'_c$
   within each other's transmission range, form an edge.
6: For the relay nodes sorted arbitrarily, do the following
   (starting at $i = 1$):
     • Remove node $i$ (and all adjacent edges).
     • Check for $k$-vertex connectivity between the terminals.
     • If the graph is $k$-vertex connected, repeat for $i = i+1$,
       else put back the node $i$ and corresponding edges, and
       repeat for $i = i + 1$.
     • Stop when all relay nodes have been considered.
7: Output the resulting graph.
```

A. Proof of Approximation Ratio

We now analyze the algorithm to provide with approximation guarantees. We provide the analysis for terminals distributed in the Euclidean plane.

We start with some notation. Let $\mathcal{T}$ be the set of terminals, and $\mathcal{S}$ be the set of optimally placed Steiner nodes (relay nodes) needed to achieve k-vertex connectivity among the terminal nodes. Let s be the number of Steiner nodes needed when we place them optimally, i.e, $s = |\mathcal{S}|$. In the proof, we will call the relay nodes placed on straight lines between terminals (as in our algorithm) beads and the optimally placed relay nodes Steiner nodes.

As a recap of our algorithm, it forms a k-vertex connected network among the terminal nodes by placing additional links between them, and if two terminal nodes are more than unit distance apart, it adds beads (relay nodes) to form that link. When we add such a link of length l, it consists of $\lceil l \rceil - 1$ beads.

We first prove the following lemma, and then present the main result of this section.

Lemma 3.1: A network that is k-vertex connected on terminal nodes using the minimum number of beads contains at most $(3\lceil k/2 \rceil(\lceil k/2 \rceil + 1) - 1)s$ beads, where s is the minimum number of Steiner nodes needed.

Proof: Let $G_0 = (V_0, E_0)$ be the optimal k-vertex connected network on terminals (having the minimum number of Steiner nodes).

We follow the procedure of Algorithm 2 to construct a k-vertex connected network that has beads and no Steiner nodes. We will prove that this network does not contain more than $(3\lceil k/2 \rceil(\lceil k/2 \rceil + 1) - 1)s$ beads.

Algorithm 2 starts by finding the connected components (SC_i) of Steiner nodes in the graph constructed on the Steiner nodes. It constructs a minimum-degree minimum spanning tree (MST) on Steiner nodes for each connected component, starting with any Steiner node in that component as the root. Let the trees be $ST_1, .., ST_m$. The algorithm then removes Steiner nodes of a connected component SC_j from G_{j-1} and adds beads between the terminals connected to those Steiner nodes to get G_j, which is also k-vertex connected between terminal nodes (Step 4). The process is repeated for all connected components, and the resulting graph has zero Steiner nodes and is k-vertex connected on the terminals.

Let us now explain the procedure to construct G_j from G_{j-1} by adding beads between the terminal nodes and removing Steiner nodes. Consider the graph formed by the Steiner nodes in ST_j and the terminal nodes within the transmission range of these Steiner nodes. Denote this graph by H_j. We form a graph on terminals that is similar to a *Harary graph*, in which we form a cycle between the terminal nodes using beaded (and direct) links between the terminals contained in H_j in G_{j-1} and delete the Steiner nodes of ST_j to get G_j. Then, we connect each terminal node to the preceding $\lceil k/2 \rceil$ terminals on the cycle and the successive $\lceil k/2 \rceil$ terminals on the cycle. If the total number of terminal nodes in H_j is less than $k+1$, we form a complete graph.

We now prove that the graph G_m constructed using the procedure described above is k-vertex connected on the terminals. The proof is based on mathematical induction and is similar

Algorithm 2 Construction of k-vertex connected network with beads

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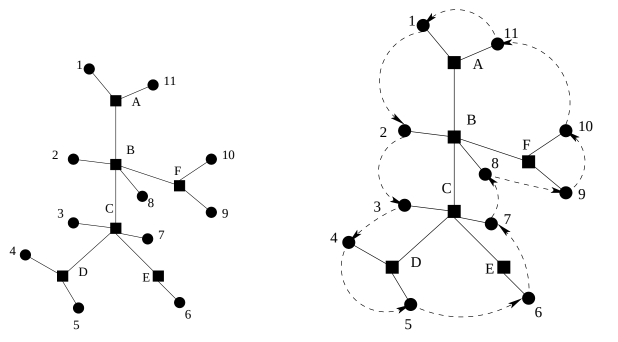
1: Define a graph $G_S = (S, E_S)$ on the Steiner nodes, where an edge $(u, v)$ is in $E_S$ if it is an edge between the Steiner nodes $u, v$ in $G_0$.
