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Washout Filters in Feedback Control: Benefits, Limitations and Extensions

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Washout Filters in Feedback Control: Benefits, Limitations and Extensions

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***Abstract*— Advantages and limitations of washout filters in feedback control of both continuous-time and discrete-time systems are discussed and generalizations that alleviate the limitations are presented. Some previously unpublished results in the Ph.D. dissertation of one of the authors (Lee, 1991) are presented in the context of their relation to the generalized results and to recent publications on delayed feedback control. We show that delayed feedback control (for discrete time systems) extensively used in control of chaos is a special case of washout filter-aided feedback. Moreover, the limitations of delayed feedback control can be overcome by the use of washout filter-aided feedback, which gives rise to the possibility of stabilizing a much larger class of systems.**

I. Introduction

It is a common practice in the analysis of nonlinear systems and in feedback control design to assume that the equilibrium point (or the operating point) of the system is accurately known or does not change over the operating regime. However, models of physical dynamical systems are in general uncertain. Therefore, static feedback control is ineffective in addressing problems where the operating point is not accurately known or there is parameter drift.

Consider the nonlinear system described by

$$\dot{x} \quad = \quad f(x,u) \qquad \text{(continuous-time)} \qquad (1)$$

or

$$x(k+1) \quad = \quad f(x(k),u(k)) \quad \text{(discrete-time)} \qquad (2)$$

where $f(\cdot,\cdot)$ is uncertain, u is the scalar input and $x \in \Re^n$ is the state vector. Due to the uncertainty in f, the equilibrium points (if any) of the system (1) and the fixed points (if any) of (2) are also in general uncertain. Despite the uncertainty in the location of the equilibria, the objective in terms of control design centers around stabilization of some equilibrium condition. Typically, one expands $f(\cdot,\cdot)$ about the operating point of interest, say x_o, and then applies linear feedback design techniques to the linearized model. Static state feedback, however, does not apply to problems in which the dynamics and the targeted operating point are uncertain. Moreover, static state feedback changes the operating conditions of the open-loop system. This results in wasted control effort and may also result in degrading system performance.

To overcome these problems, washout filters have been used in many applications (e.g., [1], [2], [3], [4], [5], [6], [7], [8]). A washout filter (also sometimes called a washout circuit) is a high pass filter that washes out (rejects) steady state inputs, while passing transient inputs [1]. The main benefit of using washout filters is that all the equilibrium points of the open-loop system are preserved (i.e., their location isn't changed). Thus, one can concentrate on the design of controllers emphasizing the increase in performance achieved for a particular operating point, without the potential for affecting the location of other equilibria. In addition, washout filters facilitate automatic following of a targeted operating point, which results in vanishing control energy once stabilization is achieved and steady state is reached.

Although washout filters have been successfully used in many control applications, there is no systematic way for choosing the constants of the washout filters and the control parameters. Recently, Bazanella, Kokotovic and Silva [9] proposed a technique to control continuous-time systems with unknown operating point. The operating point (or equilibrium point) was treated as an uncertain parameter and a certainty equivalence adaptive controller was proposed. In this work, we discuss benefits and limitations of washout filter-aided feedback for both continuous-time and discrete-time systems . We also discuss extensions of washout filter-aided feedback to overcome the limitations of washout filters and at the same time maintain their benefits. Our extensions are similar to that of [9], although we do not invoke a singular perturbation framework.

The paper proceeds as follows. In Sec. II, we discuss washout filters for both continuous-time and discrete-time systems. In Sec. III, we discuss linear washout filter-aided feedback control and present limitations of feedback through stable washout filters. In Sec. IV, we discuss delayed feedback control for discrete-time systems and its relation to washout filter-aided feedback. In Secs. V and VI, generalizations of washout filters are presented.

II. Washout Filters

A washout filter is a high pass filter that washes out (rejects) steady state inputs, while passing transient inputs [1]. In continuous-time setting, the transfer function of a typical washout filter is

$$\begin{aligned} G(s) &= \frac{y(s)}{x(s)} = \frac{s}{s+d} \\ &= 1 - \frac{d}{s+d}. \end{aligned} \tag{3}$$

Here, d is the reciprocal of the filter time constant which is positive for a stable filter and negative for an unstable filter. With the notation

$$z(s) := \frac{1}{s+d} x(s) \tag{4}$$

the dynamics of the filter can be written as

$$\dot{z} = x - dz, \tag{5}$$

along with the output equation

$$y = x - dz. \tag{6}$$

In discrete-time, the dynamics of a washout filter can be written as

$$z(k+1) = x(k) + (1-d)z(k), \tag{7}$$

along with the output equation

$$y(k) = x(k) - dz(k). \tag{8}$$

For a stable washout filter, the filter constant satisfies $0 < d < 2$.

Note that the output of the washout filter (for both continuous-time and discrete-time cases) vanishes in steady state. Therefore, using washout filters in feedback control does not move the equilibrium points of the open-loop system. As will be discussed below, there are limitations in using stable washout filters in feedback control, and some of these limitations can be overcome using unstable washout filters.

III. Linear Feedback Through Washout Filters

Below, we consider linear feedback through washout filters for both continuous-time and discrete-time systems, and we mention limitations of using stable washout filters. Some of these limitations, such as Lemma 3, are being reported in the current literature, although the results date back to the thesis of H.-C. Lee [4]. The results for the discrete-time case are new.

A. Continuous-time case

Suppose x_o is an unstable operating condition for system (1). In a small neighborhood of x_o, system (1) can be rewritten as

$$\dot{x} = Ax + bu + h(x,u) \tag{9}$$

where x now denotes $x - x_o$ (is the state vector referred to x_o), u is a scalar input, A is the Jacobian matrix of f evaluated at x_o, b is the derivative of f with respect to u evaluated at x_o, and $h(\cdot,\cdot)$ represents higher order terms, i.e., $h(0,0) = 0$ and $\frac{\partial h(0,0)}{\partial x} = 0$.

