On d-panconnected tournaments with large semidegrees

Samvel Kh. Darbinyan
Institute for Informatics and Automation Problems, NAS RA
Gregory Z. Gutin
Department of Compute Science, Royal Holloway, University of London, UK
E-mails: samdarbin@iiap.sci.am, gutin@cs.rhul.ac.uk

Abstract

We prove the following new results.

(a) Let T be a regular tournament of order $2n+1 \geq 11$ and S a subset of $V(T)$. Suppose that $|S| \leq \frac{1}{2}(n-2)$ and x, y are distinct vertices in $V(T) \setminus S$. If the subtournament $T-S$ contains an (x,y)-path of length r, where $3 \leq r \leq |V(T) \backslash S|-2$, then $T-S$ also contains an (x,y)-path of length $r+1$.

(b) Let T be an m-irregular tournament of order p, i.e., $|d^+(x) - d^-(x)| \leq m$ for every vertex x of T. If $m \leq \frac{1}{3}(p-5)$ (respectively, $m \leq \frac{1}{5}(p-3)$), then for every pair of vertices x and y, T has an (x,y)-path of any length k, $4 \leq k \leq p-1$ (respectively, $3 \leq k \leq p-1$ or T belongs to a family $\mathcal{G}$ of tournaments, which is defined in the paper). In other words, (b) means that if the semidegrees of every vertex of a tournament T of order p are between $\frac{1}{3}(p+1)$ and $\frac{2}{3}(p-2)$ (respectively, between $\frac{1}{5}(2p-1)$ and $\frac{1}{5}(3p-4)$), then the claims in (b) hold.

Our results improve in a sense related results of Alspach (1967), Jacobsen (1972), Alspach et al. (1974), Thomassen (1978) and Darbinyan (1977, 1978, 1979), and are sharp in a sense.

Keywords: tournaments; arc pancyclicity; irregularity; paths; panconnected tournaments; oudegree; indegree.

1 Introduction

In this paper, we consider finite digraphs (directed graphs) without loops and multiple arcs. We use standard notation and terminology, cf. [3] and [4]. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $\mathcal{A}(D)$, respectively. The *order* of D is the number of its vertices. A subdigraph of D induced by a subset $A \subseteq V(D)$ is denoted by $D\langle A\rangle$. If $X \subseteq V(D)$, then $D-X$ is the subdigraph induced by $V(D) \setminus X$, i.e., $D-X = D\langle V(D)\setminus X\rangle$. Every cycle and path are assumed to be simple and directed. Let m and n, $m \le n$, be integers. By $[m,n]$ we denote the set $\{m, m+1, \dots, n\}$.

A digraph D of order p is *arc pancyclic* (respectively, *d-arc pancyclic*, where $d \in [3,p]$) if D has a k-cycle containing uv for every arc $uv \in \mathcal{A}(D)$ and every $k \in [3,p]$ (respectively, $k \in [d,p]$). We say that a digraph D of order p is *strongly panconnected* (respectively, *d-strongly panconnected*, where $d \in [3,p-1]$) if there is an (x,y)- and a (y,x)-path in D, both of length k, for any two vertices x, y of D and each $k \in [3,p-1]$ (respectively, $k \in [d,p-1]$).

An *oriented graph* is a digraph with no cycle of length two. A *tournament* is an oriented graph where every pair of distinct vertices are adjacent. The *outdegree* $d^+(x)$ (respectively, *indegree* $d^-(x)$) of a vertex x of a digraph D is the number of vertices y such that $xy \in \mathcal{A}(D)$ (respectively, $yx \in \mathcal{A}(D)$). The *irregularity* $i(T)$ of a tournament T is the maximum of $|d^+(x) - d^-(x)|$ over all vertices x of T. If $i(T) = 0$, then T is *regular* and if $i(T) = 1$, then T is *almost regular*. If $i(T) = m$, T is *m-irregular*. Observe that every vertex of a tournament T of order p has outdegree between $\frac{1}{2}(p-1-i(T))$ and $\frac{1}{2}(p-1+i(T))$. The outdegree and indegree of x are its *semidegrees*. Further digraph terminology and notation are given in the next section.

There are a number of conditions which guarantee that a tournament is arc pancyclic or strongly panconnected (see, e.g., [1]-[18]). In particular, Alspach [1] proved that every regular tournament is arc pancyclic. Jacobsen [12] proved that every almost regular tournament of order $p \ge 8$ is 4-arc pancyclic. Alspach et al. [2] proved that every regular tournament of order $p \ge 7$ is strongly panconnected. Darbinyan [9] proved that every almost regular tournament of order $p \ge 10$ is strongly panconnected.

Thomassen [17] generalized these results as follows:

Theorem 1.1. *Let T be an m-irregular tournament of order p. If $m \le \frac{1}{5}(p-9)$, then T is strongly panconnected. If $m \le \frac{1}{5}(p-3)$, then T is 4-arc pancyclic.*

In [8] and [10], Darbinyan obtained the following:

Theorem 1.2. *Let T be a regular tournament of order $2n+1$ and let $S \subset V(T)$.*

(i) [8] If $2 \le |S| \le \frac{1}{3}(n-2)$, then $T-S$ is strongly panconnected.

(ii) [10] Let $|S| \le \frac{1}{2}(n-3)$ and $x, y \in V(T)\setminus S$ be two distinct vertices. If $T-S$ contains an (x,y)-path of length r, where $r \in [3, 2n-|S|-1]$, then $T-S$ also contains an (x,y)-path of length $r+1$.

We will use the following result of Moon [14].

Theorem 1.3. *Let H be an m-irregular tournament of order $p \ge 2$. Then there is a regular tournament T of order $p+m$ such that T contains H as a subtournament.*

From Theorems 1.2(ii) and 1.3 it is not difficult to obtain the following:

Corollary 1.4. *[10] Let T be an m-irregular tournament of order $p \geq 7$, where $m \leq \frac{1}{3}(p-7)$, and x, y are two distinct vertices. If T contains an (x,y)-path of length r, where $r \in [3, p-2]$, then T also contains an (x,y)-path of length $r+1$.*

In this paper, we prove the following theorems, which improve in a sense the above-mentioned results of Alspach, Jacobsen, Alspach et al., Thomassen and Darbinyan. The following theorem is our main result.

Theorem 1.5. *Let T be a regular tournament of order $2n+1 \geq 11$ and let S be a subset in $V(T)$. Suppose that $|S| \leq \frac{1}{2}(n-2)$ and x, y are two distinct vertices in $V(T) \setminus S$. If $T - S$ contains an (x,y)-path of length r, where $r \in [3, 2n - |S| - 1]$, then $T - S$ also contains an (x,y)-path of length $r+1$.*

Note that Theorem 1.2(i) was only announced in [8] and its proof has never been published. The preprint [10] gave only an outline of the proof of Theorem 1.2(ii) and its complete proof has never been published either. Also note that the main arguments in the proof of Theorem 1.5 are different from those in the proof of Theorem 1.1 by Thomassen.

Remark 1. Let H be a tournament of order $p \geq 5$ with irregularity $m \leq \frac{1}{3}(p-5)$. Then by Theorem 1.3, there is a regular tournament T of order $p+m = 2n+1$ such that T contains H as a subtournament. Let $S = V(T) \setminus V(H)$. Note that $|S| = m$. Then we have $p + \frac{1}{3}(p-5) \geq 2n+1$ implying $p \geq \frac{1}{2}(3n+4)$. Therefore, $m = 2n+1-p \leq 2n+1-\frac{1}{2}(3n+4) = \frac{1}{2}(n-2)$. Thus, $|S| = m \leq \frac{1}{2}(n-2)$.

Theorem 1.6. *Let T be an m-irregular tournament of order p such that $p+m \geq 11$. If $m \leq \frac{1}{3}(p-5)$ (respectively, $m \leq \frac{1}{5}(p-3)$), then T is 4-strongly panconnected (respectively, T is strongly panconnected or belongs to the family $\mathcal{G}$ of tournaments defined in Section 3).*

Proof. Use the construction of Remark 1 to obtain a regular tournament $\hat{T}$ containing T. Let $2n+1 = p+m$ be the order of $\hat{T}$. Let $S = V(\hat{T}) \setminus V(T)$.

Let us first prove the case of $m \leq \frac{1}{3}(p-5)$. By Remark 1, $m = |S| \leq \frac{1}{2}(n-2)$ and hence by Lemma 3.2(ii), T has an (x,y)-path of length 3 or 4 for every pair x, y of distinct vertices. Also, by Theorem 1.5, T has an (x,y)-path of any length from 4 to $|V(T)|-1$. Thus, T is 4-strongly panconnected.

Now we prove the case of $m \leq \frac{1}{5}(p-3)$. Since $p+m \geq 11$ and $m \leq \frac{1}{5}(p-3)$, we have $p \geq 10$. Hence, $m \leq \frac{1}{5}(p-3) \leq \frac{1}{3}(p-5)$ and we can use Lemma 3.2 to show that T either has an (x,y)-path of length 3 for every pair x, y of distinct vertices, or T belongs to the family $\mathcal{G}$. If T does not belong to $\mathcal{G}$, by Theorem 1.5, T is strongly panconnected. $\square$

Remark 2. Using the above observation, we can formulate Theorem 1.6 in terms of semidegrees as follows: *If the semidegrees of every vertex of a tournament T of order p*

are between $\frac{1}{3}(p+1)$ *and* $\frac{2}{3}(p-2)$ *(respectively, between* $\frac{1}{5}(2p-1)$ *and* $\frac{1}{5}(3p-4)$*), then the assertion of Theorem 1.6 holds.*

From Theorem 1.6 (Theorem 1.5) it follows that

(i) if T is regular (i.e, $m = 0$), then for $p \geq 11$ we have a result by Alspach that any regular tournament is arc pancyclic (observe that any arc of a regular tournament is on 3-cycle) and for $p \geq 11$ a result by Alspach et al. that any regular tournament is strongly panconnected.

(ii) if T is almost regular (i.e, $m = 1$), then for $p \geq 10$ we have a result by Jacobsen that any almost regular tournament is 4-arc pancyclic and a result by Darbinyan that any almost regular tournament of order $p \geq 10$ is strongly panconnected.

(iii) from Theorem 1.6 (Theorem 1.5) also we have Theorems 1.1 and 1.2.

The following remark shows that Theorem 1.5 is sharp in a sense.

Remark 3. If in Theorem 1.5 we replace $\frac{1}{2}(n-2)$ with $\frac{1}{2}(n-1)$ (respectively, $\frac{n}{2}$), then there is a regular tournament of order $2n+1 = 11$ (respectively, $2n+1 = 13$), which contains a subset $S \subset V(T)$ with $|S| = \frac{1}{2}(n-1)$ (respectively, $|S| = \frac{n}{2}$) and two distinct vertices $x, y \in V(T) \setminus S$ such that $T - S$ contains an (x, y)-path of length 3, but $T - S$ contains no (x, y)-path of length 4.

Here we define only a regular tournament T of order $2n+1 = 11$ as follows:

$V(T) = \{x_0, x_1, x_2, x_3\} \cup A \cup S$, where $A = \{u_1, u_2, u_3, u_4, z\}$ and $S = \{v_1, v_2\}$. The arc set of T is defined by in-neighborhoods of vertices as follows: $N^-(x_0) = A$, $N^-(x_1) = \{x_0, z, u_3, u_4, v_2\}$, $N^-(x_2) = \{x_0, x_1, z, u_3, u_4\}$, $N^-(x_3) = \{x_0, x_1, x_2\} \cup S$, $N^-(z) = \{x_3, u_1, u_2, u_3, u_4\}$, $N^-(u_1) = \{x_1, x_2, x_3, u_4, v_2\}$, $N^-(u_2) = \{x_1, x_2, x_3, u_1, v_1\}$, $N^-(u_3) = \{x_3, u_1, u_2, v_1, v_2\}$, $N^-(u_4) = \{x_3, u_2, u_3, v_1, v_2\}$, $N^-(v_1) = \{x_0, x_1, x_2, z, u_1\}$ and $N^-(v_2) = \{x_0, x_2, z, v_1, u_2\}$.