2: Find all the connected components ($SC_i$) in $G_S$.
3: Construct a minimum-degree MST in each connected component, and call the trees $ST_1, .., ST_m$.
4: Set $j = 1$. While $j \leq m$:
    1) Remove the Steiner nodes contained in $ST_j$ from $G_{j-1}$.
    2) Add beads between terminals to get the graph $G_j$, which is also $k$-vertex connected on the terminals. The procedure for adding beads and removing Steiner nodes is explained later.
    3) Set $j = j + 1$.
5: Output the resulting graph $G_m$.
```

to the proof of k-vertex connectivity in Bredin et al. [2].

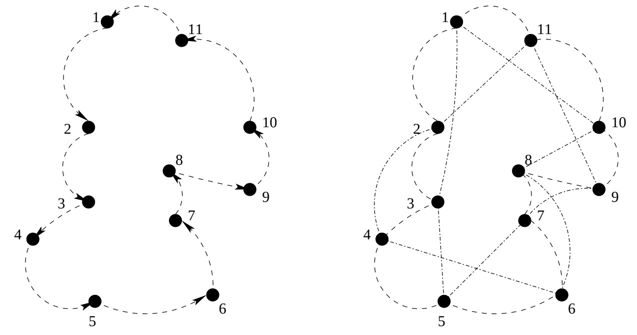
G_0 is the optimal Steiner graph, that is k-vertex connected on the terminals. Let G_{i-1} be k-vertex connected on the terminals. Thus, removal of any set C of $k-1$ vertices does not disconnect the terminals in G_{i-1}. We prove by contradiction that all terminals are connected in $G_i - C$ as well. Let u and v be the two terminals which are disconnected in $G_i - C$. All terminal pairs (u, v) have a path in $G_{i-1} - C$. If the path does not use more than one terminal connected to component SC_i, u and v are connected in $G_i - C$ as well. If the path uses at least two terminal vertices connected to SC_i (u_1, v_1 being the first and last terminals connected to SC_i on the path), that path exists as well if there are at least $k+1$ terminals connected to SC_i, since we form a Harary graph (that is k-vertex connected) between all terminals connected to SC_i. If there are less than $k+1$ terminals (which will be u_1, v_1), a direct edge exists between them (since we formed a complete graph in that case) and thus a path exists between u and v in $G_i - C$. Thus, G_i is k-vertex connected on terminals. Therefore, by induction, G_m is k-vertex connected on terminals.

We now describe the procedure of constructing the Harary graph on terminals connected to each Steiner component SC_j. Algorithm 3 describes the algorithm for construction of the k-vertex connected graph between terminal nodes connected to the Steiner nodes of ST_j. The algorithm works as follows: Start at the root of ST_j (call the root st_1, dropping subscript j for simplicity). Connect to st_1 all terminal nodes within its transmission range, and mark them. Let the set of marked terminal nodes be $\{t_1, .., t_l\}$. Start a Depth First Search (DFS) traversal of the tree formed by $ST_j \cup \{t_1, .., t_l\}$ (rooted at st_1), starting with any child of st_1. The children (both Steiner nodes and terminals) of a node are traversed in an anti-clockwise manner, i.e., the next child to traverse is the first child encountered in an anti-clockwise sweep around the Steiner node, starting from the last child traversed. If no child of the Steiner node has been traversed yet, the child traversed



(a) Tree on Steiner and terminal nodes

(b) Depth first traversal and cycle creation



(c) Cycle after removal of Steiner nodes

(d) Constructed Harary graph

Fig. 1. Example for removal of Steiner nodes and addition of beads for $k = 3$

is the one encountered in the sweep starting from the parent node. Whenever a new Steiner node st_j is encountered in the traversal, mark all unmarked terminal nodes in the Steiner node's transmission range and connect them to it (thus l increases at this step). Figure 1(a) shows an example tree constructed using this procedure for $k = 3$. While doing the DFS traversal, add required number of beads to form a link between each terminal with the next terminal encountered in the DFS traversal. Complete the cycle by connecting the last added terminal to the first terminal encountered in the DFS traversal[2]. Figure 1(b) shows the cycle created between the terminal nodes in the example, starting at terminal 1. Remove the Steiner nodes. Form the Harary graph by connecting each terminal to preceding and successive $\lceil k/2 \rceil$ vertices on the cycle, forming a complete graph if there are less than $k+1$ terminals. The edges longer than unit length are added using the required number of beads. Figure 1(c) shows the constructed cycle after removal of Steiner nodes, and Figure 1(d) shows the final topology (beaded Harary graph) on these terminals nodes.

