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Lyapunov-Based Feedback Control of Border Collision Bifurcations in Piecewise Smooth Systems

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***Abstract*— Feedback control of piecewise smooth discrete-time systems that undergo border collision bifurcations is considered. These bifurcations occur when a fixed point or a periodic orbit of a piecewise smooth system crosses or collides with the border between two regions of smooth operation as a system parameter is quasistatically varied. The goal of the control effort in this work is to modify the bifurcation so that the bifurcated steady state is locally attracting and locally unique. To achieve this, Lyapunov-based techniques are used. A sufficient condition for nonbifurcation with persistent stability in piecewise smooth maps of dimension n that depend on a parameter is derived. The derived condition is in terms of linear matrix inequalities. This condition is then used as a basis for the design of feedback controls to eliminate border collision bifurcations in piecewise smooth maps and to produce desirable behavior.**

I. Introduction

Stabilizing feedback control laws for piecewise smooth discrete-time systems exhibiting border collision bifurcations are developed. The class of piecewise smooth systems considered constitutes systems that are smooth everywhere except along borders separating regions of smooth behavior where the system is only continuous. Border collision bifurcations are bifurcations that occur when a fixed point (or a periodic orbit) of a piecewise smooth system crosses or collides with the border between two regions of smooth operation. The term border collision bifurcation was coined by Nusse and Yorke [1]. Border collision bifurcation had been studied in the Russian literature under the name C-bifurcations by Feigin [2], [3]. The results of Feigin were introduced to the Western literature in [4].

Border collision bifurcations (BCBs) include bifurcations that are reminiscent of the classical bifurcations in smooth systems such as the fold and period doubling bifurcations. Despite such resemblances, the classification of border collision bifurcations is far from complete, and certainly very preliminary in comparison to the results available in the smooth case. In smooth maps, a bifurcation occurs from a one-parameter family of fixed points when a real eigenvalue or a complex conjugate pair of eigenvalues crosses the unit circle. In piecewise smooth (PWS) maps, on the other hand, a discontinuous change in the eigenvalues of the Jacobian matrix evaluated at the fixed point (or at a periodic point) occurs when the fixed point hits the border. As a result, border collision bifurcations for piecewise smooth systems in which the one-sided derivatives on the border are finite are classified based on the linearizations of the system on both sides of the border at criticality. The phenomenon of border collision bifurcations has been observed both numerically and experimentally in many systems, in applications such as power electronic devices [5], [6], [7] and in studies of grazing impact in mechanical oscillators [8], [9].

There is little past work on control of BCBs [10], [11], [12]. The control method of [10], [11] is based on the classification scheme of BCBs that was given by Feigin [4]. However, since Feigin didn't give conditions for specific scenarios, the results of [10], [11] do not address stabilization. Moreover, references [10], [11] use a trial and error approach that doesn't provide analytical conditions for existence of controllers. In our recent work [12], feedback control of BCBs in one- and two-dimensional PWS maps has been considered.

In the present paper, a sufficient condition for *nonbifurcation with persistent stability* in piecewise smooth maps of dimension n that depend on a parameter is derived. That is, a condition under which the PWS map possesses a locally asymptotically stable fixed point which is also the locally unique attractor for all values of the bifurcation parameter in a neighborhood of the critical value. This condition is derived using Lyapunov-based methods and is given in terms of linear matrix inequalities (LIMs). The derived condition is then used as a basis for the design of feedback controls to eliminate BCBs in piecewise smooth maps and to produce desirable behavior. The analysis and control methodology presented in this paper have recently been applied to a model of cardiac arrhythmia [13].

The paper proceeds as follows. In Sec. II, brief background material on BCBs is given. In Sec. III, Lyapunov-based analysis of PWS maps undergoing BCBs is presented and a sufficient condition for nonbifurcation with persistent stability is derived. In Sec. IV, the results of Sec. III are used in the synthesis of stabilizing feedback control laws and numerical examples that demonstrate the results are given. Concluding remarks are collected in Sec. V.

II. Background on Border Collision Bifurcations

Consider the one-parameter family of piecewise smooth maps

$$f(x,\mu) \quad = \quad \begin{cases} f_A(x,\mu), & x \in R_A \\ f_B(x,\mu), & x \in R_B \end{cases} \tag{1}$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is piecewise smooth in x; f is smooth in x everywhere except on the border (hypersurface Γ) separating R_A and R_B where it is only continuous, f is smooth in μ and R_A, R_B are the two (nonintersecting) regions of smooth behavior. Of great interest is the study of the dynamics of f at a fixed point (or a periodic orbit) near or at the border Γ. If the fixed point (or periodic orbit) is in R_A (respectively R_B) and is away from the border, then the local dynamics is determined by the single map f_A (respectively f_B). If, on the other hand, the fixed point is close to the border, then jumps across the border can occur except in an extremely small neighborhood of the fixed point. Therefore, for operation close to the border, both f_A and f_B are needed in the study of the possible behavior. For a fixed point at or near the border, the dynamics is determined by the linearizations of the map on both sides of the border.

Various types of BCBs occur in (1) as the bifurcation parameter μ is varied through a critical value [1], [14], [15], [4]. Such bifurcations occurring in the map (1) can be studied using the piecewise-linearized representation [4]

$$x(k+1) := F(x(k),\mu) \quad = \quad \begin{cases} Ax(k) + b\mu, & x_1(k) \leq 0 \\ Bx(k) + b\mu, & x_1(k) > 0 \end{cases} \tag{2}$$

where A is the linearization of the PWS map f in R_A at a fixed point on the border approached from points in R_A near the border and B is the linearization of f at a fixed point on the border approached from points in R_B and b is the derivative of the map f with respect to μ. The coordinate system is chosen such that the sign of the first component of the vector x determines whether x is in R_A or R_B (a transformation to the form (2) is given in [16]). If $x_1 = 0$, then x is on the border separating R_A and R_B. The continuity of F at the border implies that A and B differ only in their first columns.

