TECHNICAL RESEARCH REPORT

On a Random Sum Formula for the Busy Period of the M|G|
Infinity Queue with Applications

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CSHCN TR 2001-4
(ISR TR 2001-9)

On a random sum formula for the busy period of the $M|G|\infty$ queue with applications

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Abstract

A random sum formula is derived for the forward recurrence time associated with the busy period length of the $M|G|\infty$ queue. This result is then used to (i) provide a necessary and sufficient condition for the subexponentiality of this forward recurrence time, and (ii) establish a stochastic comparison in the convex increasing (variability) ordering between the busy periods in two $M|G|\infty$ queues with service times comparable in the convex increasing ordering.

Key words: Random sums, $M|G|\infty$ queues, busy period, subexponentiality, stochastic orderings
AMS: 60K25, 60E15

1 Introduction

Random sums, and geometric random sums in particular, are a common occurence in applied probability models [4, 6]. For instance, it is well known

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that the stationary waiting time of a stable $M|GI|1$ queue with Poisson arrival rate λ and generic service time σ can be represented (in distribution) as a geometric sum of i.i.d. rvs distributed like the forward recurrence time $\sigma^\star$ associated with σ [7]. A similar representation holds for the stationary waiting time of a stable $GI|GI|1$ queue in terms of ladder height rvs [10]. Such random sum representations have proved useful for establishing various properties of interest [4, 6].

Perhaps less well known is the fact that similar geometric sums can also be found in the context of $M|GI|\infty$ queues. Indeed, consider a standard $M|G|\infty$ queue with arrival rate λ and generic service time σ, and let B denote its generic busy period length. Using some classical results on the Laplace transform of B, we show that the forward recurrence time $B^\star$ associated with the busy period of an $M|GI|\infty$ can be represented as a geometric sum of i.i.d. rvs whose common distribution is derived from that of the forward recurrence time $\sigma^\star$. This random sum representation is presented in Section 2, and its proof is given in Section 5.

We give two applications for this representation result: In Section 3, we show that the subexponentiality of $\sigma^\star$ is equivalent to that of $B^\star$, with a simple relation between the tail of their distributions. This result was originally derived by Boxma [1] for regularly varying σ, and further generalized in the form given here (but with a different proof) by Jelenkovic and Lazar [5] to the case where $\sigma^\star$ is subexponential. In Section 4, we investigate the monotonicity properties of $M|GI|\infty$ queues under the (increasing) convex ordering. Using the random sum representation we show that the busy period distribution is monotone in the increasing convex ordering when the service time distribution is increased in the increasing convex ordering. To the best of the author's knowledge this result is new [14, p. 147].

A word on the notation used in this paper: For any integrable $\mathbb{R}_+$–valued rv X, the forward recurrence time $X^\star$ is defined as the rv with integrated tail distribution given by

$$\mathbf{P}\left[X^\star > x\right] := \frac{1}{\mathbf{E}\left[X\right]} \int_x^\infty \mathbf{P}\left[X > t\right] dt, \quad x \geq 0. \tag{1}$$

We shall find it useful to use the equivalent representation

$$\mathbf{P}\left[X^\star > x\right] := \frac{\mathbf{E}\left[(X - x)^+\right]}{\mathbf{E}\left[X\right]}, \quad x \geq 0 \tag{2}$$

(where we write $x^+ = \max(x, 0)$ for any scalar x). For mappings $f, g : \mathbb{R}_+ \to \mathbb{R}$, the relation $f(x) \sim g(x)$ is understood as $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, the qualifier $(x \to \infty)$ being omitted for the sake of notational simplicity.

2 A random sum in the $M|G|\infty$ queue

Consider a standard $M|G|\infty$ queue with arrival rate λ and generic service time σ; we refer to this queueing system as the $M|G|\infty$ queue (λ, σ). Let B denote its generic busy period, i.e., the length of time that elapses between the arrival of a customer finding an empty system and the departure of the first customer thereafter which leaves the system empty.

In Section 5 we show that the forward recurrence time $B^\star$ associated with B admits a random sum representation: In order to present this result, let ν denote the $\mathbb{N}$–valued rv which is geometrically distributed according to

$$\mathbf{P}[\nu = \ell] = (1 - K)K^{\ell-1}, \quad \ell = 1, 2, \ldots \tag{3}$$

with

$$\rho := \lambda \mathbf{E}[\sigma] \quad \text{and} \quad K := 1 - e^{-\rho}. \tag{4}$$

Next, we introduce the $\mathbb{R}_+$-valued rv U distributed according to

$$\mathbf{P}[U \leq x] := \frac{1}{K}\left(1 - e^{-\rho \mathbf{P}[\sigma^\star \leq x]}\right), \quad x \geq 0 \tag{5}$$

where $\sigma^\star$ is the forward recurrence time associated with the generic service time σ. We are now ready to formulate the key observation of the paper:

Theorem 1 *Consider a sequence of $\mathbb{R}_+$–valued i.i.d. rvs $\{U_n,\ n = 1, 2, \ldots\}$ distributed according to (5), and an $\mathbb{N}$–valued rv ν distributed according to (12). Then, with the sequence $\{U_n,\ n = 1, 2, \ldots\}$ taken to be independent of the rv ν, it holds that*

$$B^\star =_{st} U_1 + \ldots + U_\nu. \tag{6}$$

where $=_{st}$ denotes equality in distribution.

