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Border Collision Bifurcation Control of Cardiac Alternans

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Abstract

The quenching of alternans is considered using a nonlinear cardiac conduction model. The model consists of a nonlinear discrete-time piecewise smooth system. Several authors have hypothesized that alternans arise in the model through a period doubling bifurcation. In this work, it is first shown that the alternans exhibited by the model actually arise through a period doubling *border collision* bifurcation. No smooth period doubling bifurcation occurs in the parameter region of interest. Next, recent results of the authors on feedback control of border collision bifurcation are applied to the model, resulting in control laws that quench the bifurcation and hence result in alternan suppression.

1 Introduction

In this paper, we give a bifurcation analysis of the cardiac conduction model proposed by Sun, Amellal, Glass and Billette [Sun *et al.*, 1995] and, based on the conclusions of the analysis, we investigate control design for suppression of predicted cardiac alternans. Sun et al. formulated their model as a two-dimensional piecewise smooth map. The model incorporates physiological concepts of recovery, facilitation and fatigue. It predicts a variety of experimentally observed complex rhythms of nodal conduction [Sun *et al.*, 1995]. In particular, alternans, in which there is an alternation in conduction time from beat to beat, were associated in [Sun *et al.*, 1995] to period doubling bifurcation in the model. In the present study, we first show that the instability mechanism giving rise to cardiac alternans in the model is in fact not a smooth period doubling bifurcation as has been earlier hypothesized, but rather a related bifurcation that occurs in nonsmooth systems, the period doubling border collision bifurcation. We then proceed to apply our recent work on control of border collision bifurcations [Hassouneh, 2003; Hassouneh & Abed, 2003] to the design of control laws for alternan quenching.

A border collision bifurcation (BCB) is a bifurcation that occurs when a fixed point (or a periodic orbit) of a piecewise smooth system crosses or collides with the border between two regions of smooth operation [Nusse & Yorke, 1992; di Bernardo *et al.*, 1999]. Border collision bifurcations include bifurcations that are reminiscent of the classical bifurcations in smooth systems such as the fold and period doubling bifurcations. Despite such resemblances, the classification of border collision bifurcations is far from complete, and certainly very preliminary in comparison to the results available in the smooth case. In smooth maps, a bifurcation occurs from a one-parameter family of fixed points when a real eigenvalue or a complex conjugate pair of eigenvalues crosses the unit circle. In piecewise smooth (PWS) maps, on the other hand, a border collision bifurcation can occur when a fixed point (or a periodic orbit) crosses or collides with the border between two regions of smooth behavior. This involves a discontinuous change in the eigenvalues of the Jacobian matrix evaluated at the fixed point (or at a periodic point) when the fixed point hits the border. As a result, border collision bifurcations for piecewise smooth systems in which the one-sided derivatives on the border are finite are classified based on the linearizations of the system on both sides of the border at criticality.

Several researchers have studied the model of [Sun *et al.*, 1995] and developed control techniques to eliminate the period-2 rhythm and stabilize the underlying period-1 rhythm (e.g., [Christini & Collins, 1996; Brandt *et al.*, 1997; Chen *et al.*, 1998]). With the exception of [Chen *et al.*, 1998], all the studies of this model reported in the literature viewed the border collision period doubling bifurcation in this system as if it were an ordinary period doubling bifurcation in a smooth dynamical system. Chen, Wang and Chin [1998] identified the bifurcation in the cardiac model as a border collision bifurcation based on numerical evidence. However, they didn't provide analysis to prove this claim. The authors of [Chen *et al.*, 1998] also investigated the feedback control of the BCB detected in the alternan model, but the feedback design was largely based on trial and error, and did not involve a detailed consideration of the border collision bifurcation. In [Brandt *et al.*, 1997], the authors propose the use of delayed linear feedback to suppress the period doubling bifurcation. In [Christini & Collins, 1996], the authors apply a technique for control of chaos to suppress the alternation resulting from the period doubling bifurcation. In [Hall *et al.*, 1997], a smooth one dimensional map was used as a model for cardiac conduction. A form of linear dynamic feedback where the unstable fixed point corresponding to the unstable rhythm is estimated as the average value of two consecutive beats was used to achieve alternan quenching [Hall *et al.*, 1997]. The control gain was determined by trial and error.

Another approach to the quenching of alternans by feedback is given in the paper of Christini et al. [2001], which takes an experimental approach, and determines a stabilizing controller in a fashion similar to that used in the OGY method [Ott, Grebogi & Yorke, 1990] for chaos control. The work of Christini et al. [2001] did not use a model, and certainly did not use any results on border collision bifurcations. So one might ask how this is possible, and if this implies the role of understanding border collision bifurcations in this context is diminished. We make several comments with regard to the first question. The authors of [Christini *et al.*, 2001] determine a control law for a given set of system parameters that does not include the actual point of bifurcation. Thus, the controller could conceivably fail closer to the bifurcation point. Also, since no model is available, the desired unstable fixed point is determined by an approximation based on the stable period-2 orbit that is experienced in the experiment. The extreme values of the period-2 orbit are averaged to obtain an approximation for the unstable fixed point. This discussion implies that the control law of [Christini *et al.*, 2001] takes action when an alternan is experienced, whereas a successful model-based controller would not allow the alternan to occur in the first place. The second question was on the need for understanding the border collision bifurcation. Clearly, a better understanding of and appreciation for the system dynamics will lead to more confidence in the control design. It is of course possible that designs obtained by other means, such as trial and error with some engineering and medical basis, will also do the job in some cases. However, there is no substitute for a design based on a correct understanding of the dynamics. Moreover, in piecewise smooth systems where a border collision bifurcation exists, the bifurcation is generic. That is, it is maintained under small perturbations of the model. Using a smooth system approximation does not result in any benefit, because the difficult calculations of what bifurcation actually occurs simply will need to be done in a less local region than if they were done at the fixed point on the border of the nonsmooth model.

In this paper, we use recent results of the authors on feedback control of border collision bifurcations [Hassouneh, 2003; Hassouneh & Abed, 2003] to quench the period doubling border collision bifurcation which consequently suppresses the alternans. The feedback can be either linear or piecewise linear. Both static and washout filter-aided feedbacks are considered. Washout filter-aided feedback has certain advantages over static feedback: it maintains the fixed points of the open-loop system even in the presence of model uncertainty, and it provides automatic following of the fixed point to be stabilized which alleviates the need for providing an estimate of the unstable fixed point to the controller. This is particularly useful in situations where the system model is uncertain and/or cases where there is parameter drift.

It is important to realize that, since border collision bifurcations arise at the border separating regions of smooth operation, a linear feedback that seems to “delay” a border collision bifurcation to occur away from the border actually does no such thing. If a BCB seems to have been delayed by feedback, what actually is happening is that the feedback has changed the BCB to a type that replaces the nominal fixed point by a new one (fixed point to fixed point BCB), and a new smooth bifurcation has been created elsewhere (away from the border). Thus, concepts and methods developed in the control of smooth bifurcations cannot be carried over in a direct way to the nonsmooth case.

The paper proceeds as follows. In Sec. 2, needed results on border collision bifurcations in PWS maps are recalled. In Sec. 3, results on control of BCBs in two-dimensional and n-dimensional PWS maps are given. The results of Sec. 2 and Sec. 3 are applied to the cardiac conduction model in Sec. 4.

2 Results on Border Collision Bifurcations

In this section, needed results on border collision bifurcation (BCB) in two-dimensional and n-dimensional PWS maps are collected from recent work of the authors [Hassouneh, 2003; Hassouneh & Abed, 2003]. Further background on BCBs in PWS maps can be found in [Nusse & Yorke, 1992; Benerjee & Grebogi, 1999; di Bernardo *et al.*, 1999].