It is not difficult to check that $T - S$ contains an (x_0, x_3)-path of length 3, but contains no (x_0, x_3)-path of length 4.

Remark 4. One of the main ideas of the paper is that there are two kinds of conditions for various kinds of path-connectedness, where the first one is that of "relatively small irregularity" (Theorem 1.1), and the second one is that of "being a relatively large part of a regular tournament" (Theorem 1.2). The equivalence of these two kinds of conditions is set up through Theorem 1.3. The main contribution of the this paper is the result of Theorem 1.5 which not only improves Theorem 1.2(ii) but also with a complete proof which was not given with Theorem 1.2(ii). And from Theorem 1.5 we are able to prove Theorem 1.6, which improves Theorem 1.1. (Of course more former results are improved as pointed out before Remark 3.)

2 Further terminology and notation

If xy is an arc of a digraph D, then we say that x *dominates* y. For disjoint subsets B and C of $V(D)$, let $\mathcal{A}(B \to C) = \{xy \in \mathcal{A}(D) \,|\, x \in B, y \in C\}$. We write $B \to C$ if every vertex of B dominates every vertex of C. If $A \subseteq V(D) \setminus (B \cup C)$, then $A \to B \to C$ is a shortcut for $A \to B$ and $B \to C$. If $x \in V(D)$ and $A = \{x\}$, we write x instead of $\{x\}$.

The *out-neighborhood* (respectively, *in-neighborhood*) of a vertex x is the set $N^+(x) = \{y \in V(D) \,|\, xy \in \mathcal{A}(D)\}$ (respectively, $N^-(x) = \{y \in V(D) \,|\, yx \in \mathcal{A}(D)\}$). Similarly, if $B \subseteq V(D)$, then $N^+(x, B) = \{y \in B \,|\, xy \in \mathcal{A}(D)\}$ and $N^-(x, B) = \{y \in B \,|\, yx \in \mathcal{A}(D)\}$. We define $N^{+2}(x, B)$ and $N^{-2}(x, B)$ as follows:

$$N^{+2}(x, B) = \{z \in B \setminus (\{x\} \cup N^+(x, B)) \mid yz \in \mathcal{A}(D), y \in N^+(x, B)\},$$

$$N^{-2}(x, B) = \{z \in B \setminus (\{x\} \cup N^-(x, B)) \mid zy \in \mathcal{A}(D), y \in N^-(x, B)\}.$$

Note that for a vertex x, we have $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$. Similarly, let $d^+(x, B) = |N^+(x, B)|$ and $d^-(x, B) = |N^-(x, B)|$.

The *path* (respectively, the *cycle*) in D consisting of distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). The *length* of a cycle or a path is the number of its arcs. A *k-cycle* is a cycle of length k. We say that $x_1 x_2 \ldots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-path. The digraph $\overleftarrow{D}$ obtained from a digraph D by replacing every arc $xy \in \mathcal{A}(D)$ with the arc yx is called the *converse* of D. We will use *the principle of digraph duality*: Let D be a digraph and $\overleftarrow{D}$ the converse of D. Then D contains a subdigraph H if and only if $\overleftarrow{D}$ contains the converse $\overleftarrow{H}$ of H.

3 Preliminaries

The following lemma states several well-known and simple claims which are the basis of our results and other theorems on directed cycles and paths in tournaments. The claims will be used extensively in the proof of our result.

Lemma 3.1. *Let T be a tournament of order $p \geq 2$. Then the following statements are true.*

(i) The tournament T contains two distinct vertices x and y (respectively, u and v) such that $d^-(x) \leq \frac{1}{2}(p-1)$ and $d^-(y) \geq \frac{1}{2}(p-1)$ (respectively, $d^+(u) \leq \frac{1}{2}(p-1)$ and $d^+(v) \geq \frac{1}{2}(p-1)$).

(ii) If T is regular, then $p = 2n+1$ and for any vertex $x \in V(T)$, $d^-(x) = d^+(x) = n$.

(iii) If T is not regular, then T contains two distinct vertices x and y (respectively, u and v) such that $d^-(x) \leq \frac{1}{2}(p-2)$ and $d^-(y) \geq \frac{1}{2}p$ (respectively, $d^+(u) \leq \frac{1}{2}(p-2)$ and $d^+(v) \geq \frac{1}{2}p$).

(iv) If T is almost regular, then $p = 2n$ and n vertices have indegrees equal to n and the other n vertices have outdegrees equal to n.

(v) Let T be a non-regular tournament. If for all $v \in V(T)$, $d^-(v) < \frac{1}{2}(p+1)$ (or $d^+(v) < \frac{1}{2}(p+1)$), then T is almost regular.

(vi) Let T be a tournament. If for all $v \in V(T)$, $d^-(v) < \frac{1}{2}p$ (or $d^+(v) < \frac{1}{2}p$), then T is regular.

To formulate Lemma 3.2, we need the following definition.

Definition. *By $\mathcal{G}$ we denote the set of regular tournaments, each of which has order $6k+3 \geq 9$ and vertex set $\{x, y, z\} \cup A \cup B \cup C \cup S$ with the properties $|A| = |C| = 2k-1$, $|B| = k+2$, $|S| = k$, the subtournaments induced by the subsets A, C, $\{z\} \cup B \cup S$ are regular, $A \to B \cup S \to C$, $C \to A$, $C \to z \to A$, $x \to \{y, z\} \cup A \cup S$, $\{x, z\} \cup C \cup S \to y$, $y \to A \cup B$ and $B \cup C \to x$.*

Let $G \in \mathcal{G}$. Observe that $G - S$ has no (x, y)-path of length 3.

Remark 5. It is interesting that Thomassen [17] used tournaments of the form $G - S$, where $G \in \mathcal{G}$ to show that there are many tournaments of order p with irregularity equal to $\frac{1}{5}(p-3)$, which are not 3-strongly panconnected.

Lemma 3.2. *Let T be a regular tournament of order $2n+1$. Suppose that $S \subseteq V(T)$ and x, y are two distinct vertices in $V(T) \setminus S$. Then the following hold:*

(i) If $n \geq 3$ and $k = |S| \leq \frac{1}{3}(n-1)$, then $T - S$ contains an (x, y)-path of length 3, unless T is isomorphic to a tournament from $\mathcal{G}$.

(ii) If $n \geq 5$ and $k = |S| \leq \frac{1}{2}n$ and $T - S$ contains no (x, y)-path of length 3, then $T - S$ contains an (x, y)-path of length 4.

Proof. Let $A = V(T) \setminus (\{x, y\} \cup S)$ and $R = N^+(x, A) \cap N^-(y, A)$. If $T - S$ contains no (x, y)-path of length 3, then $|R| \leq 1$, $|N^+(x, A)| \geq n - k - 1$ and

$$\mathcal{A}(N^+(x, A) \to N^-(y, A)) = \emptyset, \quad \text{i.e.,} \quad N^-(y, A) \to N^+(x, A) \tag{1}$$

and observe that

$$|N^+(x, A) \setminus R| \geq n - k - 2 \quad \text{and} \quad |N^-(y, A) \setminus R| \geq n - k - 2. \tag{2}$$

(i) Suppose that $T - S$ contains no (x, y)-path of length 3. Note that $T' = T\langle N^+(x, A) \setminus R\rangle$ must contain a vertex u with $d^-(u, T') \geq (|V(T')| - 1)/2$, the average indegree in T'. Using (1) and (2), we obtain that the following holds for $d^-(u)$:

$$n = d^-(u) \geq |\{x, y\}| + |N^-(y, A)| + (|N^+(x, A) \setminus R| - 1)/2$$

$$\geq n-k+1+(n-k-3)/2 = n+(n-3k-1)/2. \tag{3}$$

Therefore, $n-3k-1 \leq 0$, i.e., $3k = n-1$ as $3k \leq n-1$, and all the inequalities that led to (3) in fact are equalities. This means that $|N^+(x,A)\setminus R| = n-k-2$ (by the digraph duality, $|N^-(y,A)\setminus R| = n-k-2$), $|R|=1$, $|B|=k+2$, where $B = A\setminus(N^+(x,A)\cup N^-(y,A))$, $y\to B\to x$, $x\to S\to y$, $T\langle N^+(x,A)\setminus R\rangle$ is a regular tournament (by the digraph duality, $T\langle N^-(y,A)\setminus R\rangle$ is also a regular tournament), $N^+(x,A)\setminus R\to S\cup B\to N^-(y,A)\setminus R$, $y\to N^+(x,A)\setminus R$, $N^-(y,A)\setminus R\to x$, $xy\in\mathcal{A}(T)$ and $T\langle S\cup B\cup R\rangle$ is a regular tournament. Therefore, T is isomorphic to a tournament of the type $\mathcal{G}$.

(ii) By contradiction, suppose that $T-S$ contains no (x,y)-path of lengths 3 and 4. By Lemma 3.1, we know that there is a vertex $u\in N^+(x,A)\setminus R$ such that $d^+(u,N^+(x,A)\setminus R)\leq(|N^+(x,A)\setminus R|-1)/2$. Consider the out-neighbors of u. Since $d^+(u)=n$ and u can not dominate x and y, the number of out-neighbors of u in A is at least $n-k$. Further the out-neighbors of u which are contained in $N^+(x,A)\setminus R$ are all in $N^{+2}(x,A)$. This together with (1) implies that

$$|N^{+2}(x,A)|\geq n-k-(|N^+(x,A)\setminus R|-1)/2 = (2n-2k-|N^+(x,A)\setminus R|+1)/2.$$

Since $T-S$ contains no (x,y)-path of length 4, it follows that $N^-(y,A)\setminus R\to N^{+2}(x,A))$. This together with (1), (2) and the last inequality implies that for some vertex $v\in N^-(y,A)\setminus R$ the following holds

$$d^+(v)\geq|\{x,y\}|+|N^+(x,A)|+(|N^{+2}(x,A)|+(|N^-(y,A)\setminus R|-1)/2\geq 2n-2k+1.$$

Hence, $2k\geq n+1$, which is a contradiction. This completes the proof of the lemma. □

For $n=3$ and $n=4$, it is not difficult to construct tournaments for which Lemma 3.2(ii) is not true. To see this, we consider the following examples of regular tournaments of order 7 and 9.

Let T be a regular tournament of order 7 (of order 9) with vertex set $V(T)=\{x,y,z\}\cup B\cup S$, where $|S|=1$, $|B|=3$ (respectively, $|S|=2$, $|B|=4$). The tournament T satisfies the following conditions:

$\mathcal{A}(T)$ contains the arcs xy, xz and zy, $x\to S\to y$, $y\to B\to x$ and $T\langle\{z\}\cup B\cup S\rangle$ is a regular tournament. It is easy to check that $|S|\leq n/2$ and $T-S$ contains no (x,y)-path of lengths 3 and 4.

Note that if we replace $\frac{1}{2}n$ with $\frac{1}{2}(n+1)$ in Lemma 3.2(ii), then there is a regular tournament T of order $2n+1=11$ ($2n+1=15$) which contains a subset $S\subseteq V(T)$ with $|S|=k=\frac{1}{2}(n+1)$ and two distinct vertices $x,y\in V(T-S)$ such that $T-S$ contains no path of lengths 3 and 4. To see this we define regular tournaments T of order (a) $2n+1=11$ and (b) $2n+1=15$ as follows:

(a) Let T be a regular tournament of order 11 with $V(T)=\{x,y,z\}\cup A\cup S$ such that $|A|=5$, $k=|S|=3$, $x\to S\to y$, $y\to A\to x$, the arcs xy,xz,zy are in $\mathcal{A}(T)$ and $T\langle\{z\}\cup A\cup S\rangle$ is a regular tournament.

It is easy to check that $T-S$ contains no (x,y)-path of length greater or equal to 3.

