TECHNICAL RESEARCH REPORT

The Effect of Positive Correlations on Buffer Occupancy: Comparison and Lower Bounds via Supermodular Ordering

by Sarut Vanichpun

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ABSTRACT

Title of Thesis: THE EFFECTS OF POSITIVE CORRELATIONS ON BUFFER OCCUPANCY: COMPARISON AND LOWER BOUNDS VIA SUPERMODULAR ORDERING

Degree candidate: Sarut Vanichpun

Degree and year: Master of Science, 2001

Thesis directed by: Professor Armand Makowski
Department of Electrical and Computer Engineering and
Institute for Systems Research

We use recent advances from the theory of multivariate stochastic orderings to formalize the "folk theorem" to the effect that positive correlations leads to increased buffer occupancy and larger buffer levels at a discrete time multiplexer queue of infinite capacity. We do so by comparing input sequences in the supermodular (sm) ordering and the corresponding buffer contents in the increasing convex (icx) ordering, respectively.

Three popular classes of (discrete-time) traffic models are considered here, namely, the fractional Gaussian noise (FGN) traffic model, the on-off source model and the $M|G|\infty$ traffic model. The independent version of an input process in each of these classes of traffic models is a member of the same class. We show

that this independent version is smaller than the input sequence itself and that the corresponding buffer content processes are ordered in the same direction. For each traffic model, we show by simulations that the first and second moments of buffer levels are ordered in agreement with the comparison results.

The more general version of the folk theorem, namely "the larger the positive correlations of input traffic, the higher the buffer occupancy levels" is established in some cases. For the FGN traffic models, we show that the process with higher Hurst parameter is larger than the process with smaller Hurst parameter. In the case of the $M|G|\infty$ model, the effect of session-duration variability is discussed and the comparison result is obtained in the bivariate case.

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Sarut Vanichpun

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Advisory Committee:

Professor Armand Makowski, Chair
Professor Prakash Narayan
Professor Richard La

DEDICATION

To my parents

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Chapter 1

Introduction

1.1 Buffer provisioning

A basic design problem in the engineering of store-and-forward networks is buffer provisioning, namely the determination of buffer sizes at various network nodes. This question is often addressed through the analysis of an appropriate queueing system. The simplest of models operates in discrete time and considers a flow of packets arriving in a finite buffer with a capacity of at most B packets; packets are transmitted out of the buffer in order of arrival over a communication link of constant rate. More precisely, with time organized in contiguous slots of identical duration, let Q_t^B denote the number of packets still present in the system at the beginning of time slot $[t, t+1)$ and let A_t denote the number of new packets arriving into the buffer during that slot. If the buffer output link can transmit c packets/slot, then the buffer content evolves according to the recursion

$$Q_{t+1}^B = \min\left\{B, [Q_t^B + A_t - c]^+\right\}, \quad t = 0, 1, \ldots \tag{1.1}$$

for some given intial condition Q_0^B; we take $Q_0^B = 0$ for concreteness [1]. If the input sequence $\{A_t,\ t = 0, 1, \ldots\}$ is stationary and ergodic, then the system eventually reaches a statistical equilibrium or steady-state regime in that [2] $Q_t^B \Longrightarrow_t Q^B$ for some random variable (rv) Q^B [3].

Determining the distribution of Q^B is a natural step towards the evaluation of key design quantities such as the blocking probability and the packet loss rate. Indeed, in the stationary ergodic framework, these quantities are readily given by

$$\lim_{t\to\infty} \frac{1}{t+1} \sum_{s=0}^{t} \mathbf{1}\left[Q_s^B = B\right] = \mathbf{P}\left[Q^B = B\right] \tag{1.2}$$

and

$$\lim_{t\to\infty} \frac{\sum_{s=0}^{t} [[Q_s^B + A_s - c]^+ - B]^+}{\sum_{s=0}^{t} A_s} = \frac{\mathbf{E}\left[[[Q^B + A_1 - c]^+ - B]^+\right]}{\mathbf{E}\left[A_1\right]}, \tag{1.3}$$

respectively. Evaluating the right handside of (1.2) and (1.3) is often a very difficult task; closed form solutions are available in only a few instances of input sequences $\{A_t,\ t = 0, 1, \ldots\}$, and numerical techniques need to be developed to handle most cases of practical interest.

However, in many situations (e.g., ATM networks), the blocking probability and cell loss rate assume acceptable levels only when B is large. With this in mind, it is reasonable to look instead at the corresponding infinite buffer system ($B = \infty$) associated with (1.1). The evolution of the buffer content sequence

[1] Other models of buffer behavior are possible but will not be pursued here.

[2] The notation used in this thesis is collected in Section 2.1.

[3] The existence of Q^B is always guaranteed in the stationary and ergodic framework, but additional assumptions are required to have uniqueness and independence with respect to the initial condition.

$\{Q_t,\ t = 0, 1, \ldots\}$ is now governed by the Lindley recursion

$$Q_{t+1} = [Q_t + A_t - c]^+, \quad t = 0, 1, \ldots \tag{1.4}$$

for some given initial condition Q_0, say $Q_0 = 0$ for concreteness. It is well known [27] that if the input sequence $\{A_t,\ t = 0, 1, \ldots\}$ is stationary and ergodic with $\mathbf{E}[A_1] < c$, then the system will reach statistical equilibrium, i.e., $Q_t \Longrightarrow_t Q$ for some $\mathbb{R}_+$-valued rv Q.

The relevance of this approach is reinforced by the observation that the upper bounds $\mathbf{P}\left[Q_t^B = B\right] \leq \mathbf{P}[Q_t \geq B]$ are valid for all $t = 0, 1, \ldots$ and all B [4]. This fact translates to steady-state under the appropriate conditions, so that

$$\mathbf{P}\left[Q^B = B\right] \leq \mathbf{P}[Q \geq B], \quad B \geq 0. \tag{1.5}$$

As argued earlier, we need secure reasonably good approximations to the blocking probability $\mathbf{P}\left[Q^B = B\right]$ only for large B. Hence, as engineering designs tend to be conservative, (1.5) suggests that this objective can be achieved by evaluating the upper bound $\mathbf{P}[Q \geq B]$ for large B.

1.2 Dependencies in network traffic

In the past decade, this evaluation task has been the subject of intense investigations in the wake of several traffic measurement studies which have concluded to the "failure of Poisson modeling [42]" of traditional traffic models. As the data set collected at BellCore [22] and a large number of following measurement studies have by now indicated, network traffic exhibits time dependencies at much higher

[4] A direct sample path comparison can be used to show recursively that $Q_t^B \leq Q_t$ for all $t = 0, 1, \ldots$ provided $Q_0^B \leq Q_0$.

time scales than had been traditionally observed. This *long-range dependence* has been detected in a wide range of networking applications and over multiple networking infrastructures, e.g., Ethernet LANs [14, 22, 54], VBR traffic [9, 16, 36], Web traffic [11] and WAN traffic [42].

Long-range dependence amounts to the correlations in the packet stream spanning over multiple timescales. More precisely, these long-range dependent processes have hyperbolic decaying correlation structures (slow decaying) which are non-summable. This is expected to affect performance in a manner drastically different from that predicted by (traditional) summable correlation structures which typically arise in Markovian traffic models and Poisson-like source. This state of affairs has generated a strong interest in a number of alternative traffic models which capture the (long-range) dependencies; good surveys are available in [15, 30]. Suggested models are on-off sources [17, 22, 19], fractional Brownian motion (FBM) processes [8, 35, 34], fractional Auto Regressive Integrated Moving Average (f-ARIMA) processes [18], $M|G|\infty$ input processes [38, 39, 42] and etc.

Under these new models, the corresponding buffer distribution displays much heavier tails than the exponential tails which typically appear in short-range dependent Markovian models. Thus, from these analyses emerges the recommendations that buffers in networks carrying long-range dependent traffic should be provisioned more generously than in networks with short-range dependent traffic.

1.3 Positive correlations

The recommendation above is based on asymptotic results of the form

$$\lim_{B \to \infty} \frac{1}{v(B)} \ln \mathbf{P}[Q > B] = -\gamma \tag{1.6}$$

with constant $\gamma > 0$ and monotone function $v : (0, \infty) \to (0, \infty)$ increasing at infinity. Of course, γ and v are determined through the statistics of the input sequence $\{A_t, t = 0, 1, \ldots\}$ to the buffer [36] – Typical examples include $v(B) = B, v(B) = B^\beta$ $(0 < \beta < 1)$ and $v(B) = \ln B$.

Thus, (1.6) implies tails of the form

$$\mathbf{P}[Q > B] \sim e^{-v(B)(\gamma + o(1))} \quad (B \to \infty) \tag{1.7}$$

but more detailed information on the tail of Q is usually not available as closed-form expressions are simply not known, or hard to come by due to the inherent computational complexity of these models. However, for the traffic models for which (1.6) has been developed, these asymptotics already suggest the following: Assume the input process $\{A_t, t = 0, 1, \ldots\}$ to be *positively correlated*, say associated [Definition 4.5.1], and let $\{\hat{A}_t, t = 0, 1, \ldots\}$ denote its independent version [Definition 4.3.2]. Then, the corresponding buffer content processes $\{Q_t, t = 0, 1, \ldots\}$ and $\{\hat{Q}_t, t = 0, 1, \ldots\}$ are "ordered" in some suitable sense, and $\hat{Q}$ is "smaller" than Q where $\hat{Q}$ and Q denote the steady state versions (whenever appropriate). In other words, positive correlations lead to increased buffer occupancy and larger buffer levels.

This "folk theorem" has been observed by others, e.g., the simulation study in [26] with the help of the TES modeling tool. When Large Deviations arguments are used to validate (1.6) with $v(B) = B$, then γ can often be related to the

Large Deviations functional of the input sequence $\{A_t, t = 0, 1, \ldots\}$, and under association, it is easy to see that

$$\lim_{B\to\infty} \frac{1}{B} \ln \mathbf{P}\left[\hat{Q} > B\right] = -\hat{\gamma} \tag{1.8}$$

with $\gamma \le \hat{\gamma}$. Consequently, $\mathbf{P}\left[\hat{Q} > B\right]$ is less than $\mathbf{P}\left[Q > B\right]$ for large values of B.

1.4 Overview

In this thesis, we consider the "folk theorem" on a more formal basis with the help of recent advances from the theory of multivariate stochastic orderings [29, 47]: First, we compare the input sequence and its independent version using the supermodular (sm) ordering [Definition 4.3.1] which is well suited for capturing the positive dependence in the components of a random vector. From this comparison, we then can compare the corresponding buffer contents in the increasing convex (icx) ordering [Definition 4.1.2]. Unlike the sm ordering, the icx ordering formalizes comparability in terms of size and variability.

In our discussion, we consider three versatile, mathematically convenient and widely used models, namely, fractional Gaussian noise (FGN) model, on-off sources and $M|G|\infty$ input processes. The results we obtain for these classes of traffic models can be briefly described as follows: If $\{A_t, t = 0, 1, \ldots\}$ and $\{\hat{A}_t, t = 0, 1, \ldots\}$ denote the input traffic process and its independent version, then we can conclude that

$$\{\hat{A}_t, t = 0, 1, \ldots\} \le_{sm} \{A_t, t = 0, 1, \ldots\} \tag{1.9}$$

where the independent version is also a member of the *same* class of traffic models as the input traffic process. This comparison (1.9) implies a similar comparison in the increasing directionally convex (idcx) ordering [Definition 4.3.1 and (4.2)] and thus by our main theorem [Theorem 5.1.1], the corresponding buffer content processes $\{Q_t, t = 0, 1, \ldots\}$ and $\{\hat{Q}_t, t = 0, 1, \ldots\}$ are icx ordered with

$$\hat{Q}_t \leq_{icx} Q_t, \quad t = 0, 1, \ldots, \tag{1.10}$$

provided $\hat{Q}_0 = Q_0$. Furthermore, under the stationarity assumption of the input traffic and stability condition ($\mathbf{E}[A_0] < c$), the steady state comparison

$$\hat{Q} \leq_{icx} Q \tag{1.11}$$

can be derived through (1.10) provided $\hat{Q}_0 = Q_0 = 0$ [Theorem 5.2.1]. In other words, the independent version does act as a lower bound process, and (1.9), (1.10) and (1.11) yield a formalization of the "folk theorem" mentioned above for these classes of traffic models.

The passage from (1.9) to (1.10) is simply a consequence of the properties of the sm ordering (and of its close cousin the idcx ordering) [Theorem 5.1.1]. The key idea behind the comparison (1.9) is the property of positive dependence, known as *sequentially stochastically increasing* (SSI). As shown by Meester and Shantihikumar [29], this property provided a sufficient condition for (1.9) to hold [Theorem 4.5.1]. While these ideas are applied without too much difficulties to the FGN traffic model, the analysis for the on-off sources and $M|G|\infty$ traffic models is more elaborate. In the case of on-off sources, conditions on the distribution of the on- and off-periods are needed to achieve (1.9) (and resp. (1.10)). For the $M|G|\infty$ process, it is not clear whether the process is SSI or not. However, by the properties of sm and idcx orderings, a decomposition of the $M|G|\infty$ process into independent components allows us to obtain (1.9).

Besides the folk theorem mentioned above, we also consider a more general version of the folk theorem, namely "the larger the positive correlations of input traffic, the higher the buffer occupancy levels." This problem can be formalized as followed: For a given traffic model, if the input traffic $\{A_t^2, t = 0, 1, \ldots\}$ is more "correlated" than the input traffic $\{A_t^1, t = 0, 1, \ldots\}$ (in some sense), then it is desirable to establish the order relation

$$\{A_t^1, t = 0, 1, \ldots\} \leq_{sm} \{A_t^2, t = 0, 1, \ldots\} \tag{1.12}$$

as suggested by the main theorem [Theorem 5.1.1] in order to obtain the comparison for the corresponding buffer levels. For FGN traffic models, we show that higher Hurst parameter results in higher size and variability of the buffer levels. In the case of $M|G|\infty$ processes, if the session duration of $\{A_t^2, t = 0, 1, \ldots\}$ is more variable than that of $\{A_t^1, t = 0, 1, \ldots\}$, then it is reasonable to expect that (1.12) holds. Unfortunately, we are unable to establish (1.12) in this case and the problem remains open for future research. However, we do get some insight into this problem by proving the comparison in the bivariate case, i.e,

$$(A_0^1, A_t^1) \leq_{sm} (A_0^2, A_t^2), \quad t = 1, 2, \ldots. \tag{1.13}$$

Lastly, we conjecture for the case of on-off sources that the comparison (1.12) should hold if the on-period duration distribution of $\{A_t^2, t = 0, 1, \ldots\}$ is more variable than that of $\{A_t^1, t = 0, 1, \ldots\}$.

1.5 Thesis organization

The thesis is organized as follows: Chapter 2 collects the notation used in this thesis and basic facts on $\mathbb{N}$-valued rvs, exponential rvs and discrete-Pareto rvs. Chapter 3 summarizes basic definitions and facts of three traffic models, namely, fractional Gaussian noise (FGN) traffic model, on-off sources and $M|G|\infty$ input traffic. These traffic models are well suited to model the long-range dependent traffic, e.g. FGN with Hurst parameter is $0.5 < H < 1$ [1], the on-off sources with Pareto-like activity periods [19], and $M|G|\infty$ with Pareto-like session duration [38].

In Chapter 4, we introduce the notion of stochastic orderings of random vectors and random sequences with an emphasis on multivariate orderings that capture the dependence structure among the components of random vectors. Then, the key notion of sequentially stochastically increasing (SSI) property is presented and its relationship with the sm ordering is shown. In Chapter 5, we formulate the buffer sizing problem and apply the property of the idcx ordering to obtain the main theorem [Theorem 5.1.1]. The translation of the results from transient to steady state is also discussed.

We give the comparison and simulation results of FGN traffic model in Chapter 6. Chapter 7 and 8 contain the discussions of SSI conditions for stationary on-off sources and non-stationary on-off sources, respectively. We then confirm the comparison results of on-off sources by simulations in Chapter 9. Moreover, in Chapter 10, we extend the result of a single on-off source to the superposition of N independent on-off sources. Under some enforced assumptions, as N goes to infinity, the limiting process of the superposition of N i.i.d. on-off sources converges in distribution to an $M|G|\infty$ process. From this approach, we establish the comparison between the $M|G|\infty$ process and its independent version under the condition that the session duration rv and its forward recurrence are DFR rvs.

Finally, in Chapter 11 we develop the comparison between the $M|G|\infty$ input process and its independent version using an *independent decomposition*.

We conclude to the same result as in Chapter 10 but without any condition on the session duration rv. Furthermore, the effect of session-duration variability is discussed and the comparison in the case of two-dimensional rvs (1.13) is established. Lastly, we confirm the comparison between the $M|G|\infty$ input process and its independent version using simulations in Section 11.4.

Chapter 2

Notation and Basic Facts

2.1 Notation

A few words on the notation used in this thesis:

A scalar x in $\mathbb{R}$ is denoted by a regular font, while a vector $\mathbf{x}$ in $\mathbb{R}^n$ is written in a bold font. For any scalar x in $\mathbb{R}$, we write x^+ to denote $\max(0, x)$. For any two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, let $\leq$ and $<$ denote the coordinatewise orderings in $\mathbb{R}^n$ such that if $\mathbf{x} \leq \mathbf{y}$, then $x_i \leq y_i$ for $i = 1, \ldots, n$, and similarly, if $\mathbf{x} < \mathbf{y}$, then $x_i < y_i$ for $i = 1, \ldots, n$. Moreover, for $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ in $\mathbb{R}^n$, we write $[\mathbf{x}, \mathbf{y}] \leq \mathbf{z}$ if $\mathbf{x} \leq \mathbf{z}$ and $\mathbf{y} \leq \mathbf{z}$. Also, let $\wedge$ and $\vee$ denote the coordinatewise minimum and maximum, respectively, i.e., for any two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, $\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$ (respectively, $\mathbf{x} \vee \mathbf{y} = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$).

All random variables (rvs) are defined on some probability triple $(\Omega, \mathcal{F}, \mathbf{P})$, with $\mathbf{E}$ denoting the corresponding expectation operator. Usually, unless specified otherwise, we use upper case letters (e.g., X, Y) to denote rvs. Moreover, random vectors are denoted by bold upper case letters (e.g., $\mathbf{X}, \mathbf{Y}$). Two rvs X and Y are said to be *equal in law* if they have the same distribution, a fact we denote by $X =_{st} Y$. Weak convergence is denoted by $\Longrightarrow_N$ (with N going to infinity).

Finally, for any vector μ in $\mathbb{R}^n$ and for any symmetric non-negative definite $n \times n$ matrix $\mathbf{\Sigma} = (\Sigma_{ij})$, we write $\mathbf{X} =_{st} \mathcal{N}(\mu, \mathbf{\Sigma})$ to indicate that the $\mathbb{R}^n$-valued rv $\mathbf{X}$ is normally distributed with mean vector μ and covariance matrix $\mathbf{\Sigma}$.

2.2 Basic facts on $\mathbb{N}$-valued rvs

We begin with a few definitions: For any $\mathbb{N}$-valued rv X, we define

$$\mathcal{S}(X) := \{t = 0, 1, \ldots : \ \mathbf{P}[X \geq t] > 0\}.$$

Obviously, $\mathcal{S}(X)$ is not empty as it contains $t = 0$, and is of the form $\{0, 1, \ldots, T_X\}$ for some integer T_X (possibly infinite). More precisely, we have

$$T_X = \sup\{t = 0, 1, \ldots : \mathbf{P}[X \geq t] > 0\}.$$

Recall that the failure rate function and the residual life function of an $\mathbb{N}$-valued rv X are defined by

$$h_X(t) := \frac{\mathbf{P}[X = t]}{\mathbf{P}[X \geq t]}, \quad t = 1, \ldots, T_X$$

and

$$r_X(t) := \frac{\mathbf{P}[X \geq t + 1]}{\mathbf{P}[X \geq t]} = 1 - h_X(t), \quad t = 1, \ldots, T_X,$$

respectively[1]. We say that the rv X is increasing failure rate (IFR) (resp. decreasing failure rate (DFR)) if the mapping $\{1, \ldots, T_X\} \to \mathbb{R}_+ : t \to h_X(t)$ is increasing (resp. decreasing).

If the $\mathbb{N}$-valued rv X has finite mean, we define the forward recurrence time $\hat{X}$ to be the $\mathbb{N}$-valued rv with pmf given by

$$\mathbf{P}\left[\hat{X} = t\right] = \frac{\mathbf{P}[X \geq t]}{\mathbf{E}[X]}, \quad t = 0, 1, \ldots \tag{2.1}$$

[1]In most cases of interest, X is a $\{1, 2, \ldots\}$-valued rv. Therefore, we define the domain of h_X and r_X as the set $\{1, \ldots, T_X\}$.

Note that

$$\mathbf{P}\left[\hat{X} \geq t\right] = \mathbf{E}\left[X\right]^{-1} \sum_{s=t}^{\infty} \mathbf{P}\left[X \geq s\right], \quad t = 0, 1, \ldots \tag{2.2}$$

so that $\mathbf{P}\left[\hat{X} \geq t\right] = 0$ if and only if $\mathbf{P}\left[X \geq t\right] = 0$, and we conclude $\mathcal{S}(\hat{X}) = \mathcal{S}(X)$.

The next lemma provides a simple characterization of the DFR (resp. IFR) property of $\hat{X}$.

Lemma 2.2.1 *For any* $\mathbb{N}$*-valued rv* X *with finite mean, the corresponding* $\mathbb{N}$*-valued rv* $\hat{X}$ *is DFR (resp. IFR) if and only if*

$$h_{\hat{X}}(t+1) \leq (\text{resp. } \geq)\ h_X(t), \quad t = 1, \ldots, T_X - 1. \tag{2.3}$$

Proof. Fix $t = 1, \ldots, T_X - 1$. Combining the definition of $h_{\hat{X}}(t)$ with (2.1) and (2.2), we first obtain

$$h_{\hat{X}}(t) = \frac{\mathbf{P}\left[X \geq t\right]}{\sum_{s=t}^{\infty} \mathbf{P}\left[X \geq s\right]} \tag{2.4}$$

so that

$$\begin{aligned} \mathbf{P}\left[X \geq t\right] &= \mathbf{P}\left[X = t\right] + \mathbf{P}\left[X \geq t+1\right] \\ &= h_X(t)\mathbf{P}\left[X \geq t\right] + h_{\hat{X}}(t+1) \sum_{s=t+1}^{\infty} \mathbf{P}\left[X \geq s\right]. \end{aligned} \tag{2.5}$$

Upon substituting (2.5) into (2.4), we get

$$h_{\hat{X}}(t) \sum_{s=t}^{\infty} \mathbf{P}\left[X \geq s\right] = h_X(t)\mathbf{P}\left[X \geq t\right] + h_{\hat{X}}(t+1) \sum_{s=t+1}^{\infty} \mathbf{P}\left[X \geq s\right]$$

and rearranging we find

$$(h_{\hat{X}}(t+1) - h_X(t))\mathbf{P}\left[X \geq t\right] = (h_{\hat{X}}(t+1) - h_{\hat{X}}(t)) \sum_{s=t}^{\infty} \mathbf{P}\left[X \geq s\right].$$

The desired conclusion now follows. ■

2.3 Geometric and discrete-Pareto rvs

In this section, we discuss facts and properties of two well-known rvs namely, geometric and discrete-Pareto rvs.

For $0 < \rho < 1$, an $\{1, 2, \ldots\}$-valued rv X is said to be a geometric rv with parameter ρ if it is distributed according to the pmf

$$\mathbf{P}\left[X = k\right] = \rho^{k-1}(1-\rho), \quad k = 1, 2, \ldots,$$

in which case we write $X =_{st} \mathcal{G}(\rho)$. It is plain that

$$\mathbf{E}\left[X\right] = \frac{1}{1-\rho} \quad \text{and} \quad h_X(t) = 1 - \rho, \quad t = 1, 2, \ldots. \tag{2.6}$$

Let $\hat{X}$ be the forward recurrence associated with X. By its definition in (2.1), the rv $\hat{X} =_{st} \mathcal{G}(\rho)$, thus $h_{\hat{X}}(t) = 1 - \rho$. Therefore, both X and $\hat{X}$ are DFR.

A rv Y is said to have a discrete-Pareto distribution with parameter $1 < \alpha \leq 2$ if its pmf is given by

$$\mathbf{P}\left[Y = k\right] = k^{-\alpha} - (k+1)^{-\alpha}, \quad k = 1, 2, \ldots, \tag{2.7}$$

so that

$$\mathbf{P}\left[Y \geq k\right] = k^{-\alpha}, \quad k = 1, 2, \ldots. \tag{2.8}$$

We denote this rv Y by $\mathcal{P}(\alpha)$. It is known that when $1 < \alpha < 2$, the discrete-Pareto rv has finite mean but infinite variance. From (2.7) and (2.8), we have

$$\mathbf{E}\left[Y\right] = \sum_{k=1}^{\infty} k^{-\alpha} \quad \text{and} \quad h_Y(t) = 1 - (\frac{t}{t+1})^{\alpha}, \quad t = 1, 2, \ldots. \tag{2.9}$$

Moreover, its forward recurrence rv $\hat{Y}$ has pmf

$$\mathbf{P}\left[\hat{Y} = t\right] = \frac{t^{-\alpha}}{\sum_{k=1}^{\infty} k^{-\alpha}}, \quad t = 1, 2, \ldots, \tag{2.10}$$

and hazard rate function

$$h_{\hat{Y}}(t) = \frac{t^{-\alpha}}{\sum_{k=t}^{\infty} k^{-\alpha}}, \quad t = 1, 2, \ldots. \tag{2.11}$$

Clearly, the rv Y is DFR from the expression (2.9) of $h_Y(t)$. In order to show the DFR property of $\hat{Y}$, define the sequence of mappings $\{f_K : \mathbb{R}_+ \to \mathbb{R}, K = 1, 2, \ldots\}$ such that for each $K = 1, 2, \ldots$,

$$f_K(t) = \frac{t^{-\alpha}}{\sum_{k=0}^{K} (t+k)^{-\alpha}}, \quad t > 0. \tag{2.12}$$

For fixed $K = 1, 2, \ldots$, the mapping f_K is decreasing since by evaluating the first derivative of f_K, we have

$$\begin{aligned} \frac{df_K(t)}{dt} &= \frac{\alpha t^{-\alpha}}{(\sum_{k=0}^{K} (t+k)^{-\alpha})^2} [-\sum_{k=0}^{K} t^{-1}(t+k)^{-\alpha} + \sum_{k=0}^{K} (t+k)^{-\alpha-1}] \\ &= \frac{\alpha t^{-\alpha}}{(\sum_{k=0}^{K} (t+k)^{-\alpha})^2} \sum_{k=0}^{K} (t+k)^{-\alpha}((t+k)^{-1} - t^{-1}) \leq 0, \quad t > 0. \end{aligned}$$

Thus, it holds that for all $K = 1, 2, \ldots$,

$$f_K(t+1) \leq f_K(t), \quad t = 1, 2, \ldots. \tag{2.13}$$

Note from (2.11) that $h_{\hat{Y}}(t) = \lim_{K \to \infty} f_K(t)$ for all $t = 1, 2, \ldots$. By letting K go to infinity in (2.13), we conclude that $\hat{Y}$ is DFR.

Chapter 3

Traffic Models

Over the past decade, network traffic in both LANs and WANs, has been shown to possess long-range dependence [22, 42]. As a result, traditional traffic models, which mostly characterize short-range dependent traffic, cannot be used for evaluating the performance of today's networks. Many new traffic models have been proposed [22, 36, 42] to capture the long-range dependence property. In this thesis, we consider three traffic models, namely, the fractional Gaussian noise model, the on-off source model, and the $M|G|\infty$ input traffic model. These three models are described in details in Sections 3.1-3.3.

3.1 Fractional Gaussian noise (FGN)

The fractional Gaussian noise is a Gaussian process which is strictly self-similar (Appendix A). A detailed treatment of fractional Gaussian noise (and its close cousin, fractional Brownian motion) can be found in the monograph [44], and their use for traffic modeling is discussed in [34, 49, 55]. For $0 < H < 1$, any zero-mean Gaussian random process $\{B_H(t), t \in \mathbb{R}\}$ with autocorrelation function

$$R_H(t,s) \equiv Cor(B_H(t), B_H(s))$$

$$= \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}Var(B_H(1)), \quad t, s \in \mathbb{R} \qquad (3.1)$$

is called a *fractional Brownian motion (FBM)* with Hurst parameter H. *Fractional Gaussian noise (FGN)* $\{N_t^H, t = 0, 1, \ldots\}$ is now defined by

$$N_t^H \equiv B_H(t+1) - B_H(t), \quad t = 0, 1, \ldots.$$

Since FBM has stationary increments, the rvs $\{N_t^H, t = 0, 1, \ldots\}$ form a zero-mean stationary Gaussian sequence with autocovariance function

$$\begin{aligned} \Gamma_H(k) &\equiv cov(N_t^H, N_{t+k}^H) \\ &= \frac{\sigma^2}{2}(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}), \quad k = 0, 1, \ldots \qquad (3.2) \end{aligned}$$

where $\sigma^2 \equiv Var(N_t^H) = Var(B_H(1))$. We refer to the sequence $\{N_t^H, t = 0, 1, \ldots\}$ by FGN(H).

In this thesis, we consider H only in the range $0.5 \le H < 1$, which corresponds to positive correlations as was found appropriate for network traffic modeling. It is easy to see that when $H = 0.5$, $\Gamma_H(k) = 0$ for all $k = 1, 2, \ldots$, and $\{N_t^H, t = 0, 1, \ldots\}$ is then a sequence of i.i.d. Gaussian random variables. However, when $0.5 < H < 1$, the asymptotics [44]

$$\Gamma_H(k) \sim \sigma^2 H(2H-1)k^{2H-2} \quad \text{as} \quad k \to \infty$$

show that FGN(H) exhibits long-range dependence. It is also clear from condition (A.5), (A.6) and (3.2) that FGN(H) is an exactly second-order self-similar process, thus a self-similar process, since it is a Gaussian process.

The FGN(H) traffic model we use as the input traffic $\{A_t^H, t = 0, 1, \ldots\}$ in this thesis will be of the form

$$A_t^H = m + N_t^H, \quad t = 0, 1, \ldots$$

where m is the average rate of the traffic. The process $\{A_t^H, t = 0, 1, \ldots\}$ is also a Gaussian process with $\mathbf{E}\left[A_t^H\right] = m$ for all $t = 0, 1, \ldots$, and its autocovariance function is still given by (3.2). Therefore, $\{A_t^H, t = 0, 1, \ldots\}$ is a self-similar process.

3.2 On-off sources

A discrete time independent on-off source is defined as a succession of cycles where each cycle contains an off period followed by an on period. In the off period, the source is silent while during the on period the source is active and produces traffic at a constant rate. For simplicity, we set this rate to be unity, say one cell/slot. The first on and off period durations, denoted B_0 and I_0, are $\mathbb{N}$-valued rvs. The following on period durations $\{B_n, n = 1, 2, \ldots\}$ and off period durations $\{I_n, n = 1, 2, \ldots\}$ form i.i.d. sequences of $\{1, 2, \ldots\}$-valued rvs distributed according to generic rvs B and I, respectively. We will assume that the rvs $B_0, I_0, \{B_n, n = 1, 2, \ldots\}$ and $\{I_n, n = 1, 2, \ldots\}$ are independent and that both B and I have finite first moments, i.e., $0 < \mathbf{E}[B], \mathbf{E}[I] < \infty$. We refer to the independent on-off process just defined as the on-off source (I, B).

To mark the beginning of cycles, we define a sequence of epochs $\{T_n, n = 0, 1, \ldots\}$ by $T_0 := 0$ and $T_{n+1} := \sum_{k=0}^{n}(I_k + B_k)$ for all $n = 0, 1, \ldots$. Hence, at time T_n the process begins the $(n+1)$st cycle with an off period of duration I_n followed by an on period of duration B_n. Furthermore, the first cycle will start with timeslot [0,1). The traffic process $\{A_t, t = 0, 1, \ldots\}$ indicates the level of activity of the source and can be represented by

$$A_t = \sum_{n=0}^{\infty} \mathbf{1}\left[T_n + I_n \leq t < T_{n+1}\right], \quad t = 0, 1, \ldots. \tag{3.3}$$

In general, the discrete-time on-off process $\{A_t, t = 0, 1, \ldots\}$ as defined above is not stationary, and additional assumptions on the sequence of rvs $\{I_n, B_n, n = 0, 1, \ldots\}$ are needed to obtain the stationary version. As in the continuous-time version [2, 17], we require the following:

(i) The rvs (I_0, B_0), $\{I_n, n = 1, \ldots\}$ and $\{B_n, n = 1, \ldots\}$ are mutually independent families of rvs;

(ii) The rvs $\{I_n, n = 1, \ldots\}$ (respectively, $\{B_n, n = 1, \ldots\}$) are i.i.d. $\{1, 2, \ldots\}$-valued rvs distributed according to a generic off period rv I (respectively, on period rv B).

