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Formation Dynamics under a Class of Control Laws

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Formation Dynamics Under A Class of Control Laws[1]

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Abstract

A system of two earth satellites is analyzed as a controlled mechanical system. The orbit of an earth satellite can be represented by a point in the vector space of ordered pairs of angular momentum and Laplace vectors. Control laws are obtained by introducing a Lyapunov function on this space. Formations of two satellites are achieved asymptotically by the controlled dynamics.

1 Introduction

There is increasing interest in the behaviors of a cluster of relatively small satellites in space. Such satellites are usually inter-connected by wireless radio or laser links for communication. By keeping a cluster of such satellites in a certain geometrical form, one can acquire benefits for scientific observations. The information sharing across the cluster will allow the satellites to work cooperatively to perform tasks impossible or difficult for a single satellite. Compared to a single satellite providing the functionality of a cluster, a member of a cluster can be smaller/lighter. Building and launch costs will then be reduced. In addition, A cluster can also be reconfigured according to different mission goals or in the case that a member of the cluster fails.

The size and shape of a cluster or formation are usually determined by the required functionality of the formation. With J_2 and higher order terms of the earth's gravitational field ignored, the solution of the Kepler problem tells us that a satellite will track an elliptical or circular orbit. To determine the orbit of each satellite in a cluster such that the size and shape of the cluster is kept unchanged over sufficiently long period is not a trivial problem. Because the amount of fuel on board is limited for each satellite, one needs to find a set of orbits that demand the minimum control effort. This problem becomes more challenging when the effect of J_2 and other disturbances are considered. In [3] the authors proposed a ring of evenly distributed satellites on the same circular orbit for communication purpose. The stability of such ring is proved in [4]. In [2] and [5] the investigations of Clohessy-Wiltshire equations revealed possible formations with constant apparent distributions. The effects of perturbations are calculated and possible station keeping strategy are proposed. In [7], the authors proposed that in the presence of J_2, a set of constraints on the orbital elements shall be satisfied to prevent the orbits from drifting apart. However, extra station keeping is still necessary due to the complicated nature of the disturbances. The adjustment can be performed periodically when the drifting error exceeds certain threshold.

The initialization of a formation is another important problem. The whole cluster can be launched together by a space shuttle or rocket. Satellites will be first placed in a parking orbit before transferring to the final orbits. The final orbits must be such that the formation will be achieved. The orbital transfer can happen individually. One idea is to develop a Lyapunov function which will achieve its minimum when correct orbit is reached. In [6], a Lyapunov function is expressed as a quadratic function of the differences of orbital elements between current orbit and the destination orbit. In [1], the authors proved that an elliptic orbit can be represented uniquely as a point on the linear space formed by the angular momentum vectors and Laplace vectors. A Lyapunov function was built naturally from the Euclidean metric on this space. It has been suggested that compared to Hohmann transfer, this approach will consume less fuel.

In this paper, we will show that cooperative orbit transfer is possible in the setting of pairs of satellites. The satellites in a cluster can make the transfer together and the cluster relationship can be established asymptotically. The same technique can be used to perform

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station keeping task when the relative position errors of the satellites exceed a certain threshold. By extending the method in [1] to multi-satellite case, we will design a Lyapunov function and show that correct cluster relationship between orbits can be established by the controlled dynamics which will minimize this function.

In section (2), we will review the Kepler two body problem and introduce the concept of the shape space of elliptic Keplerian orbits. In section (3), the Lyapunov function for a pair of satellites is developed and the case of two satellites making a cooperative orbit transfer is studied. Simulation results are given in section (4).

2 The Kepler two body problem

For a system of two small satellites, one can make the following assumptions: (a) The gravitational attraction between the satellites can be omitted. (b) The total mass of the satellites satisfies $m_1 + m_2 \ll M$ where M is the mass of the Earth. Under these assumptions, the three body system can be approximated by two uncoupled two body problems. Each of the two body problems can be further simplified to a one center problem with the center of Earth being the center of mass.

The gravitational potential function V depends only on the distance $\|q\|$ of the satellite from the center. Then we have

$$m\ddot{q} = -\nabla V \tag{1}$$

Let $p = m\dot{q}$ be the momentum of the satellite. The angular momentum $l = q \times p$ and the energy $W = \frac{1}{2}m\|\dot{q}\|^2 + V$ are integrals of motion (i.e. they are conserved). Since

$$\begin{aligned} \dot{l} &= \dot{q} \times p + q \times \dot{p} \\ &= 0 + q \times m\ddot{q} \\ &= -q \times m\frac{q}{\|q\|}\frac{\partial V}{\partial \|q\|} = 0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} \dot{W} &= \dot{q} \cdot m\ddot{q} + \dot{V} \\ &= -\dot{q} \cdot \nabla V + \dot{V} = 0 \end{aligned} \tag{3}$$

If we make further assumptions that the shape of the earth is a perfect homogeneous ball, then $V = -m\mu/\|q\|$ where $\mu = kM$ and k is the gravity constant, the Laplace vector $A = p \times l - m^2\mu\frac{q}{\|q\|}$ is also conserved given $q(t) \neq 0$ for all $t > 0$. To see this,

$$\begin{aligned} \dot{A} &= \dot{p} \times l - m^2\mu\frac{d}{dt}(\frac{q}{\|q\|}) \\ &= m\ddot{q} \times (q \times p) - m^2\mu(\frac{\dot{q}}{\|q\|} - \frac{(q \cdot \dot{q})q}{\|q\|^3}) \end{aligned} \tag{4}$$