2.1 Border collision bifurcation in 2-D PWS maps

Consider a general two-dimensional PWS map of the form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = f(x_k, y_k, \mu)$$

where

$$f(x, y, \mu) = \begin{cases} f_A(x, y, \mu), & x \leq 0 \\ f_B(x, y, \mu), & x > 0 \end{cases} \tag{1}$$

Here μ is the bifurcation parameter and $R_A := \{(x, y) : x \leq 0\}$, $R_B := \{(x, y) : x > 0\}$ are two regions of smooth behavior separated by the border $x = 0$. The map $f(\cdot, \cdot, \cdot)$ is assumed to be PWS: $f_A(x, y, \mu)$ is smooth on R_A, $f_B(x, y, \mu)$ is smooth on R_B and f is continuous in (x, y) but not differentiable at the border and depends smoothly on μ everywhere. Let $(x_0(\mu), y_0(\mu))$ be a path of fixed points of f; this path depends continuously on μ. Suppose also that the fixed point hits the border at a critical parameter value μ_b. Assume without loss of generality that $\mu_b = 0$. Thus, $(x_0(0), y_0(0)) = (x_b, y_b)$. Suppose that the coordinate system is chosen such that $(x_b, y_b) = (0, 0)$.

Expanding (1) in a Taylor series near the fixed point $(0, 0, 0)$ and ignoring the higher order terms gives

$$f(x, y, \mu) = \begin{cases} A \begin{pmatrix} x \\ y \end{pmatrix} + b\mu, & (x, y) \in R_A \\ B \begin{pmatrix} x \\ y \end{pmatrix} + b\mu, & (x, y) \in R_B \end{cases} \tag{2}$$

where $A := \begin{pmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{pmatrix}$ is the limiting Jacobian of f at (x, y, μ) close to $(0, 0, 0)$ with $(x, y) \in R_A$, $B := \begin{pmatrix} a_{21} & a_{22} \\ a_{23} & a_{24} \end{pmatrix}$ is the limiting Jacobian of f at (x, y, μ) close to $(0, 0, 0)$ with $(x, y) \in R_B$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is the derivative of f with respect to μ. We assume that the elements of A and B are finite. Since the map f is not differentiable at the border $x = 0$, $A \neq B$. The continuity of f at the border implies that the second column of A equals the second column of B, i.e., $a_{12} = a_{22} =: a_2$ and $a_{14} = a_{24} =: a_4$. Let $\tau_A :=\text{trace}(A) = a_{11} + a_4$, $\delta_A :=\det(A) = a_{11}a_4 - a_2a_{13}$, $\tau_B :=\text{trace}(B) = a_{21} + a_4$ and $\delta_B :=\det(B) = a_{21}a_4 - a_2a_{23}$.

Border collision bifurcations are classified based on the eigenvalues or, equivalently, by the trace and the determinant of the Jacobian matrices on both sides of the border [Nusse & Yorke, 1992; Benerjee & Grebogi, 1999]. Next, we give two propositions on border collision bifurcations in two-dimensional PWS maps.

Denote the eigenvalues of A by λ_A^+, λ_A^- and the eigenvalues of B by λ_B^+, λ_B^-. The following proposition gives sufficient conditions for nonbifurcation with persistent stability in two-dimensional PWS maps.

Proposition 1 [Hassouneh, 2003] (Sufficient Conditions for Nonbifurcation with Persistent Stability in 2-D PWS Maps) *If the eigenvalues of the Jacobian matrices A and B on either side of the border of the two-dimensional PWS map (2) are real and satisfy any of the following conditions:*

(i) $$0 < \lambda_A^- < \lambda_A^+ < 1 \text{ and } 0 < \lambda_B^- < \lambda_B^+ < 1$$

(ii) $$-1 < \lambda_A^- < 0 < \lambda_A^+ < 1 \text{ and } -1 < \lambda_B^- < 0 < \lambda_B^+ < 1$$

(iii) $$0 < \lambda_A^- < \lambda_A^+ < 1\text{and } -1 < \lambda_B^- < 0 < \lambda_B^+ < 1 \text{ with } \lambda_B^+ + \lambda_B^- > 0$$
or $$0 < \lambda_B^- < \lambda_B^+ < 1\text{and } -1 < \lambda_A^- < 0 < \lambda_A^+ < 1 \text{ with } \lambda_A^+ + \lambda_A^- > 0$$

then a locally unique and stable fixed point on one side of the border leads to a locally unique and stable fixed point on the other side of the border as μ is varied through zero.

For PWS maps of dimension two or higher, having the eigenvalues of the Jacobian matrices on both sides of the border within the unit circle does not imply that the fixed points are the only attractors as μ is increased (decreased) through its critical value. For example, if the eigenvalues of one of the Jacobian matrices or both Jacobian matrices are within the unit circle but nonreal, then higher period periodic attractors may exist on one side or both sides of the border in addition to the stable fixed points [Benerjee & Grebogi, 1999; Hassouneh, 2003].

The following proposition gives a sufficient condition for the occurrence of a supercritical period doubling BCB in two-dimensional PWS maps.

Proposition 2 *Suppose that the fixed point of (2) to the left of the border is stable for $\mu < 0$ (i.e., $|\delta_A| < 1$ and $-(1+\delta_A) < \tau_A < (1+\delta_A)$) and that it crosses the border and becomes unstable as μ is increased through zero. If*

$$\begin{aligned} |\delta_A \delta_B| <& \; 1, \\ -(1-\delta_B)(1-\delta_A) < \tau_B \tau_A <& \; (1+\delta_B)(1+\delta_A). \end{aligned}$$

then a supercritical period doubling border collision bifurcation occurs as μ is increased through zero. That is, a stable fixed point to the left of the border for $\mu < 0$ crosses the border and becomes unstable and a period two attractor is born as μ is increased through zero.

For a proof and further details, see [Hassouneh, 2003]. Regarding the stability condition in Prop. 2, it can be understood by noting that the stability of the bifurcated period-2 orbit is determined by the eigenvalues of the Jacobian matrix of the second iterate map with one point in R_A and the other point in R_B, i.e., by the matrix product AB. The stability condition follows by a straightforward application of the Jury Test for second order systems to AB.

2.2 Border collision bifurcations in n-dimensional PWS maps

Next, a sufficient condition for nonbifurcation with persistent stability in n-dimensional PWS maps is given. This result, which is based on using a quadratic Lyapunov function and linear matrix inequalities, is derived in [Hassouneh, 2003; Hassouneh & Abed, 2003].

Consider the one-parameter family of piecewise smooth maps

$$f(x,\mu) \;=\; \begin{cases} f_A(x,\mu), & x_1 \le 0 \\ f_B(x,\mu), & x_1 > 0 \end{cases} \tag{3}$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is piecewise smooth in x (f is smooth everywhere except on the border $\{x \in \mathbb{R}^n : x_1 = 0\}$ where it is only continuous), f is smooth in μ and $R_A := \{x \in \mathbb{R}^n : x_1 \le 0\}$, $R_B := \{x \in \mathbb{R}^n : x_1 > 0\}$ are two regions of smooth behavior. The notation x_1 denotes the first component of the vector x. Suppose that at $\mu = \mu_b$, a fixed point of f is at the border separating R_A and R_B. Assume without loss of generality that $\mu_b = 0$ and $x_0(0) = 0$. Border collision bifurcations occurring in (3) can be studied using the piecewise-linearized representation [di Bernardo *et al.*, 1999]

$$x(k+1) \;=\; f^1(x(k),\mu) \tag{4}$$

where

$$f^1(x(k),\mu) = \begin{cases} Ax(k) + b\mu, & \text{if } \; x_1(k) \le 0 \\ Bx(k) + b\mu, & \text{if } \; x_1(k) > 0 \end{cases}$$

Here, A is the linearization of the PWS map f in R_A at a fixed point on the border approached from points in R_A near the border and B is the linearization of f at a fixed point on the border approached from points in R_B and b is the derivative of the map f with respect to μ.