Next, washout filters are used in the feedback loop. The dynamic equations of the washout filters can be written as

$$\dot{z}_i = -d_i z_i + \sum_{j=1}^{n} c_{ij} x_j \tag{10}$$

where z_i is the state of the ith washout filter, $i = 1, \cdots, m$, and $m \leq n$ is a positive integer. Note that (10), where more than one state is used as an input to the washout filter, is more general than (5). The relationship between the operating point of interest of the open-loop system and the operating point of the washout filters is as follows:

$$z_{oi} = \frac{1}{d_i} \sum_{j=1}^{n} c_{ij} x_{oj} \tag{11}$$

In vector form, the closed-loop system can therefore be written as

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix} u + \begin{pmatrix} h(x,u) \\ 0 \end{pmatrix} \tag{12}$$

where $C = [c_{ij}]$ is an $m \times n$ matrix, which consists of nonzero row vectors, $D = diag(d_i)$, $i = 1, \cdots, m$.

The control input u is taken as a linear function of the washout filter' outputs obtained from the right side of (10)

$$y_i = -d_i z_i + \sum_{j=1}^{n} c_{ij} x_j. \tag{13}$$

The following two lemmas give general guidelines for choosing the matrices C and D based on controllability considerations.

Lemma 1: ([4]) If any two diagonal entries of the matrix D are the same, the linearization of the closed-loop system (12) is not controllable regardless of the controllability of the pair (A, b).

Proof: From the Popov-Belevitch-Hautus (PBH) rank test [10], the linearization of system (12) is controllable if and only if

$$\rho \begin{pmatrix} \lambda I - A & 0 & b \\ C & \lambda I - D & 0 \end{pmatrix} = n + m$$

for each complex number λ. Here, ρ denotes the rank of a matrix. Letting λ_1 be an eigenvalue of D with multiplicity greater than one, we have

$$\rho \begin{pmatrix} 0 \\ \lambda_1 I - D \end{pmatrix} < m - 1.$$

Since

$$\rho \begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} \leq n + 1,$$

we have

$$\rho\begin{pmatrix} \lambda_1 I - A & 0 & b \\ C & \lambda_1 I - D & 0 \end{pmatrix} \leq \rho\begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} + \rho\begin{pmatrix} 0 \\ \lambda_1 I - D \end{pmatrix} < n+m.$$

Thus, the linearization of the closed-loop system is not controllable. ■

Note that controllability of the closed-loop system (12) does not imply that the eigenvalues of system (9) can be arbitrarily assigned by feedback through washout filters.

Lemma 2: ([4]) Suppose that λ_1 is an eigenvalue of both A and D, and that

$$\rho\begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} \leq n. \tag{14}$$

Then, the linearization of the closed-loop system (12) is not controllable.

Proof: Using the PBH test,

$$\rho\begin{pmatrix} \lambda_1 I - A & 0 & b \\ C & \lambda_1 I - D & 0 \end{pmatrix} \leq n+m-1. \tag{15}$$

Thus, the PBH fails and we conclude that the closed-loop system is uncontrollable. ■

Since washout filter-aided feedback can be viewed as a form of output feedback (see Appendix B), where the outputs of the washout filters instead of the open-loop system states are used in the feedback, some of the capabilities of direct state feedback are lost. This is due to the restriction that $d_i \neq 0$. The following lemma summarizes some of the capability limitations of feedback through stable washout filters.

Lemma 3: ([4]) If A has an odd number of eigenvalues with positive real part, then (9) cannot be stabilized using stable washout filters. This holds even if the eigenvalues of A with positive real part are linearly controllable.

Proof: Only linear feedback control is considered since nonlinear terms in the feedback control would not change the linearization of the system. Using linear feedback $u = Ky$, where K is a $1 \times m$ vector and y is the vector of washout filter outputs, results in a closed-loop system with linearization

$$\begin{aligned}\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A+bKC & -bKD \\ C & -D \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} \\ &=: A_c \begin{pmatrix} x \\ z \end{pmatrix}\end{aligned} \tag{16}$$

where

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_m \end{pmatrix} \tag{17}$$

with $d_i > 0$, $i = 1, \cdots, m$. Next,

$$\begin{aligned}\det(A_c) &= \det\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} A_c \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\ &= \det\begin{pmatrix} -D & C \\ -bKD & A+bKC \end{pmatrix} \\ &= \det(-D)\det(A+bKC-bKDD^{-1}C) \\ &= \det(-D)\det(A)\end{aligned}$$

where the next to last equality follows by the Schur complement. Suppose that A has an odd number, say q, of unstable eigenvalues. Then

$$\begin{aligned}sign(\det(A_c)) &= sign(\det(-D))sign(\det(A)) \\ &= (-1)^m(-1)^{n-q} \\ &= (-1)^{n+m}(-1)^q \\ &= (-1)^{n+m+1}\end{aligned}$$

By way of contradiction, suppose that A_c has no unstable eigenvalues. Then, $sign(\det(A_c)) = (-1)^{n+m}$, which contradicts (18). Thus, the closed-loop system possesses at least one unstable eigenvalue and cannot be stabilized using stable washout filters. ■

Lemma 3 implies that if the linearization of the open-loop system possesses an odd number of unstable eigenvalues, then in order to stabilize the system, it is necessary to use an odd number of unstable washout filters in the feedback loop.

Corollary 1: If the open-loop system possesses a zero eigenvalue, it cannot be moved using washout filter-aided feedback.

Proof: Follows from the proof of Lemma 3. ■

B. Discrete-time case

Suppose x_o is an unstable operating condition for system (2). In a small neighborhood of x_o, system (2) can be rewritten as

$$x(k+1) = Ax(k) + bu(k) + h(x(k), u(k)) \tag{18}$$

where x now denotes $x - x_o$ (is the state vector referred to x_o), u is a scalar input, A is the Jacobian matrix of f evaluated at x_o, b is the derivative of f with respect to u evaluated at x_o, and $h(\cdot,\cdot)$ represents higher order terms, i.e., $h(0,0) = 0$ and $\frac{\partial h(0,0)}{\partial x} = 0$.