The classification of BCBs depends on the eigenvalues of A and B [4]. A complete classification of BCBs is only available for one-dimensional PWS maps [17], [18]. For two dimensional PWS maps, some results are available that only address a class of two-dimensional PWS maps [1], [19], [15], [20].

Although Feigin [4] studied general n-dimensional PWS maps exhibiting border collisions, only very general conditions for existence of a fixed point and period-2 solutions before and after the border were given. The classification scheme of [4] does not give any information about stability or uniqueness of fixed points or period-2 orbits involved in the border collision bifurcation nor does it give information about higher period periodic orbits or chaos that might be involved in a border collision bifurcation. Therefore, in this paper, one of the main goals is to develop a sufficient condition for nonbifurcation with persistent stability that can be used in the design of stabilizing feedback control laws. This is done in the next section using Lyapunov-based techniques.

III. Lyapunov-Based Analysis of Piecewise Smooth Maps

Recently, many researchers have studied stability of a fixed point of switched discrete-time linear systems (e.g., [21], [22], [23]) as well as continuous time switched systems (e.g., [24]). In all the referenced studies, Lyapunov techniques were used to obtain sufficient conditions for stability of the fixed point (or equilibrium point) of a piecewise linear system. For instance, in [24], [22], quadratic as well as piecewise quadratic Lyapunov functions were used in the analysis of stability of switched systems and also in the synthesis of stabilizing controls. The author is unaware of any previous study using Lyapunov methods to analyze the dynamics of switched systems depending on a parameter. Here, a quadratic Lyapunov function is used to study border collision bifurcations in PWS maps and to obtain sufficient conditions for nonbifurcation with persistent stability in such maps.

Consider the piecewise-linearized representation of PWS maps given in (2). The sign of the first component of the vector x determines whether x is in R_A or in R_B. If $x_1 = 0$, then x is on the border separating R_A and R_B. The continuity of F at the border implies that A and B differ only in their first columns. That is, $a_{ij} = b_{ij}$, for $j \neq 1$, where $A = [a_{ij}]$ and $B = [b_{ij}]$.

Assume that $1 \notin \sigma(A)$, $1 \notin \sigma(B)$ (i.e., both $I-A$, $I-B$ are nonsingular). Formally solving for the fixed points of (2) yields $x_A(\mu) = (I-A)^{-1}b\mu$ and $x_B(\mu) = (I-B)^{-1}b\mu$. For $x_A(\mu)$ to actually occur, the first component of $x_A(\mu)$ must be nonpositive. That is,

$$x_{A_1}(\mu) \quad = \quad (e^1)^T \mu (I-A)^{-1} b \leq 0 \tag{3}$$

where $(e^1)^T = (1 \ 0 \ \cdots \ 0)$. Similarly, for $x_B(\mu)$ to actually occur, one needs

$$x_{B_1}(\mu) \quad = \quad (e^1)^T \mu (I-B)^{-1} b > 0. \tag{4}$$

If on the other hand, the first component of $x_A(\mu)$ is positive (the first component of $x_B(\mu)$ is nonpositive), then the fixed point is called a virtual fixed point. Virtual fixed points are important in studying the dynamics of a PWS map at or near the border.

Let $p_A(\lambda)$ and $p_B(\lambda)$ be the characteristic polynomials of A and B, respectively. Then, $p_A(\lambda) = \det(\lambda I - A)$ and $p_B(\lambda) = \det(\lambda I - B)$.

The fixed points can be written as

$$\begin{aligned} x_A(\mu) &= (I-A)^{-1} b\mu \\ &= \frac{\operatorname{adj}(I-A) b\mu}{\det(I-A)} \\ &= \frac{\bar{b}_A}{p_A(1)}\mu, \end{aligned} \tag{5}$$

and

$$\begin{aligned} x_B(\mu) &= (I-B)^{-1} b\mu \\ &= \frac{\mathrm{adj}(I-B)b\mu}{\det(I-B)} \\ &= \frac{\bar{b}_B}{p_B(1)}\mu, \end{aligned} \quad (6)$$

where $\bar{b}_A = \mathrm{adj}(I-A)b$ and $\bar{b}_B = \mathrm{adj}(I-B)b$. It can be shown that $\bar{b}_{A_1} = \bar{b}_{B_1} =: \bar{b}_1$ [4]. To see this, recall that A and B differ only in their first columns and $\mathrm{adj}(I-A) = (cof(I-A))^T$. Thus, the first row of $\mathrm{adj}(I-A)$ is equal to the first row of $\mathrm{adj}(I-B)$, which implies that $(e^1)^T adj(I-A)b = (e^1)^T adj(I-B)b =: \bar{b}_1$. Thus, the first component of $x_A(\mu)$ is $x_{A_1}(\mu) = \frac{\bar{b}_1}{p_A(1)}\mu$ and the first component of $x_B(\mu)$ is $x_{B_1}(\mu) = \frac{\bar{b}_1}{p_B(1)}\mu$. For the fixed point $x_A(\mu)$ to occur for $\mu \le 0$, it is required that $x_{A_1}(\mu) \le 0$, i.e., $\frac{\bar{b}_1}{p_A(1)}\mu \le 0 \Longleftrightarrow \frac{\bar{b}_1}{p_A(1)} > 0$. Similarly, for the fixed point $x_B(\mu)$ to occur for $\mu > 0$, it is required that $x_{B_1}(\mu) > 0$, i.e., $\frac{\bar{b}_1}{p_B(1)}\mu > 0 \Longleftrightarrow \frac{\bar{b}_1}{p_B(1)} > 0$. Therefore, a necessary and sufficient condition to have a fixed point for all μ is $p_A(1)p_B(1) > 0$ which is assumed to be in force in the remainder of the discussion.