(b) Let T be a regular tournament of order 15 with $V(T)=\{x,y,z,u,v\}\cup A\cup B\cup S$ such that $|A|=|B|=3$, $k=|S|=4$, $S=\{a_1,a_2,b_1,b_2\}$, the arcs $xy, xz, zy, xu, vy, vu, vz, zu, a_1a_2, b_1b_2, a_2b_2, a_2b_1, a_1b_1, b_2a_1$ are in $\mathcal{A}(T)$, $x\to S\to y$, $\{y,z,u,v\}\to A$, $B\to\{x,u,v,z\}$, $y\to\{u\}\cup B$, $A\to\{x\}\cup B$, $v\to\{x\}\cup A$, $S\to v$, $u\to A\cup S$, the induced subtournaments $T\langle A\rangle$ and $T\langle B\rangle$ are regular tournaments, $\{b_1,b_2\}\to A\to\{a_1,a_2\}$, $\{a_1,a_2\}\to B\to\{b_1,b_2\}$ and $\{b_1,b_2\}\to z\to\{a_1,a_2\}$.

It is not difficult to check that $T-S$ contains no (x,y)-path of length 3 and 4. Note that $T-S$ contains an (x,y)-path of every length $5,6,\ldots,10$.

In Lemmas 3.3 and 3.4, we assume that $P:=x_0x_1\ldots x_r$ is an (x_0,x_r)-path of length r in a tournament T and z is a vertex in $V(T)\setminus V(P)$ such that $\{x_{\alpha+1},x_{\alpha+2},\ldots,x_r\}\to z\to\{x_0,x_1,\ldots,x_\alpha\}$, where $\alpha\in[2,r-3]$. Moreover, any (x_0,x_r)-path of length $r+1$ with vertex set $\{z\}\cup V(P)$ is denoted by Q, and we assume that T contains no such a path Q.

Lemma 3.3. *Suppose that $x_sx_t\in\mathcal{A}(T)$ with $s\in[1,\alpha-1]$ and $t\in[\alpha+3,r]$. Then $\mathcal{A}(\{x_0,x_1,\ldots,x_{s-2}\}\to\{x_{\alpha+2},x_{\alpha+3},\ldots,x_{t-1}\})=\emptyset$ when $s\geq 2$, and*

$$\mathcal{A}(x_{s-1}\to\{x_{\alpha+2},x_{\alpha+3},\ldots,x_{t-1}\})=\emptyset,\quad \textit{when}\quad t-s\neq 5.$$

Proof. By contradiction, suppose that there exist integers $a\in[0,s-1]$ and $b\in[\alpha+2,t-1]$ such that $x_ax_b\in\mathcal{A}(T)$. Observe that $\{x_{b-1},x_{t-2},x_{t-1}\}\to z\to\{x_{a+1},x_{a+2},x_{s+1}\}$. Note that $t-s\geq 4$. Now we prove the following facts.

Fact 3.1.

(i) $x_{a+1}x_{b-1}\in\mathcal{A}(T)$.

(ii) If $a\leq s-2$, then $x_{s+1}x_{a+1}\in\mathcal{A}(T)$ and if $a=s-1$, then $x_{s+2}x_s\in\mathcal{A}(T)$ and $t-s\geq 6$.

Proof. (i). Indeed, if $x_{b-1}x_{a+1}\in\mathcal{A}(T)$, then $Q=x_0x_1\ldots x_ax_b\ldots x_{t-1}zx_{s+1}\ldots x_{b-1}x_{a+1}\ldots x_sx_t\ldots x_r$, a contradiction.

(ii). If $a\leq s-2$ and $x_{a+1}x_{s+1}\in\mathcal{A}(T)$, then

$$Q=x_0x_1\ldots x_{a+1}x_{s+1}\ldots x_{t-1}zx_{a+2}\ldots x_sx_t\ldots x_r.$$

If $a=s-1$ and $x_sx_{s+2}\in\mathcal{A}(T)$, then either $x_{s+1}x_{t-1}\in\mathcal{A}(T)$ and

$$Q=x_0x_1\ldots x_sx_{s+2}\ldots x_{t-2}zx_{s+1}x_{t-1}\ldots x_r$$

or $x_{t-1}x_{s+1}\in\mathcal{A}(T)$ and $Q=x_0\ldots x_{s-1}x_b\ldots x_{t-1}x_{s+1}\ldots x_{b-1}zx_sx_t\ldots x_r$. Thus, in both cases we have a contradiction.

It is easy to see that from $x_sx_{b-1}\in\mathcal{A}(T)$ and $x_{s+2}x_s\in\mathcal{A}(T)$ it follows that $b-1\geq s+3$ and hence, $t-s\geq 6$ since $t-s\neq 5$ when $a=s-1$.

Fact 3.2. If $x_ix_{a+1}\in\mathcal{A}(T)$ with $i\in[s+1,b-3]$, then $x_{i+2}x_{a+1}\in\mathcal{A}(T)$.

Proof. To prove it by contradiction, suppose that $x_ix_{a+1} \in \mathcal{A}(T)$ with $i \in [s+1, b-3]$ and $x_{a+1}x_{i+2} \in \mathcal{A}(T)$. If $x_{t-1}x_{i+1} \in \mathcal{A}(T)$, then

$$Q = x_0x_1 \dots x_ax_b \dots x_{t-1}x_{i+1} \dots x_{b-1}zx_{s+1} \dots x_ix_{a+1} \dots x_sx_t \dots x_r,$$

and if $x_{i+1}x_{t-1} \in \mathcal{A}(T)$, then $Q = x_0x_1 \dots x_{a+1}x_{i+2} \dots x_{t-2}zx_{a+2} \dots x_{i+1}x_{t-1} \dots x_r$, a contradiction.

Using Fact 3.2, it is not difficult to see that there is no $i \in [s+1, b-3]$ such that $\{x_i, x_{i+1}\} \to x_{a+1}$, i.e., $d^-(x_{a+1}, \{x_i, x_{i+1}\}) \leq 1$ (for otherwise, we obtain that $x_{b-1}x_{a+1} \in \mathcal{A}(T)$, contradicting Fact 3.1 that $x_{a+1}x_{b-1} \in \mathcal{A}(T)$). By Fact 3.1 we have that if $a \leq s-2$, then $x_{s+1}x_{a+1} \in \mathcal{A}(T)$, and $a = s-1$, then $x_{s+2}x_s \in \mathcal{A}(T)$. This together with $d^-(x_{a+1}, \{x_i, x_{i+1}\}) \leq 1$ and Fact 3.2 implies that

$$\begin{aligned} &\text{if } a \leq s-2, \text{ then } \{x_{s+1}, x_{s+3}, \dots, x_{b-2}\} \to x_{a+1} \to \{x_{s+2}, x_{s+4}, \dots, x_{b-1}\}, \\ &\text{if } a = s-1, \text{ then } \{x_{s+2}, x_{s+4}, \dots, x_{b-2}\} \to x_s \to \{x_{s+1}, x_{s+3}, \dots, x_{b-1}\}. \end{aligned} \tag{4}$$

Thus, in both cases we have $x_{b-2}x_{a+1} \in \mathcal{A}(T)$. Now using (4), we obtain, if $x_{t-1}x_{b-1} \in \mathcal{A}(T)$, then $b \leq t-2$ and $Q = x_0 \dots x_ax_b \dots x_{t-1}x_{b-1}zx_{s+1} \dots x_{b-2}x_{a+1} \dots x_sx_t \dots x_r$, and if $b \leq t-2$ and $x_{b-1}x_{t-1} \in \mathcal{A}(T)$, then $Q = x_0 \dots x_ax_b \dots x_{t-2}zx_{a+1} \dots x_{b-1}x_{t-1} \dots x_r$, thus in both cases we have a contradiction. We may therefore assume that $b = t-1$. In this case, if $x_bx_{s+1} \in \mathcal{A}(T)$, then $Q = x_0x_1 \dots x_ax_bx_{s+1} \dots x_{b-1}zx_{a+1} \dots x_sx_t \dots x_r$, a contradiction. Therefore, we may assume that $x_{s+1}x_b \in \mathcal{A}(T)$. This follows that if $a \leq s-2$, then $Q = x_0 \dots x_{a+1}x_{s+2} \dots x_{b-1}zx_{a+2} \dots x_{s+1}x_b \dots x_r$, a contradiction. Assume that $a = s-1$. Then by Fact 3.1(ii), $t - s \geq 6$. Let $t \geq \alpha + 4$. It is easy to see that if $x_{b-1}x_{s+2} \in \mathcal{A}(T)$, then by (4), $x_{a+1}x_{s+2} \in \mathcal{A}(T)$ and $Q = x_0x_1 \dots x_sx_{b-1}x_{s+2} \dots x_{b-2}zx_{s+1}x_b \dots x_r$, and if $x_{s+2}x_{b-1} \in \mathcal{A}(T)$, then $Q = x_0x_1 \dots x_sx_{s+3} \dots x_{b-2}zx_{s+1}x_{s+2}x_{b-1} \dots x_r$. Let now $t = \alpha+3$. Then $b = \alpha + 2$ and $z \to \{x_{s+1}, x_{s+2}, \dots, x_\alpha\}$. From $t - s \geq 6$ and (4) it follows that $t - s \geq 7$ and $\alpha \geq s + 4$. From (4) we also have that $x_{b-4}x_s \in \mathcal{A}(T)$. Therefore, if $x_bx_{b-3} \in \mathcal{A}(T)$, then $Q = x_0x_1 \dots x_{s-1}x_bx_{b-3}x_{b-2}x_{b-1}zx_{s+1} \dots x_{b-4}x_sx_t \dots x_r$, a contradiction. We may therefore assume that $x_{b-3}x_b \in \mathcal{A}(T)$. If $x_{b-2}x_{s+1} \in \mathcal{A}(T)$, then $Q = x_0x_1 \dots x_sx_{b-1}zx_{b-2}x_{s+1} \dots x_{b-3}x_b \dots x_r$, a contradiction. Thus, we have that $x_{b-3}x_b \in \mathcal{A}(T)$ and $x_{s+1}x_{b-2} \in \mathcal{A}(T)$. Therefore, $Q = x_0x_1 \dots x_{s+1}x_{b-2}x_{b-1}zx_{s+2} \dots x_{b-3}x_b \dots x_r$, a contradiction. Thus, in all cases we have a contradiction. □

Lemma 3.4. *Suppose that $x_sx_t \in \mathcal{A}(T)$ with $s \in [\alpha, r-2]$ and $t \in [s+2, r]$. If $k = \lfloor \frac{1}{2}(t-s) \rfloor$, then $\mathcal{A}(\{x_0, x_1, \dots, x_{\alpha-1}\} \to \{x_{s+1}, x_{s+2}, \dots, x_{s+k}\}) = \emptyset$.*

Proof. The proof by induction on $k = \lfloor \frac{1}{2}(t-s) \rfloor$. Observe that $\{x_{s+1}, x_{t-1}\} \to z$. For the base step, it is easy to see that if $x_ix_{s+1} \in \mathcal{A}(T)$ with $i \in [0, \alpha-1]$, then $Q = x_0x_1 \dots x_ix_{s+1} \dots x_{t-1}zx_{i+1} \dots x_sx_t \dots x_r$, a contradiction. We may therefore assume that $\mathcal{A}(\{x_0, x_1, \dots, x_{\alpha-1}\} \to x_{s+1}) = \emptyset$. This means that if $2 \leq t - s \leq 3$, then the lemma is true. Assume that $t-s\geq 4$. For the inductive step, we assume that if $x_{s_1}x_{t_1}\in\mathcal{A}(T)$ with $s_1\in[\alpha,r-2]$, $t_1\in[s_1+2,r]$ and $t_1-s_1<t-s$, then $\mathcal{A}(\{x_0,x_1,\dots,x_{\alpha-1}\}\to\{x_{s_1+1},x_{s_1+2},\dots,x_{s_1+k_1}\})=\emptyset$, where $k_1=\lfloor\frac{1}{2}(t_1-s_1)\rfloor$.