(iii) The relations

$$[(I_0, B_0)|I_0 > 0] =_{st} (\hat{I}, B) \quad \text{and} \quad [(I_0, B_0)|I_0 = 0] =_{st} (0, \hat{B}) \tag{3.4}$$

hold where $\hat{I}$ is independent of B. In addition, the average rate p of the source is given by

$$p := \mathbf{P}[I_0 = 0] = \frac{\mathbf{E}[B]}{\mathbf{E}[B] + \mathbf{E}[I]}. \tag{3.5}$$

Under the assumption (i)-(iii), we have

$$[I_0 + B_0|I_0 > 0] =_{st} \hat{I} + B \quad \text{and} \quad [I_0 + B_0|I_0 = 0] =_{st} \hat{B}. \tag{3.6}$$

We note that every cycle always starts with an off period with the possibility that $I_0 = 0$, so that every cycle contains an on period. Throughout the thesis, we refer to the stationary version of the discrete-time on-off source (I, B) as the stationary on-off source (I, B). The non-stationary on-off source (I, B) is defined in the same way as the stationary on-off source but with $(I_0, B_0) =_{st} (I, B)$

(instead of (3.6)), i.e., the non-stationary on-off source (I, B) always starts with the off-period with off-period duration distributed according to I followed by the on-period with on-period duration distributed according to B.

3.3 $M|G|\infty$ input traffic

The $M|G|\infty$ input traffic is simply the number of busy servers in the infinite server system fed by a discrete-time Poisson process with rate λ (customers per timeslot) and with generic service time S (expressed in timeslots). A more detailed treatment of $M|G|\infty$ input processes can be found in [37, 40, 52]. For the continuous-time version, we refer the reader to [19, 24]. This process is a versatile class of input traffic since both short-range and long-range dependent traffic can be generated by properly selecting the service distribution of S. In this section, we start with the description of the discrete-time $M|G|\infty$ process and of its stationary version, and then give an explicit representation of these processes.

3.3.1 System description

Consider a system of infinitely many servers. Suppose there are B_t customers arriving to the system in timeslot $[t-1, t), t = 1, 2, \ldots$. Customer i, $i = 1, 2, \ldots, B_t$, is assigned its own server from which it starts receiving service with duration $S_{t,i}$ (number of slots) in timeslot $[t, t+1)$. If there are B initial customers present in the system at time $t = 0$ (i.e., $A_0 = B$), then initial customer i, $i = 1, 2, \ldots, B$, will have service time duration $S_{0,i}$ (starting at $t = 0$). Let A_t be the number of busy servers, or equivalently, the number of customers still present at the beginning of the timeslot [t,t+1). The busy server process $\{A_t, t = 0, 1, \ldots\}$ defines the $M|G|\infty$ input process.

To define the stationary and ergodic version of the $M|G|\infty$ input process, we need to make the following assumptions on the $\mathbb{N}$-valued rvs $B, \{B_t, t = 1, 2, \ldots\}, \{S_{t,i}, t = 1, 2, \ldots, i = 1, 2, \ldots\}$ and $\{S_{0,i}, i = 1, 2, \ldots\}$:

(i) These rvs are mutually independent;

(ii) The rv B is Poisson distributed with mean $\lambda \mathbf{E}[S]$;

(iii) The rvs $\{B_t, t = 1, 2, \ldots\}$ are i.i.d. Poisson rvs with mean $\lambda > 0$;

(iv) The rvs $\{S_{t,i}, t = 1, 2, \ldots, i = 1, 2, \ldots\}$ are i.i.d. with common pmf G on $\{1, 2, \ldots\}$. Let S be a generic rv distributed according to the pmf G and assume throughout that this pmf G has a finite first moment, or equivalently, that $\mathbf{E}[S] < \infty$;

(v) The rvs $\{S_{0,i}, i = 1, 2, \ldots\}$ are i.i.d. $\{1, 2, \ldots\}$-valued rvs distributed according to pmf $\hat{G}$ which is the forward recurrence pmf associated with G.

Hereafter, by an $M|G|\infty$ input process we mean the stationary and ergodic version, still denoted $\{A_t, t = 0, 1, \ldots\}$ and identified by the conditions above. In that case, we will write $\{\hat{S}_i, i = 1, 2, \ldots\}$ instead of $\{S_{0,i}, i = 1, 2, \ldots\}$. Since the process can be characterized by two parameters, namely λ and S, we refer to this $M|G|\infty$ process as the $M|G|\infty$ input process (λ, S). The next proposition summarizes key properties of such a stationary $M|G|\infty$ input process.

Proposition 3.3.1 *Under assumptions (i)-(v) above, the $M|G|\infty$ input process (λ, S) $\{A_t, t = 0, 1, \ldots\}$ is a (strictly) stationary and ergodic process with the following properties:*

(i) For each $t = 0, 1, \ldots$, the rv A_t is a Poisson rv with parameter $\lambda\mathbf{E}[S]$.

(ii) Its covariance function is given by

$$cov(A_t, A_{t+h}) = \lambda\mathbf{E}\left[(S-h)^+\right] = \lambda\mathbf{E}[S]\,\mathbf{P}\left[\hat{S} > h\right], \quad t, h = 0, 1, \ldots.$$

(iii) Its index of dispersion of counts (IDC) is given by

$$IDC \equiv \sum_{h=0}^{\infty} cov(A_t, A_{t+h}) = \lambda\mathbf{E}[S]\sum_{h=0}^{\infty}\mathbf{P}\left[\hat{S} > h\right] = \frac{\lambda}{2}\mathbf{E}[S(S+1)],$$

and the process is short-range dependent (i.e., has finite IDC) if and only if $\mathbf{E}[S^2]$ is finite.

3.3.2 Mathematical representation

Fix $t = 0, 1, \ldots$. From the system description in Section 3.3.1, we can write

$$A_t = A_t^{(0)} + A_t^{(a)} \tag{3.7}$$

where $A_t^{(0)}$ and $A_t^{(a)}$ are the numbers of busy servers in the system at the beginning of the timeslot $[t, t+1)$ contributed by the initial customers and new arrivals during $[0, t)$, respectively. From the B initial customers, customer i, $i = 1, 2, \ldots, B$, will be in the system at the beginning of timeslot $[t, t+1)$ if and only if $\hat{S}_i > t$, whence

$$A_t^{(0)} = \sum_{i=1}^{B}\mathbf{1}\left[\hat{S}_i > t\right]. \tag{3.8}$$

The arrival portion $A_t^{(a)}$ can be viewed as the number of customers at the beginning of timeslot $[t, t+1)$ of the system with no initial customer ($B = A_0 = 0$). Let $A_t^{(a,s)}$ be the number of customers still in the system at the beginning of timeslot $[t, t+1)$ having arrived in the timeslot $[s-1, s)$ with $s = 1, \ldots, t$. These

B_s customers who arrive in $[s-1, s)$ will be in the system at the beginning of timeslot $[t, t+1)$ if their service time durations exceed $t-s$ timeslots. Therefore,

$$A_t^{(a,s)} = \sum_{i=1}^{B_s} \mathbf{1}\left[S_{s,i} > t-s\right], \quad s = 1, \ldots, t.$$

Since $A_t^{(a)}$ is the sum of the *independent rvs* $\{A_t^{(a,s)}, s = 1, 2, \ldots, t\}$, we have

$$A_t^{(a)} = \sum_{s=1}^{t} \sum_{i=1}^{B_s} \mathbf{1}\left[S_{s,i} > t-s\right]. \tag{3.9}$$

It can be shown via Laplace transforms [37] that the rv $A_t^{(a,s)}$ is Poisson distributed with rate $\lambda \mathbf{P}\left[S > t-s\right]$ so that the rv $A_t^{(a)}$ is also Poisson distributed with rate $\lambda \sum_{s=1}^{t} \mathbf{P}\left[S > s\right]$.

From (3.7),(3.8) and (3.9), the stationary version of the $M|G|\infty$ input process has the expression

$$\begin{aligned} A_t &= A_t^{(0)} + A_t^{(a)} \\ &= \sum_{i=1}^{B} \mathbf{1}\left[\hat{S}_i > t\right] + \sum_{s=1}^{t} \sum_{i=1}^{B_s} \mathbf{1}\left[S_{s,i} > t-s\right]. \end{aligned} \tag{3.10}$$

Sometimes, it is useful to consider the cumulative arrival process on $[0, t]$, namely,

$$\sum_{s=1}^{t} A_s = \sum_{s=1}^{t} A_s^{(0)} + \sum_{s=1}^{t} A_s^{(a)}. \tag{3.11}$$

Explicit expressions are available for these quantities; they are summarized in the next proposition proved in [37].

Proposition 3.3.2 *For each $t = 1, 2, \ldots$, we have the expressions*

$$\sum_{s=1}^{t} A_s^{(0)} = \sum_{i=1}^{B} \min(t, \hat{S}_i - 1), \tag{3.12}$$

and

$$\sum_{s=1}^{t} A_s^{(a)} = \sum_{s=1}^{t} \sum_{i=1}^{B_s} \min(t-s+1, S_{s,i}). \tag{3.13}$$

Chapter 4

Stochastic Orderings and Positive Dependence

4.1 Integral stochastic orderings

In this section, we summarize some important definitions and facts about the stochastic orderings of random vectors. Additional information can be found in the monographs by Shaked and Shanthikumar [45] and by Stoyan [48]. The basic definition of integral stochastic orderings can be stated as follows:

Definition 4.1.1 *Let $\mathcal{F}$ be a class of Borel measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$. We say that the two $\mathbb{R}^n$-valued rvs $\mathbf{X}$ and $\mathbf{Y}$ satisfy the order relation $\mathbf{X} \leq_{\mathcal{F}} \mathbf{Y}$ if*

$$\mathbf{E}\left[\varphi(\mathbf{X})\right] \leq \mathbf{E}\left[\varphi(\mathbf{Y})\right] \tag{4.1}$$

for all functions φ in $\mathcal{F}$, whenever the expectations exist.

This generic definition has been specialized in the literature. Here are some important examples.

Definition 4.1.2 *The $\mathbb{R}^n$-valued rvs $\mathbf{X}$ and $\mathbf{Y}$ are said to be ordered according to* *(i) the usual stochastic ordering, written* $\mathbf{X} \leq_{st} \mathbf{Y}$*, if (4.1) holds for all increasing functions* $\varphi : \mathbb{R}^n \to \mathbb{R}$*, provided the expectations exist;*

(ii) the convex ordering, written $\mathbf{X} \leq_{cx} \mathbf{Y}$*, if (4.1) holds for all convex functions* $\varphi : \mathbb{R}^n \to \mathbb{R}$*, provided the expectations exist; and*

(iii) the increasing convex ordering, written $\mathbf{X} \leq_{icx} \mathbf{Y}$*, if (4.1) holds for all increasing convex functions* $\varphi : \mathbb{R}^n \to \mathbb{R}$*, provided the expectations exist.*

It is known [45] that for $\mathbb{R}$-valued rvs X and Y, if $X \leq_{cx} Y$ then $\mathbf{E}[X] = \mathbf{E}[Y]$ and $Var(X) \leq Var(Y)$, thus X has the same mean as Y but less variability than Y. In addition, when $X \leq_{icx} Y$, we find $\mathbf{E}[X] \leq \mathbf{E}[Y]$, hence Y is greater than X in both "size and variability." Consequently, the orderings cx and icx are appropriate for comparing the variability of rvs. However, in the case of random vectors, it is also desirable to compare their "dependence" structures. In the following sections, we investigate stochastic orderings which are well suited for comparing the dependence structures of random vectors and sequences.

4.2 Directional convexity and supermodularity

The stochastic ordering based on directional convexity has been introduced by Shaked and Shanthikumar [46] and Meester and Shanthikumar [29]. Recently, the supermodular ordering, which is closely related to the directionally convex ordering, has been used in a number of queueing and reliability applications [6, 7, 47]. We begin by introducing the classes of functions associated with these two orderings. We then state some important lemmas and theorems that will be useful in the buffer sizing problem.

Definition 4.2.1 *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be supermodular (sm) if*

$$\varphi(\mathbf{x} \vee \mathbf{y}) + \varphi(\mathbf{x} \wedge \mathbf{y}) \geq \varphi(\mathbf{x}) + \varphi(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Definition 4.2.2 *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be directionally convex (dcx) if for any $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}$ in $\mathbb{R}^n$ such that*

$$\mathbf{x_1} \leq [\mathbf{x_2}, \mathbf{x_3}] \leq \mathbf{x_4} \quad \textit{and} \quad \mathbf{x_1} + \mathbf{x_4} = \mathbf{x_2} + \mathbf{x_3},$$

it holds that

$$\varphi(\mathbf{x_1}) + \varphi(\mathbf{x_4}) \geq \varphi(\mathbf{x_2}) + \varphi(\mathbf{x_3}).$$

With $\varepsilon > 0$ and $\mathbf{e_i}$ the ith unit vector in $\mathbb{R}^n$, $i = 1, 2, \ldots, n$, we define the difference operator

$$\triangle_i^\varepsilon \varphi(\mathbf{x}) = \varphi(\mathbf{x} + \varepsilon \mathbf{e_i}) - \varphi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n$$

for a function $\varphi : \mathbb{R}^n \to \mathbb{R}$.

The followings contain well-known equivalent conditions for directionally convex functions [46].

Proposition 4.2.1 *For a function $\varphi : \mathbb{R}^n \to \mathbb{R}$, the following conditions (i)-(iv) are equivalent, where*

(i) φ is directionally convex;

(ii) φ is supermodular and convex in each coordinate;

(iii) For all $\varepsilon, \delta > 0$ and $1 \leq i, j \leq n$, it holds that

$$\triangle_i^\varepsilon \triangle_j^\delta \varphi(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathbb{R}^n;$$

(iv) For any $\mathbf{x_1}, \mathbf{x_2}, \mathbf{y}$ in $\mathbb{R}^n$ with $\mathbf{x_1} \leq \mathbf{x_2}$ and $\mathbf{y} > \mathbf{0}$, it holds that

$$\varphi(\mathbf{x_1} + \mathbf{y}) - \varphi(\mathbf{x_1}) \leq \varphi(\mathbf{x_2} + \mathbf{y}) - \varphi(\mathbf{x_2}).$$

We note that directional convexity does not imply nor is it implied by convexity. However, it is plain from condition (iv) of Proposition 4.2.1 that directional convexity can be viewed as a natural extension of univariate convexity. Condition (iii) also implies that any twice differentiable function with non-negative second derivatives is dcx. A function is said to be increasing directionally convex (idcx) (resp. increasing supermodular (ism)) when it is *increasing* in addition to being dcx (resp. sm).

4.3 Directionally convex and supermodular orderings

We now are ready to define the dependence orderings based on the supermodular and directionally convex functions.

Definition 4.3.1 *The $\mathbb{R}^n$-valued rvs $\mathbf{X}$ and $\mathbf{Y}$ are said to be ordered according to*

(i) the supermodular ordering, written $\mathbf{X} \leq_{sm} \mathbf{Y}$, if (4.1) holds for all supermodular functions $\varphi : \mathbb{R}^n \to \mathbb{R}$, provided the expectations exist;

(ii) the directionally convex ordering, written $\mathbf{X} \leq_{dcx} \mathbf{Y}$, if (4.1) holds for all directionally convex functions $\varphi : \mathbb{R}^n \to \mathbb{R}$, provided the expectations exist;

(iii) the increasing supermodular ordering, written $\mathbf{X} \leq_{ism} \mathbf{Y}$*, if (4.1) holds for all increasing supermodular functions* $\varphi : \mathbb{R}^n \to \mathbb{R}$*, provided the expectations exist; and*

(iv) the increasing directionally convex ordering, written $\mathbf{X} \leq_{idcx} \mathbf{Y}$*, if (4.1) holds for all increasing directionally convex functions* $\varphi : \mathbb{R}^n \to \mathbb{R}$*, provided the expectations exist.*

From condition (ii) in Proposition 4.2.1, the class of directionally convex functions is a subclass of the class of supermodular functions, so that the supermodular ordering is stronger than the directionally convex ordering. Moreover, the dcx ordering also implies the idcx ordering and the following implications thus hold: For any $\mathbb{R}^n$-valued rvs $\mathbf{X}$ and $\mathbf{Y}$,

$$\mathbf{X} \leq_{sm} \mathbf{Y} \Rightarrow \mathbf{X} \leq_{dcx} \mathbf{Y} \Rightarrow \mathbf{X} \leq_{idcx} \mathbf{Y}. \tag{4.2}$$

It can be shown that for non-negative $\mathbb{R}^n$-valued rvs $\mathbf{X}$ and $\mathbf{Y}$, if $\mathbf{X} \leq_{idcx} \mathbf{Y}$, then $\mathbf{E}[X_i X_j] \leq \mathbf{E}[Y_i Y_j]$ for all $i, j = 1, \ldots, n$. Lastly, we note the equivalence between the sm and ism orderings given in [32].

Proposition 4.3.1 *For* $\mathbb{R}^n$*-valued rvs* $\mathbf{X}$ *and* $\mathbf{Y}$*, the conditions (i)-(ii) below are equivalent, where*

(i) $\mathbf{X} \leq_{sm} \mathbf{Y}$*;*

(ii) $\mathbf{X} \leq_{ism} \mathbf{Y}$ *and* $X_i =_{st} Y_i, i = 1, \ldots, n.$

Additional information on the supermodular ordering can be found in [5, 6, 7, 32, 47]. For some properties and applications of the dcx ordering, we refer the reader to [7, 29, 51].

The next two lemmas are due to Meester and Shanthikumar [29], and point to relations between the directionally convex and convex orderings.

Lemma 4.3.1 *If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is an idcx function and the mapping $g : \mathbb{R} \to \mathbb{R}$ is increasing and convex, then the composition $g \circ \varphi : \mathbb{R}^n \to \mathbb{R}$ is idcx.*

Lemma 4.3.2 *If $\mathbf{X} \leq_{idcx} \mathbf{Y}$, then $\varphi(\mathbf{X}) \leq_{icx} \varphi(\mathbf{Y})$ for any idcx function $\varphi : \mathbb{R}^n \to \mathbb{R}$.*

In addition, we shall use the fact that the sm, dcx and idcx orderings are closed under convolution.

Lemma 4.3.3 *Let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ be independent $\mathbb{R}^n$-valued rvs. With $\leq$ denoting either $\leq_{sm}, \leq_{dcx}$ or $\leq_{idcx}$, if $\mathbf{X} \leq \mathbf{Y}$, then $\mathbf{X} + \mathbf{Z} \leq \mathbf{Y} + \mathbf{Z}$.*

Proof. We give the proof for the idcx ordering as a similar argument can be used to establish the result for the sm and dcx orderings. By definition, $\mathbf{X} \leq_{idcx} \mathbf{Y}$ means that for any idcx function $\psi : \mathbb{R}^n \to \mathbb{R}$, we have

$$\mathbf{E}\left[\psi(X_1, \ldots, X_n)\right] \leq \mathbf{E}\left[\psi(Y_1, \ldots, Y_n)\right] \tag{4.3}$$

whenever the expectations exist. Now, for an idcx function $\varphi : \mathbb{R}^n \to \mathbb{R}$, we define the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}$ by

$$\Phi(\mathbf{x}) = \mathbf{E}\left[\varphi(\mathbf{x} + \mathbf{Z})\right], \quad \mathbf{x} \in \mathbb{R}^n,$$

and note that Φ is also an idcx function. Under the independence assumption, we can write

$$\begin{aligned} \mathbf{E}\left[\varphi(X_1 + Z_1, \ldots, X_n + Z_n)\right] &= \mathbf{E}\left[\Phi(X_1, \ldots, X_n)\right] \\ &\leq \mathbf{E}\left[\Phi(Y_1, \ldots, Y_n)\right] \\ &= \mathbf{E}\left[\varphi(Y_1 + Z_1, \ldots, Y_n + Z_n)\right] \end{aligned}$$

where the inequality follows from (4.3) (when applied to Φ). ■

We also note the following convergence result [32, Thm. 3.1, p. 112].

Lemma 4.3.4 *Let $\{\mathbf{X}_i, i = 1, 2, \ldots\}$ and $\{\mathbf{Y}_i, i = 1, 2, \ldots\}$ denote two sequences of $\mathbb{R}^n$-valued rvs such that $\mathbf{X}_n \Longrightarrow_n \mathbf{X}_\infty$ and $\mathbf{Y}_n \Longrightarrow_n \mathbf{Y}_\infty$. If $\mathbf{X}_n \leq_{sm} \mathbf{Y}_n$ for each $n = 1, 2, \ldots$, then $\mathbf{X}_\infty \leq_{sm} \mathbf{Y}_\infty$.*

It is not known whether either the dcx or idcx ordering is stable under weak convergence.

Iterating Lemma 4.3.3 with the help of Lemma 4.3.4 leads to the following two special cases, but first, a definition:

Definition 4.3.2 *For $\mathbb{R}^n$-valued rvs $\mathbf{X}$ and $\hat{\mathbf{X}}$, we say that $\hat{\mathbf{X}} = (\hat{X}_1, \ldots, \hat{X}_n)$ is an independent version of $\mathbf{X} = (X_1, \ldots, X_n)$ if the rvs $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$ are mutually independent with $\hat{X}_k =_{st} X_k$, $k = 1, \ldots, n$.*

Corollary 4.3.1 *Let $\{\mathbf{X}_i, i = 1, 2, \ldots\}$ and $\{\hat{\mathbf{X}}_i, i = 1, 2, \ldots\}$ denote two sequences of mutually independent $\mathbb{R}^n$-valued rvs where for each $i = 1, 2, \ldots$, $\hat{\mathbf{X}}_i = (\hat{X}_{i1}, \ldots, \hat{X}_{in})$ is an independent version of $\mathbf{X}_i$. With $\leq$ denoting either $\leq_{dcx}$ or $\leq_{idcx}$, if $\hat{\mathbf{X}}_i \leq \mathbf{X}_i$ for all $i = 1, 2, \ldots$, then for each $N = 1, 2, \ldots$, the rv $\sum_{i=1}^N \hat{\mathbf{X}}_i$ is an independent version of $\sum_{i=1}^N \mathbf{X}_i$ and*

$$\sum_{i=1}^{N} \hat{\mathbf{X}}_i \leq \sum_{i=1}^{N} \mathbf{X}_i, \quad N = 1, 2, \ldots.$$

Proof. We give the proof only for the idcx ordering as the dcx ordering can be established by the same argument. Without loss of generality, we can always assume that the sequences $\{\mathbf{X}_i, i = 1, 2, \ldots\}$ and $\{\hat{\mathbf{X}}_i, i = 1, 2, \ldots\}$ are mutually independent. The proof proceeds by induction: When $N = 2$, by a repeated use of Lemma 4.3.3, we have

$$\hat{\mathbf{X}}_1 + \hat{\mathbf{X}}_2 \leq_{idcx} \mathbf{X}_1 + \hat{\mathbf{X}}_2 \leq_{idcx} \mathbf{X}_1 + \mathbf{X}_2,$$

and the basis step is established. Suppose now that the result holds for some $N = m$. For $N = m + 1$, again by repeatedly using Lemma 4.3.3, we can write

$$\sum_{i=1}^{m} \hat{\mathbf{X}}_i + \hat{\mathbf{X}}_{m+1} \leq_{idcx} \sum_{i=1}^{m} \mathbf{X}_i + \hat{\mathbf{X}}_{m+1} \leq_{idcx} \sum_{i=1}^{m} \mathbf{X}_i + \mathbf{X}_{m+1}.$$

We complete the proof by noting that $\sum_{i=1}^{N} \hat{\mathbf{X}}_i$ is the independent version of $\sum_{i=1}^{N} \mathbf{X}_i$ since the rvs $\sum_{i=1}^{N} \hat{X}_{i1}, \sum_{i=1}^{N} \hat{X}_{i2}, \ldots, \sum_{i=1}^{N} \hat{X}_{in}$ are independent with

$$\sum_{i=1}^{N} \hat{X}_{ik} =_{st} \sum_{i=1}^{N} X_{ik}, \quad k = 1, 2, \ldots, n.$$

■

Corollary 4.3.2 *Let* $\{\mathbf{X}_i, i = 1, 2, \ldots\}$ *and* $\{\hat{\mathbf{X}}_i, i = 1, 2, \ldots\}$ *denote two sequences of mutually independent* $\mathbb{R}^n$*-valued rvs where for each* $i = 1, 2, \ldots$, $\hat{\mathbf{X}}_i = (\hat{X}_{i1}, \ldots, \hat{X}_{in})$ *is an independent version of* $\mathbf{X}_i$. *If* $\hat{\mathbf{X}}_i \leq_{sm} \mathbf{X}_i$ *for all* $i = 1, 2, \ldots$, *then:*

(i) For $N = 1, 2, \ldots$, *the rv* $\sum_{i=1}^{N} \hat{\mathbf{X}}_i$ *is an independent version of* $\sum_{i=1}^{N} \mathbf{X}_i$ *and*

$$\sum_{i=1}^{N} \hat{\mathbf{X}}_i \leq_{sm} \sum_{i=1}^{N} \mathbf{X}_i;$$

(ii) If $\sum_{i=1}^{N} \mathbf{X}_i \Longrightarrow_N \sum_{i=1}^{\infty} \mathbf{X}_i$, *then* $\sum_{i=1}^{N} \hat{\mathbf{X}}_i \Longrightarrow_N \sum_{i=1}^{\infty} \hat{\mathbf{X}}_i$ *where the rv* $\sum_{i=1}^{\infty} \hat{\mathbf{X}}_i$ *is an independent version of* $\sum_{i=1}^{\infty} \mathbf{X}_i$ *and*

$$\sum_{i=1}^{\infty} \hat{\mathbf{X}}_i \leq_{sm} \sum_{i=1}^{\infty} \mathbf{X}_i.$$

Proof. For Claim (i), we apply the same argument as in the proof of Corollary 4.3.1, this time with the sm ordering. That $\sum_{i=1}^N \hat{\mathbf{X}}_i$ is an independent version of $\sum_{i=1}^N \mathbf{X}_i$ has already been established in the proof of Corollary 4.3.1.

To establish Claim (ii), we invoke the weak convergence property of the sm ordering [Lemma 4.3.4]. By Claim (i), for each $N = 1, 2, \ldots$, we have $\sum_{i=1}^N \hat{\mathbf{X}}_i \leq_{sm} \sum_{i=1}^N \mathbf{X}_i$ and $\sum_{i=1}^N \hat{\mathbf{X}}_i$ is an independent version of $\sum_{i=1}^N \mathbf{X}_i$. By Lemma 4.3.4, we need only show $\sum_{i=1}^N \hat{\mathbf{X}}_i \Longrightarrow_N \sum_{i=1}^\infty \hat{\mathbf{X}}_i$ in order to establish $\sum_{i=1}^\infty \hat{\mathbf{X}}_i \leq_{sm} \sum_{i=1}^\infty \mathbf{X}_i$. Since $\sum_{i=1}^N \mathbf{X}_i \Longrightarrow_N \sum_{i=1}^\infty \mathbf{X}_i$, we have for each $k = 1, \ldots, n$, $\sum_{i=1}^N X_{ik} \Longrightarrow_N (\sum_{i=1}^\infty \mathbf{X}_i)_k$, whence

$$\sum_{i=1}^N \hat{X}_{ik} =_{st} \sum_{i=1}^N X_{ik} \Longrightarrow_N (\sum_{i=1}^\infty \mathbf{X}_i)_k.$$

Upon noting that $(\sum_{i=1}^\infty \mathbf{X}_i)_k =_{st} \sum_{i=1}^\infty \hat{X}_{ik}$ for all $k = 1, 2, \ldots, n$ and that the rvs $\sum_{i=1}^\infty \hat{X}_{i1}, \sum_{i=1}^\infty \hat{X}_{i2}, \ldots$ and $\sum_{i=1}^\infty \hat{X}_{in}$ are independent rvs, we conclude that

$$\sum_{i=1}^N \hat{\mathbf{X}}_i \Longrightarrow_N \sum_{i=1}^\infty \hat{\mathbf{X}}_i$$

and $\sum_{i=1}^\infty \hat{\mathbf{X}}_i$ is an independent version of $\sum_{i=1}^\infty \mathbf{X}_i$. ■

Finally, it is useful to extend some of these definitions to sequences of rvs.

Definition 4.3.3 *With $\leq$ denoting either $\leq_{sm}$, $\leq_{dcx}$ or $\leq_{idcx}$, we say that the two $\mathbb{R}$-valued sequences $\mathbf{X} = \{X_n, n = 1, 2, \ldots\}$ and $\mathbf{Y} = \{Y_n, n = 1, 2, \ldots\}$ satisfy the relation $\mathbf{X} \leq \mathbf{Y}$ if for all $n = 1, 2, \ldots$, it holds that*

$$(X_1, \ldots, X_n) \leq (Y_1, \ldots, Y_n).$$

Definition 4.3.4 *For sequences of* $\mathbb{R}$*-valued rvs* $\mathbf{X} = \{X_n, n = 1, 2, \ldots\}$ *and* $\hat{\mathbf{X}} = \{\hat{X}_n, n = 1, 2, \ldots\}$*, we say that* $\hat{\mathbf{X}}$ *is an independent version of* $\mathbf{X}$ *if the rvs* $\{\hat{X}_n, n = 1, 2, \ldots\}$ *are mutually independent with* $\hat{X}_n =_{st} X_n, n = 1, 2, \ldots$*.*

4.4 The orthant orderings

In addition to the supermodular and directionally convex orderings, we consider another class of orderings, called the orthant orderings, defined as follows:

Definition 4.4.1 *The* $\mathbb{R}^n$*-valued rvs* $\mathbf{X}$ *and* $\mathbf{Y}$ *are said to be ordered according to*

(i) the upper orthant ordering, written $\mathbf{X} \leq_{uo} \mathbf{Y}$*, if for any* $\mathbf{t}$ *in* $\mathbb{R}^n$*,*
$\mathbf{P}[\mathbf{X} > \mathbf{t}] \leq \mathbf{P}[\mathbf{Y} > \mathbf{t}]$*; and*

(ii) the lower orthant ordering, written $\mathbf{X} \leq_{lo} \mathbf{Y}$*, if for any* $\mathbf{t}$ *in* $\mathbb{R}^n$*,*
$\mathbf{P}[\mathbf{X} \leq \mathbf{t}] \geq \mathbf{P}[\mathbf{Y} \leq \mathbf{t}]$*.*

The orthant orderings has been treated by Shaked an Shanthikumar [45]. However, the definition of the lower orthant ordering is not consistent in the literature. For example, some authors [32] define the lower orthant ordering in the opposite way. Here, we use the definition found in [45].

Now, we show the relationship between the orthant orderings and supermodular ordering but first, a definition:

Definition 4.4.2 *A function* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is said to be* Δ*-monotone if for all* $\mathbf{x}$ *in* $\mathbb{R}^n$*,* $k = 1, \ldots, n$*, any subset* $J = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ *and every* $\varepsilon_1, \ldots, \varepsilon_k > 0$*, it holds that*

$$\triangle_{i_1}^{\varepsilon_1} \ldots \triangle_{i_k}^{\varepsilon_k} \varphi(\mathbf{x}) \geq 0.$$

It has been shown by Rüschendorf [43] that the integral ordering generated by Δ-monotone functions is equivalent to the upper orthant ordering.

Proposition 4.4.1 *For $\mathbb{R}^n$-valued rvs $\mathbf{X}$ and $\mathbf{Y}$, the conditions (i)-(ii) below are equivalent, where*

(i) $\mathbf{X} \leq_{uo} \mathbf{Y}$;

(ii) The inequality (4.1) holds for all Δ-monotone functions $\varphi : \mathbb{R}^n \to \mathbb{R}$, provided the expectations exist.

For $\mathbf{t}$ in $\mathbb{R}^n$, the indicator functions $\mathbb{R}^n \to \{0,1\} : \mathbf{x} \to \mathbf{1}[\mathbf{x} > \mathbf{t}]$ and $\mathbb{R}^n \to \{0,1\} : \mathbf{x} \to \mathbf{1}[\mathbf{x} \leq \mathbf{t}]$ are supermodular. Hence, we have the implications

$$\mathbf{X} \leq_{sm} \mathbf{Y} \Rightarrow \mathbf{X} \leq_{uo} \mathbf{Y} \quad \text{and} \quad \mathbf{X} \leq_{sm} \mathbf{Y} \Rightarrow \mathbf{X} \geq_{lo} \mathbf{Y}. \tag{4.4}$$

If $\mathbf{X} \leq_{uo} \mathbf{Y}$ and $\mathbf{Y} \leq_{lo} \mathbf{X}$, then the marginals of $\mathbf{X}$ and $\mathbf{Y}$ must be equal, i.e., $X_i =_{st} Y_i, i = 1, \ldots, n$. Thus, $\mathbf{X}$ and $\mathbf{Y}$ are ordered according to the supermodular ordering only if $\mathbf{X}$ and $\mathbf{Y}$ have the same marginals.