Notice that

$$m\ddot{q} = -m\mu\frac{q}{\|q\|^3} \tag{5}$$

Thus

$$m\ddot{q} \times (q \times p) = -m\mu\frac{(q \cdot p)q}{\|q\|^3} + m\mu\frac{p}{\|q\|} \tag{6}$$

Comparing (6) with (4) we have $\dot{A} = 0$

Knowing l , W and A, we have seven integrals for the two-body problem. They are not all independent because there are two relations connecting them. They are:

$$\begin{aligned} A \cdot l &= 0 \\ \| A \|^2 &= m^4\mu^2 + 2mW \| l \|^2 \end{aligned} \tag{7}$$

The space of ordered pairs (l, A) is $R^3 \times R^3$ on which we define the metric:

$$d((l_1, A_1), (l_2, A_2)) = (\| l_1 - l_2 \|^2 + \| A_1 - A_2 \|^2)^{\frac{1}{2}} \tag{8}$$

Let P denote the phase space of the satellite, define a mapping $\pi : P \to R^3 \times R^3$, $(q, p) \mapsto (l, A)$. Let the set Σ_e be defined as

$$\Sigma_e = \{(q, p) \in P \,|\, W(q, p) < 0, l \neq 0\} \tag{9}$$

and let the set D be defined as

$$D = \{(l, A) \in R^3 \times R^3 \,|\, A \cdot l = 0, l \neq 0, \| A \| < m^2\mu\} \tag{10}$$

In [1], the authors proved the following results:

Theorem 2.1 *(Chang-Chichka-Marsden)The following hold:*

1. *Σ_e is the union of all elliptic Keplerian orbits.*

2. *$\pi(\Sigma_e) = D$ and $\Sigma_e = \pi^{-1}(D)$.*

3. *The fiber $\pi^{-1}(l, A)$ is a unique (oriented) elliptic Keplerian orbit for each $(l, A) \in D$.*

The mapping π is a continuous mapping because l and A are continuous with respect to (q, p). Furthermore, the following corollary hold.

Corollary 2.2 *$\pi^{-1}(K)$ is compact for any compact set $K \subset D$*

Proof: Since D is a metric space, K is compact implies that K is closed and bounded. By the continuity of π, $\pi^{-1}(K)$ is closed. The only thing left to prove is the boundedness of $\pi^{-1}(K)$.

K is closed and bounded implies that there exist $r_0 > 0$, $r_1 > 0$ and $0 \le r_2 < m^2\mu$ s.t.

$$r_0 \le \| l \| \le r_1 \quad \| A \| \le r_2 \tag{11}$$

This is true since we already know $\| A \| < m^2\mu$ and $\| l \| > 0$. Because $\| \cdot \|$ is continuous on the compact set K, we declare that r_0 and r_2 exist and can be achieved. The existence of r_1 is based on the fact that $\| l \|^2 + \| A \|^2$ is bounded.

Without loss of generality, assume the satellite has unit mass. From

$$\| A \|^2 = \mu^2 + 2W \| l \|^2 \tag{12}$$

We get

$$|W| = \frac{\mu^2 - \| A \|^2}{2 \| l \|^2} \tag{13}$$

Hence,

$$0 < \frac{\mu^2 - r_2^2}{2r_1^2} \le |W| \le \frac{\mu^2}{2r_0^2} \tag{14}$$

Let a denote the semi-major axis of an elliptic Keplerian orbit and e denote the eccentricity. It is well known that

$$a = -\frac{2\mu}{W} \quad e = \frac{\| A \|}{\mu} \tag{15}$$

Thus a,e are bounded. For any $(q(t), p(t)) \in \pi^{-1}(K)$, on an elliptic orbit $q(t)$ is bounded below by $a(1-e)$ and bounded above by $a(1+e)$. Furthermore, because $\| l \| = \| p(t) \| \| q(t) \| \sin\theta$, $\| p(t) \|$ is bounded given the fact that $\| l \| \ne 0$. Hence $(q(t), p(t))$ is bounded which implies that $\pi^{-1}(K)$ is bounded. ■

In the case of two satellites, let $(l_i, A_i, P_i, \Sigma_{ei}, D_i, \pi_i, d_i)$ denote the corresponding objects defined for the ith satellite. Let $\widehat{P} = P_1 \times P_2$, $\widehat{q}(t) = (q_1(t), q_2(t))$, $\widehat{p}(t)) = (p_1(t), p_2(t))$. Let $\widehat{\Sigma}_e = \Sigma_{e1} \times \Sigma_{e2}$. Let $\widehat{D} = D_1 \times D_2$ and $\widehat{l} = (l_1, l_2)$, $\widehat{A} = (A_1, A_2)$. Let

$$\widehat{d}((\widehat{l}_1, \widehat{A}_1), (\widehat{l}_2, \widehat{A}_2)) = \sqrt{d_1^2 + d_2^2} \tag{16}$$

and $\widehat{\pi} = \pi_1 \times \pi_2$. We define *the shape space of elliptic Keplerian orbits* to be the set $\widehat{D}$ with the distance function $\widehat{d}$.

