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Resequencing delays under multipath routing -- Asymptotics in a simple queueing model

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Resequencing delays under multipath routing – Asymptotics in a simple queueing model

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***Abstract*— We study the resequencing delay caused by multipath routing. We use a queueing model which consists of parallel queues to model the network routing behavior. We define a new metric, denoted by γ, to study the impact of resequencing on the customer end-to-end delay. Our results characterize some properties of γ with respect to different service time distributions. In particular, the resequencing delay can be negligible when the delay along each path is light-tailed, but can be of major concern when it is heavy-tailed.**

I. Introduction

A. Resequencing delay

Multipath routing has recently received a lot of attention in the context of both wired and wireless communication networks. By sending data packets along different paths, multipath routing can potentially help balance the traffic load and reduce congestion levels in the network, in the process resulting in lower end-to-end delay and higher throughput. This path diversity also increases the ability of the network to adapt to link failures, an important issue in wireless ad hoc networks where the topology often changes [10, 16].

There are already network layer protocols that provide multipath routing, e.g., TORA [12] for MANETs, and transport layer protocols supporting multipath routing are currently under development. For example, the Stream Control Transmission Protocol (SCTP) with multihoming is one approach for supporting stream applications, such as multimedia, using multipath routing. Under SCTP, multiple IP addresses are associated with one client. The connection between these different IP and the server IP are established on different routes. As with TCP, SCTP is responsible for congestion control and assuring packets arrival order.

Since consecutive packets travel possibly along different paths from source to destination, they can easily be misordered, i.e., received out-of-order, at the destination. If the application requires the packets to be processed in a certain order at the destination, e.g., say the order in which they were sent, then the mis-ordered packets have to wait an additional amount of time, known as the resequencing delay, before being consumed. If this resequencing delay is too large, it can significantly degrade the performance of some real time streaming applications. The throughput provided by the network layer may be high, but the instantaneous throughput measured by applications may not be satisfactory and may result in a poor end-to-end user experience.

Usually, the impact of mis-ordering is evaluated by comparing the expected resequencing delay against the expected (system) end-to-end delay [14, 7]. These first order statistics give some information concerning the average behavior, but are only crude indicators of the more subtle interactions taking place. Here, we seek to evaluate the relative importance of the resequencing and end-to-end delays from a different perspective. We do so by means of a new metric that measures the conditional probability of the resquencing delay given the end-to-end delay; see below for a more detailed description.

B. A simple model of mis-ordering and multi-path routing

We shall evaluate this new metric in the context of a *simple* queueing model with mis-ordering, namely a set of K parallel single server queueing stations $\cdot|GI|1$ fed by Poisson arrivals under probabilistic state-independent routing. In short, the discussion will be given for a queueing system comprising a set of K parallel $M|GI|1$ queues.

This model constitutes an *ersatz* of the very complex situation we seek to investigate: The multiple paths between a given source/destination pair correspond to a set of parallel stations with each source/destination path represented by a single server queue. While many of the details of the protocol have been eliminated, the essence of network behavior (i.e., mis-ordering) has been preserved. By controlling the service time distribution at a station, we can model the delay of the packets traveling along that path. When a packet is generated at the source, it is sent out along one of the paths. Upon reaching the destination, out-of-order packets are stored in a resequencing buffer until they satisfy the ordering condition of the application, at which point they leave the resequencing buffer.

The study of resequencing problem in queueing networks is not without history [5, 8, 2, 4, 13, 7]. Recently, large deviations

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results have been obtained for the size of the resequencing buffer for a system of two parallel $M|M|1$ queues [15]. The analysis given here has its point of departure in the work of [8] where the distributions of the resequencing and end-to-end delays were derived for a system of parallel $M|GI|1$ queues. Unfortunately, most of the closed-form results rely on the service times being exponentially distributed, while general service time distributions do not always give nice form solutions. For our purpose this limits somewhat the usefulness of the results from [8] since in many circumstances, it may not be appropriate to assume that the service times are exponentially distributed. For instance, in the Internet, traffic coming to a router arises as the aggregation of many flows from many different sources, and the impact of the admission control and scheduling policies is far from understood. It is therefore unrealistic to assume that packet delays along a path are exponentially distributed. Thus, in the name of model robustness (and ignorance), it seems appropriate to make as few assumptions on the service time distributions as possible.

C. A new metric

Let R and D denote the (stationary) packet resequencing and end-to-end delays, respectively. The question of interest here is as follows: If a packet experiences a long end-to-end delay, is this caused by resequencing? One way to approach this issue would be to study the joint distribution of R and D, in the process going one step further than the results on the individual distributions currently available in the literature. However, as should be clear from earlier comments, such a joint distribution depends heavily on the service time distributions and is likely to be in quite a cumbersome form unsuited for comparison.

For these reasons, we focus instead on a new metric that reveals the relationship between the tail behaviors of R and D. More precisely, define

$$\gamma = \lim_{x\to\infty} \frac{\mathbf{P}\left[R > x\right]}{\mathbf{P}\left[D > x\right]} \ .$$

Since $R \le D$, we have $0 \le \gamma \le 1$ and it is plain that γ is asymptotically the conditional probability

$$\gamma = \lim_{x\to\infty} \frac{\mathbf{P}\left[R > x, D > x\right]}{\mathbf{P}\left[D > x\right]} = \lim_{x\to\infty} \mathbf{P}\left[R > x | D > x\right] \ .$$

Thus γ could also be interpreted as the asymptotic fraction of customers that given their end-to-end delay exceeds x, their resequencing delay also exceeds x (for large x). In general, the smaller value γ takes, the smaller impact the resequencing delay has on the end-to-end delay. In some extreme cases, for example:

(i) If $\gamma = 0$, then the tail of R is much lighter than the tail of D – If a customer experiences a long end-to-end delay, it is very likely due to the delay along the path, not to resequencing; and

(ii) If $\gamma = 1$, then R and D are equivalent in the tail – Thus, the long resequencing delay is a major contributor to the long end-to-end delay suffered by the customer.

The tail probability of the service time distributions is crucial in determining γ. Hence we will examine γ under a wide range of service time distributions from light-tail distribution, e.g., exponential distribution, to heavy-tail distribution. Whether these service time distributions can be found in existing networks is a subject for further study.

D. Outline of the paper

The paper is organized as follows: In Section II, we describe the model and state some preliminary results from [8]. Some simple but useful asymptotics that will be repeatedly used are given in Section III. Sections IV – IX contain the main results on the evaluation of γ. In Section IV we derive the tail behavior of D under any service distributions. We examine the tail behavior of R for different service time distributions in Sections V and VII – IX. Section VI clarifies the definitions and summarizes the properties of tails of distributions, which are used in the rest of the paper. The two extreme cases, i.e., exponential services vs. heavy tailed services, are compared in Section X and Section XI concludes the paper with a few discussions on the future work.

A word on the notation used in this paper. Two $\mathbb{R}$-valued rvs X and Y are said to be equal in law if they have the same distribution, a fact we denote by $X =_{st} Y$. The rv X is stochastically smaller the rv Y if $\mathbf{P}\left[X > x\right] \le \mathbf{P}\left[Y > x\right]$ for all x in $\mathbb{R}$, a fact we denote by $X \le_{st} Y$. We assume the reader to be familiar with the classes $\mathcal{S}$ and $\mathcal{L}$ of subexponential and long-tailed probbaility distributions, respectively. Additional material on these classes of distributions is available in the monograph [6].

II. The model and preliminaries

The system of interest, depicted in Figure 1, consists of K queueing stations operating in parallel, followed by a resequencing buffer. Upon arrival, each customer[1] is routed to

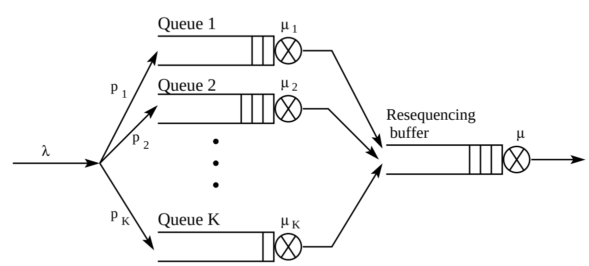

Fig. 1. System model

one of the K stations. When it completes service, the customer leaves the system immediately only if all prior customers[2] have already left the system; otherwise, the customer awaits in the resequencing buffer until all prior outstanding customers complete their service, at which point it leaves the resequencing buffer. This ensures that each customer leaves the system in the

[1]From now on we adopt the generic terminology of Queueing Theory and refer to packets as customers.

[2]Prior customers refer to the customers which have entered the system before that particular customer.

order in which it arrived, possibly experiencing in the process an additional delay in the resequencing buffer. Of particular interest is the *end-to-end* delay experienced by customers; this quantity is defined as the sum of the customer *sojourn* time (i.e., the time spent at the station it joined, and composed of the time waiting in buffer for service and of the time in service) and of the customer *resequencing* delay (i.e., the time spent in the resequencing buffer).