Proposition 3 [Hassouneh, 2003] (Sufficient Condition for Persistent Stability in n-Dimensional PWS Maps) *Consider the system (4). If there is a* $P = P^T > 0$ *such that*

$$A^T PA - P \;<\; 0, \tag{5}$$

$$B^T PB - P \;<\; 0, \tag{6}$$

then system (4) has a globally stable fixed point for all $\mu \in \mathbb{R}$.

3 Feedback Control of Border Collision Bifurcation

The normal form for BCBs contains only linear terms in the state [Nusse & Yorke, 1992; di Bernardo *et al.*, 1999]. This leads us to seek linear or piecewise linear feedback controllers to modify the system's bifurcation characteristics. The feedback can either be applied on only one side of the border or on both sides. However, using switching feedback with switching depending on the location of the state requires accurate knowledge of where the border lies, which is not necessarily available in practice. To achieve robustness to uncertainties in the border itself, stabilization is pursued using the same stabilizing feedback acting on both sides of the border. We call this the "simultaneous stabilization" problem for BCBs. Below, the design of simultaneous static feedback is considered followed by design of simultaneous washout filter-aided feedback.

3.1 Simultaneous static feedback

In this method, the same linear state feedback control is applied additively on both the left and right sides of the border. This leads to the closed-loop system

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{cases} A\begin{pmatrix} x_k \\ y_k \end{pmatrix} + b\mu + bu_k, & x_k \leq 0 \\ B\begin{pmatrix} x_k \\ y_k \end{pmatrix} + b\mu + bu_k, & x_k > 0 \end{cases} \tag{7}$$

$$u_k = (\gamma_1 \ \gamma_2)\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \gamma_1 x_k + \gamma_2 y_k \tag{8}$$

where γ_1, γ_2 are the control gains to be chosen. The following proposition gives sufficient conditions for the existence of a stabilizing control policy as in (7)-(8) above. The conditions are given in terms of linear inequalities. The existence of a solution can be easily checked numerically.

Proposition 4 [Hassouneh, 2003] *Suppose that the fixed point in R_A for $\mu < 0$ is stable—that is, assume $|\delta_A| < 1$ and $-(1+\delta_A) < \tau_A < (1+\delta_A)$. Suppose also that a border collision bifurcation occurs as μ is increased through zero. A simultaneous control that renders the BCB to be from a locally unique stable fixed point to a locally unique stable fixed point exists if there is a (γ_1, γ_2) and $0 < \epsilon < 1$ such that the following inequalities are satisfied:*

$$(b_1 a_4 - a_2 b_2)\gamma_1 < -(a_{11} b_2 - a_{13} b_1)\gamma_2 - \delta_A + \epsilon \tag{9}$$

$$(b_1 a_4 - a_2 b_2)\gamma_1 > -(a_{11} b_2 - a_{13} b_1)\gamma_2 - \delta_A \tag{10}$$

$$b_1 \gamma_1 > -b_2 \gamma_2 + 2\sqrt{\epsilon} - \tau_A \tag{11}$$

$$(b_1 a_4 - a_2 b_2 - b_1)\gamma_1 > -(-b_2 + a_{11} b_2 - a_{13} b_1)\gamma_2 - (1 - \tau_A + \delta_A) \tag{12}$$

and

$$(b_1 a_4 - a_2 b_2)\gamma_1 < -(a_{21} b_2 - a_{23} b_1)\gamma_2 - \delta_B \tag{13}$$

$$
\begin{aligned}
(b_1a_4 - a_2b_2)\gamma_1 &> -(a_{21}b_2 - a_{23}b_1)\gamma_2 - \delta_B - 1 && (14)\\
b_1\gamma_1 &> -b_2\gamma_2 - \tau_B && (15)\\
(b_1a_4 - a_2b_2 - b_1)\gamma_1 &> -(-b_2 + a_{21}b_2 - a_{23}b_1)\gamma_2 - (1 - \tau_B + \delta_B) && (16)
\end{aligned}
$$

Any (γ_1, γ_2) satisfying these inequalities is stabilizing.

The assertion of Prop. 4 follows by choosing the control gains so that the eigenvalues of the closed-loop system satisfy Prop. 1 (iii).

Let $g := (\gamma_1\ \gamma_2)$. The following proposition gives a sufficient condition for stabilization of border collision bifurcation in terms of linear matrix inequalities (LMIs) [Hassouneh, 2003; Hassouneh & Abed, 2003]. If the LMIs are feasible, a stabilizing control gain can be calculated using any LMI solver such as the LMI toolbox in MATLAB.

Proposition 5 *If there exist a $P = P^T > 0$ and a feedback gain (row) vector g such that*

$$
\begin{aligned}
P - (A + bg)^T P(A + bg) &> 0 && (17)\\
P - (B + bg)^T P(B + bg) &> 0, && (18)
\end{aligned}
$$

then any border collision bifurcation that occurs in the open-loop system ($u \equiv 0$) of (7) can be eliminated using simultaneous feedback (8). Equivalently, if there exist a Q and y such that

$$
\begin{pmatrix} Q & AQ + by \\ (AQ + by)^T & Q \end{pmatrix} > 0, \qquad (19)
$$

$$
\begin{pmatrix} Q & BQ + by \\ (BQ + by)^T & Q \end{pmatrix} > 0, \qquad (20)
$$

then any border collision bifurcation that occurs in (7) can be eliminated using simultaneous feedback (8). Here $Q = P^{-1}$ and the feedback gain is given by $g = yP$.

3.2 Simultaneous washout filter-aided feedback

System (7) augmented with a washout filter-aided simultaneous feedback control is given by the closed-loop system

$$
\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{cases} A \begin{pmatrix} x_k \\ y_k \end{pmatrix} + b\mu + bu_k, & x_k \leq 0 \\ B \begin{pmatrix} x_k \\ y_k \end{pmatrix} + b\mu + bu_k, & x_k > 0 \end{cases} \qquad (21)
$$

$$
\begin{aligned}
w_{k+1} &= x_k + (1 - d)w_k && (22)\\
z_k &= x_k - dw_k && (23)\\
u_k &= \gamma_1 z_k && (24)
\end{aligned}
$$

Here, w_k is the state of the washout filter, z_k is the output of the washout filter and d is the washout filter constant ($d \in (0, 2)$ for a stable washout filter). Note that only one washout filter was used in this feedback—this will be found in the next section to be sufficient for application to the alternan model. The closed-loop system can be rewritten as

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ w_{k+1} \end{pmatrix} = \begin{cases} \tilde{A} \begin{pmatrix} x_k \\ y_k \\ w_k \end{pmatrix} + \tilde{b}\mu, & x_k \leq 0 \\ \tilde{B} \begin{pmatrix} x_k \\ y_k \\ w_k \end{pmatrix} + \tilde{b}\mu, & x_k > 0 \end{cases} \tag{25}$$

where

$$\tilde{A} = \begin{pmatrix} a_{11} + b_1\gamma_1 & a_2 & -b_1 d\gamma_1 \\ a_{13} + b_2\gamma_1 & a_4 & -b_2 d\gamma_1 \\ 1 & 0 & 1 - d \end{pmatrix}, \tag{26}$$

$$\tilde{B} = \begin{pmatrix} a_{21} + b_1\gamma_1 & a_2 & -b_1 d\gamma_1 \\ a_{23} + b_2\gamma_1 & a_4 & -b_2 d\gamma_1 \\ 1 & 0 & 1 - d \end{pmatrix}, \tag{27}$$

and

$$\tilde{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}. \tag{28}$$

The following proposition gives a sufficient condition for the existence of a stabilizing washout filter-aided feedback.