Next, a change of variables is performed on (2) to simplify the analysis.

Case 1): $\underline{\mu \le 0}$. The fixed point of F is $x_A(\mu) = (I-A)^{-1}b\mu$. Changing the state variable in (2) to $z = x - x_A(\mu)$ yields after simplification

$$z(k+1) = \begin{cases} Az(k), & \text{if } z_1(k) \le -x_{A_1}(\mu) \\ Bz(k) + c\mu, & \text{if } z_1(k) > -x_{A_1}(\mu) \end{cases} \quad (7)$$

where $c = (B-A)(I-A)^{-1})b$. In the new coordinates, $z = 0$ is a fixed point for all $\mu \le 0$. (Note that the border $z_{border} = \{z : z_1 = -x_{A_1}(\mu)\}$, varies as a function of μ.) Note that since B and A differ only in their first columns, all elements of $B-A$ are zero except for the first column. Thus, $c\mu = (B-A)(I-A)^{-1}b\mu = (B-A)x_A(\mu) = x_{A_1}(\mu)(B^1 - A^1)$, where the notation A^i means the i-th column of the matrix A.

Consider the quadratic Lyapunov function candidate

$$V(z) = z^T P z, \qquad \text{where } P = P^T > 0 \quad (8)$$

The forward difference of V along trajectories of (7) is $\Delta V(z(k)) = V(z(k+1) - V(z(k))$. There are two cases: $z_1(k) \le -x_{A_1}(\mu)$ and $z_1(k) > -x_{A_1}(\mu)$.

Case 1.1): $z_1(k) \le -x_{A_1}(\mu)$

$$\begin{aligned} \Delta V_L(z(k)) &= V(z(k+1)) - V(z(k)) \\ &= (Az(k))^T PAz(k) - z(k)^T Pz(k) \\ &= z(k)^T (A^T PA - P) z(k) \end{aligned} \quad (9)$$

Case 1.2): $z_1(k) > -x_{A_1}(\mu)$

$$\begin{aligned} \Delta V_R(z(k)) &= V(z(k+1)) - V(z(k)) \\ &= (Bz(k) + c\mu)^T P(Bz(k) + c\mu) - z(k)^T Pz(k) \\ &= z(k)^T (B^T PB - P) z(k) + 2\mu c^T PBz(k) + \mu^2 c^T Pc \\ &= z(k)^T (B^T PB - P) z(k) + 2x_{A_1}(\mu)(B^1 - A^1)^T PBz(k) \\ &\quad + x_{A_1}^2(\mu)(B^1 - A^1)^T P(B^1 - A^1) \end{aligned} \quad (10)$$

Combining (9) and (10) yields

$$\Delta V(z(k)) = \begin{cases} \Delta V_L(z(k)), & \text{if } z_1(k) \le -x_{A_1}(\mu) \\ \Delta V_R(z(k)), & \text{if } z_1(k) > -x_{A_1}(\mu) \end{cases} \quad (11)$$

From (9) and (10), a necessary condition for $\Delta V(z(k))$ to be negative definite is that the following two matrix inequalities hold:

$$A^T PA - P < 0, \quad (12)$$
$$B^T PB - P < 0. \quad (13)$$

Moreover, the following claim, which asserts sufficiency of (12),(13) for negative definiteness of $\Delta V(z(k))$ is stated and proved.

Claim: (Sufficiency of LMIs (12)-(13) for Decreasing Lyapunov Function)
If the matrix inequalities (12)-(13) are satisfied with $P = P^T > 0$, then $\Delta V(z(k))$ given by (11) is negative definite.

Proof: Assume that there is a $P = P^T > 0$ such that (12)-(13) are satisfied. Then, $\Delta V_L = z^T(A^T PA - P)z < 0\ \forall z \ne 0$. It remains to show that $\Delta V_R < 0$. Let $z = (z_1, z_2)^T$, where $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}^{n-1}$. Note that ΔV is continuous for all z. Continuity of ΔV follows from the continuity of V and continuity of the map (7). Since $\Delta V_L < 0$ ($\Delta V_L = 0$ if and only if $z = 0$) and ΔV is continuous for all z, it follows that $\Delta V_R < 0$ at the border $\{z_1 = -x_{A_1}(\mu)\}$ (since $\lim_{(z_1,z_2)\to(-x_{A1}^-(\mu),z_2)} \Delta V_L = \lim_{(z_1,z_2)\to(-x_{A1}^+(\mu),z_2)} \Delta V_R$). It remains to show that $\Delta V_R < 0$ for all z in the region $z_1 > -x_{A_1}(\mu)$ (note that $-x_{A_1}(\mu) > 0$). Completing the squares in (10) allows us to write $\Delta V_R(z)$ as follows:

$$\begin{aligned} \Delta V_R(z) &= z^T(B^T PB - P)z + 2x_{A_1}(\mu)(B^1 - A^1)^T PBz \\ &\quad + x_{A_1}^2(\mu)(B^1 - A^1)^T P(B^1 - A^1) \\ &= (z - \alpha)^T (B^T PB - P)(z - \alpha) - \alpha^T (B^T PB - P)\alpha \\ &\quad + x_{A_1}^2(\mu)(B^1 - A^1)^T P(B^1 - A^1) \end{aligned} \quad (14)$$