The discussion is carried out under the following assumptions: For each $k = 1, \ldots, K$, the k^{th} queueing station is modeled as an infinite capacity buffer attended by a single server which serves customers in the FCFS order; hereafter we refer to the k^{th} queueing station as queue k. Consecutive service time durations provided at queue k are assumed to be i.i.d. $\mathbb{R}_+$-valued rv, and let σ_k denote the corresponding generic service time. Customers arrive into the system according to a Poisson process with parameter λ, and each incoming customer is routed probabilistically to queue k with probability p_k, $k = 1, \ldots, K$, with $p_k > 0$ and $\sum_{k=1}^{K} p_k = 1$. Routing decisions are made independently across customers, and can therefore be modeled as a sequence of i.i.d. thinning rvs. Throughout, we assume that the Poisson arrival process, the routing process and the service duration processes at the K queues are mutually independent.

Under these assumptions, by standard properties concerning the Bernoulli thinning of Poisson processes, it is plain that the K single server systems can be interpreted as K *independent* $M|G|1$ queues operating in parallel. For each $k = 1, \ldots, K$, queue k is fed by Poisson arrivals with parameter λp_k and generic service time σ_k, and its stability is characterized by

$$\rho_k := \lambda p_k \mathbf{E}[\sigma_k] < 1. \tag{1}$$

Under (1), there exist $\mathbb{R}_+$-valued rvs W_k and T_k which denote the (customer) stationary waiting and sojourn times in queue k.

The system is stable if and only if all queues are stable, namely

$$\rho_k < 1, \quad k = 1, \ldots, K. \tag{2}$$

Then, under (2), there exist $\mathbb{R}_+$-valued rvs T, R and D which denote the (customer) stationary sojourn time, resequencing delay and end-to-end delay, respectively. The next two results summarize the distributional properties of these rvs, which can be found as Theorems 3.1 and 3.2 in [8] with slightly different notation: Let ν be an $\{1, \ldots, K\}$-valued rv distributed according to

$$\mathbf{P}[\nu = k] = p_k, \quad k = 1, \ldots, K.$$

In what follows, the rvs W_k and σ_k, $k = 1, \ldots, K$, and ν are assumed to be mutually independent rvs defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and

$$T_k = W_k + \sigma_k, \quad k = 1, \ldots, K. \tag{3}$$

Strictly speaking, we only have $T_k =_{st} W_k + \sigma_k$ for each $k = 1, \ldots, K$. However, we shall engage in the common and somewhat innocuous practice of taking $W_k + \sigma_k$ to be the customer stationary sojourn time T_k in the stable $M|G|1$ queue introduced earlier.

Theorem 2.1: *With*

$$\widetilde{W} = \max_{k \neq \nu} W_k, \tag{4}$$

it holds that

$$T = T_\nu, \tag{5}$$

$$D = \max\left(\widetilde{W}, T_\nu\right) \tag{6}$$

and

$$R = \left(\widetilde{W} - T_\nu\right)^+. \tag{7}$$

The resequencing delay of a customer, if nonzero, is the additional time it waits for all prior outstanding customers to complete their service at all queues other than selected queue ν. As a result, from (5) and (6) we get

$$R = D - T = \max\left(\widetilde{W}, T_\nu\right) - T_\nu$$

and the form of (7) follows.

We can now make use of these representations to get the distributions of D and R in terms of those of the rvs W_k and T_k, $k = 1, \ldots, K$. With

$$\widetilde{W}_k = \max_{\ell \neq k} W_\ell, \quad k = 1, \ldots, K \tag{8}$$

it is a simple matter to conclude the following:

Theorem 2.2: *The distribution functions of the rvs R and D are given by*

$$\mathbf{P}[D \leq x] = \sum_{k=1}^{K} p_k \mathbf{P}\left[\max\left(\widetilde{W}_k, T_k\right) \leq x\right] \tag{9}$$

and

$$\mathbf{P}[R \leq x] = \sum_{k=1}^{K} p_k \mathbf{P}\left[\widetilde{W}_k \leq x + T_k\right] \tag{10}$$

for all $x \geq 0$.

As pointed out in the introduction, we seek to evaluate the ratio

$$\gamma = \lim_{x \to \infty} \frac{\mathbf{P}[R > x]}{\mathbf{P}[D > x]}. \tag{11}$$

Our strategy for doing so is to identify the tail behavior of the distributions of D and R, respectively. The next section present some of the basic tools to do just that. This is followed in Section IV by a general result of the tail behavior of D.

III. Some easy but useful asymptotics

We shall have repeated use for the next elementary lemma where asymptotic equivalence is defined as follows: For mappings $f, g : \mathbb{R}_+ \to \mathbb{R}$, we write $f(x) \sim g(x)$ $(x \to \infty)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. In what follows, without further mention, all asymptotics are understood in the regime where x is large, and the qualifier $x \to \infty$ is now dropped from the notation.

Lemma 3.1: *Consider a finite family of functions* $f_1, \ldots, f_L, g_1, \ldots, g_L : \mathbb{R}_+ \to \mathbb{R}$. *If* $f_\ell(x) \sim g_\ell(x)$ *for each* $\ell = 1, \ldots, L$, *then we have*

$$\sum_{\ell=1}^{L} f_\ell(x) \sim \sum_{\ell=1}^{L} g_\ell(x).$$

Now consider a finite collection of $\mathbb{R}_+$-valued sequences, say $\{a_{1,n},\ n = 1, 2, \ldots\}$, …, $\{a_{L,n},\ n = 1, 2, \ldots\}$ for some integer L.

Lemma 3.2: *If* $\lim_{n\to\infty} a_{\ell,n} = 0$ *for each* $\ell = 1, \ldots, L$, *then it holds that*

$$\lim_{n\to\infty} \frac{1 - \prod_{\ell=1}^{L}(1 - a_{\ell,n})}{\sum_{\ell=1}^{L} a_{\ell,n}} = 1. \tag{12}$$

Before giving a proof, recall some elementary facts from calculus: As $x \downarrow 0$, we have

$$\log(1 - x) = -x + o(x), \quad 0 \le x < 1 \tag{13}$$

and

$$1 - e^x = -x + o(x), \quad x \ge 0. \tag{14}$$

Proof. Fix $n = 1, 2, \ldots$ and write

$$u_n = \sum_{\ell=1}^{L} \log(1 - a_{\ell,n}).$$

Then, we note that

$$\begin{aligned} 1 - \prod_{\ell=1}^{L}(1 - a_{\ell,n}) &= 1 - \prod_{\ell=1}^{L} e^{\log(1-a_{\ell,n})} \\ &= 1 - e^{u_n} \\ &= -u_n + o(u_n) \\ &= -u_n(1 + o(1)) \end{aligned} \tag{15}$$

as we make use of (14) since $\lim_{n\to\infty} u_n = 0$ under the enforced assumptions. On the other hand, (13) readily yields

$$\begin{aligned} u_n &= -\sum_{\ell=1}^{L} a_{\ell,n}(1 + o(1)) \\ &= -\left(\sum_{\ell=1}^{L} a_{\ell,n}\right)(1 + o(1)). \end{aligned} \tag{16}$$

Combining (15) and (16) yields

$$1 - \prod_{\ell=1}^{L}(1 - a_{\ell,n}) = \left(\sum_{\ell=1}^{L} a_{\ell,n}\right)(1 + o(1))$$

and the desired conclusion follows. □

IV. The tail behavior of D

Taking (9) as the point of departure, we first deal with the tail behavior of the rvs $\widetilde{W}_1, \ldots, \widetilde{W}_K$.

Lemma 4.1: *For each* $k = 1, \ldots, K$, *it holds that*

$$\mathbf{P}\left[\widetilde{W}_k > x\right] \sim \sum_{\ell \ne k} \mathbf{P}\left[W_\ell > x\right]. \tag{17}$$

Proof. This is a straightforward consequence of Lemma 3.2 once we note that

$$\begin{aligned} \mathbf{P}\left[\widetilde{W}_k > x\right] &= 1 - \prod_{\ell \ne k} \mathbf{P}\left[W_\ell \le x\right] \\ &= 1 - \prod_{\ell \ne k}(1 - \mathbf{P}\left[W_\ell > x\right]), \quad x \ge 0 \end{aligned}$$

under the independence assumptions. □

Lemma 4.1 readily leads to the next asymptotic equivalence.