Proposition 6 *If there exist $P = P^T > 0$, $\gamma_1 \in \mathbb{R}$ and $d \in (0, 2)$ such that*

$$\tilde{A}^T P \tilde{A} - P < 0 \tag{29}$$

$$\tilde{B}^T P \tilde{B} - P < 0 \tag{30}$$

then any border collision bifurcation that occurs in the open-loop system (21) (with $u \equiv 0$) can be eliminated by simultaneous washout filter-aided feedback.

Note that the matrix inequalities (29)-(30) used in Prop. 6 are equivalent, respectively, to

$$\begin{pmatrix} P & \tilde{A}^T P \\ P\tilde{A} & P \end{pmatrix} > 0, \tag{31}$$

$$\begin{pmatrix} P & \tilde{B}^T P \\ P\tilde{B} & P \end{pmatrix} > 0. \tag{32}$$

Next, the results on border collision bifurcation and its control are applied to the cardiac conduction model that undergoes a period doubling border collision bifurcation.

4 The Cardiac Conduction Model

In this section, we consider the cardiac conduction model of [Sun *et al.*, 1995]. The model incorporates physiological concepts of recovery, facilitation and fatigue. It is formulated as a two-dimensional PWS map. Two factors determine the atrioventricular (AV) nodal conduction time: the time interval from the atrial activation to the activation of the Bundle of His and the history of activation of the node. The model predicts a variety of experimentally observed complex rhythms of nodal conduction. In particular, alternans, in which there is an alternation in conduction time from beat to beat, were associated with period-doubling bifurcation in the theoretical model.

The authors of [Sun *et al.*, 1995] first define the atrial His interval, A, to be the time interval between cardiac impulse excitation of the lower interatrial septum and activation of the Bundle of His. (See [Sun *et al.*, 1995] for definitions.) The model is

$$\begin{pmatrix} A_{k+1} \\ R_{k+1} \end{pmatrix} = f(A_k, R_k, H_k) \tag{33}$$

where

$$f(A_k, R_k, H_k) =$$
$$\begin{cases} \begin{pmatrix} A_{min} + R_{k+1} + (201 - 0.7A_k)e^{-\frac{H_k}{\tau_{rec}}} \\ R_k e^{-\frac{A_k + H_k}{\tau_{fat}}} + \gamma e^{-\frac{H_k}{\tau_{fat}}} \end{pmatrix}, & \text{for } A_k \leq 130 \\ \begin{pmatrix} A_{min} + R_{k+1} + (500 - 3.0A_k)e^{-\frac{H_k}{\tau_{rec}}} \\ R_k e^{-\frac{A_k + H_k}{\tau_{fat}}} + \gamma e^{-\frac{H_k}{\tau_{fat}}} \end{pmatrix}, & \text{for } A_k > 130 \end{cases}$$

with $R_0 = \gamma exp(-H_0/\tau_{fat})$. Here H_0 is the initial H interval and the parameters A_{min}, τ_{fat}, γ and τ_{rec} are positive constants. The variable H_k represents the time interval between bundle of His activation and the subsequent activation (the AV nodal recovery time) and is usually taken as the bifurcation parameter. The variable R_k represents a drift in the nodal conduction time, and is sometimes taken to be constant. In this work, we consider R_k as a variable as in [Sun *et al.*, 1995]. Note that the map f is piecewise smooth and is continuous at the border $A_b := 130$ms.

4.1 Analysis of the border collision bifurcation

Numerical simulations indicate that the map (33) undergoes (some type of) supercritical period doubling bifurcation as the bifurcation parameter $S := H_k$ is decreased through a critical value (see Fig. 1). We show that this bifurcation is in fact a supercritical period doubling BCB which occurs when the fixed point of the map hits the border $A_b = 130$.

Let the fixed points of the map (33) be given by $(A_-^*(S), R_-^*(S))$ for $A_k < A_b$ and $(A_+^*(S), R_+^*(S))$ for $A_k > A_b$. Under normal conditions, the fixed point $(A_-^*(S), R_-^*(S))$ is stable and it loses stability as S is decreased through a critical value $S = S_b$ where $A_-^* = A_b$. Then, at criticality, the value of R_-^* is denoted by R_b.

Next, we calculate the limiting Jacobians on both sides of the border:

$$A = \begin{pmatrix} -0.7e^{\frac{-S_b}{\tau_{rec}}} - \frac{R_b}{\tau_{fat}} e^{\frac{-(130+S_b)}{\tau_{fat}}} & e^{\frac{-(130+S_b)}{\tau_{fat}}} \\ -\frac{R_b}{\tau_{fat}} e^{\frac{-(130+S_b)}{\tau_{fat}}} & e^{\frac{-(130+S_b)}{\tau_{fat}}} \end{pmatrix} \tag{34}$$

and

$$B = \begin{pmatrix} -3.0e^{\frac{-S_b}{\tau_{rec}}} - \frac{R_b}{\tau_{fat}} e^{\frac{-(130+S_b)}{\tau_{fat}}} & e^{\frac{-(130+S_b)}{\tau_{fat}}} \\ -\frac{R_b}{\tau_{fat}} e^{\frac{-(130+S_b)}{\tau_{fat}}} & e^{\frac{-(130+S_b)}{\tau_{fat}}} \end{pmatrix} \tag{35}$$

Also, the derivative of f with respect to S at (A_b, R_b, S_b) is

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -\frac{110}{\tau_{rec}} e^{\frac{-S_b}{\tau_{rec}}} - \frac{\gamma}{\tau_{fat}} e^{\frac{-S_b}{\tau_{fat}}} - \frac{R_b}{\tau_{fat}} e^{\frac{-(130+S_b)}{\tau_{fat}}} \\ -\frac{\gamma}{\tau_{fat}} e^{\frac{-S_b}{\tau_{fat}}} - \frac{R_b}{\tau_{fat}} e^{\frac{-(130+S_b)}{\tau_{fat}}} \end{pmatrix} \tag{36}$$

Next, the following parameter values are assumed (borrowed from [Sun *et al.*, 1995]): $\tau_{rec} = 70$ms, $\tau_{fat} = 30000$ms, $A_{min} = 33$ms, $\gamma = 0.3$ms. For these parameter values, $S_b = 56.9078$ms, $R_b = 48.2108$ms,

$$A = \begin{pmatrix} -0.3121 & 0.9938 \\ -0.0016 & 0.9938 \end{pmatrix}, \quad B = \begin{pmatrix} -1.3322 & 0.9938 \\ -0.0016 & 0.9938 \end{pmatrix}$$

$$\text{and} \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -0.69861 \\ -0.00161 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_{A1} = -0.3109$, $\lambda_{A2} = 0.9926$ ($\tau_A = 0.6817$, $\delta_A = -0.3086$) and those of B are $\lambda_{B1} = -1.3315$, $\lambda_{B2} = 0.9931$ ($\tau_B = -0.3384$ and $\delta_B = -1.3224$). Note that there is a discontinuous jump in the eigenvalues of the Jacobian matrix when the fixed point hits the border at the critical parameter values $S = S_b$. The occurrence of a period-doubling border collision bifurcation at S_b is now ascertained by applying Prop 2. The stability of the period-2 orbit with one point in R_A and the other in R_B is determined by the eigenvalues of AB. These eigenvalues are $\lambda_{AB_1} = 0.4135$ and $\lambda_{AB_2} = 0.9867$. This implies that a stable period-2 orbit is born after the border collision. The supercritical period doubling BCB is shown in the bifurcation diagram in Fig. 1. In the figure, the bifurcated solution departs in a nonsmooth way from the nominal fixed point branch.