Next, washout filters are used in the feedback loop. The dynamic equations of the washout filters can be written as

$$z_i(k+1) = (1-d_i)z_i(k) + \sum_{j=1}^{n} c_{ij}x_j(k) \tag{19}$$

where z_i is the state of the ith washout filter, $i = 1, \cdots, m$, and $m \leq n$ is a positive integer. The relationship between the operating point of the open-loop system and the operating point of the washout filters is as follows:

$$z_{oi} = \frac{1}{d_i}\sum_{j=1}^{n} c_{ij}x_{oj} \tag{20}$$

In vector form, the closed-loop system can therefore be written as

$$\begin{pmatrix} x(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & I-D \end{pmatrix} \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} h(x(k),u(k)) \\ 0 \end{pmatrix} \quad (21)$$

where $C = [c_{ij}]$ is an $m \times n$ matrix, which consists of nonzero row vectors, $D = diag(d_i)$, $i = 1, \cdots, m$.

The control input u is taken as a linear function of the washout filter' outputs obtained from the right side of (19)

$$y_i(k) \quad = \quad -d_i z_i(k) + \sum_{j=1}^{n} c_{ij} x_j(k). \quad (22)$$

The following two lemmas give general guidelines for choosing the matrices C and D based on controllability considerations. The results are analogous to the continuous-time results presented in the previous section.

Lemma 4: If any two diagonal entries of the matrix D are the same, the linearization of the closed-loop system (21) is not controllable regardless of the controllability of the pair (A, b).

Proof: From the PBH rank test, the linearization of system (21) is controllable if and only if

$$\rho \begin{pmatrix} \lambda I - A & 0 & b \\ C & \lambda I - (I-D) & 0 \end{pmatrix} = n + m$$

for each complex number λ. Here, ρ denotes the rank of a matrix. Let d_1 be an eigenvalue of D with multiplicity greater than one. Letting $\lambda_1 = 1 - d_1$, we have

$$\rho \begin{pmatrix} 0 \\ \lambda_1 I - (I-D) \end{pmatrix} < m - 1.$$

Since

$$\rho \begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} \leq n + 1,$$

we have

$$\rho \begin{pmatrix} \lambda_1 I - A & 0 & b \\ C & \lambda_1 I - (I-D) & 0 \end{pmatrix} \leq \rho \begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} + \rho \begin{pmatrix} 0 \\ \lambda_1 I - (I-D) \end{pmatrix} < n + m.$$

Thus, the linearization of the closed-loop system is not controllable. ■

Note that controllability of the closed-loop system (21) does not imply that the eigenvalues of system (18) can be arbitrarily assigned by feedback through washout filters.

Lemma 5: Suppose that λ_1 is an eigenvalue of both A and $I - D$, and that

$$\rho \begin{pmatrix} \lambda_1 I - A & b \\ C & 0 \end{pmatrix} \leq n. \quad (23)$$

Then, the linearization of the closed-loop system (21) is not controllable.

Proof: Using the PBH test,

$$\rho \begin{pmatrix} \lambda_1 I - A & 0 & b \\ C & \lambda_1 I - (I-D) & 0 \end{pmatrix} \leq n + m - 1. \quad (24)$$

Thus, the PBH fails and we conclude that the closed-loop system is uncontrollable. ■

Since washout filter-aided feedback can be viewed as a form of output feedback (see Appendix B), some of the capabilities of direct state feedback are lost. This is due to the restriction that $d_i \neq 0$. The following lemma summarizes some of the capability limitations of feedback through stable washout filters.

Lemma 6: If A possesses an odd number of real eigenvalues (counting multiplicities) in $(1, \infty)$ (i.e., if $\det(I - A) < 0$) then it cannot be stabilized using stable washout filters.

Proof: Only linear feedback control is considered since nonlinear terms in the feedback control would not change the linearization of the system. Using linear feedback $u(k) = gy(k)$, where g is a $1 \times m$ vector and $y(k)$ is the vector of washout filter outputs, results in a closed-loop system with linearization

$$\begin{pmatrix} x(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} A + bgC & -bgD \\ C & I - D \end{pmatrix} \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} =: A_c \begin{pmatrix} x(k) \\ z(k) \end{pmatrix}$$

where

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_m \end{pmatrix}$$

with $d_i \in (0, 2)$, $i = 1, \cdots, m$. Next,

$$\begin{aligned} \det(I - A_c) &= \det \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} (I - A_c) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} D & -C \\ bgD & I - A - bgC \end{pmatrix} \\ &= \det(D) \det(I - A - bgC + bgDD^{-1}C) \\ &= \det(D) \det(I - A) \end{aligned}$$

where the next to last equality follows by the Schur complement. If all the washout filters are stable, i.e., $d_i \in (0, 2)$, then $\det(D) = d_1 d_2 \ldots d_n > 0$. Thus, $\det(I - A_c) < 0$ if $\det(I - A) < 0$. Hence, the closed-loop system possesses an odd number of real unstable eigenvalues in $(1, \infty)$. ■

Lemma 6 implies that if the open-loop state dynamics matrix A possesses an odd number of real eigenvalues in $(1, \infty)$, then in order to stabilize the system, an odd number of unstable washout filters (with $d_i < 0$) must be used in the feedback loop.

Corollary 2: If the linearization of the open-loop state dynamics matrix A possesses an eigenvalue of 1 (i.e., $I - A$

is singular), then this eigenvalue cannot be moved using washout filter-aided feedback.

Proof: Follows from the proof of Lemma 6. ■

IV. Delayed Feedback Control as a Special Case of Washout Filters

Delayed feedback control (DFC) was introduced by Pyragas [11] as a technique for control of chaos. Since its introduction, DFC has been used in many applications. It has been shown in [12] that DFC for discrete-time systems has two limitations. The first limitation is known as the odd number of real eigenvalues greater than 1 limitation. That is, DFC cannot be used to stabilize systems whose linearization possesses an odd number of real eigenvalues in $(1,\infty)$. The second limitation is that DFC can be used to stabilize only a class of unstable systems; it cannot stabilize highly unstable systems [13]. For example, for the one dimensional map (25) with system dynamics coefficient a, DFC can be used to stabilize the map if and only if $-3 < a \leq -1$ (see below). A discussion of two-dimensional discrete time systems that can be stabilized using DFC is given in [12]. Many researchers proposed different techniques to overcome the odd number limitation (e.g.,[14], [15], [16], [17]). Extended delayed feedback control (EDFC), where many previous states of the system are used in the feedback, was proposed to extend the range of systems that can be stabilized (e.g., [13]). However, the analysis of the EDFC method is cumbersome and it also suffers from the odd number limitation as DFC [18].

