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Feedback Control of Border Collision Bifurcations in Piecewise Smooth Systems

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Feedback Control of Border Collision Bifurcations in Piecewise Smooth Systems

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Abstract

Feedback controls that stabilize border collision bifurcations are designed for piecewise smooth systems undergoing border collision bifurcations. The paper begins with a summary of the main results on border collision bifurcations, and proceeds to a study of stabilization of these bifurcations for one-dimensional systems using both static and dynamic feedback. The feedback can be applied on one side of the border, or on both sides. To achieve robustness to uncertainty in the border itself, a simultaneous stabilization problem is stated and solved. In this problem, the same control is applied on both sides of the border. Dynamic feedback employing washout filters to maintain fixed points is shown to lead to stabilizability for a greater range of systems than static feedback. The results are obtained with a focus on systems in normal form.

1 Introduction

Recently, several researchers have studied bifurcations in piecewise smooth (PWS) systems [20, 21, 23, 28, 6, 7, 8]. PWS systems occur as models for switched systems, such as power electronic circuits. They are usually modeled by piecewise smooth maps. PWS systems can of course exhibit classical smooth bifurcations, for example at a fixed point in a neighborhood of which the system is smooth. What is of interest therefore is the study of bifurcations in PWS systems that occur at the boundaries between regions of smooth behavior, or that involve motions that include more than one such region. These bifurcations have been termed border collision bifurcations [20, 21].

The studies of border collision bifurcations (BCBs) have to-date resulted in a basic understanding of these bifurcations, and, in turn, an identification of basic distinctions with smooth bifurcations. As will be seen below, there are BCBs that are analogous to the saddle-node (or fold) bifurcation and to the period doubling bifurcation. There are also special BCBs such as a bifurcation from a fixed point directly to chaotic behavior (named instant chaos [20, 23]).

BCBs can occur when a fixed point hits the border between regions of smooth behavior. Note that BCBs have been studied for systems which, while being piecewise smooth, are also continuous. This assumption is in force in the present work as well.

In the next section, some results on BCB in one dimensional PWS maps are summarized. In Section 3, some background on bifurcation control is given, including a summary of the use of washout filters in bifurcation control. In Section 4, the control of BCBs in one-dimensional PWS systems is considered. Brief concluding remarks are collected in Section 5.

2 Background on Border Collision Bifurcation in One Dimensional Maps

This section begins with a summary of results on border collision bifurcation for 1D systems. The presentation closely follows [27, 7, 8, 6], with only cosmetic modifications.

The analysis of border collision bifurcations in one dimensional piecewise smooth (PWS) maps is straightforward. There are two main ingredients in the analysis: (i) an observation about normal forms being affine (for fixed points on borders), and (ii) sketches that clarify how fixed points and periodic points depend on the bifurcation parameter for the scenarios associated with the various cases. For simplicity, a PWS map is considered that involves only two regions of smooth behavior.

Consider the 1-D PWS system

$$x_{k+1} = f(x_k, \mu) \tag{1}$$

where the map $f(x, \mu)$ is of the form

$$f(x,\mu) = \begin{cases} g(x,\mu), & x \le x_b \\ h(x,\mu), & x \ge x_b \end{cases} \tag{2}$$

and where μ is the bifurcation parameter. Since the system is one-dimensional, the border is just the point x_b. The map $f : R \times R \to R$ is assumed to be PWS: f depends smoothly on x everywhere except at x_b, where it is continuous in x. It is also assumed that f depends smoothly on μ everywhere. Denote by R_L and R_R the two regions in state space separated by the border: $R_L := \{x : \ x \le x_b\}$ and $R_R := \{x : \ x \ge x_b\}$.

Let $x_0(\mu)$ be a possible path of fixed points of f; this path depends continuously on μ. Suppose also that the fixed point hits the boundary at a critical parameter value μ_b: $x_0(\mu_b) = x_b$. Below, conditions are recalled for the occurrence of various types of BCBs from x_b for μ near μ_b.

The normal form for the PWS (1) at a fixed point on the border is a piecewise affine approximation of the map in the neighborhood of the border point x_b , in scaled coordinates [27, 28, 6, 7, 8]. For completeness, a derivation of the 1-D normal form is now recalled

[27, 7]. Letting $\bar{x} = x - x_b$ and $\bar{\mu} = \mu - \mu_b$, Eq. (2) becomes

$$\tilde{f}(\bar{x}, \bar{\mu}) := f(\bar{x} + x_b, \bar{\mu} + \mu_b) = \begin{cases} g(\bar{x} + x_b, \bar{\mu} + \mu_b), & \bar{x} \leq 0 \\ h(\bar{x} + x_b, \bar{\mu} + \mu_b), & \bar{x} \geq 0 \end{cases} \tag{3}$$

In these variables, the border is at $\bar{x} = 0$, and the state space is divided into two halves, $\Re_- = (-\infty, 0]$ and $\Re_+ = [0, \infty)$. Also, the fixed point of (1) is at the border for the parameter value $\bar{\mu} = 0$.

Expanding $\tilde{f}$ to first order about $(0, 0)$ gives

$$\tilde{f}(\bar{x}, \bar{\mu}) = \begin{cases} a\bar{x} + \bar{\mu}v + o(\bar{x}, \bar{\mu}), & \bar{x} \leq 0 \\ b\bar{x} + \bar{\mu}v + o(\bar{x}, \bar{\mu}), & \bar{x} \geq 0 \end{cases} \tag{4}$$

where

$$\begin{aligned} a &= \lim_{\bar{x} \to 0^-} \frac{\partial}{\partial x} \tilde{f}(\bar{x}, 0), \\ b &= \lim_{\bar{x} \to 0^+} \frac{\partial}{\partial x} \tilde{f}(\bar{x}, 0), \\ v &= \lim_{\bar{x} \to 0} \frac{\partial}{\partial \mu} \tilde{f}(\bar{x}, 0). \end{aligned}$$

(The last limit doesn't depend on the direction of approach of 0 by x, due to the assumed smoothness of f in μ.) Suppose $v \neq 0$, $|a| \neq 1$ and $|b| \neq 1$. The assumption $|a| \neq 1$ and $|b| \neq 1$ implies that the nonlinear terms are negligible close to the border. The 1-D normal form is therefore obtained by defining a new parameter $\bar{\bar{\mu}} = \bar{\mu}v$ and dropping the higher order terms [27, 7]:

$$G_1(\bar{x}, \bar{\bar{\mu}}) = \begin{cases} a\bar{x} + \bar{\bar{\mu}}, & \bar{x} \leq 0 \\ b\bar{x} + \bar{\bar{\mu}}, & \bar{x} \geq 0 \end{cases}$$

(The convention in [27, 7] of using the subscript 1 in the map G_1 is followed, indicating that this is the normal form for 1D systems. Later, G_2 will be used for the normal form of 2D systems.) The normal form map $G_1(\cdot, \cdot)$ can be used to study local bifurcations of the original map $f(\cdot, \cdot)$ [27, 7].

For simplicity of notation, below (x, μ) is used instead of $(\bar{x}, \bar{\bar{\mu}})$. The normal form is therefore

$$x_{k+1} = G_1(x_k, \mu) = \begin{cases} ax_k + \mu, & x_k \leq 0 \\ bx_k + \mu, & x_k \geq 0 \end{cases} \tag{5}$$

Denote by x_R^* and x_L^* possible fixed points of the system near the border to the right ($x > x_b$) and left ($x < x_b$) of the border, respectively. Then in the normal form (5), $x_R^* = \frac{\mu}{1-b}$ and $x_L^* = \frac{\mu}{1-a}$. For these fixed points to actually occur, we need $x_L^* \leq 0$ and $x_R^* \geq 0$, i.e., $b < 1$ and $a < 1$.

Various combinations of the parameters a and b lead to different kinds of bifurcation behavior as μ is varied. Since the map G_1 is invariant under the transformation $x \to -x$, $\mu \to -\mu$, $a \rightleftharpoons b$, it suffices to consider only the case $a \geq b$.

The possible bifurcation scenarios are recalled next. Some of the language used below to describe the BCBs is introduced here to more easily convey the ideas.

2.1 Scenario A. Persistent fixed point (nonbifurcation)

Two situations lead to this scenario:

Scenario A1 (Persistence of Stable Fixed Point)[1]:

$$\text{If} \quad -1 < b \leq a < 1 \tag{6}$$

then a stable fixed point for $\mu < 0$ persists and remains stable for $\mu > 0$.

In this case, the fixed point changes continuously as a function of the bifurcation parameter, but the eigenvalue associated with the system linearization at the fixed point changes discontinuously from a to b at $\mu = 0$.

Figure 1 illustrates the dependence of the map G_1 and its fixed points on μ near the border. The system has a single eigenvalue at the fixed point, which changes discontinuously at the border. The distinct eigenvalues are the slopes of the map on the two sides of the vertical axis in Figure 1.

Scenario A2 (Persistence of Unstable Fixed Point)[2]:

$$\text{If} \quad \text{i)} \qquad 1 < b \leq a \tag{7}$$

$$\text{or} \quad \text{ii)} \qquad b \leq a < -1 \tag{8}$$

then an unstable fixed point for $\mu < 0$ persists and remains unstable for $\mu > 0$.

In this case, there is one unstable fixed point for both positive and negative values of μ and no local attractors exist. As before, the fixed point depends continuously on μ. The system trajectory diverges for all initial conditions. Figure 2 shows typical bifurcation diagrams for Scenario A.

2.2 Scenario B. Border collision pair bifurcation

For other values of the parameters a and b, there are two main kinds of border collision bifurcation, namely, *border collision pair bifurcation* and *border crossing bifurcation.* Border collision pair bifurcation is similar to saddle node bifurcation in smooth systems, where

[1]In [27, 7], the terminology *Period-1* $\to$ *Period-1* is used to describe this case.

[2]In [27, 7], the terminology *No Attractor* $\to$ *No Attractor* is used to describe this case.

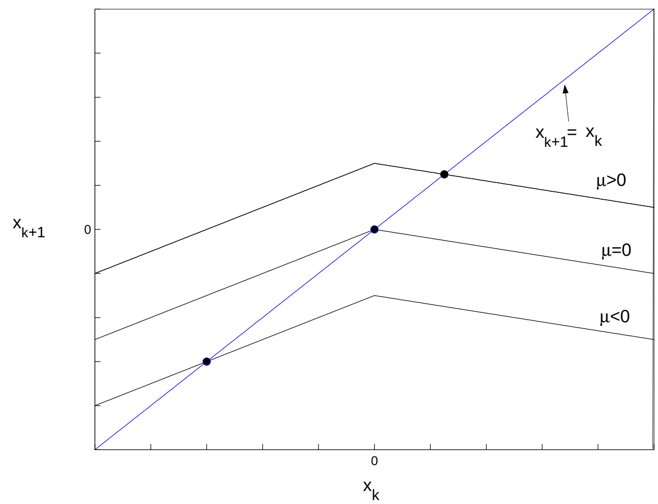

Figure 1: Dependence of first return map and its fixed point on μ for Scenario A ($-1 < b < a < 1$) is shown here. Intersections of the map with the line $x_{k+1} = x_k$ are the fixed points.

two fixed points of the system collide and disappear at the bifurcation. Border crossing bifurcation, on the other hand, has some similarities with period doubling bifurcation in smooth maps as discussed below.

In border collision pair bifurcation, the map has two fixed points for positive (respectively, negative) values of μ, and no fixed points for negative (respectively, positive) values of μ. For the μ range that two fixed points exist, one fixed point is on one side of the border and the other fixed point is on the opposite side. This is analogous to the saddle-node bifurcation in smooth systems. The border collision pair bifurcation occurs if $b < 1 < a$. There are three situations that lead to this scenario. These are summarized next (see also Figure 3).