If $x_{t-1}x_{s+1}\in\mathcal{A}(T)$, then for all $i\in[1,\alpha-1]$ and $j\in[s+2,t-1]$, $x_ix_j\notin\mathcal{A}(T)$ (for otherwise, if $x_ix_j\in\mathcal{A}(T)$, then $Q=x_0x_1\dots x_ix_j\dots x_{t-1}x_{s+1}\dots x_{j-1}z$ $x_{i+1}\dots x_sx_t\dots x_r$, a contradiction). Therefore, $\mathcal{A}(\{x_0,x_1,\dots,x_{\alpha-1}\to\{x_{s+1},x_{s+2},\dots,$ $x_{t-1}\})=\emptyset$ and we are done. Now assume that $x_{s+1}x_{t-1}\in\mathcal{A}(T)$. Then $t-1-(s+1)<t-s$ and, by the induction hypothesis, $\mathcal{A}(\{x_0,x_1,\dots,x_{\alpha-1}\}\to\{x_{s+2},\dots,x_{s+1+m}\})=\emptyset$, where $m=\lfloor(t-s-2)/2\rfloor$. Thus, $\mathcal{A}(\{x_0,x_1,\dots,x_{\alpha-1}\}\to\{x_{s+1},x_{s+2},\dots,x_{s+1+m}\})=\emptyset$. This implies that Lemma 3.4 is true since $m+1=\lfloor\frac{1}{2}(t-s)\rfloor$. □

4 Proof of the main result

For convenience of the reader, we restate it here.

Theorem 1.5. *Let T be a regular tournament of order $2n+1\geq 11$ and let S be a subset in $V(T)$. Let $|S|\leq\frac{1}{2}(n-2)$ and x,y be two distinct vertices in $V(T)\setminus S$. If $T-S$ contains an (x,y)-path of length r, where $r\in[3,2n-|S|-1]$, then $T-S$ also contains an (x,y)-path of length $r+1$.*

Proof. Recall that $P=x_0x_1\dots x_r$ is a path of length r in T and $k=|S|$.

Observe that for any vertex $x\in V(T-S)$, $d^-(x,V(T-S))\geq n-k$ and $d^+(x,V(T-S))\geq n-k$. Now by Q we denote any (x_0,x_r)-path of length $r+1$ in $T-S$. Suppose that $T-S$ has no such path Q. Let $A=V(T)\setminus(V(P)\cup S)$. We will prove a series of claims (Claims 1-13).

Claim 1: $N^-(x_r,A)=N^+(x_0,A)=\emptyset$, i.e., $d^-(x_r,A)=d^+(x_0,A)=0$.

Proof: By the digraph duality, it suffices to prove that $N^-(x_r,A)=\emptyset$. Suppose, on the contrary, that $N^-(x_r,A)\neq\emptyset$. Let $z\in N^-(x_r,A)$, i.e., $zx_r\in\mathcal{A}(T)$. Then $z\to\{x_0,x_1,\dots,x_r\}$, $r\leq n-1$ and $|N^-(z,A)|\geq n-k$. It is easy to see that

$$\mathcal{A}(\{x_0,x_1,\dots,x_{r-2}\}\to N^-(z,A)\cup\{z\})=\emptyset. \tag{5}$$

We distinguish the following two cases depending on r.

Case 1.1. $r\geq n-k$. We know that $N^-(z,A)$ contains a vertex u such that $d^-(u,N^-(z,A))\leq 0.5(d^-(z,A)-1)$. This together with (5) implies that

$$|N^{-2}(z,A)|\geq|N^{-2}(z,A)\cap N^-(u)|\geq n-k-2-0.5(d^-(z,A)-1). \tag{6}$$

It is easy to see that $\mathcal{A}(\{x_0,x_1,\dots,x_{r-3}\}\to N^{-2}(z,A))=\emptyset$. This together with (5) implies that $\{z\}\cup N^-(z,A)\cup N^{-2}(z,A)\to\{x_0,x_1,\dots,x_{r-3}\}$. Therefore, $\{x_0,x_1,\dots,x_{r-3}\}$ contains a vertex v such that $d^-(v)\geq|N^-(z,A)|+|N^{-2}(z,A)|+1+0.5(r-3)$ and unless

$\{x_0, x_1, \dots, x_{r-3}\}$ induces a regular tournament, we can find a vertex v so that equality does not hold. Now using (6), $r \geq n-k$ and the fact that $|N^-(z,A)| \geq n-k$, we obtain

$$n = d^-(v) \geq d^-(z,A) + n - k - 2 - 0.5(d^-(z,A) - 1) + 0.5(r-1) \geq 2n - 2k - 2.$$

Therefore, $2k = n-2$ (as $2k \leq n-2$) and all inequalities which were used in the last inequality in fact are equalities, in particular, $T\langle\{x_0, x_1, \dots, x_{r-3}\}\rangle$ is a regular tournament. Therefore, $\{x_0, x_1, \dots, x_{r-3}\} \to x_{r-2}$. Hence,

$$\begin{aligned} d^-(x_{r-2}) &\geq |N^-(z,A)| + |\{z\}| + |\{x_0, x_1, \dots, x_{r-3}\}| \\ &\geq n - k + 1 + r - 2 \\ &\geq n - k + 1 + n - k - 2 \\ &= 2n - 2k - 1 \\ &\geq 2n - (n-2) - 1 \\ &= n + 1, \end{aligned}$$

a contradiction.

Case 1.2. $r \leq n-k-1$. Then $N^+(x_0,A) \neq \emptyset$. It is easy to see that

$$N^+(x_0,A) \cap N^-(z,A) = \mathcal{A}(N^+(x_0,A) \to N^-(z,A) \cup \{z, x_1, x_2, \dots, x_r\}) = \emptyset. \quad (7)$$

We know that $N^+(x_0,A)$ contains a vertex u such that $d^+(u, N^+(x_0,A)) \leq 0.5(d^+(x_0,A) - 1)$. From this and (7) it follows that $|N^{+2}(x_0,A)| \geq n-k-0.5(d^+(x_0,A)-1)$. It is not difficult to see that $N^-(x_r,A) \to \{x_0, x_1, \dots, x_r\} \cup N^+(x_0,A) \cup N^{+2}(x_0,A)$. Therefore, $N^-(x_r,A)$ contains a vertex y such that

$$n = d^+(y) \geq |N^+(x_0,A)| + |N^{+2}(x_0,A)| + 0.5(|N^-(x_r,A)| - 1) + r + 1.$$

Now, since $\min\{|N^+(x_0,A)|, |N^-(x_r,A)|\} \geq n-k-r$, we obtain that $n = d^+(y) \geq 2n - 2k + 1$. Hence, $2k \geq n+1$, which contradicts that $2k \leq n-2$. Claim 1 is proved. □

Using Claim 1, $n \geq 5$ and $2k \leq n-2$, we obtain that $|A| \leq n$, $r = 2n - k - |A| \geq n - k \geq 3.5$ (i.e., $r \geq 4$) and

$$\min\{|N^+(x_0, V(P))|, |N^-(x_r, V(P))|\} \geq n-k. \quad (8)$$

Claim 2: $N^+(x_1,A) = N^-(x_{r-1},A) = \emptyset$.

Proof: By the digraph duality, it suffices to prove that $N^-(x_{r-1},A) = \emptyset$. Suppose, on the contrary, that $N^-(x_{r-1},A) \neq \emptyset$. Let z be a vertex in $N^-(x_{r-1},A)$. Then $zx_{r-1} \in \mathcal{A}(T)$, $z \to \{x_0, x_1, \dots, x_{r-1}\}$ and $|N^-(z,A)| \geq n-k-1$.

Case 2.1. $N^-(z,A) \to x_{r-1}$. Then

$$\mathcal{A}(\{x_0, x_1, \dots, x_{r-3}\} \to \{z\} \cup N^-(z,A) \cup N^{-2}(z,A)) = \emptyset. \quad (9)$$

Since $N^-(z,A)$ contains a vertex u such that $d^-(u,N^-(z,A)) \leq 0.5(|N^-(z,A)|-1)$, it follows that $|N^{-2}(z,A)| \geq n-k-1-0.5(|N^-(z,A)|-1)$. Now using (9) and $r \geq n-k$, we obtain that $\{x_0,x_1,\dots,x_{r-3}\}$ contains a vertex v such that

$$n = d^-(v) \geq 0.5(r-3)+n-k-1-0.5(|N^-(z,A)|-1)+|N^-(z,A)|+1 \geq 2n-2k-1.5,$$

a contradiction to $2k \leq n-2$.

Case 2.2. $\mathcal{A}(x_{r-1} \to N^-(z,A)) \neq \emptyset$. Since $\{z\} \cup N^-(z,A) \subseteq A$ and $|N^-(z,A)| \geq n-k-1$, we have that $|A| \geq n-k$. Observe that

$$N^-(z,A) \cup N^-(x_{r-1},A) \to \{x_0,x_1,\dots,x_{r-3}\}. \tag{10}$$

Hence, $\{z\} \cup N^-(z,A) \to x_1$, as $r \geq 4$. Therefore, for some $y \in N^-(z,A)$, $x_{r-1} \to y \to x_1$. If $x_i x_r \in \mathcal{A}(T)$ with $i \in [1,r-2]$ (respectively, with $i \in [1,r-3]$), then $x_0 x_{i+1} \notin \mathcal{A}(T)$ (respectively, $x_0 x_{i+2} \notin \mathcal{A}(T)$), for otherwise $Q = x_0 x_{i+1} \dots x_{r-1} y x_1 \dots x_i x_r$ (respectively, $Q = x_0 x_{i+2} \dots x_{r-1} y z x_1 \dots x_i x_r$), which is a contradiction. From this and (8) we have,

$$d^-(x_0,\{x_2,x_3,\dots,x_{r-1}\}) \geq n-k-2. \tag{11}$$

This together with $|A| \geq n-k$ and $A \to x_0$ implies that $n = d^-(x_0) \geq |A| + d^-(x_0,\{x_2, x_3,\dots,x_{r-1}\}) \geq 2n-2k-2$, which in turn implies $2k = n-2$, $|A| = n-k$ (i.e., $A = \{z\} \cup N^-(z,A)$), $r = n \geq 6$, $k \geq 2$ (as $n \geq 5$ and n is even and $r = 2n-k-|A|$) and $d^-(x_0,\{x_2,x_3,\dots,x_{r-1}\}) = n-k-2$. Now by the above arguments, it is not difficult to see that $N^-(x_r,\{x_1,x_2,\dots,x_{r-2}\}) = \{x_{k+1},x_{k+2},\dots,x_{n-2}\}$. Since $yx_2 \in \mathcal{A}(T)$ (by (10) and $r \geq 6$), it follows that if $x_1 x_{i+1} \in \mathcal{A}(T)$ with $i \in [k+1,n-2]$, then $Q = x_0 x_1 x_{i+1} \dots x_{n-1} y x_2 \dots x_i x_n$, a contradiction. We may therefore assume that $d^+(x_1,\{x_{k+2},x_{k+3},\dots,x_{n-1}\}) = 0$. This together with $d^+(x_1,\{x_0\} \cup A) = 0$ implies that $d^-(x_1) \geq |A|+n-k-1 \geq 2n-2k-1$, a contradiction. Claim 2 is proved. □

From Claim 2 it follows that $d^-(x_{r-1},\{x_0,x_1,\dots,x_{r-2}\}) \geq n-k$. Hence, $r \geq n-k+1$. Using (11) and Claim 1, we obtain, $d^-(x_0) \geq n-k-2+|A|$. Therefore, $|A| \leq k+2$.

Claim 3: $|A| \leq k+1$.