Lemma 4.2: *For each* $k = 1, \ldots, K$, *it holds that*

$$\begin{aligned} &\mathbf{P}\left[\max\left(\widetilde{W}_k, T_k\right) > x\right] \\ \sim\ &\mathbf{P}\left[T_k > x\right] + \sum_{\ell \ne k} \mathbf{P}\left[W_\ell > x\right]. \end{aligned} \tag{18}$$

Proof. Fix $k = 1, \ldots, K$ and $x \ge 0$. By independence of $\widetilde{W}_k$ and T_k, we find

$$\begin{aligned} &\mathbf{P}\left[\max\left(\widetilde{W}_k, T_k\right) > x\right] \\ =\ &\mathbf{P}\left[T_k > x\right] + \mathbf{P}\left[\widetilde{W}_k > x\right]\mathbf{P}\left[T_k \le x\right] \end{aligned}$$

and the result is immediate from Lemma 4.1 as we note that $\mathbf{P}\left[\widetilde{W}_k > x\right]\mathbf{P}\left[T_k \le x\right] \sim \mathbf{P}\left[\widetilde{W}_k > x\right]$. □

We are now in a position to characterize the tail behavior of the end-to-end delay D.

Proposition 4.3: *It holds that*

$$\begin{aligned} &\mathbf{P}\left[D > x\right] \\ \sim\ &\sum_{k=1}^{K} p_k \mathbf{P}\left[T_k > x\right] + (1 - p_k)\mathbf{P}\left[W_k > x\right]. \end{aligned} \tag{19}$$

Proof. From (9), upon using Lemma 4.2, we get

$$\begin{aligned}\mathbf{P}\left[D > x\right] &= \sum_{k=1}^{K} p_k \mathbf{P}\left[\max\left(\widetilde{W}_k, T_k\right) > x\right] \\ &\sim \sum_{k=1}^{K} p_k \left(\mathbf{P}\left[T_k > x\right] + \sum_{\ell \neq k} \mathbf{P}\left[W_\ell > x\right]\right) \\ &\sim \sum_{k=1}^{K} p_k \mathbf{P}\left[T_k > x\right] \\ &\quad + \sum_{k=1}^{K} p_k \left(\sum_{\ell \neq k} \mathbf{P}\left[W_\ell > x\right]\right). \end{aligned} \quad (20)$$

Easy algebra leads to desired conclusion (19) since

$$\begin{aligned}\sum_{k=1}^{K} p_k \sum_{\ell \neq k} \mathbf{P}\left[W_\ell > x\right] &= \sum_{\ell=1}^{K} \left(\sum_{k \neq \ell} p_k\right) \mathbf{P}\left[W_\ell > x\right] \\ &= \sum_{\ell=1}^{K} (1 - p_\ell) \mathbf{P}\left[W_\ell > x\right]\end{aligned}$$

for all $x \geq 0$. □

The tail behavior of R is more delicate to obtain and depends very much on the distributional assumptions made on the service time durations. We explore this point in turn for (i) exponential services [Section V], (ii) services with exponential tails [Section VII], (iii) subexponential services [Section VIII] and (iv) a combination of service times with exponential and heavy tails [Section IX].

V. Exponential services

We begin with the simple situation where the service times at each queue are exponentially distributed, i.e., for each $k = 1, \ldots, K$, we have

$$\mathbf{P}\left[\sigma_k \leq x\right] = 1 - e^{-\mu_k x}, \quad x \geq 0 \quad (21)$$

for some $\mu_k > 0$. Queue k is an $M|M|1$ queue and closed-form expressions are thus available for the distributions of the waiting time W_k and of the sojourn time T_k. We write

$$\alpha_k := \mu_k - \lambda p_k = \mu_k(1 - \rho_k)$$

where, as customary, we have set $\rho_k = \frac{\lambda p_k}{\mu_k}$. It is known [9, pp. 202-203] that

$$\mathbf{P}\left[W_k \leq x\right] = 1 - \rho_k e^{-\alpha_k x}, \quad x \geq 0 \quad (22)$$

and

$$\mathbf{P}\left[T_k \leq x\right] = 1 - e^{-\alpha_k x}, \quad x \geq 0. \quad (23)$$

In this case, we can compute γ directly. Set

$$\alpha^\star := \min\left(\alpha_1, \ldots, \alpha_K\right) \quad (24)$$

and introduce

$$\mathcal{M} := \left\{k = 1, \ldots, K : \ \alpha_k = \alpha^\star\right\}. \quad (25)$$

Proposition 5.1: *For exponentially distributed services (21), it holds that*

$$\mathbf{P}\left[D > x\right] \sim \left(\sum_{k \in \mathcal{M}} (p_k + (1 - p_k)\rho_k)\right) e^{-\alpha^\star x}. \quad (26)$$

Proof. Substituting (22) and (23) into (19) leads to

$$\mathbf{P}\left[D > x\right] \sim \sum_{k=1}^{K} \left(p_k + (1 - p_k)\rho_k\right) e^{-\alpha_k x},$$

and the conclusion (26) is now straightforward. □

Next, we tackle the tail behavior of R. To state the result, for each $k = 1, \ldots, K$, define

$$\alpha_k^\star := \min\left(\alpha_\ell, \ \ell \neq k\right) \quad (27)$$

and

$$\Gamma_k := \left(\sum_{\ell \in \mathcal{M}_k} \rho_\ell\right) \quad (28)$$

with

$$\mathcal{M}_k := \left\{\ell \neq k : \ \alpha_\ell = \alpha_k^\star\right\}. \quad (29)$$

Proposition 5.2: *For exponentially distributed services (21), it holds that*

$$\mathbf{P}\left[R > x\right] \sim \left(\sum_{k \in \mathcal{M}^\star} p_k \frac{\alpha_k}{\alpha^\star + \alpha_k} \Gamma_k\right) e^{-\alpha^\star x} \quad (30)$$

where

$$\mathcal{M}^\star := \left\{k = 1, \ldots, K : \ \alpha_k^\star = \alpha^\star\right\}. \quad (31)$$

A proof of Proposition 5.2 is given in Appendix I. The evaluation of γ is now within easy reach by combining Propositions 5.1 and 5.2.

Proposition 5.3: *For exponentially distributed services (21), it holds that*

$$\gamma = \frac{\sum_{k \in \mathcal{M}^\star} p_k \frac{\alpha_k}{\alpha^\star + \alpha_k} \Gamma_k}{\sum_{k \in \mathcal{M}} (p_k + (1 - p_k)\rho_k)}. \quad (32)$$

The expression (32) takes a particularly simple form when the waiting time and sojourn time distributions at all the queues have the same decaying rate; this can be seen by direct substitution.

Corollary 5.4: *For exponentially distributed services (21) with $\alpha_1 = \ldots = \alpha_K$, we have*

$$\gamma = \frac{1}{2} \frac{\Gamma}{1 + \Gamma}$$

where

$$\Gamma := \sum_{k=1}^{K} \rho_k (1 - p_k).$$

In this special case, we have $\gamma < \frac{1}{2}$ for any finite value K, which implies that a long end-to-end delay is more likely to result from factors other than resequencing. However, it is still possible for $\gamma \simeq 1$ under certain conditions. For instance, when the service rates of the stations are drastically different among themseleves, the resequencing delays of mis-ordered customers become a large portion of their end-to-end delays. We illustrate this point with the following numerical example with three stations in parallel, i.e., $K = 3$. The customer arrival rate is $\lambda = 5$, and the service rates are $\mu_1 = 0.3$, $\mu_2 = 8$ and $\mu_3 = 12$, respectively. Stability is ensured by taking the routing probabilities to be $p_1 = 0.05$, $p_2 = 0.35$ and $p_3 = 0.6$. Direct calculations based on (32) yield $\gamma = 0.9346$. Station 1 has the slowest service rate, which limits the amount of work that can be routed to it. Thus, a large number of customers served by stations 2 and 3 are very likely to spend a long time in the resequencing buffer waiting for a small number of customers arrived earlier to finish their service from station 1.

VI. Tails of distributions

In this section we have collected some of the basic definitions and properties pertaining to tails of distributions which we shall use in the remainder of this paper.

A. Exponential tails

Definition 6.1: *The $\mathbb{R}_+$-valued rv X has an exponential(ly decaying) tail, denoted $X \in \mathcal{E}$, if*

$$\theta^\star := \sup\left(\theta \in \mathbb{R} : \mathbf{E}\left[e^{\theta X}\right] < \infty\right) > 0. \tag{33}$$

Note that

$$\mathbf{E}\left[e^{\theta X}\right] = \infty, \quad \theta^\star < \theta.$$

Markov's inequality (applied to $e^{\theta X}$ with $\theta > 0$) yields

$$\mathbf{P}[X > x] \le e^{-\theta x}\mathbf{E}\left[e^{\theta X}\right], \quad x \ge 0 \tag{34}$$

whenever $0 < \theta < \theta^\star$, and the rv X satisfying (33) indeed displays an exponentially decaying tail.