It is known that for bivariate rvs ($\mathbb{R}^2$-valued rvs) $\mathbf{X}$ and $\mathbf{Y}$, if the marginals of $\mathbf{X}$ and $\mathbf{Y}$ are equal, then the supermodular ordering is equivalent to the orthant orderings [32].

Lemma 4.4.1 *Let $\mathbf{X}$ and $\mathbf{Y}$ be $\mathbb{R}^2$-valued rvs with equal marginals, i.e., $X_1 =_{st} Y_1$ and $X_2 =_{st} Y_2$. Then, the conditions (i)-(iii) below are equivalent, where*

(i) $\mathbf{X} \leq_{sm} \mathbf{Y}$;

(ii) $\mathbf{X} \leq_{uo} \mathbf{Y}$;

(iii) $\mathbf{X} \geq_{lo} \mathbf{Y}$.

Proof. It is clear from (4.4) that (i) implies (ii) and (iii). The implication from (ii) to (i) is shown by Tchen [50] for the case of rvs with finite-lattice pmf. However, we use another method to prove this implication. For any $\mathbb{R}^2$-valued rvs $\mathbf{X}$ and $\mathbf{Y}$, assume that $\mathbf{X} \leq_{uo} \mathbf{Y}$. By Proposition 4.3.1, in order to show (i), it is enough to show that $\mathbf{X} \leq_{ism} \mathbf{Y}$ because $\mathbf{X}$ and $\mathbf{Y}$ have same marginals. To do so, we recall from Proposition 4.4.1 that the uo ordering can be generated via Δ-monotone functions. Thus, if every ism function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is Δ-monotone, then

$$\mathbf{X} \leq_{uo} \mathbf{Y} \Rightarrow \mathbf{X} \leq_{ism} \mathbf{Y}. \tag{4.5}$$

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be an ism function. By Definition 4.4.2, we must show that for any subset $J = \{i_1, i_2\} \subset \{1, 2\}$ and $\varepsilon_1, \varepsilon_2 > 0$,

$$\triangle_{i_1}^{\varepsilon_1} \triangle_{i_2}^{\varepsilon_2} \varphi(\mathbf{x}) \geq 0, \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2. \tag{4.6}$$

Since $\triangle_1^{\varepsilon_1} \triangle_2^{\varepsilon_2} \varphi(\mathbf{x}) = \triangle_2^{\varepsilon_2} \triangle_1^{\varepsilon_1} \varphi(\mathbf{x})$, we need only consider the following three cases:

(a) $J = \{1\}$

(b) $J = \{2\}$

(c) $J = \{1, 2\}$

Using the definition of difference operator, we can rewrite (4.6) in Cases (a) and (b) as

$$\varphi(x_1 + \varepsilon_1, x_2) - \varphi(x_1, x_2) \geq 0 \quad \text{and} \quad \varphi(x_1, x_2 + \varepsilon_2) - \varphi(x_1, x_2) \geq 0, \tag{4.7}$$

respectively. Since φ is increasing, the conclusion (4.7) is satisfied. For Case (c), we again use the definition of difference operator to rewrite (4.6) as

$$\varphi(x_1 + \varepsilon_1, x_2 + \varepsilon_2) + \varphi(x_1, x_2) \geq \varphi(x_1 + \varepsilon_1, x_2) + \varphi(x_1, x_2 + \varepsilon_2). \tag{4.8}$$

Clearly, (4.8) holds since φ is supermodular. Therefore, φ is also Δ-monotone and the conclusion (4.5) holds in the bivariate case. Applying Proposition 4.3.1, we obtain the implication (ii) to (i).

It remains to show that (ii) and (iii) are equivalent. Fix (x_0, x_1) in $\mathbb{R}^2$. If the marginals of $\mathbf{X}$ and $\mathbf{Y}$ are equal, then we have

$$\begin{aligned} & \mathbf{P}\left[X_0 > x_0, X_1 > x_1\right] + \mathbf{P}\left[X_0 > x_0, X_1 \leq x_1\right] \\ = \ & \mathbf{P}\left[Y_0 > x_0, Y_1 > x_1\right] + \mathbf{P}\left[Y_0 > x_0, Y_1 \leq x_1\right] \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} & \mathbf{P}\left[X_0 \leq x_0, X_1 \leq x_1\right] + \mathbf{P}\left[X_0 > x_0, X_1 \leq x_1\right] \\ = \ & \mathbf{P}\left[Y_0 \leq x_0, Y_1 \leq x_1\right] + \mathbf{P}\left[Y_0 > x_0, Y_1 \leq x_1\right]. \end{aligned} \tag{4.10}$$

Upon combining (4.9) and (4.10), it holds that

$$\begin{aligned} & \mathbf{P}\left[Y_0 > x_0, Y_1 > x_1\right] - \mathbf{P}\left[X_0 > x_0, X_1 > x_1\right] \\ = \ & \mathbf{P}\left[X_0 > x_0, X_1 \leq x_1\right] - \mathbf{P}\left[Y_0 > x_0, Y_1 \leq x_1\right] \\ = \ & \mathbf{P}\left[Y_0 \leq x_0, Y_1 \leq x_1\right] - \mathbf{P}\left[X_0 \leq x_0, X_1 \leq x_1\right] \end{aligned} \tag{4.11}$$

and we conclude that $\mathbf{X} \leq_{uo} \mathbf{Y}$ if and only if $\mathbf{X} \geq_{lo} \mathbf{Y}$.

■

4.5 Positive dependence

Positive dependence in a collection of rvs can be captured in several ways. Association of rvs is one of the most useful such characterizations. It was introduced by Esary, Proschan and Walkup [13] and has proved useful in various settings [3, 4, 10, 20].

Definition 4.5.1 *The* $\mathbb{R}$*-valued rvs* $\{X_1, \ldots, X_n\}$ *are said to be associated if, with* $\mathbf{X} = (X_1, \ldots, X_n)$*, the inequality*

$$\mathbf{E}\left[f(\mathbf{X})g(\mathbf{X})\right] \geq \mathbf{E}\left[f(\mathbf{X})\right]\mathbf{E}\left[g(\mathbf{X})\right]$$

holds for all non-decreasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$ *for which the expectations exist and are finite.*

Here, we focus on a stronger notion of positive dependence, known as stochastic monotonicity in sequence (SSI). The concept of positive dependence using the SSI property can by found in [4, 29, 33].

Definition 4.5.2 *The* $\mathbb{R}$*-valued rvs* $\{X_1, \ldots, X_n\}$ *are said to be sequentially stochastically increasing (SSI) if for each* $k = 1, 2, \ldots, n-1$*, the family of conditional distributions* $[X_{k+1}|X_1 = x_1, \ldots, X_k = x_k]$*,* $(x_1, \ldots, x_k) \in \mathbb{R}^k$*, is stochastically increasing in* $(x_1, \ldots, x_k)$*.*

More precisely, this definition states that for each $k = 1, \ldots, n-1$, for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$ with $\mathbf{x} \leq \mathbf{y}$, it holds that

$$\mathbf{E}\left[\varphi(X_{k+1})|(X_1, \ldots, X_k) = \mathbf{x}\right] \leq \mathbf{E}\left[\varphi(X_{k+1})|(X_1, \ldots, X_k) = \mathbf{y}\right] \tag{4.12}$$

for all increasing functions $\varphi : \mathbb{R} \to \mathbb{R}$. In particular, if the rvs $\{X_1, \ldots, X_n\}$ are $\mathbb{N}$-valued rvs, then the SSI property requires that for each $k = 1, \ldots, n-1$, for

$\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{N}^k$ with $\mathbf{x} \leq \mathbf{y}$, it holds that

$$\mathbf{P}\left[X_{k+1} > m | (X_1, \ldots, X_k) = \mathbf{x}\right] \leq \mathbf{P}\left[X_{k+1} > m | (X_1, \ldots, X_k) = \mathbf{y}\right]. \quad (4.13)$$

for all m in $\mathbb{N}$. Note that in the SSI definition, we need to consider only $(x_1, \ldots, x_k)$ for which $\mathbf{P}\left[X_1 = x_1, \ldots, X_k = x_k\right] > 0$.

These definitions can be extended to sequences in a natural way along the lines of Definition 4.3.3:

Definition 4.5.3 *A sequence of $\mathbb{R}$-valued rvs $\{X_n, n = 1, 2, \ldots\}$ is said to be SSI (resp. associated) if for each $n = 1, 2, \ldots$, the rvs $\{X_1, \ldots, X_n\}$ are SSI (resp. associated).*

It is well known that if the $\mathbb{R}$-valued rvs $\{X_1, \ldots, X_n\}$ are SSI, then they are necessarily associated [4, Thm. 4.7, p. 146] but the converse is not true. The next theorem was established in [29], and relates the SSI property of rvs to the supermodular ordering. This fact will prove crucial for subsequent developments in this thesis.

Theorem 4.5.1 *If $\{X_n, n = 1, 2, \ldots\}$ is SSI and $\{\hat{X}_n, n = 1, 2, \ldots\}$ is the independent version of $\{X_n, n = 1, 2, \ldots\}$, then*

$$(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_k) \leq_{sm} (X_1, X_2, \ldots, X_k), \quad k = 1, 2, \ldots \quad (4.14)$$

i.e., for any supermodular function $\varphi : \mathbb{R}^k \to \mathbb{R}$,

$$\mathbf{E}\left[\varphi(\hat{X}_1, \ldots, \hat{X}_k)\right] \leq \mathbf{E}\left[\varphi(X_1, \ldots, X_k)\right].$$

The following consequence of Theorem 4.5.1 is immediate in view of (4.2).

Corollary 4.5.1 *Under the assumptions of Theorem 4.5.1, the comparison (4.14) also holds in the dcx and idcx orderings.*

Chapter 5

The Buffer Sizing Problem

Consider a discrete-time single server queue with infinite buffer capacity and constant service rate of c cells/slot (packets/slot). Under the first-come first-serve discipline, this queueing system can be used to represent an infinite-buffer multiplexer. We are interested in the effect of positive correlations in a stationary input stream $\{A_t, t = 0, 1, \ldots\}$ on the buffer occupancy level Q_t at the end of timeslot $[t-1, t)$ for $t = 1, 2, \ldots$. By formalizing this problem with the help of the notion of stochastic orderings, we can show the following: For $i = 1, 2$, let $\{A_t^i, t = 0, 1, \ldots\}$ and $\{Q_t^i, t = 0, 1, \ldots\}$ be the input traffic i and its corresponding buffer content sequence. If the input traffic 1 is less "dependence" than the input traffic 2 in the sense of the idcx ordering (or its close cousin, the sm ordering), i.e.,

$$\{A_t^1, t = 0, 1, \ldots\} \leq_{idcx} \{A_t^2, t = 0, 1, \ldots\},$$

then $Q_t^1 \leq_{icx} Q_t^2$ for each $t = 1, 2, \ldots$. This last statement implies that for each $t = 1, 2, \ldots$, Q_t^1 is less than Q_t^2 in both "size and variability." More generally, since Q_t^1 and Q_t^2 are non-negative rvs, for $k \geq 1$, the kth moment of Q_t^1 is less than the kth moment of Q_t^2.

In this chapter, we first discuss the Lindley recursion and the aforementioned comparison result in the transient state ($t = 1, 2, \ldots$). We then translate the transient result into the steady state comparison of the buffer sizes ($t \to \infty$).

5.1 The Lindley recursion

Let $\{A_t, t = 0, 1, \ldots\}$ be a stationary and ergodic input traffic feeding into a discrete-time single server queue with infinite buffer capacity and let Q_t be the number of cells remaining in the buffer at the end of timeslot $[t-1, t)$. At the start of timeslot $[t, t+1)$, A_t new cells have arrived so that there are $Q_t + A_t$ cells ready for transmission in that slot. With the multiplexer releasing c cells/slot, the sequence of buffer contents $\{Q_t, t = 0, 1, \ldots\}$ satisfies the Lindley recursion

$$Q_0 = q; \quad Q_{t+1} = (Q_t + A_t - c)^+, \quad t = 0, 1, \ldots \tag{5.1}$$

for some fixed initial condition q. It is well known [27] that if the mean input rate is less than the service rate, i.e., $\mathbf{E}[A_0] < c$, then this queueing system will be stable in the sense that $Q_t \Longrightarrow_t Q$ for some $\mathbb{R}$-valued rv Q. If the input traffic is a reversible sequence, then the rv Q can be represented as

$$Q =_{st} [\sup_{t=0,1,\ldots} \{\sum_{s=0}^{t} A_s - c(t+1)\}]^+. \tag{5.2}$$

From (5.1), it is plain that for each $t = 1, 2, \ldots$, the buffer content Q_t is a function of the input traffic $A_0, \ldots, A_{t-1}$ (and of the initial condition q). Thus, there exists a mapping $T_t : \mathbb{R}^t \times \mathbb{R} \to \mathbb{R}$ such that $Q_t = T_t(A_0, \ldots, A_{t-1}, Q_0)$. This function T_t is readily obtained by iterating the mapping $T : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$T(a, q) := (q + a - c)^+, \quad (a, q) \in \mathbb{R}^2, \tag{5.3}$$

through the Lindley recursion since

$$Q_{t+1} = T(A_t, Q_t), \quad t = 0, 1, \ldots.$$

As we would like to apply Theorem 4.5.1 (in fact its Corollary 4.5.1) to the buffer sizing problem, we need to establish that the mappings $\{T_t, t = 1, 2, \ldots\}$ are idcx.

Proposition 5.1.1 *For each* $t = 1, 2, \ldots$ *and* $q \in \mathbb{R}$*, the mapping* $\mathbb{R}^t \to \mathbb{R} : (a_0, \ldots, a_{t-1}) \to T_t(a_0, \ldots, a_{t-1}, q)$ *is idcx.*

Proof. We establish the proof by induction (on t). For $t = 1$, $T_1 = T$ and from (5.3), $a \to T(a, q)$ is increasing and directionally convex in a since convexity in one dimension implies directional convexity.

Suppose we have for some $t = 1, 2, \ldots$ that the mapping $(a_0, \ldots, a_{t-1}) \to T_t(a_0, \ldots, a_{t-1})$ is idcx where we omit q for simplicity. By the Lindley recursion, we obtain

$$T_{t+1}(a_0, \ldots, a_t) = (T_t(a_0, \ldots, a_{t-1}) + a_t - c)^+. \tag{5.4}$$

Obviously, the mapping $h_t : (a_0, \ldots, a_t) \to a_t$ is idcx. Thus, the mapping $(a_0, \ldots, a_t) \to f_{t+1}(a_0, \ldots, a_t)$, given by

$$\begin{aligned} f_{t+1}(a_0, \ldots, a_t) &= T_t(a_0, \ldots, a_{t-1}) + a_t - c \\ &= T_t(a_0, \ldots, a_{t-1}) + h_t(a_0, \ldots, a_t) - c, \end{aligned}$$

is idcx as the sum of idcx functions is still idcx. The function $g : x \to x^+$ is a convex function and by Lemma 4.3.1, the mapping T_{t+1} is therefore an idcx function of $(a_0, \ldots, a_t)$ since

$$T_{t+1}(a_0, \ldots, a_t) = g(f_{t+1}(a_0, \ldots, a_t)).$$

The proof of the induction step is now completed. ■

In conclusion, by virtue of Lemma 4.3.2 and Proposition 5.1.1, we have the following comparison in the transient state.

Theorem 5.1.1 *Let* $\{A_t^1, t = 0, 1, \ldots\}$ *and* $\{A_t^2, t = 0, 1, \ldots\}$ *be input traffic processes to the discrete-time single server queue (5.1). If*

$$\{A_t^1, t = 0, 1, \ldots\} \leq_{idcx} \{A_t^2, t = 0, 1, \ldots\},$$

then their corresponding buffer contents $\{Q_t^1, t = 0, 1, \ldots\}$ *and* $\{Q_t^2, t = 0, 1, \ldots\}$ *are ordered in the icx ordering, i.e., for any fixed initial condition* $Q_0^1 = Q_0^2 = q$, *we have*

$$Q_t^1 \leq_{icx} Q_t^2, \quad t = 1, 2, \ldots.$$

5.2 Steady-state results

Under some conditions on the initial condition Q_0 and on the input processes, the transient results of Theorem 5.1.1 can be translated into a steady state result. Before doing so, we begin with a lemma on the stability of the icx ordering under weak convergence [48].

Lemma 5.2.1 *Let* $\{X_n, n = 1, 2, \ldots\}$ *and* $\{Y_n, n = 1, 2, \ldots\}$ *denote two sequences of* $\mathbb{R}$*-valued rvs such that* $X_n \Longrightarrow_n X$ *and* $Y_n \Longrightarrow_n Y$ *with* $X_n \leq_{icx} Y_n$ *for each* $n = 1, 2, \ldots$. *If* $\lim_{n\to\infty} \mathbf{E}[X_n^+] = \mathbf{E}[X^+]$ *and* $\lim_{n\to\infty} \mathbf{E}[Y_n^+] = \mathbf{E}[Y^+]$, *then* $X \leq_{icx} Y$.

Proof. It is well known [48] that for any $\mathbb{R}$-valued rvs ξ and ξ', $\xi \leq_{icx} \xi'$ if and only if

$$\mathbf{E}\left[(\xi - a)^+\right] \le \mathbf{E}\left[(\xi' - a)^+\right], \quad a \in \mathbb{R}. \tag{5.5}$$

For $n = 1, 2, \ldots$, rewriting (5.5) for X_n and Y_n, we readily get

$$\mathbf{E}\left[X_n^+\right] - \int_0^a \mathbf{P}\left[X_n > t\right] dt \le \mathbf{E}\left[Y_n^+\right] - \int_0^a \mathbf{P}\left[Y_n > t\right] dt, \quad a \in \mathbb{R}. \tag{5.6}$$

Now let n go to infinity in (5.6). For the first term in (5.6), we use the assumptions $\lim_{n\to\infty} \mathbf{E}\left[X_n^+\right] = \mathbf{E}\left[X^+\right]$ and $\lim_{n\to\infty} \mathbf{E}\left[Y_n^+\right] = \mathbf{E}\left[Y^+\right]$. While for the second term, we simply apply the Bounded Convergence Theorem to conclude that $\lim_{n\to\infty} \int_0^a \mathbf{P}\left[X_n > t\right] dt = \int_0^a \mathbf{P}\left[X > t\right] dt$ and $\lim_{n\to\infty} \int_0^a \mathbf{P}\left[Y_n > t\right] dt = \int_0^a \mathbf{P}\left[Y > t\right] dt$ for each a in $\mathbb{R}$ since by the assumed weak convergence, we have $\lim_{n\to\infty} \mathbf{P}\left[X_n > t\right] = \mathbf{P}\left[X > t\right]$ and $\lim_{n\to\infty} \mathbf{P}\left[Y_n > t\right] = \mathbf{P}\left[Y > t\right]$. Thus, we obtain

$$\mathbf{E}\left[X^+\right] - \int_0^a \mathbf{P}\left[X > t\right] dt \le \mathbf{E}\left[Y^+\right] - \int_0^a \mathbf{P}\left[Y > t\right] dt, \quad a \in \mathbb{R},$$

or equivalently,

$$\mathbf{E}\left[(X - a)^+\right] \le \mathbf{E}\left[(Y - a)^+\right], \quad a \in \mathbb{R},$$

and the desired result follows. ■

The case of non-negative rvs is of special interest here.

Lemma 5.2.2 *Let $\{X_n, n = 1, 2, \ldots\}$ and $\{Y_n, n = 1, 2, \ldots\}$ denote two sequences of $\mathbb{R}_+$-valued rvs such that $X_n \Longrightarrow_n X$ and $Y_n \Longrightarrow_n Y$ with $X_n \le_{icx} Y_n$ for each $n = 1, 2, \ldots$. If $\lim_{n\to\infty} \mathbf{E}\left[X_n\right] = \mathbf{E}\left[X\right]$ and $\lim_{n\to\infty} \mathbf{E}\left[Y_n\right] = \mathbf{E}\left[Y\right]$, then $X \le_{icx} Y$.*

By virtue of Theorem 2.2.9 in Stoyan [48], it is now possible to show that the steady state comparison of the buffer levels holds under the sole stationarity assumption of the input traffic.

Theorem 5.2.1 *For each $i = 1, 2$, let $\{Q_t^i, t = 0, 1, \ldots\}$ with $Q_0^i = 0$ be the buffer content sequence of the discrete-time single server queue (5.1) fed by the stationary input traffic $\{A_t^i, t = 0, 1, \ldots\}$ with $\mathbf{E}[A_0^i] < c$. If for each $t = 0, 1, \ldots$, $Q_t^1 \leq_{icx} Q_t^2$, then $Q^1 \leq_{icx} Q^2$ where for each $i = 1, 2$, Q^i is the steady state buffer contents of the sequence $\{Q_t^i, t = 0, 1, \ldots\}$.*

Proof. Fix $i = 1, 2$. Recall the mapping $T : \mathbb{R}^2 \to \mathbb{R}$ defined in Section 5.1 by

$$T(a, q) = (q + a - c)^+.$$

Clearly, T is increasing in both arguments and $Q_{t+1}^i = T(A_t^i, Q_t^i)$. Since the input traffic $\{A_t^i, t = 0, 1, \ldots\}$ is stationary and $Q_0^i = 0$, Theorem 2.2.9 in Stoyan [48] implies

$$0 = Q_0^i \leq_{st} Q_t^i \leq_{st} Q_{t+1}^i, \quad t = 1, 2, \ldots. \tag{5.7}$$

More precisely, (5.7) is equivalent [45] to $\mathbf{P}[Q_t^i > x] \leq \mathbf{P}\left[Q_{t+1}^i > x\right]$ for all $t = 0, 1, \ldots$ and $x \geq 0$. From the fact that $\{A_t^i, t = 0, 1, \ldots\}$ is stationary with $\mathbf{E}[A_0^i] < c$, we have $Q_t^i \Longrightarrow_t Q^i$ [27], whence $\lim_{t\to\infty} \mathbf{P}[Q_t^i > x] = \mathbf{P}[Q^i > x]$ for any point x of continuity of Q^i. Applying the Monotone Convergence Theorem,

$$\mathbf{E}\left[Q^i\right] = \int_0^\infty \mathbf{P}\left[Q^i > x\right] dx = \lim_{t\to\infty} \int_0^\infty \mathbf{P}\left[Q_t^i > x\right] dx = \lim_{t\to\infty} \mathbf{E}\left[Q_t^i\right]$$

and the desired result follows by Lemma 5.2.2. ■

The steady state result seems very attractive since the only assumptions required on the input process are stationarity and the negative drift condition ($\mathbf{E}[A_0] < c$). However, the comparison will be trivial if the first moment of the steady buffer size is infinite. Hence, it is desirable to find conditions on the input traffic which ensure the finiteness of this moment. We take on this issue in the next section.

5.3 Finiteness of the first moment of steady state buffer levels

We begin with the classical result on the standard $GI|GI|1$ queue by Kiefer and Wolfowitz [21].

Theorem 5.3.1 *Consider a $GI|GI|1$ queue with a sequence of i.i.d. service time $\{\sigma_n, n = 0, 1, \ldots\}$ with generic rv σ and a sequence of i.i.d. interarrival time $\{\tau_{n+1}, n = 0, 1, \ldots\}$ with generic rv τ. If $\mathbf{E}[\sigma] < \mathbf{E}[\tau]$, then $\mathbf{E}[W] < \infty$ if and only if $\mathbf{E}[\sigma^2] < \infty$, where*

$$W = [\sup_{t=0,1,\ldots} \sum_{i=0}^{t} (\sigma_i - \tau_{i+1})]^+ \tag{5.8}$$

is the stationary waiting time of the $GI|GI|1$ queue.

In the sequel, we write $W(\sigma, \tau)$ to denote the stationary waiting time rv (5.8) associated with the standard $GI|GI|1$ queue with generic service time σ and interarrival time τ.

Consider a discrete-time single server queue (5.1) fed by the i.i.d. sequence $\{A_t, t = 0, 1, \ldots\}$. From (5.2) and (5.8), the input traffic $\{A_t, t = 0, 1, \ldots\}$ is identified with the sequence of i.i.d. service time in the $GI|GI|1$ queue, thus $Q =_{st} W(A_0, c)$. Hence, by Theorem 5.3.1, the moment $\mathbf{E}[Q] < \infty$ if and only if $\mathbf{E}[A_0^2] < \infty$. Since the independent version of any stationary input traffic is i.i.d., the first moment of its steady state buffer levels is finite if and only if the second moment of the input traffic is finite. For other input processes, we refer to the conditions for finite moments of waiting time of a $G|G|1$ queue in [12], however, these conditions are not very useful here with the traffic models of interest.

In the following sections, we consider conditions for the finiteness of the first moment of steady state buffer levels under each of the three traffic models, namely FGN(H) traffic models, on-off sources, and $M|G|\infty$ input processes.

5.3.1 FGN

Let $\{A_t^H, t = 0, 1, \ldots\}$ be the FGN(H) traffic model defined in Section 3.1. Since the FGN(H) traffic model is reversible, by using the representation (5.2) for the steady state buffer levels, we have for fixed $x > 0$,

$$\begin{aligned}
\mathbf{P}[Q > x] &= \mathbf{P}\left[\sup_{t=0,1,\ldots} \left(\sum_{s=0}^{t} A_s^H - c(t+1)\right) > x\right] \\
&\leq \sum_{t=0}^{\infty} \mathbf{P}\left[\sum_{s=0}^{t} A_s^H - c(t+1) > x\right] \\
&\leq \sum_{t=0}^{\infty} e^{-\theta(x+c(t+1))} \mathbf{E}\left[e^{\theta \sum_{s=0}^{t} A_s^H}\right], \quad \theta > 0,
\end{aligned}$$

where the last inequality follows by Chernoff bound argument. Therefore, for some $\theta > 0$, we obtain

$$\begin{aligned}
\mathbf{E}[Q] = \int_0^\infty \mathbf{P}[Q > x]\, dx &\leq \int_0^\infty \sum_{t=0}^{\infty} e^{-\theta(x+c(t+1))} \mathbf{E}\left[e^{\theta \sum_{s=0}^{t} A_s^H}\right] dx \\
&= \frac{C(\theta)}{\theta}
\end{aligned} \tag{5.9}$$

where we have set

$$C(\theta) = \sum_{t=0}^{\infty} e^{-\theta c(t+1)} \mathbf{E}\left[e^{\theta \sum_{s=0}^{t} A_s^H}\right]$$

and $\mathbf{E}[Q] \leq \infty$ if $C(\theta) < \infty$.

Recall that for each $t = 0, 1, \ldots$, $A_t^H = m + N_t^H$ where m is the average traffic rate, $N_t^H = B_{t+1}^H - B_t^H$ and $\{B_t^H, t \geq 0\}$ indicates the FBM with Hurst parameter

H. Thus,

$$\sum_{s=0}^{t} A_s^H = m(t+1) + B_{t+1}^H, \quad t = 0, 1, \ldots,$$

and because $\{B_t^H, t \geq 0\}$ is a Gaussian process, we simply have

$$\begin{aligned} C(\theta) &= \sum_{t=0}^{\infty} e^{-\theta c(t+1)} \mathbf{E}\left[e^{\theta(m(t+1)+B_{t+1}^H)}\right] \\ &= \sum_{t=0}^{\infty} e^{-\theta(c-m)(t+1)} e^{-\frac{1}{2}\theta^2\sigma^2(t+1)^{2H}}. \end{aligned} \tag{5.10}$$

Under the stability assumption $m < c$, it can be shown that for $\theta > 0$ and $0.5 \leq H < 1$,

$$\lim_{t\to\infty} \exp\{-\theta(c-m) - \frac{1}{2}\theta^2\sigma^2[(t+1)^{2H} - t^{2H}]\} = 0 < 1,$$

whence by d'Alembert's test, the series in (5.10) converges and $C(\theta) < \infty$ for $\theta > 0$. Applying this finding to (5.9), we conclude the following:

Lemma 5.3.1 *Under the stability assumption* $\mathbf{E}[A_0] < c$, *if the input traffic* $\{A_t, t = 0, 1, \ldots\}$ *fed to the discrete-time single server queue (5.1) is the FGN(H) traffic model with* $0.5 \leq H < 1$, *then* $\mathbf{E}[Q] < \infty$ *where* Q *is the steady state buffer level.*

5.3.2 On-off sources

Consider the stationary version $\{A_t, t = 0, 1, \ldots\}$ of the on-off sources (I, B) described in Section 3.2. Assume that the server capacity is smaller than peak rate of the source (i.e., $c < 1$) so that Q_t is not identically zero for all $t = 1, 2, \ldots$. It is well-known that the stationary on-off process is reversible, thus we can write (5.2) for the steady state buffer level of the on-off source (I, B) as

$$Q =_{st} (\sup_{t=0,1,\ldots} \{\sum_{s=0}^{t} A_s - c(t+1)\})^+. \tag{5.11}$$

We first show that Q can be related to the stationary waiting time of a $GI|GI|1$ queue [28]. Define a sequence of mutually independent $\mathbb{R}$-valued rvs $\{X_n, n = 0, 1, \ldots\}$ by

$$X_l := (1-c)B_l - cI_l, \quad l = 0, 1, \ldots, \tag{5.12}$$

and set

$$M := (\sup_{n=1,2,\ldots} \sum_{l=1}^{n} X_l)^+ \tag{5.13}$$

where the rvs $\{I_l, B_l, l = 0, 1, \ldots\}$ are as specified in the construction of the stationary on-off source $\{A_t, t = 0, 1, \ldots\}$. While $X_l =_{st} (1-c)B - cI$ for all $l = 1, 2, \ldots$, we have

$$X_0 =_{st} (1-c)\hat{B}U + ((1-c)B - c\hat{I})(1-U) \tag{5.14}$$

where U is a Bernoulli rv with parameter p independent of $B, \hat{B}$ and $\hat{I}$. Note from its definition (5.13) that M is identified with the stationary waiting time $W((1-c)B, cI)$.

Proposition 5.3.1 *It holds that*

$$Q =_{st} (X_0 + M)^+ \tag{5.15}$$

with the rv X_0 taken independent of the rv M.

Proof. Recall the sequence of epochs $\{T_n = 0, 1, \ldots\}$ marking the beginning of the $(n+1)$th on-off cycle. Fix $n = 0, 1, \ldots$, from (5.12), we readily get

$$\begin{aligned} X_n &= \sum_{s=T_n}^{T_{n+1}-1} A_s - c(T_{n+1} - T_n) \\ &\geq \sum_{s=T_n}^{t} A_s - c(t - T_n + 1), \quad T_n \leq t < T_{n+1}, \end{aligned} \tag{5.16}$$

where the last inequality follows from the specification that each cycle consists of an idle period followed by an active period. The inequality (5.16) also implies that

$$\sum_{l=0}^{n} X_l = \sum_{s=0}^{T_{n+1}-1} A_s - cT_{n+1} \geq \sum_{s=0}^{t} A_s - c(t+1), \quad T_n \leq t < T_{n+1}. \tag{5.17}$$

Combining the last inequality with (5.11) yields

$$\begin{aligned} Q &=_{st} (\sup_{t=0,1,\ldots} \{\sum_{s=0}^{t} A_s - c(t+1)\})^+ \\ &=_{st} (\sup_{n=0,1,\ldots} (\sup_{T_n \leq t < T_{n+1}} \{\sum_{s=0}^{t} A_s - c(t+1)\}))^+ \\ &=_{st} (\sup_{n=0,1,\ldots} \sum_{l=0}^{n} X_l)^+. \end{aligned}$$

It is easy to show that

$$Q = (X_0 + \max(0, \sup_{n=1,2,\ldots} \sum_{l=1}^{n} X_l))^+$$

and the conclusion (5.15) thus follows. ■

From the relationship (5.15) between Q, X_0 and $M =_{st} W((1-c)B, cI)$, we obtain the following:

Lemma 5.3.2 *Under the stability assumption* $\mathbf{E}[A_0] < c$, *if the input traffic* $\{A_t, t = 0, 1, \ldots\}$ *fed to the discrete-time single server queue (5.1) is the stationary on-off source* (I, B), *then* $\mathbf{E}[Q] < \infty$ *if and only if* $\mathbf{E}[B^2] < \infty$, *where* Q *is the steady state buffer level.*

Proof. Suppose that $\mathbf{E}[Q] < \infty$. Using the relationship (5.15), both $\mathbf{E}[X_0]$ and $\mathbf{E}[M]$ must be finite. Since $M =_{st} W((1-c)B, cI)$, we have from Theorem 5.3.1 that $\mathbf{E}[B^2] < \infty$.