Proof: The proof is by contradiction. Suppose that $|A| \geq k+2$. Then, $|A| = k+2 \geq 2$ and $r = 2n-2k-2$ since $|A| \leq k+2$. This implies that r is even, $r \geq n$ and $r \geq 6$ since $n \geq 5$ and $2k \leq n-2$. Since $x_{r-1} \to A \to x_1$ (Claim 2), it is not difficult to see that if $x_i x_r \in \mathcal{A}(T)$ with $i \in [1,r-2]$, then $x_0 x_{i+1} \notin \mathcal{A}(T)$ (for otherwise, $Q = x_0 x_{i+1} \dots x_{r-1} z x_1 \dots x_i x_r$, where $z \in A$). This together with $A \to x_0$, $d^-(x_r,\{x_1,x_2,\dots,x_{r-2}\}) \geq n-k-2$ and $|A| = k+2$ implies that $n = d^-(x_0) \geq |A| + d^-(x_r,\{x_1,x_2,\dots,x_{r-2}\}) \geq k+2+n-k-2 = n$. This means that $d^-(x_r,\{x_1,x_2,\dots,x_{r-1}\} = n-k-2$ and the in-neighbors of x_0 in $\{x_2,x_3,\dots,x_{r-1}\}$ are only those vertices x_i for which $x_{i-1}x_r \in \mathcal{A}(T)$. From this we conclude that there is no $i \in [1,r-3]$ such that $x_i x_r \in \mathcal{A}(T)$ and $x_{i+1}x_r \notin \mathcal{A}(T)$ since in the converse case $x_0 x_{i+2} \in \mathcal{A}(T)$ and $Q = x_0 x_{i+2} \dots x_{r-1} a_1 a_2 x_1 \dots x_i x_r$, where $a_1, a_2 \in A$

and $a_1a_2 \in \mathcal{A}(T)$. Therefore, $N^-(x_r, \{x_1, x_2, \dots, x_{r-2}\}) = \{x_{n-k-1}, x_{n-k}, \dots, x_{r-2}\}$ and $N^+(x_0, \{x_2, x_3, \dots, x_{r-1}\}) = \{x_2, x_3, \dots, x_{n-k-1}\}$. In particular, $x_rx_1 \in \mathcal{A}(T)$ as $n \geq 5$ and $n-k \geq 3$.

Assume first that for some $y \in A$, $yx_2 \in \mathcal{A}(T)$. Using the above equalities, we obtain $\{x_{n-k}, x_{n-k+1}, \dots, x_{r-1}\} \to x_1$ (for otherwise, $Q = x_0x_1x_j \dots x_{r-1}yx_2 \dots x_{j-1}x_r$, where $j \in [n-k, r-1]$). Now, since $A \cup \{x_0, x_r\} \to x_1$ and $|A| = k+2$, we obtain $d^-(x_1) \geq n+2$, a contradiction.

Assume next that $\mathcal{A}(A \to x_2) = \emptyset$, i.e., $x_2 \to A$. Then $\{x_2, x_3, \dots, x_r\} \to A$ and there is a vertex $u \in A$ such that $d^-(u) \geq 2n - 2k - 3 + (k+1)/2$. Therefore, $2k = n-2$ as $2k \leq n-2$. Thus $k \leq 1$, which is a contradiction, since $2k = n-2$ and $n \geq 6$. Claim 3 is proved. □

Since $r = 2n - k - |A|$, $|A| \leq k+1$ and $2k \leq n-2$, we have $r \geq n+1 \geq 6$ as $n \geq 5$.

Claim 4: $N^+(x_2, A) = N^-(x_{r-2}, A) = \emptyset$.

Proof: By the digraph duality, it suffices to prove that $N^+(x_2, A) = \emptyset$. Suppose, on the contrary, that $N^+(x_2, A) \neq \emptyset$. Let z be a vertex in $N^+(x_2, A)$. Then $\{x_2, x_3, \dots, x_r\} \to z$. This together with $z \to \{x_0, x_1\}$ implies that $|N^+(z, A)| \geq n-k-2$ and $|A| \geq n-k-1$. Now using the facts that $|A| \leq k+1$ (Claim 3) and $2k \leq n-2$, we obtain that $2k = n-2$, $|A| = k+1$ and $r = n+1$. From $2k = n-2$ it follows that n is even, $n \geq 6$ (as $n \geq 5$) and $k \geq 2$. Since $\{x_0\} \cup A \to x_1$ and $r = n+1$, it follows that there is a vertex $x_j \in \{x_4, x_5, \dots, x_{n+1}\}$ such that $x_1x_j \in \mathcal{A}(T)$. Therefore, if $x_0x_2 \in \mathcal{A}(T)$, then $Q = x_0x_2 \dots x_{j-1}zx_1x_j \dots x_r$, and if $x_0x_3 \in \mathcal{A}(T)$, then for some $y \in N^+(z, A)$ we have $Q = x_0x_3 \dots x_{j-1}zyx_1x_j \dots x_r$, which is a contradiction. We may therefore assume that $\{x_2, x_3\} \to x_0$. Since $\{x_2, x_3\} \cup A \to x_0$, $\{x_2, x_3, \dots, x_r\} \to z$ and $r = n+1$, it follows that $d^+(x_0, \{x_4, x_5, \dots, x_n\}) \geq n-k-2 \geq 2$ and $d^+(z, A) = k$. Let $x_0x_m \in \mathcal{A}(T)$ with $m \in [4, n]$ and m the minimum with these properties. Since $A \to x_1$, $x_nz \in \mathcal{A}(T)$ and $T\langle A\rangle$ contains a path zyu of length two, it is easy to see that $d^-(x_{n+1}, \{x_{m-3}, x_{m-2}, x_{m-1}\}) = 0$ and if $x_0x_j \in \mathcal{A}(T)$ with $j \in [4, n]$, then $x_{j-1}x_{n+1} \notin \mathcal{A}(T)$. These together with $d^-(x_{n+1}, A) = 0$ and $|A| = k+1$ imply that $d^+(x_{n+1}) \geq |A| + n - k - 2 + 2 = n+1$, a contradiction. Claim 4 is proved. □

From Claims 1, 2 and 4 it follows that $\{x_r, x_{r-1}, x_{r-2}\} \to A \to \{x_0, x_1, x_2\}$.

Let $\alpha := \max\{i \in [2, r-3] \mid A \to x_i\}$. From this and the assumption that T contains no Q path it follows that $A \to \{x_0, x_1, \dots, x_\alpha\}$, $\mathcal{A}(x_j \to A) \neq \emptyset$ for all $j \in [\alpha+1, r]$ and A contains a vertex z such that

$$\{x_{\alpha+1}, x_{\alpha+2}, \dots, x_r\} \to z \to \{x_0, x_1, \dots, x_\alpha\}. \tag{12}$$

Fact 1: (i) If $2k = n-2$, then n is even (i.e., $n \geq 6$, as $n \geq 5$), $n - k - |A| \geq 1$ and

$r \geq n+1 \geq 7$.

(ii) If $2k \leq n-3$, then $n-k-|A| \geq 2$ and $r \geq n+2 \geq 7$.

Proof: Indeed, $n-k-|A| \geq n-2k-1$ and $r = 2n-k-|A| \geq 2n-2k-1$ as $|A| \leq k+1$. Now, if $2k = n-2$, then $n-k-|A| \geq 1$ and $r \geq n+1 \geq 7$, and if $2k \leq n-3$, then $n-k-|A| \geq 2$ and $r \geq n+2 \geq 7$. $\square$

Fact 2: $n-k-|A| \leq \alpha \leq n-1-\lfloor \frac{1}{2}|A| \rfloor \leq n-1$.

Proof: Since $\{x_{\alpha+1}, x_{\alpha+2}, \ldots, x_r\} \to z$, we have $n = d^-(z) \geq r-\alpha = 2n-k-|A|-\alpha$ and $\alpha \geq n-k-|A|$. From $A \to \{x_0, x_1, \ldots, x_\alpha\}$ it follows that there exists a vertex $x \in A$ such that $n = d^+(x) \geq \alpha+1+\lfloor \frac{1}{2}|A| \rfloor$. Hence, $\alpha \leq n-1-\lfloor \frac{1}{2}|A| \rfloor \leq n-1$. $\square$

Let $B := \{x_i \mid i \in [1, \alpha-1],\ \mathcal{A}(x_i \to \{x_{\alpha+2}, \ldots, x_r\}) \neq \emptyset\}$.

Proposition 1: (i) If $x_i \in B$, then $\mathcal{A}(\{x_0, x_1, \ldots, x_{i-1}\} \to x_{i+1}) = \emptyset$.

(ii) If $x_j x_r \in \mathcal{A}(T)$ with $j \in [\alpha, r-2]$, then $\mathcal{A}(\{x_0, x_1, \ldots, x_{\alpha-1}\} \to x_{j+1}) = \emptyset$.

Proof: (i) $x_i \in B$ means that $i \in [1, \alpha-1]$ and there is an integer $b \in [\alpha+2, r]$ such that $x_i x_b \in \mathcal{A}(T)$. Assume that for some $a \in [0, i-1]$, $x_a x_{i+1} \in \mathcal{A}(T)$. By (12), $z x_{a+1}, x_{b-1} z \in \mathcal{A}(T)$. Therefore, $Q = x_0 \ldots x_a x_{i+1} \ldots x_{b-1} z x_{a+1} \ldots x_i x_b \ldots x_r$, a contradiction. So, $\mathcal{A}(\{x_0, x_1, \ldots, x_{i-1}\} \to x_{i+1}) = \emptyset$.

(ii) Now assume that $x_j x_r \in \mathcal{A}(T)$ with $j \in [\alpha, r-2]$ and $x_i x_{j+1} \in \mathcal{A}(T)$ with $i \in [0, \alpha-1]$. Again using (12), we obtain $Q = x_0 x_1 \ldots x_i x_{j+1} \ldots x_{r-1} z x_{i+1} \ldots x_j x_r$, a contradiction. $\square$

Now, we divide the proof of Theorem 1.5 into two cases. Note that, by digraph duality and Lemma 3.3, the second case easily follows from the first case.

Case I. $\mathcal{A}(\{x_1, x_2, \ldots, x_{\alpha-1}\} \to x_r) = \emptyset$, i.e., $x_r \to \{x_1, x_2, \ldots, x_{\alpha-1}\}$.

For this case first we need to show the following claims below (Claims 5-13).

Claim 5: $B \neq \{x_1, x_2, \ldots, x_{\alpha-1}\}$.

Proof: Suppose, on the contrary, that $B = \{x_1, x_2, \ldots, x_{\alpha-1}\}$. Let $\alpha \geq 3$. Then using Proposition 1(i), we see that $d^+(x_1, \{x_0, x_3, \ldots, x_\alpha, x_r\}) = 0$. From $d^-(x_r, V(P)) \geq n-k$ and $x_r \to \{x_1, x_2, \ldots, x_{\alpha-1}\}$ it follows that $d^-(x_r, \{x_\alpha, x_{\alpha+1}, \ldots, x_{r-2}\}) \geq n-k-2$.