Scenario B1 (Merging and Annihilation of Stable and Unstable Fixed Points)[3]:

$$\text{If} \quad -1 < b < 1 < a \tag{9}$$

then there is a bifurcation from no fixed point to two period-1 fixed points. In this case, there is no fixed point for $\mu < 0$ while there are two fixed points x_L^* (unstable) and x_R^* (stable) for $\mu \geq 0$ (see Figure 3 (a)). This is analogous to saddle-node bifurcation (or tangent bifurcation) in smooth maps.

Scenario B2 (Merging and Annihilation of Two Unstable Fixed Points, Plus Chaos)[4]:

$$\text{If} \quad a > 1 \quad \text{and} \quad -\frac{a}{a-1} < b < -1 \tag{10}$$

[3]In [27, 7], the terminology *No Fixed Point* → *Period-1* is used to describe this case.

[4]In [27, 7], the terminology *No Fixed Point* → *Chaos* is used to describe this case.

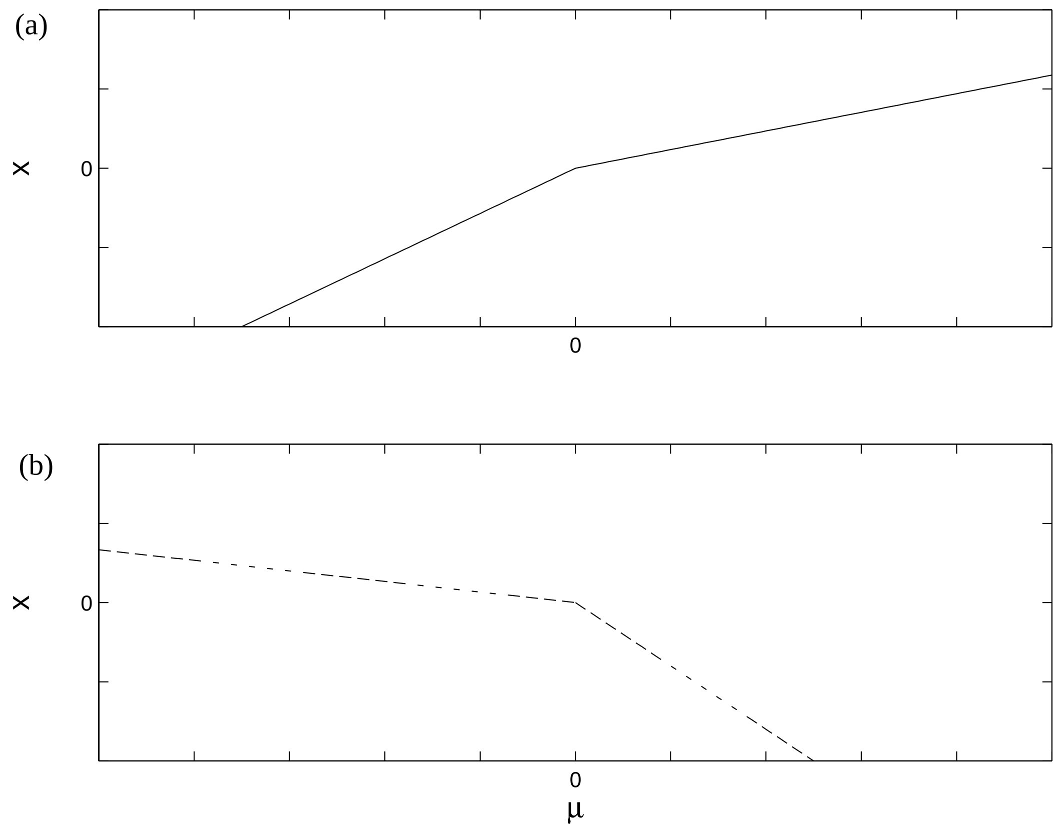

Figure 2: Bifurcation diagrams for Scenario A. A solid line represents a stable fixed point whereas a dashed line represents an unstable fixed point. (a) A typical bifurcation diagram for Scenario A1 (b) A typical bifurcation diagram for Scenario A2 i) and ii).

then there is a bifurcation from no fixed point to two unstable fixed points plus a growing chaotic attractor as μ is increased through zero (see Figure 3 (b)).

Scenario B3 (Merging and Annihilation of Two Unstable Fixed Points)[5]:

$$\text{If} \quad a > 1 \quad \text{and} \quad b < -\frac{a}{a-1} \tag{11}$$

then there is a bifurcation form no fixed point to two unstable fixed points as μ is increased through zero (see Figure 3 (c)). The system trajectory diverges for all initial conditions. Figure 3 shows typical bifurcation diagrams for Scenario B.

2.3 Scenario C. border crossing bifurcation

In border crossing bifurcation, the fixed point persists and crosses the border as μ is varied through zero. Other attractors or repellers may appear or disappear as a result of the bifurcation. Border crossing bifurcation occurs if $a > -1$ and $b < -1$. There are three situations that lead to this scenario. These are summarized next.

Scenario C1 (Supercritical Border Collision Period Doubling)[6]:

$$\text{If} \quad b < -1 < a < 0 \quad \text{and} \quad ab < 1 \tag{12}$$

then there is a bifurcation from a stable fixed point to an unstable fixed point plus a stable period-2 orbit.

Note that the condition $b < -1 < a < 0$ implies that there is a bifurcation from a stable period-1 fixed point to an unstable period-1 fixed point. The additional condition $ab < 1$ implies the emergence of a stable period-2 solution for $\mu > 0$. This bifurcation is analogous to supercritical period doubling bifurcation in smooth maps with one distinction. In smooth maps, the period doubled orbit emerges at an angle perpendicular to the path of the bifurcating fixed point, whereas in border collision period doubling, the period doubled orbit emerges from the path of the bifurcating fixed point at an acute angle.

Scenario C2 (Subcritical Border Collision Period Doubling)[7]*:*

$$\text{If} \quad b < -1 < a < 0 \quad \text{and} \quad ab > 1 \tag{13}$$

there is a bifurcation from a stable fixed point to an unstable fixed point. In this case, there is a period-1 attractor and an unstable period-2 orbit (see the proof below) for $\mu < 0$ and an unstable fixed point for $\mu > 0$. This is analogous to subcritical period doubling bifurcation in smooth maps. The system trajectory diverges to infinity for $\mu > 0$.

[5] In [27, 7], the terminology *No Fixed Point* $\rightarrow$ *No attractor* is used to describe this case.

[6] In [27, 7], the terminology *Period-1* $\rightarrow$ *Period-2* is used to describe this case.

[7] In [27, 7], the terminology *Period-1* $\rightarrow$ *No Attractor* is used to describe this case, and the bifurcation of an unstable period-2 orbit is not mentioned.

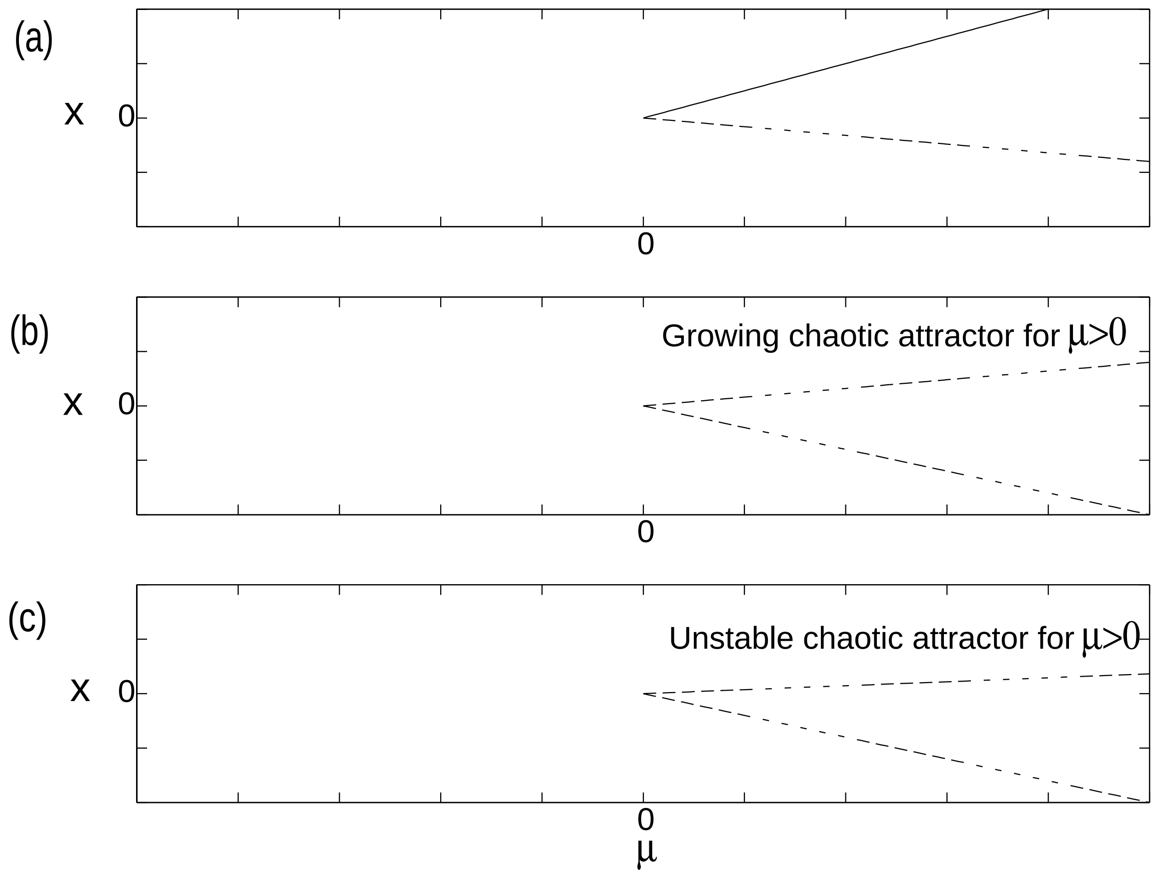

Figure 3: Bifurcation diagrams for Scenario B. A solid line represents a stable fixed point whereas a dashed line represents an unstable fixed point. (a) A typical bifurcation diagram for Scenario B1 (b) A typical bifurcation diagram for Scenario B2 (c) A typical bifurcation diagram for Scenario B3.

To show the bifurcation of a subcritical period-2 orbit for $\mu < 0$, consider the first and second return maps (for $\mu < 0$), given by

$$x_{k+1} = \begin{cases} ax_k + \mu, & x_k \leq 0 \\ bx_k + \mu, & x_k \geq 0 \end{cases} \tag{14}$$

$$x_{k+2} = \begin{cases} abx_k + \mu(1+b), & x_k \leq -\frac{\mu}{a} \\ a^2x_k + \mu(1+a), & -\frac{\mu}{a} \leq x_k \leq 0 \\ abx_k + \mu(1+a), & x_k \geq 0 \end{cases} \tag{15}$$

respectively. The first return map has a stable fixed point, $x_L^* = \frac{\mu}{1-a}$. The second return map has three fixed points one of which coincides with x_L^*. The other two fixed points are given by $x_1^* = \frac{\mu(1+b)}{1-ab}$ and $x_2^* = \frac{\mu(1+a)}{1-ab}$. The fixed points x_1^* and x_2^* are unstable (since the slope of the second return map at both fixed points is $ab > 1$). Since x_1^* and x_2^* form a period-2 orbit for the first return map, it is concluded that the normal form has an unstable period-2 orbit in addition to the stable fixed point before the border. A plot of the first and second return maps that demonstrates the existence of an unstable period-2 orbit in addition to the stable fixed point for $\mu < 0$ is shown in Figure 4.