A little more can be said under (33): It is plain from (34) that

$$\lim_{x\to\infty} \frac{\mathbf{P}[X > x]}{e^{-\theta x}} = 0, \quad 0 < \theta < \theta^\star \tag{35}$$

On the other hand, for each $\theta > 0$, we have

$$\begin{aligned}
\mathbf{E}\left[e^{\theta X}\right] &= 1 + \mathbf{E}\left[\int_0^X \theta e^{\theta x} dx\right] \\
&= 1 + \mathbf{E}\left[\int_0^\infty \mathbf{1}[X > x]\, \theta e^{\theta x} dx\right] \\
&= 1 + \int_0^\infty \theta e^{\theta x}\mathbf{P}[X > x]\, dx
\end{aligned} \tag{36}$$

by Tonelli's Theorem. With the help of this last relationship, it is then not too difficult to show that

$$\lim_{x\to\infty} \frac{\mathbf{P}[X > x]}{e^{-\theta x}} = \infty, \quad \theta^\star < \theta. \tag{37}$$

In general, the limit

$$\lim_{x\to\infty} \frac{\mathbf{P}[X > x]}{e^{-\theta^\star x}} \tag{38}$$

may *not* exist, as can be seen through the example

$$\mathbf{P}[X \le x] = 1 - \frac{K + \cos x}{K+1} e^{-x}, \quad x \ge 0$$

with $K > 2$.[3] Obviously $\theta^\star = 1$ here but

$$\frac{\mathbf{P}[X > x]}{e^{-\theta^\star x}} = \frac{K + \cos x}{K+1}, \quad x \ge 0$$

and the limit does not exist owing to the oscillatory nature of the cosine function.

Yet, as we shall see shortly, there are natural circumstances where the limit (38) exists, i.e., there exist constants $\alpha > 0$ and $C > 0$ such that

$$\mathbf{P}[X > x] \sim C e^{-\alpha x}. \tag{39}$$

In that case, it is easy to check from (36) that $\alpha = \theta^\star$ with $\mathbf{E}\left[e^{\theta X}\right] < \infty$ (resp. $\mathbf{E}\left[e^{\theta X}\right] = \infty$) whenever $\theta < \alpha$ (resp. $\alpha < \theta$).

B. Heavy tails

In order to capture the notion of heavy tailed distributions, we shall rely on the following definitions.

Definition 6.2: *The $\mathbb{R}_+$-valued rv X has a long tail, denoted $X \in \mathcal{L}$, if $\mathbf{P}[X > x] > 0$ all $x \ge 0$ and*

$$\lim_{x\to\infty} \frac{\mathbf{P}[X > x - y]}{\mathbf{P}[X > x]} = 1, \quad y \in \mathbb{R}. \tag{40}$$

Definition 6.3: *The $\mathbb{R}_+$-valued rv X has a subexponential tail, denoted $X \in \mathcal{S}$, if $\mathbf{P}[X > x] > 0$ all $x \ge 0$ and*

$$\lim_{x\to\infty} \frac{\mathbf{P}[X + Y > x]}{\mathbf{P}[X > x]} = 2 \tag{41}$$

where Y is an independent copy of X.

The inclusion $\mathcal{S} \subset \mathcal{L}$ is known to hold [6, p. 50] (and references therein). Moreover, $\mathcal{S}$ contains many well-known distributions such as Weibull, log-normal, Pareto and regularly varying distributions. Some useful facts concerning the classes $\mathcal{L}$ and $\mathcal{S}$ are presented below;

Proposition 6.4: *Assume $X \in \mathcal{L}$.*

(i): For every $\theta > 0$, it holds that

$$\lim_{x\to\infty} e^{\theta x}\mathbf{P}[X > x] = \infty$$

whence $\mathbf{E}\left[e^{\theta X}\right] = \infty$;

(ii): If $0 < \mathbf{E}[X] < \infty$, then

$$\mathbf{P}[X > x] = o(\mathbf{P}[X^\star > x])$$

where $X^\star$ denotes the forward recurrence time associated with X; it is the rv with integrated tail distribution given by

$$\mathbf{P}[X^\star > x] := \mathbf{E}[X]^{-1} \int_x^\infty \mathbf{P}[X > t]\, dt, \quad x \ge 0; \tag{42}$$

(iii): For any $\mathbb{R}_+$-valued rv Y which is independent of X, the equivalence

$$\mathbf{P}[X > x + Y] \sim \mathbf{P}[X > x]$$

[3] This will ensure that the right handside is indeed increasing and therefore a *bona fide* distribution.

holds.

A proof is available in Appendix II. The following fact is a simple consequence of (35) and of Claim (i) of Proposition 6.4.

Corollary 6.5: *For $\mathbb{R}_+$-valued rvs X and Y, it holds that*

$$\mathbf{P}[X > x] = o(\mathbf{P}[Y > x]) \tag{43}$$

whenever $X \in \mathcal{E}$ and $Y \in \mathcal{L}$.

VII. The exponential tail case

We generalize the setup of Section VII to service time distributions with exponentially decaying tails: For each $k = 1, \ldots, K$, we assume $\sigma_k \in \mathcal{E}$, namely

$$\theta_k := \sup\left(\theta \in \mathbb{R} : \ \mathbf{E}\left[e^{\theta \sigma_k}\right] < \infty\right) > 0. \tag{44}$$

At this level of generality, closed form expressions such as (22) and (23) are usually not available for the rvs W_k and T_k. However, the tail behavior of W_k can be characterized through the classic Cramér-Lundberg approximation [1, Example 5.5, p. 366]. For easy reference, we briefly discuss this approximation in the context of $M|G|1$ queues considered here: The Lundberg equation takes here the special form

$$\mathbf{E}\left[e^{\theta \sigma_k}\right] = 1 + (\lambda p_k)^{-1}\theta, \quad \theta > 0 \tag{45}$$

and under the stability condition (1) it has a unique solution which we denote by α_k. It is plain that $\alpha_k \leq \theta_k$. As we are in the non-lattice case for the underlying random walk, it follows [1, Thm. 5.3, p. 365] that

$$\mathbf{P}[W_k > x] \sim C_k e^{-\alpha_k x} \tag{46}$$

for some constant C_k given by

$$C_k := \frac{1 - \rho_k}{\lambda p_k \mathbf{E}\left[\sigma_k e^{\alpha_k \sigma_k}\right] - 1} \tag{47}$$

with ρ_k given by (1). Details are available in [1, p. 367].

The Cramér-Lundberg approximation leaves open the question of the tail behavior of the sojourn time T_k. Given that $W_k \leq_{st} T_k$, it is plain from (46) that *if* the asymptotics

$$\mathbf{P}[T_k > x] \sim D_k e^{-\beta_k x} \tag{48}$$

were to hold with constants $D_k > 0$ and $\beta_k > 0$, then necessarily $\beta_k \leq \alpha_k$. In the case of exponentially distributed services, we do in fact have (48) with $\beta_k = \alpha_k$. However, the asymptotics (48) appear not to hold in the same level of generality as the Cramér-Lundberg approximation (46). Still, for our purpose, some pertinent information can be deduced from (46).

As already pointed out in Section VI, under (46), we have $\mathbf{E}\left[e^{\theta W_k}\right] < \infty$ (resp. $\mathbf{E}\left[e^{\theta W_k}\right] = \infty$) if $\theta < \alpha_k$ (resp. $\alpha_k < \theta$). The independence of the rvs W_k and σ_k yields

$$\mathbf{E}\left[e^{\theta T_k}\right] = \mathbf{E}\left[e^{\theta W_k}\right] \cdot \mathbf{E}\left[e^{\theta \sigma_k}\right], \quad \theta \in \mathbb{R} \tag{49}$$

and from earlier remarks we conclude that

$$\mathbf{E}\left[e^{\theta T_k}\right] < \infty, \quad \theta < \min(\theta_k, \alpha_k) = \alpha_k \tag{50}$$

while

$$\mathbf{E}\left[e^{\theta T_k}\right] = \infty, \quad \alpha_k < \theta. \tag{51}$$

Using Markov's inequality again we get

$$\mathbf{P}[T_k > x] \leq e^{-\theta x}\mathbf{E}\left[e^{\theta T_k}\right], \quad x \geq 0 \tag{52}$$

whenever $0 < \theta < \alpha_k$. In short, under (44), we have

$$W_k \in \mathcal{E} \quad \text{and} \quad T_k \in \mathcal{E}. \tag{53}$$

These observations can now be put to work towards characterizing the tail behavior of D and R.

Proposition 7.1: *If (44) holds for each $k = 1, \ldots, K$, then*

$$\begin{aligned}\mathbf{P}[D > x] \quad \sim \quad & \sum_{k \in \mathcal{M}} p_k \mathbf{P}[T_k > x] \\ & + \left(\sum_{k \in \mathcal{M}} (1 - p_k) C_k\right) e^{-\alpha^\star x}\end{aligned} \tag{54}$$

with $\alpha^\star$ and $\mathcal{M}$ defined at (24) and (25), respectively.