In this work, we note that delayed feedback control for discrete-time systems is a special case of washout filter-aided feedback (it corresponds to washout filter-aided feedback with all washout filters' constants equal 1). We then proceed to show that the limitations of DFC mentioned above can be overcome by the use of washout filter-aided feedback, giving rise to the possibility of stabilizing a much larger class of systems than is possible with DFC.

To illustrate DFC and its limitations, consider the simple one dimensional discrete-time system

$$x(k+1) = ax(k) + u(k) \tag{25}$$

where $u(k)$ is the control input. In DFC, the control is taken to be $u(k) = \gamma(x(k) - x(k-1))$, where γ is the control gain.

The closed-loop system can be written as

$$\begin{aligned} x(k+1) &= ax(k) + \gamma(x(k) - z(k)) && (26)\\ z(k+1) &= x(k) && (27) \end{aligned}$$

A necessary and sufficient condition for DFC to be stabilizing is $-3 < a < 1$ [19], [12], [20]. This can be seen from (26)-(27) as follows: The linearization is $J := \begin{pmatrix} a+\gamma & -\gamma \\ 1 & 0 \end{pmatrix}$. By Jury's test for second order systems, J is Schur stable if and only if

$$\begin{aligned} -1 < \gamma &< 1 \\ a+\gamma &< 1+\gamma \\ a+\gamma &> -1-\gamma \end{aligned}$$

The second inequality implies that $a < 1$ and the first and third inequalities imply that $a > -3$. Of course, the system is already stable if $-1 < a < 1$. Thus, DFC can stabilize an unstable plant (25) iff $-3 < a \leq -1$.

Next, we show that DFC for the one dimensional system (25) is a special case of washout filter-aided feedback. We then show that using washout filter-aided feedback, any one-dimensional system can be stabilized.

Consider the same system as before but with washout filter-aided feedback:

$$\begin{aligned} x(k+1) &= ax(k) + u(k) && (28)\\ z(k+1) &= x(k) + (1-d)z(k) && (29)\\ u(k) &= \gamma(x(k) - dz(k)) && (30) \end{aligned}$$

Here d is the washout filter constant and γ is the control gain. Note that the washout filter-aided feedback reduces to DFC by setting $d = 1$.

The fixed point of the closed-loop system is asymptotically stable if the eigenvalues of the closed-loop system are within the unit circle. Note that if $|a| < 1$, the uncontrolled system (i.e., (28) with $u(k) = 0$) is asymptotically stable.

Proposition 1: Denote the system dynamics coefficient by a. Suppose that $a \leq -1$ or $a > 1$, i.e., the open-loop system is unstable.
Case 1: If $a \leq -1$, then a stabilizing washout filter-aided feedback exists (using a stable washout filter). Indeed, washout filter-aided linear feedbacks with gain γ and washout filter constant d satisfying

$$-1 - a + \frac{d}{2}(1+a) < \gamma < 1 - a(1-d),\ 0 < d < \frac{4}{1-a} \tag{31}$$

are stabilizing.
Case 2: If $a > 1$, then a stabilizing washout filter-aided feedback exists (using an unstable washout filter). Indeed, washout filter-aided linear feedbacks with gain γ and washout filter constant d satisfying

$$-1 - a + \frac{d}{2}(1+a) < \gamma < 1 - a(1-d),\ \frac{4}{1-a} < d < 0 \tag{32}$$

are stabilizing.

Proof: See Appendix A.

For the n-dimensional case, DFC takes the form

$$\begin{aligned} x(k+1) &= Ax(x) + bu(k) && (33)\\ u(k) &= g(x(k) - x(k-1)) && (34) \end{aligned}$$

This corresponds to washout filter-aided feedback with n uncoupled washout filters (i.e., in Eq. (19), $c_{ij} = 1$ if $i = j$ and $c_{ij} = 0$ if $i \neq j$) and all washout filters having constants equal to 1 (i.e., in Eq. (19), $d_i = 1, i = 1, \cdots, n$):

$$\begin{aligned} x(k+1) &= Ax(x) + bu(k) \\ z(k+1) &= x(k) \\ y(k) &= x(k) - z(k) \\ u(k) &= g(x(k) - z(k)). \end{aligned}$$

Although the case of one-dimensional systems was handled in a systematic way above, systems of higher dimension do not lend themselves to a general analytical design procedure. After presenting a two-dimensional example, we will proceed in the next section to give a generalization of washout filter-aided feedback that does permit development of a systematic design method.

Example 1: Consider the two-dimensional map [21]

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 1.9 & 1 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} - \begin{pmatrix} x_1^3(k) \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k) \quad (35)$$

The uncontrolled system (35) has three fixed points $x_{o1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x_{o2} = \begin{pmatrix} \sqrt{1.4} \\ \sqrt{1.4}/2 \end{pmatrix}$ and $x_{o3} = -x_{o2}$, and indeed displays chaotic motion (see [21]). The fixed point x_{o1} is unstable: the eigenvalues of the linearization at the origin are $\lambda_1 = 2.1343$ and $\lambda_2 = -0.2343$. Since $\lambda_2 > 1$, the origin cannot be stabilized using DFC nor using stable washout filters. We will show that the origin can be stabilized using one unstable washout filter

$$\begin{aligned} z(k+1) &= x_1(k) + (1-d)z(k) \quad (36) \\ y(k) &= x_1(k) - dz(k) \quad (37) \end{aligned}$$

where $z(k)$ is the washout filter state, $y(k)$ is the washout filter output and d is the washout filter constant the value of which determines the stability of the washout filter. Let $u(k) = \gamma y(k)$, where γ is the control gain to be chosen.