By running j from α to $r-2$ in Proposition 1(ii), we have

$$d^-(x_r, \{x_\alpha, x_{\alpha+1}, \ldots, x_{r-2}\}) + d^+(x_1, \{x_{\alpha+1}, x_{\alpha+2}, \ldots, x_{r-1}\}) \leq r-1-\alpha.$$

Therefore,

$$\begin{aligned} d^-(x_1, \{x_{\alpha+1}, x_{\alpha+2}, \ldots, x_{r-1}\}) &= r-1-\alpha-d^-(x_1, \{x_{\alpha+1}, x_{\alpha+2}, \ldots, x_{r-1}\}) \\ &\geq d^-(x_r, \{x_\alpha, x_{\alpha+1}, \ldots, x_{r-2}\}) \\ &\geq n-k-2. \end{aligned}$$

Hence, by this inequality and $\alpha \geq n-k-|A|$ (Fact 2) we have

$$n = d^-(x_1) \geq |A| + |\{x_0, x_3, \dots, x_\alpha, x_r\}| + d^-(x_1, \{x_{\alpha+1}, x_{\alpha+2}, \dots, x_{r-1}\})$$

$$\geq |A| + \alpha + n - k - 2 \geq 2n - 2k - 2.$$

Let now $\alpha = 2$. From $2 = \alpha \geq n-k-|A|$ it follows that $|A| \geq n-k-2$. Therefore,

$$n = d^-(x_1) \geq |A| + |\{x_0, x_r\}| + d^-(x_1, \{x_{\alpha+1}, x_{\alpha+2}, \dots, x_{r-1}\})$$

$$\geq |A| + 2 + n - k - 2 \geq 2n - 2k - 2.$$

Since $k = |S| \leq (n-2)/2$, in both cases we have equalities, that is, $2k = n-2$, $k \geq 2$, $\alpha = n-k-|A|$ and $d^-(x_r, \{x_\alpha, x_{\alpha+1}, \dots, x_{r-2}\}) = n-k-2$. From these we obtain that $|\{x_{\alpha+1}, x_{\alpha+2}, \dots, x_r\}| = r - \alpha = 2n-k-|A| - (n-k-|A|) = n$ and if $x_l x_r \notin \mathcal{A}(T)$ with $l \in [\alpha, r-2]$, then $x_1 x_{l+1} \in \mathcal{A}(T)$. Since $\mathcal{A}(x_\alpha \to \{x_{\alpha-1}\} \cup A) = \emptyset$, there exists $j \in [\alpha+2, r]$ such that $x_\alpha x_j \in \mathcal{A}(T)$. If $x_\alpha x_r \notin \mathcal{A}(T)$, then $j \leq r-1$, $x_1 x_{\alpha+1} \in \mathcal{A}(T)$ (by the above observation) and $Q = x_0 x_1 x_{\alpha+1} \dots x_{j-1} z x_2 \dots x_\alpha x_j \dots x_r$, a contradiction. We may therefore assume that $x_\alpha x_r \in \mathcal{A}(T)$. Since $d^-(x_r, \{x_\alpha, x_{\alpha+1}, \dots, x_{r-2}\}) = n-k-2$ and $|\{x_\alpha, x_{\alpha+1}, \dots, x_{r-4}\}| = n-3 \geq n-k-2$, it follows that there is an integer $i \in [\alpha, r-4]$ such that $x_i x_r \in \mathcal{A}(T)$ and $x_{i+1} x_r \notin \mathcal{A}(T)$. By our observation, $x_1 x_{i+2} \in \mathcal{A}(T)$. Therefore, if $x_{i+1} x_{r-1} \in \mathcal{A}(T)$, then $Q = x_0 x_1 x_{i+2} \dots x_{r-2} z x_2 \dots x_{i+1} x_{r-1} x_r$, and if $x_{r-1} x_{i+1} \in \mathcal{A}(T)$, then $Q = x_0 x_1 x_{i+2} \dots x_{r-1} x_{i+1} z x_2 \dots x_i x_r$. In both cases we have a contradiction. Claim 5 is proved. $\square$

From now on, by T^1 we denote the subtournament $T\langle\{x_1, x_2, \dots, x_{\alpha-1}\}\rangle$.

Claim 6: $B \neq \emptyset$.

Proof: By contradiction, suppose that $B = \emptyset$. This means that $\mathcal{A}(\{x_1, x_2, \dots, x_{\alpha-1}\} \to \{x_{\alpha+2}, x_{\alpha+3}, \dots, x_r\}) = \emptyset$, i.e., for any $i \in [1, \alpha-1]$, $d^-(x_i, \{x_{\alpha+2}, x_{\alpha+3}, \dots, x_r\}) = 2n-k-|A|-\alpha-1$.

Case 6.1. The subtournament T^1 is not regular. Then by Lemma 3.1(iii), there is a vertex $x \in V(T^1)$ such that $d^-(x, V(T^1)) \geq 0.5(\alpha-1)$. Therefore, since $r = 2n-k-|A|$ and $\alpha \leq n - \lfloor 0.5|A| \rfloor - 1$, we have

$$n = d^-(x) \geq |A| + |\{x_{\alpha+2}, x_{\alpha+3}, \dots, x_r\}| + 0.5(\alpha-1) \geq n + 0.5(n-2k-2+\lfloor 0.5|A| \rfloor).$$

Since $2k \leq n-2$ and $|A| \geq 1$, we have that all equalities in the last inequality must hold, i.e., $|A| = 1$, $2k = n-2$ (n is even) and $\alpha = n - 0.5\lfloor |A| \rfloor - 1 = n-1$. If $x_i x_r \in \mathcal{A}(T)$ with $i \in [\alpha, r-2]$, then, since $\alpha = n-1$ and $|A| = 1$, we have $d^+(x_{i+1}) \geq |A| + |\{x_0, x_1, \dots, x_{\alpha-1}, x_{i+2}\}| \geq n+1$, a contradiction. We may therefore assume that $x_r \to \{x_\alpha, x_{\alpha+1}, \dots, x_{r-2}\}$. This together with $x_r \to A \cup \{x_1, x_2, \dots, x_{\alpha-1}\}$ and $2k = n-2$ implies that $d^+(x_r) \geq 2n-k-2 \geq n+1$, a contradiction.

Case 6.2. The subtournament T^1 is regular. Then $d^-(x_1, V(T^1)) = 0.5(\alpha-2)$. Since $B = \emptyset$, $2k \leq n-2$ and $r = 2n-k-|A|$, we have that $n = d^-(x_1) \geq |A| +$

$|\{x_0, x_{\alpha+2}, x_{\alpha+3}, \dots, x_r\}| + 0.5(\alpha - 2) \geq n + 0.5(n - \alpha)$, a contradiction since $\alpha \leq n - 1$. Claim 6 is proved. $\square$

By Claim 4, $\alpha \geq 2$. If $\alpha = 2$, then $B = \emptyset$ or $B = \{x_1\} = \{x_1, \dots, x_{\alpha-1}\}$, violating Claim 5 or Claim 6. Therefore, $\alpha \geq 3$ and $A \to \{x_0, x_1, x_2, x_3\}$. By the digraph duality, $\{x_{r-3}, x_{r-2}, x_{r-1}, x_r\} \to A$.

Claim 7: $|B| \geq 2n - 2k - \alpha - 3$.

Proof: By Claim 5, $\{x_1, x_2, \dots, x_{\alpha-1}\} \setminus B \neq \emptyset$.

Assume first that the subtournament $T^2 := T\langle\{x_1, x_2, \dots, x_{\alpha-1}\} \setminus B\rangle$ is not regular. By Lemma 3.1(iii), $V(T^2)$ contains a vertex x such that $d^-(x, V(T^2)) \geq 0.5|V(T^2)|$. On the other hand, from the definition of B it follows that $A \cup \{x_{\alpha+2}, x_{\alpha+3}, \dots, x_r\} \to x$. Therefore,

$$n = d^-(x) \geq |A| + r - \alpha - 1 + 0.5(\alpha - 1 - |B|) = n + 0.5(2n - 2k - \alpha - 3 - |B|).$$

Hence, $|B| \geq 2n - 2k - \alpha - 3$.

Assume next that T^2 is regular. Then for all $x_i \in V(T^2)$, $d^-(x_i, V(T^2)) = 0.5(\alpha - |B| - 2)$. Let $x_q \in V(T^2)$ and q is the minimum with this property. Then

$$n = d^-(x_q) \geq |A \cup \{x_{q-1}, x_{\alpha+2}, x_{\alpha+3}, \dots, x_r\}| + 0.5(\alpha - |B| - 2)$$

$$= n + 0.5(2n - 2k - \alpha - 2 - |B|).$$

Hence, $|B| \geq 2n - 2k - \alpha - 2$. Claim 7 is proved. $\square$

Let $M := \{x_j \mid j \in [\alpha + 2, r - 1] \,\text{and}\, \mathcal{A}(\{x_0, x_1, \dots, x_{\alpha-1}\} \to x_j) \neq \emptyset\}$.

Proposition 2: If $x_j \in M$, then $x_{j-1}x_r \notin \mathcal{A}(T)$.

Proof: If $x_j \in M$ and $x_{j-1}x_r \in \mathcal{A}(T)$, by the definition of M there is a vertex x_i with $i \in [0, \alpha - 1]$ such that $x_ix_j \in \mathcal{A}(T)$. Therefore, $Q = x_0x_1 \dots x_ix_j \dots x_{r-1}zx_{i+1} \dots x_{j-1}x_r$, a contradiction. $\square$

Claim 8: $M \neq \{x_{\alpha+2}, x_{\alpha+3}, \dots, x_{r-1}\}$.

Proof: Suppose, on the contrary, that $M = \{x_{\alpha+2}, x_{\alpha+3}, \dots, x_{r-1}\}$. From $d^-(x_r, \{x_1, x_2, \dots, x_{\alpha-1}\}) = 0$ (by the condition of Case I) and Proposition 2 it follows that

$$n = d^+(x_r) \geq |A| + |\{x_1, x_2, \dots, x_{\alpha-1}, x_{\alpha+1}, x_{\alpha+2}, \dots, x_{r-2}\}| = 2n - k - 3.$$

Hence, $n - k \leq 3$. Since $2k \leq n - 2$, we obtain that $n \leq 4$, which contradicts that $n \geq 5$. Claim 8 is proved. $\square$

From Claim 8 it follows that $\{x_{\alpha+2}, x_{\alpha+3}, \dots, x_{r-1}\} \setminus M \neq \emptyset$.

Claim 9: $2|M| \geq 2n - 2k - \alpha - 3$.

Proof: Assume first that the subtournament T^1 is not regular. Then $V(T^1)$ contains a vertex x such that $d^-(x, V(T^1)) \geq 0.5(\alpha-1)$. This together with $A\cup(\{x_{\alpha+2}, x_{\alpha+3}, \dots, x_r\}\setminus M) \to x$ and $r = 2n - k - |A|$ implies that

$$n = d^-(x) \geq |A| + 0.5(\alpha - 1) + r - \alpha - 1 - |M| = n + 0.5(2n - 2k - 2|M| - \alpha - 3).$$

Hence, $2|M| \geq 2n - 2k - \alpha - 3$. Assume next that T^1 is regular. Then, $d^-(x_1, V(T^1)) = 0.5(\alpha - 2)$. Now by a similar argument as above, we obtain

$$n = d^-(x_1) \geq |A| + |\{x_0\}| + 0.5(\alpha - 2) + r - \alpha - 1 - |M| = n + 0.5(2n - 2k - 2|M| - \alpha - 2).$$

Hence, $2|M| \geq 2n - 2k - \alpha - 2$. Claim 9 is proved. □

From Claim 9, $2k \leq n - 2$ and $\alpha \leq n - \lfloor 0.5|A| \rfloor - 1$ it follows that $2|M| \geq 2n - 2k - \alpha - 3 \geq \lfloor 0.5|A| \rfloor$. Hence, if $|M| = 0$ (i.e., $M = \emptyset$), then $2k = n - 2$, $|A| = 1$ and $\alpha = n - \lfloor 0.5|A| \rfloor - 1 = n - 1$. In particular, $|M| = 0$ means that $x_{r-1} \to \{x_0, x_1, \dots, x_{\alpha-1}\}$. This together with $x_{r-1} \to \{x_r\} \cup A$ implies that $d^+(x_{r-1}) \geq n + 1$, which is a contradiction. Therefore, $M \neq \emptyset$.

Let $W := \{x_i \text{ with } i \in [\alpha + 1, r - 1] \,|\, \text{there exists } j \in [\alpha, i - 1] \text{ such that } x_j x_r \in \mathcal{A}(T) \text{ and } |\{x_{j+1}, \dots, x_i\}| \leq \lfloor 0.5|\{x_{j+1}, \dots, x_r\}| \rfloor\}$.

By Lemma 3.4, $\mathcal{A}(\{x_0, x_1, \dots, x_{\alpha-1}\} \to W) = \emptyset$, i.e., $W \to \{x_0, x_1, \dots, x_{\alpha-1}\}$.

Let $x_{\alpha+s}x_r \in \mathcal{A}(T)$, $s \geq 0$ and s is the minimum with this property.

Claim 10: $|W| \geq n - k - 2 + \lfloor 0.5(n - |A| - \alpha - s + 1) \rfloor$.