Conversely, assume that $\mathbf{E}[B^2] < \infty$. This condition ensures that $\mathbf{E}[B]$, $\mathbf{E}\left[\hat{B}\right]$ and $\mathbf{E}[M]$ are finite (from [40] and Theorem 5.3.1). Upon noting that B and M are non-negative rvs, from (5.15) and (5.14), we have

$$\begin{aligned}
\mathbf{E}[Q] &= \int_0^\infty (p\mathbf{P}\left[(1-c)\hat{B} + M > x\right] + (1-p)\mathbf{P}\left[(1-c)B - c\hat{I} + M > x\right])dx \\
&= p((1-c)\mathbf{E}\left[\hat{B}\right] + \mathbf{E}[M]) \\
&\quad +(1-p)\sum_{y=1}^{\infty}(\int_0^\infty \mathbf{P}\left[(1-c)B + M > x + cy\right]dx)\frac{\mathbf{P}[I \geq y]}{\mathbf{E}[I]} \\
&= p((1-c)\mathbf{E}\left[\hat{B}\right] + \mathbf{E}[M]) \\
&\quad +(1-p)\sum_{y=1}^{\infty}(\int_{cy}^\infty \mathbf{P}\left[(1-c)B + M > x\right]dx)\frac{\mathbf{P}[I \geq y]}{\mathbf{E}[I]} \\
&\leq p((1-c)\mathbf{E}\left[\hat{B}\right] + \mathbf{E}[M]) + (1-p)((1-c)\mathbf{E}[B] + \mathbf{E}[M])
\end{aligned}$$

and the conclusion $\mathbf{E}[Q] < \infty$ follows. ■

5.3.3 $M|G|\infty$ input traffic

Consider a discrete-time single server queue (5.1) fed by an $M|G|\infty$ input traffic (λ, S) $\{A_t, t = 0, 1, \ldots\}$ with $\mathbf{E}[A_0] < c$. Since the $M|G|\infty$ process is reversible [37], we have the representation (5.2). Upon combining the decomposition (3.11) and Proposition 3.3.2 via (5.2), we obtain

$$\begin{aligned}
Q &=_{st} (\sup_{t=0,1,\ldots}\{\sum_{s=0}^{t} A_s^{(0)} + \sum_{s=0}^{t} A_s^{(a)} - c(t+1)\})^+ \\
&=_{st} (\sup_{t=0,1,\ldots}\{\sum_{i=1}^{B}\min(t+1, \hat{S}_i) + \sum_{s=1}^{t}\sum_{i=1}^{B_s}\min(t-s+1, S_{s,i}) - c(t+1)\})^+ \\
&\leq_{st} (\sup_{t=0,1,\ldots}\{\sum_{i=1}^{B}\hat{S}_i + \sum_{s=1}^{t}\sum_{i=1}^{B_s} S_{s,i} - c(t+1)\})^+. \qquad (5.18)
\end{aligned}$$

The last inequality follows from the fact that $\min(t+1, \hat{S}_i) \leq_{st} \hat{S}_i$ and $\min(t - s + 1, S_{s,i}) \leq_{st} S_{s,i}$.

We again show that Q can be related to the stationary waiting time of a $GI|GI|1$ queue [25]. Upon defining the $\mathbb{R}$-valued rvs $\{Y_n, n = 0, 1, \ldots\}$ and Z by

$$Y_0 := \sum_{i=1}^{B} \hat{S}_i - c, \quad Y_n := \sum_{i=1}^{B_n} S_{n,i} - c, \quad n = 1, 2, \ldots,$$

and

$$Z := (\sup_{t=1,\ldots} \sum_{s=1}^{t} Y_s)^+, \tag{5.19}$$

respectively, we rewrite (5.18) as

$$Q \leq_{st} (\sup_{t=0,1,\ldots} \{Y_0 + \sum_{s=1}^{t} Y_s\})^+ =_{st} (Y_0 + Z)^+. \tag{5.20}$$

It is clear from (5.19) that Z is identified with the stationary waiting time $W(S, c)$, i.e., $Z =_{st} W(S, c)$.

Using the relationship (5.20) between Q, Y_0 and Z, we obtain a sufficient condition for the finiteness of the first moment of Q.

Lemma 5.3.3 *Under the stability assumption* $\mathbf{E}[A_0] < c$*, if the input traffic* $\{A_t, t = 0, 1, \ldots\}$ *fed to the discrete-time single server queue (5.1) is the* $M|G|\infty$ *input process* (λ, S) *with* $\mathbf{E}[S^2] < \infty$*, then* $\mathbf{E}[Q] < \infty$ *where* Q *is the steady state buffer level.*

Proof. If $\mathbf{E}[S^2]$ is finite, then $\mathbf{E}[S]$ and $\mathbf{E}\left[\hat{S}\right]$ are finite, and from Theorem 5.3.1, we obtain $\mathbf{E}[Z] = \mathbf{E}[W(S, c)] < \infty$. Note that the inequality (5.20) implies

$$\begin{aligned} \mathbf{E}[Q] \leq \mathbf{E}\left[(Y_0 + Z)^+\right] &\leq \mathbf{E}[|Y_0|] + \mathbf{E}[Z] \\ &\leq \lambda \mathbf{E}[S]\,\mathbf{E}\left[\hat{S}\right] + c + \mathbf{E}[Z], \end{aligned}$$

where we have used the facts that Z, S and $\hat{S}$ are non-negative rvs and $\mathbf{E}[|Y_0|] < \mathbf{E}[B]\,\mathbf{E}\left[\hat{S}\right] + c$ with $\mathbf{E}[B] = \lambda \mathbf{E}[S]$. The conclusion $\mathbf{E}[Q] < \infty$ readily follows. ■

Chapter 6

FGN

6.1 Main results

From its definition in Chapter 3, the FGN(H) traffic model with Hurst parameter $0.5 \leq H < 1$ is a Gaussian process $\{A_t^H, t = 0, 1, \ldots\}$ with average traffic rate m, and autocovariance function

$$\Gamma_H(k) = \frac{\sigma^2}{2}(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}), \quad k = 0, 1, \ldots. \tag{6.1}$$

It is plain that the independent version of FGN(H) traffic models must be an i.i.d. Gaussian process $\{\hat{A}_t, t = 0, 1, \ldots\}$ with $\mathbf{E}\left[\hat{A}_t\right] = m$ for each $t = 0, 1, \ldots$. Since $\Gamma_H(0) = \sigma^2$, its autocovariance function is given by $\hat{\Gamma}(k) = \sigma^2\delta(k)$ where $\delta(k) = 1$ when $k = 0$ and $\delta(k) = 0$ when $k \neq 0$. Equivalently, this independent process is simply the FGN(0.5) traffic model.

It is easily seen that for $t = 1, 2, \ldots$, $[A_t^H | A_0^H, \ldots, A_{t-1}^H]$ is normally distributed. Since a Gaussian rv is stochastically increasing in the mean [45], we can establish the SSI property, if we can show that the conditional mean $\mathbf{E}\left[A_t^H | A_0^H = a_0, \ldots, A_{t-1}^H = a_{t-1}\right]$ is an increasing function in $(a_0, \ldots, a_{t-1})$ for each $t = 1, 2, \ldots$. Although the autocovariance function is explicitly given, we were unable to obtain a usable closed-form expressions for these conditional means due to the complicated structure of the involved matrices (and of their inverses). Instead, we turn to the comprehensive characterization of stochastic orderings given by Müller [31, Thm. 3.8] for Gaussian rvs.

Theorem 6.1.1 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be* $\mathbb{R}^n$*-valued rvs such that* $\mathbf{X} =_{st} \mathcal{N}(\mu, \mathbf{\Sigma})$ *and* $\mathbf{Y} =_{st} \mathcal{N}(\mu', \mathbf{\Sigma}')$*. Then* $\mathbf{X} \leq_{dcx} \mathbf{Y}$ *if and only if* $\mu = \mu'$ *and* $\Sigma_{ij} \leq \Sigma'_{ij}$ *for all* $1 \leq i, j \leq n$*.*

From (6.1), with $0.5 \leq H < 1$, it follows that $\Gamma_H(0) = \sigma^2$ and $\Gamma_H(k) \geq 0$ for all $k = 1, 2, \ldots$. Moreover, $\mathbf{E}\left[\hat{A}_t\right] = \mathbf{E}\left[A_t\right] = m$ for all $t = 0, 1, \ldots$. As a direct application of (4.2) and Theorem 6.1.1, we conclude that the independent version (i.e., the FGN(0.5) traffic model) is indeed a lower bound process for the FGN(H) traffic model with $0.5 \leq H < 1$.

Theorem 6.1.2 *Let* $\{A_t^H, t = 0, 1, \ldots\}$ *be a FGN traffic model with parameter* $0.5 \leq H < 1$*. Its independent version* $\{\hat{A}_t, t = 0, 1, \ldots\}$ *coincides with the FGN(0.5) traffic model, and satisfies*

$$\{\hat{A}_t, t = 0, 1, \ldots\} \leq_{idcx} \{A_t^H, t = 0, 1, \ldots\}.$$

Moreover, their corresponding buffer contents $\{Q_t^H, t = 0, 1, \ldots\}$ *and* $\{\hat{Q}_t, t = 0, 1, \ldots\}$ *are ordered in the icx ordering, i.e., for any fixed initial condition* $\hat{Q}_0 = Q_0 = q$*, we have*

$$\hat{Q}_t \leq_{icx} Q_t^H, \quad t = 0, 1, \ldots.$$

Furthermore, by virtue of Theorem 6.1.1, it is possible to compare two FGN traffic models with Hurst parameter H and H' in [0.5,1) such that $H' < H$. To do so, we need to verify that when $H' < H$, $\Gamma_{H'}(k) \leq \Gamma_H(k)$ for all $k = 0, 1, \ldots$ as established in the next lemma.

Lemma 6.1.1 *For each $k = 0, 1, \ldots$, the mapping $H \to \Gamma_H(k)$ given by (6.1) is monotone increasing in H on [0.5,1).*

Proof. For $k = 0$, $\Gamma_H(0) = \sigma^2$ for all $0.5 \le H < 1$. Fix $k = 1, 2, \ldots$. Suppose $0.5 \le H' < H < 1$. We will show that $\Gamma_{H'}(k) \le \Gamma_H(k)$, or equivalently,

$$|k+1|^{2H} - |k+1|^{2H'} + |k-1|^{2H} + |k-1|^{2H'} \ge 2|k|^{2H} - 2|k|^{2H'}. \tag{6.2}$$

Clearly, (6.2) holds for $k = 1$.

Now, for $k > 1$, define the mapping $f : \mathbb{R}_+ \to \mathbb{R} : x \to x^{2H} - x^{2H'}$ and note that (6.2) can be rewritten as

$$f(k+1) + f(k-1) \ge 2f(k), \quad k > 1. \tag{6.3}$$

Thus, it is enough to show that f is convex on $[1, \infty)$ in order to show that $\Gamma_{H'}(k) \le \Gamma_H(k), k = 2, 3, \ldots$. To do so, we evaluate the second derivative of f, namely

$$\frac{df^2(x)}{dx^2} = 2x^{-2}[H(2H-1)x^{2H} - H'(2H'-1)x^{2H'}], \quad x \ge 1. \tag{6.4}$$

As $0.5 \le H' < H < 1$, $H(2H-1) > H'(2H'-1)$ and $2H > 2H'$. Therefore, we obtain $\frac{df^2(x)}{dx^2} > 0$ whenever $x \ge 1$ and the mapping f is indeed convex on $[1, \infty)$. In conclusion, (6.2) (or equivalently, $\Gamma_{H'}(k) \le \Gamma_H(k)$) is satisfied for all $k = 1, 2, \ldots$, whence, $\Gamma(k)$ is monotone increasing on [0.5,1). ■

By Lemma 6.1.1 and Theorem 6.1.1, we conclude that

$$\{A_t^{H'}, t = 0, 1, \ldots\} \le_{idcx} \{A_t^H, t = 0, 1, \ldots\} \quad \text{if} \quad 0.5 \le H' < H < 1. \tag{6.5}$$

The following theorem is now a simple consequence of (6.5) and of Theorem 5.1.1.

Theorem 6.1.3 *Let* $\{A_t^{H'}, t = 0, 1, \ldots\}$ *and* $\{A_t^H, t = 0, 1, \ldots\}$ *be the FGN(H') and FGN(H) traffic models, respectively, with* $0.5 \le H' < H < 1$*. Then, we have the comparison*

$$\{A_t^{H'}, t = 0, 1, \ldots\} \le_{idcx} \{A_t^H, t = 0, 1, \ldots\},$$

and their corresponding buffer contents $\{Q_t^{H'}, t = 0, 1, \ldots\}$ *and* $\{Q_t^H, t = 0, 1, \ldots\}$ *are ordered in the icx ordering, i.e., for any fixed initial condition* $Q_0^{H'} = Q_0^H = q$*, we have*

$$Q_t^{H'} \le_{icx} Q_t^H, \quad t = 0, 1, \ldots.$$

From Theorem 6.1.2 we can conclude, as expected, that the long-range dependent traffic ($0.5 < H < 1$) requires more buffer space than the short-range dependent traffic ($H = 0.5$). Moreover, when $H' < H$, $\Gamma_{H'}(k) \le \Gamma_H(k)$, i.e., FGN($H$) is more correlated than FGN(H'), and by Theorem 6.1.3, the more correlated the traffic, the more buffer space is required to meet the same QoS requirement.

6.2 Simulation results

In this section, we use simulations to verify the comparison results of Theorem 6.1.2 and 6.1.3. To do so, we begin with the description of the experiments which will also be used in Chapter 9 and Section 11.4. In order to illustrate "size and variability" concepts of the icx ordering, we compare the first and second moments of the buffer levels corresponding to the input traffics. For each simulation, we generate N independent sample paths of the input traffic and feed them through the discrete-time single server queue (5.1) with multiplexer release rate c packets/slot to obtain the buffer content sequences $\{q_{t,i}, t = 0, 1, \ldots\}$ for each sample path $i = 1, \ldots, N$ where we have set $q_{0,i} = 0$ for all $i = 1, \ldots, N$. For fixed $t = 0, 1, \ldots$, the first and second moments of buffer occupancy levels are calculated by $\frac{1}{N}\sum_{i=1}^{N} q_{t,i}$ and $\frac{1}{N}\sum_{i=1}^{N} q_{t,i}^2$, respectively. In the sequel, we will refer to the sequences $\{\frac{1}{N}\sum_{i=1}^{N} q_{t,i}, t = 0, 1, \ldots\}$ and $\{\frac{1}{N}\sum_{i=1}^{N} q_{t,i}, t = 0, 1, \ldots\}$ as the first and second moments of buffer sizes.

In the case of FGN(H) traffic models, we verify the comparison results by showing that the first and second moments of the buffer sizes corresponding to the FGN(H) traffic models are monotone in the Hurst parameter. Throughout this section, we fix the multiplexer release rate $c = 6$ and the number of sample paths N=10,000 and generate the $FGN(H)$ traffic models for $H = 0.5, 0.6, 0.7, 0.8$ and 0.9 using the method described in [41]. Regardless of the value of H, each sample path has mean traffic rate $m = 5$ and variance $\sigma^2 = 5$. Figure 1-4 compare 4 pairs of the first moment of buffer sizes for $FGN(H)$ traffic models with $H = 0.5, 0.6, 0.7, 0.8$ and 0.9. It is clear that the first moments are monotone in H and the FGN(0.5) traffic model provides the lower bound as it is an independent version. By the same manner, we compare 4 pairs of second moments of buffer sizes for $FGN(H)$ traffic models with $H = 0.5, 0.6, 0.7, 0.8$ and 0.9 in Figure 5-8. Again, the monotonicity in H of the second moments holds and the FGN(0.5) traffic model indeed yields the smallest second moment.

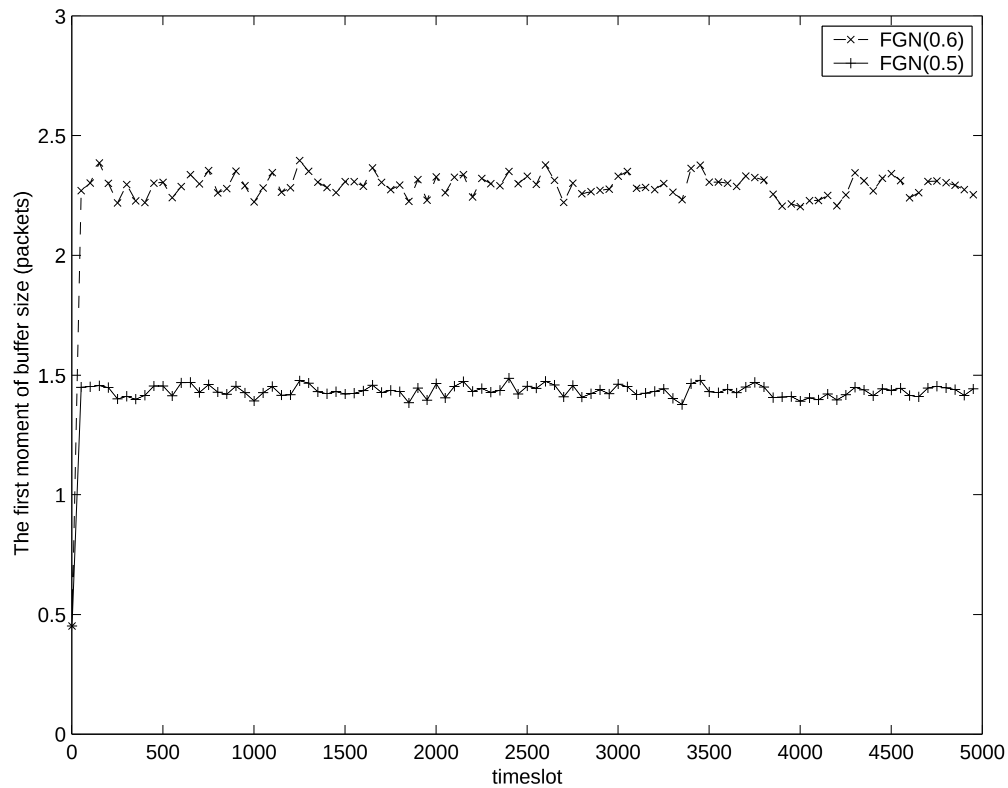

Figure 1: The first moments of the buffer sizes for the FGN(0.5) (the independent version) and FGN(0.6) traffic models

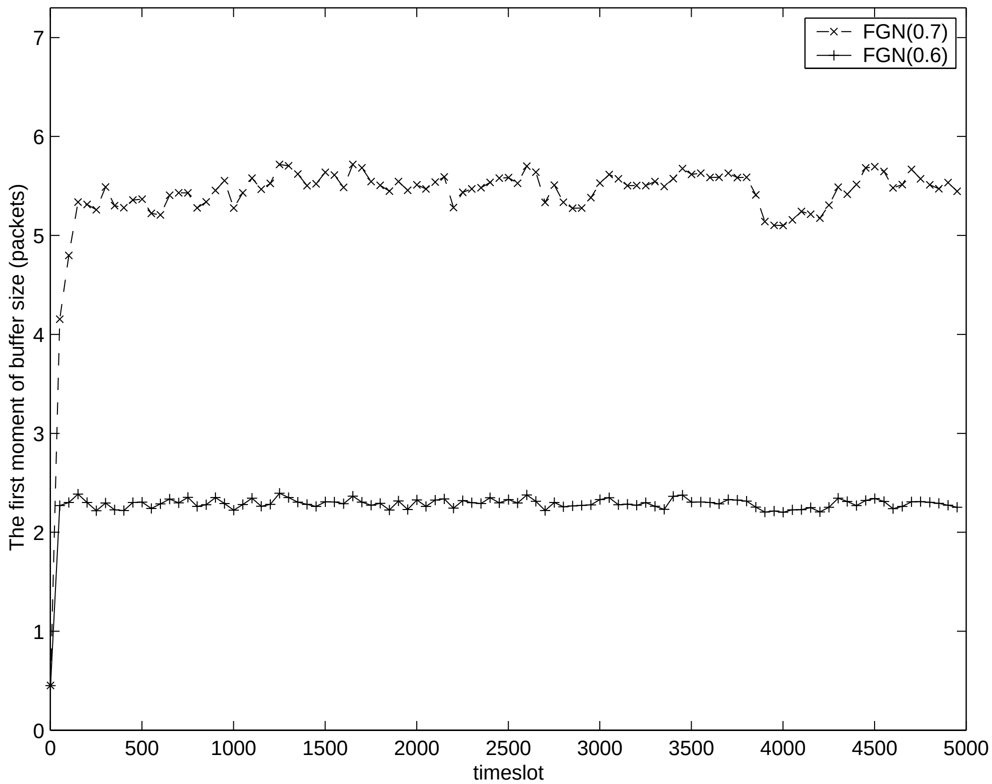

Figure 2: The first moments of the buffer sizes for the FGN(0.6) and FGN(0.7) traffic models

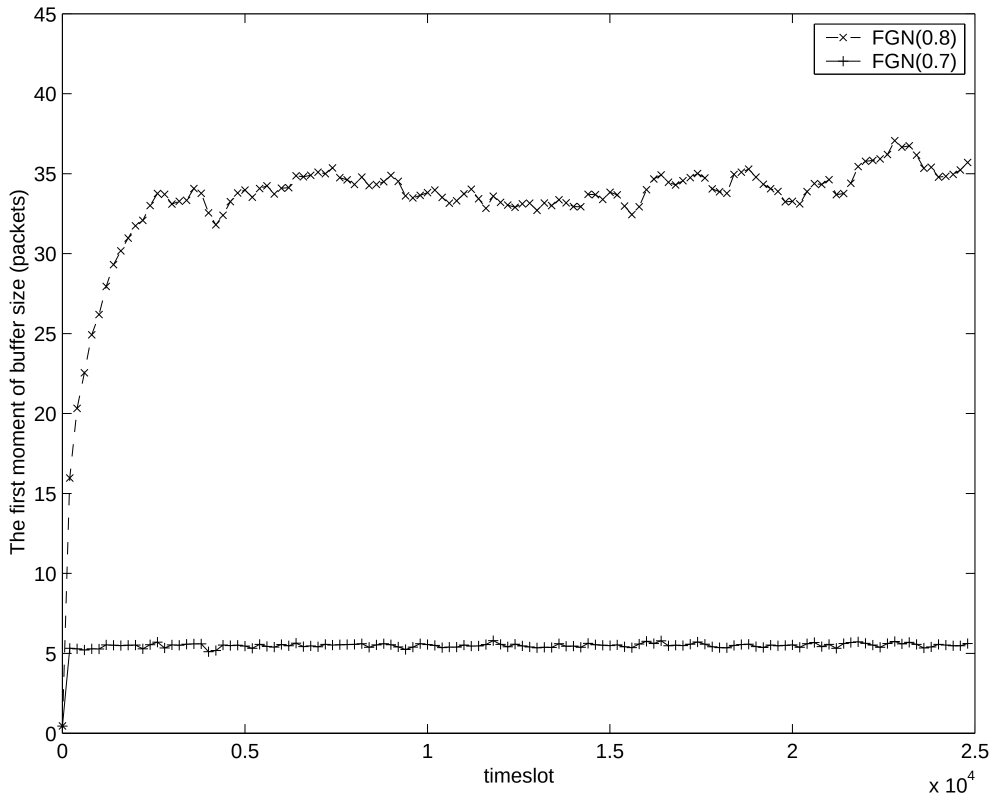

Figure 3: The first moments of the buffer sizes for the FGN(0.7) and FGN(0.8) traffic models

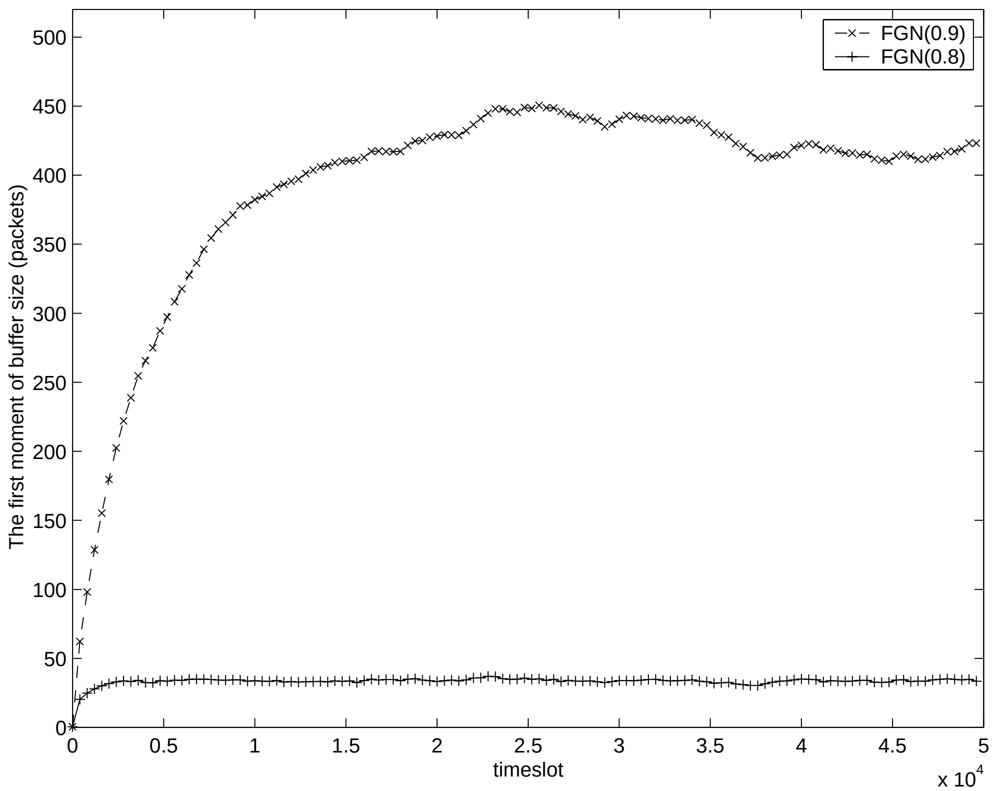

Figure 4: The first moments of the buffer sizes for the FGN(0.8) and FGN(0.9) traffic models

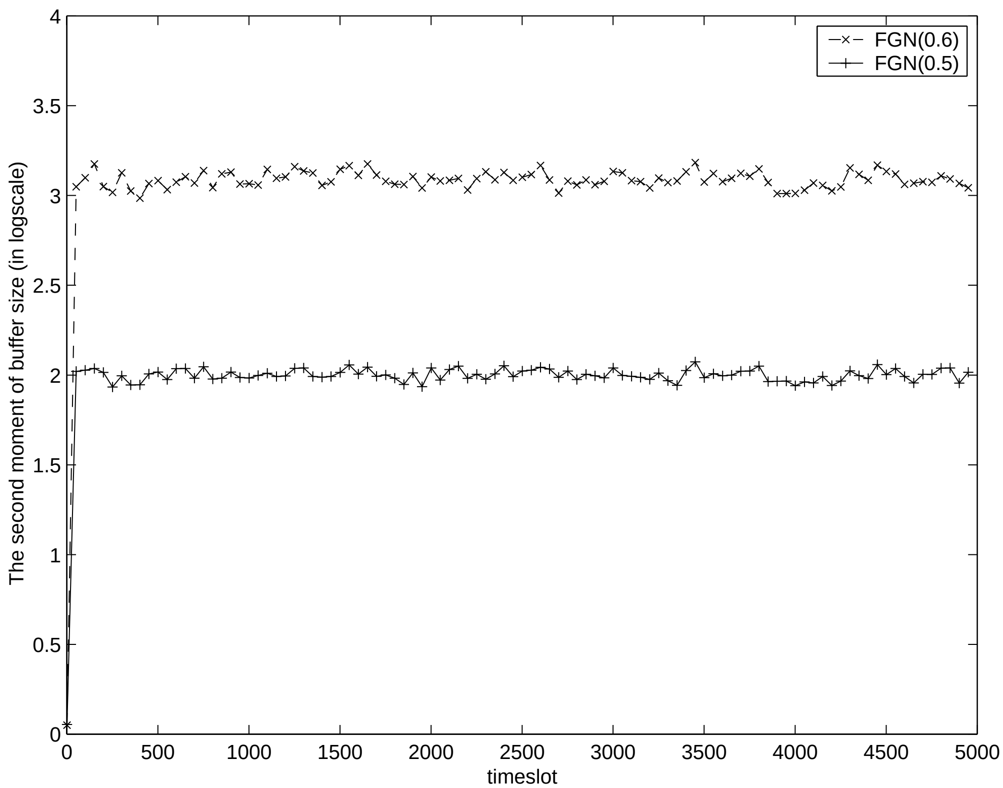

Figure 5: The second moments of the buffer sizes (in logscale) for the FGN(0.5) (the independent version) and FGN(0.6) traffic models

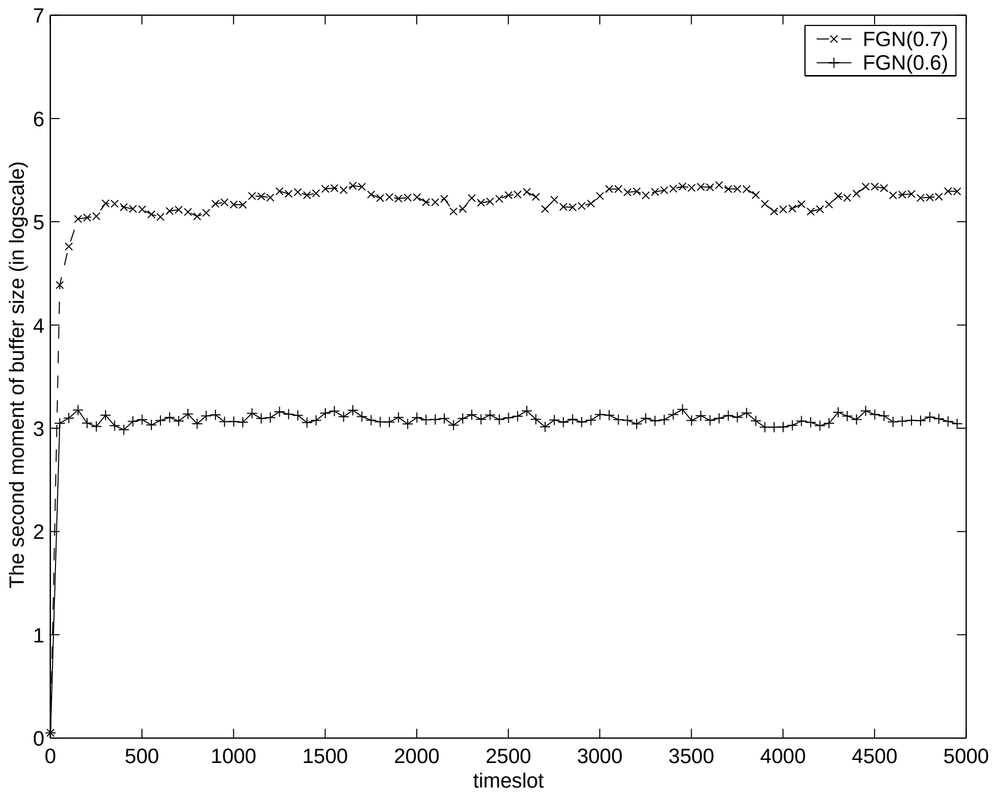

Figure 6: The second moments of the buffer sizes (in logscale) for the FGN(0.6) and FGN(0.7) traffic models

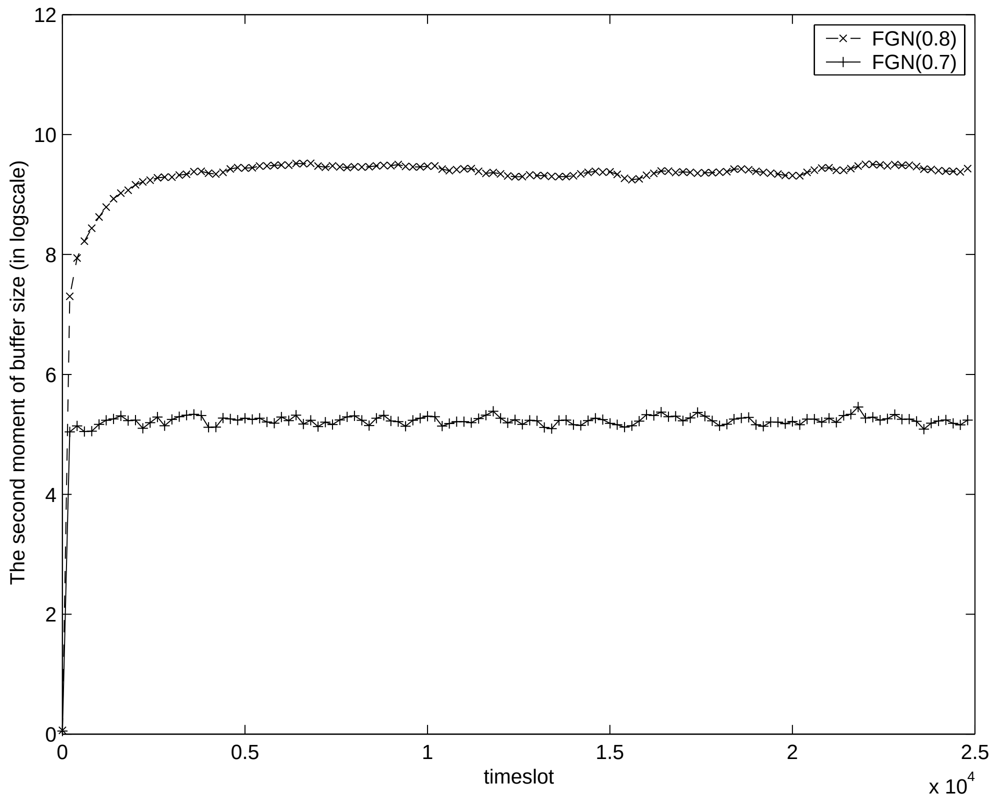

Figure 7: The second moments of the buffer sizes (in logscale) for the FGN(0.7) and FGN(0.8) traffic models

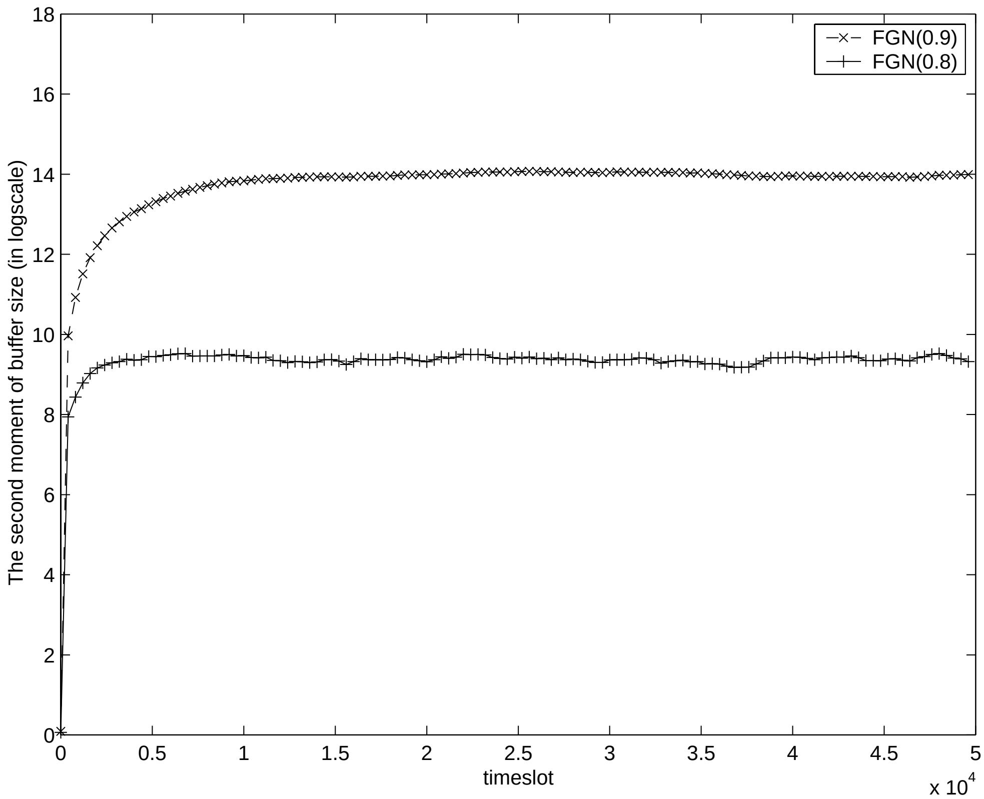

Figure 8: The second moments of the buffer sizes (in logscale) for the FGN(0.8) and FGN(0.9) traffic models

Chapter 7

Stationary On-off Sources

The discrete-time on-off source $\{A_t, t = 0, 1, \ldots\}$ is a $\{0, 1\}$-process with $A_t = 0$ (respectively, $A_t = 1$) if there is no packet (respectively, a packet) generated during timeslot $[t, t+1)$. Then, the independent version $\{\hat{A}_t, t = 0, 1, \ldots\}$ of the on-off process is simply an independent sequence of $\{0, 1\}$-valued rvs with

$$\mathbf{P}\left[\hat{A}_t = 1\right] = 1 - \mathbf{P}\left[\hat{A}_t = 0\right] = p, \quad t = 0, 1, \ldots$$

where p is the average rate of source given in (3.5). It is easily seen that $\{\hat{A}_t, t = 0, 1, \ldots\}$ is also an on-off process with geometric on period and off period, i.e., the corresponding on period duration rv B (respectively, off period duration rv I) is geometrically distributed with parameter p (respectively, $1-p$), i.e.,

$$B =_{st} \mathcal{G}(p) \quad \text{and} \quad I =_{st} \mathcal{G}(1-p).$$

In other words, $\{\hat{A}_t, t = 0, 1, \ldots\}$ can be interpreted as the discrete-time stationary on-off process $(\mathcal{G}(1-p), \mathcal{G}(p))$.