It has been proved in Kashyap et al. [3] that for terminals distributed in the Euclidean plane, the cycle constructed in step 6 of Algorithm 3 contains maximum $5s_j$ beads for each

[2]Note that there will be at least two terminals connected to the Steiner nodes of ST_j. If there were only one terminal node, the Steiner nodes of ST_j could be deleted from the optimal Steiner graph without affecting the connectivity. In case of two terminal nodes, adding one edge between the two makes it a complete graph, which suffices, as shown before.

Algorithm 3 Removal of Steiner nodes and addition of beads in ST_j

```
1: Start at root $st_1$ of $ST_j$.
2: Connect to it all terminals within its transmission range, and mark them.
3: Construct a tree $T_j$, with the vertex set as the Steiner nodes in $ST_j$ and a leaf vertex corresponding to each marked terminal vertex. The edges are the edges of $ST_j$ and the edges between each Steiner node and the marked terminal vertices connected to it.
4: Do a Depth First Search (DFS) traversal of $T_j$ rooted at $st_1$, starting with any child of $st_1$. For each node, traverse its children in an anti-clockwise manner.
5: Each time a new Steiner node $st_i$ is encountered, connect it to all unmarked terminal vertices in its range, and mark them. Update $T_j$ by adding these terminal vertices, and continue DFS traversal around $st_i$ from the edge between $st_i$ and its parent.
6: Connect all the terminal vertices in order of their DFS traversal and complete the cycle between them.
7: Connect each vertex to preceding and successive $\lceil k/2 \rceil$ vertices on the cycle. Form a complete graph if there are less than $k+1$ terminals.
8: Add beads to all added edges of length greater than one.
9: Add the newly added edges to $G_{j-1}$, and remove the Steiner nodes of $ST_j$ and all incident edges from $G_{j-1}$. The resulting graph is $G_j$.
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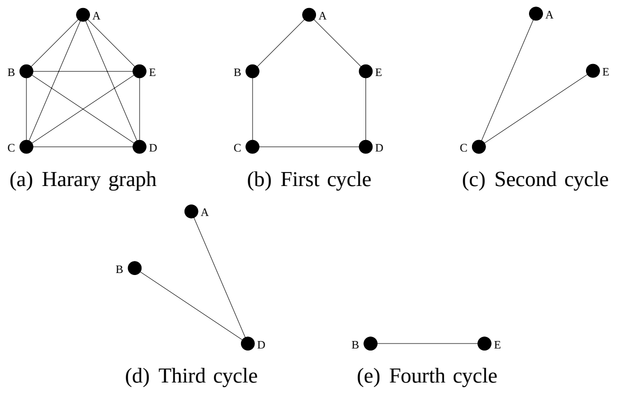

Fig. 2. Example for decomposition of a Harary graph for $k = 3$

Steiner component ST_j with s_j Steiner nodes. The analysis only uses the property that the terminals are connected to ST_j in the Steiner solution. Thus, this result holds for any subset of terminals connected to ST_j, as long as the cycle is constructed in an anti-clockwise manner.

Let the number of terminal nodes in H_j (connected to the Steiner component in consideration) be N. We consider the case $N \geq k+1$, so that we can construct the Harary graph[3]. We decompose the constructed Harary graph into complete and incomplete cycles (we call all of them cycles), and use the result of Kashyap et al. [3] to compute its cost. Let us explain it with an example of Figure 2. Figure 2(a) shows the Harary graph constructed for $k = 3$, with each terminal connecting to preceding and successive two nodes on the cycle. We decompose the graph into multiple cycles as follows:

- **Type I cycle**: First cycle is the cycle formed between all the terminals, as constructed in steps 1-6 of Algorithm 3.