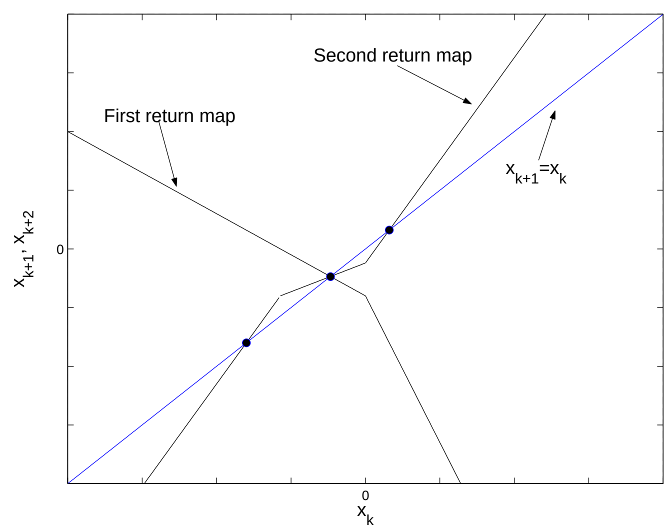

Figure 4: First and second return maps for Scenario C2, subcritical border collision period doubling ($(a, b, \mu) = (-0.7, -2.5, -0.04)$). Intersections of the maps with the line $x_{k+1} = x_k$ are the fixed points.

Scenario C3 (Emergence of Periodic or Chaotic Attractor from Stable Fixed Point)[8]:

[8]In [27, 7], the terminology *Period-1 → Periodic or Chaotic Attractor* is used to describe this case.

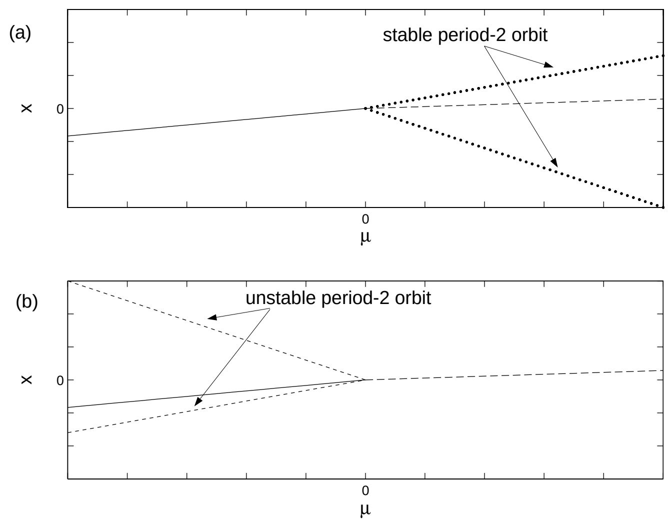

Figure 5: Typical bifurcation diagrams for Scenario C1 and Scenario C2. Solid line represents a stable fixed point whereas a dashed line represents an unstable fixed point. Dotted lines represent period-2 orbits: stable (dark) and unstable (light). (a) Supercritical border collision period doubling (Scenario C1, $b < -1 < a < 0$ and $ab < 1$) (b) Subcritical border collision period doubling (Scenario C2, $b < -1 < a < 0$ and $ab > 1$).

$$\text{If} \quad 0 < a < 1 \quad \text{and} \quad b < -1 \tag{16}$$

then there is a bifurcation from a stable fixed point to an unstable fixed point plus a period-n attractor, $n \geq 2$ or a chaotic attractor as μ is increased through zero. The specific scenario depends on the pair (a, b) as shown in Figure 6 (see [22] for details).

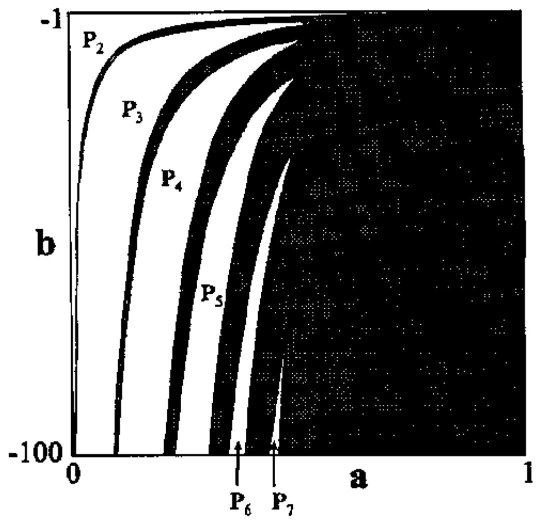

Figure 6: The bifurcation behavior describing Scenario C3 ($0 < a < 1$ and $b < -1$). Shaded regions indicate the existence of a chaotic attractor and a P_n : $n = 2, 3, \cdots, 7$, indicates the existence of a stable period-n attractor.

3 Background on Bifurcation Control and Washout Filters

A bifurcation is a qualitative change in steady state behavior resulting from small parameter changes. The parameter being varied is referred to as the bifurcation parameter. A value of the bifurcation parameter at which a bifurcation occurs is called a critical value of the bifurcation parameter. A nonlinear system operating at an equilibrium point undergoes a bifurcation when a quasistatic change in parameters causes the equilibrium to lose stability.

Bifurcation control refers to the task of designing a controller to modify the bifurcation properties of a given system to achieve some desirable dynamical behavior. Bifurcation control in smooth systems was first considered by Abed and Fu [1, 2]. Originally, their control design was based on static feedback which locally stabilized both stationary and Hopf bifurcations. Due to certain advantages it has over static feedback, washout filter-aided feedback was later used in the control design (see for example, [3, 17, 16, 26]). The main advantage of using washout filters is equilibrium preservation. Other advantages include, automatic operating point following and facilitation of the design of robust controllers. A review of bifurcation and bifurcation control can be found in [4].

In this work, washout filters are used for the first time in the control of BCB in PWS maps. A washout filter is a high frequency filter, it washes out (rejects) steady state inputs, while passes transient inputs. The dynamics of a discrete time washout filter is given by:

$$\begin{aligned} w_{k+1} &= x_k + (1-d)w_k && (17)\\ z_k &= x_k - dw_k && (18)\\ u_k &= u(z_k) && (19) \end{aligned}$$

where x_k is a state of a dynamical system to be controlled, w_k is the state of the washout filter, z_k is the output of the washout filter, and $0 < d < 2$ is the washout filter constant for a stable filter. In general, the number of washout filters needed is equal to the order of the system, although often fewer washout filters suffice.

The control law is taken as $u_k = u(z_k)$ with $u(0) = 0$. Notice that since z_k vanishes at steady state, the fixed points of the open loop system are not moved by the control. Since the normal form for BCBs is affine in the state, linear static feedback is sought to modify the bifurcation characteristics.

4 Control of Border Collision Bifurcation in 1-D Maps

In this section, control of BCBs in PWS maps of dimension one is discussed. Consider a general 1-D PWS map of the form

$$f(x,\mu) = \begin{cases} g(x,\mu), & x \le x_b \\ h(x,\mu), & x \ge x_b \end{cases} \tag{20}$$

The fact that the normal form for BCBs contains only linear terms in the state leads one to seek linear feedback controllers to modify the system's bifurcation characteristics. The linear feedback can either be applied on one side of the border and not the other, or on both sides of the border. Both approaches are considered below. The issue of which approach to take and with what constraints is a delicate one. There are practical advantages to applying a feedback on only one side of the border, say the stable side. However, this requires knowledge of where the border lies, which is not necessarily the case in practice. The purpose of pursuing stabilizing feedback acting on both sides of the border is to ensure robustness with respect to modeling uncertainty. This is done below by investigating the use of simultaneous stabilization as an option — that is, controls are sought that function in exactly the same way on both sides of the border, while stabilizing the system's behavior. Not surprisingly, the conditions for existence of simultaneously stabilizing controls are more restrictive than for the existence of one sided controls.

The notion of applying different controls on the two sides of the border was previously considered by Bernardo [10]. This method provides some flexibility in controlling both sides of the border to any desirable behavior, but knowledge of the border is needed in the design.

Here, the concept of applying the control to the stable side of the border only is considered. This method will be shown to facilitate stabilization of the system to a period-2 orbit after the BCB in cases where the uncontrolled system bifurcates to an unstable fixed point or to a chaotic attractor.

Two methods of feedback are discussed. In the first method, linear static feedback is used. This method is simple to analyze but it results in moving the system equilibria which may be viewed as a drawback. In the second method, washout filter-aided linear feedback is considered. The use of washout filters has the advantage of maintaining the fixed points of the system at the expense of increasing the system order by one.

All the developed control laws are developed for use with the normal forms of BCBs. To apply these control laws to a map not in normal form, inverse transformations need to be performed, which is straightforward for 1-D maps.

First, static feedback is considered, followed by washout filter-aided feedback.

4.1 Control of BCB in 1-D maps using static feedback

Consider the one-dimensional normal form (5) for a BCB, repeated here for convenience

$$x_{k+1} = \begin{cases} ax_k + \mu, & x_k \leq 0 \\ bx_k + \mu, & x_k \geq 0 \end{cases} \tag{21}$$

where μ is the bifurcation parameter. System (21) undergoes a variety of border collision bifurcations depending on the values of the parameters a and b as summarized in Section 2 above.

4.1.1 Method 1: Control applied on one side of border

In this control scheme, the feedback control is applied only on one side of the border. Suppose that the system is operating at a stable fixed point on one side of the border, locally as the parameter approaches a its critical value. Without loss of generality, assume this region of stable operation is $\{x : x < 0\}$— that is, assume $-1 < a < 1$. Since the control is applied only on one side of the border, the linear feedback can be applied either on the unstable side or the stable side of the border.

Method (1a): Linear feedback applied in the unstable side of the border
Recall that the fixed point is stable if $x^* \in \Re_-$ and unstable if $x^* \in \Re_+$. Applying additive linear state feedback only for $x \in \Re_+$ leads to the closed-loop system

$$x_{k+1} = \begin{cases} ax_k + \mu, & x_k \leq 0 \\ bx_k + \mu + u_k, & x_k \geq 0 \end{cases} \tag{22}$$

$$u_k = \gamma x_k \tag{23}$$

The following proposition asserts stabilizability of the border collision bifurcation with this type of control policy.

Proposition 1 *Suppose that the fixed point of (21) is stable in $\Re_-$ and unstable in $\Re_+$ (i.e., $|a| < 1$ and $|b| > 1$). Then there is a stabilizing linear feedback on the right side of the border. That is, a linear feedback exists resulting in a stable fixed point to the left and right of the border (i.e., achieving Scenario A1). Indeed, precisely those linear feedbacks $u_k = \gamma x_k$ with gain γ satisfying*

$$-1 - b < \gamma < 1 - b \tag{24}$$

are stabilizing.

Proof:
With $u_k = \gamma x_k$, the closed loop system is

$$x_{k+1} = \begin{cases} ax_k + \mu, & x_k \leq 0 \\ (b+\gamma)x_k + \mu, & x_k \geq 0 \end{cases} \tag{25}$$

For $\mu > 0$, the fixed point is $\frac{\mu}{1-(b+\gamma)} \in \Re_+$. The stability of this fixed point is determined by checking the associated eigenvalue, which is the slope of the system map in the one-dimensional case. That slope, for a fixed point in $\Re_+$, is $b + \gamma$. Thus, stability is achieved for gains γ such that $|b + \gamma| < 1$. ■

Method (1b): Linear feedback applied in the stable side of the border
For a linear feedback applied on the stable side of the border to be effective in ensuring an acceptable bifurcation, it turns out that one must assume that the open-loop system supports an unstable fixed point on the right side of the border. This is tantamount to assuming $b < -1$. Of course, the assumption $-1 < a < 1$ is still in force. Now, applying additive linear feedback in the $x < 0$ region yields the closed-loop system

$$x_{k+1} = \begin{cases} ax_k + \mu + u_k, & x_k \leq 0 \\ bx_k + \mu, & x_k \geq 0 \end{cases} \tag{26}$$

$$u_k = \gamma x_k \tag{27}$$

Notice that such a control scheme does not stabilize the unstable fixed point on the right side of the border. This is because the control has no direct effect on the system for $x > 0$. All is not lost, however. The next proposition asserts that such a control scheme may be used to stabilize the system to a period-2 solution after the border.