The closed-loop system takes the form

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} 1.9+\gamma & 1 & -\gamma d \\ 0.5 & 0 & 0 \\ 1 & 0 & 1-d \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ z(k) \end{pmatrix} - \begin{pmatrix} x_1^3(k) \\ 0 \\ 0 \end{pmatrix} \quad (38)$$

The origin is locally asymptotically stable if the linearization of (38) is asymptotically stable. Since the uncontrolled system has one eigenvalue greater than 1, we need to choose d such that the washout filter is unstable but the closed-loop system is stable. It is an easy calculation to show that a controller with $d = -0.05$ and $\gamma = -1.8$ stabilizes the origin locally. With such parameters, the eigenvalues of the Jacobian matrix of the closed-loop system are $\{-0.6345, 0.8923 \pm j0.1767\}$. Figure 1 demonstrates the effectiveness of the controller.

The authors of [21] use dynamic delayed feedback control and reduced order dynamic DFC to stabilize the origin of (35). Dynamic DFC results in tripling the dimension of the system and reduced order dynamic DFC results in a closed-loop system dimension more than twice the dimension of the open-loop system. On the other hand, washout filter-aided feedback results in increasing the dimension to at most twice the dimension of the open-loop system. In the example above, only one washout filter was needed to stabilize the system.

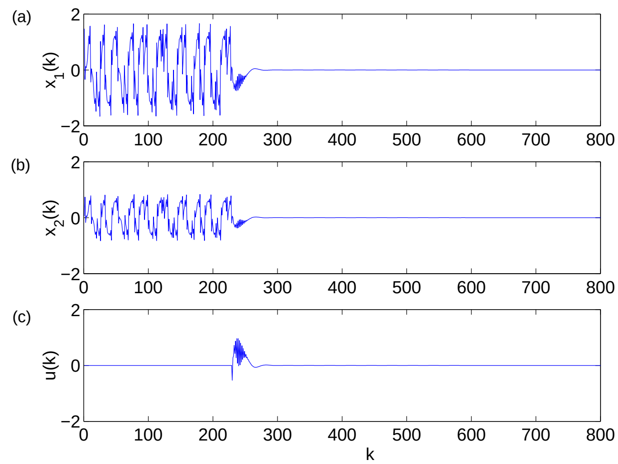

Fig. 1. Time series (with initial condition (1.4725,0.0557)) of (a) x_1 (b) x_2 and (c) control input u. The control is applied when the trajectory of the open-loop system enters the neighborhood $\{x = (x_1, x_2) \in \Re^2 : \|x\| < 0.1\}$ of the origin.

V. Generalization of Continuous-Time Washout Filter-Aided Feedback

Next, we consider a generalization of washout filters in which the individual washout filters are coupled through a constant coupling matrix. Consider system (1) with x_o as the operating condition. System (1) can be rewritten as follows in a small neighborhood of x_o:

$$\dot{x} = Ax + Bu + h(x, u) \quad (39)$$

We are interested in designing a control law that stabilizes this system while maintaining all its equilibrium points.

The generalized washout filter-aided feedback proposed here results in the closed-loop system

$$\begin{aligned} \dot{x} &= Ax + Bu + h(x, u) \quad (40) \\ \dot{z} &= P(x - z) \quad (41) \\ u &= K(x - z) \quad (42) \end{aligned}$$

Here P is a nonsingular matrix and K is a feedback gain matrix. Since in steady state, the control input vanishes (i.e., $u \equiv 0$), the equilibrium points of the open-loop system are not shifted by this type of feedback control. Suppose that the pair (A, B) is stabilizable. Are there matrices K and P such that the closed-loop system is stable?

To answer this question, we consider the effect of matrices K and P on the linearization of the closed-loop system

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A+BK & -BK \\ P & -P \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \\ &=: A_c \begin{pmatrix} x \\ z \end{pmatrix} \end{aligned} \tag{43}$$

Proposition 2: ([9]) The determinant of the closed-loop state dynamics matrix A_c satisfies

$$\det(A_c) = \det(A)\det(-P) \tag{44}$$

Proof: Using the fact that similarity transformations do not change the eigenvalues and the Schur complement of a matrix, we have that

$$\begin{aligned} \det(A_c) &= \det \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} A_c \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\ &= \det \begin{pmatrix} -P & P \\ -BK & A+BK \end{pmatrix} \\ &= \det(-P)\det(A+BK-BKPP^{-1}) \\ &= \det(-P)\det(A) \end{aligned}$$

■

Corollary 3: If the matrix A has a zero eigenvalue, then the closed-loop system state dynamics matrix A_c will also have a zero eigenvalue.
Proof: Follows from Proposition 2. ■

Since this type of feedback doesn't shift equilibria, it shouldn't be surprising that it can't modify a zero eigenvalue (this would also modify any stationary bifurcation in the system).

The following result gives some conditions on the controller matrix P for the controller to be stabilizing. This result is akin to Lemma 3 pertaining to washout filter-aided feedback. In words, the result means that if the open-loop system possesses an odd number of unstable eigenvalues, then a necessary condition for the closed-loop system to be stable is that the controller must also have an odd number of unstable eigenvalues.

Lemma 7: [9] Let the number of unstable eigenvalues of A be odd. Then, for the closed-loop matrix A_c to be Hurwitz, $-P$ must also have an odd number of unstable eigenvalues.
Proof: Recall that $\det(A_c) = \det(A)\det(-P)$. Thus,

$$sign(\det(A_c)) = sign(\det(A))sign(\det(-P)) \tag{45}$$

Let q be the number of unstable eigenvalue of A, and r be the number of unstable eigenvalues of $-P$. Then using the fact that A_c is Hurwitz, Eq. (45) becomes

$$(-1)^{2n} = (-1)^{n-q}(-1)^{n-r} = (-1)^{2n}(-1)^{q+r} \tag{46}$$

Thus, $q+r$ must be even. ■

We will show that if A is nonsingular and (A,B) is stabilizable, then there is a pair P, K such that A_c is Hurwitz. Recall that eigenvalues are preserved under similarity transformations.