Proposition 2.3 *The following hold:*

1. *$\widehat{\Sigma}_e$ is the union of all pairs of elliptic Keplerian orbits.*

2. *$\widehat{\pi}(\widehat{\Sigma}_e) = \widehat{D}$ and $\widehat{\Sigma}_e = \widehat{\pi}^{-1}(\widehat{D})$.*

3. *The fiber $\widehat{\pi}^{-1}(\widehat{l}, \widehat{A})$ is a unique pair of (oriented) elliptic Keplerian orbit for each $(\widehat{l}, \widehat{A}) \in \widehat{D}$.*

4. *$\widehat{\pi}^{-1}(\widehat{K})$ is compact for any compact set $\widehat{K} \subset \widehat{D}$*

Proof: Use the definitions of $(\widehat{l}, \widehat{A}, \widehat{\Sigma}_e, \widehat{D}, \widehat{\pi}, \widehat{d})$ and apply Theorem (2.1) and corollary (2.2) to $(l_i, A_i, \Sigma_{ei}, D_i, \pi_i, d_i)$ for $i = 1, 2$. The statements are immediately proved. ■

3 Control Strategies to Achieve Formations

Based on the distance function on the shape space of elliptic Keplerian orbits, a Lyapunov function can be introduced on the phase space for the Hamiltonian system of two satellites. By applying LaSalle's invariance principle, we prove that the system can be driven to an invariant manifold of the phase space where formation is achieved.

The system of two satellites is a Hamiltonian (Control) system with $H = \sum_i^2 H_i$ where

$$H_i = \frac{1}{2m_i} \| p_i \|^2 - V(\| q_i \|) \tag{17}$$

For each satellite the dynamics is:

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H_i}{\partial q_i} + u_i = -m_i\mu \frac{q_i}{\|q_i\|^3} + u_i \\ \dot{q}_i &= \frac{\partial H_i}{\partial p_i} = \frac{p_i}{m_i} \end{aligned} \tag{18}$$

where $i = 1, 2$. Here u_i are controls.

Using the notations at the end of the last section, let $J_1(\widehat{q}, \widehat{p}) = (l_1, A_1)$, $J_2(\widehat{q}, \widehat{p}) = (l_2, A_2)$ and $J(\widehat{q}, \widehat{p}) = (J_1, J_2)$. Define a Lyapunov function as

$$\begin{aligned} V(J(\widehat{q}, \widehat{p})) &= \frac{1}{2}[\|l_1 - l_2 - \delta l\|^2 + \|A_1 - A_2 - \delta A\|^2 \\ &\quad + \| l_1 - l_d \|^2 + \| A_1 - A_d \|^2] \end{aligned} \tag{19}$$

Here, δl , δA are constant vectors which specify the desired final difference between (l_1, A_1) and (l_2, A_2). l_d and A_d are also constant vectors which specify the reference orbit of the formation. It can be verified that V is continuously differentiable and bounded from below by 0.

To investigate the derivation of the control law, we can rewrite (18) as:

$$\begin{aligned} \dot{\widehat{p}} &= -\frac{\partial H}{\partial \widehat{q}} + u = B(\widehat{q}, \widehat{p}) + u \\ \dot{\widehat{q}} &= \frac{\partial H}{\partial \widehat{p}} \end{aligned} \tag{20}$$

and $J(\widehat{q}, \widehat{p})$ is conserved when $u = 0$. Then we have

$$\begin{aligned} \dot{V}(J(\widehat{q}, \widehat{p})) &= \frac{\partial V}{\partial J} \cdot \frac{\partial J}{\partial \widehat{q}} \dot{\widehat{q}} + \frac{\partial V}{\partial J} \cdot \frac{\partial J}{\partial \widehat{p}} \dot{\widehat{p}} \\ &= \frac{\partial V}{\partial J} \cdot \frac{\partial J}{\partial \widehat{q}} \dot{\widehat{q}} + \frac{\partial V}{\partial J} \cdot \frac{\partial J}{\partial \widehat{p}} B(\widehat{q}, \widehat{p}) + \frac{\partial V}{\partial J} \cdot \frac{\partial J}{\partial \widehat{p}} u \end{aligned}$$

$$= \frac{\partial V}{\partial J}\cdot \dot{J}|_{u=0} + \frac{\partial V}{\partial J}\cdot\frac{\partial J}{\partial \widehat{p}}u$$
$$= \frac{\partial V}{\partial J}\cdot\frac{\partial J}{\partial \widehat{p}}u = (\frac{\partial J}{\partial \widehat{p}})^T\frac{\partial V}{\partial J}\cdot u \quad (21)$$

By letting

$$u = -\lambda((\frac{\partial J}{\partial \widehat{p}})^T\frac{\partial V}{\partial J}) \quad (22)$$

i.e

$$u_i = -\lambda_i\sum_{j=1}^{2}(\frac{\partial V}{\partial J_j}\frac{\partial J_j}{\partial p_i}) \quad (23)$$

where $\lambda = diag(\lambda_i)$ and $\lambda_i > 0$ for $i = 1, 2$, we have $\dot{V} \leq 0$ along the integral curves of the system. In [1] the authors used this observation for the setting of a single satellite to derive a Lyapunov based orbit transfer. The following theorem based on LaSalle's invariance principle holds.