4.2 Feedback control of the period doubling border collision bifurcation

In past studies of control of the cardiac conduction model considered here, the control is usually applied as a perturbation to the bifurcation parameter (the nodal recovery time) S [Christini & Collins, 1996; Chen *et al.*, 1998]. The state A_k has been used in the feedback loop by other researchers who developed control laws for this model (e.g., [Hall *et al.*, 1997;

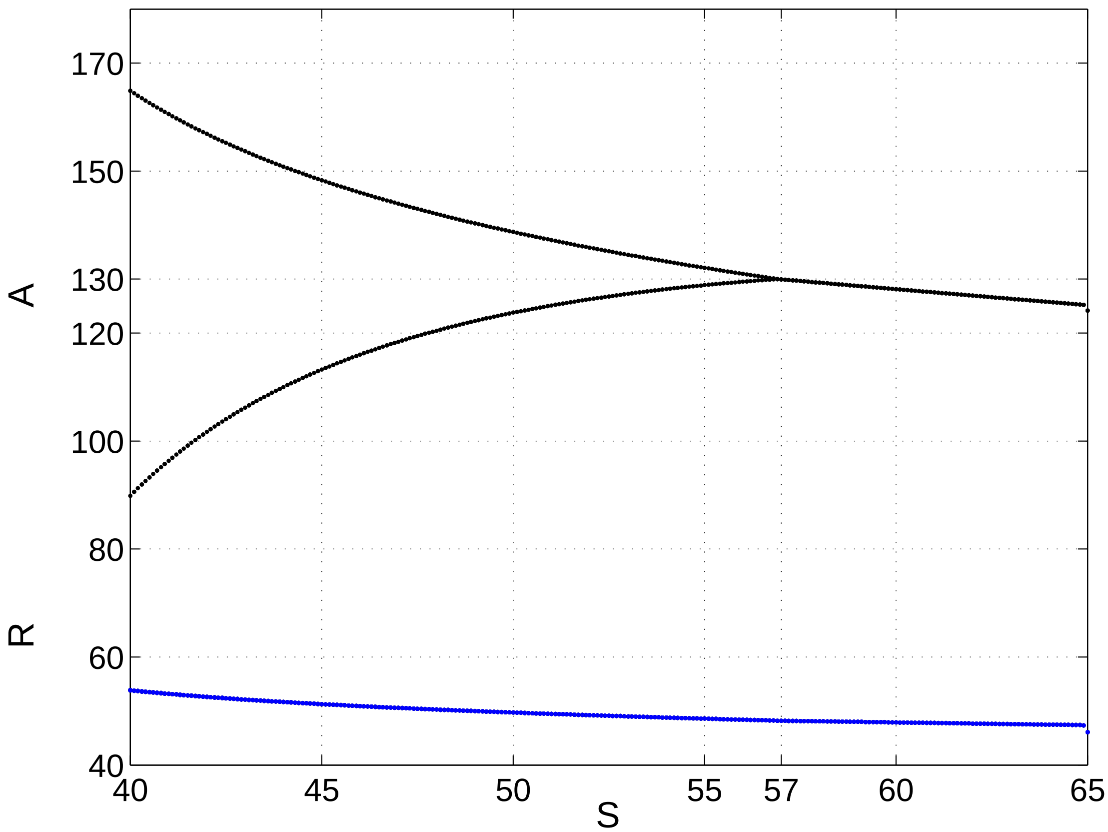


Figure 1: Joint bifurcation diagram for A_k and for R_k for (33) with $S := H_k$ as bifurcation parameter and $\tau_{rec} = 70$ms, $\tau_{fat} = 30000$ms, $A_{min} = 33$ms and $\gamma = 0.3$ms.

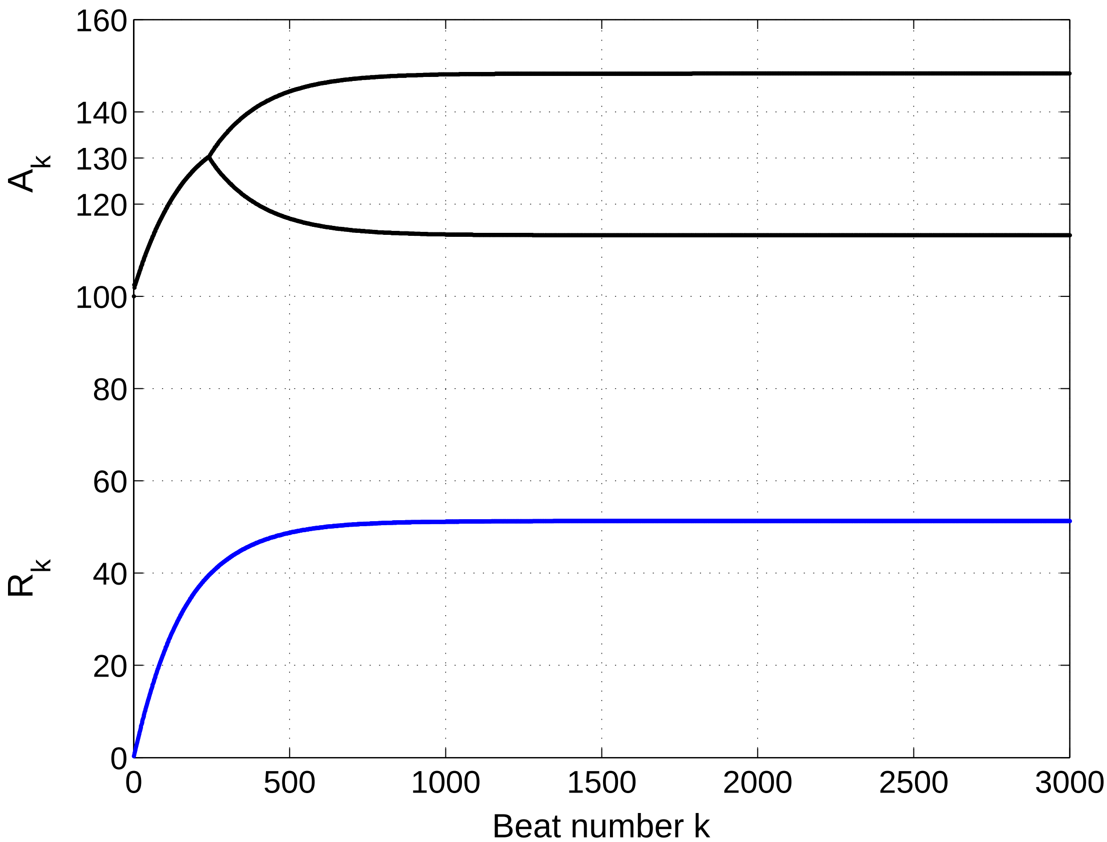


Figure 2: Iterations of map (33) showing the alternation in A_k as a result of a supercritical period doubling BCB (τ_{rec} = 70ms, τ_{fat} = 30000ms, A_{min} = 33ms, γ = 0.3ms and S = 45ms).