Proof. In view of the observations made earlier, it is clear that

$$\mathbf{P}[T_k > x] = o(e^{-\alpha^\star x}), \quad k \notin \mathcal{M} \tag{55}$$

since $\alpha^\star < \alpha_k$ for k not in $\mathcal{M}$. Combining the Cramér-Lundberg approximation (46) with the asymptotics (19) now leads to

$$\begin{aligned}& \mathbf{P}[D > x] \\ \sim \ & \sum_{k=1}^{K} p_k \mathbf{P}[T_k > x] + \sum_{k=1}^{K} (1 - p_k) C_k e^{-\alpha_k x}\end{aligned}$$

and (54) follows as we make use of (55). □

Under the enforced assumptions, there is no guarantee *a priori* that the limits

$$\lim_{x \to \infty} \frac{\mathbf{P}[T_k > x]}{e^{-\alpha^\star x}}, \quad k \in \mathcal{M}$$

exist.

We now turn to the tail behavior of R: For each $k = 1, \ldots, K$, define $\alpha_k^\star$ and $\mathcal{M}_k$ as before by (27) and (29), respectively, and set

$$\Gamma_k := \sum_{\ell \in \mathcal{M}_k} C_\ell. \tag{56}$$

This is the analog of (28) for service time distributions with exponentially decaying tails. A proof of the next result is available in Appendix III.

Proposition 7.2: *If (44) holds for each $k = 1, \ldots, K$, then*

$$\mathbf{P}[R > x] \sim \left(\sum_{k \in \mathcal{M}^\star} p_k \Gamma_k \mathbf{E}\left[e^{-\alpha_k^\star T_k}\right]\right) e^{-\alpha^\star x} \tag{57}$$

with $\mathcal{M}^\star$ given by (31).

VIII. THE SUBEXPONENTIAL CASE

In this section, we consider the situation where the service times at all the queues are "heavy tailed" in the following technical sense: For each $k = 1, \ldots, K$,

$$\sigma_k \in \mathcal{L} \quad \text{and} \quad \sigma_k^\star \in \mathcal{S} \tag{58}$$

where $\sigma_k^\star$ denotes the forward recurrence time (42) associated with the service time rv σ_k. Sufficient conditions on σ_k to ensure that $\sigma_k^\star$ is subexponential can be found in [6, p. 52].

The tail behavior of the rvs W_k and T_k is summarized here for future reference.

Proposition 8.1: *Under the asumptions (58), it holds that*

$$\mathbf{P}[W_k > x] \sim \frac{\rho_k}{1-\rho_k}\mathbf{P}[\sigma_k^\star > x] \quad \textit{with} \quad W_k \in \mathcal{S} \tag{59}$$

and

$$\mathbf{P}[T_k > x] \sim \mathbf{P}[W_k > x] \quad \textit{with} \quad T_k \in \mathcal{S}. \tag{60}$$

Proof. By interpreting the Pollaczeck-Khintchine formula in stable $M|GI|1$ queues, we can express the stationary waiting time as a geometric sum of i.i.d. forward recurrence time associated with the service time [1, p. 296] [9, p. 201]. More precisely, let ν_k be a geometric rv with parameter ρ_k, i.e.,

$$\mathbf{P}[\nu_k = n] = (1-\rho_k)\rho_k^n, \quad n = 0, 1, \ldots$$

and let $\{\sigma_{k,n}^\star,\ n = 1, 2, \ldots\}$ denote a collection of i.i.d. rvs distributed according to $\sigma_k^\star$. Then, it holds

$$W_k =_{st} \sum_{n=1}^{\nu_k} \sigma_{k,n}^\star \tag{61}$$

where the rv ν_k is taken to be independent of the sequence $\{\sigma_{k,n}^\star,\ n = 1, 2, \ldots\}$. Then, (59) follows by the Random Sum Theorem for subexponential rvs [1, Lemma 9.2., p. 296] [6, Thm. A3.20, p. 580].

To obtain the tail behavior of T_k, observe from Claim (ii) of Lemma 6.4 that $\sigma_k \in \mathcal{L}$ implies

$$\mathbf{P}[\sigma_k > x] = o(\mathbf{P}[\sigma_k^\star > x]),$$

and the equivalence $\mathbf{P}[T_k > x] \sim \mathbf{P}[W_k > x]$ follows from (59) upon making use of Lemma A3.23 in [6, p. 582]. The statement $T_k \in \mathcal{S}$ follows by tail equivalence from the statement $W_k \in \mathcal{S}$ [6, Lemma A3.15, p. 572]. □

Under the asumptions (58), reporting the asymptotic equivalence (60) into Proposition 4.3 yields

$$\mathbf{P}[D > x] \sim \sum_{k=1}^{K} \mathbf{P}[W_k > x] \tag{62}$$

and the asymptotic equivalence (59) leads to the following

Proposition 8.2: *If (58) holds for each $k = 1, \ldots, K$, then*

$$\mathbf{P}[D > x] \sim \sum_{k=1}^{K} \frac{\rho_k}{1-\rho_k}\mathbf{P}[\sigma_k^\star > x]. \tag{63}$$

The behavior of R is examined next, with a proof given in Appendix IV.

Proposition 8.3: *If (58) holds for each $k = 1, \ldots, K$, then we have*

$$\mathbf{P}[R > x] \sim \sum_{k=1}^{K} (1-p_k)\mathbf{P}[W_k > x]. \tag{64}$$

The asymptotics (62) and (64) already suggest

$$\gamma = \lim_{x\to\infty} \frac{\sum_{k=1}^{K}(1-p_k)\mathbf{P}[W_k > x]}{\sum_{k=1}^{K}\mathbf{P}[W_k > x]} \tag{65}$$

provided the limit exists. This observation can be exploited as follows when uniform routing is used.

Corollary 8.4: *If (58) holds for each $k = 1, \ldots, K$, with $p_1 = \ldots = p_K = \frac{1}{K}$, then the limit (65) exists with*

$$\gamma = \frac{K-1}{K}. \tag{66}$$

This result is *independent* of the specific form of the service times $\sigma_1, \ldots, \sigma_K$ under condition (58). Moreover, γ will become increasignly close to 1 as K becomes large.

Under non-uniform routing, the existence of the limit at (65) is not automatically guaranteed. We explore the issue by strengthening the underlying assumptions used thus far. Specifically, in addition to the assumptions (58) holding for each $k = 1, \ldots, K$, there exists an $\mathbb{R}_+$-valued rv $Z \in \mathcal{S}$ such that

$$\mathbf{P}[\sigma_k^\star > x] \sim c_k \mathbf{P}[Z > x] \tag{67}$$

for some constant $c_k \geq 0$. The case $c_k > 0$ coresponds to the rvs $\sigma_k^\star$ and Z being tail equivalent, and (67) does imply $\sigma_k^\star \in \mathcal{S}$. On the other hand, $c_k = 0$ represents the situation where the rv $\sigma_k^\star$ has a weaker tail than Z.[4] In either case, under (58) the equivalence (59) yields

$$\mathbf{P}[W_k > x] \sim d_k \mathbf{P}[Z > x] \quad \text{with} \quad d_k = c_k \frac{\rho_k}{1-\rho_k} \tag{68}$$

and (65) easily leads to the following asymptotics.

Proposition 8.5: *If (58) holds for each $k = 1, \ldots, K$ with (67), then the limit (65) exists with*

$$\gamma = \frac{\sum_{k=1}^{K}(1-p_k)d_k}{\sum_{k=1}^{K} d_k} \tag{69}$$

provided

$$\sum_{k=1}^{K} d_k > 0.$$

[4] The property $\sigma_k^\star \in \mathcal{S}$ is not necessarily implied, and therefore needs to be assumed.

IX. The general case

Assume now that the set of K queues is partitioned into two non-empty subsets, say

$$\{1, \ldots, K\} = \mathcal{E} \cup \mathcal{H} \tag{70}$$

with $\mathcal{E}$ and $\mathcal{H}$ containing the queues with exponentially decaying services times and heavy-tailed service times, respectively. More precisely, we assume that (44) holds for each k in $\mathcal{E}$, while (58) holds for k in $\mathcal{H}$. In other words,

$$\theta_k := \sup\left(\theta \in \mathbb{R} : \ \mathbf{E}\left[e^{\theta\sigma_k}\right] < \infty\right) > 0, \quad k \in \mathcal{E} \tag{71}$$

and

$$\sigma_k \in \mathcal{L} \quad \text{and} \quad \sigma_k^\star \in \mathcal{S}, \quad k \in \mathcal{H}. \tag{72}$$

To eliminate situations previously encountered in earlier sections, we assume that both $\mathcal{E}$ and $\mathcal{H}$ are not empty, i.e., $|\mathcal{E}| \geq 1$ and $|\mathcal{H}| \geq 1$. Although the symbol $\mathcal{E}$ has been given two different meanings, this will not create any confusion.