- **Type II cycles**: We now consider the edges needed to connect nodes with preceding and successive nodes i hops away (number of edges between the nodes in the Type I cycle) on the Type I cycle into multiple cycles. We start with any terminal node (node A in the example), and form a complete or incomplete cycle by starting with the node and traversing edges that connect nodes i hops away, in an anti-clockwise manner. The cycle ends before or at the node we started at. Figure 2(c) shows the constructed cycle for $i = 2$. We repeat the procedure for the $i-1$ nodes successive to the node we started at (node B in the example, for $i = 2$), obtaining one complete or incomplete cycle in each case. Figure 2(d) shows this cycle for the example. Each cycle contains $\lfloor N/i \rfloor$ edges, and there are i Type II cycles. Each node is connected to one preceding node and one successive node i hops away. Thus, N edges are required to connect all nodes with neighbors i hops away on the Type I cycle. The total edges covered by Type II cycles is $i\lfloor N/i \rfloor$. Thus, to cover the $N - i\lfloor N/i \rfloor$ uncovered edges, we form Type III cycles, which are just single edges.
- **Type III cycles**: These cycles are single edges, each pertaining to one of the $N - i\lfloor N/i \rfloor$ uncovered edges. Figure 2(e) shows the cycle for the example.

There is one Type I cycle in the Harary graph, and i Type II and $N - i\lfloor N/i \rfloor$ Type III cycles for each $i = 2, 3, .., \lceil k/2 \rceil$. These cycles cover all the edges in the Harary graph, and thus the number of beads needed for these cycles is the same as needed for the Harary graph. The Type I cycle uses at most $5s_j$ beads. The following lemma bounds the number of beads needed for the Type II edges.

Lemma 3.2: A Type II cycle constructed from edges connecting nodes i hops apart requires at most $5s_j$ beads.

Proof: Let the set of terminals in the Harary graph be $\mathcal{T}_j$. Let the set of terminals in the Type II cycle be $\mathcal{T}'_j \subset \mathcal{T}_j$. Consider another instance of the problem, in which only the terminals of $\mathcal{T}'_j$ are connected to the Steiner node MST ST_j. Follow steps 1-6 of Algorithm 3 on this instance to form a cycle. This cycle has at most $5s_j$ beads, Kashyap et al. [3]. The only difference between this instance and the original instance is that the terminals $\mathcal{T}_j \backslash \mathcal{T}'_j$ have been removed. The order of children traversal in the DFS traversal is anti-clockwise. Removing the terminals $\mathcal{T}_j \backslash \mathcal{T}'_j$ does not change the order in which the terminals $\mathcal{T}'_j$ are encountered (compared to the original instance). Thus, the cycle constructed cycle is the same as the Type II cycle in consideration (or has one extra edge if the Type II cycle is not complete). Thus, the Type II

[3]Else, we construct a complete graph, which is a subset of the set of edges in the Harary graph (since $N < k+1$). Thus, the analysis for Harary graph is an upper bound for this case.

cycle has at most $5s_j$ beads. ■

Now, we consider the Type III cycles. Each Type III cycle is just an edge. Thus, the required number of beads is at most the number of Steiner nodes in the DFS path between the end-terminals of this edge. Thus, a Type III edge requires at most s_j beads.

Thus, the total number of beads (b_j) required by the Harary graph is as given in Equation 2.

$$\begin{aligned} b_j &\leq 5s_j + \sum_{i=2}^{\lceil k/2 \rceil} (5i + N - i\lfloor N/i \rfloor)s_j \\ &= 5s_j + \sum_{i=2}^{\lceil k/2 \rceil} (5i + i(N/i - \lfloor N/i \rfloor))s_j \\ &\leq 5s_j + \sum_{i=2}^{\lceil k/2 \rceil} (6i)s_j \\ &= (6\sum_{i=1}^{\lceil k/2 \rceil} i - 1)s_j \\ &= (3\lceil k/2 \rceil(\lceil k/2 \rceil + 1) - 1)s_j \qquad (2) \end{aligned}$$

Since the Steiner components SC_j do not have common Steiner nodes, the number of beads is at most $(3\lceil k/2 \rceil(\lceil k/2 \rceil + 1) - 1)s$. Thus, a solution with minimum number of beads requires at most $(3\lceil k/2 \rceil(\lceil k/2 \rceil + 1) - 1)s$ beads. ■

Theorem 3.3 states the main result of this section.