Proof: Under the condition of Case I, we have that $x_r \to \{x_1, x_2, \dots, x_{\alpha-1}\}$ and $d^-(x_r, \{x_{\alpha+s}, x_{\alpha+s+1}, \dots, x_{r-2}\}) \geq n - k - 2 \geq 2$. Without loss of generality, we may assume that $\{x_{\alpha+s+1}, x_{\alpha+s+2}, \dots, x_{\alpha+s+m}, x_{\alpha+s+m+1}, \dots, x_{\alpha+s+m+t}\} \subseteq W$, where $m \geq 0$, $x_{\alpha+s+m}x_r \in \mathcal{A}(T)$, $x_{\alpha+s+m+1}x_r \notin \mathcal{A}(T)$, $x_{\alpha+s+m+t+1} \notin W$ and $t = \lfloor (r - m - \alpha - s)/2 \rfloor$. Let $Y := \{x_{i+1} \,|\, i \in [\alpha + s + m + t + 1, r - 2],\ x_i x_r \in \mathcal{A}(T)\}$. It is not difficult to see that $Y \subseteq W$, $m \geq n - k - 3 - |Y|$ and $|W| \geq |Y| + m + t$. Since $r = 2n - k - |A|$, it is not hard to check that

$$|W| \geq \lfloor \frac{2|Y| + m + r - \alpha - s}{2} \rfloor \geq n - k - 2 + \lfloor \frac{n - |A| + |Y| - \alpha - s + 1}{2} \rfloor.$$

Therefore, Claim 10 is true, since $|Y| \geq 0$. □

We know that

$$\{x_r\} \cup A \cup W \to \{x_1, x_2, \dots, x_{\alpha-1}\}. \tag{13}$$

Claim 11: $x_\alpha x_r \notin \mathcal{A}(T)$, i.e., $s \geq 1$.

Proof: By contradiction, suppose that $x_\alpha x_r \in \mathcal{A}(T)$, i.e., $s = 0$.

Case 11.1. The subtournament T^1 is regular. Using Claim 10 (when $s = 0$) and $2k \leq n-2$, we obtain $d^-(x_1) \geq |\{x_0, x_r\} \cup A \cup W| + 0.5(\alpha-2) \geq 0.5(3n-2k-2+|A|) > n$, which is a contradiction.

Case 11.2. The subtournament T^1 is not regular. If $d^-(x_j, V(T^1)) \geq \alpha/2$ for some $x_j \in V(T^1)$, then using (13), Claim 10 and $2k \leq n-2$, we obtain

$$d^-(x_j) \geq |\{x_r\} \cup A \cup W| + \alpha/2 \geq |A| + 1 + \alpha/2 + n - k - 2 + 0.5(n - |A| - \alpha)$$

$$= 0.5(3n - 2k + |A| - 2) > n,$$

a contradiction. We may therefore assume that for all $x_i \in V(T^1)$, $d^-(x_i, V(T^1)) < \alpha/2$. Therefore by Lemma 3.1(v), T^1 is almost regular. Then for all $x_i \in V(T^1)$, $d^-(x_i, V(T^1)) \geq (\alpha - 3)/2$. Again using (13), we obtain

$$n = d^-(x_1) \geq |\{x_0, x_r\} \cup A \cup W| + 0.5(\alpha - 3) \geq n + 0.5(n - 2k - 3 + |A|).$$

Hence, $2k = n - 2$, $|A| = 1$ ($A = \{z\}$), $d^-(x_1, V(T^1)) = (\alpha - 3)/2$, $N^-(x_1) = \{x_0, x_r\} \cup A \cup W \cup N^-(x_1, V(T^1))$. Since $\alpha \geq 3$ and $x_\alpha \notin W$, we have that $x_1 x_\alpha \in \mathcal{A}(T)$. If $x_{\alpha-1} x_j \in \mathcal{A}(T)$ with $j \in [\alpha+2, r]$, then $Q = x_0 x_1 x_\alpha \ldots x_{j-1} z x_2 \ldots x_{\alpha-1} x_j \ldots x_r$, if $x_{\alpha-1} x_{\alpha+1} \in \mathcal{A}(T)$, then $Q = x_0 x_1 \ldots x_{\alpha-1} x_{\alpha+1} \ldots x_{r-1} z x_\alpha x_r$. Thus, in both cases we have a contradiction. Therefore, $\{x_{\alpha+1}, x_{\alpha+2}, \ldots, x_r\} \to x_{\alpha-1}$. Now, since $|A| = 1$ and $2k = n - 2$, we have

$$n = d^-(x_{\alpha-1}) \geq |\{z\}| + |\{x_{\alpha+1}, x_{\alpha+2}, \ldots, x_r\}| + d^-(x_{\alpha-1}, V(T^1)) \geq n + (n - \alpha - 1)/2.$$

Therefore, $\alpha = n-1$ since $\alpha \leq n-1$. Then, $x_{\alpha+1} \to \{z, x_{\alpha+2}, x_0, x_1, \ldots, x_{\alpha-1}\}$ by Proposition 1(ii). This means that $d^+(x_{\alpha+1}) \geq n+1$, a contradiction. Claim 11 is proved. □

Let $F := \{x_i \,|\, i \in [\alpha + 1, \alpha + s] \text{ and } d^-(x_i, \{x_1, x_2, \ldots, x_{\alpha-1}\}) \geq 1\}$.

Then, $\{x_{\alpha+1}, \ldots, x_{\alpha+s}\} \setminus F \to \{x_1, \ldots, x_{\alpha-1}\}$.

Claim 12: $|F| \geq \lfloor 0.5(s + 1) \rfloor$.

Proof: Recall that $x_{\alpha+s} x_r \in \mathcal{A}(T)$, $s \geq 1$ and $x_r \to \{x_1, \ldots, x_{\alpha-1}\}$. Suppose, on the contrary, that $|F| < \lfloor 0.5(s+1) \rfloor$. Then $|\{x_{\alpha+1}, \ldots, x_{\alpha+s}\} \setminus F| \geq \frac{1}{2}(s+1)$ (for this it suffices to consider when s is even or not). Therefore, for every vertex $x_i \in \{x_1, x_2, \ldots, x_{\alpha-1}\}$, $d^-(x_i, \{x_{\alpha+1}, \ldots, x_{\alpha+s}\}) \geq \frac{1}{2}(s + 1)$. Assume first that the subtournament T^1 is not regular. Then $V(T^1)$ contains a vertex x such that $d^-(x, V(T^1)) \geq 0.5(\alpha - 1)$. Now using Claim 10, $2k \leq n - 2$ and the fact that $\{x_r\} \cup A \cup W \to x$, we obtain

$$d^-(x) \geq |A \cup \{x_r\} \cup W| + 0.5(\alpha - 1) + 0.5(s + 1) \geq n + 0.5|A|,$$

a contradiction. Assume next that the subtournament T^1 is regular. Then $d^-(x_1, V(T^1)) = 0.5(\alpha - 2)$. Therefore, as the above, we obtain

$$d^-(x_1) \geq |A \cup \{x_0, x_r\} \cup W| + 0.5(\alpha - 2) + 0.5(s + 1) \geq n + 0.5(|A| + 1),$$

a contradiction. Claim 12 is proved. $\square$

Claim 13: $\mathcal{A}(A \to x_{\alpha+s+1}) \neq \emptyset$.

Proof: Suppose, on the contrary, that $\mathcal{A}(A \to x_{\alpha+s+1}) = \emptyset$. Then $\{x_{\alpha+s+1}, \dots, x_r\} \to A$. Since $W \subseteq \{x_{\alpha+s+1}, \dots, x_{r-1}\}$ it follows that $W \to A$. Recall that $x_{\alpha+s}x_r \in \mathcal{A}(T)$, $s \geq 1$ and $\alpha + s$ is the minimum with this property. By Proposition 1(i), we know that if $x_i \in B$ with $i \geq 2$, then $x_{i+1}x_1 \in \mathcal{A}(T)$ and $x_{i+1} \notin W$. By the definition of F, $\{x_{\alpha+1}, \dots, x_{\alpha+s}\} \setminus F \to x_1$. Now using Claims 7 and 10, we obtain

$$n = d^-(x_1) \geq |A| + |B| - 1 + |W| + |\{x_0, x_r\}| + |\{x_{\alpha+1}, \dots, x_{\alpha+s}\} \setminus F|$$

$$\geq |A| + 2n - 2k - \alpha - 4 + n - k - 2 + 0.5(n - |A| - \alpha - s) + 2 + s - |F|$$

$$= n + 0.5(|A| + s - 2|F| + 5n - 6k - 3\alpha - 8).$$

Therefore, $0 \geq |A| + s - 2|F| + 5n - 6k - 3\alpha - 8$.

Now we consider the set W. We know that $W \to \{x_0, x_1, \dots, x_{\alpha-1}\}$ (by Lemma 3.4). Note that if $x_i \in W$, then $i \geq \alpha + s + 1$ and $x_{i-1}z \in \mathcal{A}(T)$. Using this, it is not difficult to show that if $x_j \in F$ and $x_i \in W$, then $x_i x_{j-1} \in \mathcal{A}(T)$. Thus we have, if $x_j \in F$, then $W \to x_{j-1}$. If $T\langle W\rangle$ is not regular, then for some $x \in W$, $d^+(x, W) \geq 0.5|W|$, if $T\langle W\rangle$ is regular, then for $x_l \in W$ with the maximum index we have, $d^+(x_l, W \cup \{x_{l+1}\}) \geq 0.5(|W| + 1)$. In both cases, for some $x_j \in W$, $d^+(x_j, W \cup \{x_{l+1}\}) \geq 0.5|W|$. Therefore there is a vertex $x_j \in W$ such that

$$n = d^+(x_j) \geq |A| + |F| + 0.5|W| + |\{x_0, x_1, \dots, x_{\alpha-1}\}|$$

$$\geq |A| + |F| + \alpha + 0.5(n - k - 2 + 0.5(n - |A| - \alpha - s))$$

$$= (3|A| + 3\alpha + 4|F| + 3n - 2k - 4 - s)/4.$$

Therefore, $0 \geq 3|A| + 3\alpha + 4|F| - n - 2k - 4 - s$. Together with $0 \geq |A| + s - 2|F| + 5n - 6k - 3\alpha - 8$ and $n \geq 2k + 2$, we obtain $0 \geq 2|A| + |F| - 2$, which is a contradiction since $|A| \geq 1, |F| \geq 1$, and thus $2|A| + |F| - 2 \geq 1$. Claim 13 is proved. $\square$

Now we are ready to complete the discussion of Case I. Let $q := \max\{i \,|\, \mathcal{A}(A \to x_i) \neq \emptyset\}$. Let now $zx_q \in \mathcal{A}(A \to x_q)$. Then $z \to \{x_0, x_1, \dots, x_q\}$ and $q \leq n - 1$. Note that $3 \leq \alpha \leq q \leq r - 4$ and $\{x_{q+1}, x_{q+2}, \dots, x_r\} \to A$. In this case (Case I) we have $\mathcal{A}(\{x_1, x_2, \dots, x_\alpha\} \to x_r) = \emptyset$ as $s \geq 1$. From $\mathcal{A}(A \to x_{\alpha+s+1}) \neq \emptyset$ it follows that $q \geq \alpha + s + 1$. Observe that $|A| \geq 2$ since $\alpha + s > \alpha$, $\mathcal{A}(A \to x_{\alpha+s}) \neq \emptyset$ and $\mathcal{A}(x_{\alpha+s} \to A) \neq \emptyset$. From $q \geq \alpha + s + 1 \geq \alpha + 2$ and $x_{\alpha+s}x_r \in \mathcal{A}(T)$ it follows that there is an integer $p \in [\alpha + s, q - 1]$ such that $x_p x_r \in \mathcal{A}(T)$. Let p be the maximum with this property. Now using the first part of Lemma 3.3, we obtain $\mathcal{A}(\{x_0, x_1, \dots, x_{p-2}\} \to \{x_{q+2}, x_{q+3}, \dots, x_{r-1}\}) = \emptyset$, which in turn implies that (since $p - 2 \geq \alpha - 1$)

$$\mathcal{A}(\{x_0, x_1, \dots, x_{\alpha-1}\} \to \{x_{q+2}, x_{q+3}, \dots, x_{r-1}\}) = \emptyset. \qquad (14)$$

Let $m := \max\{i \in [1, r-2] \mid x_i x_r \in \mathcal{A}(T)\}$ and consider two cases depending on m.