where $\alpha = -x_{A_1}(\mu)(B^T PB - P)^{-1} B^T P(B^1 - A^1)$. Let $\mathcal{N} \subset \mathbb{R}^n$ such that $\mathcal{N}$ is convex and contains the origin (for example, a ball). Since the fixed point $x_A(\mu)$ is close to the origin for small μ, the hyperplane $z_1 = -x_{A_1}(\mu)$ slices the neighborhood $\mathcal{N}$. Consider $\Delta V_R(z)$ restricted to $\mathcal{N}$. The second derivative of $\Delta V_R(z)$ with respect to z (i.e., its Hessian matrix) is $\nabla^2 \Delta V_R = 2(B^T PB - P) < 0$. Thus, $\Delta V_R(z)$ is strictly concave on $\mathcal{N}$, i.e., for every z, $y \in \mathcal{N}$, and $\theta \in (0,1)$, $\Delta V_R(\theta z + (1-\theta)y) > \theta \Delta V_R(z) + (1-\theta)\Delta V_R(y)$. Note that $\Delta V_R(0) = x_{A_1}^2(\mu)(B^1 - A^1)^T P(B^1 - A^1) > 0$. Now, we show that $\Delta V_R < 0\ \forall z \in \mathcal{N}$ with $z_1 > -x_{A_1}(\mu)$. By way of contradiction, suppose there is a $y \in \mathcal{N}$, with $y_1 > -x_{A1}(\mu)$, such that $\Delta V_R(y) > 0$. Since $\Delta V_R(z)$ is strictly concave, it follows that $\Delta V_R(z)$ is positive along the line segment connecting 0 and y: $\Delta V_R(\theta \cdot 0 + (1-\theta)y) > \theta \underbrace{\Delta V_R(0)}_{>0} + (1-\theta)\underbrace{\Delta V_R(y)}_{>0} > 0$, $\forall \theta \in (0,1)$. But, along the line connecting $z = 0$ with $z = y$, there is a point z^* with $z_1^* = -x_{A_1}(\mu)$ where

$\Delta V_R(z^*) < 0$, which is a contradiction. Thus, $\Delta V_R(z) < 0$ for all $z \in \mathcal{N}$ with $z_1 > -x_{A_1}(\mu) > 0$. ■

The following proposition summarizes the results so far.

Proposition 1: The forward difference of $V = z^T Pz$, with $P = P^T > 0$, along trajectories of (7) with $\mu \leq 0$ is negative definite (i.e., $\Delta V(z) < 0$) if and only if the following matrix inequalities hold:

$$A^T PA - P < 0, \quad (15)$$
$$B^T PB - P < 0. \quad (16)$$

Case 2): $\underline{\mu > 0}$. The fixed point of F is $x_B(\mu) = (I-B)^{-1} b\mu$. Changing the state variable in (2) to $z = x - x_B(\mu)$ yields after simplification

$$z(k+1) = \begin{cases} Az(k) + c\mu, & \text{if } z_1(k) \leq -x_{B_1}(\mu) \\ Bz(k), & \text{if } z_1(k) > -x_{B_1}(\mu) \end{cases} \quad (17)$$

where $c = (A-B)(I-B)^{-1})b$. In the new coordinates, $z = 0$ is a fixed point for all $\mu > 0$. (Note that the border $z_{border} = \{z : z_1 = -x_{B_1}(\mu)\}$, varies as a function of μ.) Note that since B and A differ only in their first columns, all elements of $A-B$ are zero except for the first column. Thus, $c\mu = (A-B)(I-B)^{-1} b\mu = (A-B)x_B(\mu) = x_{B_1}(\mu)(A^1 - B^1)$.

Consider the same quadratic Lyapunov function candidate as in (8) above:

$$V(z) = z^T Pz, \qquad \text{where } P = P^T > 0$$

The forward difference of V along trajectories of (17) is $\Delta V(z(k)) = V(z(k+1) - V(z(k))$. There are two cases: $z_1(k) \leq -x_{B_1}(\mu)$ and $z_1(k) > -x_{B_1}(\mu)$. (Note that $x_{B1}(\mu) > 0$ from (4).)

Case 2.1): $z_1(k) \leq -x_{B_1}(\mu)$

$$\begin{aligned} \Delta V_L(z(k)) &= V(z(k+1)) - V(z(k)) \\ &= (Az(k) + c\mu)^T P(Az(k) + c\mu) - z(k)^T Pz(k) \\ &= z(k)^T (A^T PA - P)z(k) + 2\mu c^T PAz(k) + \mu^2 c^T Pc \\ &= z(k)^T (A^T PA - P)z(k) + 2x_{B_1}(\mu)(A^1 - B^1)^T PAz(k) \\ &\quad + x_{B_1}^2(\mu)(A^1 - B^1)^T P(A^1 - B^1) \end{aligned} \quad (18)$$

Case 2.2): $z_1(k) > -x_{B_1}(\mu)$

$$\begin{aligned} \Delta V_R(z(k)) &= V(z(k+1)) - V(z(k)) \\ &= (Bz(k))^T PBz(k) - z(k)^T Pz(k) \\ &= z(k)^T (B^T PB - P)z(k) \end{aligned} \quad (19)$$

Combining (18) and (19) yields

$$\Delta V(z(k)) = \begin{cases} \Delta V_L(z(k)), & \text{if } z(k) \leq -x_{B_1}(\mu) \\ \Delta V_R(z(k)), & \text{if } z(k) > -x_{B_1}(\mu) \end{cases} \quad (20)$$

Proposition 2: (Necessary and Sufficient Conditions for Decreasing Lyapunov Function)
The forward difference of $V = z^T Pz$, with $P = P^T > 0$, along trajectories of (17) with $\mu \geq 0$ is negative definite (i.e., $\Delta V(z) < 0$) if and only if the following matrix inequalities hold:

$$A^T PA - P < 0, \quad (21)$$
$$B^T PB - P < 0. \quad (22)$$

Proof: Necessity follows from (18) and (19), and the proof for sufficiency is similar to that for the case $\mu \leq 0$ above. ■

By combining Proposition 1 and Proposition 2, the main result of this paper is obtained.