Theorem 3.1 *Given the system as defined in (20), Let* $V : \widehat{P} \to R$ *be a continuously differentiable function which is bounded from below. Suppose there exists a compact set* $\Omega \subset \widehat{\Sigma}_e$ *on which* $\dot{V} \leq 0$. *Let* $E \subset \Omega$ *be the set on which* $\dot{V} = 0$. *Let* M *be the largest invariant set in E. Then by applying the control u as defined (22), every solution of the system starting within* Ω *approaches* M *as* $t \to \infty$. *Furthermore, Let* Γ *denote the subset of* Ω *where* $u(t) \equiv 0$ *holds for all t, then* $M = \Gamma$

Proof: It is easy to verify that the conditions for LaSalle's invariance principle are all satisfied. Hence $(q(t), p(t)) \to M$ as $t \to \infty$.

The set Γ where $u(t) \equiv 0$ is an invariant set contained in E. This is because we must have $\dot{V} = u \cdot u \equiv 0$ on Γ. Hence $\Gamma \subset M$. On the other hand, M is invariant implies $\dot{V} \equiv 0$ on M. Furthermore, $u(t) \equiv 0$ on M. Hence $M \subset \Gamma$. We conclude that $M = \Gamma$. ■

From this theorem we know that the largest invariant set is $M = \Omega \cap C$ where $C = \{(\widehat{q}, \widehat{p}) \in \widehat{\Sigma}_e | u(t) \equiv 0\}$. Then two questions should be answered. The first question is whether a suitable compact set Ω exists in $\widehat{\Sigma}_e$. The second question is to determine C.

The following proposition gives the answer to the first question.

Proposition 3.2 *Given*

$$\begin{aligned} \| l_d \| &\neq 0 \\ \| A_d \| &< m_1^2\mu \\ A_d \cdot l_d &= 0 \end{aligned} \quad (24)$$

Let c,c_1,c_2,c_3,c_4 *be positive numbers which satisfy*

$$\begin{aligned} c_1 &= \frac{1}{4}\| l_d - \delta l \|^2 \\ c_2 &= \frac{1}{2}\| l_d \|^2 \\ c_3 &= \frac{1}{2}(m_1^2\mu - \| A_d \|)^2 \\ c_4 &= \frac{1}{4}(m_2^2\mu - \| A_d - \delta A \|)^2 \\ c &< min\{c_1, c_2, c_3, c_4\} \end{aligned} \quad (25)$$

Let the set

$$G = \{(\widehat{l}, \widehat{A}) \in \mathcal{R}^6 \times \mathcal{R}^6 | V(\widehat{l}, \widehat{A}) \leq c\} \quad (26)$$

where V *is defined in equation (19).*

Then the set $\Omega = \widehat{\pi}^{-1}(G \cap \widehat{D})$ *is a compact subset of* $\widehat{\Sigma}_e$.

Proof: According to proposition (2.3), all we need to show is that the set $G \cap \widehat{D}$ is a compact subset of the set $\widehat{D}$. According to our definitions,

$$\begin{aligned} \widehat{D} = \ & \{(\widehat{l}, \widehat{A}) \in R^6 \times R^6 | l_i \cdot A_i = 0, \ \widehat{l}_i \neq 0, \\ & \| A_i \| < m_i^2\mu, \ i = 1, 2\} \end{aligned} \quad (27)$$

Obviously, this set $\widehat{D}$ is not a closed subset of $\mathcal{R}^6 \times \mathcal{R}^6$. If we let

$$K = \{(\widehat{l}, \widehat{A}) \in R^6 \times R^6 | l_i \cdot A_i = 0, \ i = 1, 2\} \quad (28)$$

This set K is a closed subset of $\mathcal{R}^6 \times \mathcal{R}^6$. The set $\widehat{D}$ is a subset of K. It is not difficult to see that the set $(K - \widehat{D})$ is a closed subset of $\mathcal{R}^6 \times \mathcal{R}^6$.

In the next step we want to show that if the value of c is chosen as proposed, we must have

$$G \cap (K - \widehat{D}) = \emptyset \quad (29)$$

Suppose this is not true. Notice that since $G \cap (K - \widehat{D})$ is a compact subset of $\mathcal{R}^6 \times \mathcal{R}^6$, the function V has a minimum and a maximum value on this set. The maximum value shall not exceed c. The minimum value can be found by solving a constrained minimization problem.One should compare the unconstrained minimum value with the minimum value achieved on the boundary of $(K - \widehat{D})$. In this case, the unconstrained minimum value of V is 0 which can not be achieved since $(l_d, A_d) \in \widehat{D}$.On the other hand, c_1 can be achieved on the boundary of $(K - \widehat{D})$ where $l_1 = 0$. c_2 can be achieved on the boundary where $l_2 = 0$. c_3 can be achieved on the boundary where $A_1 = m_1^2\mu$. c_4 can be achieved on the boundary where $A_2 = m_2^2\mu$. However, if $c < min\{c_1, c_2, c_3, c_4\}$, then the maximum is less than the minimum. We have a contradiction. Thus the set G must have no intersection with the set $(K - \widehat{D})$.