Case I.1. $m \neq p$. Then $m \geq q$ (since $p < q$). Using the facts that $\{x_{q+1}, x_{q+2}, \ldots, x_r\} \to A$ and $z \to \{x_0, x_1, \ldots, x_q\}$, we obtain $x_{m+1} \to \{x_0, x_1, \ldots, x_{q-1}\}$ (for otherwise, $x_j x_{m+1} \in \mathcal{A}(T)$ with $j \in [0, q-1]$ and $Q = x_0 x_1 \ldots x_j x_{m+1} \ldots x_{r-1} z x_{j+1} \ldots x_m x_r$, a contradiction). Therefore,

$$n = d^+(x_{m+1}) \geq |A| + |\{x_0, x_1, \ldots, x_{q-1}, x_{m+2}\}| = |A| + q + 1.$$

Hence, $q \leq n - |A| - 1$, which in turn implies that $|\{x_{q+2}, x_{q+3}, \ldots, x_{r-1}\}| = r - q - 2 \geq n - k - 1$. By the definition of the set M and (14), we have that $M \subseteq \{x_{\alpha+2}, x_{\alpha+3}, \ldots, x_{q+1}\}$. Let $x_{j+1} \in M$. Then $\alpha + 1 \leq j \leq q$, and for all $i \in [q+2, r]$, $x_j x_i \notin \mathcal{A}(T)$ (for otherwise, if $x_j x_i \in \mathcal{A}(T)$ for some $i \in [q+2, r]$, then by the definition of M there is a vertex x_g with $g \in [0, \alpha - 1]$ such that $x_g x_{j+1} \in \mathcal{A}(T)$, and hence $Q = x_0 x_1 \ldots x_g x_{j+1} \ldots x_{i-1} z x_{g+1} \ldots x_j x_i \ldots x_r$, a contradiction). Therefore, $\{x_{q+2}, x_{q+3}, \ldots, x_{r-1}\}$ contains a vertex x such that $n = d^+(x) \geq |A| + |\{x_0, x_1, \ldots, x_{\alpha-1}\}| + |M| + 0.5(r - q - 3)$. Now using Claim 9 and the facts that $r - q - 2 \geq n - k - 1$ and $\alpha \geq n - k - |A|$, we obtain

$$n = d^+(x) \geq |A| + \alpha + 0.5(2n - 2k - \alpha - 3) + 0.5(n - k - 2)$$

$$\geq n + 0.5(|A| + 2n - 4k - 5) \geq n + 0.5(|A| - 1),$$

which is a contradiction since $|A| \geq 2$.

Case I.2. $m = p < q$. Recall that $s \geq 1$ and $d^-(x_r, \{x_{\alpha+s}, x_{\alpha+s+1}, \ldots, x_{m-2}\}) \geq n - k - 4 \geq 0$. By the first part of Lemma 3.3 (when in Lemma 3.3, $s = m$ and $t = r$) we have

$$\mathcal{A}(\{x_0, x_1, \ldots, x_{m-2}\} \to \{x_{q+2}, x_{q+3}, \ldots, x_{r-1}\}) = \emptyset. \quad (15)$$

It is easy to see that $|\{x_0, x_1, \ldots, x_{m-2}\}| \geq n - k + \alpha - 3$ since $d^-(x_r, \{x_1, x_2, \ldots, x_\alpha\}) = 0$. We have that $|\{x_{q+1}, \ldots, x_{r-1}\}| \geq 2n - k - |A| - q - 1 \geq n - k - |A|$ as $q \leq n - 1$. Since $\alpha + s + 1 \leq m \leq q - 1$ and $|A| \geq 2$, it follows that for some $y \in A \setminus \{z\}$, $x_{\alpha+s} y \in \mathcal{A}(T)$. If $q = n - 1$, then $yz \in \mathcal{A}(T)$. Therefore, $Q = x_0 x_1 \ldots x_{\alpha+s} y z x_{\alpha+s+2} \ldots x_r$, a contradiction. From now on, assume that $q \leq n - 2$. Then $|\{x_{q+2}, \ldots, x_{r-1}\}| \geq n - k - |A|$.

Assume first that the subtournament $T^3 := T\langle\{x_{q+2}, \ldots, x_{r-1}\}\rangle$ is regular. Then, $d^+(x_{r-1}, V(T^3)) \geq 0.5(n - k - |A| - 1)$. This together with $m \geq n - k + \alpha - 2$, (15) and $\alpha \geq n - k - |A|$ implies that

$$n = d^+(x_{r-1}) \geq |A| + |\{x_0, x_1, \ldots, x_{m-2}, x_r\}| + 0.5(n - k - |A| - 1)$$

$$= |A| + m + 0.5(n - k - |A| - 1) \geq |A| + n - k + \alpha - 2 + 0.5(n - k - |A| - 1)$$

$$\geq |A| + n - k - 2 + n - k - |A| + 0.5(n - k - |A| - 1)$$

$$= 2n - 2k - 2 + 0.5(n - k - |A| - 1).$$

If $|A| \leq k$ or $2k \leq n-3$, then the last inequality gives a contradiction. We may therefore assume that $2k = n-2$ and $|A| = k+1$. Then from $\alpha \geq 3$ and $\alpha \geq n-k-|A|$ it follows that $\alpha \geq n-k-|A|+2$. Now it is not difficult to check that $d^+(x_{r-1}) \geq n-k+\alpha-2+|A|+0.5(n-k-|A|-1) \geq n+2$, which is a contradiction.

Assume next that the subtournament T^3 is not regular. Then $V(T^3)$ contains a vertex x such that $d^+(x, V(T^3)) \geq 0.5(n-k-|A|)$. Hence by (15),

$$d^+(x) \geq n-k+\alpha-3+|A|+0.5(n-k-|A|). \tag{16}$$

If $2k \leq 2n-3$, then using the last inequality, $|A| \leq k+1$ and $\alpha \geq n-k-|A|$, we obtain that $d^+(x) \geq n+1$, which is a contradiction. We may therefore assume that $2k = n-2$. Observe that when $2k = n-2$, then from $\alpha \geq 3$ and $\alpha \geq n-k-|A|$ it follows that if $|A| = k$, then $\alpha \geq n-k-|A|+1$, and if $|A| = k+1$, then $\alpha \geq n-k-|A|+2$. If $|A| \leq k-1$, then using (16), $2k = n-2$ and $\alpha \geq n-k-|A|$, we obtain $d^+(x) > n$, a contradiction. We may therefore assume that $k \leq |A| \leq k+1$. Then by the above observation, $\alpha \geq n-k-|A|+1$, which together with (16) implies that $d^+(x) > n$, which is a contradiction. The discussion of Case I.2 is completed.

Case II. $\mathcal{A}(\{x_1, x_2, \dots, x_{\alpha-1}\} \to x_r) \neq \emptyset$. Let $x_p x_r \in \mathcal{A}(T)$ with $p \in [1, \alpha-1]$. If $\mathcal{A}(x_0 \to \{x_{\alpha+2}, x_{\alpha+3}, \dots, x_{r-1}\}) = \emptyset$, then by considering the converse tournament of T we reduce the case to Case I since for some $\beta \in [\alpha+1, r-3]$, $\{x_{\beta+1}, x_{\beta+2}, \dots, x_r\} \to A$. By the digraph duality, we may assume that $\mathcal{A}(x_0 \to \{x_{\alpha+2}, x_{\alpha+3}, \dots, x_{r-1}\}) \neq \emptyset$. Let $x_0 x_i \in \mathcal{A}(T)$ with $i \in [\alpha+2, r-1]$. From (12) and the first part of Lemma 3.3 (when in Lemma 3.3, $s = p$ and $t = r$) it follows that $p = 1$. Since $r \geq 7$ (Fact 1), we have that $t-s \geq 6$. Thus, $x_1 x_r \in \mathcal{A}(T)$ and $x_0 x_i \in \mathcal{A}(T)$ with $i \in [\alpha+2, r-1]$, which contradicts the second part of Lemma 3.3. This contradiction completes the proof of Theorem 1.5. $\square$

Remark 6: The following example of a tournament of order $2n+1 = 9$ shows that Theorem 1.5 for $n = 4$ is not true. Consider a tournament H such that $V(H) = \{x, y, u, v, z\} \cup A \cup B$, where $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, with the properties $x \to \{u, v, z\} \to y$, $B \to \{u, v\} \to A$, $y \to A \cup B \to x$, $A \to z \to B$ and the following $xy, uv, vz, zu, a_1a_2, b_1b_2, a_1b_1$, a_2b_2, b_2a_1, a_2b_1 arcs are in $\mathcal{A}(H)$.

It is easy to see that $H - z$ contains an (x, y)-path of length 3, but contains no (x, y)-path of length 4.

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References

[1] B. Alspach, "Cycles of each length in regular tournaments", *Canadian Mathematical Bulleten*, vol. 10, no. 2, pp. 283-286, 1967.

[2] B. Alspach, K.B. Reid and D.P. Roselle, "Bypasses in Asymmetric Digraphs," *Journal of Combinatorial Theory*, vol. 17, no.1, pp. 11-18, 1974.

[3] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London, 2000.

[4] J. Bang-Jensen and F. Havet, *Tournaments and semicomplete digraphs, in Classes of Directed Graphs (J. Bang-Jensen and G. Gutin, eds)* , Springer-Verlag, London, 2018.

[5] J. Bang-Jensen and G. Gutin, "Paths, Trees and Cycles in tournaments", *Congressus Numerantium*, vol. 115, pp. 131-170, 1996.

[6] J.-C. Bermond and C. Thomassen, "Cycles in Digraphs-A survey", *Journal of Graph Theory*, vol. 5, pp. 1-43, 1981.

[7] M. Darah, Y.-P. Liu, C.-Q. Zang, Cycles of all lengths in arc-3-cyclic semi-complete digraphs. *Discrete Mathematics,* 173(1-3) 23-33, 1997.

[8] S. Kh. Darbinyan, "Bypasses in tournaments", *Akad. Nauk Armyan. SSR Dokl.*, vol. 66, no. 3, pp. 138-141, 1978.

[9] S. Kh. Darbinyan, "Bypasses in regular tournaments", *Thesises of reports All-union conference of the problem theoretical cybernetics*, pp. 133-134, Novosibirsk, 1977.

[10] S. Kh. Darbinyan, "On panconnected tournaments", *Preprint of Computer Center of Akad. Nauk Armyan. SSR*, pp. 15, 1979.

[11] F. Havet, "Pancyclic arcs and connectivity in tournaments", *Journal of Graph Theory*, vol. 47, no. 2, pp. 87-110, 2004.

[12] O.S. Jacobsen, "Cycles and paths in tournaments", *PhD Thesis, University of Aarhus*, 1972.

[13] X. Lu, "A note on Min-Max pair in Tournaments" *Graphs and Combinatorics*, vol. 35, pp. 1139-1145, 2019.

[14] J. Moon, "Some mapping problems for theory tournaments", *Graph Theory and its Applications (B,Harris, editor)*, Academic Press, New York, pp. 133-148, 1970.

[15] T. Yao, Y. Guo and K. Zhang, "Pancyclic out-arcs of a vertex in tournaments", *Discrete Applied Mathematics*, vol. 99, 245-249, 2000.

[16] A. Yeo, "The number of pancyclic arcs in a k-strong tournament", *Journal of Graph Theory*, vol.50, pp. 212-219, 2005.

[17] C. Thomassen, "Hamiltonian-Connected Tournaments", *Journal of Combinatorial Theory Ser. B*, vol. 28, pp. 142-163, 1980.

[18] K.-M. Zhang, "Completely Path-Connected Tournaments", *Journal of Combinatorial Theory Ser. B*, vol. 33, pp. 166-177, 1982.