Theorem 3.3: If the optimal network uses s Steiner nodes so that terminals distributed in the Euclidean plane are k-vertex connected, Algorithm 1 forms a network with maximum of $c(3\lceil k/2 \rceil(\lceil k/2 \rceil + 1) - 1)s$ beads and zero Steiner nodes, in which the terminal nodes are k-vertex connected.

Proof: The algorithm for finding a k-vertex connected subgraph is a c-approximation for finding the minimum cost k-vertex connected subgraph (cost of each edge being number of beads required to form it). Thus, according to Lemma 3.1, the number of beads required is at most $c(3\lceil k/2 \rceil(\lceil k/2 \rceil + 1) - 1)s$. The last step of Algorithm 1 (sequential removal step) removes beads from the network by allowing them to connect to all nodes within the transmission range, so the resulting network after sequential removal also has maximum of $c(3\lceil k/2 \rceil(\lceil k/2 \rceil + 1) - 1)s$ relay nodes. ■

For $k = 2$, c is 2, and the algorithm is 10-approximation, which is consistent with the results of Kashyap et al. [3].

IV. Algorithm for k-edge connectivity

We follow the algorithm proposed by Kashyap et al. [4]. To connect two terminal nodes outside each other's transmission range, the relay nodes are placed on the straight line connecting the two nodes. The algorithm proceeds by forming a multi-graph G_c on the terminal nodes. There are k edges between each pair of terminal nodes in G_c. We use the weight function of Equation 1 to weight the edges. We do not allow the relay nodes to have edges other than the ones required to form the edge they are placed on. Then we compute an approximate minimum cost spanning k-edge connected subgraph (G'_c) of the multi-graph G_c.

The problem of finding the minimum cost spanning k-edge connected subgraph of a graph is NP-Hard (Gary and Johnson [13]). Thus, we use an approximation algorithm for the problem, proposed by Khuller and Vishkin [18]. The algorithm achieves an approximation ratio of 2 for the problem, and takes $O((kn)^2)$ time for a graph with n nodes. The algorithm uses the matroid intersection based algorithm of Gabow [19], which finds k edge-disjoint spanning trees from a root vertex in a directed graph. It is worth noting that the weight function of Equation 1 is not a metric as it does not satisfy triangle inequality. Thus, the approximation algorithm of Khuller and Vishkin [18] is the best known for the problem. In the resulting subgraph from the approximation algorithm of Khuller and Vishkin [18], the relay nodes are placed to form the links (of length greater than one) of the subgraph. In the next section, we prove that this algorithm has an approximation ratio of $2M\lceil k/2 \rceil s$. The solution is then improved by removing some relays. The relays are allowed to form edges with all nodes in their transmission range and sequentially removed if k-edge connectivity is preserved. We call this step the *sequential removal step*, and it takes $O(n'((n + n')m))$ time, where n' is the number of relays before the sequential removal step, and m is the number of edges in the network formed by the terminals and relays. Thus, the first part of the algorithm takes $O((kn)^2)$ time, while the complete algorithm takes $O((kn)^2 + n'm(n + n'))$ time. Algorithm 4 describes the algorithm. For a network in a cuboid of length L, the maximum number of relays on any edge in G_c is $O(L)$, and the number of edges in the graph at the output of Step 3 of Algorithm 4 (G'_c) is $k(n-1)$, thus, $n' = O(knL)$ and $m = O((knL)^2)$. Therefore, the algorithm takes $O((knL)^4)$ time.

A. Proof of Approximation Ratio

We now analyze the algorithm to provide with approximation guarantees. Let the terminal nodes be placed in any metric space with MST number M [6]. MST number is defined as the maximum node degree in a minimum-degree Minimum Spanning Tree (MST) spanning points from the space. The MST number for the Euclidian plane is 5 [5], three-dimensional Euclidian space is 12, and rectilinear plane (two-dimensional space with metric defined by L_1 norm) is 4. The approximation ratio for the MST based algorithm of [9] for connecting terminals using minimum relays has been shown to be $M - 1$ in [10]. We prove that the algorithm is a $2M\lceil k/2 \rceil$-approximation.

We start with some notation. Let $\mathcal{T}$ be the set of terminals, and $\mathcal{S}$ be the set of optimally placed Steiner nodes (relay nodes) needed to achieve k-edge connectivity among the terminal nodes. Let s be the number of Steiner nodes needed when we place them optimally, i.e, $s = |\mathcal{S}|$. In the proof, we will call the relay nodes placed on straight lines between terminals (as in our algorithm) beads and the optimally placed relay nodes Steiner nodes.