On the other hand, we know that $G \cap K \neq \emptyset$. This means $G \cap K \subset \widehat{D}$. The compactness of $G \cap K$ comes from the fact that G is compact and K is closed in $\mathcal{R}^6 \times \mathcal{R}^6$

■

Next we want to characterize the invariant set. By equation (23), without loss of generality, we let $\lambda_1 = \lambda_2 = 1$, the controls can be calculated as

$$\begin{aligned} u_1 &= -[\frac{\partial V}{\partial l_1}\frac{\partial l_1}{\partial p_1} + \frac{\partial V}{\partial A_1}\frac{\partial A_1}{\partial p_1}] \\ &= -[(l_1 - l_2 - \delta l + l_1 - l_d)\times q_1 \\ &\quad + l_1\times(A_1 - A_2 - \delta A + A_1 - A_d) \\ &\quad + ((A_1 - A_2 - \delta A + A_1 - A_d)\times p_1)\times q_1] \end{aligned} \tag{30}$$

$$\begin{aligned} u_2 &= -[\frac{\partial V}{\partial l_2}\frac{\partial l_2}{\partial p_2} + \frac{\partial V}{\partial A_2}\frac{\partial A_2}{\partial p_2}] \\ &= (l_1 - l_2 - \delta l)\times q_2 + l_2\times(A_1 - A_2 - \delta A) \\ &\quad + ((A_1 - A_2 - \delta A)\times p_2)\times q_2 \end{aligned} \tag{31}$$

We need the following lemma to solve the equations obtained by letting $u_1(t) \equiv 0$ and $u_2(t) \equiv 0$.

Lemma 3.3 *Suppose a single satellite has external control $u(t) \equiv 0$. Let x, y be time invariant unknown vectors. Suppose $(q(t), p(t)) \in \Sigma_e$, the solution of equation*

$$x\times q + l\times y + (y\times p)\times q \equiv 0 \tag{32}$$

is

$$x = \alpha A \quad y = \alpha l \tag{33}$$

For some $\alpha \in R$

Proof: Take the inner product with $q(t)$ on both sides of equation (32) we get:

$$(l\times y)\cdot q(t) \equiv 0 \tag{34}$$

Since the orbit of the satellite is elliptic, $q(t) \neq 0$. Furthermore, $q(t)$ would stay in the orbital plane $l^{\perp}$ which is perpendicular to l. Notice because $l\times y$ is time invariant and $q(t)$ is not 0, we conclude that $l\times y$ must be either parallel to l or vanish. However, since the relation $(l\times y)\cdot l = 0$ must be satisfied, the only possibility is to have $l\times y = 0$. Thus, there exists a number $\alpha \in R$ such that $y = \alpha l$. Equation (32) can be simplified to

$$(x + \alpha l\times p)\times q \equiv 0 \tag{35}$$

Since

$$p\times l = A + m^2\mu\frac{q}{\| q \|} \tag{36}$$

we now have,

$$(x - \alpha A)\times q(t) \equiv 0 \tag{37}$$

Again, since $q(t)$ is tracking an elliptic orbit, from the fact that x and A are time invariant, we must have $x = \alpha A$. ■

Problem 3.4 *Suppose two satellites are controlled by (30) and (31) respectively. If we select $l_d \neq 0$ and $\| A_d \| < m_1^2\mu$, we want to calculate the maximal set $C \subset \widehat{\Sigma}_e$ where $u_1(t) \equiv 0$ and $u_2(t) \equiv 0$ are satisfied.*

Solution: From $u_1(t) \equiv 0$, we have

$$\begin{aligned} &(l_1 - l_2 - \delta l + l_1 - l_d)\times q_1 + l_1\times(A_1 - A_2 - \delta A + A_1 \\ &\quad - A_d) + ((A_1 - A_2 - \delta A + A_1 - A_d)\times p_1)\times q_1 \equiv 0 \end{aligned} \tag{38}$$

From $u_2(t) \equiv 0$, we have

$$\begin{aligned} &(l_1 - l_2 - \delta l)\times q_2 + l_2\times(A_1 - A_2 - \delta A) \\ &\quad + ((A_1 - A_2 - \delta A)\times p_2)\times q_2 \equiv 0 \end{aligned} \tag{39}$$

We want to solve these two equations for l_1,l_2,A_1 and A_2. Notice that these unknowns are time invariant.

By applying the results in the lemma to equation (38), we get

$$\begin{aligned} l_1 - l_2 - \delta l + l_1 - l_d &= \alpha A_1 \\ A_1 - A_2 - \delta A + A_1 - A_d &= \alpha l_1 \end{aligned} \tag{40}$$

From equation (38) we get

$$\begin{aligned} A_1 - A_2 - \delta A &= \beta l_2 \\ l_1 - l_2 - \delta l &= \beta A_2 \end{aligned} \tag{41}$$

Hence,

$$\begin{aligned} l_1 - l_d &= \alpha A_1 - \beta A_2 \\ A_1 - A_d &= \alpha l_1 - \beta l_2 \end{aligned} \tag{42}$$

The maximal invariant set C within $\widehat{\Sigma}_e$ is where equation (41) and (42) are satisfied.