Brandt *et al.*, 1997; Chen *et al.*, 1998]). We use the same measured signal in our feedback design. Below, the control methods of Sec. 3 are used to quench the period doubling border collision bifurcation, replacing the period doubled orbit by a stable fixed point. First, simultaneous static feedback control is considered followed by simultaneous washout filter-aided feedback control.

4.2.1 Simultaneous static feedback control

Next, a feedback control $u_k = \gamma_1(A_k - A_b) + \gamma_2(R_k - R_b)$ is applied as a perturbation to S on both sides of the border.

The limiting Jacobians of the controlled system to the left and right of the border are given by

$$\tilde{A} = \underbrace{\begin{pmatrix} -0.3121 & 0.9938 \\ -0.0016 & 0.9938 \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} -0.6986 \\ -0.00161 \end{pmatrix}}_{b} \begin{pmatrix} \gamma_1 & \gamma_2 \end{pmatrix}$$

and

$$\tilde{B} = \underbrace{\begin{pmatrix} -1.3322 & 0.9938 \\ -0.0016 & 0.9938 \end{pmatrix}}_{B} + \underbrace{\begin{pmatrix} -0.6986 \\ -0.00161 \end{pmatrix}}_{b} \begin{pmatrix} \gamma_1 & \gamma_2 \end{pmatrix}$$

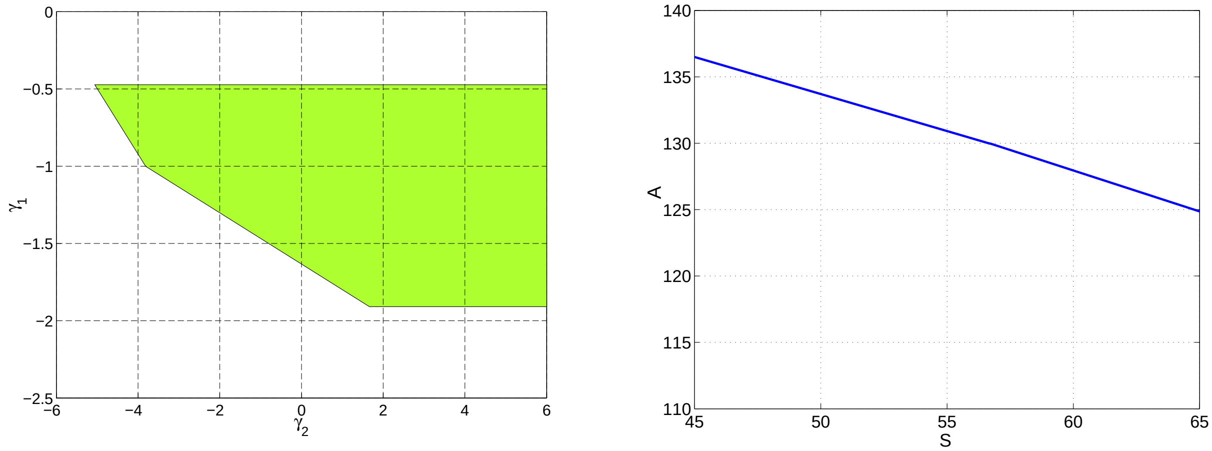
respectively. Using the results of Sec. 3.1, stabilizing control gains (γ_1, γ_2) are obtained by solving (9)-(16). Figure 3 (a) shows all stabilizing gains (γ_1, γ_2) that satisfy (9)-(16), and Fig. 3 (b) shows the bifurcation diagram of the controlled system with $(\gamma_1, \gamma_2) = (-1, 0)$. Figure 4 (a) shows the effectiveness of the control in quenching the period-2 orbit and simultaneously stabilizing the unstable fixed point. The robustness of the control law with respect to noise is demonstrated in Fig. 4 (b).

4.2.2 Washout filter-aided feedback

In the previous section, control of period doubling border collision bifurcation using static feedback was considered. Static state feedback changes the operating conditions (fixed points) of the open-loop system. This results in wasted control effort and may also result in degrading system performance. Washout filter-aided linear feedback, on the other hand, does not change the value of the fixed points of the open-loop system since the control vanishes by nature at steady state. Adding a washout filter in the feedback loop provides automatic tracking of the fixed point to be stabilized even in the presence of model uncertainty or small parameter drift. This is valuable in applications where the parameters may vary, which is particularly useful for the cardiac arrhythmia model considered in this paper.

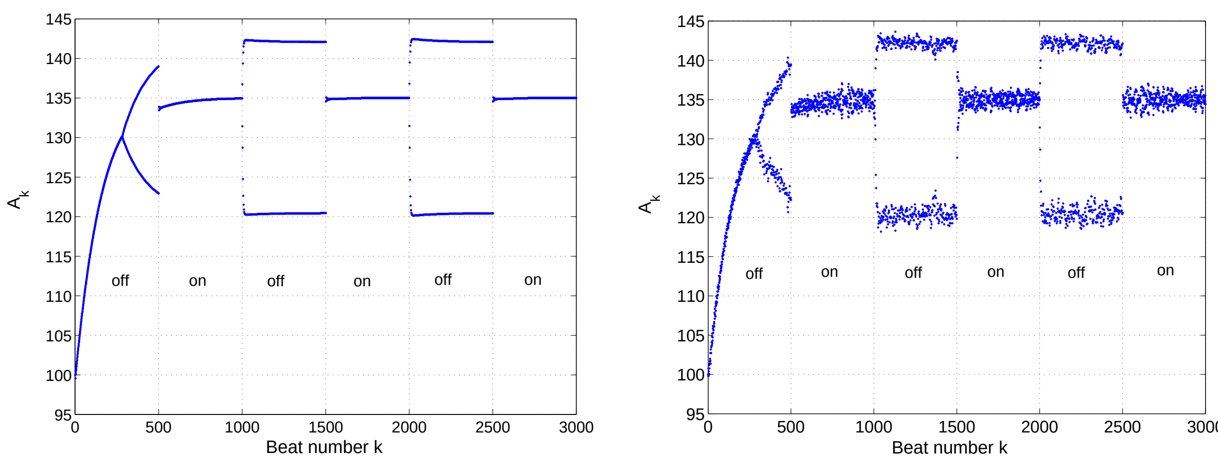
Consider the cardiac model with simultaneous washout filter-aided feedback

$$\begin{pmatrix} A_{k+1} \\ R_{k+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} A_{min} + R_{k+1} + (201 - 0.7A_k)e^{-(S+u_k)/\tau_{rec}} \\ R_k e^{-(A_k+(S+u_k))/\tau_{fat}} + \gamma e^{-(S+u_k)/\tau_{fat}} \end{pmatrix}, & \text{for } A_k \leq 130 \\ \begin{pmatrix} A_{min} + R_{k+1} + (500 - 3.0A_k)e^{-(S+u_k)/\tau_{rec}} \\ R_k e^{-(A_k+(S+u_k))/\tau_{fat}} + \gamma e^{-(S+u_k)/\tau_{fat}} \end{pmatrix}, & \text{for } A_k > 130 \end{cases} \quad (37)$$



(a) (b)

Figure 3: (a) Stabilizing simultaneous control gain pairs satisfying (9)-(16) are within the shaded region in the figure; (b) Bifurcation diagram of the controlled system using simultaneous state feedback with control gains $(\gamma_1, \gamma_2) = (-1, 0)$.