Proposition 2 *Suppose that the fixed point of (21) is stable in $\Re_-$ and unstable in $\Re_+$ (i.e., $|a| < 1$ and $b < -1$). Then there is a linear feedback that when applied to the left of the border (i) maintains a stable fixed point to the left of the border, and (ii) produces a stable period-2 orbit to the right of the border (i.e., the feedback achieves Scenario C1). Indeed, precisely those linear feedbacks $u_k = \gamma x_k$ with gain γ satisfying*

$$\frac{1}{b} - a < \gamma < -\frac{1}{b} - a \tag{28}$$

are stabilizing.

Proof:
The closed-loop system is given by

$$x_{k+1} = \begin{cases} (a+\gamma)x_k + \mu, & x_k \leq 0 \\ bx_k + \mu, & x_k \geq 0 \end{cases} \tag{29}$$

The fixed point to the left of the border remains stable if and only if

$$\begin{aligned} & |a+\gamma| < 1 \\ \Longleftrightarrow \quad & -1-a < \gamma < 1-a \end{aligned} \tag{30}$$

The fixed point to the right of the border remains unstable since the control is applied only in the $x < 0$ region. The closed-loop system bifurcates to a period-2 orbit as μ is increased through zero if the fixed point of the second return map x_{k+2} for $\mu > 0$, which form a period-2 orbit for the first return map, is stable. That is, if

$$\begin{aligned} & |(a+\gamma)b| < 1 \\ \Longleftrightarrow \quad & \frac{1}{b} - a < \gamma < -\frac{1}{b} - a \end{aligned} \tag{31}$$

Combining conditions (30) and (31) yields

$$\max\left\{\frac{1}{b} - a, -1-a\right\} < \gamma < \min\left\{-\frac{1}{b} - a, 1-a\right\} \tag{32}$$

Since $b < -1$, condition (32) is equivalent to

$$\frac{1}{b} - a < \gamma < -\frac{1}{b} - a \tag{33}$$

which completes the proof. ■

4.1.2 Method 2: Simultaneous stabilization

In this method, the same linear feedback control is applied additively in both the $x < 0$ and $x > 0$ regions. This leads to the closed-loop system

$$x_{k+1} = \begin{cases} ax_k + \mu + u_k, & x_k \leq 0 \\ bx_k + \mu + u_k, & x_k \geq 0 \end{cases} \tag{34}$$

$$u_k = \gamma x_k \tag{35}$$

The result is given in the following proposition.

Proposition 3 *The fixed points of the closed-loop system (34)-(35) on both sides of the border can be simultaneously stabilized using linear feedback control $u_k = \gamma x_k$ if and only if*

$$|a-b| < 2 \tag{36}$$

Indeed, precisely those linear feedbacks $u_k = \gamma x_k$ with gain γ satisfying

$$-1-b < \gamma < 1-a \tag{37}$$

are stabilizing.

Proof:
The fixed points of the closed-loop system on both sides of the border are stabilized by the feedback control $u_k = \gamma x_k$ if and only if

$$-1 < \gamma + a < 1 \qquad \text{and} \qquad -1 < \gamma + b < 1 \tag{38}$$

$$\begin{aligned} &\iff && (-1-a,\ 1-a)\bigcap(-1-b,\ 1-b) \neq \emptyset \\ &\iff && |a-b| < 2. \end{aligned} \tag{39}$$

■

Clearly, this condition is not met by all values of a and b. This condition might or might not be met for all scenarios of BCBs discussed in Section 2 above, except scenarios B2 and B3, in which it is definitely not met because $|a-b| \geq 2$.

Next, cases in which $|a-b| \geq 2$ are considered. Recall that, because of symmetry, $a-b \geq 2$ can be assumed to hold. The next proposition asserts that in this case a simultaneous linear feedback control exists that ensures the border collision bifurcation is from a stable fixed point to a stable period-2 solution (i.e., the feedback achieves Scenario C1, supercritical border collision period doubling).

Proposition 4 *Suppose $a - b \geq 2$. Then, there is a simultaneous control law that renders the BCB in the system (34)-(35) a supercritical border collision period doubling (Scenario C1). To achieve this, the control gain must be chosen to satisfy*

$$-1 < \gamma + a < 1 \qquad \textit{and} \qquad -1 < (\gamma + a)(\gamma + b) < 1 \tag{40}$$

A specific class of such control gains is $\gamma = -a + \epsilon$, with ϵ sufficiently small.

Proof:
The closed-loop system is given by

$$x_{k+1} = \begin{cases} (a+\gamma)x_k + \mu, & x_k \leq 0 \\ (b+\gamma)x_k + \mu, & x_k \geq 0 \end{cases} \tag{41}$$

The fixed point to the left of the border is stable if and only if

$$-1 < a + \gamma < 1 \tag{42}$$

Suppose the control gain γ is chosen such that (42) is satisfied. The closed loop system bifurcates to a period-2 orbit as μ is increased through zero if (i) the fixed point to the right of the border for $\mu > 0$ is unstable, and (ii) the fixed points of the second return map x_{k+2} for $\mu > 0$, which form a period-2 orbit for the first return map, is stable. That is, if

$$|b + \gamma| \ > \ 1 \tag{43}$$

and

$$-1 < (a+\gamma)(b+\gamma) \quad < \quad 1 \tag{44}$$

Condition (43) is satisfied since $a-b \geq 2$ and $-1 < a+\gamma < 1$. Thus, the closed-loop system undergoes a bifurcation from a stable fixed point to a period-2 orbit at $\mu = 0$ if

$$-1 < \gamma + a < 1 \quad \text{and} \quad -1 < (\gamma + a)(\gamma + b) < 1 \tag{45}$$

Finally, if the control gain $\gamma = -a + \epsilon$, then

$$a + \gamma = \epsilon, \tag{46}$$
$$(a+\gamma)(b+\gamma) = \epsilon(b - a + \epsilon) \tag{47}$$

Thus, the stabilizability condition (45) is satisfied for a sufficiently small ϵ. ■

The next example illustrates the use of Proposition 4.

Example 1 [21]
Consider the following simple example in normal form for border collision bifurcation

$$x_{k+1} = \begin{cases} 0.5x_k + \mu, & x_k \leq 0 \\ bx_k + \mu, & x_k \geq 0 \end{cases} \tag{48}$$

A BCB occurs as μ is increased through zero. The resulting BCB depends on the value of b [21]. For $b = -3.5$, there is a period-1 to period-3 border collision bifurcation (an instance of Scenario C3). For $b = -4.15$, there is a bifurcation from a period-1 fixed point to a "six-piece " [21] chaotic attractor. For $b = -4.44$, there is a bifurcation from a period-1 fixed point to a "three-piece " chaotic attractor. Finally, for $b = -5.5$, a period-1 fixed point produces a one-piece chaotic attractor.

Figure 7 shows the bifurcation diagrams for different values of b together with those of the controlled map using simultaneous control (with $\gamma = -0.51$ in all cases). ■

4.2 Control of BCB in 1-D maps using washout filter-aided linear feedback

Consider the one-dimensional normal form (5) for a BCB, repeated here for convenience

$$x_{k+1} = \begin{cases} ax_k + \mu, & x_k \leq 0 \\ bx_k + \mu, & x_k \geq 0 \end{cases} \tag{49}$$

where μ is the bifurcation parameter.

In the pervious section, control of border collision bifurcation using static linear feedback was considered. Static linear feedback changes the operating conditions of the open-loop

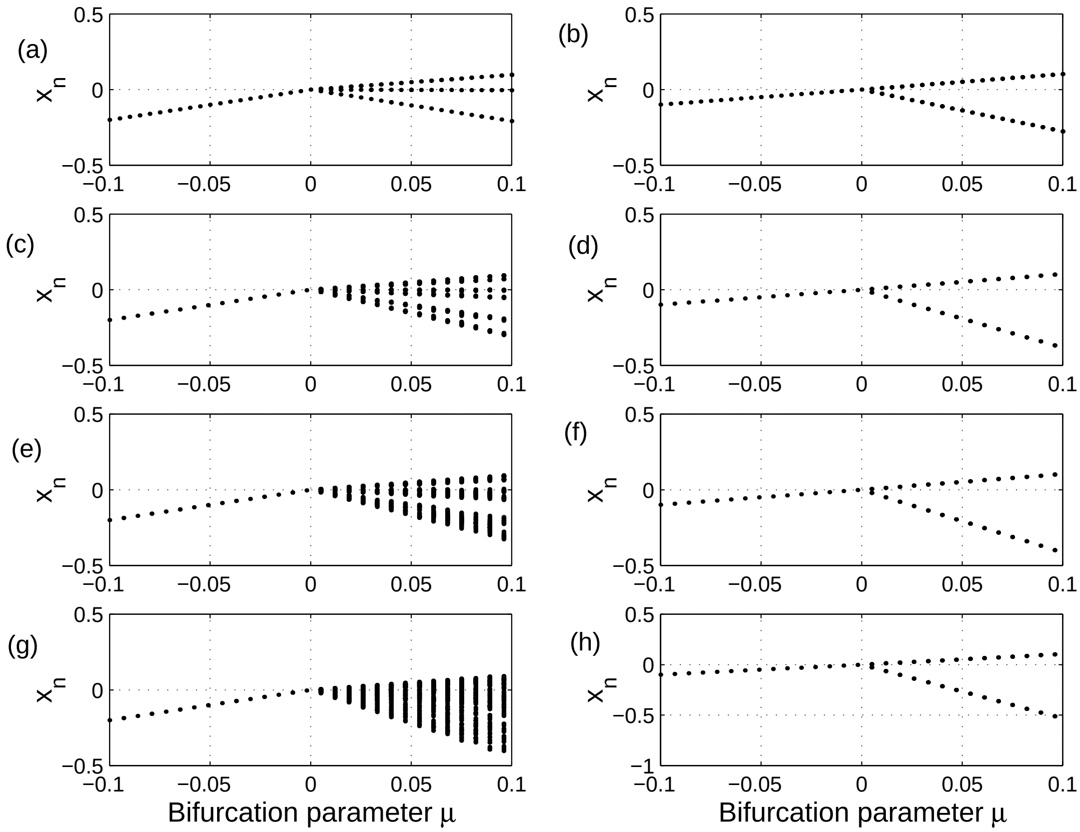

Figure 7: Bifurcation diagrams for Example 1. (a) $b = -3.5$, (c) $b = -4.15$ (e) $b = -4.44$, (g) $b = -5.5$, (b), (d), (f) and (h) are bifurcation diagrams for the corresponding closed-loop system using the same control gain $\gamma = -0.51$ in all cases.

system. This results in wasted control effort and may also result in degrading system performance. Washout filter-aided linear feedback, on the other hand, does not change the value of the fixed points of the open-loop system. Two control methods are considered as in the static linear feedback case discussed in the previous section.

Below, the Jury test for second order systems is recalled (see, for instance [18]) which will be used in the sequel. It gives necessary and sufficient conditions for Schur stability of characteristic polynomials of degree two.

Lemma 1 (Jury's Test for Second Order Systems [18]).
A necessary and sufficient condition for the zeros of the polynomial

$$p(\lambda) = a_2\lambda^2 + a_1\lambda + a_0 \tag{50}$$

($a_2 > 0$*) to lie within the unit circle is*

$$\begin{aligned} p(1) &> 0 &(51)\\ p(-1) &> 0 &(52) \end{aligned}$$

and

$$|a_0| < a_2 \tag{53}$$

4.2.1 Method 1: Control applied only on one side of the border

In this control scheme, it is assumed that the system is operating at a stable fixed point in one of the regions where the system is smooth. Without loss of generality, assume the region of stable operation is $\{x : x < 0\}$— that is, assume $-1 < a < 1$. Assume also that the system possesses an unstable fixed point in the region $x > 0$. This is tantamount to assuming $b < -1$. Two methods of applying the control are considered next. First, washout filter-aided linear feedback is applied to the unstable side, followed by the application of the control to the stable side.