Notice that although we have introduced two unknown variables α and β, we can still solve for (l_1, l_2, A_1, A_2) in terms of $(l_d, A_d, \delta l, \delta A)$ because we have two "extra" equations

$$l_1 \cdot A_1 = 0 \quad l_2 \cdot A_2 = 0 \tag{43}$$

Let C_1 denote the set where $\alpha = 0$ and $\beta = 0$. Let C_2 denote the set where $\alpha = 0$ and $\beta \neq 0$. Let C_3 denote the set where $\alpha \neq 0$ and $\beta = 0$. Let C_4 denote the set where $\alpha \neq 0$ and $\beta \neq 0$. Then

$$C = C_1 \cup C_2 \cup C_3 \cup C_4 \tag{44}$$

Among all the possible invariant sets we have calculated, only C_1 is the one we want the system to approach. Thus, we shall pick suitable values for δl, δA, l_d, A_d and initial conditions so that C_1 is the only possible invariant set within Ω. The following proposition gives a set of sufficient conditions to achieve this goal.

Proposition 3.5 *Let δl, δA, l_d,A_d satisfies all the assumptions in proposition (3.2) and*

$$\delta l \cdot \delta A = 0$$

$$\begin{aligned} \delta l \cdot (l_d - \delta l) &= 0 \\ \delta A \cdot (A_d - \delta A) &= 0 \\ (l_d - \delta l) \cdot (A_d - \delta A) &= 0 \end{aligned} \tag{45}$$

and one of the following inequalities

$$\begin{aligned} l_d &\neq \delta l - \sigma \delta A \\ A_d &\neq \delta A - \sigma \delta l \end{aligned} \tag{46}$$

where $\sigma = \pm\sqrt{2}$.

Select the initial condition $(\widehat{q}_0, \widehat{p}_0)$ *s.t*

$$V(\widehat{q}_0, \widehat{p}_0) < c \tag{47}$$

where c is defined in proposition (3.2). The system will be controlled to the set C_1 *where*

$$\begin{aligned} l_1 - l_2 - \delta l &= 0 \\ A_1 - A_2 - \delta A &= 0 \\ l_1 &= l_d \\ A_1 &= A_d \end{aligned} \tag{48}$$

are satisfied.

Proof: We will show that the maximal invariant subset of the compact set determined by the given initial conditions contains only the set C_1. Other invariant sets will vanish or will be impossible to reach.

On C_2, the following equations are satisfied

$$\begin{aligned} l_1 - l_2 &= \delta l \\ A_1 - A_2 &= \delta A \end{aligned} \tag{49}$$

$$\begin{aligned} l_1 - \alpha A_1 &= l_d \\ A_1 - \alpha l_1 &= A_d \end{aligned} \tag{50}$$

from (50), since $l_d \cdot A_d = 0$ and $l_1 \cdot A_1 = 0$ we have

$$\alpha(\| l_1 \|^2 + \| A_1 \|^2) = 0 \tag{51}$$

Because $\alpha \neq 0$, this implies that l_1 and A_1 vanish. Thus equation (50) will not be satisfied. C_2 vanishes.

On C_3, the following equations are satisfied

$$\begin{aligned} l_1 - l_2 &= \delta l + \beta A_2 \\ A_1 - A_2 &= \delta A + \beta l_2 \end{aligned} \tag{52}$$

$$\begin{aligned} l_1 + \beta A_2 &= l_d \\ A_1 + \beta l_2 &= A_d \end{aligned} \tag{53}$$

From (52) we can solve for l_1 and A_1 and put them in (53). We get

$$\begin{aligned} l_2 + 2\beta A_2 &= l_d - \delta l \\ A_2 + 2\beta l_2 &= A_d - \delta A \end{aligned} \tag{54}$$

Since $(l_d - \delta l) \cdot (A_d - \delta A) = 0$ and $l_2 \cdot A_2 = 0$ we have

$$2\beta(\| l_2 \|^2 + \| A_2 \|^2) = 0 \tag{55}$$

Because $\beta \neq 0$, this implies that l_2 and A_2 vanish. Hence C_3 vanishes.

On C_4, the following equations are satisfied

$$\begin{aligned} l_1 - l_2 &= \delta l + \beta A_2 \\ A_1 - A_2 &= \delta A + \beta l_2 \end{aligned} \tag{56}$$

$$\begin{aligned} l_1 - \alpha A_1 + \beta A_2 &= l_d \\ A_1 - \alpha l_1 + \beta l_2 &= A_d \end{aligned} \tag{57}$$

Replace βA_2 and βl_2 of equation (57), we have

$$\begin{aligned} 2l_1 - l_2 - \alpha A_1 &= l_d + \delta l \\ 2A_1 - A_2 - \alpha l_1 &= A_d + \delta A \end{aligned} \tag{58}$$

From (56) we can solve for l_1 and A_1 and put them in (58). We get

$$\begin{aligned} (1 - \alpha\beta)l_2 + (2\beta - \alpha)A_2 &= l_d - \delta l + \alpha \delta A \\ (2\beta - \alpha)l_2 + (1 - \alpha\beta)A_2 &= A_d - \delta A + \alpha \delta l \end{aligned} \tag{59}$$