Let $T_1 = \begin{pmatrix} I & 0 \\ 0 & P^{-1} \end{pmatrix}$. Then we have

$$A_{c1} := T_1 A_c T_1^{-1} = \begin{pmatrix} A+BK & -BKP \\ I & -P \end{pmatrix} \tag{47}$$

Next, let $T_2 = \begin{pmatrix} I & M \\ 0 & I \end{pmatrix}$. It is easy to see that $T_2^{-1} = \begin{pmatrix} I & -M \\ 0 & I \end{pmatrix}$. Applying the transformation T_2 to A_{c1} gives

$$\begin{aligned} A_{c2} &:= T_2 A_{c1} T_2^{-1} \\ &= \begin{pmatrix} A+BK+M & -AM-BKM-M^2-BKP-MP \\ I & -M-P \end{pmatrix} \end{aligned}$$

Consider the $(1,2)$ block term of A_{c2}. Suppose $P = \epsilon P_1$ and $M = M_0 + \epsilon M_1 + O(\epsilon^2)$ with $\epsilon > 0$ and sufficiently small. It is straightforward to show, after setting the $(1,2)$ block of A_{c2} to zero, i.e.,

$$AM + BKM + M^2 + BKP + MP = 0 \tag{48}$$

and collecting terms with same power in ϵ, that $O(1)$ terms:

$$(A+BK+M_0)M_0 = 0. \tag{49}$$

This holds if $M_0 = 0$ or $M_0 = -A - BK$. Taking $M_0 = -A - BK$ and finding the ϵ^1 terms gives

$$M_1(A+BK) + AP_1 = 0 \tag{50}$$

Since $A+BK$ can be guaranteed invertible (by restricting K so that $0 \notin \sigma(A+BK)$), we find that $M_1 = -AP_1(A+BK)^{-1}$. Since M_1 can be determined uniquely through matrix inversion, it is clear that the Implicit Function Theorem implies that (48) has a locally unique solution $M(\epsilon) = M_0+\epsilon M_1+O(\epsilon^2)$ near M_0. Therefore, $M = M_0+\epsilon M_1 + O(\epsilon^2) = -A - BK - \epsilon AP_1(A+BK)^{-1} + O(\epsilon^2)$. Substituting M and P in A_{c2} yields

$$A_{c2} = \begin{pmatrix} -\epsilon AP_1(A+BK)^{-1} + O(\epsilon^2) & 0 \\ I & A_{c2}(2,2) \end{pmatrix}$$

where $A_{c2}(2,2) = A + BK + \epsilon(AP_1(A+BK)^{-1} - P_1) + O(\epsilon^2)$.

Assume that A has no zero eigenvalues. To make A_{c2} Hurwitz, we need to choose P_1 such that $-AP_1(A+BK)^{-1}$ is Hurwitz. Clearly such a P_1 exists (e.g., $P_1 = A^{-1}(A+BK)$). Also we need to choose K such that $A+BK$ is Hurwitz with eigenvalues away from zero such that the perturbation $\epsilon(AP_1(A+BK)^{-1} - P_1)$ does not cause the eigenvalues of $A+BK$ to become unstable. Such a K is guaranteed to exist since the pair (A,B) is assumed to be stabilizable.

Proposition 3: Consider the closed-loop system (43). Suppose that the matrix A has no eigenvalues at 0. Suppose also that the pair (A,B) is stabilizable. Then there exists a $P \in R^{n\times n}$ and $K \in R^{m\times n}$ such that $(x_o^T, x_o^T)^T$ is asymptotically stable equilibrium point of (43).

VI. Generalization of Discrete-Time Washout Filter-Aided Feedback

The results of this section are counterparts of the continuous-time results of the previous section for the discrete-time case. Consider system (2) with x_o as the operating condition. System (2) can be rewritten as follows in a small neighborhood of x_o:

$$x(k+1) = Ax(k) + Bu(k) + h(x(k), u(k)) \quad (51)$$

We are interested in designing a control law that stabilizes this system while maintaining all its equilibrium points. The generalized washout filter-aided feedback proposed here results in the closed-loop system

$$x(k+1) = Ax(k) + Bu(k) + h(x(k), u(k)) \quad (52)$$
$$z(k+1) = Px(k) + (I-P)z(k) \quad (53)$$
$$u(k) = G(x(k) - z(k)) \quad (54)$$

Here P is a nonsingular matrix and G is a feedback gain matrix. Since in steady state, the control input vanishes (i.e., $u \equiv 0$), the equilibrium points of the open-loop system are not shifted by this type of feedback control. Suppose that the pair (A, B) is stabilizable. Are there matrices G and P such that the closed-loop system is stable?

To answer this question, we consider the effect of matrices P and G on the linearization of the closed-loop system

$$\begin{pmatrix} x(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} A+BG & -BG \\ P & I-P \end{pmatrix} \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} =: A_c \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} \quad (55)$$

Proposition 4: The determinant of $I - A_c$ satisfies

$$\det(I - A_c) = \det(I-A)\det(P) \quad (56)$$

Proof: Using the fact that similarity transformations do not change the eigenvalues and the Schur complement of a matrix, we have that

$$\begin{aligned} \det(I-A_c) &= \det\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \det(I-A_c) \det\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\ &= \det\begin{pmatrix} P & -P \\ BG & I-A-BG \end{pmatrix} \\ &= \det(P)\det(I-A-BG+BGP^{-1}P) \\ &= \det(P)\det(I-A) \end{aligned}$$

■

Corollary 4: If the open-loop matrix A has an eigenvalue of 1 (i.e., if $(I-A)$ is singular), this eigenvalue cannot be shifted using this type of dynamic feedback.
Proof: Follows from Proposition 4.