Method (1a): Control applied in the unstable side of the border
Recall that the fixed point is stable if $x^* \in \Re_-$ and unstable if $x^* \in \Re_+$. Applying additive washout filter-aided linear feedback only for $x \in \Re_+$ leads to the closed-loop system

$$\begin{aligned} x_{k+1} &= \begin{cases} ax_k + \mu, & x_k \le 0 \\ bx_k + \mu + u_k, & x_k \ge 0 \end{cases} &(54)\\ w_{k+1} &= x_k + (1-d)w_k &(55)\\ z_k &= x_k - dw_k &(56)\\ u_k &= \gamma z_k &(57) \end{aligned}$$

where w_k is the washout filter state, z_k is the washout filter output, d is the washout filter constant and γ is a control parameter. Recall from Section 3 that $0 < d < 2$ for a stable washout filter.

The stability of the fixed point of the system to the right of the border is determined by the eigenvalues of the Jacobian matrix J_R in the $x > 0$ region where

$$J_R = \begin{pmatrix} b+\gamma & -\gamma d \\ 1 & 1-d \end{pmatrix} \tag{58}$$

The following proposition asserts stabilizability of the border collision bifurcation using a control policy as in (54)-(57) above.

Proposition 5 *Suppose that the fixed point of (49) is stable in $\Re_-$ and unstable in $\Re_+$ (i.e., assume $|a| < 1$ and $b < -1$). Then there is a stabilizing washout filter-aided linear feedback on the right side of the border. Such a feedback results in a stable fixed point to the left and right of the border. Indeed, precisely those washout filter-aided linear feedbacks with gain γ and washout filter constant d satisfying*

$$-1 - b + \frac{d}{2}(1+b) < \gamma \;\; < \;\; 1 - b(1-d) \quad \text{and} \quad 0 < d < \frac{4}{1-b} \tag{59}$$

are stabilizing.

Proof:
The fixed point of the closed-loop system (54)-(57) in the $x < 0$ remains stable since the control is applied in the $x > 0$ region. The fixed point of the system in the region $x > 0$ is stable if and only if the eigenvalues of the linearization in the $x > 0$ region are inside the unit circle. The characteristic equation of the Jacobian of the controlled system in the region $x > 0$ is given by

$$p(\lambda_R) := \lambda_R^2 - (b + \gamma + 1 - d)\lambda_R + b(1-d) + \gamma = 0 \tag{60}$$

By the Jury's test for second order systems, both eigenvalues are within the unit circle if and only if

$$|b(1-d) + \gamma| < 1 \tag{61}$$
$$p(1) > 0 \tag{62}$$
$$p(-1) > 0 \tag{63}$$

Conditions (61)-(63) imply

$$-1 < b(1-d) + \gamma \;\; < \;\; 1 \tag{64}$$
$$d(1-b) \;\; > \;\; 0 \tag{65}$$
$$2 + 2b + 2\gamma - d - bd \;\; > \;\; 0 \tag{66}$$

respectively. Note that inequality (65) is trivially satisfied since d is positive and $b < -1$ by hypothesis. Inequalities (64) and (66) translate to an explicit condition on γ and d as follows:

$$\max\left\{-1-b(1-d),\ \ -1-b+\frac{d}{2}(1+b)\right\} < \gamma < 1-b(1-d) \tag{67}$$

Now, $\max\left\{-1-b(1-d),\ \ -1-b+\frac{d}{2}(1+b)\right\} = -1-b+\frac{d}{2}(1+b)$, which is seen as follows:

$$\begin{aligned} & -1-b(1-d) < && -1-b+\frac{d}{2}(1+b) \\ \Longleftrightarrow \quad & bd < && \frac{d}{2}(1+b) \\ \Longleftrightarrow \quad & b < && 1 \qquad \text{(true by hypothesis, } b<-1\text{)} \end{aligned} \tag{68}$$

Hence, inequality (67) reduces to:

$$-1-b+\frac{d}{2}(1+b) < \gamma < 1-b(1-d) \tag{69}$$

Finally, for a γ satisfying (69) to exist, the upper limit in (69) must be greater than the lower limit. This is shown to be true as follows:

$$\begin{aligned} & 1-b(1-d) > && -1-b+\frac{d}{2}(1+b) \\ \Longleftrightarrow \quad & 2+bd > && \frac{d}{2}(1+b) \\ \Longleftrightarrow \quad & 2+\frac{bd}{2} > && \frac{d}{2} \\ \Longleftrightarrow \quad & d < && \frac{4}{1-b} \end{aligned} \tag{70}$$

However, $\frac{4}{1-b} < 2$, since $b < -1$. Therefore, the allowed range for d is $(0, \frac{4}{1-b})$. This completes the proof. ■

The washout filter-aided linear feedback $u_k = \gamma(x_k - dw_k)$ reduces to the static feedback $u_k = \gamma x_k$ if $d = 0$. Comparing static feedback and washout filter-aided feedback, it is easy to see that the stabilizability results obtained for the washout filter-aided feedback (59) reduce to those for static feedback for $d = 0$.

Method (1b): Control applied in the stable side of the border

Recall that the fixed point is stable in the $x < 0$ region and unstable in the $x > 0$ region (i.e., $-1 < a < 1$ and $b < -1$). Applying additive washout filter-aided linear feedback only in the $x < 0$ region leads to the closed-loop system

$$x_{k+1} = \begin{cases} ax_k + \mu + u_k, & x_k \le 0 \\ bx_k + \mu, & x_k \ge 0 \end{cases} \tag{71}$$

$$w_{k+1} = x_k + (1-d)w_k \tag{72}$$

$$z_k = x_k - dw_k \tag{73}$$

$$u_k = \gamma z_k \tag{74}$$

Such a control scheme does not stabilize the unstable fixed point to the right of the border. The reason is that the control has no direct effect on the system for $x > 0$.

The following proposition asserts that a control scheme nevertheless exists that stabilizes the system to a period-2 solution after the border.

Proposition 6 *Suppose that the fixed point of (49) is stable in $\Re_-$ and unstable in $\Re_+$ (i.e., $|a| < 1$ and $b < -1$). Then there is a washout filter-aided linear feedback, that when applied to the left of the border as in (71)-(74),*
(i) maintains a stable fixed point to the left of the border, and
(ii) produces a stable period-2 orbit to the right of the border.

Part 1 ($0 < d < 1$): *Precisely those linear feedbacks through a washout filter with gain γ satisfying*

$$\gamma_{min_1} < \gamma < \gamma_{max_1} \tag{75}$$

are stabilizing, where γ_{min_1} and γ_{max_1} are given by

$$\gamma_{min_1} = \max\left\{-1 - a + \frac{d}{2}(1+a),\ \frac{1-ab(1-d)^2}{b(1-d)},\ \frac{(1-ab)(2-d)}{b-1}\right\}, \tag{76}$$

$$\gamma_{max_1} = \min\left\{1 - a(1-d),\ \frac{(1+ab)(-1-(1-d)^2)}{b(2-d)-d},\ \frac{-1-ab(1-d)^2}{b(1-d)}\right\}. \tag{77}$$

Existence of a γ satisfying (75) is guaranteed for all d < 1 but sufficiently close to 1.

Part 2 ($d = 1$): *Precisely those linear feedbacks through a washout filter with gain γ satisfying*

$$\frac{1-ab}{b-1} < \gamma < \frac{-(1+ab)}{b-1} \tag{78}$$

are stabilizing.

Part 3 ($1 < d < 2$): *Precisely those linear feedbacks through a washout filter with gain γ satisfying*

$$\gamma_{min_2} < \gamma < \gamma_{max_2} \tag{79}$$

are stabilizing, where γ_{min_2} and γ_{max_2} are given by

$$\gamma_{min_2} = \max\left\{-1 - a + \frac{d}{2}(1+a),\ \frac{-1-ab(1-d)^2}{b(1-d)},\ \frac{(1-ab)(2-d)}{b-1}\right\}, \tag{80}$$

$$\gamma_{max_2} = \min\left\{1 - a(1-d), \frac{1-ab(1-d)^2}{b(1-d)}, \frac{(1+ab)(-1-(1-d)^2)}{b(2-d)-d}\right\}. \tag{81}$$

Existence of a γ satisfying (79) is guaranteed for all d > 1 but sufficiently close to 1.

Proof: See Appendix B. ■

Stabilizability by washout filter-aided linear feedback and static feedback are now compared. From the derivation of Proposition 6, it is easy to see that when $d = 0$, the conditions on the control gain γ reduce to

$$
\begin{aligned}
b\gamma &< 1 - ab \Longrightarrow \gamma > \frac{1}{b} - a && (82)\\
b\gamma &> -(1 + ab) \Longrightarrow \gamma < \frac{-1}{b} - a && (83)
\end{aligned}
$$

This is the stabilizability condition for the static feedback case given in Proposition 2.

4.2.2 Method 2: Simultaneous stabilization

In this method, the same washout filter-aided linear feedback control is applied in both the $x < 0$ and $x > 0$ regions. This leads to the closed-loop system

$$
\begin{aligned}
x_{k+1} &= \begin{cases} ax_k + \mu + u_k, & x_k \leq 0 \\ bx_k + \mu + u_k, & x_k \geq 0 \end{cases} && (84)\\
w_{k+1} &= x_k + (1 - d)w_k && (85)\\
z_k &= x_k - dw_k && (86)\\
u_k &= \gamma z_k && (87)
\end{aligned}
$$

Assume that the uncontrolled system (49) possesses a stable fixed point x_L^* in the region $x < 0$ and an unstable fixed point x_R^* in the region $x > 0$ (i.e., $b < -1 < a < 1$).

The following proposition gives a sufficient condition for the existence of a simultaneous washout filter-aided linear feedback control that stabilizes the fixed points of the closed loop system (84)- (87) on both sides of the border.

Remark: Simultaneous stabilization through washout filter-aided linear feedback can stabilize systems with $a - b > 2$ which cannot be done using static feedback. For example, $a = 0.9$, $b = -2.4$, $d = 0.94$ and $\gamma = 0.83$ results in a stabilizing the bifurcation to period-1 to period-1.

Proposition 7 (Simultaneous stabilization through washout filters)
Suppose $b < -1 < a < 1$. A sufficient condition for the existence of a simultaneous washout filter-aided linear feedback control that stabilizes the fixed points of the closed loop system (84)- (87) on both sides of the border is

$$
d\left(a - \frac{b}{2} - \frac{1}{2}\right) > a - b - 2 \qquad \text{and} \quad d \in (0, 1) \tag{88}
$$

The sufficient condition above is equivalent to:
If $-1 < a < 1$ and $-3 < b < -1$, then there exists a $d \in (0, 1)$ such that (88) is satisfied.