According to (45), we have

$$(l_d - \delta l + \alpha \delta A) \cdot (A_d - \delta A + \alpha \delta l) = 0$$

so

$$(1 - \alpha\beta)(2\beta - \alpha)(\| l_2 \|^2 + \| A_2 \|^2) = 0$$

Because $\| l_2 \|$ and $\| A_2 \|$ can not vanish, there are three possibilities:

(1) $\alpha = 2\beta$ but $\alpha\beta \neq 1$

From equation (59) we have

$$\begin{aligned} l_2 &= \frac{1}{1 - 2\beta^2}(l_d - \delta l + 2\beta \delta A) \\ A_2 &= \frac{1}{1 - 2\beta^2}(A_d - \delta A + 2\beta \delta l) \end{aligned} \tag{60}$$

Solve for l_1,A_1 from equation (56), we have

$$\begin{aligned} l_1 &= \frac{1}{1 - 2\beta^2}(l_d + \beta(A_d + \delta A)) \\ A_1 &= \frac{1}{1 - 2\beta^2}(A_d + \beta(l_d + \delta l)) \end{aligned} \tag{61}$$

Now, since $l_1 \cdot A_1 = 0$ we have

$$(l_d + \beta(A_d + \delta A)) \cdot (A_d + \beta(l_d + \delta l)) = 0$$

which implies

$$\beta(\| l_d \|^2 + \| A_d \|^2 + \| \delta l \|^2 + \| \delta A \|^2) = 0$$

This result is impossible.

(2) $\alpha\beta = 1$ but $\alpha \neq 2\beta$

From equation (59) we have

$$l_2 = \frac{1}{2\beta - \alpha}(A_d - \delta A + \alpha \delta l)$$

$$A_2 = \frac{1}{2\beta - \alpha}(l_d - \delta l + \alpha \delta A) \quad (62)$$

Solve for l_1,A_1 from equation (56), we have

$$\begin{aligned} l_1 &= \frac{1}{2\beta - \alpha}(A_d + \beta(l_d + \delta l)) \\ A_1 &= \frac{1}{2\beta - \alpha}(l_d + \beta(A_d + \delta A)) \end{aligned} \quad (63)$$

Now, since $l_1 \cdot A_1 = 0$ we have

$$(A_d + \beta(l_d + \delta l)) \cdot (l_d + \beta(A_d + \delta A)) = 0$$

which implies

$$\beta(\| l_d \|^2 + \| A_d \|^2 + \| \delta l \|^2 + \| \delta A \|^2) = 0$$

This result is also impossible.

(3) $\alpha\beta = 1$ and $\alpha = 2\beta$.

This implies that $\alpha = \pm\sqrt{2}$. Equation (59) will be

$$\begin{aligned} l_d - \delta l + \alpha \delta A &= 0 \\ A_d - \delta A + \alpha \delta l &= 0 \end{aligned} \quad (64)$$

This is impossible because it violated the conditions in (46).

Hence C_4 vanishes. ∎

A special case of this proposition is when we choose $\delta l = 0$ and $\delta A = 0$. This is the case where two satellites are driven to the same orbit.

If we let the Lyapunov function $V(J(q,p)) = \frac{1}{2}[\| l_1 - l_d \|^2 + \| A_1 - A_d \|^2]$, we can control a single satellite to transfer between elliptic orbits. This case is first analyzed in [1]. In this case, the invariant set is simply the set where $l_1 = l_d$ and $A_1 = A_d$ are satisfied. Comparing to this case, the invariant set of two satellites case is more complicated.

To completely set up the formation completely, we need to know the final positions of the satellites on their orbits. This can be computed off-line by simulation. Then we can choose proper starting separation between two satellites.

4 Simulation Results

In order to verify the control algorithm, we wrote a simulation of controlling two satellites into formation on MATLAB. We take the Lyapunov function as

$$V(J(q,p)) = \frac{1}{2}(b_1\|l_1 - l_2 - \delta l\|^2 + b_2\|A_1 - A_2 - \delta A\|^2 + b_1\|l_1 - l_d\|^2 + b_2\|A_1 - A_d\|^2) \quad (65)$$

here, we put factors b_1 and b_2 into the Lyapunov function so that the control is weighted. Our control is

$$\begin{aligned} u_1 &= -\xi[b_1(l_1 - l_2 - \delta l + l_1 - l_d)\times q_1 \\ &\quad + l_1 \times b_2(A_1 - A_2 - \delta A + A_1 - A_d) \\ &\quad + (b_2(A_1 - A_2 - \delta A)\times p_1)\times q_1] \\ u_2 &= \eta[b_1(l_1 - l_2 - \delta l)\times q_2 + l_2 \times b_2(A_1 - A_2 - \delta A) \\ &\quad + (b_2(A_1 - A_2 - \delta A)\times p_2)\times q_2] \end{aligned} \quad (66)$$

where ξ , η are positive numbers which can be adjusted for numerical performance.