Proposition 3: (Sufficient Condition for Nonbifurcation with Persistent Stability in n-Dimensional PWS Maps)
The PWS map (2) has a globally asymptotically stable fixed point for all $\mu \in \mathbb{R}$ if there is a $P = P^T > 0$ such that

$$A^T PA - P < 0,$$
$$B^T PB - P < 0.$$

Corollary 1: If at $\mu = 0$ the origin of the map (2) is quadratically stable, i.e., using a quadratic Lyapunov function $V = x^T Px$, with $P > 0$, then the fixed point depending on μ on both sides of the border is attracting and no bifurcation occurs from the origin as μ is varied through zero.

Below, a numerical example is given to demonstrate how the Lyapunov-based techniques considered in the previous section can be used in the stability and bifurcation analysis.

Example 1: Consider the three-dimensinal PWS map

$$x(k+1) = \begin{cases} Ax(k) + b\mu, & x_1(k) \leq 0 \\ Bx(k) + b\mu, & x_1(k) > 0 \end{cases} \quad (23)$$

where

$$A = \begin{pmatrix} 0.4192 & 0.3514 & 0.3473 \\ 0.2840 & -0.2733 & -0.3107 \\ 0.1852 & -0.2224 & -0.3974 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.60 & 0.3514 & 0.3473 \\ 0.56 & -0.2733 & -0.3107 \\ -0.90 & -0.2224 & -0.3974 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvalues of A and B are $\sigma(A) = \{0.5653, -0.7413, -0.0755\}$ and $\sigma(B) = \{0.0395, -0.6551 \pm j \ 0.4246\}$, respectively. Although both A and B are Schur stable matrices, it cannot be concluded that no bifurcation for (23) occurs at $\mu = 0$.

A common quadratic Lyapunov function $V = x^T Px$, with $P = P^T > 0$ that satisfies the conditions of Proposition 3 exists for this example. To wit:

$$P = \begin{pmatrix} 1.6304 & 0.1559 & -0.1313 \\ 0.1559 & 1.3200 & 0.4436 \\ -0.1313 & 0.4436 & 1.3266 \end{pmatrix}$$

is obtained using the MATLAB LMI toolbox. Thus, the PWS map (23) has a unique attracting fixed point for all μ (see Figure 1).

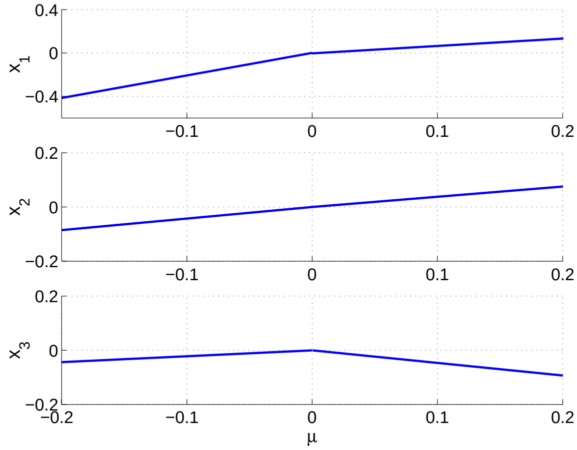

Fig. 1. Bifurcation diagram for Example 1. Each solid line represents a path of stable fixed points.

IV. Lyapunov-Based Feedback Control Design

In this section, the results of Section III are used in the design stabilizing feedback control laws. It is important to emphasize that for our approach to apply, the control action should not introduce discontinuity in the map. This is because, as discussed in the introduction, the definition of BCBs requires that the system map be continuous at the border, and thus our results on nonbifurcation with persistent stability also apply only under this condition. Therefore, to maintain continuity of the map after control is applied, it is assumed that the input vectors on both sides of the border are equal. In this work, the input vectors are taken to be equal to b (the derivative of the map with respect to the bifurcation parameter.)

Simultaneous feedback control is considered first, followed by switched feedback control design.

A. Simultaneous Feedback Control Design

In this control method, the same control is applied on both sides of the border. The purpose of pursuing stabilizing feedback acting on both sides of the border is to ensure robustness with respect to modeling uncertainty. Moreover, transformation to the normal form is not required when simultaneous control is used. All that is needed is a good estimate of the Jacobian matrices on both sides of the border.

Consider the closed-loop system using static linear state feedback

$$x(k+1) = \begin{cases} Ax(k)+b\mu+bu(k), & \text{if } x_1(k)\le 0 \\ Bx(k)+b\mu+bu(k), & \text{if } x_1(k)>0 \end{cases} \quad (24)$$

$$u(k) = gx(k) \quad (25)$$

where g is the control gain (row) vector.

The following proposition gives stabilizability condition for the border collision bifurcation with this type of control policy.

Proposition 4: If there exists a $P=P^T>0$, and a feedback gain (row) vector g such that

$$P-(A+bg)^T P(A+bg) > 0 \quad (26)$$

$$P-(B+bg)^T P(B+bg) > 0 \quad (27)$$

then, any border collision bifurcation that occurs in the open-loop system ($u\equiv 0$) of (24) can be eliminated and persistent stability is guaranteed using simultaneous feedback (25). Equivalently, if there exists a Q and y such that

$$\begin{pmatrix} Q & AQ+by \\ (AQ+by)^T & Q \end{pmatrix} > 0, \quad (28)$$

$$\begin{pmatrix} Q & BQ+by \\ (BQ+by)^T & Q \end{pmatrix} > 0, \quad (29)$$

then any border collision bifurcation that occurs in (24) can be eliminated using simultaneous feedback (25). Here $Q=P^{-1}$ and the feedback gain is given by $g=yP$.