The value of d depends on the pair (a, b). For (a, b) in the region ABD of Figure 8, any $d \in (0, 1)$ is allowed. For (a, b) in the region ADE of Figure 8, only values of $d < 1$ but sufficiently close to 1 are allowed. If (88) is satisfied, then precisely those washout filter-aided linear feedbacks with gain γ satisfying

$$-1 - b + \frac{d}{2}(1 + b) < \gamma \quad < \quad 1 - a(1 - d) \tag{89}$$

are stabilizing

Proof: See Appendix C. ∎

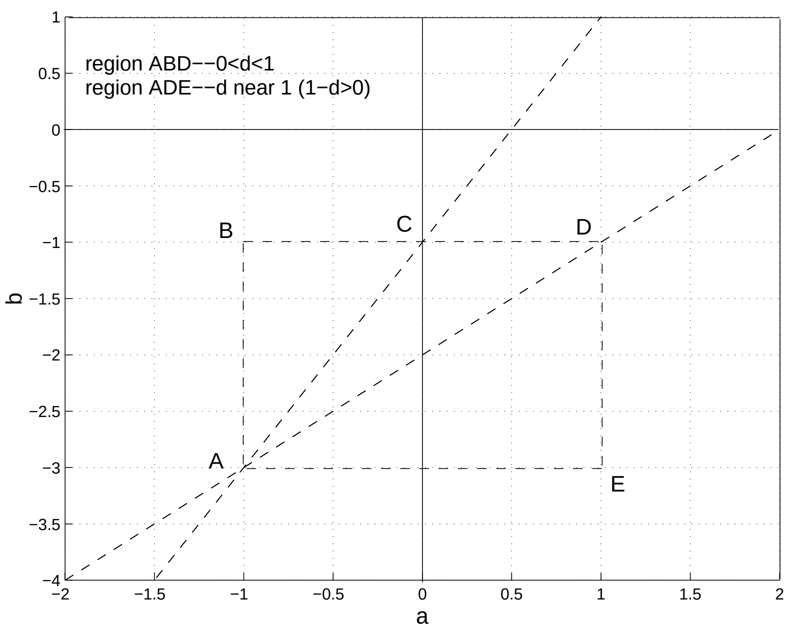

Figure 8: The region in (a, b) parameter space where a simultaneous stabilizing washout filter-aided linear feedback is guaranteed to exist is the rectangle ABDE. In the triangle ABD, such a feedback is guaranteed to exist for any d between 0 and 1. In the triangle ADE, the guarantee holds for all d between 0 and 1 but sufficiently close to 1.

Comparing the simultaneous control stabilizability results for washout filter-aided feedback with those for static feedback, it is easy to see that the conditions in Proposition 7 reduce to those of Proposition 3. In particular, setting $d = 0$, the sufficient condition for simultaneous stabilization becomes $a - b < 2$ and the control gain γ is bounded by $-1 - b < \gamma < 1 - a$ which is exactly the static feedback result.

Next, cases in which the fixed points on both sides of the border cannot be simultaneously stabilized are considered. The following proposition gives conditions under which a simultaneous linear feedback control exists that ensures the border collision bifurcation is from a stable fixed point to a stable period-2 solution (i.e., the feedback renders the BCB a supercritical border collision period doubling).

Proposition 8 *Suppose $b < -1 < a < 1$. Then a simultaneous control law through a washout filter renders the BCB in the system (84)-(87) a supercritical border collision period doubling if there exist a (γ, d) that satisfy the following inequalities*

$$-1 - a + \frac{d}{2}(1+a) < \gamma \;<\; 1 - a(1-d), \tag{90}$$

$$\gamma^2 + (a+b)(1-d)\gamma + ab(1-d)^2 - 1 \;<\; 0, \tag{91}$$

$$\gamma \;>\; \frac{(ab-1)(d-2)}{a+b-2}, \tag{92}$$

$$2\gamma^2 + ((a+b)(2-d) - 2d)\gamma + (ab+1)(1+(1-d)^2) \;>\; 0. \tag{93}$$

Part 1 ($b < -1,\ 0 \le a < 1$): *The existence of a γ satisfying (90)-(93)is guaranteed for all sufficiently small d.*

Part 2 ($-5 < b < -1,\ -1 < a < 0$): *The existence of a γ is guaranteed for all sufficiently small d.*

Part 3 ($b \le -5,\ -1 < a < 0$): *The existence of a γ is guaranteed if $b > \frac{2}{a} + a - 2$ and d is sufficiently small.*

For Parts 1-3 above, a specific class of such control gains is $\gamma = -a$ and d sufficiently small.

Proof: See Appendix D. ∎

5 Concluding Remarks

In this paper, a detailed development of possible stabilizing feedback controls for border collision bifurcations has been given. The paper focused on one-dimensional piecewise smooth systems undergoing various types of border collision bifurcations. A careful summary of the basic results on border collision bifurcations was given, which is hoped to be of value independent of the control results. The feedback control designs included static feedback as well as washout filter-aided dynamic feedbacks. The design of the two types of feedback was detailed, and the differences between them were studied. Robustness with respect to uncertainty in the border was addressed using a simultaneous stabilization formulation of the problem. The results are obtained with a focus on systems in normal form. Further work is needed to extend the calculations to systems before transformation of variables to the normal form, and to systems of higher order.

Appendices

A A Transformation to the 2-D Normal Form for BCB

It is straightforward to show that any system of the form

$$\begin{pmatrix} \bar{x}_{k+1} \\ \bar{y}_{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}}_{A} \begin{pmatrix} \bar{x}_k \\ \bar{y}_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu \tag{94}$$

with $a_3 \neq 0$ can be transformed to the 2-D normal form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \tau & 1 \\ -\delta & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu \tag{95}$$

using the following simple transformation

$$x_k = T\bar{x}_k \tag{96}$$

$$T = \begin{pmatrix} 1 & \frac{a_4}{a_3} \\ 0 & -\frac{\delta}{a_3} \end{pmatrix} \tag{97}$$

where τ :=trace$(A) = a_1 + a_4$ and δ :=det$(A) = a_1 a_4 - a_2 a_3$.

B Proof of Proposition 6

The closed loop system (71)-(74) can be written in vector form as follows:

$$\begin{pmatrix} x_{k+1} \\ w_{k+1} \end{pmatrix} = \begin{cases} \underbrace{\begin{pmatrix} a+\gamma & -\gamma d \\ 1 & 1-d \end{pmatrix}}_{A} \begin{pmatrix} x_k \\ w_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \leq 0 \\ \underbrace{\begin{pmatrix} b & 0 \\ 1 & 1-d \end{pmatrix}}_{B} \begin{pmatrix} x_k \\ w_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \geq 0 \end{cases} \tag{98}$$

Let τ_L :=trace$(A) = a + \gamma + 1 - d$, δ_L :=det$(A) = a(1-d) + \gamma$, τ_R :=trace$(B) = b + 1 - d$ and δ_R =det$(B) = b(1-d)$. Recall that the convention is to use the subscript L for left and R for right side of the border.

Conditions on the gain γ and the washout filter constant d are obtained from conditions on the stability of the fixed point to the left of the border and the stability of a period-2 orbit in the $x > 0$ region. To facilitate the analysis, system (98) can be transformed to a simpler form using the similarity transformation given in Appendix A. The system becomes[9]

$$\begin{pmatrix} \tilde{x}_{k+1} \\ \tilde{w}_{k+1} \end{pmatrix} = \begin{cases} J_L \begin{pmatrix} \tilde{x}_k \\ \tilde{w}_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & \tilde{x}_k \leq 0 \\ J_R \begin{pmatrix} \tilde{x}_k \\ \tilde{w}_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & \tilde{x}_k \geq 0 \end{cases} \tag{99}$$

where

$$J_L = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \tag{100}$$

and

$$J_R = \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix} \tag{101}$$

The characteristic equation of J_L is

$$\lambda_L^2 - \tau_L \lambda_L + \delta_L = 0 \tag{102}$$

By the Jury's test for second order systems, the fixed point to the left of the border remains stable if and only if

$$\begin{aligned} -1 < \delta_L &< 1 \Longrightarrow -1 < a(1-d) + \gamma < 1 & (103) \\ 1 - \tau_L + \delta_L &> 0 \Longrightarrow d(1-a) > 0 & (104) \\ 1 + \tau_L + \delta_L &> 0 \Longrightarrow 2 + 2\gamma + 2a - d - ad > 0 & (105) \end{aligned}$$

Inequality (104) is trivially satisfied and inequalities (103) and (105) translate to conditions on the parameters (γ, d) as follows:

$$\max\left\{-1 - a(1-d), \ -1 - a + \frac{d}{2}(1+a)\right\} < \gamma \ < \ 1 - a(1-d) \tag{106}$$

It is easy to show that $\max\left\{-1 - a(1-d), \ -1 - a + \frac{d}{2}(1+a)\right\} = -1 - a + \frac{d}{2}(1+a)$ by the following reasoning:

$$\begin{aligned} & -1 - a(1-d) < && -1 - a + \tfrac{d}{2}(1+a) \\ \Longleftrightarrow \quad & ad < && \tfrac{d}{2}(1+a) \\ \Longleftrightarrow \quad & a < && 1 \quad \text{(true by hypothesis, } -1 < a < 1) \end{aligned} \tag{107}$$

Therefore, condition (106) reduces to:

$$-1 - a + \frac{d}{2}(1+a) < \gamma \ < \ 1 - a(1-d) \tag{108}$$

[9]Since similarity transformations do not change eigenvalues, the stability conditions are independent of the system coordinates used.

Also, the system has a stable period-2 solution for $\mu > 0$ if the fixed points of the second return map which form a period-2 orbit for the first return map are stable. The Jacobian of the second return map is the product of Jacobians evaluated at a point in $\Re_-$ and another in $\Re_+$:

$$J = J_L J_R = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix} = \begin{pmatrix} \tau_L \tau_R - \delta_R & \tau_L \\ -\delta_L \tau_R & -\delta_L \end{pmatrix} \tag{109}$$

Let $\delta :=\det(J) = \delta_R \delta_L$ and $\tau :=\text{trace}(J) = \tau_L \tau_R - \delta_R - \delta_L$. Applying the Jury's test for second order systems, the fixed points of the second return map that form a period-2 orbit for the first return map are stable if and only if

$$\begin{aligned} -1 < \delta_R \delta_L &< 1 && (110)\\ \tau_R \tau_L &< (1+\delta_R)(1+\delta_L) && (111)\\ \tau_R \tau_L &> -(1-\delta_R)(1-\delta_L) && (112) \end{aligned}$$

Substituting the expressions for δ_L, τ_L, δ_R and τ_R in (110)-(112) yields the following conditions on the parameters γ and d

$$\begin{aligned} -1 - ab(1-d)^2 < b(1-d)\gamma &< 1 - ab(1-d)^2 && (113)\\ (b-1)\gamma &< (1-ab)(2-d) && (114)\\ (b(2-d)-d)\gamma &> (1+ab)(-1-(1-d)^2) && (115) \end{aligned}$$

Consider the cases $0 < d < 1$, $d = 1$ and $1 < d < 2$ separately.
<u>**Case 1** $(0 < d < 1)$:</u>
Since $b < -1$, the coefficients of γ in (113)-(115) are all negative: $b(1-d) < 0$, $b - 1 < 0$ and $b(2-d) - d < 0$. Therefore, inequalities (113)-(115) can be written as:

$$\begin{aligned} \frac{1-ab(1-d)^2}{b(1-d)} < \gamma &< \frac{-1-ab(1-d)^2}{b(1-d)} && (116)\\ \gamma &> \frac{(1-ab)(2-d)}{b-1} && (117)\\ \gamma &< \frac{(1+ab)(-1-(1-d)^2)}{b(2-d)-d} && (118) \end{aligned}$$

Combining conditions (108) and (116)-(118) yields the condition

$$\gamma_{min_1} < \gamma < \gamma_{max_1} \tag{119}$$

where

$$\begin{aligned} \gamma_{min_1} &= \max\left\{-1-a+\frac{d}{2}(1+a),\ \frac{1-ab(1-d)^2}{b(1-d)},\ \frac{(1-ab)(2-d)}{b-1}\right\}, && (120)\\ \gamma_{max_1} &= \min\left\{1-a(1-d),\ \frac{(1+ab)(-1-(1-d)^2)}{b(2-d)-d},\ \frac{-1-ab(1-d)^2}{b(1-d)}\right\}. && (121) \end{aligned}$$