Under these assumptions, it is plain from Corollary 6.5 that

$$\mathbf{P}\left[W_\ell > x\right] = o(\mathbf{P}\left[W_k > x\right]), \quad \ell \in \mathcal{E}, \ k \in \mathcal{H} \tag{73}$$

and

$$\mathbf{P}\left[T_\ell > x\right] = o(\mathbf{P}\left[W_k > x\right]), \quad \ell \in \mathcal{E}, \ k \in \mathcal{H} \tag{74}$$

as we recall that $W_\ell \in \mathcal{E}$ and $T_\ell \in \mathcal{E}$ while $W_k \in \mathcal{S}$. Reporting these observations into (19) gives

$$\mathbf{P}\left[D > x\right] \sim \sum_{k \in \mathcal{H}} \mathbf{P}\left[W_k > x\right] \tag{75}$$

upon using (60). Finally, the explicit asymptotics (59) leads to

Proposition 9.1: *If (71) and (72) hold under the decomposition (70), then*

$$\mathbf{P}\left[D > x\right] \sim \sum_{k \in \mathcal{H}} \frac{\rho_k}{1 - \rho_k} \mathbf{P}\left[\sigma_k^\star > x\right]. \tag{76}$$

The behavior of R is examined next, with a proof given in Appendix V.

Proposition 9.2: *If (71) and (72) hold under the decomposition (70), then*

$$\mathbf{P}\left[R > x\right] \sim \sum_{k \in \mathcal{H}} (1 - p_k) \mathbf{P}\left[W_k > x\right]. \tag{77}$$

This time, the asymptotics (75) and (77) suggest

$$\gamma = \lim_{x \to \infty} \frac{\sum_{k \in \mathcal{H}} (1 - p_k) \mathbf{P}\left[W_k > x\right]}{\sum_{k \in \mathcal{H}} \mathbf{P}\left[W_k > x\right]} \tag{78}$$

provided the limit exists. This constitutes a natural generalization of (65).

A situation where the limit (78) exists is presented next; it parallels the uniform routing case discussed in Corollary 8.4: Assume that

$$p := p_k = \frac{1 - \sum_{k \in \mathcal{E}} p_k}{|\mathcal{H}|}, \quad k \in \mathcal{H}. \tag{79}$$

Under this assumption, (75) and (77) lead to

$$\mathbf{P}\left[R > x\right] \sim (1 - p) \mathbf{P}\left[D > x\right]$$

with

$$\mathbf{P}\left[D > x\right] \sim \sum_{k \in \mathcal{H}} \mathbf{P}\left[W_k > x\right].$$

Corollary 9.3: *Assume that (71) and (72) hold under the decomposition (70). With (79), the limit at (78) exists and is given by*

$$\gamma = 1 - p \tag{80}$$

This result is *independent* of the specific form of the service times $\sigma_1, \ldots, \sigma_K$ under conditions (71) and (72). Moreover, Corollary 9.3 automatically applies when $|\mathcal{H}| = 1$, in which case $\gamma = 1 - p_1$ under the convention $\mathcal{H} = \{1\}$.

When (79) fails to hold, then the existence of the limit at (78) is not automatically guaranteed. Here, as we did in the previous sections, we need to strengthen the underlying assumptions. Thus, we complement (72) by requiring the existence of an $\mathbb{R}_+$-valued rv $Z \in \mathcal{S}$ such that for each k in $\mathcal{H}$,

$$\mathbf{P}\left[\sigma_k^\star > x\right] \sim c_k \mathbf{P}\left[Z > x\right] \tag{81}$$

for some constant $c_k \geq 0$. We close with the following analog of Proposition 8.5.

Proposition 9.4: *Assume that (71) and (72) hold under the decomposition (70). If (81) holds, then the limit (78) exists with*

$$\gamma = \frac{\sum_{k \in \mathcal{H}} (1 - p_k) d_k}{\sum_{k \in \mathcal{H}} d_k} \tag{82}$$

provided

$$\sum_{k \in \mathcal{H}} d_k > 0.$$

X. Discussion

To get a better sense of the impact of the service distributions on the value of γ, we compare the results in two extreme situations, namely the one where the service times are all exponentially distributed and the one where they are all heavy-tailed.

As seen earlier, when the workload is evenly distributed with $p_1 = \ldots = p_K = \frac{1}{K}$, then $\gamma < \frac{1}{2}$ in the exponential case with $\mu_1 = \ldots = \mu_K$, while γ still goes to 1 as K increases for the heavy-tailed case regardless of the exact distributions of the service time.

Now, in order to go beyond this uniform allocation, we consider the following situation. The number K of stations, the customer arrival rate λ and the load allocation vector $\boldsymbol{p} = (p_1, \ldots, p_K)$ are given and will be the same in both cases. In the heavy-tailed case, we take the service times to have a Pareto distribution, e.g., for each $k = 1, \ldots, K$, we take

$$\mathbf{P}\left[\sigma_k > x\right] = \left(\frac{B_k}{B_k + x}\right)^{\beta_k}, \quad x \geq 0$$

with $B_k > 0$ and $\beta_k > 1$. In that case, $\mathbf{E}[\sigma_k]$ is finite and given by

$$\mathbf{E}[\sigma_k] = \frac{B_k}{\beta_k - 1}.$$

It is well known that σ_k and $\sigma_k^\star$ are both in $\mathcal{S}$ [6].

We shall take $\beta_1 = \ldots = \beta_K = \beta$ to simplify matters somewhat. In order to make the two situations comparable, we require that the expected service times at each station coincide, i.e.,

$$\frac{B_k}{\beta - 1} = \mathbf{E}[\sigma_k] = \frac{1}{\mu_k}, \quad k = 1, \ldots, K.$$

Thus,

$$B_k = \frac{\beta - 1}{\mu_k}, \quad k = 1, \ldots, K.$$

Let $\Gamma_{\mathcal{E}}(\boldsymbol{p})$ and $\Gamma_{\mathcal{S}}(\boldsymbol{p})$ denote the value of γ in the exponential and Pareto cases, respectively, under the load allocation $\boldsymbol{p}$. These quantities were evaluated in a large number of scenarios (some of which are reported below). The findings can be summarized as follows:

(i) In all situations considered, we had $\Gamma_{\mathcal{E}}(\boldsymbol{p}) \leq \Gamma_{\mathcal{S}}(\boldsymbol{p})$. In other words, everything else being equal, resequencing is felt more strongly in the heavy tail case;

(ii) When $\mu_1 = \ldots = \mu_K$ (in which case $B_1 = \ldots = B_K$), we found

$$\Gamma_{\mathcal{S}}(\boldsymbol{p}') \leq \Gamma_{\mathcal{S}}(\boldsymbol{p}) \tag{83}$$

whenever $\boldsymbol{p}$ is "more balanced" than $\boldsymbol{p}'$. A formal way to express this situation is to say that $\boldsymbol{p}$ is majorized by $\boldsymbol{p}'$, written $\boldsymbol{p} \prec \boldsymbol{p}'$ [11]. This is equivalent to

$$\sum_{\ell=1}^{k} p'_{(\ell)} \leq \sum_{\ell=1}^{k} p_{(\ell)}, \;\; k = 1, \ldots, K-1 \tag{84}$$

and[5]

$$\sum_{\ell=1}^{K} p'_{(\ell)} = \sum_{\ell=1}^{K} p_{(\ell)} \tag{85}$$

with $p_{(1)} \leq p_{(2)} \leq \ldots \leq p_{(K)}$ and $p'_{(1)} \leq p'_{(2)} \leq \ldots \leq p'_{(K)}$ denoting the components of $\boldsymbol{p}$ and $\boldsymbol{p}'$ arranged in increasing order, respectively. The comparison (83) is certainly in line with intuition since as $\boldsymbol{p}$ gets more skewed, more customers are routed to a smaller set of queues and these customers will not be mis-ordered; and

(iii) The property (83) does not hold in the exponential case. In many cases, $\Gamma_{\mathcal{E}}(\boldsymbol{p})$ appears to be less sensitive to changes in load allocation compared to $\Gamma_{\mathcal{S}}(\boldsymbol{p})$.

We illustrate these findings through the following numerical example displayed in Figure 2. The parameters used are $K = 3$, $\lambda = 5$, $\beta = 3$ and $\mu_1 = \mu_2 = \mu_3 = 4$. Calculations were performed for the four load allocations $\boldsymbol{p}_1 = (0.33, 0.33, 0.34)$, $\boldsymbol{p}_2 = (0.25, 0.25, 0.5)$, $\boldsymbol{p}_3 = (0.1, 0.35, 0.55)$ and $\boldsymbol{p}_4 = (0.05, 0.35, 0.6)$. The data points in cross and circle are for $\Gamma_{\mathcal{E}}(\boldsymbol{p})$ and $\Gamma_{\mathcal{S}}(\boldsymbol{p})$, respectively. Note that $\boldsymbol{p}_1 \prec \boldsymbol{p}_2 \prec \boldsymbol{p}_3 \prec \boldsymbol{p}_4$, and that the corresponding data points have been arranged in that order on the x-axis at $x = 1$, $x = 2$, $x = 3$ and $x = 4$, repectively. As shown in the figure, $\Gamma_{\mathcal{S}}(\boldsymbol{p})$ is indeed monotone decreasing.