Algorithm 4 Relay placement for k-edge connectivity

```
1: Make a multi-graph $G_c = (V, E_c)$ by adding $k$ edges between each pair of vertices of graph $G$.
2: Weight the edges of the graph as follows. $|e|$ represents the length of edge $e$.
       $c_e = \lceil |e| \rceil - 1$
3: Compute an approximate minimum cost spanning $k$-edge connected subgraph from this graph $G_c$ using the approximation algorithm proposed by Khuller and Vishkin [18]. Let the resulting graph be $G'_c$.
4: Place relay nodes (number equal to the weight of the edge) on the edges in $G'_c$ with link costs greater than zero.
5: For all pairs of nodes (including the relay nodes) in $G'_c$ within each other's transmission range, form an edge.
6: For the relay nodes sorted arbitrarily, do the following (starting at $i = 1$):
     • Remove node $i$ (and all adjacent edges).
     • Check for $k$-edge connectivity between the terminals.
     • If the graph is $k$-edge connected, repeat for $i = i+1$, else put back the node $i$ and corresponding edges, and repeat for $i = i + 1$.
     • Stop when all relay nodes have been considered.
7: Output the resulting graph.
```

As a recap of our algorithm, it forms a k-edge connected network among the terminal nodes by placing additional links between them, and if two terminal nodes are more than unit distance apart, it adds beads (relay nodes) to form that link. When we add such a link of length l, it consists of $\lceil l \rceil - 1$ beads.

We first prove the following lemma, and then the main result of this section.

Lemma 4.1: A k-edge connected network on terminal nodes using minimum number of beads contains at most $M\lceil k/2 \rceil s$ beads, where s is the minimum number of Steiner nodes needed.

Proof: Let $G_0 = (V_0, E_0)$ be the optimal k-edge connected network on terminals (having the minimum number of Steiner nodes).

We follow a slightly modification of Algorithm 2 to construct a k-edge connected network that has beads and no Steiner nodes. We will prove that this network does not contain more than Mk beads.

The algorithm starts by finding the connected components (SC_i) of Steiner nodes in the graph constructed on the Steiner nodes. It constructs a Breadth First Search (BFS) spanning tree on Steiner nodes for each connected component, starting with any Steiner node in that component as the root. Let the trees be $ST_1, .., ST_m$. The algorithm then removes Steiner nodes of a connected component SC_j from G_{j-1} and adds beads between the terminals connected to those Steiner nodes to get G_j which is also k-edge connected between terminal nodes (Step 4). The process is repeated for all connected components, and the resulting graph has zero Steiner nodes and is k-edge connected on the terminals.

Let us now explain the procedure to construct G_j from G_{j-1} by adding beads between the terminal nodes and removing Steiner nodes. Consider the graph formed by the Steiner nodes in ST_j and the terminal nodes within the transmission range of these Steiner nodes. Denote this graph by H_j. We form a cycle among the terminals in H_j, and replicate the edges to have $\lceil k/2 \rceil$ copies of each. We use beads to form the edges longer than unit length. The terminals in H_j are k-edge connected since deleting any set of $k-1$ edges does not disconnect the terminals from each other. This procedure maintains k-edge connectivity between the terminal nodes that were k-edge connected because of the Steiner nodes in ST_j. As we do this for all trees $ST_1, .., ST_m$[4], and do not create any $(k-1)$-edge cut in any step, the resulting network is k-edge connected on the terminals.

It has been proved in Kashyap et al. [4] that the constructed cycle contains at most Ms_j beads for each Steiner component ST_j with s_j Steiner nodes. We replicate the edges to include $\lceil k/2 \rceil - 1$ additional copies of each edge, and thus the graph uses $M\lceil k/2 \rceil s_j$ beads. Since the Steiner components do not have common Steiner nodes, total number of beads required is bounded by $M\lceil k/2 \rceil s$. ■

Theorem 4.2 states the main result of this section.

Theorem 4.2: If the optimal network uses s Steiner nodes so that terminals distributed in metric space of MST number M are k-edge connected, Algorithm 4 forms a network with maximum of $2M\lceil k/2 \rceil s$ beads and zero Steiner nodes, in which the terminal nodes are k-edge connected.