One must show that there exists a $d \in (0,1)$ such that γ_{max_1} is greater than γ_{min_1}. To this end, let $d = 1-\epsilon$ with $\epsilon > 0$ and small. Substituting $1-d=\epsilon$ in (120)-(121) gives:

$$\begin{aligned}
\gamma_{min_1} &= \max\left\{-\frac{1}{2}(1+a)(1+\epsilon),\ \frac{1}{\epsilon b}-\epsilon a,\ \frac{(1-ab)(1+\epsilon)}{b-1}\right\}, && (122)\\
\gamma_{max_1} &= \min\left\{1-\epsilon a, \frac{-(1+ab)(1+\epsilon^2)}{b(1+\epsilon)-1+\epsilon},\ -\frac{1}{\epsilon b}-\epsilon a\right\}. && (123)
\end{aligned}$$

The equations for γ_{min_1} and γ_{max_1} can be simplified by comparing the terms inside the max / min operators. Since $b < -1 < a < 1$, $\frac{1}{\epsilon b} - \epsilon a < -\frac{1}{2}(1+a)(1+\epsilon)$ for a sufficiently small ϵ. Also $-\frac{1}{2}(1+a)(1+\epsilon) < \frac{(1-ab)(1+\epsilon)}{b-1}$ which can be seen as follows:

$$\begin{aligned}
& -\frac{1}{2}(1+a)(1+\epsilon) < \frac{(1-ab)(1+\epsilon)}{b-1}\\
\iff & -\tfrac{1}{2}(1+a) < \frac{(1-ab)}{b-1}\\
\iff & (b-1)(1+a) < -2+2ab\\
\iff & \underbrace{(b+1)}_{<0}\underbrace{(1-a)}_{>0} < 0
\end{aligned} \tag{124}$$

Therefore, the equation for γ_{min_1} reduces to:

$$\gamma_{min_1} = \frac{(1-ab)(1+\epsilon)}{b-1}, \tag{125}$$

Next, by comparing the terms of γ_{max_1}, it is easy to see that $1-\epsilon a < -\frac{1}{\epsilon b} - \epsilon a$ for a sufficiently small ϵ. Also it is straightforward to show that $\dfrac{-(1+ab)(1+\epsilon^2)}{b(1+\epsilon)-1+\epsilon} < 1-\epsilon a$ for a sufficiently small ϵ. This follows from:

$$\begin{aligned}
& \frac{-(1+ab)(1+\epsilon^2)}{b(1+\epsilon)-1+\epsilon} < 1-\epsilon a\\
\iff & -1-ab-\epsilon^2-\epsilon^2 ab > b(1+\epsilon)-1+\epsilon-\epsilon ab-\epsilon^2 ab+\epsilon a-\epsilon^2 a\\
\iff & -a(b+\epsilon)(1-\epsilon) > (b+\epsilon)(1+\epsilon)\\
\iff & \underbrace{(b+\epsilon)}_{<0}\underbrace{(1+\epsilon+a(1-\epsilon))}_{>0} < 0
\end{aligned} \tag{126}$$

Therefore, the equation for γ_{max_1} reduces to:

$$\gamma_{max_1} = \frac{-(1+ab)(1+\epsilon^2)}{b(1+\epsilon)-1+\epsilon}. \tag{127}$$

Finally, it can be shown that γ_{min_1} as given in (125) is smaller than γ_{max_1} in (127) for a sufficiently small ϵ:

$$\begin{aligned}
& & \frac{(1-ab)(1+\epsilon)}{b-1} &< \frac{-(1+ab)(1+\epsilon^2)}{b(1+\epsilon)-1+\epsilon} \\
&\iff & (1+\epsilon\underbrace{\frac{b+1}{b-1}}_{\in(0,1)})(1-ab) &> -(1+ab)\underbrace{\frac{1+\epsilon^2}{1+\epsilon}}_{\approx 1} \qquad (128)\\
&\iff & 1-ab &> -1-ab
\end{aligned}$$

Case 2 ($d=1$):

Consider inequalities (108) and (113)-(115). Substituting $d=1$, inequality (108) becomes

$$-\frac{1}{2}(1+a) < \gamma \quad < \quad 1 \qquad (129)$$

Also inequality (113) is satisfied for any γ and inequalities (114)-(115) reduce to

$$\frac{1-ab}{b-1} \quad < \quad \gamma < \frac{-(1+ab)}{b-1} \qquad (130)$$

Inequalities (129)-(130) translate to a single inequality as follows:

$$\max\left\{-\frac{1}{2}(1+a),\ \frac{1-ab}{b-1}\right\} < \gamma \quad < \quad \min\left\{1,\ \frac{-(1+ab)}{b-1}\right\} \qquad (131)$$

By recalling that $\frac{-(1+ab)}{b-1} < 1$ (see (126) with $\epsilon = 0$) and $\frac{1-ab}{b-1} > -\frac{1}{2}(1+a)$ (see (124)), inequality (131) reduces to:

$$\frac{1-ab}{b-1} \quad < \quad \gamma < \frac{-(1+ab)}{b-1} \qquad (132)$$

Finally, the upper limit in (132) on γ is greater than the lower limit using the logic in (128) with $\epsilon = 0$.

It is concluded that for $d = 1$, a control gain γ satisfying (132) renders the bifurcation a supercritical border collision period doubling.

Case 3 ($1 < d < 2$):

Consider inequalities (108) and (113)-(115). Since $b < -1$, the coefficient of γ in (113) is positive and the coefficients of γ in (114)-(115) are both negative. Therefore, inequalities (113)-(115) can be written as:

$$\frac{-1-ab(1-d)^2}{b(1-d)} < \gamma \quad < \quad \frac{1-ab(1-d)^2}{b(1-d)} \qquad (133)$$

$$\gamma \quad > \quad \frac{(1-ab)(2-d)}{b-1} \qquad (134)$$

$$\gamma \quad < \quad \frac{(1+ab)(-1-(1-d)^2)}{b(2-d)-d} \qquad (135)$$

Combining condition (108) with (133)-(135) yields the condition

$$\gamma_{min_2} < \gamma < \gamma_{max_2} \tag{136}$$

where

$$\begin{aligned}
\gamma_{min_2} &= \max\left\{-1-a+\frac{d}{2}(1+a),\ \frac{-1-ab(1-d)^2}{b(1-d)},\ \frac{(1-ab)(2-d)}{b-1}\right\}, &(137)\\
\gamma_{max_2} &= \min\left\{1-a(1-d), \frac{1-ab(1-d)^2}{b(1-d)}, \frac{(1+ab)(-1-(1-d)^2)}{b(2-d)-d}\right\}. &(138)
\end{aligned}$$

Next, one must show that there is a washout filter constant $d \in (1,2)$ such that the upper limit in (136) is greater than the lower limit. To this end, let $d = 1+\epsilon$ with $\epsilon > 0$ and small enough. Substituting $1-d = -\epsilon$ in (137)-(138) gives:

$$\begin{aligned}
\gamma_{min_2} &= \max\left\{-\frac{1}{2}(1+a)(1-\epsilon),\ \frac{1}{\epsilon b}+\epsilon a,\ \frac{(1-ab)(1-\epsilon)}{(b-1)}\right\}, &(139)\\
\gamma_{max_2} &= \min\left\{1+\epsilon a, \frac{-1}{\epsilon b}+\epsilon a, \frac{-(1+ab)(1+\epsilon^2)}{b(1-\epsilon)-1-\epsilon}\right\}. &(140)
\end{aligned}$$

The terms in braces in Eq. (139) of γ_{min_2} can be seen to satisfy $\frac{1}{\epsilon b}+\epsilon a < -\frac{1}{2}(1+a)(1-\epsilon)$ for a sufficiently small ϵ and $-\frac{1}{2}(1+a)(1-\epsilon) < \frac{(1-ab)(1-\epsilon)}{(b-1)}$ (see (124)). Thus, γ_{min_2} becomes:

$$\gamma_{min_2} = \frac{(1-ab)(1-\epsilon)}{b-1}. \tag{141}$$

Similarly, the terms in braces in Eq. (140) of γ_{max_2} can be seen to satisfy $1+\epsilon a < \frac{-1}{\epsilon b}+\epsilon a$ for small ϵ since b is negative ($b < -1$). Also, $\frac{-(1+ab)(1+\epsilon^2)}{b(1-\epsilon)-1-\epsilon} < 1+\epsilon a$ (the proof is similar to the logic in (126)). Thus, γ_{max_2} becomes:

$$\gamma_{max_2} = \frac{-(1+ab)(1+\epsilon^2)}{b(1-\epsilon)-1-\epsilon}. \tag{142}$$

It remains to show that, $\gamma_{min_2} < \gamma_{max_2}$ for the expressions in (141) and (142), for sufficiently small ϵ.

$$\begin{aligned}
\frac{(1-ab)(1-\epsilon)}{b-1} &< \frac{-(1+ab)(1+\epsilon^2)}{b(1-\epsilon)-1-\epsilon}\\
\iff \quad (1-\epsilon\underbrace{\frac{b+1}{b-1}}_{\in(0,1)})(1-ab) &> -(1+ab)\underbrace{\frac{1+\epsilon^2}{1-\epsilon}}_{\approx 1} &(143)\\
\iff \quad 1-ab &> -1-ab
\end{aligned}$$

C Proof of Proposition 7

The closed loop system (84)-(87) can be written in vector form as follows:

$$\begin{pmatrix} x_{k+1} \\ w_{k+1} \end{pmatrix} = \begin{cases} \underbrace{\begin{pmatrix} a+\gamma & -\gamma d \\ 1 & 1-d \end{pmatrix}}_{A} \begin{pmatrix} x_k \\ w_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \le 0 \\ \underbrace{\begin{pmatrix} b+\gamma & -\gamma d \\ 1 & 1-d \end{pmatrix}}_{B} \begin{pmatrix} x_k \\ w_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & x_k \ge 0 \end{cases} \tag{144}$$

Let $\tau_L :=$trace$(A) = a+\gamma+1-d$, $\delta_L :=$det$(A) = a(1-d)+\gamma$, $\tau_R :=$trace$(B) = b+\gamma+1-d$ and $\delta_R :=$det$(B) = b(1-d)+\gamma$.

To facilitate the analysis, system (144) can be transformed to a simpler form using the similarity transformation given in Appendix A. The system becomes[10]

$$\begin{pmatrix} \tilde{x}_{k+1} \\ \tilde{w}_{k+1} \end{pmatrix} = \begin{cases} J_L \begin{pmatrix} \tilde{x}_k \\ \tilde{w}_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & \tilde{x}_k \le 0 \\ J_R \begin{pmatrix} \tilde{x}_k \\ \tilde{w}_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & \tilde{x}_k \ge 0 \end{cases} \tag{145}$$

where

$$J_L = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}, \tag{146}$$

and

$$J_R = \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}. \tag{147}$$

The characteristic equations of J_L and J_R are given by

$$\lambda_L^2 - (a+\gamma+1-d)\lambda_L + a(1-d) + \gamma = 0, \tag{148}$$

$$\lambda_R^2 - (b+\gamma+1-d)\lambda_R + b(1-d) + \gamma = 0 \tag{149}$$

respectively. Applying the Jury's test for second order systems, the left and right fixed points are seen to be stable if and only if

$$-1 < a(1-d) + \gamma \;<\; 1 \tag{150}$$

$$d(1-a) \;>\; 0 \tag{151}$$

$$2 + 2a + 2\gamma - d - ad \;>\; 0 \tag{152}$$

[10]Since similarity transformations do not change eigenvalues, the stability conditions are independent of the system coordinates used.

and

$$
\begin{aligned}
-1 < b(1-d) + \gamma &< 1 & (153)\\
d(1-b) &> 0 & (154)\\
2 + 2b + 2\gamma - d - bd &> 0 & (155)
\end{aligned}
$$

respectively.