In order to establish the comparison between the on-off source and its independent version, we are interested in finding conditions for the SSI property, i.e., conditions on the rvs I and B so that for all $t = 0, 1, \ldots$, the inequalities

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] \leq \mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{y}^t\right] \tag{7.1}$$

hold whenever $\mathbf{x}^t \leq \mathbf{y}^t$ in $\{0,1\}^{t+1}$ with

$$\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] > 0 \quad \text{and} \quad \mathbf{P}\left[\mathbf{A}^t = \mathbf{y}^t\right] > 0. \tag{7.2}$$

As we proceed by evaluating the relevant conditional probabilities, in all cases we rely on the basic observation that

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] = \frac{\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t; A_{t+1} = 1\right]}{\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right]} \tag{7.3}$$

for every $\mathbf{x}^t$ in $\{0,1\}^{t+1}$ for which $\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] > 0$.

7.1 Expressions for stationary on-off sources

In this section, we focus on evaluating (7.3) when $\{A_t, t = 0, 1, \ldots\}$ is a stationary on-off source. Let $h_B(t)$ and $r_B(t), t = 1, 2, \ldots, T_B$, be the failure rate function and residual life function of rv B, respectively. Similarly for rv $\hat{B}$, I and $\hat{I}$, we define $h_{\hat{B}}(t)$, $r_{\hat{B}}(t), t = 1, 2, \ldots, T_B$, $h_I(t)$, $r_I(t)$, $h_{\hat{I}}(t)$ and $r_{\hat{I}}(t), t = 1, 2, \ldots, T_I$. We first find the expression (7.3) of the stationary on-off source for the case $t = 0$.

Lemma 7.1.1 *For the stationary on-off source (I, B), we have*

$$\mathbf{P}\left[A_1 = 1 | A_0 = 0\right] = h_{\hat{I}}(1) \tag{7.4}$$

and

$$\mathbf{P}\left[A_1 = 1 | A_0 = 1\right] = r_{\hat{B}}(1) = 1 - h_{\hat{B}}(1). \tag{7.5}$$

Proof. The conclusions (7.4) and (7.5) are easy consequences of the facts

$$\mathbf{P}\left[A_1 = 1 | A_0 = 0\right] \quad = \quad \frac{\mathbf{P}\left[A_0 = 0, A_1 = 1\right]}{\mathbf{P}\left[A_0 = 0\right]}$$

$$
\begin{aligned}
&= \frac{\mathbf{P}[I_0 = 1, B_0 \geq 1]}{\mathbf{P}[I_0 \geq 1]} \\
&= \frac{\mathbf{P}[I_0 = 1, B_0 \geq 1 | I_0 > 0]}{\mathbf{P}[I_0 \geq 1 | I_0 > 0]} \\
&= \frac{\mathbf{P}\left[\hat{I} = 1\right]}{\mathbf{P}\left[\hat{I} \geq 1\right]} \mathbf{P}[B \geq 1]
\end{aligned}
$$

with $\mathbf{P}[B \geq 1] = 1$, and

$$
\begin{aligned}
\mathbf{P}[A_1 = 1 | A_0 = 1] &= \frac{\mathbf{P}[A_0 = 1, A_1 = 1]}{\mathbf{P}[A_0 = 1]} \\
&= \frac{\mathbf{P}[I_0 = 0, B_0 \geq 2]}{\mathbf{P}[I_0 = 0, B_0 \geq 1]} \\
&= \frac{\mathbf{P}[B_0 \geq 2 | I_0 = 0]}{\mathbf{P}[B_0 \geq 1 | I_0 = 0]} \\
&= \frac{\mathbf{P}\left[\hat{B} \geq 2\right]}{\mathbf{P}\left[\hat{B} \geq 1\right]}.
\end{aligned}
$$

■

To describe the results when $t = 1, 2, \ldots$, we associate with any $\mathbf{x}^t$ in $\{0, 1\}^{t+1}$ the index $\ell(\mathbf{x}^t)$ of "last change" given by

$$
\ell(\mathbf{x}^t) := \min \{r = 0, 1, \ldots, t : \; x_r = \ldots = x_t\}.
$$

If $\ell(\mathbf{x}^t) > 0$, then

$$
x_{\ell(\mathbf{x}^t)-1} \neq x_{\ell(\mathbf{x}^t)} = \ldots = x_t, \tag{7.6}
$$

while if $\ell(\mathbf{x}^t) = 0$, then

$$
x_0 = x_1 = \ldots = x_t.
$$

Fix $t = 1, 2, \ldots$ throughout.

Proposition 7.1.1 *For the stationary on-off source (I, B), for each $\mathbf{x}^t$ in $\{0, 1\}^{t+1}$*

with $x_t = 1$, *we have*

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] = \begin{cases} r_B(t - \ell(\mathbf{x}^t) + 1) & \textit{if } \ell(\mathbf{x}^t) > 0 \\ \\ r_{\hat{B}}(t+1) & \textit{if } \ell(\mathbf{x}^t) = 0 \end{cases} \tag{7.7}$$

provided $\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] > 0$.

Proof. With $x_t = 1$, we already note the relations

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t),\ A_{\ell(\mathbf{x}^t)} = \ldots = A_{t+1} = 1\right] \end{aligned}$$

and

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t),\ A_{\ell(\mathbf{x}^t)} = \ldots = A_t = 1\right] \end{aligned}$$

If $\ell(\mathbf{x}^t) > 0$, then with some rv B independent of $\{A_s, 0 \le s < \ell(\mathbf{x}^t)\}$, we conclude that

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t), B \ge t - \ell(\mathbf{x}^t) + 2\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t)\right] \mathbf{P}\left[B \ge t - \ell(\mathbf{x}^t) + 2\right] \end{aligned} \tag{7.8}$$

and

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t), B \ge t - \ell(\mathbf{x}^t) + 1\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t)\right] \mathbf{P}\left[B \ge t - \ell(\mathbf{x}^t) + 1\right]. \end{aligned} \tag{7.9}$$

The first half of (7.7) follows readily by combining (7.8) and (7.9) through (7.3).

On the other hand, if $\ell(\mathbf{x}^t) = 0$, then $\mathbf{x}^t = (1, \ldots, 1)$ and it holds that

$$\begin{aligned} \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = A_{t+1} = 1\right] \\ &= \mathbf{P}\left[\hat{B} \geq t+2\right] \end{aligned} \tag{7.10}$$

and

$$\begin{aligned} \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = 1\right] \\ &= \mathbf{P}\left[\hat{B} \geq t+1\right]. \end{aligned} \tag{7.11}$$

The second half of (7.7) is obtained by combining (7.10) and (7.11) via (7.3). ■

Proposition 7.1.2 *For the stationary on-off source* (I, B)*, for each* $\mathbf{x}^t$ *in* $\{0,1\}^{t+1}$ *with* $x_t = 0$*, we have*

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] = \begin{cases} h_I(t - \ell(\mathbf{x}^t) + 1) & \text{if } \ell(\mathbf{x}^t) > 0 \\ \\ h_{\hat{I}}(t+1) & \text{if } \ell(\mathbf{x}^t) = 0 \end{cases} \tag{7.12}$$

provided $\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] > 0$.

Proof. The proof follows a pattern similar to that of Proposition 7.1.1. With $x_t = 0$, we obtain the relations

$$\begin{aligned} &\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] \\ = \ &\mathbf{P}\left[A_s = x_s, 0 \leq s < \ell(\mathbf{x}^t),\ A_{\ell(\mathbf{x}^t)} = \ldots = A_t = 0,\ A_{t+1} = 1\right] \end{aligned}$$

and

$$
\begin{aligned}
& \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] \\
= \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t),\ A_{\ell(\mathbf{x}^t)} = \ldots = A_t = 0\right].
\end{aligned}
$$

If $\ell(\mathbf{x}^t) > 0$, then with some pair of independent rvs I and B which are independent of $\{A_s, 0 \le s < \ell(\mathbf{x}^t)\}$, we conclude that

$$
\begin{aligned}
& \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] \\
= \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t), I = t - \ell(\mathbf{x}^t) + 1,\ B \ge 1\right] \\
= \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t)\right] \mathbf{P}\left[I = t - \ell(\mathbf{x}^t) + 1\right] \mathbf{P}\left[B \ge 1\right] \qquad (7.13)
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] \\
= \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t), I \ge t - \ell(\mathbf{x}^t) + 1\right] \\
= \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t)\right] \mathbf{P}\left[I \ge t - \ell(\mathbf{x}^t) + 1\right]. \qquad (7.14)
\end{aligned}
$$

Combining (7.13) and (7.14) through (7.3) we get the first half of (7.12).

On the other hand, if $\ell(\mathbf{x}^t) = 0$, then $\mathbf{x}^t = (0, \ldots, 0)$ and it holds that

$$
\begin{aligned}
\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = 0,\ A_{t+1} = 1\right] \\
&= \mathbf{P}\left[I_0 = t + 1, B_0 \ge 1\right] \\
&= \mathbf{P}\left[I_0 > 0\right] \mathbf{P}\left[\hat{I} = t + 1\right] \mathbf{P}\left[B \ge 1\right] \qquad (7.15)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = 0\right] \\
&= \mathbf{P}\left[I_0 \ge t + 1\right] \\
&= \mathbf{P}\left[I_0 > 0\right] \mathbf{P}\left[\hat{I} \ge t + 1\right]. \qquad (7.16)
\end{aligned}
$$

We conclude to the second half of (7.12) by combining (7.15) and (7.16) via (7.3).

■

7.2 The SSI conditions

With the help of results from Section 7.1, we are ready to find the SSI conditions for the stationary on-off source. The following proposition states conditions on I and B for a discrete-time stationary on-off source to have the SSI property.

Proposition 7.2.1 *The discrete-time stationary on-off source (I, B) satisfies the SSI property if the conditions (i)-(vi) below hold, where*

(i) The rvs I and B are DFR;

(ii) For all $s = 1, 2, \ldots, T_I$ and $t = 1, 2, \ldots, T_B$,

$$h_I(s) + h_B(t) \leq 1; \tag{7.17}$$

(iii) For all $s = 1, 2, \ldots, T_I$ and $t = 1, 2, \ldots, T_B$,

$$h_{\hat{I}}(s) + h_{\hat{B}}(t) \leq 1; \tag{7.18}$$

(iv) The rvs $\hat{I}$ and $\hat{B}$ are DFR;

(v) For all $t = 1, 2, \ldots, T_I - 1$,

$$h_{\hat{I}}(t+1) \leq 1 - h_B(1); \tag{7.19}$$

(vi) For all $t = 1, 2, \ldots, T_B - 1$,

$$h_{\hat{B}}(t+1) \leq 1 - h_I(1). \tag{7.20}$$

The following consequences of the conditions (i)-(vi) are worth noting before we embark on a proof of Proposition 7.2.1. By Lemma 2.2.1, condition (iv) implies that

$$h_{\hat{B}}(t+1) \leq h_B(t), \quad t = 1, 2, \ldots, T_B - 1$$

and

$$h_{\hat{I}}(t+1) \leq h_I(t), \quad t = 1, 2, \ldots, T_I - 1.$$

Together, condition (i) and the last remarks yield

$$h_{\hat{B}}(t+1) \leq h_B(s) \quad s = 1, \ldots, t \quad \text{with } t < T_B \tag{7.21}$$

and

$$h_{\hat{I}}(t+1) \leq h_I(s), \quad s = 1, \ldots, t \quad \text{with } t < T_I. \tag{7.22}$$

Moreover, for $t < T_I$, conditions (i) and (v) imply

$$h_{\hat{I}}(t+1) + h_B(s) \leq 1, \quad s = 1, \ldots, T_B, \tag{7.23}$$

while for $t < T_B$, conditions (i) and (vi) give

$$h_{\hat{B}}(t+1) + h_I(s) \leq 1, \quad s = 1, \ldots, T_I. \tag{7.24}$$

Proof. For each $t = 0, 1, \ldots$, we need to show that (7.1) holds for distinct elements $\mathbf{x}^t$ and $\mathbf{y}^t$ in $\{0,1\}^{t+1}$ such that $\mathbf{x}^t \leq \mathbf{y}^t$ and (7.2) is satisfied.

For $t = 0$, (7.2) automatically holds here since $\mathbf{P}[A_0 = 1] = 1 - \mathbf{P}[A_0 = 0] = p$ with $0 < p < 1$. By Lemma 7.1.1 we see that (7.1) reduces to $h_{\hat{I}}(1) \leq 1 - h_{\hat{B}}(1)$ which is equivalent to (iii) with $s = t = 1$.

For $t = 1, 2, \ldots$, three cases present themselves, depending on whether (a) $x_t = y_t = 1$; (b) $x_t = y_t = 0$; and (c) $x_t = 0 < y_t = 1$. Recall that in all cases, we are only interested in the event that (7.2) is satisfied. We consider each one of three cases in turn:

Case (a) – With $x_t = y_t = 1$, the condition $\mathbf{x}^t \leq \mathbf{y}^t$ implies $\ell(\mathbf{y}^t) \leq \ell(\mathbf{x}^t)$. If $\ell(\mathbf{y}^t) > 0$, then $\ell(\mathbf{x}^t) > 0$ as well. By Proposition 7.1.1, the inequality (7.1) reduces to

$$h_B(t - \ell(\mathbf{y}^t) + 1) \leq h_B(t - \ell(\mathbf{x}^t) + 1) \tag{7.25}$$

with $t - \ell(\mathbf{x}^t) + 1 \leq t - \ell(\mathbf{y}^t) + 1 \leq T_B$. The inequality (7.25) does hold when B is DFR. If $\ell(\mathbf{y}^t) = 0$, then $\ell(\mathbf{x}^t) > 0$ (for otherwise $\mathbf{x}^t = \mathbf{y}^t$) and Proposition 7.1.1 this time shows that (7.1) is equivalent to

$$h_{\hat{B}}(t+1) \leq h_B(t - \ell(\mathbf{x}^t) + 1) \quad \text{with} \quad 1 \leq t - \ell(\mathbf{x}^t) + 1 \leq t < T_B.$$

This last inequality is satisfied as a consequence of (i) and (iv) (as indicated by (7.21)).

Case (b) – With $x_t = y_t = 0$, the condition $\mathbf{x}^t \leq \mathbf{y}^t$ now implies $\ell(\mathbf{x}^t) \leq \ell(\mathbf{y}^t)$. If $\ell(\mathbf{x}^t) > 0$, then $\ell(\mathbf{y}^t) > 0$ and by Proposition 7.1.2, the inequality (7.1) reduces to

$$h_I(t - \ell(\mathbf{x}^t) + 1) \leq h_I(t - \ell(\mathbf{y}^t) + 1) \tag{7.26}$$

with $t - \ell(\mathbf{y}^t) + 1 \leq t - \ell(\mathbf{x}^t) + 1 \leq T_I$. The inequality (7.26) is implied by the fact that the rv I is DFR under (i). If $\ell(\mathbf{x}^t) = 0$, then $\ell(\mathbf{y}^t) > 0$ (for otherwise $\mathbf{x}^t = \mathbf{y}^t$) and Proposition 7.1.2 shows that (7.1) is equivalent to

$$h_{\hat{I}}(t+1) \leq h_I(t - \ell(\mathbf{y}^t) + 1) \quad \text{with} \quad 1 \leq t - \ell(\mathbf{y}^t) + 1 \leq t < T_I.$$

This last inequality is satisfied as a result of (i) and (iv) (as indicated by (7.22)).

Case (c) – With $x_t = 0 < y_t = 1$, four possible scenarios need to be considered when invoking Propositions 7.1.1 and 7.1.2 to rewrite the inequality (7.1) in reduced form: First, if $\ell(\mathbf{x}^t) = \ell(\mathbf{y}^t) = 0$, then (7.1) can be rewritten as

$$h_{\hat{I}}(t+1) \leq 1 - h_{\hat{B}}(t+1), \quad t < \min(T_I, T_B), \tag{7.27}$$

and this inequality does hold by virtue of (iii). If $\ell(\mathbf{x}^t) = 0$ and $\ell(\mathbf{y}^t) > 0$, then (7.1) becomes

$$h_{\hat{I}}(t+1) \leq 1 - h_B(t - \ell(\mathbf{y}^t) + 1) \quad \text{with} \quad 1 \leq t - \ell(\mathbf{y}^t) + 1 \leq t < T_I. \tag{7.28}$$

If $\ell(\mathbf{x}^t) > 0$ and $\ell(\mathbf{y}^t) = 0$, then (7.1) reads

$$h_I(t - \ell(\mathbf{x}^t) + 1) \leq 1 - h_{\hat{B}}(t+1), \quad \text{with} \quad 1 \leq t - \ell(\mathbf{x}^t) + 1 \leq t < T_B. \tag{7.29}$$

Under the enforced assumptions, the validity of (7.28) and (7.29) is guaranteed under the observations (7.23) and (7.24) that flow from conditions (i), (v) and (vi). If $\ell(\mathbf{x}^t) > 0$ and $\ell(\mathbf{y}^t) > 0$, then (7.1) is equivalent to

$$h_I(t - \ell(\mathbf{x}^t) + 1) \leq 1 - h_B(t - \ell(\mathbf{y}^t) + 1) \tag{7.30}$$

with $1 \leq t - \ell(\mathbf{x}^t) + 1 \leq \min(T_I, t)$ and $1 \leq t - \ell(\mathbf{y}^t) + 1 \leq \min(T_B, t)$. The last inequality is satisfied by condition (ii). The proof is now complete. ■

Upon combining Proposition 7.2.1 and Corollary 4.5.1 with Theorem 5.1.1, we have

Theorem 7.2.1 *Let $\{A_t, t = 0, 1, \ldots\}$ be a discrete-time stationary on-off source (I, B) satisfying the conditions of Proposition 7.2.1. Its independent version $\{\hat{A}_t, t = 0, 1, \ldots\}$ is a sequence of i.i.d. $\{0, 1\}$-valued rvs with $\mathbf{P}\left[\hat{A}_t = 1\right] = p$ for all $t = 0, 1, \ldots$, and we have the comparison*

$$\{\hat{A}_t, t = 0, 1, \ldots\} \leq_{idcx} \{A_t, t = 0, 1, \ldots\}.$$

Moreover, their corresponding buffer contents $\{\hat{Q}_t, t = 0, 1, \ldots\}$ *and* $\{Q_t, t = 0, 1, \ldots\}$ *are ordered in the icx ordering, i.e., for any fixed initial condition* $\hat{Q}_0 = Q_0 = q$*, we have*

$$\hat{Q}_t \leq_{icx} Q_t, \quad t = 0, 1, \ldots.$$

7.3 Equivalent conditions

The conditions (i)-(vi) in Proposition 7.2.1 are stated so as to simplify the proof of the SSI property of the stationary on-off source (I, B). In fact, these conditions can be rewritten in a more compact way as will be shown in Lemma 7.3.1. In addition, we can relax some conditions to achieve another set of weaker conditions in Lemma 7.3.2 that still ensures the SSI property. These two sets of conditions will prove useful when applying the SSI conditions to specific distributions in Chapter 9.

Lemma 7.3.1 *The conditions (i)-(vi) in Proposition 7.2.1 are equivalent to the following conditions:*

(A.1) *The rvs* I *and* B *are DFR;*

(A.2) $\mathbf{P}[I = 1] + \mathbf{P}[B = 1] \leq 1$*;*

(A.3) *The rvs* $\hat{I}$ *and* $\hat{B}$ *are DFR;*

(A.4) $\frac{1}{\mathbf{E}[I]} + \frac{1}{\mathbf{E}[B]} \leq 1$.

Proof. First, conditions (A.2) and (A.4) are simply

$$h_I(1) + h_B(1) \leq 1 \tag{7.31}$$

and

$$h_{\hat{I}}(1) + h_{\hat{B}}(1) \leq 1, \tag{7.32}$$

respectively. Hence, it is easily seen that conditions (A.1)-(A.4) are implied by conditions (i)-(vi).

Now, we show that conditions (A.1)-(A.4) imply conditions (i)-(vi). By (7.31) and (A.1), condition (ii) holds. In the same way, combining (A.3) with (7.32) implies condition (iii). Fix $t = 1, 2, \ldots, T_I - 1$, by Lemma 2.2.1, we have

$$h_{\hat{I}}(t+1) \leq h_I(t) \leq h_I(1) \leq 1 - h_B(1)$$

upon using (7.31), whence condition (v) holds. Using the same argument, it is a simple matter to that show condition (vi) holds since for fixed $t = 1, 2, \ldots, T_B - 1$,

$$h_{\hat{B}}(t+1) \leq h_B(t) \leq h_B(1) \leq 1 - h_I(1)$$

from Lemma 2.2.1 and (7.31). ■

When T_B and T_I are larger than 1 (as is the case in most situations of interest), we have a weaker set of conditions as demonstrated below.

Lemma 7.3.2 *The following conditions (B.1)-(B.5) ensure the SSI property of the stationary on-off source (I, B), where*

(B.1) The rvs I and B are DFR;

(B.2) $h_I(1) + h_B(2) \leq 1$ and $h_I(2) + h_B(1) \leq 1$;

(B.3) The rvs $\hat{I}$ and $\hat{B}$ are DFR;

(B.4) $\frac{1}{\mathbf{E}[I]} + \frac{1}{\mathbf{E}[B]} \leq 1$;

(B.5) $h_{\hat{I}}(2) \le 1 - h_B(1)$ *and* $h_{\hat{B}}(2) \le 1 - h_I(1)$.

Proof. The proof here is a simple modification of the proof of Proposition 7.2.1. We first note as in the proof of Lemma 7.3.1 that conditions (B.3) and (B.4) imply condition (iii). Therefore, the SSI property holds for $t = 0$,

Consider three cases when $t = 1, 2, \ldots$ as in the proof of Proposition 7.2.1. Case (a) and Case (b) hold with the DFR properties of $I, B, \hat{I}$ and $\hat{B}$. For Case (c) that $x_t = 0 < y_t = 1$, since conditions (B.3) and (B.4) imply condition (iii), the inequality (7.27) does hold. Upon combining (B.5) and the DFR properties of $I, B, \hat{I}$ and $\hat{B}$, we have (7.28) and (7.29).

Lastly, it remains to show the last requirement of Case (c), i.e., when $\ell(\mathbf{x}^t) > 0$ and $\ell(\mathbf{y}^t) > 0$, which is summarized in the inequality (7.30) as

$$h_I(t - \ell(\mathbf{x}^t) + 1) \le 1 - h_B(t - \ell(\mathbf{y}^t) + 1)$$

with $1 \le t - \ell(\mathbf{x}^t) + 1 \le \min(T_I, t)$ and $1 \le t - \ell(\mathbf{y}^t) + 1 \le \min(T_B, t)$. However, it is not possible to have $\ell(\mathbf{x}^t) = \ell(\mathbf{y}^t) = k$ for any $k > 0$. Hence, from (7.30), the requirement $h_I(1) + h_B(1) \le 1$ which occurs when $\ell(\mathbf{x}^t) = \ell(\mathbf{y}^t) = t$ is unnecessary and can be eliminated. It can be verified that the inequality (7.30) without the case $\ell(\mathbf{x}^t) = \ell(\mathbf{y}^t) = t$ is implied by invoking the DFR properties of I and B with conditions (B.2). Therefore, conditions (B.1)-(B.5) ensure the SSI property of the stationary on-off source (I, B). ■

Chapter 8

Non-stationary On-off Sources

We now consider the non-stationary on-off source (I, B) and show that the SSI conditions are much weaker than those of the stationary on-off source (I, B). In analogy with the stationary on-off source (I, B), we first find the expression (7.3) for the non-stationary on-off source (I, B) and then derive the corresponding SSI conditions.

8.1 Expressions for non-stationary on-off sources

Here, we evaluate (7.3) when $\{A_t, t = 0, 1, \ldots\}$ is a non-stationary on-off source (I, B). As described in Section 3.2, we always have $I_0 =_{st} I$ so that $\mathbf{P}[A_0 = 0] = 1$. This observation leads to the following analog of Lemma 7.1.1.

Lemma 8.1.1 *For the non-stationary on-off source (I, B), we have*

$$\mathbf{P}[A_1 = 1 | A_0 = 0] = h_I(1). \tag{8.1}$$

Proof. As in the proof of Lemma 7.1.1, the conclusion (8.1) is an easy consequence of the facts

$$\begin{aligned}\mathbf{P}\left[A_1 = 1 | A_0 = 0\right] &= \frac{\mathbf{P}\left[I_0 = 1, B_0 \geq 1\right]}{\mathbf{P}\left[I_0 \geq 1\right]} \\ &= \mathbf{P}\left[I_0 = 1\right] = h_I(1)\end{aligned}$$

with $\mathbf{P}\left[B_0 \geq 1\right] = \mathbf{P}\left[I_0 \geq 1\right] = 1$ ∎

In the non-stationary case, the analogs of Propositions 7.1.1 and 7.1.2 can be expressed more compactly as the next proposition shows:

Proposition 8.1.1 *Fix $t = 1, 2, \ldots$. For the non-stationary on-off source (I, B), for each $\mathbf{x}^t$ in $\{0,1\}^{t+1}$ with $\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] > 0$, we have the following: If $x_t = 1$, then*

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] = r_B(t - \ell(\mathbf{x}^t) + 1), \quad \ell(\mathbf{x}^t) \geq 1 \tag{8.2}$$

and if $x_t = 0$, then

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] = h_I(t - \ell(\mathbf{x}^t) + 1), \quad \ell(\mathbf{x}^t) \geq 0. \tag{8.3}$$

Proof. A careful inspection of the proofs of Propositions 7.1.1 and 7.1.2 shows that both (8.2) and (8.3) hold when $\ell(\mathbf{x}^t) > 0$. Hence, only the case $\ell(\mathbf{x}^t) = 0$ needs to be considered.

With $\ell(\mathbf{x}^t) = 0$ and $x_t = 1$, $\mathbf{x}^t = (1, \ldots, 1)$. This event cannot occur since $I_0 =_{st} I$ implies $A_0 =_{st} 0$. With $\ell(\mathbf{x}^t) = 0$ and $x_t = 0$, $\mathbf{x}^t = (0, \ldots, 0)$, thus (7.15) and (7.16) now become

$$\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] = \mathbf{P}\left[A_0 = \ldots = A_t = 0,\ A_{t+1} = 1\right]$$

$$\begin{aligned} &= \mathbf{P}\left[I_0 = t+1, B_0 \geq 1\right] \\ &= \mathbf{P}\left[I = t+1\right]\mathbf{P}\left[B \geq 1\right] \end{aligned} \tag{8.4}$$

and

$$\begin{aligned} \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = 0\right] \\ &= \mathbf{P}\left[I_0 \geq t+1\right] \\ &= \mathbf{P}\left[I \geq t+1\right]. \end{aligned} \tag{8.5}$$

We conclude to the desired result by combining (8.4) and (8.5) via (7.3). ■

8.2 The SSI conditions

We now turn to the SSI property for the non-stationary on-off source (I, B). The analog of Proposition 8.2.1 relies on Proposition 8.1.1 and is given next.

Proposition 8.2.1 *The non-stationary on-off source (I, B) satisfies the SSI property if the conditions (i)-(ii) below hold, where*

(i) The rvs I and B are DFR;

(ii) For all $s = 1, 2, \ldots, T_I$ and $t = 1, 2, \ldots, T_B$,

$$h_I(s) + h_B(t) \leq 1. \tag{8.6}$$

Proof. For each $t = 0, 1, \ldots,$ we need to show that (7.1) holds for distinct elements $\mathbf{x}^t$ and $\mathbf{y}^t$ in $\{0, 1\}^{t+1}$ satisfying (7.2) and such that $\mathbf{x}^t \leq \mathbf{y}^t$.

For $t = 0$, $x_0 = 0$ and $y_0 = 1$ and there is no need for comparison here since $\mathbf{P}[A_0 = y_0] = 0$.

For $t = 1, 2, \ldots$, as in the proof of Proposition 7.2.1, three cases present themselves, depending on whether (a) $x_t = y_t = 1$; (b) $x_t = y_t = 0$; and (c) $x_t = 0 < y_t = 1$.