Proof: The algorithm of Khuller and Vishkin [18] is a 2-approximation for finding the minimum cost (cost of each edge being number of beads required to form it) k-edge connected subgraph. Thus, the number of beads required is at most $2M\lceil k/2 \rceil s$. The last step of Algorithm 4 (sequential removal step) removes beads from the network by allowing them to connect to all nodes within the transmission range, so the resulting network after sequential removal also has at most $2M\lceil k/2 \rceil s$ relay nodes. Note that for $k = 2$, the algorithm is a $2M$-approximation, which is consistent with the ratio proved in Kashyap et al. [4]. ■

V. Generalization to Restricted Relay Placement

We extend our results for terminals distributed in a Euclidian plane to the scenario where relays cannot be placed in certain polygonal regions of the network . We call these regions forbidden regions. We assume that two nodes can communicate if they are within each other's transmission range even when there is a forbidden region between them. We modify the edge and vertex connectivity algorithms to work with the same approximation guarantees for this generalization.

We follow the same algorithms as before for both edge connectivity and vertex connectivity. It may not be possible

[4]If two terminal nodes are adjacent in multiple cycles formed while removing the Steiner components, we form maximum k beaded links between them. This suffices for maintaining k-edge connectivity.

to connect two terminals by placing relay nodes on the straight line between them due to the forbidden regions. Thus, Equation 1, which represents the number of relays needed to connect two terminals by placing relays on the line between them, cannot be used to weight the edges of the network formed on terminal nodes in our algorithms. Recently, a polynomial time algorithm has been proposed for placing the minimum number of relay nodes needed to form a link between two nodes with the presence of polygonal forbidden regions between them [20] . The problem is called the puddle-jumper problem. We modify our edge weights by running the algorithm given in [20] on each pair of terminals in the network to find the minimum number of relay nodes needed for each link, and using that as the weight of each edge. We then run our edge connectivity and vertex connectivity algorithms on a network with these edge weights. Then, for the selected links, we place the relays according to the algorithm given in [20].

A. Proof of Approximation Ratio

We first prove that the approximation ratio for the k-vertex connectivity algorithm is $O(k^2)$ for terminals distributed in the Euclidian plane. We follow the same construction as before, the only change being that beads (relay nodes) are not placed on straight lines between terminal nodes now; instead they are placed optimally taking forbidden regions into account. The only part of the proof that needs reconsideration to take forbidden regions into account is when Steiner nodes on a tree (ST_j) are removed from the optimal Steiner solution and beads are placed to make the cycle (and the Harary graph) between terminal nodes connected to tree ST_j (see Algorithm 3). We argue that the number of relays needed to form a beaded link between two terminals is still upper bounded by the number of Steiner nodes encountered in the depth first traversal between the two terminals: Take any two terminals being connected using beads, and let a be the number of Steiner nodes on the DFS path between them. Thus, there is a placement of Steiner nodes to connect the two terminal nodes using a Steiner nodes. As even Steiner nodes could not be placed in forbidden regions, and we connect the terminals using beads placed according to the optimal algorithm of [20], the number of beads required is upper bounded by a. Thus, each bead can still be charged to a different Steiner node on the DFS path between the terminals. We showed in Section IV that each Steiner node is charged at most $(3\lceil k/2\rceil(\lceil k/2\rceil+1)-1)$ times ($M=5$ for Euclidian plane), so the total number of beads required for replacing the Steiner node tree ST_j is still $(3\lceil k/2\rceil(\lceil k/2\rceil+1)-1)s_j$, s_j being the number of Steiner nodes in ST_j. Thus the total number of beads required in the network is at most $(3\lceil k/2\rceil(\lceil k/2\rceil+1)-1)s$ for the beaded network using minimum number of beads, s being the number of optimal Steiner nodes. As our algorithms use c-approximations ($c=2$ for edge-connectivity) for finding the optimal beaded network, the algorithms are $c(3\lceil k/2\rceil(\lceil k/2\rceil+1)-1)s$-approximations.

The same arguments can be used to prove that the edge-connectivity algorithm is $10\lceil k/2\rceil$-approximation for this generalization ($c=2$ for edge-connectivity, and $M=5$ for Euclidean plane).

VI. Full k-Connectivity

We now present the algorithms and analysis for achieving k-vertex and k-edge connectivity between terminals and added relays. We first discuss k-vertex connectivity.