Inequalities (151) and (154) are trivially satisfied since $d > 0$ and $b < -1 < a < 1$ (by hypothesis). Conditions (150),(152) and (153),(155) are equivalent to

$$
-1 - a + \frac{d}{2}(1+a) < \gamma < 1 - a(1-d), \quad 0 < d < \min\left\{2, \frac{4}{1-a}\right\} = 2 \tag{156}
$$

and

$$
-1 - b + \frac{d}{2}(1+b) < \gamma < 1 - b(1-d), \quad 0 < d < \min\left\{2, \frac{4}{1-b}\right\} = \frac{4}{1-b} \tag{157}
$$

respectively. Recall (see the proof of Proposition 5) that the upper limits on d are required to ensure that the upper limits on γ in (156) and (157) are greater than the corresponding lower limits on γ. Combining (156) and (157) gives

$$
\begin{aligned}
\max\{-1 - a + \frac{d}{2}(1+a), -1 - b + \frac{d}{2}(1+b)\} < \ &\gamma < \min\{1 - a(1-d), 1 - b(1-d)\}, & (158)\\
0 < \ &d < \frac{4}{1-b} & (159)
\end{aligned}
$$

Since $b < -1 < a < 1$, it is straightforward to show that conditions (158)-(159) reduce to

$$
-1 - b + \frac{d}{2}(1+b) < \gamma < \gamma_{max} \quad \text{and} \quad 0 < d < \frac{4}{1-b} \tag{160}
$$

where

$$
\gamma_{max} = \begin{cases} 1 - a(1-d), & 0 < d \le 1 \\ 1 - b(1-d), & 1 \le d < 2 \end{cases} \tag{161}
$$

Finally, for a control gain γ satisfying (160) to exist, one must find a condition under which the upper limit in (160) is greater than the lower limit. It is straightforward to show that $-1 - b + \frac{d}{2}(1+b) < 1 - a(1-d)$ if $(a-b)(1-d) + \frac{d}{2}(1-b) < 2$ and $1 - b(1-d) > -1 - b + \frac{d}{2}(1+b)$ if $0 < d < \frac{4}{1-b}$. Thus, the following condition is a sufficient condition for the existence of a simultaneously stabilizing control law:

$$
(a-b)(1-d) + \frac{d}{2}(1-b) < 2 \quad \text{and} \quad d \in (0,1) \tag{162}
$$

Next, the sufficient condition (162) is written explicitly in terms of a and b. Note that (162) can be written as:

$$d(a - \frac{b}{2} - \frac{1}{2}) > a - b - 2 \qquad \text{and} \quad d \in (0,1) \tag{163}$$

Case 1: If $a - \dfrac{b}{2} - \dfrac{1}{2} > 0$, then inequality (163) is equivalent to $d > \dfrac{a-b-2}{a - \frac{b}{2} - \frac{1}{2}}$. Since d must lie in $\in (0, 1)$, the following must be satisfied:

$$\frac{a-b-2}{a - \frac{b}{2} - \frac{1}{2}} < 1 \tag{164}$$

Case 1.1: Inequality (164) is satisfied if $\dfrac{a-b-2}{a - \frac{b}{2} - \frac{1}{2}} < 0$, which is valid if $a - b - 2 < 0$. In this case, any $d \in (0, 1)$ is allowed. The constraints $a - \frac{b}{2} - \frac{1}{2} > 0$, $a - b - 2 < 0$ and $b < -1 < a < 1$ correspond to the open region ACD in Figure 8.
Case 1.2: If $\dfrac{a-b-2}{a - \frac{b}{2} - \frac{1}{2}} < 1$, then values of d near 1 are allowed. Note that, $\dfrac{a-b-2}{a - \frac{b}{2} - \frac{1}{2}} < 1$ $\iff a - b - 2 < a - \dfrac{b}{2} - \dfrac{1}{2} \iff b > -3$. The constraints $b < 2a - 1$, $b > a - 2$ and $-3 < b < -1 < a < 1$ correspond to the open region $ACDE$ in Figure 8.
Case 2: If $a - \frac{b}{2} - \frac{1}{2} = 0$, then for (163) to be satisfied, one needs $a - b - 2 < 0$. Therefore, the intersection of the regions satisfied by $a - \frac{b}{2} - \frac{1}{2} = 0$ and $a - b - 2 < 0$ in the (a, b) parameter space is the line segment connecting the points AC in Figure 8.
Case 3: Suppose that $a - \frac{b}{2} - \frac{1}{2} < 0$. Then, inequality (163) is equivalent to

$$d < \frac{a-b-2}{a - \frac{b}{2} - \frac{1}{2}} \tag{165}$$

Case 3.1: Inequality (165) is satisfied if $\dfrac{a-b-2}{a - \frac{b}{2} - \frac{1}{2}} > 0$, which is valid if $a - b - 2 < 0$. In this case, values of d near 0 are allowed. The constraints $a - \frac{b}{2} - \frac{1}{2} < 0$, $a - b - 2 < 0$ and $b < -1 < a < 1$ correspond to the open region ABC in Figure 8.
Case 3.2: If $\dfrac{a-b-2}{a - \frac{b}{2} - \frac{1}{2}} \geq 1$, then all values of $d \in (0, 1)$ are allowed. Note that, $\dfrac{a-b-2}{a - \frac{b}{2} - \frac{1}{2}} \geq 1$ $\iff a - b - 2 \leq a - \frac{b}{2} - \frac{1}{2} \iff b \geq -3$. The constraints $b < 2a - 1$, $b > a - 2$ and $-3 \leq b < -1 < a < 1$ correspond to the region ABC as in Figure 8.

D Proof of Proposition 8

Consider the closed-loop system (84)-(87) written in a new coordinates (see Appendix C)

$$\begin{pmatrix} \tilde{x}_{k+1} \\ \tilde{w}_{k+1} \end{pmatrix} = \begin{cases} J_L \begin{pmatrix} \tilde{x}_k \\ \tilde{w}_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & \tilde{x}_k \leq 0 \\ J_R \begin{pmatrix} \tilde{x}_k \\ \tilde{w}_k \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, & \tilde{x}_k \geq 0 \end{cases} \tag{166}$$

where

$$J_L = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}, \tag{167}$$

and

$$J_R = \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}. \tag{168}$$

The parameters of J_L and J_R are given by: $\tau_L = a + \gamma + 1 - d$, $\delta_L = a(1-d) + \gamma$, $\tau_R = b + \gamma + 1 - d$ and $\delta_R = b(1-d) + \gamma$.

Conditions on the gain γ and the washout filter constant d are obtained from conditions on the stability of the fixed point in the $x < 0$ region and the stability of a period-2 orbit in the $x > 0$ region.

Recall (see the derivation of Eq. (156) above) that the fixed point of the closed loop system to the left of the border remains stable if and only if

$$-1 - a + \frac{d}{2}(1+a) < \gamma \quad < \quad 1 - a(1-d), \qquad 0 < d < 2 \tag{169}$$

Also, the system has a stable period-2 solution for $\mu > 0$ if the fixed points of the second return map that correspond to a period-2 orbit of the first return map are stable. The Jacobian of the second return map is the product of Jacobians evaluated at a point in $\Re_-$ and another in $\Re_+$:

$$J = J_L J_R = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix} = \begin{pmatrix} \tau_L\tau_R - \delta_R & \tau_L \\ -\delta_L\tau_R & -\delta_L \end{pmatrix} \tag{170}$$

Let $\delta := \det(J) = \delta_R\delta_L$ and $\tau := \mathrm{trace}(J) = \tau_L\tau_R - \delta_R - \delta_L$. Applying the Jury's test for second order systems, the eigenvalues of J are inside the unit circle if and only if

$$\begin{aligned} -1 < \delta_R\delta_L &< 1 && (171) \\ \tau_R\tau_L &< (1+\delta_R)(1+\delta_L) && (172) \\ \tau_R\tau_L &> -(1-\delta_R)(1-\delta_L) && (173) \end{aligned}$$

Substituting the expressions for δ_L, τ_L, δ_R and τ_R in (171)-(173) gives

$$-2 < \gamma^2 + (a+b)(1-d)\gamma + ab(1-d)^2 - 1 \quad < \quad 0, \tag{174}$$

$$\gamma > \frac{(ab-1)(d-2)}{a+b-2} := \gamma_3, \tag{175}$$

$$2\gamma^2 + ((a+b)(2-d) - 2d)\gamma + (ab+1)(1+(1-d)^2) \quad > \quad 0. \tag{176}$$

The discriminant of the quadratic equation in (174) is given by

$$\begin{aligned} q \quad &:= \quad ((a+b)(1-d))^2 - 4(ab(1-d)^2 - 1) &(177)\\ &= \quad ((a-b)(1-d))^2 + 4 &(178) \end{aligned}$$

which is always positive. Thus, inequality (174) implies

$$\gamma_1 < \gamma < \gamma_2 \tag{179}$$

where

$$\begin{aligned} \gamma_1 \quad &= \quad -\frac{1}{2}(a+b)(1-d) - \frac{1}{2}\sqrt{((a-b)(1-d))^2 + 4}, &(180)\\ \gamma_2 \quad &= \quad -\frac{1}{2}(a+b)(1-d) + \frac{1}{2}\sqrt{((a-b)(1-d))^2 + 4}. &(181) \end{aligned}$$

Combining (169), (175)-(176) and (179)-(181), yields the following conditions on the parameters (γ, d)

$$\max\{\gamma_1, \gamma_3, -1-a+\frac{d}{2}(1+a)\} < \gamma < \min\{\gamma_2, 1-a(1-d)\} \tag{182}$$

and

$$\gamma^2 + \frac{1}{2}((a+b)(2-d)-2d)\gamma + \frac{1}{2}(ab+1)(1+(1-d)^2) > 0, \quad 0 < d < 2 \tag{183}$$

Next, conditions on d are sought under which a γ exists such that (169), (174)-(176) (equivalently (182)-(183)) are satisfied. Equations (174)-(176) can be written as:

$$-2 < \gamma^2 + (a+b)\gamma - d(a+b)\gamma + ab - 2dab + d^2ab - 1 \quad < \quad 0, \tag{184}$$

$$\gamma > \frac{(ab-1)(d-2)}{a+b-2}, \tag{185}$$

$$2\gamma^2 + 2(a+b)\gamma - d(a+b)\gamma - 2d\gamma + 2ab - 2dab + d^2ab + 2 - 2d + d^2 \quad > \quad 0. \tag{186}$$

Let $\gamma = -a$ and d be sufficiently small. Then, inequalities (169), (184) and (186) are satisfied as shown below:

$$\left.\begin{array}{lr} \text{Inequality (169)} \implies & -a\underbrace{-1+\frac{d}{2}(1+a)}_{<0} < -a < \quad -a + \underbrace{1+ad}_{>0} \\ \text{Inequality (184)} \implies & \underbrace{da(a-b+db)}_{small} -1 < \quad 0 \\ \text{Inequality (186)} \implies & \underbrace{d(a^2-ab+2a+dab-2+d)}_{small} +2 > \quad 0 \end{array}\right\} \tag{187}$$

Finally, it remains to check whether inequality (185) is satisfied or not:

$$-a > \frac{(ab-1)(d-2)}{a+b-2} \Longleftrightarrow -a\underbrace{(a-b)}_{>0} - \underbrace{2(1-a)}_{>0} - \underbrace{d(ab-1)}_{small} < 0 \tag{188}$$

Clearly, inequality (188) is satisfied for $0 \leq a < 1$. For $-1 < a < 0$, inequality (188) can be written as

$$\begin{aligned} & b\underbrace{a(1-d)}_{<0} \;<\; a^2 + 2(1-a) - d \\ \Longleftrightarrow \quad & b \;>\; \frac{a^2+2(1-a)-d}{a(1-d)} \approx \frac{2}{a} + a - 2 \quad \text{for a sufficiently small } d \end{aligned} \tag{189}$$

Since $\sup_{-1<a<0} \frac{2}{a} + a - 2 = -5$, a sufficient condition for (189) to be satisfied is $b > -5$. Thus, a stabilizing control law that renders the bifurcation a supercritical border collision period doubling exists for $0 \leq a < 1$, $b < -1$ and $-1 < a < 0$, $\frac{a^2+2(1-a)-d}{a(1-d)} < b < -1$.

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