Case (a) – With $x_t = y_t = 1$, the condition $\mathbf{x}^t \leq \mathbf{y}^t$ implies $\ell(\mathbf{y}^t) \leq \ell(\mathbf{x}^t)$. By Proposition 8.1.1, the inequality (7.1) can be rewritten as

$$h_B(t - \ell(\mathbf{y}^t) + 1) \leq h_B(t - \ell(\mathbf{x}^t) + 1) \tag{8.7}$$

with $t - \ell(\mathbf{x}^t) + 1 \leq t - \ell(\mathbf{y}^t) + 1 \leq T_B$. It is plain that (8.7) holds because B is assumed DFR.

Case (b) – With $x_t = y_t = 0$, the condition $\mathbf{x}^t \leq \mathbf{y}^t$ implies $\ell(\mathbf{x}^t) \leq \ell(\mathbf{y}^t)$. By Proposition 8.1.1, the inequality (7.1) reduces to

$$h_I(t - \ell(\mathbf{x}^t) + 1) \leq h_I(t - \ell(\mathbf{y}^t) + 1) \tag{8.8}$$

with $t - \ell(\mathbf{y}^t) + 1 \leq t - \ell(\mathbf{x}^t) + 1 \leq T_I$, and the validity of (8.8) is implied by the fact that the rv I is DFR under (i).

Case (c) – With $x_t = 0 < y_t = 1$, invoking Proposition 8.1.1 we can rewrite (7.1) in reduced form as

$$h_I(t - \ell(\mathbf{x}^t) + 1) \leq 1 - h_B(t - \ell(\mathbf{y}^t) + 1) \tag{8.9}$$

with $1 \leq t - \ell(\mathbf{x}^t) + 1 \leq \min(T_I, t + 1)$ and $1 \leq t - \ell(\mathbf{y}^t) + 1 \leq \min(T_B, t)$. The validity of (8.9) is guaranteed under (ii). The proof is now complete. ■

It is easy to verify that condition (ii) can be implied by invoking the DFR properties of I and B with

$$h_I(1) + h_B(1) \leq 1. \tag{8.10}$$

Thus, a set of equivalent conditions emerges as we recall that (8.10) is equivalent to the condition (C.2) below.

Lemma 8.2.1 *The conditions (i)-(ii) in Proposition 8.2.1 are equivalent to the conditions (C.1)-(C.2) below, where*

(C.1) The rvs I and B are DFR;

(C.2) $\mathbf{P}[I = 1] + \mathbf{P}[B = 1] \leq 1$*;*

Moreover, in Case (c) of the proof of Proposition 8.2.1, $\ell(\mathbf{x}^t) \neq \ell(\mathbf{y}^t) = k$ for all $k \geq 0$. In analogy with Lemma 7.3.2 when $T_I, T_B > 1$, the requirement (8.9) can be relaxed by eliminating the event $\ell(\mathbf{x}^t) = \ell(\mathbf{y}^t) = t$. As a result, we get a weaker set of conditions that still ensure the SSI property.

Lemma 8.2.2 *The following conditions (D.1)-(D.2) ensure the SSI property of the non-stationary on-off source (I, B), where*

(D.1) The rvs I and B are DFR;

(D.2) $h_I(1) + h_B(2) \leq 1$ *and* $h_I(2) + h_B(1) \leq 1$*;*

The proof of Lemma 8.2.2 is omitted as it is similar to that of Lemma 7.3.2.

Chapter 9

Simulation Results for On-off Sources

In this chapter, we show simulation results comparing the first and second moments of the buffer levels of a single on-off source with the SSI property and those of its independent version. The on- and off-period durations used here are two specific types of distributions, namely the geometric and discrete-Pareto distributions, defined in Section 2.3.

The following sections discuss the SSI conditions and show simulation results of three on-off sources models, namely the on-off source $(\mathcal{G}(\rho_I), \mathcal{G}(\rho_B))$, $(\mathcal{G}(\rho_I), \mathcal{P}(\alpha_B))$ and $(\mathcal{P}(\alpha_I), \mathcal{P}(\alpha_B))$. It is known [19] that on-off sources with a discrete-Pareto distributed on-period exhibits long-range dependence. For all simulations in this chapter, the simulation descriptions are specified in Section 6.2.

9.1 The on-off source $(\mathcal{G}(\rho_I), \mathcal{G}(\rho_B))$

From Section 2.3, the geometrically distributed rvs $I, \hat{I}, B$ and $\hat{B}$ are DFR. Applying conditions (A.1)-(A.4), it remains to show that conditions (A.2) and (A.4) are satisfied. Since $\mathbf{P}[I = 1] = \frac{1}{\mathbf{E}[I]}$ and $\mathbf{P}[B = 1] = \frac{1}{\mathbf{E}[B]}$, both (A.2) and (A.4) reduce to

$$\rho_I + \rho_B \geq 1. \tag{9.1}$$

Thus, (9.1) is the only required SSI condition for the on-off source $(\mathcal{G}(\rho_I), \mathcal{G}(\rho_B))$. Note that using conditions (B.1)-(B.5) yields the same conclusion (9.1).

By selecting $\rho_I = \rho_B = 0.8$, (9.1) is satisfied and we have $\mathbf{E}[B] = \mathbf{E}[I] = 5$. As a result, the traffic rate p is 0.5 and the independent version is simply the on-off source $(\mathcal{G}(0.5), \mathcal{G}(0.5))$. In this simulations, we use the number of sample path N=10,000 and fix the multiplexer release rate at $c = 0.6$. Figures 9 and 10 show the first and second moments of the buffer sizes of the on-off source $(\mathcal{G}(0.8), \mathcal{G}(0.8))$ and of its independent version. Both the first and second moments of the buffer fed by the on-off source $(\mathcal{G}(0.8), \mathcal{G}(0.8))$ are larger than those of its independent version as expected.

9.2 The on-off source $(\mathcal{G}(\rho_I), \mathcal{P}(\alpha_B))$

The rvs $I, \hat{I}, B$ and $\hat{B}$ are DFR [Section 2.3]. Again, we apply the conditions (A.1)-(A.4). It can be shown that (A.2) and (A.4) are equivalent to

$$\rho_I \geq 1 - 2^{-\alpha_B} \tag{9.2}$$

and

$$\rho_I \geq \frac{1}{\mathbf{E}[B]}, \tag{9.3}$$

respectively, where $\mathbf{E}[B] = \sum_{k=1}^{\infty} k^{-\alpha_B}$. Combining (9.2) and (9.3) gives the SSI condition

$$\rho_I \geq \max(1 - 2^{-\alpha_B}, \frac{1}{\mathbf{E}[B]}) \tag{9.4}$$

for the on-off source $(\mathcal{G}(\rho_I), \mathcal{P}(\alpha_B))$. Conclusion (9.4) can also be reached by using conditions (B.1)-(B.5).

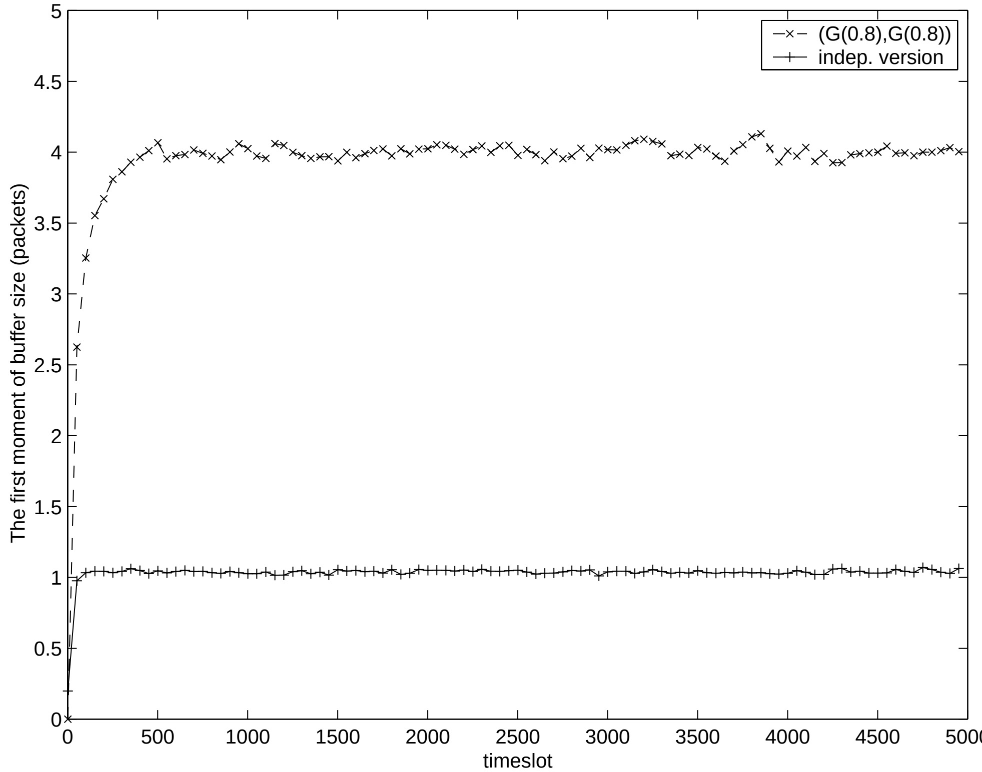

Figure 9: The first moments of the buffer sizes of the on-off source $(\mathcal{G}(0.8), \mathcal{G}(0.8))$ and of its independent version (the on-off source $(\mathcal{G}(0.5), \mathcal{G}(0.5))$)

To meet condition (9.4), we select the on-off source $(\mathcal{G}(0.678), \mathcal{P}(1.4))$ in our experiment. It can be shown that $\mathbf{E}[B] = 3.10555$ and therefore, $p = 0.5$. The independent version is again the on-off source $(\mathcal{G}(0.5), \mathcal{G}(0.5))$. We fix N=10,000 and set the multiplexer release rate at $c = 0.6$. The comparisons of the first and second moments of the buffer sizes of the on-off source $(\mathcal{G}(0.678), \mathcal{P}(1.4))$ and of its independent version are illustrated in Figures 11 and 12, respectively. It can be seen that both moments the buffer sizes of the on-off source $(\mathcal{G}(0.678), \mathcal{P}(1.4))$ grow with t corresponding to the results in Section 5.3.2 that the steady state mean buffer size $\mathbf{E}[Q]$ is infinite. The simulation results are as expected since the

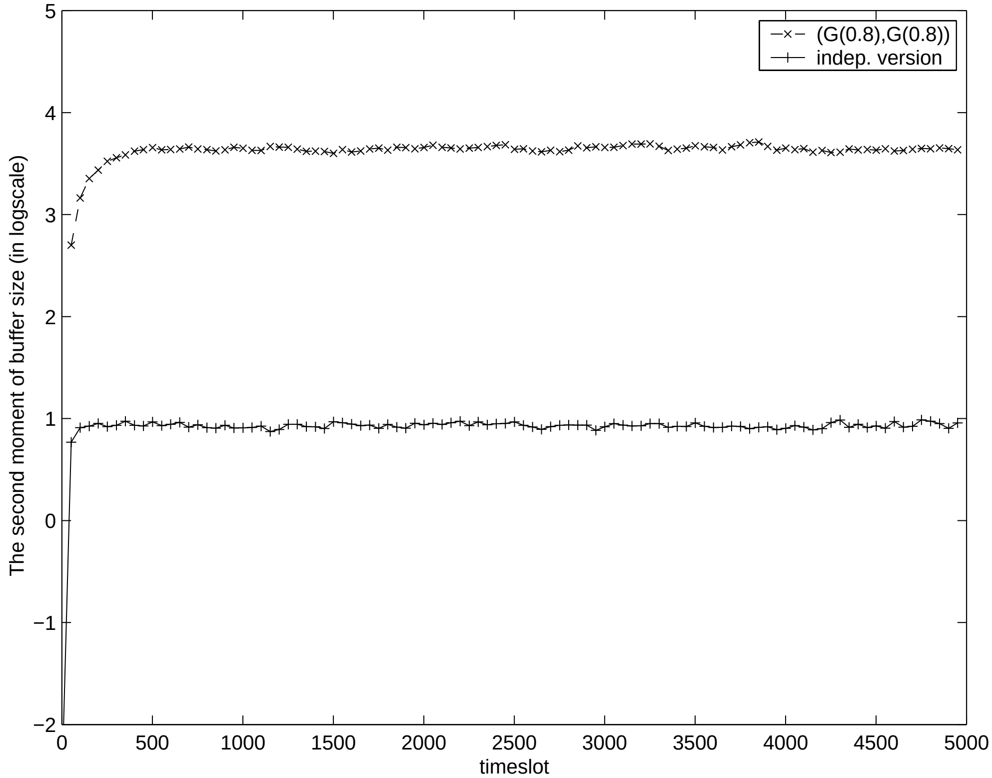

Figure 10: The second moments of the buffer sizes (in logscale) of the on-off source $(\mathcal{G}(0.8), \mathcal{G}(0.8))$ and of its independent version (the on-off source $(\mathcal{G}(0.5), \mathcal{G}(0.5))$)

first and second moments of the buffer level of the on-off source $(\mathcal{G}(0.678), \mathcal{P}(1.4))$ are greater than those of its independent version. Furthermore, both the first and second moments of the buffer level of the on-off source $(\mathcal{G}(0.678), \mathcal{P}(1.4))$ are clearly larger than those of the on-off source $(\mathcal{G}(0.8), \mathcal{G}(0.8))$ shown in Figure 9 and 10, even though both processes have the same traffic rate $p = 0.5$.

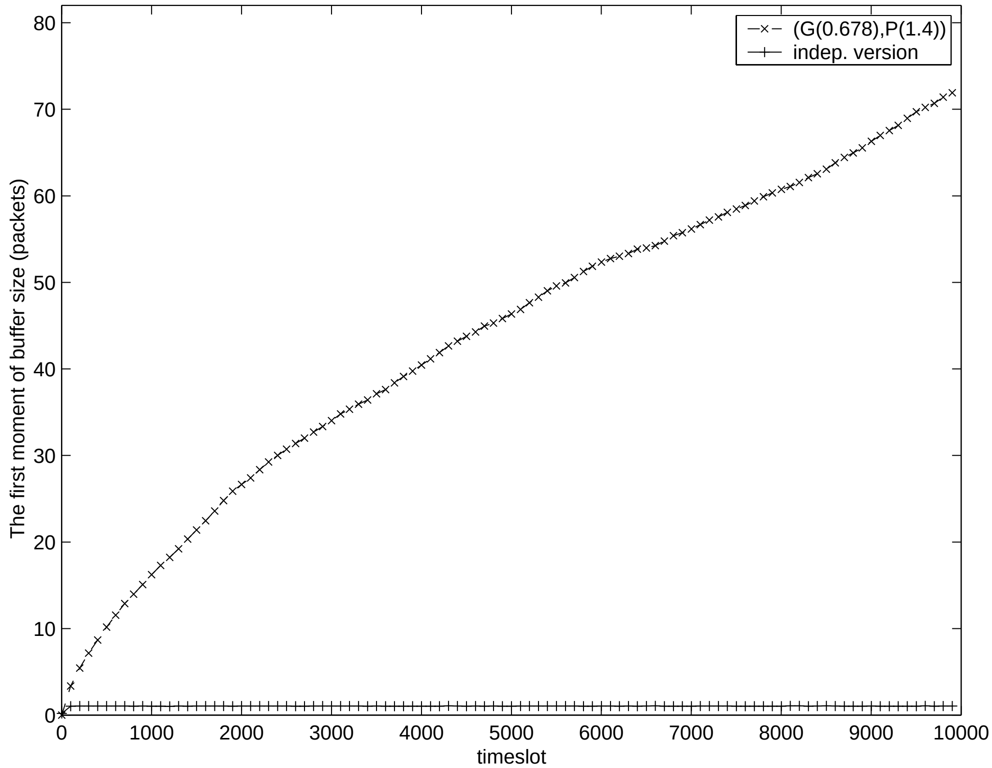

Figure 11: The first moments of the buffer sizes of the on-off source $(\mathcal{G}(0.678), \mathcal{P}(1.4))$ and of its independent version (the on-off source $(\mathcal{G}(0.5), \mathcal{G}(0.5))$)

9.3 The on-off source $(\mathcal{P}(\alpha_I), \mathcal{P}(\alpha_B))$

Since both I and B have discrete-Pareto distributions, the rvs $I, \hat{I}, B$ and $\hat{B}$ are automatically DFR. We first try to satisfy the conditions (A.1)-(A.4). The condition (A.2) implies

$$2^{-\alpha_I} + 2^{-\alpha_B} \geq 1 \tag{9.5}$$

which is not valid for $1 < \alpha_I, \alpha_B \leq 2$. Therefore, we turn to the weaker conditions (B.1)-(B.5). Using the expressions (2.9) and (2.11) of discrete-Pareto rvs, we

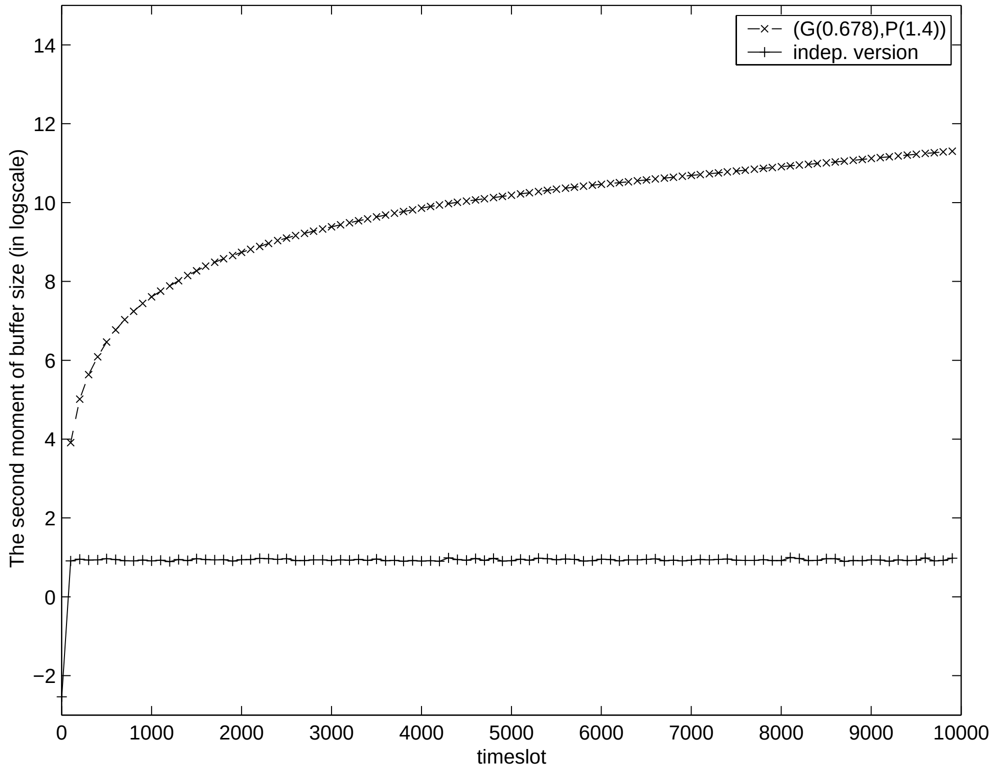

Figure 12: The second moments of the buffer sizes (in logscale) of the on-off source $(\mathcal{G}(0.678), \mathcal{P}(1.4))$ and of its independent version (the on-off source $(\mathcal{G}(0.5), \mathcal{G}(0.5))$)

rewrite (B.2) and (B.5) as

$$2^{-\alpha_I} + (\frac{3}{2})^{-\alpha_B} \geq 1 \quad \text{and} \quad 2^{-\alpha_B} + (\frac{3}{2})^{-\alpha_I} \geq 1, \tag{9.6}$$

and

$$2^{-\alpha_I} \geq \frac{2^{-\alpha_B}}{\sum_{k=2}^{\infty} k^{-\alpha_B}} \quad \text{and} \quad 2^{-\alpha_B} \geq \frac{2^{-\alpha_I}}{\sum_{k=2}^{\infty} k^{-\alpha_I}}, \tag{9.7}$$

respectively. By simple algebra, we require three following SSI conditions for the on-off source $(\mathcal{P}(\alpha_I), \mathcal{P}(\alpha_B))$:

(P.1) $\frac{1}{\mathbf{E}[B]} + \frac{1}{\mathbf{E}[I]} \geq 1$;

(P.2) $2^{-(\alpha_I+\alpha_B)} \geq \max(2^{-\alpha_I} - 3^{-\alpha_I}, 2^{-\alpha_B} - 3^{-\alpha_B})$;

(P.3) $\frac{1}{\mathbf{E}[I]-1} \leq 2^{-(\alpha_B-\alpha_I)} \leq \mathbf{E}[B] - 1$.

In the simulation of the on-off source $(\mathcal{P}(\alpha_I), \mathcal{P}(\alpha_B))$, choosing $\alpha_I = \alpha_B = 1.2$ ensures conditions (P.1)-(P.3). Since $B =_{st} I$ in this case, $p = 0.5$ and its independent version is clearly the on-off source $(\mathcal{G}(0.5), \mathcal{G}(0.5))$. We again fix the number of sample path at N=10,000 and the multiplexer release rate at $c = 0.6$. From Figures 13 and 14, the first and second moments of the buffer level fed by the on-off source $(\mathcal{P}(1.2), \mathcal{P}(1.2))$ are indeed larger than those of its independent version. Moreover, both moments of the buffer sizes the on-off source $(\mathcal{P}(1.2), \mathcal{P}(1.2))$ grow with t as expected from the results in Section 5.3.2 that $\mathbf{E}[Q] = \infty$. While the traffic rate of the on-off source $(\mathcal{P}(1.2), \mathcal{P}(1.2))$ and the on-off source $(\mathcal{G}(0.678), \mathcal{P}(1.4))$ are equal ($p = 0.5$), both the first and second moments of the buffer level of the on-off source $(\mathcal{P}(1.2), \mathcal{P}(1.2))$ are higher than those of the on-off source $(\mathcal{G}(0.678), \mathcal{P}(1.4))$ shown in Figure 11 and 12.

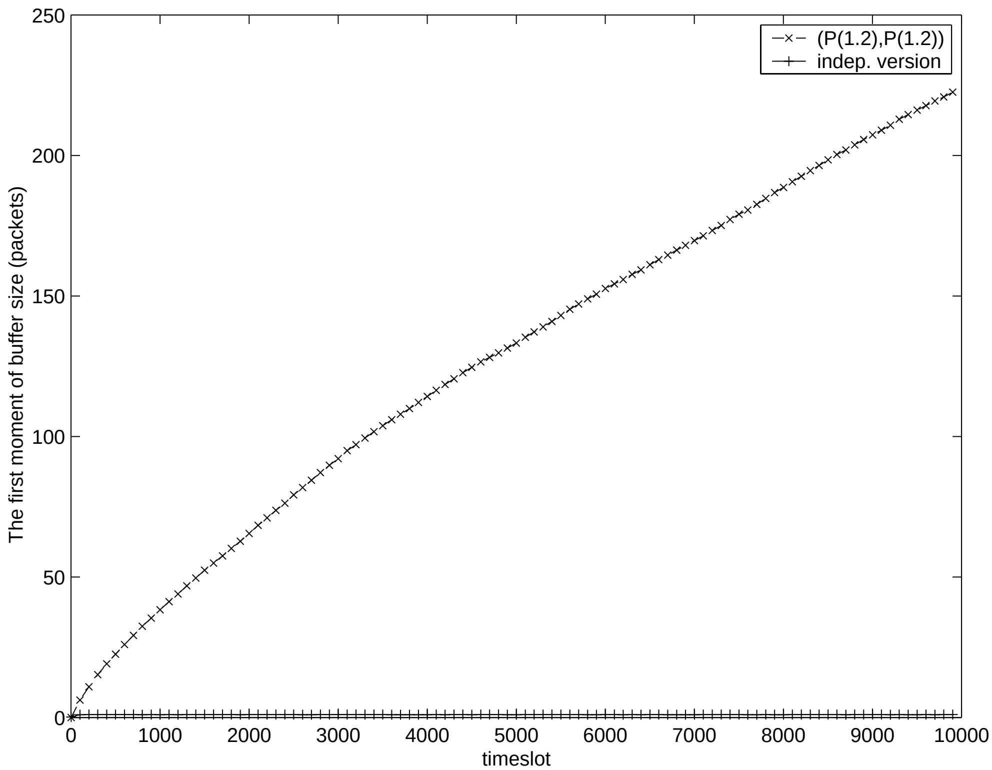

Figure 13: The first moments of the buffer sizes of the on-off source $(\mathcal{P}(1.2), \mathcal{P}(1.2))$ and of its independent version (the on-off source $(\mathcal{G}(0.5), \mathcal{G}(0.5))$)

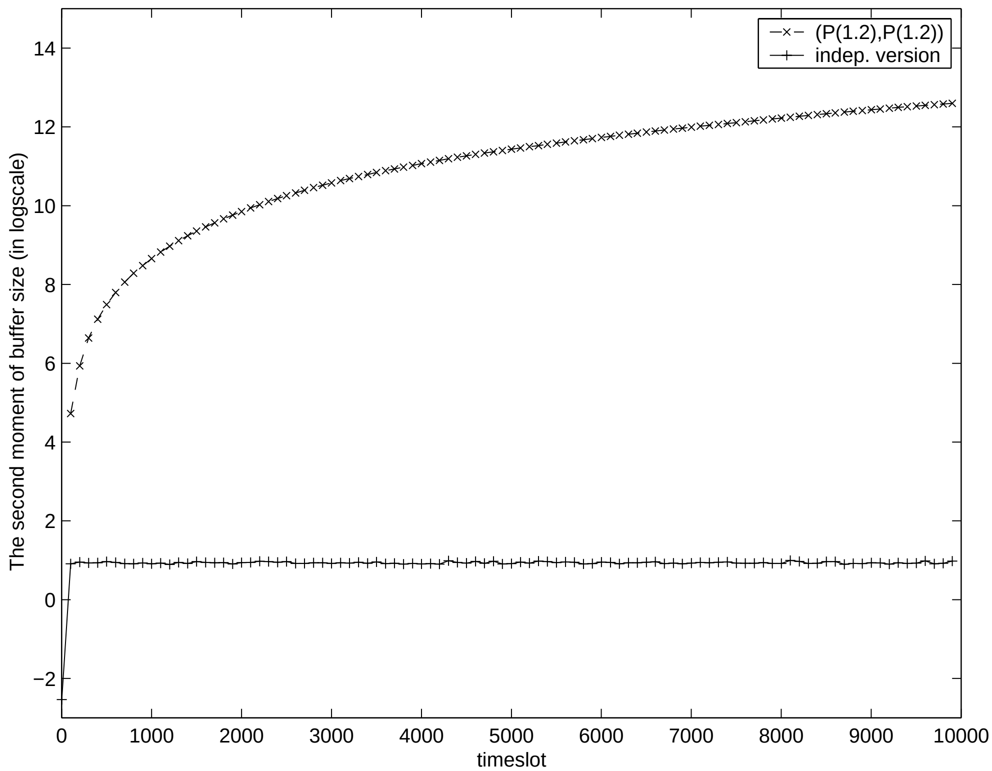

Figure 14: The second moments of the buffer sizes (in logscale) of the on-off source $(\mathcal{P}(1.2), \mathcal{P}(1.2))$ and of its independent version (the on-off source $(\mathcal{G}(0.5), \mathcal{G}(0.5))$)

Chapter 10

Multiplexing On-off Sources

As multiplexing is a major function in communication networks, multiplexed traffic processes naturally arise at routers and at multiplexer buffers. With each on-off source representing a traffic stream, we construct the multiplexed traffic by superposing the on-off sources. We show under some conditions on the on- and off-period duration distributions that the comparison between the multiplexed on-off sources and its independent version in the idcx ordering, and the comparison of their corresponding buffer levels in the icx ordering hold. We separate our discussion in two cases, namely the finite number of on-off sources and the infinite number of on-off sources.

10.1 Finite number of on-off sources

Consider N independent on-off sources but not necessarily identically distributed. For each $i = 1, 2, \ldots, N$, let $\{A_t^i, t = 0, 1, \ldots\}$ denote the stationary on-off source (I^i, B^i) with rate $p_i = \frac{\mathbf{E}[B^i]}{\mathbf{E}[B^i]+\mathbf{E}[I^i]}$. The multiplexing of these N on-off processes results in the process $\{M_t^N, t = 0, 1, \ldots\}$ where

$$M_t^N = \sum_{i=1}^{N} A_t^i, \quad t = 0, 1, \ldots. \tag{10.1}$$

The process $\{M_t^N, t = 0, 1, \ldots\}$ is also stationary with traffic intensity $\sum_{i=1}^N p_i$.

We are interested in establishing a comparison between the multiplexed process $\{M_t^N, t = 0, 1, \ldots\}$ and its independent version. For each $i = 1, 2, \ldots, N$, we assume the rvs I^i and B^i defining the on-off process $\{A_t^i, t = 0, 1, \ldots\}$ to satisfy the conditions in Proposition 7.2.1. Thus, $\{A_t^i, t = 0, 1, \ldots\}$ is SSI and the comparison

$$\{\hat{A}_t^i, t = 0, 1, \ldots\} \leq_{idcx} \{A_t^i, t = 0, 1, \ldots\} \tag{10.2}$$

holds by Theorem 7.2.1 where $\{\hat{A}_t^i, t = 0, 1, \ldots\}$ is the independent version of $\{A_t^i, t = 0, 1, \ldots\}$. This independent version is simply the sequence of i.i.d. $\{0, 1\}$-valued rvs with $\mathbf{P}\left[\hat{A}_t^i = 1\right] = p_i$ for all $t = 0, 1, \ldots$, or equivalently, the on-off process $(\mathcal{G}(1 - p_i), \mathcal{G}(p_i))$.

Since (10.2) holds for all $i = 1, 2, \ldots, N$, by applying Corollary 4.3.1, we obtain

$$\{\sum_{i=1}^N \hat{A}_t^i, t = 0, 1, \ldots\} \leq_{idcx} \{\sum_{i=1}^N A_t^i, t = 0, 1, \ldots\} \tag{10.3}$$

where $\{\sum_{i=1}^N \hat{A}_t^i, t = 0, 1, \ldots\}$ is the independent version of $\{\sum_{i=1}^N A_t^i, t = 0, 1, \ldots\}$. Upon combining (10.3) with Theorem 5.1.1, we conclude the following result.

Theorem 10.1.1 *Let $\{M_t^N, t = 0, 1, \ldots\}$ be the process (10.1) obtained by multiplexing the N independent stationary on-off sources $\{A_t^i, t = 0, 1, \ldots\}$, $i = 1, \ldots, N$, with (I^i, B^i) satisfying the conditions of Proposition 7.2.1. Its independent version $\{\hat{M}_t^N, t = 0, 1, \ldots\}$ is the sequence of i.i.d. $\{0, 1, \ldots, N\}$-valued rvs $\{\sum_{i=1}^N \hat{A}_t^i, t = 0, 1, \ldots\}$ where for each $i = 1, \ldots, N$, $\{\hat{A}_t^i, t = 0, 1, \ldots\}$ is a sequence of i.i.d. $\{0, 1\}$-valued rvs with $\mathbf{P}\left[\hat{A}_t^i = 1\right] = p_i$ for all $t = 0, 1, \ldots$ and we have the comparison*

$$\{\hat{M}_t^N, t = 0, 1, \ldots\} \leq_{idcx} \{M_t^N, t = 0, 1, \ldots\}.$$

Moreover, their corresponding buffer contents $\{\hat{Q}_t, t = 0, 1, \ldots\}$ *and* $\{Q_t, t = 0, 1, \ldots\}$ *are ordered in the icx ordering, i.e., for any fixed initial condition* $\hat{Q}_0 = Q_0 = q$*, we have*

$$\hat{Q}_t \leq_{icx} Q_t, \quad t = 0, 1, \ldots.$$

Now, we consider the special case when the processes $\{A_t^i, t = 0, 1, \ldots\}, i = 1, 2, \ldots, N$, are i.i.d., i.e., they are N independent stationary on-off processes (I, B) with a common rate of $p = \frac{\mathbf{E}[B]}{\mathbf{E}[B]+\mathbf{E}[I]}$. We refer to $\{M_t^N, t = 0, 1, \ldots\}$ as the superposition of N i.i.d. on-off sources (I, B). If I and B satisfy the conditions in Proposition 7.2.1, the comparison (10.2) holds for each $i = 1, \ldots, N$, with $\{\hat{A}_t^i, t = 0, 1, \ldots\}$ being a sequence of i.i.d. $\{0, 1\}$-valued rvs with $\mathbf{P}\left[\hat{A}_t^i = 1\right] = p$ for all $t = 0, 1, \ldots$. Using the argument in Theorem 10.1.1, $\hat{M}_t^N = \sum_{i=1}^N \hat{A}_t^i$ for all $t = 0, 1, \ldots$ and the independent version $\{\hat{M}_t^N, t = 0, 1, \ldots\}$ is therefore a sequence of i.i.d. binomial rvs with parameter (N, p). The comparison can be summarized as follows:

Corollary 10.1.1 *Let* $\{M_t^N, t = 0, 1, \ldots\}$ *be the superposition of* N *i.i.d. on-off sources* (I, B) *with* I *and* B *satisfying the conditions of Proposition 7.2.1. Its independent version* $\{\hat{M}_t^N, t = 0, 1, \ldots\}$ *is a sequence of i.i.d. binomial rvs with parameter* (N, p)*, and we have the comparison*

$$\{\hat{M}_t^N, t = 0, 1, \ldots\} \leq_{idcx} \{M_t^N, t = 0, 1, \ldots\}.$$

Moreover, their corresponding buffer contents $\{\hat{Q}_t, t = 0, 1, \ldots\}$ *and* $\{Q_t, t = 0, 1, \ldots\}$ *are ordered in the icx ordering, i.e., for any fixed initial condition* $\hat{Q}_0 = Q_0 = q$*, we have*

$$\hat{Q}_t \leq_{icx} Q_t, \quad t = 0, 1, \ldots.$$

10.2 Infinite number of on-off sources

In this section, we investigate the comparison results for the superposition of N i.i.d. on-off sources $\{M_t^N, t = 0, 1, \ldots\}$ as the number N of sources grows unboundedly large. This time, for each $N = 1, 2, \ldots$, the stationary on-off sources $\{A_t^{(N,i)}, t = 0, 1, \ldots\}, i = 1, \ldots, N$, are mutually independent with same on- and off-period durations distributions (I_N, B) as in the special case of Section 10.1. The traffic intensity of the process $\{M_t^N, t = 0, 1, \ldots\}$ is given by $\frac{N\mathbf{E}[B]}{\mathbf{E}[B]+\mathbf{E}[I_N]}$ and we define the arrival rate

$$\lambda_N = \frac{N}{\mathbf{E}\left[B\right] + \mathbf{E}\left[I_N\right]}. \tag{10.4}$$

Likhanov, Tsybakov and Georganas [24] have shown that as N goes to infinity, if B is kept unchanged and $\lim_{N\to\infty} \lambda_N = \lambda$, then the limiting process of $\{M_t^N, t = 0, 1, \ldots\}$ approaches the $M|G|\infty$ input process (λ, B).