A. Full k-vertex connectivity

We use the algorithm proposed in Bredin et al. [2]. The algorithm adds relays to find a subgraph on terminals that is k-vertex connected on terminals using the algorithm presented in Section III. Then, for each edge of the k-connected subgraph with weight greater than zero, i.e., with at least one relay, the algorithm places additional $k-1$ relays at each of the end terminal vertices of the edge, and $k-1$ relays along with each relay used on the edge. In the resulting graph, all terminals and relays have k-vertex connectivity. The algorithm has been proved to have an approximation ratio of $O(k^4)$ for terminals in Euclidean plane. We improve the analysis, and prove the algorithm to be an $O(k^3)$-approximation. Theorem 6.1 states the result. Here, c is the approximation ratio of the best algorithm for finding a k-vertex connected subgraph of a graph.

Theorem 6.1: If the optimal network uses s Steiner nodes so that terminals and Steiner nodes are k-vertex connected, the algorithm of Bredin et al. [2] forms a network with at most $3c(3\lceil k/2\rceil(\lceil k/2\rceil+1)-1)ks$ relays and zero Steiner nodes, in which the terminal and relay nodes are k-vertex connected.

Proof: The optimal network that is k-vertex connected only on terminals uses s Steiner nodes. Thus, according to Theorem 3.3, the intermediate graph that is k-vertex connected on terminals has at most $c(3\lceil k/2\rceil(\lceil k/2\rceil+1)-1)s$ relays. As shown in Bredin et al. [2], for each edge between terminals with $w \geq 1$ relays, we duplicate them to have a total of $kw+2(k-1) < 3kw$ relays on the edge. Thus, the total number of relays in the final fully k-vertex connected graph is bounded by $3c(3\lceil k/2\rceil(\lceil k/2\rceil+1)-1)ks$, which is an $O(k^3)$-approximation, an $O(k)$ improvement over the previous bounds of Bredin et al. [2]. ■

B. Full k-edge connectivity

For edge connectivity, we use the algorithm presented in Section IV. We use the algorithm to construct a k-edge connected subgraph on terminals, and then duplicate the relays needed on the edges of that subgraph. For each edge of weight greater than zero, we place additional $\lceil k/2\rceil-1$ relays at each end terminal vertex of the edge, and each relay location on that edge. The resulting graph is k-edge connected on both terminals and relays. We prove the algorithm is an $O(Mk^2)$-approximation for terminals distributed in a metric space of MST number M. Theorem 6.2 states the result.

Theorem 6.2: If the optimal network uses s Steiner nodes so that terminals and Steiner nodes are k-edge connected, our algorithm forms a network with at most $6M\lceil k/2\rceil^2 s$ relays and zero Steiner nodes, in which the terminal and relay nodes are k-edge connected.

Proof: The optimal network that is k-edge connected only on terminals uses s Steiner nodes. Thus, according to Theorem 4.2, the intermediate graph that is k-edge connected on terminals has at most $2M\lceil k/2\rceil s$ relays. Then, for each edge between terminals with $w \geq 1$ relays, we duplicate them to have a total of $\lceil k/2\rceil w + 2(\lceil k/2\rceil - 1) < 3\lceil k/2\rceil w$ relays on the edge. Thus, the total number of relays in the final fully k-edge connected graph is bounded by $6M\lceil k/2\rceil^2 s$. ■

VII. Conclusion

We considered the problem of constructing a fault-tolerant topology among sensor nodes using minimum number of additional relay nodes. We proposed $O(k^2)$-approximation algorithms for achieving k-vertex connectivity between sensor nodes distributed in the Euclidean plane. We proposed $2Mk$-approximation algorithms for achieving k-edge connectivity between terminal nodes distributed in any metric space with MST number M. We extended our algorithms to work with the same approximation guarantees for the generalization where the relay nodes cannot be placed in certain polygonal regions of the network. We analyzed an existing algorithm proposed for achieving k-vertex connectivity between sensor nodes and relays using minimum number of relays, and proved it to have an approximation guarantee of $O(k^3)$, which is an $O(k)$ improvement over previously proved bounds. We also proposed an algorithm for achieving k-edge connectivity between sensor nodes and relays using minimum number of relays, and proved it to have an approximation guarantee of $O(Mk^2)$.

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