Proof: The closed-loop system is given by

$$x(k+1) = \begin{cases} (A+bg)x(k)+\mu b, & \text{if } x_1(k)\le 0 \\ (B+bg)x(k)+\mu b, & \text{if } x_1(k)>0 \end{cases} \quad (30)$$

Using Proposition 3, a sufficient condition to eliminate the BCB is the existence of a $P=P^T>0$ such that

$$P-(A+bg)^T P(A+bg) > 0 \quad (31)$$

$$P-(B+bg)^T P(B+bg) > 0 \quad (32)$$

where g is the control gain to be chosen.

Next, inequalities (31)-(32) are shown to be equivalent to (28)-(29) using the Schur complement [25], [22]. It is straightforward to show that

$$\begin{aligned} & P-(A+bg)^T P(A+bg) > 0 \\ \Longleftrightarrow\ & P^{-1}-(A+bg)P^{-1}(A+bg)^T > 0, \end{aligned}$$

and

$$\begin{aligned} & P-(B+bg)^T P(B+bg) > 0 \\ \Longleftrightarrow\ & P^{-1}-(B+bg)P^{-1}(B+bg)^T > 0. \end{aligned}$$

The nonlinear matrix inequalities above are transformed into LMIs using the Schur complement [25], [22]:

$$\begin{aligned} & P^{-1}-(A+bg)P^{-1}(A+bg)^T \\ = \ & P^{-1}-(A+bg)P^{-1}PP^{-1}(A+bg)^T \\ = \ & P^{-1}-(AP^{-1}+bgP^{-1})P(AP^{-1}+bgP^{-1})^T > 0 \\ \Longleftrightarrow\ & \begin{pmatrix} P^{-1} & AP^{-1}+by \\ (AP^{-1}+by)^T & P^{-1} \end{pmatrix} > 0 \end{aligned}$$

Similarly,

$$\begin{aligned} & P-(B+bg)^T P(B+bg) > 0 \\ \Longleftrightarrow\ & \begin{pmatrix} P^{-1} & BP^{-1}+by \\ (BP^{-1}+by)^T & P^{-1} \end{pmatrix} > 0 \end{aligned}$$

by similar reasoning. ■

The following proposition states that if a CQLF exists in one coordinate system, another CQLF exists in a different coordinate system arrived at using a simultaneous similarity transformation applied to both A and B.

Proposition 5: (CQLF and Similarity Transformations) Suppose $V = x^T Px$ (with $P = P^T > 0$) is a common quadratic Lyapunov function for both of the matrices A and B (i.e., $A^T PA - P < 0$ and $B^T PB - P < 0$). Then $\tilde{V} = x^T \tilde{P}x$ with $\tilde{P} = (T^{-1})^T PT^{-1} = \tilde{P}^T > 0$ is a common quadratic Lyapunov function for $\tilde{A} = TAT^{-1}$ and $\tilde{B} = TBT^{-1}$ (i.e. $\tilde{A}^T \tilde{P}\tilde{A} - \tilde{P} < 0$ and $\tilde{B}^T \tilde{P}\tilde{B} - \tilde{P} < 0$). In other words, if a CQLF exists in one coordinate system, another CQLF exists if a simultaneous change of coordinates is applied to both A and B.

Proof: See [16].

Remark 1: The switching control design presented above does not depend on the border separating the two regions of smooth behavior. Thus, transformation to the normal form is not required before the control design.

B. Switched Feedback Control Design

Consider the closed-loop system using static piecewise linear state feedback

$$f_\mu(x(k)) = \begin{cases} Ax(k) + b\mu + bu(k), & \text{if} \quad x_1(k) \le 0 \\ Bx(k) + b\mu + bu(k), & \text{if} \quad x_1(k) > 0 \end{cases} \tag{33}$$

where

$$u(k) = \begin{cases} g_1 x(k), & x_1(k) \le 0 \\ g_2 x(k), & x_1(k) > 0 \end{cases} \tag{34}$$

with the restriction that g_1 and g_2 may only differ in their first component, i.e., $g_{1i} = g_{2i}$, $i = 2, 3, \cdots, n$. This condition is imposed to maintain continuity along the border $\{x : x_1 = 0\}$.

Proposition 6: If there exists a $P = P^T > 0$, and feedback gains g_1 and g_2 with $g_{1i} = g_{2i}$, $i = 2, 3, \cdots, n$ such that

$$P - (A + bg_1)^T P(A + bg_1) > 0, \tag{35}$$
$$P - (B + bg_2)^T P(B + bg_2) > 0, \tag{36}$$

then any border collision bifurcation that occurs in the open-loop system ($u \equiv 0$) of (33) can be eliminated using switching feedback (34). Equivalently, if there exists a Q, y_1 and $\alpha \in \mathbb{R}$ such that

$$\begin{pmatrix} Q & AQ + by_1 \\ (AQ + by_1)^T & Q \end{pmatrix} > 0, \tag{37}$$

$$\begin{pmatrix} Q & BQ + by_1 \\ (BQ + by_1)^T & Q \end{pmatrix} - \alpha \begin{pmatrix} 0 & b(e^1)^T Q \\ Qe^1 b^T & 0 \end{pmatrix} > 0, \tag{38}$$

then any border collision bifurcation that occurs in (33) can be eliminated using switching feedback (34). Here, $Q = P^{-1}$ and the feedback gains are given by $g_1 = y_1 P$ and $g_2 = g_1 - \alpha(e^1)^T$.