(a) (b)

Figure 4: Time series resulting from map (33) with simultaneous linear state feedback control applied at beat number $n = 500$. The control is switched off and on every 500 beats to show the effectiveness of the controller ($S = 48$ms and $(\gamma_1, \gamma_2) = (-1, 0)$). (a) without noise; (b) with zero mean, $\sigma = 0.5$ms white Gaussian noise added to S.

$$
\begin{aligned}
w_{k+1} &= A_k + (1-d)w_k & (38)\\
z_k &= A_k - dw_k & (39)\\
u_k &= \gamma_1 z_k & (40)
\end{aligned}
$$

The limiting Jacobian matrices of the controlled system to the left and right of the border are given by

$$
\tilde{A} = \begin{pmatrix} -0.31208 - 0.69860\gamma_1 & 0.99379 & 0.69860\gamma_1 d \\ -0.001597 - 0.001607\gamma_1 & 0.99379 & 0.001607\gamma_1 d \\ 1 & 0 & 1-d \end{pmatrix}
$$

and

$$
\tilde{B} = \begin{pmatrix} -1.33223 - 0.69860\gamma_1 & 0.99379 & 0.69860\gamma_1 d \\ -0.001597 - 0.001607\gamma_1 & 0.99379 & 0.001607\gamma_1 d \\ 1 & 0 & 1-d \end{pmatrix}
$$

respectively. Note that only one washout filter was used in the feedback loop. In general, the number of washout filters needed can be any number between 1 and the dimension of the system. In some cases, such as the cardiac model considered here, a single washout filter suffices.

Stabilizing washout filter-aided feedback parameters are obtained using Proposition 6. Figure 5 shows the region of stabilizing control parameters γ_1, d, which was obtained using the LMI toolbox in Matlab.

Next, simultaneous static feedback and simultaneous washout filter-aided feedback are compared. Figure 6 shows the bifurcation diagram of the closed-loop system for both simultaneous static feedback and simultaneous washout filter-aided feedback. Note that the (stabilized) fixed point of the closed-loop system using washout filter-aided feedback coincides with the open-loop (unstable) fixed point. However, the (stabilized) fixed point of the closed-loop system using static state feedback differs from the open-loop (unstable) fixed point. This is also evident from Fig. 7 and Fig. 8 which show that the control effort becomes zero in steady state when a washout filter is employed, whereas when static state feedback is used, the control effort approaches a constant value different from zero.

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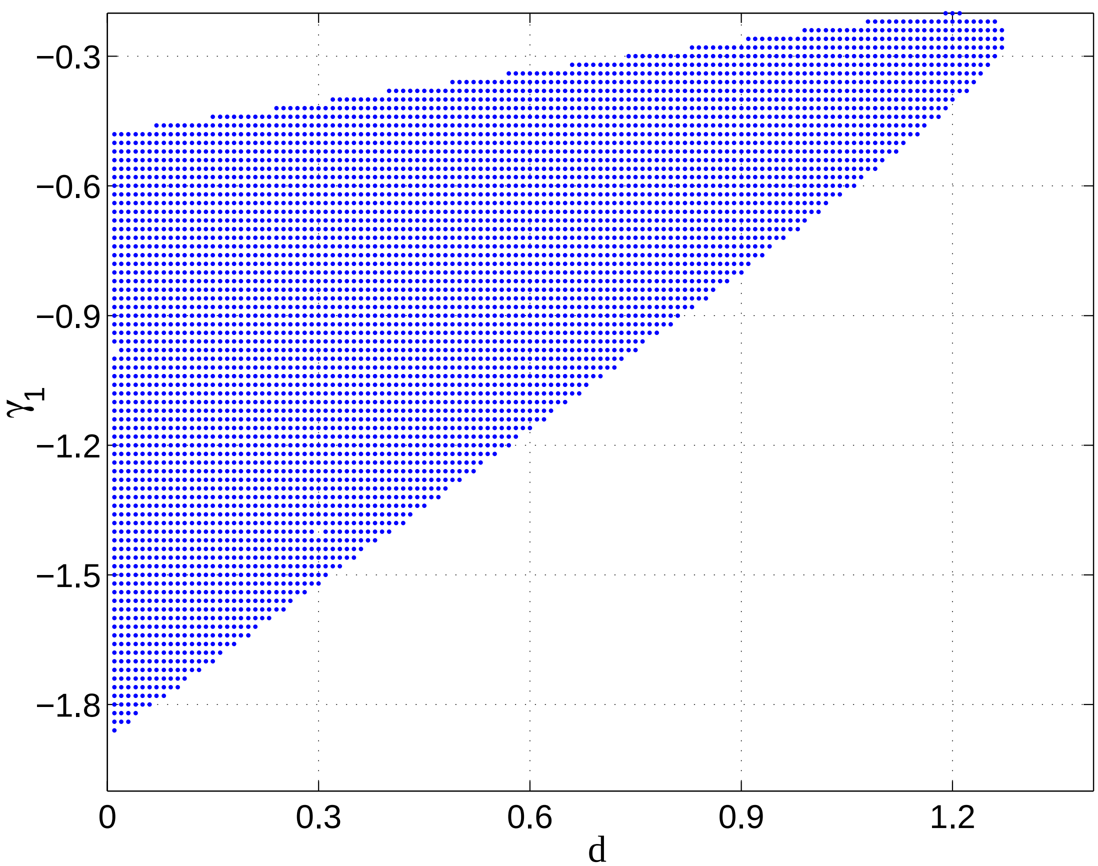

Figure 5: Stabilizing simultaneous washout filter-aided feedback control parameters are within the shaded region.