Theorem 10.2.1 *Let $\{M_t^N, t = 0, 1, \ldots\}$ be the superposition of N i.i.d. on-off sources (I_N, B). If $\lim_{N\to\infty} \lambda_N = \lambda$ and $\lim_{N\to\infty} \mathbf{P}\left[I_N \leq k\right] = 0$ for each $k = 0, 1, \ldots$, then*

$$\{M_t^N, t = 0, 1, \ldots\} \Longrightarrow_N \{M_t, t = 0, 1, \ldots\}, \tag{10.5}$$

where $\{M_t, t = 0, 1, \ldots\}$ is the $M|G|\infty$ process (λ, B).

Notice from Theorem 10.2.1 that the on-period duration B is simply the session duration in the $M|G|\infty$ process and the limiting process does not depend on the fine details of off-period duration distributions. As a result, in order to ensure the assumptions of Theorem 10.2.1, we can construct the sequence of processes $\{M_t^N, t = 0, 1, \ldots\}$, $N = 1, 2, \ldots$, that converges in distribution to the

$M|G|\infty$ process (λ, B) by fixing B and selecting I_N such that $\lambda_N = \lambda$ for all $N = 1, 2, \ldots$, and for fixed $k = 0, 1, \ldots$, $\lim_{N\to\infty} \mathbf{P}[I_N \leq k] = 0$.

By defining the $M|G|\infty$ input process using this limiting approach, we can in principle establish the lower bound comparison of the $M|G|\infty$ input process by making use of the comparison for single on-off sources (as in Corollary 10.1.1). Since the limiting process does not depend on the off-period duration distribution, we expect that in order to have the comparison with its independent version, the conditions on the $M|G|\infty$ process must be relaxed from that of the original single on-off process given in Proposition 7.2.1. This is indeed the case as the next theorem indicates.

Theorem 10.2.2 *Let $\{M_t, t = 0, 1, \ldots\}$ be an $M|G|\infty$ input process (λ, B) such that B and $\hat{B}$ are DFR rvs. Its independent version $\{\hat{M}_t, t = 0, 1, \ldots\}$ is a sequence of i.i.d. Poisson rvs with mean $\lambda\mathbf{E}[B]$ and we have the comparison*

$$\{\hat{M}_t, t = 0, 1, \ldots\} \leq_{idcx} \{M_t, t = 0, 1, \ldots\}.$$

Moreover, their corresponding buffer contents $\{\hat{Q}_t, t = 0, 1, \ldots\}$ and $\{Q_t, t = 0, 1, \ldots\}$ are ordered in the icx ordering, i.e., for any fixed initial condition $\hat{Q}_0 = Q_0 = q$, we have

$$\hat{Q}_t \leq_{icx} Q_t, \quad t = 0, 1, \ldots.$$

Proof. The key of the proof is as follows: Consider the setup of Theorem 10.2.1. If we can ensure that each on-off process (I_N, B) is SSI for all $N = 1, 2, \ldots$, then the sm comparison of the superposition of N i.i.d. on-off sources $\{M_t^N, t = 0, 1, \ldots\}$ with its independent version holds for $N = 1, 2, \ldots$. By invoking the weak convergence lemma for the sm ordering [Lemma 4.3.4], we can obtain the sm comparison of the limiting process $\{M_t, t = 0, 1, \ldots\}$ with its independent version. Throughout the proof, we will refer to the SSI conditions (A.1)-(A.4) given in Lemma 7.3.1 for a single on-off source.

Fix $N = 1, 2, \ldots$: For each $t = 0, 1, \ldots$, we have $M_t^N = \sum_{i=1}^N A_t^{(N,i)}$ where $\{A_t^{(N,i)}, t = 0, 1, \ldots\}$ is the on-off source (I_N, B). As mentioned earlier, we can construct $\{M_t^N, t = 0, 1, \ldots\}$ by choosing a sequence of off-period duration rvs $\{I_N, N = 1, 2, \ldots\}$ such that $\lambda_N = \lambda$ for all $N = 1, 2, \ldots$, and for fixed $k = 0, 1, \ldots$, $\lim_{N\to\infty} \mathbf{P}[I_N \le k] = 0$. Consequently, we take $I_N =_{st} \mathcal{G}(1 - \frac{\lambda}{N - \lambda \mathbf{E}[B]})$ for all $N = 1, 2, \ldots$. Clearly, such a sequence $\{I_N, N = 1, 2, \ldots\}$ satisfies the requirements of Theorem 10.2.1 and

$$\{M_t^N, t = 0, 1, \ldots\} \Longrightarrow_N \{M_t, t = 0, 1, \ldots\}, \tag{10.6}$$

where $\{M_t, t = 0, 1, \ldots\}$ is the $M|G|\infty$ input process (λ, B).

Now, we consider the SSI conditions (A.1)-(A.4) of the on-off processes defined above. For each $N = 1, 2, \ldots$, I_N and $\hat{I}_N$ are DFR. Thus, by taking the rvs B and $\hat{B}$ to be DFR, conditions (A.1) and (A.3) are satisfied. Conditions (A.2) and (A.4) require that

$$\mathbf{P}[I_N = 1] + \mathbf{P}[B = 1] \le 1$$

and

$$\frac{1}{\mathbf{E}[I_N]} + \frac{1}{\mathbf{E}[B]} \le 1,$$

respectively. But for fixed $k = 0, 1, \ldots$, it holds that $\lim_{N\to\infty} \mathbf{P}[I_N \le t] = 0$ so that $\lim_{N\to\infty} \frac{1}{\mathbf{E}[I_N]} = 0$, whence conditions (A.2) and (A.4) are indeed satisfied if $N > N^*$ for some $N^* > 0$.

For fixed $N > N^*$, I_N and B satisfy conditions (A.1)-(A.4), $\{A_t^{(N,i)}, t = 0, 1, \ldots\}$ is SSI for each $i = 1, \ldots, N$, and by Theorem 4.5.1, we get

$$\{\hat{A}_t^{(N,i)}, t = 0, 1, \ldots\} \le_{sm} \{A_t^{(N,i)}, t = 0, 1, \ldots\} \tag{10.7}$$

where $\{\hat{A}_t^{(N,i)}, t = 0,1,\ldots\}$ denotes the independent version of $\{A_t^{(N,i)}, t = 0,1,\ldots\}$. Upon combining (10.7) and Corollary 4.3.2, we obtain

$$\{\hat{M}_t^N, t = 0,1,\ldots\} \leq_{sm} \{M_t^N, t = 0,1,\ldots\}. \tag{10.8}$$

Before applying the weak convergence lemma for the sm ordering, it remains to show that

$$\{\hat{M}_t^N, t = 0,1,\ldots\} \Longrightarrow_N \{\hat{M}_t, t = 0,1,\ldots\}. \tag{10.9}$$

For each $t = 0,1,\ldots$, $\hat{M}_t^N$ is a binomial rv with parameter $(N, \frac{\lambda \mathbf{E}[B]}{N})$. It is well-known that $\hat{M}_t^N$ converges in distribution to a Poisson rv with mean $\lambda \mathbf{E}[B]$, which is equivalent to the marginal of the original $M|G|\infty$ process (λ, B) (See Section 3.3 and Section 11.1). Thus, (10.9) is satisfied with $\{\hat{M}_t, t = 0,1,\ldots\}$ identified as the independent version of the $M|G|\infty$ process $\{M_t, t = 0,1,\ldots\}$. By applying Lemma 4.3.4 and making use of (10.6) and (10.9), we conclude that

$$\{\hat{M}_t, t = 0,1,\ldots\} \leq_{sm} \{M_t, t = 0,1,\ldots\}.$$

Because this comparison also holds in the idcx ordering, the second half of the result is now immediate from Theorem 5.1.1 and the proof is complete.

■

Chapter 11

$M|G|\infty$ Input Traffic

11.1 Lower bounds for $M|G|\infty$ input traffic

As stated in Section 3.3, an $M|G|\infty$ input process is characterized by a pair of parameters (λ, S). We now argue that the independent version of an $M|G|\infty$ input process (λ, S) is also an $M|G|\infty$ input process, say (λ^0, S^0), where λ^0 and S^0 are properly selected. Indeed, if we take $S^0 \equiv 1$, then each customer (each session) requires exactly one timeslot of service before leaving the system at the end of that slot. Therefore, the number of customers in the system at the beginning of timeslot $[t, t+1)$ is simply the number of customers who arrive in timeslot $[t-1, t)$ independently of arrivals in past and future timeslots. Let $\{\hat{A}_t, t = 0, 1, \ldots\}$ denoted the $M|G|\infty$ input process $(\lambda^0, S^0 \equiv 1)$ as specified above. From the discussion above, the rvs $\{\hat{A}_t, t = 0, 1, \ldots\}$ are mutually independent, in agreement with Claim (ii) of Proposition 3.3.1 which yields in that case

$$cov(\hat{A}_t, \hat{A}_{t+h}) = \lambda^0 \mathbf{P}\left[\hat{S}^0 > h\right] = \lambda^0 \delta(0, h), \quad h = 0, 1, \ldots,$$

for all $t = 0, 1, \ldots$ where $\delta(s, t) = 1$ if $s = t$, otherwise $\delta(s, t) = 0$. By Claim (i) of Proposition 3.3.1, for $t = 0, 1, \ldots$, the rv $\hat{A}_t$ is a Poisson rv with parameter λ^0.

Thus, the marginals of the sequence $\{A_t, t = 0, 1, \ldots\}$ for the given $M|G|\infty$ input process (λ, S) will coincide with those of the independent version $\{\hat{A}_t, t = 0, 1, \ldots\}$ provided $\lambda^0 = \lambda \mathbf{E}[S]$. In conclusion, the independent version of $M|G|\infty$ input process (λ, S) is simply the $M|G|\infty$ input process $(\lambda \mathbf{E}[S], 1)$.

We now turn to finding conditions under which an $M|G|\infty$ input process is SSI. Unfortunately, we are unable to directly establish the SSI property of the $M|G|\infty$ input process, although this process is associated [37, 40]. However, as we shall see shortly, it is still possible to show that the independent version of $M|G|\infty$ input processes does act as a lower bound. To do so, note from Theorem 5.1.1 that the desired result will be obtained if the $M|G|\infty$ input process is shown to be greater in the idcx ordering than its independent version. By Corollary 4.3.2, the sm ordering is stable under convolution (thus independent summation), thereby suggesting the following approach: We first seek to identify an additive *independent decomposition* of the $M|G|\infty$ input process, each with SSI property. The independent version of each component then acts as a lower bound process to the corresponding component in the sm ordering. Finally, the sum of the independent versions of the decomposed processes is statistically indistinguishable from $\{\hat{A}_t, t = 0, 1, \ldots\}$ and satisfies

$$\{\hat{A}_t, t = 0, 1, \ldots\} \leq_{sm} \{A_t, t = 0, 1, \ldots\}$$

The desired comparison result is formalized through the following theorem.

Theorem 11.1.1 *Let $\{A_t, t = 0, 1, \ldots\}$ be an $M|G|\infty$ input process (λ, S). Its independent version $\{\hat{A}_t, t = 0, 1, \ldots\}$ is the $M|G|\infty$ input process $(\lambda \mathbf{E}[S], 1)$ and we have the comparison*

$$\{\hat{A}_t, t = 0, 1, \ldots\} \leq_{idcx} \{A_t, t = 0, 1, \ldots\}.$$

Moreover, their corresponding buffer contents $\{\hat{Q}_t, t = 0, 1, \ldots\}$ *and* $\{Q_t, t = 0, 1, \ldots\}$ *are ordered in the icx ordering, i.e., for any fixed initial condition* $\hat{Q}_0 = Q_0 = q$*, we have*

$$\hat{Q}_t \leq_{icx} Q_t, \quad t = 0, 1, \ldots.$$

Note that the conclusion of Theorem 11.1.1 holds for any session duration distribution S. This is in sharp contrast with Theorem 10.2.2 which requires the rvs S and $\hat{S}$ of an $M|G|\infty$ input process to be DFR for the comparison to hold. This limitation can be traced back to the method of proof of Theorem 10.2.2, namely the use of the results for on-off sources via a limiting process. The *independent decomposition* approach used in Section 11.2 yields the comparison result without any condition on S, thereby achieving the same result under a weaker condition.

11.2 Proof of Theorem 11.1.1

We first identify the independent decomposition and then use it to show the comparison of the $M|G|\infty$ input process with its independent version. Recall from (3.7) that the stationary $M|G|\infty$ input traffic (λ, S) admits the decomposition

$$A_t = A_t^{(0)} + A_t^{(a)}, \quad t = 0, 1, \ldots$$

with

$$A_t^{(0)} = \sum_{i=1}^{B} \mathbf{1}\left[\hat{S}_i > t\right] \tag{11.1}$$

where $\{\hat{S}_i, i = 1, 2, \ldots, B\}$ are i.i.d. $\{1, 2, \ldots\}$-valued rvs distributed according to the forward recurrence time associated with S and independent of the Poisson rv B with mean $\lambda\mathbf{E}[S]$.

In addition, we have the decomposition

$$A_t^{(a)} = \sum_{r=1}^{\infty} A_t^{(r)}, \quad t = 0, 1, \ldots, \tag{11.2}$$

where for each $r = 1, 2, \ldots$, $\{A_t^{(r)}, t = 0, 1, \ldots\}$ is the process corresponding to those B_r customers who arrive in timeslot $[r-1, r)$. Formally,

$$A_t^{(r)} = \mathbf{1}\left[t \geq r\right] \sum_{i=1}^{B_r} \mathbf{1}\left[S_{r,i} > t - r\right], \quad t = 0, 1, \ldots. \tag{11.3}$$

Note that $A_0^{(0)} =_{st} B$ and $A_r^{(r)} =_{st} B_r$ for all $r = 1, 2, \ldots$. The processes $\{A_t^{(0)}, t = 0, 1, \ldots\}$ and $\{A_t^{(r)}, t = 0, 1, \ldots\}$, $r = 1, 2, \ldots$, are mutually independent and display very similar structures. To exploit this observation, we shall make use of the following general result:

Proposition 11.2.1 *Let K be an $\mathbb{N}$-valued rv and let $\{\xi, \xi_i, i = 1, 2, \ldots\}$ be a sequence of i.i.d. $\{1, 2, \ldots\}$-valued rvs. If K is independent of $\{\xi_i, i = 1, 2, \ldots\}$, then the process $\{\sum_{i=1}^{K} \mathbf{1}\left[\xi_i > t\right], t = 0, 1, \ldots\}$ is SSI.*

Proof. For each $t = 0, 1, \ldots$, set $X_t = \sum_{i=1}^{K} \mathbf{1}\left[\xi_i > t\right]$. Since $\xi_i > 0$ for all $i = 1, 2, \ldots$, $X_0 =_{st} K$ and we have $[X_t | X_0 = x_0] =_{st} \sum_{i=1}^{x_0} \mathbf{1}\left[\xi_i > t\right]$ for all $t > 0$ and $x_0 = 0, 1, \ldots$. Let $\mathbf{X}^t$ denote $(X_1, \ldots, X_t)$ and set $\mathbf{x}^t = (x_1, \ldots, x_t)$ in $\mathbb{R}^t$. In order to show the SSI property, we need to consider the conditional distribution $[X_{t+1} | \mathbf{X}^t = \mathbf{x}^t, X_0 = x_0]$. It is plain that

$$\mathbf{P}\left[X_{t+1} = x | \mathbf{X}^t = \mathbf{x}^t, X_0 = x_0\right] = \frac{\mathbf{P}\left[X_{t+1} = x, \mathbf{X}^t = \mathbf{x}^t | X_0 = x_0\right]}{\mathbf{P}\left[\mathbf{X}^t = \mathbf{x}^t | X_0 = x_0\right]} \tag{11.4}$$

provided $\mathbf{P}\left[\mathbf{X}^t = \mathbf{x}^t | X_0 = x_0\right] > 0$. In particular, this requires $x_0 \geq x_1 \geq \ldots \geq x_t \geq x$.

As we now consider the evaluation of $\mathbf{P}\left[\mathbf{X}^t = \mathbf{x}^t | X_0 = x_0\right]$, we pick integers $x_0 \geq x_1 \geq \ldots \geq x_t \geq x$ and note that

$$\begin{aligned} X_s - X_{s+1} &= \sum_{i=1}^{K} \mathbf{1}\left[\xi_i > s\right] - \sum_{i=1}^{K} \mathbf{1}\left[\xi_i > s+1\right] \\ &= \sum_{i=1}^{K} \mathbf{1}\left[\xi_i = s+1\right], \quad s = 0, 1, \ldots. \end{aligned}$$

Consequently, the conditions $X_0 = x_0, X_s = x_s, s = 1, 2, \ldots, t$, together are equivalent to

$$X_0 = x_0, \quad \sum_{i=1}^{K} \mathbf{1}\left[\xi_i = s+1\right] = x_s - x_{s+1}, \quad s = 0, 1, \ldots, t-1.$$

In other words, given $X_0 = x_0$, the event $\mathbf{X}^t = \mathbf{x}^t$ will take place if $(x_0 - x_1)$ among the rvs $\xi_i, i = 1, \ldots, x_0$, take value 1, $(x_1 - x_2)$ among the rvs $\xi_i, i = 1, \ldots, x_0$, take value 2, …, and x_t among the rvs $\xi_i, i = 1, \ldots, x_0$, take value greater than t. As a result, $[\mathbf{X}^t | X_0 = x_0]$ is a multinomial distribution given by

$$\begin{aligned} &\mathbf{P}\left[\mathbf{X}^t = \mathbf{x}^t | X_0 = x_0\right] \qquad (11.5) \\ =\; &\frac{x_0!}{x_t!(x_{t-1} - x_t)! \cdots (x_0 - x_1)!} \cdot \mathbf{P}\left[\xi > t\right]^{x_t} \mathbf{P}\left[\xi = t\right]^{x_{t-1} - x_t} \cdots \mathbf{P}\left[\xi = 1\right]^{x_0 - x_1}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathbf{P}\left[X_{t+1} = x, \mathbf{X}^t = \mathbf{x}^t | X_0 = x_0\right] \qquad (11.6) \\ =\; &\frac{x_0!}{x!(x_t - x)! \cdots (x_0 - x_1)!} \cdot \mathbf{P}\left[\xi > t+1\right]^{x} \mathbf{P}\left[\xi = t+1\right]^{x_t - x} \cdots \mathbf{P}\left[\xi = 1\right]^{x_0 - x_1}. \end{aligned}$$

Upon combining (11.4), (11.5) and (11.6), we obtain

$$\begin{aligned} &\mathbf{P}\left[X_{t+1} = x | \mathbf{X}^t = \mathbf{x}^t, X_0 = x_0\right] \\ =\; &\frac{x_t!}{x!(x_t - x)!} \frac{\mathbf{P}\left[\xi > t+1\right]^{x} \mathbf{P}\left[\xi = t+1\right]^{x_t - x}}{\mathbf{P}\left[\xi > t\right]^{x_t}} \\ =\; &\frac{x_t!}{x!(x_t - x)!} \mathbf{P}\left[\xi > t+1 | \xi > t\right]^{x} \mathbf{P}\left[\xi = t+1 | \xi > t\right]^{x_t - x}, \end{aligned}$$

and $[X_{t+1}|\mathbf{X}^t = \mathbf{x}^t, X_0 = x_0]$ is a binomial distribution with parameter (N, p) where $N = x_t$ and $p = \mathbf{P}[\xi > t+1|\xi > t]$. Since for a given p, the binomial distribution is stochastically increasing in the parameter N [45] and $[X_{t+1}|\mathbf{X}^t = \mathbf{x}^t, X_0 = x_0]$ depends only on x_t, it is clear that for each $t = 1, 2, \ldots$, $[X_{t+1}|\mathbf{X}^t = \mathbf{x}^t, X_0 = x_0]$ is stochastically increasing with respect to the past sequence $(\mathbf{x}^t, x_0)$, whence $\{X_t, t = 0, 1, \ldots\}$ is SSI. ■

Consequently, we have the two following lemmas.

Lemma 11.2.1 *$\{A_t^{(0)}, t = 0, 1, \ldots\}$ is SSI.*

Proof. Recall that $A_t^{(0)} = \sum_{i=1}^{B} \mathbf{1}\left[\hat{S}_i > t\right]$ for each $t = 0, 1, \ldots$ and $A_0^{(0)} =_{st} B$ is Poisson distributed with mean $\lambda \mathbf{E}[S]$. With the rvs $\{\hat{S}_i, i = 1, 2, \ldots\}$ being i.i.d. $\{1, 2, \ldots\}$-valued rvs, $\{A_t^{(0)}, t = 0, 1, \ldots\}$ possesses the SSI property by Proposition 11.2.1 (with $K = B$). ■

Lemma 11.2.2 *For each $r = 1, 2, \ldots$, $\{A_t^{(r)}, t = 0, 1, \ldots\}$ is SSI.*

Proof. Fix $r = 1, 2, \ldots$, it can be seen from the definition of the process that $A_r^{(r)} =_{st} B_r$ and $A_t^{(r)} =_{st} \mathbf{1}[t \geq r] \sum_{i=1}^{B_r} \mathbf{1}[S_{r,i} > t - r]$ for each $t = 0, 1, \ldots$ where the rvs $\{S_{r,i}, i = 1, 2, \ldots\}$ are i.i.d. $\{1, 2, \ldots\}$-valued rvs. Because $A_t^r = 0$ when $t < r$, it is enough to consider $[A_{t+1}^{(r)}|A_t^{(r)} \ldots, A_r^{(r)}]$ as we note that $[A_{t+1}^{(r)}|A_t^{(r)}, \ldots, A_0^{(r)}] =_{st} [A_{t+1}^{(r)}|A_t^{(r)} \ldots, A_r^{(r)}]$ whenever $t \geq r$.

For $t \geq r$, $A_t^{(r)} = \sum_{i=1}^{B_r} \mathbf{1}[S_{r,i} > t - r]$ so that in the notation of Proposition 11.2.1, $A_t^{(r)}$ is equivalent to $\sum_{i=1}^{K} \mathbf{1}[\xi_i > u]$ with $u = t - r$ where K and

$\{\xi_i, i = 1, 2, \ldots\}$ are identified with B_r and $\{S_{r,i}, i = 1, 2, \ldots\}$, respectively. Hence, $\{A_t^{(r)}, t = 0, 1, \ldots\}$ is SSI by Proposition 11.2.1. ■

By virtue of Lemma 11.2.1 and 11.2.2, we can now prove Theorem 11.1.1.

Proof of Theorem 11.1.1 From Lemma 11.2.1 and Theorem 4.5.1, we have that

$$\{\hat{A}_t^{(0)}, t = 0, 1, \ldots\} \leq_{sm} \{A_t^{(0)}, t = 0, 1, \ldots\}$$

where $\{\hat{A}_t^{(0)}, t = 0, 1, \ldots\}$ is the independent version of $\{A_t^{(0)}, t = 0, 1, \ldots\}$. On the other hand, by Lemma 11.2.2 and Theorem 4.5.1, for each $r = 1, 2, \ldots$,

$$\{\hat{A}_t^{(r)}, t = 0, 1, \ldots\} \leq_{sm} \{A_t^{(r)}, t = 0, 1, \ldots\}$$

where again $\{\hat{A}_t^{(r)}, t = 0, 1, \ldots\}$ denotes the independent version of $\{A_t^{(r)}, t = 0, 1, \ldots\}$. It is always possible to construct all rvs on a single probability triple so that the independent versions are mutually independent. Hence, under the enforced independence assumptions, upon invoking Claim (ii) of Corollary 4.3.2 and the pointwise convergences $\lim_{R\to\infty}(\sum_{r=1}^R A_t^{(r)} + A_t^{(0)}) = A_t$ for all $t = 0, 1, \ldots$, we obtain the comparison

$$\{\hat{A}_t, t = 0, 1, \ldots\} \leq_{sm} \{A_t, t = 0, 1, \ldots\},$$

where for each $t = 0, 1, \ldots$, $\hat{A}_t = \lim_{R\to\infty}(\sum_{r=1}^R \hat{A}_t^{(r)} + \hat{A}_t^{(0)})$.

Recall from Corollary 4.3.2 that $\{\hat{A}_t, t = 0, 1, \ldots\}$ is the independent version of $\{A_t, t = 0, 1, \ldots\}$, whence it must be the $M|G|\infty$ input process $(\lambda \mathbf{E}[S], 1)$ described in Section 11.1. Because the sm ordering implies the idcx ordering, we have $\{\hat{A}_t, t = 0, 1, \ldots\} \leq_{idcx} \{A_t, t = 0, 1, \ldots\}$ as desired and the buffer comparison follows from Theorem 5.1.1. ■

11.3 Effects of session-duration variability

Besides the comparison with its independent version, it is also desirable to establish the comparison between two $M|G|\infty$ processes with the same marginals but different correlation structures. More precisely, we expect that if $S^{(1)} \leq_{cx} S^{(2)}$, then the $M|G|\infty$ process $(\lambda, S^{(2)})$ exhibits more dependence than the $M|G|\infty$ process $(\lambda, S^{(1)})$ in the sense of the sm ordering. In this section, we show that such a comparison can be indeed obtained in the case of two-dimensional rvs.

For $i = 1, 2$, let $\{A_t^i, t = 0, 1, \ldots\}$ be the $M|G|\infty$ input process $(\lambda, S^{(i)})$. Assuming $S^{(1)} \leq_{cx} S^{(2)}$, we have $\mathbf{E}\left[S^{(1)}\right] = \mathbf{E}\left[S^{(2)}\right]$ and $Var(S^{(1)}) \leq Var(S^{(2)})$, i.e., $S^{(1)}$ has less variability than $S^{(2)}$. Hence, for each $t = 0, 1, \ldots$, $A_t^1 =_{st} A_t^2$ from Claim (i) of Proposition 3.3.1. Moreover, for $h = 1, 2, \ldots$,

$$\mathbf{P}\left[\hat{S}^{(1)} > h\right] = \frac{\sum_{t=h+1}^{\infty} \mathbf{P}\left[S^{(1)} \geq t\right]}{\mathbf{E}\left[S^{(1)}\right]} \leq \frac{\sum_{t=h+1}^{\infty} \mathbf{P}\left[S^{(2)} \geq t\right]}{\mathbf{E}\left[S^{(2)}\right]} = \mathbf{P}\left[\hat{S}^{(2)} > h\right],$$

or equivalently, $\hat{S}^{(1)} \leq_{st} \hat{S}^{(2)}$ where for $i = 1, 2$, the rv $\hat{S}^{(i)}$ denotes the forward recurrence of rv $S^{(i)}$. Thus, by Claim (ii) of Proposition 3.3.1, we have for each $h = 1, 2, \ldots$,

$$\begin{aligned} cov(A_t^1, A_{t+h}^1) &= \lambda \mathbf{E}\left[S^{(1)}\right] \mathbf{P}\left[\hat{S^{(1)}} > h\right] \\ &\leq \lambda \mathbf{E}\left[S^{(2)}\right] \mathbf{P}\left[\hat{S^{(2)}} > h\right] = cov(A_t^2, A_{t+h}^2). \end{aligned}$$

In conclusion, the $M|G|\infty$ process $(\lambda, S^{(1)})$ has the same marginals as the $M|G|\infty$ process $(\lambda, S^{(2)})$ but its correlation function is smaller than that of the the $M|G|\infty$

process $(\lambda, S^{(2)})$, i.e., the $M|G|\infty$ process $(\lambda, S^{(2)})$ is more positively correlated than the $M|G|\infty$ process $(\lambda, S^{(1)})$.

As in the case of independent version, if we can establish the comparison in either the sm or idcx ordering between $\{A_t^1, t = 0, 1, \ldots\}$ and $\{A_t^2, t = 0, 1, \ldots\}$, i.e.,

$$\{A_t^1, t = 0, 1, \ldots\} \leq_{idcx} \{A_t^2, t = 0, 1, \ldots\}, \tag{11.7}$$

then the comparison of their corresponding buffer contents is made possible by Theorem 5.1.1. Unfortunately, we are unable to show (11.7) in this case. However, we can establish the sm comparison in the case of two-dimensional marginals, i.e.,

$$(A_0^1, A_t^1) \leq_{sm} (A_0^2, A_t^2), \quad t = 1, 2, \ldots, \tag{11.8}$$

by using the facts on orthant orderings which were developed in Section 4.4.

Since $A_0^1 =_{st} A_0^2$ and $A_t^1 =_{st} A_t^2, t = 1, 2, \ldots$, by virtue of Lemma 4.4.1, showing (11.8) is equivalent to showing

$$(A_0^1, A_t^1) \geq_{lo} (A_0^2, A_t^2), \quad t = 1, 2, \ldots, \tag{11.9}$$

or equivalently,

$$\mathbf{P}\left[A_0^1 \leq x_0, A_t^1 \leq x_1\right] \leq \mathbf{P}\left[A_0^2 \leq x_0, A_t^2 \leq x_1\right], \quad x_0, x_1 = 0, 1, \ldots, \tag{11.10}$$

for all $t = 1, 2, \ldots$.

Note that the comparison between the $M|G|\infty$ process $(\lambda, S^{(1)})$ and the $M|G|\infty$ process $(\lambda, S^{(2)})$ when $S^{(1)} \leq_{cx} S^{(2)}$ is not a generalized version of the comparison with the independent version developed in Section 11.1. The reason is that for any $M|G|\infty$ process (λ, S), its independent version is given by the $M|G|\infty$ process $(\lambda \mathbf{E}[S], 1)$, thus both processes are not identified with the assumptions of $M|G|\infty$ process $(\lambda, S^{(1)})$ and $M|G|\infty$ process $(\lambda, S^{(2)})$.

11.3.1 Expressions for $M|G|\infty$ processes

Before we proceed to establish (11.10), we first concentrate on finding the expressions of $\mathbf{P}[A_0 \le x_0, A_t \le x_1]$ for each $t = 1, 2, \ldots$, when $\{A_t, t = 0, 1, \ldots\}$ is an $M|G|\infty$ process (λ, S).