In practical applications this control law needs to be discretized. Here we use a simple technique to obtain a discrete control law from (66). Assume that the thruster of a satellite can be fired towards any direction. Let T be the time interval between two firings of the thruster pulses. Let t_0 denote the starting time of the controller. Denote $\bar{u}_i$ the discretized control. At time $t_0 + nT$, we have:

$$\bar{u}_i(n) = u_i(t_0 + nT) \cdot T \quad (67)$$

That is, we assume that during the time interval T the control u_i is constant. Of course, if T is too large the algorithm may fail to converge.

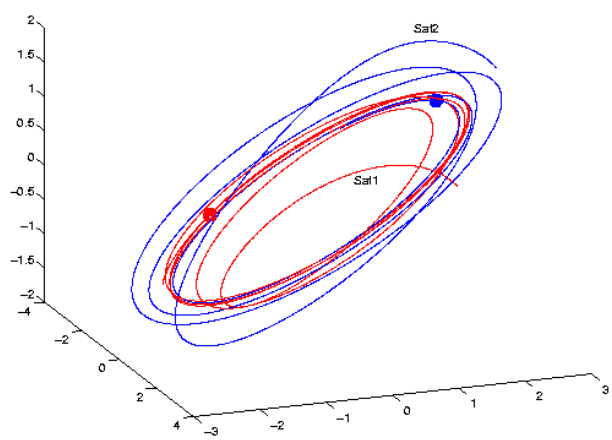

Figure 1: Two Satellites Formation

Figure 1 shows when $\delta l = 0$ and $\delta A = 0$. By applying the discretized control law we can drive two satellites Sat1 and Sat2 onto the same orbit. We take the unit length to be the radius of the earth, the unit time to be one minute and the unit mass to be 1000kg. The initial conditions of the two satellites are given by specifying their six orbital elements $(a, e, i, \omega, \Omega, \tau)$. We have $(3, 0.3, 0, \pi/2, 0, 0)$ for Sat1 and $(4, 0.2, \pi/4, \pi/4, \pi/3, 0)$ for Sat2. The destination orbit is a circular orbit given as $(3, 0, \pi/6, \pi/2, 0, 0)$. Hence, l_d and A_d are determined. During the simulation process, we noticed that by choosing $b_2 = 1000b_1$ and $\xi = \eta = 0.01$ we can get a decent result.

Figure 2 shows the change of $\| l_1 - l_2 \|$, $100 \| A_1 - A_2 \|$ and the Lyapunov function V with respect to time during the whole process. As we can see, the Lyapunov function is being reduced during the whole process.

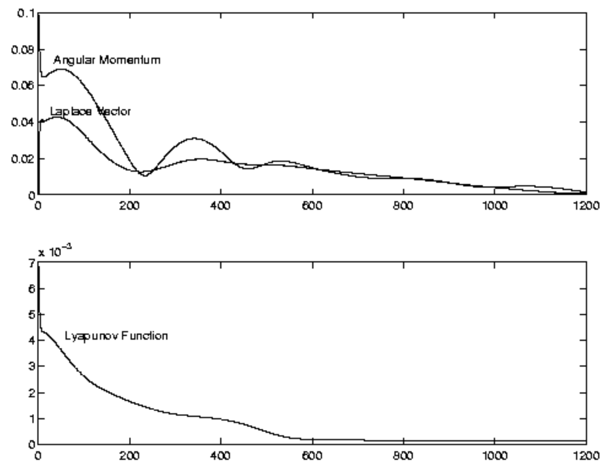

Figure 2: Above: $\| l_1 - l_2 \|$ and $\| A_1 - A_2 \|$ as a function of time; Below: the Lyapunov function as a function of time

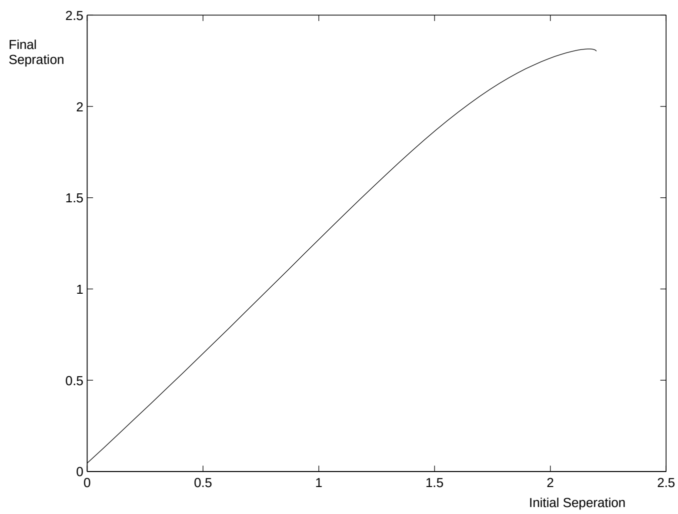

Figure 3: The final separation between two Satellites as a function of starting separation

The final relative position of the two satellites depends on the starting relative position of the two satellites. In the following experiments, two LEO satellites are first placed on the same circular orbit specified as $[1.1, 0, 0, \pi/2, 0, 0]$ with certain amount of initial separation. Then they are transfered to a final orbit of $[1.3, 0, 0, \pi/2, 0, 0]$. Figure 3 shows the final separation versus the initial separation between the two satellites. It can be seen that the separation is increased since we are transferring to a higher obit.

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