[5] Obviously, here the condition (85) automatically holds since the vectors of interest are load allocation vectors.

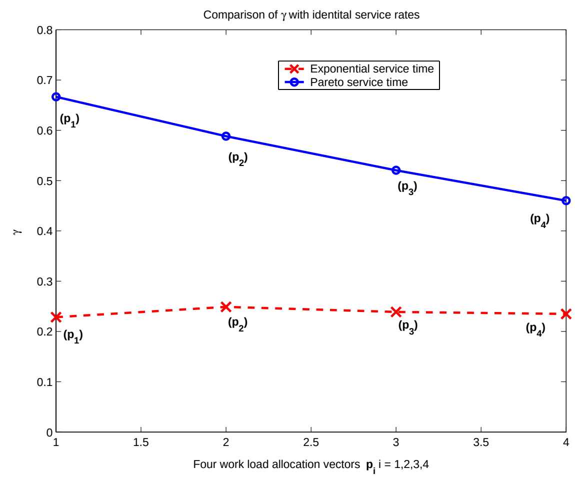

Fig. 2. Comparison of γ with equal service rates

XI. Conclusions and Future Work

We have studied the impact of packet mis-ordering due to multipath routing on their resequencing delay using a simple queueing model. A new metric γ was introduced as the asymptotic ratio between the tail probability of the resequencing delay and that of the end-to-end delay. We evaluate this parameter under different service time distributions (and hence different delay statistics along the multiple paths). By examining γ, we are now in a position to better understand whether the long end-to-end delay experienced by a customer is due to its resequencing delay.

A few words on the future work. Obviously, we can minimize γ by simply sending all of the traffic to one of the stations. Thus, minimizing γ alone is somewhat counterproductive as this would of course negate the potential benefits of path diversity. On the other hand, it is proved in [8] that when the service time distributions are identical for all queues, the uniform load allocation minimizes the sojourn time in the stochastic order and the end-to-end delay in the convex increasing order. The remaining question is how to select a load allocation vector so that the expected end-to-end delay is minimized while keeping γ below some given threshold. Admittedly our choice of service time distributions with various tail behavior was driven mainly by mathematical convenience. However, validating this tail behavior (of both service times and delays) in current networks is an interesting direction for future work.

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APPENDIX I
A PROOF OF PROPOSITION 5.2

Fix $k = 1, \ldots, K$. Lemma 4.1 yields

$$\mathbf{P}\left[\widetilde{W}_k > x\right] \sim \sum_{\ell \neq k} \rho_\ell e^{-\alpha_\ell x} \sim \Gamma_k \cdot e^{-\alpha_k^\star x} \qquad (86)$$

where $a_k^\star$ and Γ_k are given by (27) and (28), respectively.

Fix $x \geq 0$. The independence of the rvs $\widetilde{W}_k$ and T_k yields

$$\mathbf{P}\left[\widetilde{W}_k > x + T_k\right] = \alpha_k \int_0^\infty \mathbf{P}\left[\widetilde{W}_k > x + t\right] e^{-\alpha_k t}\ dt,$$

which leads to

$$\begin{aligned} & e^{a_k^\star x}\mathbf{P}\left[\widetilde{W}_k > x + T_k\right] \\ = \quad & \alpha_k \int_0^\infty \frac{\mathbf{P}\left[\widetilde{W}_k > x + t\right]}{e^{-\alpha_k^\star (x+t)}} \cdot e^{-(\alpha_k^\star + \alpha_k)t}\ dt. \qquad (87) \end{aligned}$$

From (86), for every $\varepsilon > 0$, there exists $x_k^\star = x_k^\star(\varepsilon) > 0$ such that

$$\Gamma_k(1-\varepsilon) \leq \frac{\mathbf{P}\left[\widetilde{W}_k > x\right]}{e^{-\alpha_k^\star x}} \leq \Gamma_k(1+\varepsilon)$$

whenever $x \geq x_k^\star$. Therefore, on that range, it is also the case that the bounds

$$e^{a_k^\star x}\mathbf{P}\left[\widetilde{W}_k > x + T_k\right] \leq \frac{\alpha_k}{\alpha_k^\star + \alpha_k}\Gamma_k(1+\varepsilon)$$

and

$$\frac{\alpha_k}{\alpha_k^\star + \alpha_k}\Gamma_k(1-\varepsilon) \leq e^{\alpha_k^\star x}\mathbf{P}\left[\widetilde{W}_k > x + T_k\right]$$

hold since

$$\int_0^\infty (\alpha_k^\star + \alpha_k)e^{-(\alpha_k^\star + \alpha_k)t} = 1.$$

It is now plain that

$$\mathbf{P}\left[\widetilde{W}_k > x + T_k\right] \sim \frac{\alpha_k}{\alpha_k^\star + \alpha_k}\Gamma_k e^{-\alpha_k^\star x} \qquad (88)$$

given that $\varepsilon > 0$ is arbitrary.

Next, returning to (10), we conclude that

$$\begin{aligned} \mathbf{P}[R > x] \quad & \sim \quad \sum_{k=1}^K p_k \frac{\alpha_k}{\alpha_k^\star + \alpha_k}\Gamma_k e^{-\alpha_k^\star x} \\ & \sim \quad \left(\sum_{k \in \mathcal{M}_\star} p_k \frac{\alpha_k}{\alpha_k^\star + \alpha_k}\Gamma_k\right) e^{-\alpha_\star x} \qquad (89) \end{aligned}$$

where we have set

$$\alpha_\star = \min\left(\alpha_1^\star, \ldots, \alpha_K^\star\right) \qquad (90)$$

and

$$\mathcal{M}_\star := \{k = 1, \ldots, K :\ \alpha_k^\star = \alpha_\star\}. \qquad (91)$$

We reconcile (89) with the desired result (30) upon noting that $\alpha_\star = \alpha^\star$ so that $\mathcal{M}_\star = \mathcal{M}^\star$.

APPENDIX II
A PROOF OF PROPOSITION 6.4

Claim (i) is Claim (b) of Lemma 1.3.5 [6, p. 41]. Claim (ii) is a simple consequence of the membership $X \in \mathcal{L}$ upon taking x going to infinity in the relation

$$\frac{\mathbf{P}[X^\star > x]}{\mathbf{P}[X > x]} = \mathbf{E}[X]^{-1} \int_x^\infty \frac{\mathbf{P}[X > x + t]}{\mathbf{P}[X > x]}\ dt$$

valid for all $x > 0$. Details are standard and left to the interested reader.

Finally, set

$$P(x) := \mathbf{P}[X > x], \quad x \geq 0.$$

By the independence of the rvs X and Y, we have

$$\frac{\mathbf{P}[X > x + Y]}{\mathbf{P}[X > x]} = \mathbf{E}\left[\frac{P(x+Y)}{P(x)}\right], \quad x \geq 0. \qquad (92)$$

Now, let x go to infinity in this last relation, and observe that $x \to P(x)$ is non-increasing over $\mathbb{R}_+$. The Bounded Convergence Theorem yields

$$\lim_{x\to\infty} \frac{\mathbf{P}[X > x + Y]}{\mathbf{P}[X > x]} = \mathbf{E}\left[\lim_{x\to\infty} \frac{P(x+Y)}{P(x)}\right] = 1$$

since $X \in \mathcal{L}$, and the desired conclusion follows. $\square$

Appendix III
A proof of Proposition 7.2

The proof is similar to that of Proposition 5.2 given in Appendix I: Fix $k = 1, \ldots, K$. Using the Cramér-Lundberg approximation (46) together with Lemma 4.1, we get

$$\mathbf{P}\left[\widetilde{W}_k > x\right] \sim \sum_{\ell \neq k} C_\ell e^{-\alpha_\ell x}. \tag{93}$$

Therefore, with $\alpha_k^\star$, $\mathcal{M}_k$ and Γ_k defined by (27), (29) and (56), respectively, we get

$$\mathbf{P}\left[\widetilde{W}_k > x\right] \sim \Gamma_k e^{-\alpha_k^\star x}. \tag{94}$$

Thus, for each $\varepsilon > 0$, there exists $x_k^\star = x_k^\star(\varepsilon) > 0$ such that

$$\Gamma_k (1-\varepsilon) \leq \frac{\mathbf{P}\left[\widetilde{W}_k > x\right]}{e^{-\alpha_k^\star x}} \leq \Gamma_k (1+\varepsilon) \tag{95}$$

whenever $x \geq x_k^\star$. Therefore, on that range, upon making use of the independence of the rvs $\widetilde{W}_k$ and T_k, we of the rvs $\widetilde{W}_k$ and T_k, we readily get the bounds

$$\begin{aligned}\mathbf{P}\left[\widetilde{W}_k > x + T_k\right] &\geq \Gamma_k (1-\varepsilon)\, \mathbf{E}\left[e^{-\alpha_k^\star (x+T_k)}\right] \\ &= \Gamma_k (1-\varepsilon)\, \mathbf{E}\left[e^{-\alpha_k^\star T_k}\right] e^{-\alpha_k^\star x}\end{aligned}$$

and

$$\begin{aligned}\mathbf{P}\left[\widetilde{W}_k > x + T_k\right] &\leq \Gamma_k (1+\varepsilon)\, \mathbf{E}\left[e^{-\alpha_k^\star (x+T_k)}\right] \\ &= \Gamma_k (1+\varepsilon)\, \mathbf{E}\left[e^{-\alpha_k^\star T_k}\right] e^{-\alpha_k^\star x}.\end{aligned}$$

With $\varepsilon > 0$ arbitrary, it follows that

$$\lim_{x \to \infty} \frac{\mathbf{P}\left[\widetilde{W}_k > x\right]}{e^{-\alpha_k^\star x}} = \Gamma_k \mathbf{E}\left[e^{-\alpha_k^\star (x+T_k)}\right]. \tag{96}$$

This constitutes the appropriate generalization of (88).