Lemma 8: Let the number of unstable eigenvalues of A that are real and greater than 1 be odd (i.e., $\det(I-A) < 0$). Then, for the closed-loop state dynamics matrix A_c to be Schur stable, $I - P$ must also have an odd number of real eigenvalues greater than 1 in value.

Proof: We show that for A_c to be Schur stable, it is necessary that the parity of the number of real eigenvalues of P that are negative (equivalently, number of real eigenvalues of $(I-P)$ that are greater than 1) be equal to that of A. Recall that $\det(I-A_c) = \det(I-A)\det(P)$. Thus,

$$sign(\det(I-A_c)) = sign(\det(I-A))sign(\det(P)) \quad (57)$$

Let q be the number of real eigenvalue of A with value greater than 1, and r be the number of negative real eigenvalues of P. Then using the fact that A_c is Schur stable, Eq. (57) becomes

$$1 = (-1)^q(-1)^r = (-1)^{q+r} \quad (58)$$

Thus, $q + r$ must be even. ■

We will show that there exist a P, G such that A_c is Schur stable. Recall that eigenvalues are preserved under similarity transformations.

Let $T_1 = \begin{pmatrix} I & 0 \\ 0 & P^{-1} \end{pmatrix}$. Then we have

$$\begin{aligned} A_{c1} &:= T_1 A_c T_1^{-1} \\ &= \begin{pmatrix} A+BG & -BGP \\ I & I-P \end{pmatrix}. \end{aligned}$$

Next, let $T_2 = \begin{pmatrix} I & M \\ 0 & I \end{pmatrix}$ implying $T_2^{-1} = \begin{pmatrix} I & -M \\ 0 & I \end{pmatrix}$. Applying the transformation T_2 to A_{c1} gives

$$\begin{aligned} A_{c2} &:= T_2 A_{c1} T_2^{-1} \\ &= \begin{pmatrix} A+BG+M & A_{c2}(1,2) \\ I & -M+I-P \end{pmatrix} \end{aligned}$$

where

$$A_{c2}(1,2) = -AM - BGM - M^2 - BGP + M - MP. \quad (59)$$

Consider the block term $A_{c2}(1,2)$. Suppose $P = \epsilon P_1$ and $M = M_0 + \epsilon M_1 + O(\epsilon^2)$ with $\epsilon > 0$ and sufficiently small. It is straightforward to show, after setting the $A_{c2}(1,2)$ to zero and collecting terms with same power in ϵ that the $O(1)$ terms yield

$$(A + BG - I + M_0)M_0 = 0 \quad (60)$$

This holds if $M_0 = 0$ or $M_0 = -A - BG + I$. Taking $M_0 = -A - BG + I$ and finding the ϵ^1 terms gives

$$M_1(A + BG - I) + (A - I)P_1 = 0 \quad (61)$$

Since $A + BG - I$ can be guaranteed nonsingular (by restricting K so that $1 \notin \sigma(A + BG)$), we find that $M_1 = -(A-I)P_1(A+BG-I)^{-1}$. Since M_1 can be determined uniquely through matrix inversion, it is clear that the Implicit Function Theorem implies that (59) has a locally unique solution $M(\epsilon) = M_0 + \epsilon M_1 + O(\epsilon^2)$ near M_0. Therefore, $M = M_0 + \epsilon M_1 + O(\epsilon^2) = -A - BG + I - \epsilon(A-I)P_1(A+BG-I)^{-1} + O(\epsilon^2)$. Substituting M and P in A_{c2} yields

$$A_{c2} = \begin{pmatrix} I - \epsilon(A-I)P_1(A+BG-I)^{-1} + O(\epsilon^2) & 0 \\ I & A_c(2,2) \end{pmatrix}$$

where

$$A_c(2,2) = A+BG+\epsilon((A-I)P_1(A+BG-I)^{-1}-P_1) + O(\epsilon^2).$$

Assume that $I-A$ is nonsingular (i.e., $1 \notin \sigma(A)$). To make A_{c2} Schur stable, we need to choose P_1 such that $I-\epsilon(A-I)P_1(A+BG-I)^{-1}$ is Schur stable. Clearly such a P_1 exists. Also we need to choose G such that $A+BG$ is Schur stable with eigenvalues away from 1 such that the perturbation $\epsilon((A-I)P_1(A+BG-I)^{-1}-P_1)$ does not cause the eigenvalues of $A+BG$ to become unstable. Such a G is guaranteed to exist since the pair (A,B) is assumed to be stabilizable.

Proposition 5: Consider the closed-loop system (55). Suppose that the matrix $I-A$ is nonsingular. Suppose also that the pair (A,B) is stabilizable. Then there exists a $P \in R^{n\times n}$, a $G \in R^{m\times n}$ and an $\bar{\epsilon} > 0$ such that $\forall \epsilon \in (0,\bar{\epsilon}]$, $(x_o^T, x_o^T)^T$ is an asymptotically stable fixed point of (55).

Example 2: (Design of a stabilizing controller using the generalized washout filter calculations)

We revisit Example 1 using the generalized washout filter design calculations above. We choose the gain vector G so that $A+bG$ is Schur stable. A stabilizing control gain vector is $G = [-1.6343 \ \ -0.7657]$. Choosing

$$P_1 = (A-I)^{-1}(A+BG-I) = \begin{pmatrix} -0.1674 & -0.5469 \\ -0.5837 & 0.7265 \end{pmatrix}$$

and $\epsilon = 0.1$ yields $P = \epsilon P_1$, and

$$A_c = \begin{pmatrix} A+BG & -BG \\ P & I-P \end{pmatrix} = \begin{pmatrix} 0.2657 & 0.2343 & 1.6343 & 0.7657 \\ 0.5000 & 0 & 0 & 0 \\ -0.0167 & -0.0547 & 1.0167 & 0.0547 \\ -0.0584 & 0.0727 & 0.0584 & 0.9273 \end{pmatrix}.$$

The eigenvalues of A_c are $\{-0.2343,\ 0.7277,\ 0.8164,\ 0.9000\}$. Thus, the closed-loop system is asymptotically stable. Figure 2 demonstrates the effectiveness of the controller. Note that the control input vanishes after stabilization of the origin is achieved.