Proof: The closed-loop system is given by

$$x(k+1) = \begin{cases} (A + bg_1)x(k) + \mu b, & \text{if} \quad x_1(k) \le 0 \\ (B + bg_2)x(k) + \mu b, & \text{if} \quad x_1(k) > 0 \end{cases} \tag{39}$$

Using Proposition 3, a sufficient condition to eliminate the BCB is the existence of a $P = P^T > 0$ such that

$$P - (A + bg_1)^T P(A + bg_1) > 0 \tag{40}$$
$$P - (B + bg_2)^T P(B + bg_2) > 0 \tag{41}$$

where g_1, g_2 are the control gains to be chosen. Inequalities (40),(41) are equivalent to

$$\begin{pmatrix} Q & AQ + by_1 \\ (AQ + by_1)^T & Q \end{pmatrix} > 0 \tag{42}$$
$$\begin{pmatrix} Q & BQ + by_2 \\ (BQ + by_2)^T & Q \end{pmatrix} > 0 \tag{43}$$

respectively, where $Q = P^{-1}$, $g_1 = y_1 P$ and $g_2 = y_2 P$. This equivalence can be shown using similar reasoning as that used in the proof of Proposition 4.

The restriction $g_{1i} = g_{2i}$, $i = 2, 3, \cdots, n$, can be written as

$$g_2 = g_1 - \alpha(e^1)^T \tag{44}$$

where $\alpha \in \mathbb{R}$. Therefore,

$$\begin{aligned} y_1 - y_2 &= g_1 Q - g_2 Q \\ &= (g_1 - g_2)Q \\ &= \alpha(e^1)^T Q \end{aligned} \tag{45}$$

Substituting $y_2 = y_1 - \alpha(e^1)^T Q$ in (43) yields (38). This completes the proof. ∎

Note that if $\alpha = 0$ in (38), then the switching feedback control (34) becomes simultaneous control.

Remark 2: We remark that switching control design (with no restriction on feedback gains) was used in [22] for stabilization of the origin of discrete time switching systems. No bifurcation control was considered in the referenced work.

C. Numerical Examples

Below, numerical examples that demonstrate the proposed feedback control methods are given.

Example 2: (Fixed point attractor bifurcating to instantaneous chaos)
Consider the three dimensional PWS map

$$x(k+1) = \begin{cases} Ax(k) + b\mu, & x_1(k) \le 0 \\ Bx(k) + b\mu, & x_1(k) > 0 \end{cases} \tag{46}$$

where

$$A = \begin{pmatrix} 0.0334 & 1.7874 & -0.1705 \\ -0.4588 & -0.4430 & -0.8282 \\ 0.0474 & -0.0416 & 0.8000 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.8384 & 1.7874 & -0.1705 \\ -0.8180 & -0.4430 & -0.8282 \\ 0.6602 & -0.0416 & 0.8000 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvalues of A and B are $\sigma(A) = \{0.766, -0.1878 \pm j \;\; 0.8389\}$ and $\sigma(B) = \{-0.1157, 0.6555 \pm j \;\; 1.0987\}$, respectively. Note that A is Schur stable, but B is unstable. Simulation results show that (46) undergoes a border collision bifurcation from a fixed point attractor to instantaneous chaos at $\mu = 0$ (see Figure 2).

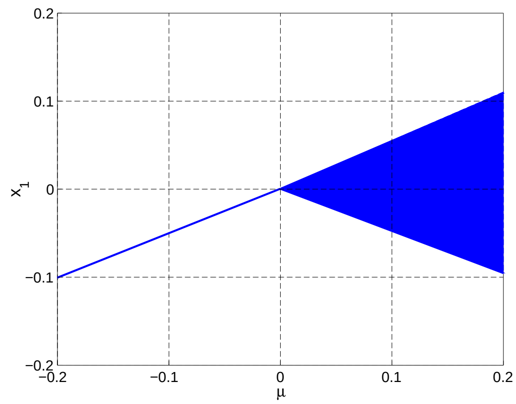

Fig. 2. Bifurcation diagram for Example 2. The solid line represents a path of stable fixed points and the shaded region represents a one piece chaotic attractor growing out of the fixed point at $\mu = 0$.

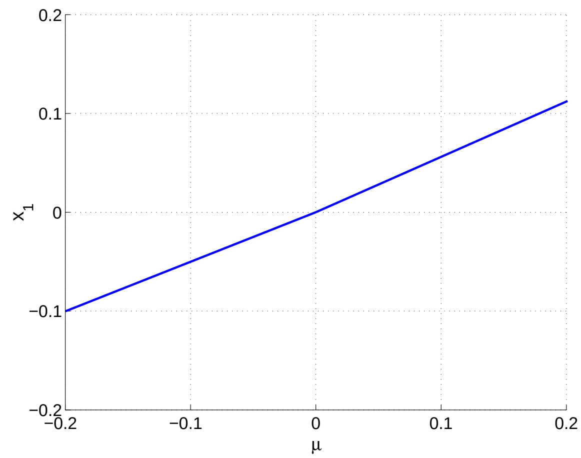

Fig. 3. Bifurcation diagram for Example 2 with simultaneous feedback control $u(k) = gx(k)$. The solid lines represent a path of stable fixed points.