Fix $t = 1, 2, \ldots$: Recall from Section 3.3 the independent decomposition

$$A_t = A_t^{(0)} + A_t^{(a)}$$

where $A_t^{(0)} = \sum_{i=1}^{B} \mathbf{1}\left[\hat{S}_i > t\right]$ and $A_t^{(a)}$ is a Poisson rv with mean $\lambda \sum_{s=1}^{t} \mathbf{P}[S > s]$. For each $i = 1, 2, \ldots$, define $\xi_i(p_t) = \mathbf{1}\left[\hat{S}_i > t\right]$ where $p_t = \mathbf{P}\left[\hat{S} > t\right]$. Clearly, the rvs $\{\xi_i(p_t), i = 1, 2, \ldots\}$ are i.i.d. Bernoulli rvs with mean p_t. Upon noting that $A_0 =_{st} B$ is Poisson distributed with mean $\lambda \mathbf{E}[S]$, we rewrite $A_t^{(0)}$ as

$$A_t^{(0)} = \sum_{i=1}^{A_0} \xi_i(p_t)$$

provided the rvs $\{\xi_i(p_t), i = 1, 2, \ldots\}$ are independent of A_0. Set $q_t = 1 - p_t = \frac{\sum_{s=1}^{t} \mathbf{P}[S>s]}{\mathbf{E}[S]}$. Thus, the rvs $A_t^{(0)}$ and $A_t^{(a)}$ are simply independent Poisson rvs with mean $\lambda \mathbf{E}[S] p_t$ and $\lambda \mathbf{E}[S] q_t$, respectively. As a result, we have the expression

$$\mathbf{P}[A_0 \le x_0, A_t \le x_1] = \mathbf{P}\left[A_0 \le x_0, \sum_{i=1}^{A_0} \xi_i(p_t) + A_t^{(a)} \le x_1\right], \tag{11.11}$$

where the rvs $A_0, \{\xi_i(p_t), i = 1, 2, \ldots\}$ and $A_t^{(a)}$ are mutually independent.

11.3.2 The comparison

From the discussion above, when $S^{(1)} \le_{cx} S^{(2)}$, we have $\hat{S}^{(1)} \le_{st} \hat{S}^{(2)}$ with

$$p_t^1 = \mathbf{P}\left[\hat{S}^{(1)} > t\right] \le \mathbf{P}\left[\hat{S}^{(2)} > t\right] = p_t^2, \quad t = 1, 2, \ldots.$$

Therefore, for each $t = 1, 2, \ldots$ if (11.11) is monotone increasing in p_t, then the comparison (11.10) holds. The desired monotonicity is made possible by the following fact:

Proposition 11.3.1 *Fix $0 \leq p \leq 1$ and write $q = 1 - p$. Let N_0 and N_1 be independent Poisson rvs with mean λ and λq, respectively. Let $\{\xi_i(p), i = 1, 2, \ldots\}$ be a sequence of i.i.d. Bernoulli rvs with mean p independent of N_0 and N_1. Then, for any $x_0, x_1 = 0, 1, 2 \ldots$, it holds that*

$$\begin{aligned} & \frac{d}{dp} \mathbf{P} \left[N_0 \leq x_0, \sum_{i=1}^{N_0} \xi_i(p) + N_1 \leq x_1 \right] \\ = \; & \lambda \mathbf{P} \left[N_0 = x_0 \right] \left(\sum_{j=0}^{x_0} \mathbf{P} \left[\sum_{i=1}^{x_0} \xi_i(p) = j \right] \mathbf{P} \left[N_1 = x_1 - j \right] \right) \end{aligned} \tag{11.12}$$

where $\mathbf{P}[N_1 = k] = 0$ for $k < 0$.

The proof of Proposition 11.3.1 is given in the next section. However, from it, we can already conclude to the following.

Lemma 11.3.1 *For any $x_0, x_1 = 0, 1, \ldots$, we have*

$$\begin{aligned} & \frac{d}{dp_t} \mathbf{P} \left[A_0 \leq x_0, A_t \leq x_1 \right] \qquad (11.13) \\ = \; & \lambda \mathbf{E}[S] \, \mathbf{P} \left[A_0 = x_0 \right] \left(\sum_{j=0}^{x_0} \mathbf{P} \left[\sum_{i=1}^{x_0} \xi_i(p_t) = j \right] \mathbf{P} \left[A_t^{(a)} = x_1 - j \right] \right) \geq 0. \end{aligned}$$

Proof. It can be seen that $\{\xi_i(p_t), i = 1, 2, \ldots\}$, A_0 and $A_t^{(a)}$ are identified with $\{\xi_i(p), i = 1, 2, \ldots\}$, N_0 and N_1 of Proposition 11.3.1, respectively, and the expression (11.13) is simply a rewrite of (11.12) and clearly non-negative. ■

From Lemma 11.3.1, it is plain that the probability $\mathbf{P}[A_0 \leq x_0, A_t \leq x_1]$ is monotone increasing in $p_t = \mathbf{P}\left[\hat{S} > t\right]$ and the comparison (11.10) holds. We summarize these findings in the next theorem.

Theorem 11.3.1 *For $i = 1, 2$, let $\{A_t^i, t = 0, 1, \ldots\}$ be an $M|G|\infty$ input process $(\lambda, S^{(i)})$. If $S^{(1)} \leq_{cx} S^{(2)}$, then it holds that*

$$(A_0^1, A_t^1) \leq_{sm} (A_0^2, A_t^2)$$

for all $t = 1, 2, \ldots$.

11.3.3 Proof of Proposition 11.3.1

We begin with some definitions and expressions. First, define the function $f : \mathbb{R}^2 \times [0,1] \to [0,1]$ by

$$f(x_0, x_1, p) = \mathbf{P}\left[N_0 \leq x_0, \sum_{i=1}^{N_0} \xi_i(p) + N_1 \leq x_1\right].$$

Next, recall that

$$\mathbf{P}[N_0 = x] = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, \ldots,$$

and that for fixed $0 \leq p \leq 1$ and $n = 1, 2, \ldots$,

$$\mathbf{P}\left[\sum_{i=1}^{n} \xi_i(p) = k\right] = \frac{n!}{(n-k)!k!} p^k q^{n-k}, \quad k = 0, \ldots, n.$$

If $k \neq 0, 1, \ldots, n$, we always have $\mathbf{P}\left[\sum_{i=1}^{n} \xi_i(p) = k\right] = 0$ and similarly if $j < 0$, we have $\mathbf{P}[N_0 = j] = \mathbf{P}[N_1 = j] = 0$.

We note two important relations: For $x = 0, 1, \ldots$,

$$\begin{aligned} \frac{d}{dp}\mathbf{P}[N_1 \leq x] &= \lambda e^{-\lambda q} \sum_{k=0}^{x} \frac{(\lambda q)^k}{k!} - \lambda e^{-\lambda q} \sum_{k=1}^{x} \frac{(\lambda q)^{k-1}}{(k-1)!} \\ &= \lambda e^{-\lambda q} \frac{(\lambda q)^x}{x!} \\ &= \lambda \mathbf{P}[N_1 = x] \end{aligned} \tag{11.14}$$

and for fixed $0 \leq p \leq 1$ and $n = 1, 2, \ldots$,

$$\frac{d}{dp}\mathbf{P}\left[\sum_{i=1}^{n} \xi_i(p) = k\right] = \frac{n!}{(n-k)!k!}[kp^{k-1}q^{n-k} - (n-k)p^k q^{n-k-1}]$$

$$
\begin{aligned}
&= \frac{n!}{(n-k)!(k-1)!}p^{k-1}q^{n-k} - \frac{n!}{(n-1-k)!k!}p^k q^{n-1-k} \\
&= n(\mathbf{P}\left[\sum_{i=1}^{n-1}\xi_i(p) = k-1\right] - \mathbf{P}\left[\sum_{i=1}^{n-1}\xi_i(p) = k\right]) \quad (11.15)
\end{aligned}
$$

for each $k = 0, 1, \ldots, n$.

Now, we are ready to establish Proposition 11.3.1. The proof proceeds by induction on x_0. Fix $0 \le p \le 1$ and $x_1 = 0, 1, \ldots$, and set $q = 1 - p$: For $x_0 = 0$, $\sum_{i=1}^{x_0}\xi_i(p) =_{st} 0$ and by independence, we have

$$
\begin{aligned}
\frac{d}{dp}f(0, x_1, p) &= \frac{d}{dp}\mathbf{P}\left[N_0 = 0\right]\mathbf{P}\left[N_1 \le x_1\right] \\
&= \lambda\mathbf{P}\left[N_0 = 0\right]\mathbf{P}\left[N_1 = x_1\right]
\end{aligned}
$$

where the last equality follows from (11.14) upon noting that $\mathbf{P}\left[N_0 = k\right], k = 0, 1, \ldots$, does not depend on p. Hence, (11.12) holds when $x_0 = 0$.

Suppose that (11.12) does hold for some $x_0 = x$, i.e.,

$$
\frac{d}{dp}f(x, x_1, p) = \lambda\mathbf{P}\left[N_0 = x\right](\sum_{j=0}^{x}\mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p) = j\right]\mathbf{P}\left[N_1 = x_1 - j\right]).
$$

For $x_0 = x + 1$, we have

$$
\begin{aligned}
f(x+1, x_1, p) &= f(x, x_1, p) + \mathbf{P}\left[N_0 = x+1, \sum_{i=1}^{N_0}\xi_i(p) + N_1 \le x_1\right] \\
&= f(x, x_1, p) + \mathbf{P}\left[N_0 = x+1\right]\mathbf{P}\left[\sum_{i=1}^{x+1}\xi_i(p) + N_1 \le x_1\right] \\
&= f(x, x_1, p) + g(x+1, x_1, p)
\end{aligned}
$$

where we have set

$$
\begin{aligned}
g(x+1, x_1, p) &= \mathbf{P}\left[N_0 = x+1\right]\mathbf{P}\left[\sum_{i=1}^{x+1}\xi_i(p) + N_1 \le x_1\right] \\
&= \mathbf{P}\left[N_0 = x+1\right]\sum_{j=0}^{x+1}\mathbf{P}\left[\sum_{i=1}^{x+1}\xi_i(p) = j\right]\mathbf{P}\left[N_1 \le x_1 - j\right].
\end{aligned}
$$

Then,

$$\frac{d}{dp}g(x+1,x_1,p) = \gamma_1(x+1,x_1,p) + \gamma_2(x+1,x_1,p)$$

where we have defined

$$\gamma_1(x+1,x_1,p) = \mathbf{P}\left[N_0 = x+1\right]\sum_{j=0}^{x+1}\mathbf{P}\left[N_1 \le x_1 - j\right]\frac{d}{dp}(\mathbf{P}\left[\sum_{i=1}^{x+1}\xi_i(p) = j\right]) \tag{11.16}$$

and

$$\gamma_2(x+1,x_1,p) = \mathbf{P}\left[N_0 = x+1\right]\sum_{j=0}^{x+1}\mathbf{P}\left[\sum_{i=1}^{x+1}\xi_i(p) = j\right]\frac{d}{dp}(\mathbf{P}\left[N_1 \le x_1 - j\right]). \tag{11.17}$$

Thus, we obtain

$$\begin{aligned}
& \frac{d}{dp}f(x+1,x_1,p) \\
= & \frac{d}{dp}f(x,x_1,p) + \frac{d}{dp}g(x+1,x_1,p) \\
= & \frac{d}{dp}f(x,x_1,p) + \gamma_1(x+1,x_1,p) + \gamma_2(x+1,x_1,p). \qquad (11.18)
\end{aligned}$$

In fact, as we show next, $\gamma_1(x+1,x_1,p) = -\frac{d}{dp}f(x,x_1,p)$ and $\gamma_2(x+1,x_1,p) = \frac{d}{dp}f(x+1,x_1,p)$: From (11.16) and (11.15), it follows that

$$\begin{aligned}
\gamma_1(x+1,x_1,p) &= \mathbf{P}\left[N_0 = x+1\right]\sum_{j=0}^{x+1}\mathbf{P}\left[N_1 \le x_1 - j\right]\cdot \\
&\quad (x+1)(\mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p) = j-1\right] - \mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p) = j\right]) \\
&= (x+1)\mathbf{P}\left[N_0 = x+1\right](\sum_{j=0}^{x+1}\mathbf{P}\left[N_1 \le x_1 - j\right]\cdot \\
&\quad \mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p) = j-1\right] - \sum_{j=0}^{x+1}\mathbf{P}\left[N_1 \le x_1 - j\right]\mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p) = j\right])
\end{aligned}$$

but since $\mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p) = -1\right] = \mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p) = x+1\right] = 0$, we obtain

$$\gamma_1(x+1,x_1,p) = (x+1)\mathbf{P}\left[N_0 = x+1\right](\sum_{j=1}^{x+1}\mathbf{P}\left[N_1 \le x_1 - j\right]\cdot$$

$$
\begin{aligned}
& \mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p)=j-1\right]-\sum_{j=0}^{x}\mathbf{P}\left[N_1\le x_1-j\right]\mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p)=j\right]) \\
= \;& (x+1)\mathbf{P}\left[N_0=x+1\right](\sum_{j=0}^{x}\mathbf{P}\left[N_1\le x_1-j-1\right]\cdot \\
& \mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p)=j\right]-\sum_{j=0}^{x}\mathbf{P}\left[N_1\le x_1-j\right]\mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p)=j\right]) \\
= \;& -(x+1)\mathbf{P}\left[N_0=x+1\right]\sum_{j=0}^{x}\mathbf{P}\left[\sum_{i=1}^{x}\xi_i(p)=j\right]\mathbf{P}\left[N_1=x_1-j\right] \\
= \;& -\frac{d}{dp}f(x,x_1,p) \qquad (11.19)
\end{aligned}
$$

where we have the last equality because $(x+1)\mathbf{P}\left[N_0=x+1\right]=\lambda\mathbf{P}\left[N_0=x\right]$. Moreover, from (11.14) and (11.17), we obtain

$$
\gamma_2(x+1,x_1,p)=\lambda\mathbf{P}\left[N_0=x+1\right]\sum_{j=0}^{x+1}\mathbf{P}\left[\sum_{i=1}^{x+1}\xi_i(p)=j\right]\mathbf{P}\left[N_1=x_1-j\right]. \quad (11.20)
$$

Finally, upon combining (11.19) and (11.20) via (11.18), we conclude to the expression

$$
\frac{d}{dp}f(x+1,x_1,p)=\lambda\mathbf{P}\left[N_0=x+1\right]\sum_{j=0}^{x+1}\mathbf{P}\left[\sum_{i=1}^{x+1}\xi_i(p)=j\right]\mathbf{P}\left[N_1=x_1-j\right]
$$

and the proof of the induction step is completed.

11.4 Simulation results

In this section, we verify the comparison in Theorem 11.1.1 (and in Theorem 10.2.2) by simulation experiments. To do so, we choose to compare the first and second moments of the buffer sizes of the $M|G|\infty$ input process (λ, S) and of its independent version ($M|G|\infty$ process $(\lambda\mathbf{E}\left[S\right],1)$) where the experiment description is specified in Section 6.2. Here, the session durations have two types of distributions, namely, geometric and discrete-Pareto. The details on the geometric rv $\mathcal{G}(\rho)$ and discrete-Pareto rv $\mathcal{P}(\alpha)$ can be found in Section 2.3. It is known that the $M|G|\infty$ process with discrete-Pareto session duration exhibits long-range dependence [38] and the steady state mean buffer size $\mathbf{E}[Q]$ is infinite [23]. For all simulations in this section, we fix the number of sample path at $N = 10,000$ and the multiplexer release rate at $c = 0.6$.

In the case $S =_{st} \mathcal{G}(\rho)$, we select $\rho = 0.8$ and $\lambda = 1$. Thus, $\lambda\mathbf{E}[S] = 5$ and its independent version is simply the $M|G|\infty$ process $(5, 1)$. The first and second moments of the corresponding buffer sizes are shown in Figure 15 and 16, respectively. The results clearly agree with the comparison in Theorem 11.1.1 since both the first and second moments of the buffer level of the $M|G|\infty$ process $(1, \mathcal{G}(0.8))$ are larger than those of its independent version.

Now, consider the case $S =_{st} \mathcal{P}(1.4)$ and $\lambda = 1.61$. Since $\alpha = 1.4$, from (2.9), we have $\mathbf{E}[S] = 3.10555$ and thus $\lambda\mathbf{E}[S] = 5$. Again, the independent version is an $M|G|\infty$ process $(5, 1)$. In Figures 17 and 18, we show the first and second moments of the buffer sizes of the $M|G|\infty$ process $(1.61, \mathcal{P}(1.4))$ and of its independent version. Both moments of the buffer sizes of the $M|G|\infty$ process $(1.61, \mathcal{P}(1.4))$ grow with t in agreement with the fact that $\mathbf{E}[Q] = \infty$. It is clear that the first and second moments of the buffer fed by the independent version is smaller than those of the $M|G|\infty$ process $(1.61, \mathcal{P}(1.4))$. Hence, we can use the independent version as a lower bound in the sense of the icx ordering. While both $M|G|\infty$ process $(1.61, \mathcal{P}(1.4))$ and $M|G|\infty$ process $(1, \mathcal{G}(0.8))$ have the same mean traffic rate at 5, the $M|G|\infty$ process $(1.61, \mathcal{P}(1.4))$ yields higher mean buffer levels than the $M|G|\infty$ process $(1, \mathcal{G}(0.8))$.

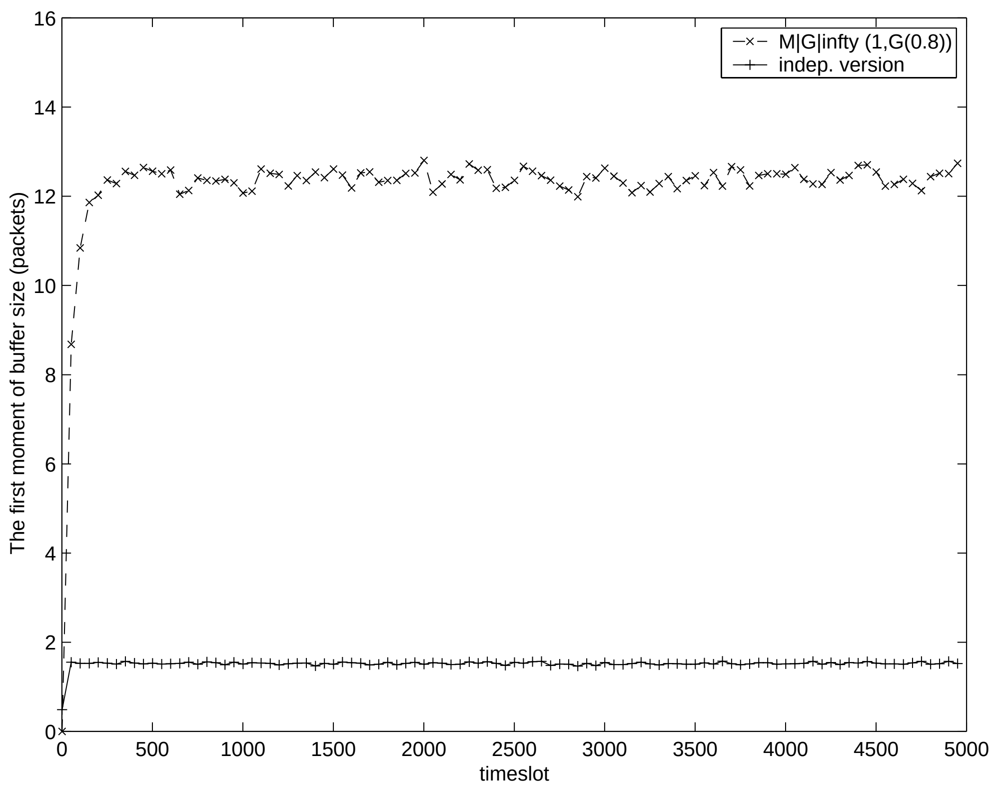

Figure 15: The first moments of the buffer sizes of the $M|G|\infty$ process $(1, \mathcal{G}(0.8))$ and of its independent version $M|G|\infty$ process $(5, 1)$

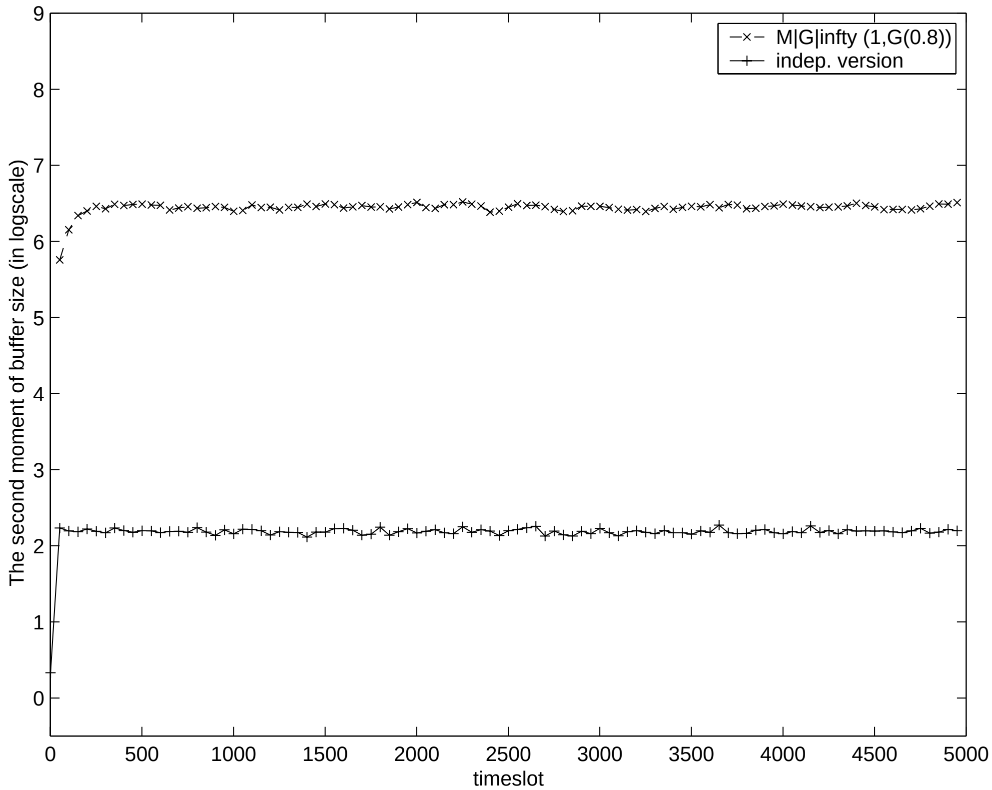

Figure 16: The second moments of the buffer sizes (in logscale) of the $M|G|\infty$ process $(1, \mathcal{G}(0.8))$ and of its independent version $M|G|\infty$ process $(5, 1)$

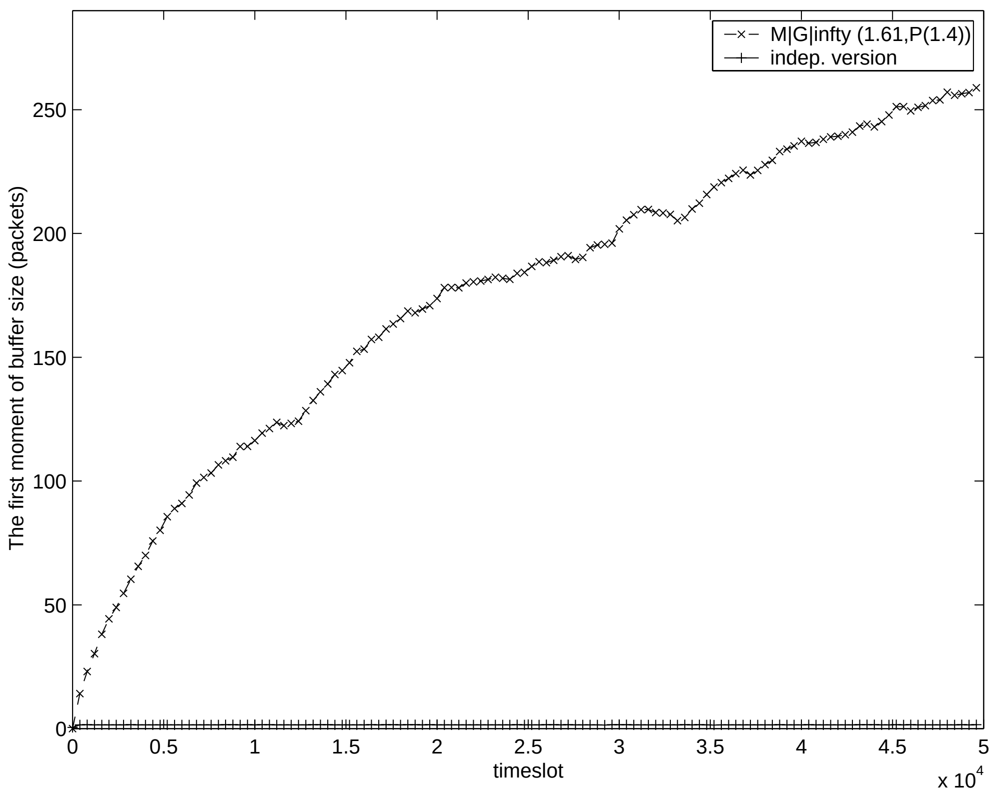

Figure 17: The first moments of the buffer sizes of the $M|G|\infty$ process $(1.61, \mathcal{P}(1.4))$ and of its independent version $M|G|\infty$ process $(5, 1)$

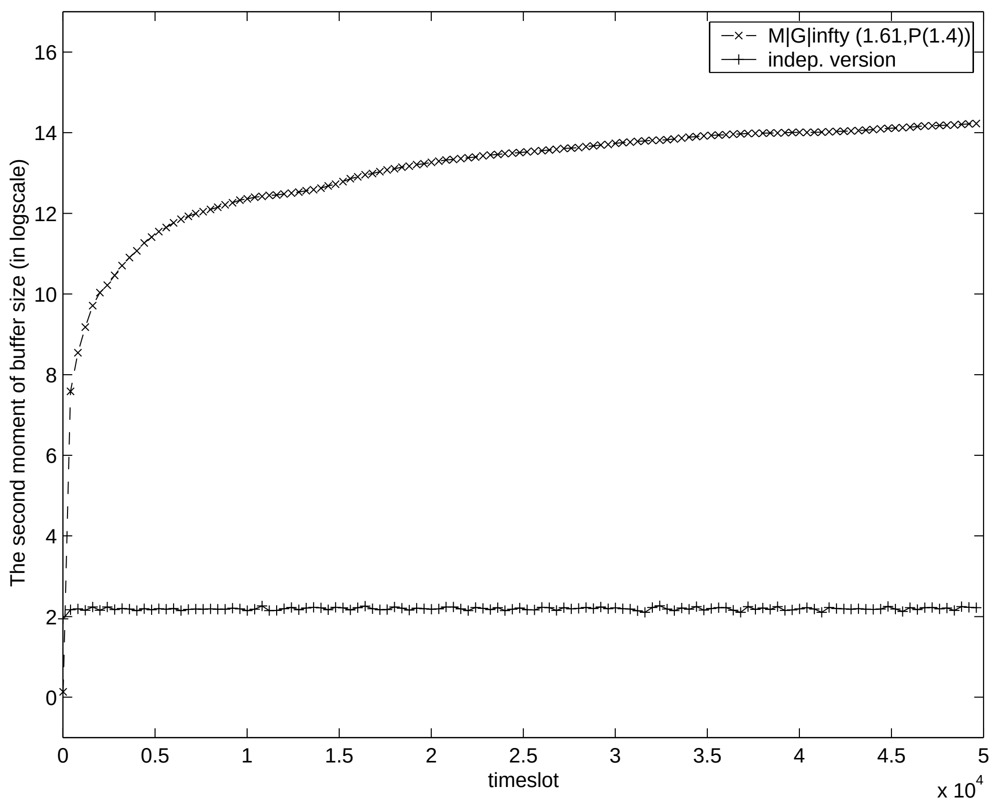

Figure 18: The second moments of the buffer sizes (in logscale) of the $M|G|\infty$ process $(1.61, \mathcal{P}(1.4))$ and of its independent version $M|G|\infty$ process $(5, 1)$

Appendix A

Self-similarity

In this appendix, we briefly discuss various definitions and properties of self-similar processes. More information about self-similarity can be found in the monograph [44].

Let $\{X_t, t = 0, 1, \ldots\}$ be any sequence of $\mathbb{R}$-valued rvs. For each $m = 1, 2, \ldots$, define the process $\{X_t^{(m)}, t = 0, 1, \ldots\}$ by

$$X_t^{(m)} \equiv \frac{1}{m} \sum_{k=0}^{m-1} X_{mt+k}, \quad t = 0, 1, \ldots, \tag{A.1}$$

to be the *m-averaged* process associated with $\{X_t, t = 0, 1, \ldots\}$. Also, let the process $\{\breve{X}_t^{(m)}, t = 0, 1, \ldots\}$ defined by

$$\breve{X}_t^{(m)} \equiv m^{1-H} X_t^{(m)} = \frac{1}{m^H} \sum_{k=0}^{m-1} X_{mt+k}, \quad t = 0, 1, \ldots, \tag{A.2}$$

be the *m-normalized* process where $0 < H < 1$ is the index of normalization.

Definition A.1 *A strictly stationary process $\{X_t, t = 0, 1, \ldots\}$ is said to be strictly self-similar with Hurst parameter H ($0 < H < 1$), if for each $m = 1, 2, \ldots$, we have*

$$\{\breve{X}_t^{(m)}, t = 0, 1, \ldots\} =_{st} \{X_t, t = 0, 1, \ldots\} \tag{A.3}$$

where $\{\breve{X}_t^{(m)}, t = 0, 1, \ldots\}$ is the m-normalized process (A.2) with index of normalization H.

From the definition, the self-similar process $\{X_t, t = 0, 1, \ldots\}$ has the same probabilistic structure as its scaled and normalized version $\{\breve{X}_t^{(m)}, t = 0, 1, \ldots\}$. If we assume that the moments $\mathbf{E}[X_t]$ exist for all $t = 0, 1, \ldots$ and $\{X_t, t = 0, 1, \ldots\}$ is strictly self-similar process, it is clear from (A.2) and (A.3) that $\mathbf{E}[X_t] = 0$ necessarily for all $t = 0, 1, \ldots$. This strictly self-similarity property is too restrictive for processes $\{X_t, t = 0, 1, \ldots\}$ which are positive and non-degenerate since neither the process itself nor the centered process $\{X_t - \mathbf{E}[X_t], t = 0, 1, \ldots\}$ can be strictly self-similar. The next definition introduces the broader class of *exactly second-order self-similar* processes.

Definition A.2 *A wide-sense stationary process $\{X_t, t = 0, 1, \ldots\}$ is said to be exactly second-order self-similar with Hurst parameter H $(0 < H < 1)$, if for each $m = 1, 2, \ldots$ we have*

$$var[\breve{X}_t^{(m)}] = var[X_t], \quad t = 0, 1, \ldots, \tag{A.4}$$

where $\{\breve{X}_t^{(m)}, t = 0, 1, \ldots\}$ is the m-normalized process (A.2) with index of normalization H.

Since $\{X_t, t = 0, 1, \ldots\}$ is wide-sense stationary, it is easy to see that both $\{X_t^{(m)}, t = 0, 1, \ldots\}$ and $\{\breve{X}_t^{(m)}, t = 0, 1, \ldots\}$ are also wide-sense stationary processes with correlation functions

$$\Gamma^{(m)}(h) \equiv cov[X_t^{(m)}, X_{t+h}^{(m)}] \quad \text{and} \quad \gamma^{(m)}(h) \equiv \frac{\Gamma^{(m)}(h)}{\Gamma^{(m)}(0)}, \quad h = 0, 1, \ldots,$$

and

$$\breve{\Gamma}^{(m)}(h) \equiv cov[\breve{X}_t^{(m)}, \breve{X}_{t+h}^{(m)}] \quad \text{and} \quad \breve{\gamma}^{(m)}(h) \equiv \frac{\breve{\Gamma}^{(m)}(h)}{\breve{\Gamma}^{(m)}(0)}, \quad h = 0, 1, \ldots,$$

respectively. Moreover, the following conditions (i)-(iii) below are equivalent [53], where

(i) $\breve{\Gamma}^{(m)}(0) = \Gamma(0), \quad m = 1, 2, \ldots \quad$ (Eq. (A.4));

(ii) $\Gamma^{(m)}(0) = \Gamma(0)m^{-2(1-H)}, \quad m = 1, 2, \ldots;$

(iii) For fixed Hurst parameter H;

$$\Gamma(h) = \Gamma(0)\gamma_H(h), \quad h = 0, 1, \ldots \tag{A.5}$$

where the mapping $\gamma_H : \mathbb{N} \to \mathbb{R}_+$ is given by

$$\gamma_H(h) \equiv \frac{1}{2}(|h+1|^{2H} - 2|h|^{2H} + |h-1|^{2H}), \quad h = 0, , \ldots. \tag{A.6}$$

With $0.5 < H < 1$, the mapping γ_H is strictly decreasing and integer-convex with $\gamma_H(0) = 1$, and behaves asymptotically as

$$\gamma_H(h) \sim H(2H-1)h^{2H-2} \quad (h \to \infty)$$

so that under (A.6) $\{X_t, t = 0, 1, \ldots\}$ exhibits long-range dependence.

Exact second-order self-similarity is sometimes too restrictive for some applications. The last definition relaxes the notion to a much larger class of processes.

Definition A.3 *A process $\{X_t, t = 0, 1, \ldots\}$ is said to be asymptotically second-order self-similar if*

$$\lim_{m\to\infty} \gamma^{(m)}(h) = \gamma_H(h), \quad h = 1, 2 \ldots, \tag{A.7}$$

for some $0 < H < 1$, in which case we still refer to H as the Hurst parameter of the process.

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