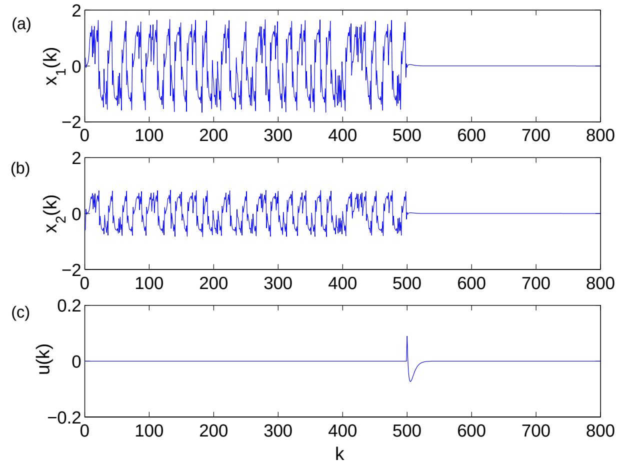

Fig. 2. Time series (with initial condition (0.3,-0.6)) of (a) x_1 (b) x_2 and (c) control input u. The control is applied when the trajectory of the open-loop system enters the neighborhood $\{x = (x_1, x_2) \in \Re^2 : \|x\| < 0.15\}$ of the origin.

Appendix A Proof of Proposition 1

The Jacobian of the closed-loop system (28)-(30) is

$$J = \begin{pmatrix} a+\gamma & -\gamma d \\ 1 & 1-d \end{pmatrix}$$

Let τ :=trace$(J) = a+1-d+\gamma$, δ :=det$(J) = a(1-d)+\gamma$. The fixed point of the closed-loop system (28)-(30) is asymptotically stable if both eigenvalues of J are within the unit circle.

The characteristic equation of J is given by

$$p(\lambda) := \lambda^2 - (a+\gamma+1-d)\lambda + a(1-d)+\gamma$$

By the Jury's test for second order systems, both eigenvalues are within the unit circle if and only if

$$-1 < p(0) < 1 \quad (62)$$
$$p(1) > 0 \quad (63)$$
$$p(-1) > 0 \quad (64)$$

Conditions (62)-(64) are equivalent to

$$-1 < a(1-d)+\gamma < 1 \quad (65)$$
$$d(1-a) > 0 \quad (66)$$
$$2+2a+2\gamma-d-ad > 0 \quad (67)$$

respectively.

Case 1: $a \le -1$

Let $d \in (0,2)$. This corresponds to a stable washout filter. Then, inequality (66) is trivially satisfied since d is positive and $a \le -1$ by hypothesis. Inequalities (65) and (67) translate to an explicit condition on γ and d as follows:

$$\max\left\{-1-a(1-d),\ -1-a+\frac{d}{2}(1+a)\right\} < \gamma < 1-a(1-d) \quad (68)$$

Now, $\max\left\{-1-a(1-d),\ -1-a+\frac{d}{2}(1+a)\right\} = -1-a+\frac{d}{2}(1+a)$, which is seen as follows:

$$\begin{aligned} -1-a(1-d) &< -1-a+\frac{d}{2}(1+a) \\ \iff \quad ad &< \frac{d}{2}(1+a) \\ \iff \quad a &< 1 \ \text{(true by hypothesis; } a \le -1) \end{aligned}$$

Hence, inequality (68) reduces to:

$$-1-a+\frac{d}{2}(1+a) < \gamma < 1-a(1-d) \quad (69)$$

Finally, for a γ satisfying (69) to exist, the upper limit in (69) must be greater than the lower limit. This is shown to be true as follows:

$$\begin{aligned} 1-a(1-d) > & \quad -1-a+\frac{d}{2}(1+a) \\ \Longleftrightarrow \qquad 2+ad > & \quad \frac{d}{2}(1+a) \\ \Longleftrightarrow \qquad 2+\frac{ad}{2} > & \quad \frac{d}{2} \\ \Longleftrightarrow \qquad d < & \quad \frac{4}{1-a} \end{aligned}$$

Case 2: $a > 1$
Similar to the previous case. From inequality (66), $d(1-a) > 0$ if and only if $d < 0$. Inequalities (65) and (67) translate to an explicit condition on γ and d as follows:

$$\max\left\{-1-a(1-d),\ -1-a+\frac{d}{2}(1+a)\right\} < \gamma < 1-a(1-d) \quad (70)$$

which reduces to

$$-1-a+\frac{d}{2}(1+a) < \gamma < 1-a(1-d) \quad (71)$$

For a γ to exist, the upper limit in (71) must be greater than the lower limit. This implies that $d > \frac{4}{1-a}$. This completes the proof. ■

APPENDIX B WASHOUT FILTER-AIDED FEEDBACK AS AN OUTPUT FEEDBACK

Below, we show that washout filter-aided feedback can be viewed as an output feedback. Consider

$$x(k+1) = Ax(k) + bu(k) \quad (72)$$

with washout filter-aided feedback

$$z(k+1) = Cx(k) + (I-D)z(k) \quad (73)$$

$$y(k) = Cx(k) - Dz(k) \quad (74)$$

Here, $C = [c_{ij}]$ is an $m \times n$ matrix and D =diag$\{d_1, d_2, \ldots, d_m\}$, $m \leq n$. Let the control input be

$$u(k) = gy(k) = g(Cx(k) - Dz(k)) \quad (75)$$

where the row vector g is the gain vector.

The closed-loop system (72)-(75) can be written as

$$\begin{aligned} \tilde{x}(k+1) &:= \begin{pmatrix} x(k+1) \\ z(k+1) \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ C & I-D \end{pmatrix} \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix} u(k) \\ &=: \tilde{A}\tilde{x}(k) + \tilde{b}u(k) \end{aligned}$$

$$\begin{aligned} y(k) &= (C \quad -D) \begin{pmatrix} x(k) \\ z(k) \end{pmatrix} \\ &=: \tilde{C}\tilde{x}(k) \\ u(k) &= gy(k) \end{aligned}$$

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