Reporting this fact into (10), we find

$$\begin{aligned}\mathbf{P}[R > x] &\sim \sum_{k=1}^K p_k \Gamma_k \mathbf{E}\left[e^{-\alpha_k^\star T_k}\right] e^{-\alpha_k^\star x} \\ &\sim \left(\sum_{k \in \mathcal{M}_\star} p_k \Gamma_k \mathbf{E}\left[e^{-\alpha_k^\star T_k}\right]\right) e^{-\alpha_\star x}\end{aligned} \tag{97}$$

with $\alpha_\star$ and $\mathcal{M}_\star$ as defined at (90) and (91), respectively. The desired result (57) follows from (97) upon noting again that $\alpha_\star = \alpha^\star$ and $\mathcal{M}_\star = \mathcal{M}^\star$. □

Appendix IV
A proof of Proposition 8.3

Fix $k = 1, \ldots, K$. Lemma 4.1 yields

$$\mathbf{P}\left[\widetilde{W}_k > x\right] \sim \sum_{\ell \neq k} \mathbf{P}[W_\ell > x] \tag{98}$$

where for each $\ell = 1, \ldots, K$, $W_\ell \in \mathcal{S}$, hence $W_\ell \in \mathcal{L}$, by Proposition 8.1. Consequently,

$$\mathbf{P}[W_\ell > x + t] \sim \mathbf{P}[W_\ell > x], \quad \ell = 1, \ldots, K$$

for each $t \geq 0$, and $\widetilde{W}_k \in \mathcal{L}$ as well. In $\widetilde{W}_k \in \mathcal{L}$ as well. In other words,

$$\lim_{x \to \infty} \frac{\mathbf{P}\left[\widetilde{W}_k > x + t\right]}{\mathbf{P}\left[\widetilde{W}_k > x\right]} = 1, \quad t > 0 \tag{99}$$

with the limit taking place from below. The rvs $\widetilde{W}_k$ and T_k being independent, the equivalence

$$\mathbf{P}\left[\widetilde{W}_k > x + T_k\right] \sim \mathbf{P}\left[\widetilde{W}_k > x\right] \tag{100}$$

follows from Claim (iii) of Proposition 6.4.

Returning to (10), we conclude from (98) and (100) that

$$\begin{aligned}\mathbf{P}[R > x] &\sim \sum_{k=1}^K p_k \mathbf{P}\left[\widetilde{W}_k > x\right] \\ &\sim \sum_{k=1}^K p_k \left(\sum_{\ell \neq k} \mathbf{P}[W_\ell > x]\right) \\ &\sim \sum_{\ell=1}^K \left(\sum_{k \neq \ell} p_k\right) \mathbf{P}[W_\ell > x]\end{aligned}$$

and (64) follows. □

Appendix V
A proof of Proposition 9.2

Fix $k = 1, \ldots, K$ and recall from Lemma 4.1 that

$$\mathbf{P}\left[\widetilde{W}_k > x\right] \sim \sum_{\ell \neq k} \mathbf{P}[W_\ell > x]. \tag{101}$$

For k in $\mathcal{E}$, it is plain from (73) and (101) that

$$\mathbf{P}\left[\widetilde{W}_k > x\right] \sim \sum_{\ell \in \mathcal{H}} \mathbf{P}[W_\ell > x]. \tag{102}$$

But for each ℓ in $\mathcal{H}$, $W_\ell \in \mathcal{S}$ by Proposition 8.1, whence $W_\ell \in \mathcal{L}$. Thus, the rvs W_ℓ and T_k being independent, we find

$$\mathbf{P}[W_\ell > x + T_k] \sim \mathbf{P}[W_\ell > x], \quad \ell \in \mathcal{H}$$

by Claim (iii) of Proposition 6.4, and it is now a simple matter to check that

$$\mathbf{P}\left[\widetilde{W}_k > x + T_k\right] \sim \sum_{\ell \in \mathcal{H}} \mathbf{P}[W_\ell > x], \quad k \in \mathcal{E}. \tag{103}$$

For k in $\mathcal{H}$, the situation is somewhat more involved: If $|\mathcal{H}| = 1$, write $\mathcal{H} = \{1\}$ and $\mathcal{E} = \{2, \ldots, K\}$ for sake of definiteness. Thus, k in $\mathcal{H}$ means $k = 1$, and (101) becomes

$$\mathbf{P}\left[\widetilde{W}_1 > x\right] \sim \sum_{\ell=2}^K \mathbf{P}[W_\ell > x] \tag{104}$$

with

$$\mathbf{P}[W_\ell > x] = o(\mathbf{P}[W_1 > x]), \quad \ell = 2, \ldots, K$$

by virtue of (73). This yields

$$\mathbf{P}\left[\widetilde{W}_1 > x\right] = o(\mathbf{P}\left[W_1 > x\right]), \tag{105}$$

whence

$$\mathbf{P}\left[\widetilde{W}_1 > x + T_1\right] = o(\mathbf{P}\left[W_1 > x\right]) \tag{106}$$

by a simple bounding argument. On the other hand, (103) specializes here to

$$\mathbf{P}\left[\widetilde{W}_k > x + T_k\right] \sim \mathbf{P}\left[W_1 > x\right], \quad k = 2, \ldots, K. \tag{107}$$

Combining (106) and (107) leads to

$$\begin{aligned} \mathbf{P}\left[R > x\right] &\sim \sum_{k=2}^{K} p_k \mathbf{P}\left[W_1 > x\right] \\ &\sim (1 - p_1)\mathbf{P}\left[W_1 > x\right] \end{aligned} \tag{108}$$

and (77) indeed holds.

Assume now that $|\mathcal{H}| > 1$. For each k in $\mathcal{H}$, we write $\mathcal{H}_k$ to denote the subset of $\mathcal{H}$ obtained by deleting k from it. The arguments, based on (73) and (101), and which lead to (102), can also be used to conclude

$$\mathbf{P}\left[\widetilde{W}_k > x\right] \sim \sum_{\ell \in \mathcal{H}_k} \mathbf{P}\left[W_\ell > x\right]. \tag{109}$$

Again, Claim (iii) of Proposition 6.4 can be used to validate the equivalence

$$\mathbf{P}\left[W_\ell > x + T_k\right] \sim \mathbf{P}\left[W_\ell > x\right], \quad \ell \in \mathcal{H}_k$$

since $W_\ell \in \mathcal{L}$ for ℓ in $\mathcal{H}_k$ and the rvs W_ℓ and T_k are independent. It is now a simple matter to conclude via (109) that

$$\mathbf{P}\left[\widetilde{W}_k > x + T_k\right] \sim \sum_{\ell \in \mathcal{H}_k} \mathbf{P}\left[W_\ell > x\right], \quad k \in \mathcal{H}. \tag{110}$$

Combining (103) and (110) yields

$$\begin{aligned} \mathbf{P}\left[R > x\right] &\sim \sum_{k \in \mathcal{E}} p_k \sum_{\ell \in \mathcal{H}} \mathbf{P}\left[W_\ell > x\right] \\ &\quad + \sum_{k \in \mathcal{H}} p_k \sum_{\ell \in \mathcal{H}_k} \mathbf{P}\left[W_\ell > x\right] \\ &\sim \left(\sum_{k \in \mathcal{E}} p_k\right) \left(\sum_{\ell \in \mathcal{H}} \mathbf{P}\left[W_\ell > x\right]\right) \\ &\quad + \sum_{k \in \mathcal{H}} p_k \left(\sum_{\ell \in \mathcal{H}} \mathbf{P}\left[W_\ell > x\right] - \mathbf{P}\left[W_k > x\right]\right) \\ &\sim \left(\sum_{k \in \mathcal{E}} p_k + \sum_{k \in \mathcal{H}} p_k\right) \sum_{\ell \in \mathcal{H}} \mathbf{P}\left[W_\ell > x\right] \\ &\quad - \sum_{k \in \mathcal{H}} p_k \mathbf{P}\left[W_k > x\right] \end{aligned}$$

and the conclusion (77) follows. □