Feedback control design: Using the results of Proposition 4, a symmetric and positive definite matrix Q and a feedback control gain vector g that satisfy the LMIs (28)-(29) are sought. A solution to (28)-(29) is obtained using the MATLAB LMI toolbox. To wit:

$$\begin{aligned} Q &= \begin{pmatrix} 0.4753 & -0.0428 & -0.1694 \\ -0.0428 & 0.8821 & -0.1647 \\ -0.1694 & -0.1647 & 0.5041 \end{pmatrix}, \\ y &= \begin{pmatrix} -0.1601 & -1.4937 & 0.3356 \end{pmatrix}, \\ g &= yQ^{-1} \\ &= \begin{pmatrix} -0.5193 & -1.7324 & -0.0747 \end{pmatrix}. \end{aligned}$$

The closed-loop matrices are given by

$$\begin{aligned} A_c &= A + bg \\ &= \begin{pmatrix} -0.4859 & 0.0550 & -0.2452 \\ -0.4588 & -0.4430 & -0.8282 \\ 0.0474 & -0.0416 & 0.8000 \end{pmatrix}, \\ B_c &= B + bg \\ &= \begin{pmatrix} 0.3191 & 0.0550 & -0.2452 \\ -0.8180 & -0.4430 & -0.8282 \\ 0.6602 & -0.0416 & 0.8000 \end{pmatrix}. \end{aligned}$$

Their eigenvalues are: $\sigma(A_c) = \{0.8141, -0.4715 \pm j\ 0.1409\}$ and $\sigma(B_c) = \{-0.4507, 0.5634 \pm j\ 0.3498\}$. The bifurcation diagram of the closed loop system is depicted in Figure 3.

Example 3: (Saddle-node border collision bifurcation) Consider the three dimensional PWS map

$$x(k+1) = \begin{cases} Ax(k) + b\mu, & x_1(k) \le 0 \\ Bx(k) + b\mu, & x_1(k) > 0 \end{cases} \tag{47}$$

where

$$A = \begin{pmatrix} 0.0350 & -0.2280 & -0.9385 \\ -0.3123 & -0.0029 & 0.9191 \\ -0.3825 & -0.5107 & 0.5553 \end{pmatrix},$$

$$B = \begin{pmatrix} 3.3000 & -0.2280 & -0.9385 \\ -0.6299 & -0.0029 & 0.9191 \\ 0.3705 & -0.5107 & 0.5553 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvalues of A and B are $\sigma(A) = \{-0.2921, 0.4397 \pm j\ 0.3470\}$ and $\sigma(B) = \{3.1739, 0.3392 \pm j\ 0.4756\}$, respectively. Note that A is Schur stable, but B is unstable. Simulation results show that (47) undergoes a saddle node border collision bifurcation where a stable and an unstable fixed point collide and disappear as μ is increased through zero (see Figure 4).

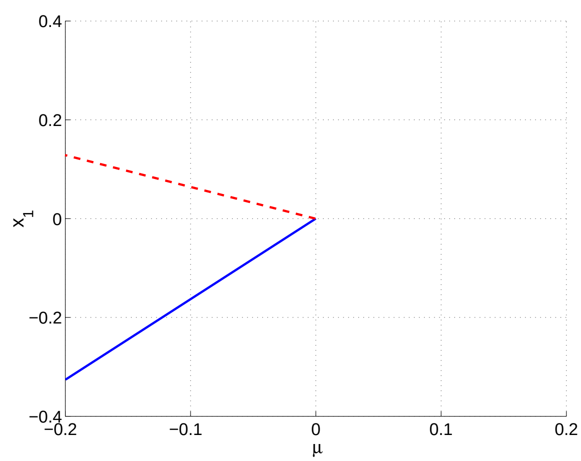

Fig. 4. Bifurcation diagram for Example 3 without control. The solid line represents a path of stable fixed points whereas the dashed line represents a path of unstable fixed points.

Feedback control design: Simultaneous stabilizing feedback control based on Proposition 4 does not exist for this example. Therefore, a stabilizing switched feedback control using Proposition 6 is sought. Using the LMI toolbox in MATLAB, a symmetric and positive definite matrix Q and a feedback control gain vectors g_1 and g_2 that satisfy the

LMIs (37)-(38) are obtained:

$$\begin{aligned}
\alpha &= 3.0972 \\
Q &= \begin{pmatrix} 25.3606 & 4.5507 & 7.9810 \\ 4.5507 & 43.0961 & 9.8713 \\ 7.9810 & 9.8713 & 30.8840 \end{pmatrix}, \\
y_1 &= \begin{pmatrix} 5.7709 & 14.8260 & 34.4887 \end{pmatrix}, \\
g_1 &= y_1 Q^{-1} \\
&= \begin{pmatrix} -0.1436 & 0.1024 & 1.1211 \end{pmatrix}, \\
g_2 &= g_1 - \alpha (e^1)^T \\
&= \begin{pmatrix} -3.2408 & 0.1024 & 1.1211 \end{pmatrix}.
\end{aligned}$$

The closed-loop matrices are given by

$$\begin{aligned}
A_c &= A + bg_1 \\
&= \begin{pmatrix} -0.1086 & -0.1256 & 0.1826 \\ -0.3123 & -0.0029 & 0.9191 \\ -0.3825 & -0.5107 & 0.5553 \end{pmatrix}, \\
B_c &= B + bg_2 \\
&= \begin{pmatrix} 0.0592 & -0.1256 & 0.1826 \\ -0.6299 & -0.0029 & 0.9191 \\ 0.3705 & -0.5107 & 0.5553 \end{pmatrix}.
\end{aligned}$$

Their eigenvalues are: $\sigma(A_c) = \{0.0011, 0.2213 \pm j\ 0.6236\}$ and $\sigma(B_c) = \{-0.0002, 0.3059 \pm j\ 0.5102\}$. The bifurcation diagram of the closed-loop system is similar to Figure 3.

V. Concluding Remarks

Lyapunov-based analysis of piecewise smooth discrete-time systems that undergo border collision bifurcations has been considered. One of the main contributions of this paper is that a sufficient condition for nonbifurcation with persistent stability in PWS maps of dimension n that depend on a parameter was derived. This condition has been used in the design of stabilizing feedback control laws to eliminate border collision bifurcations in PWS maps and produce desirable behavior. Numerical examples were given to demonstrate the efficacy of the proposed control techniques.

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