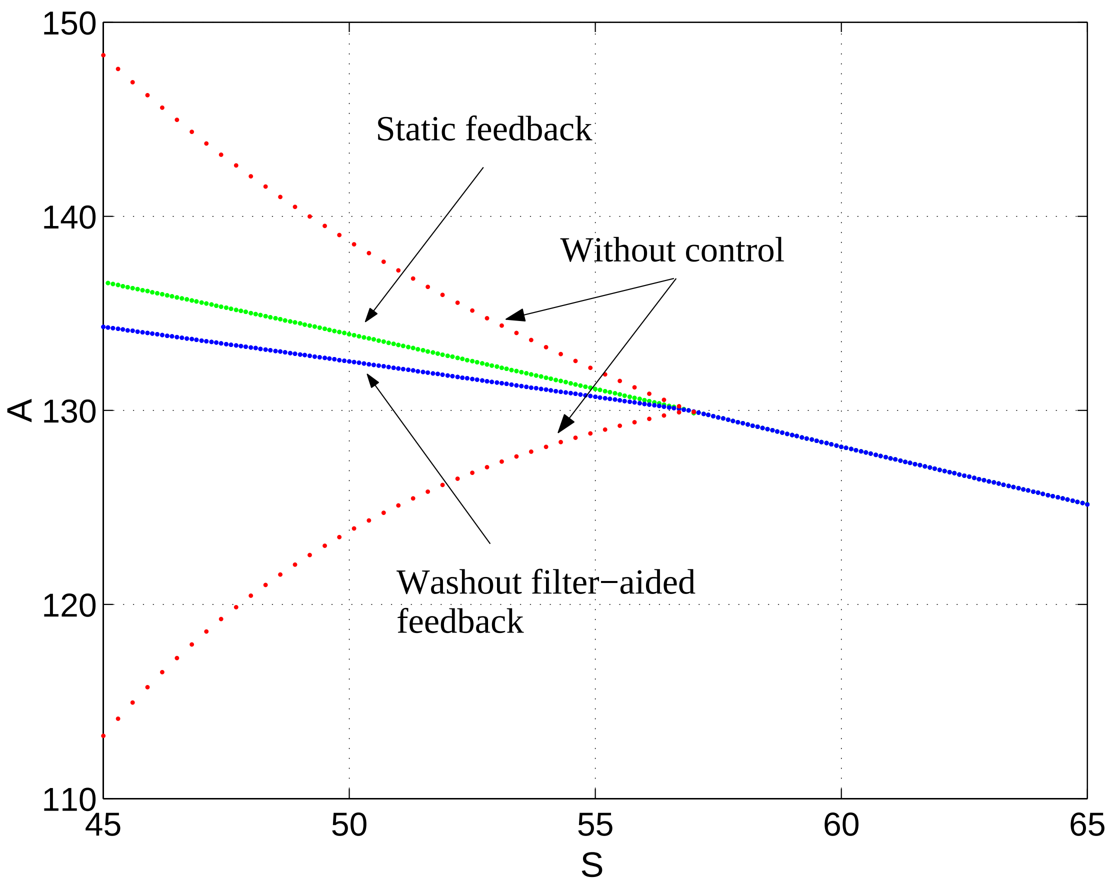

Figure 6: Bifurcation diagram of closed-loop system, comparing static feedback control ($\gamma_1 = -1$, $\gamma_2 = 0$) with washout filter-aided feedback control ($\gamma_1 = -1$, $d = 0.1$). The (red) dotted lines represent the open-loop bifurcation diagram.

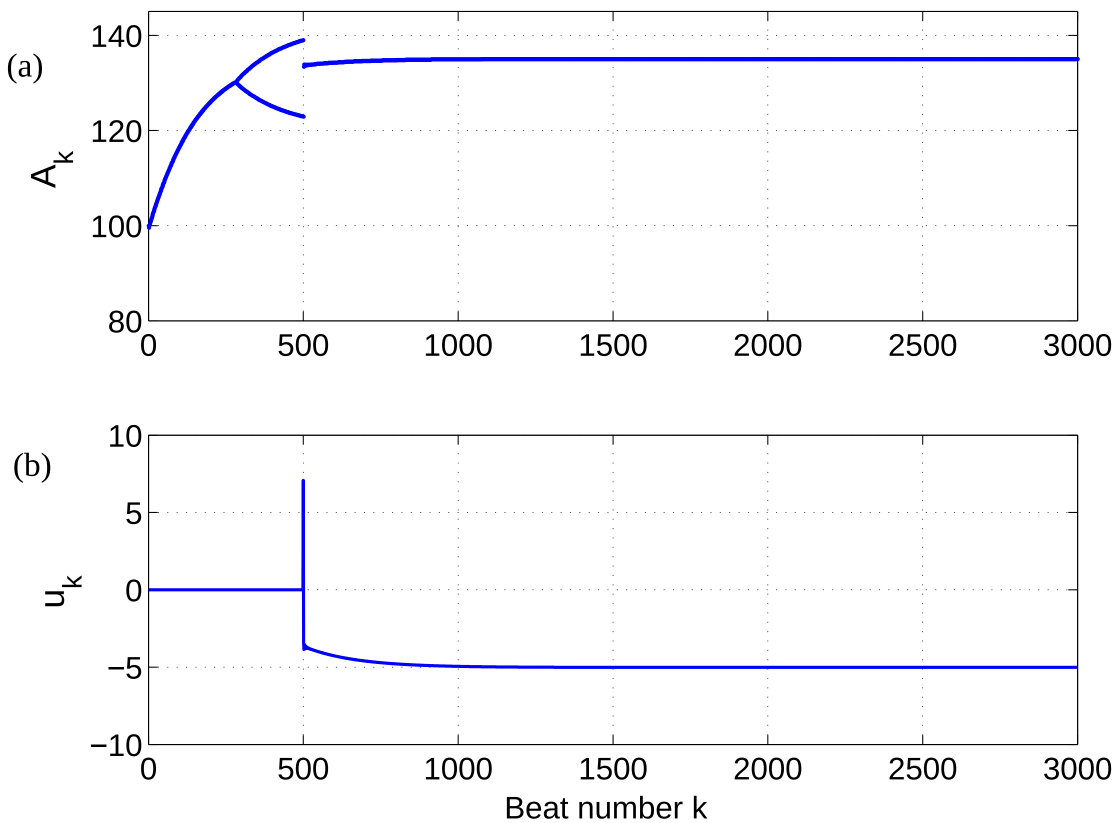

Figure 7: Time series of controlled alternan model with static state feedback applied at beat number 500 ($\gamma_1 = -1$, $\gamma_2 = 0$ and $S = 48$), (a) Conduction time A_k, (b) Control input u_k.

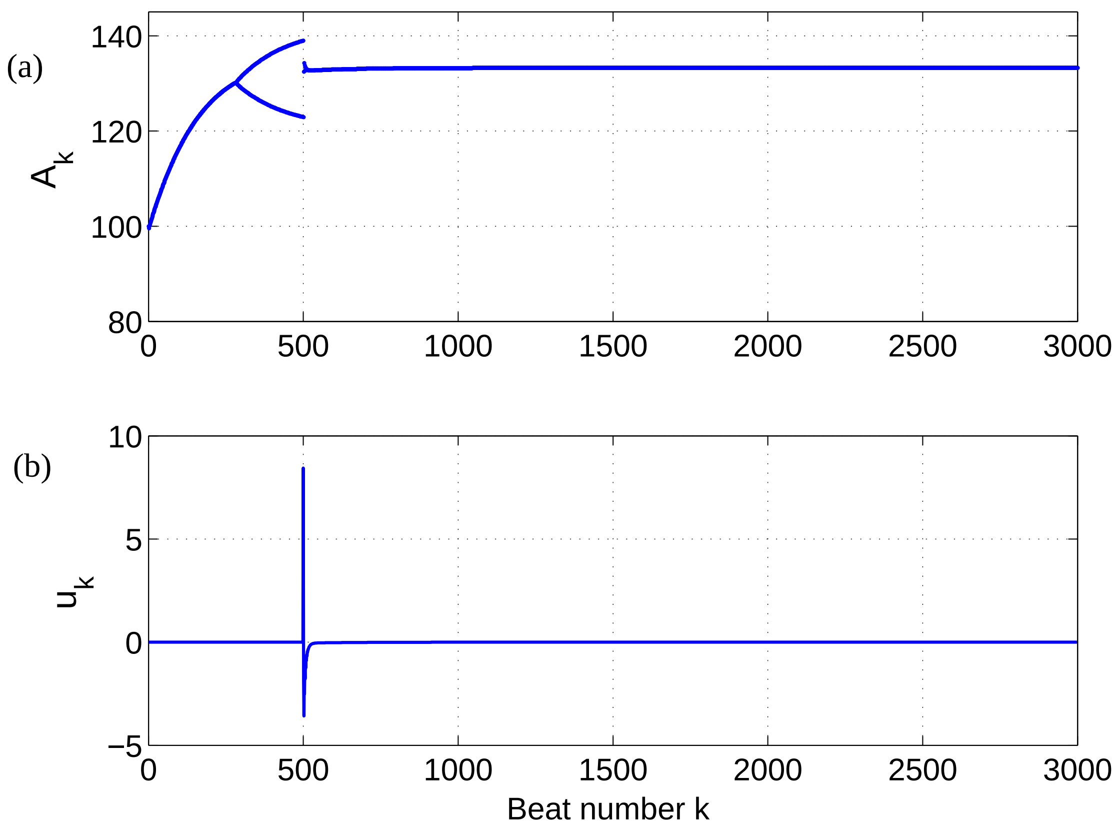

Figure 8: Time series of controlled alternan model with washout filter-aided feedback applied at beat number 500 ($\gamma_1 = -1$, $d = 0.1$ and $S = 48$), (a) Conduction time A_k, (b) Control input u_k.

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