In analogy with standard results for the $GI|GI|1$ queue [10], it is natural to wonder whether the forward recurrence time associated with the busy period in the $GI|GI|\infty$ queue also admits a representation as a geometric random sum of i.i.d. rvs whose distribution is now derived from that of ladder height rvs. To the best of the author's knowledge no results along these lines are known.

In the process of establishing Theorem 1 in Section 5 we shall show that

$$\mathbf{E}\left[B\right] = \frac{K}{\lambda(1-K)}. \tag{7}$$

We also note from (4) and (5) that

$$\mathbf{P}\left[U > x\right] = \frac{e^{-\rho}}{K}\left(e^{\rho \mathbf{P}\left[\sigma^\star > x\right]} - 1\right), \quad x \geq 0. \tag{8}$$

Moreover, using (2) we see that (8) can be rewritten as

$$\mathbf{P}\left[U > x\right] = \frac{e^{-\rho}}{K}\left(e^{\lambda \mathbf{E}\left[(\sigma - x)^+\right]} - 1\right), \quad x \geq 0. \tag{9}$$

3 Subexponentiality in the $M|G|\infty$ queue

We begin with some standard definitions and facts concerning subexponential rvs [2, 3]: The $\mathbb{R}_+$-valued rv X is said to have a *subexponential tail*, denoted $X \in \mathcal{S}$, if

$$\mathbf{P}\left[X_1 + \ldots + X_n > x\right] \sim n\mathbf{P}\left[X > x\right], \quad n = 1, 2, \ldots \tag{10}$$

where $\{X_n,\ n = 1, 2, \ldots\}$ denotes a sequence of i.i.d. rvs, each distributed like X. In fact, (10) holds for all $n = 1, 2, \ldots$ if and only if it holds for some $n \geq 2$. Under appropriate conditions, the equivalences (10) can be extended to random sums [3, Thm. 1.3.9, p. 45] (and [3, Thm. A3.20, p. 580]).

Proposition 1 *Let the* $\mathbb{N}$*–valued rv* N *be independent of the sequence of i.i.d. rvs* $\{X, X_n,\ n = 1, 2, \ldots\}$. *If* $X \in \mathcal{S}$, *then* $X_1 + \ldots + X_N \in \mathcal{S}$ *with*

$$\mathbf{P}\left[X_1 + \ldots + X_N > x\right] \sim \mathbf{E}\left[N\right]\mathbf{P}\left[X > x\right] \tag{11}$$

provided $\mathbf{E}\left[z^N\right] < \infty$ *for some* $z > 1$.

Of particular interest for the discusssion here is the case when N is distributed according to the geometric distribution

$$\mathbf{P}[N = \ell] = (1-p)p^{\ell-1}, \quad \ell = 1, 2, \ldots \tag{12}$$

for some $0 < p < 1$. Standard calculations yield $\mathbf{E}[N] = (1-p)^{-1}$, and

$$\mathbf{E}\left[z^N\right] = \sum_{\ell=1}^{\infty}(1-p)p^{\ell-1}z^{\ell} = \frac{(1-p)z}{1-pz}, \quad |z| < p^{-1} \tag{13}$$

with $p^{-1} > 1$. Thus, Proposition 1 applies, in fact, can be strenghtened as follows [3, Cor. A3.21, p. 581]:

Proposition 2 *Let the $\mathbb{N}$–valued rv N be independent of the sequence of i.i.d. rvs $\{X, X_n, \ n = 1, 2, \ldots\}$, and assume N to be distributed according to (12). Then, $X \in \mathcal{S}$ if and only if $X_1 + \ldots + X_N \in \mathcal{S}$, in which case*

$$\mathbf{P}[X_1 + \ldots + X_N > x] \sim (1-p)^{-1}\mathbf{P}[X > x] \tag{14}$$

The main result of this section is the following

Proposition 3 *Consider a standard $M|G|\infty$ queue (λ, σ) with generic busy period rv B. We have $B^\star \in \mathcal{S}$ if and only if $\sigma^\star \in \mathcal{S}$, in which case*

$$\mathbf{P}[B^\star > x] \sim \frac{\rho}{1-e^{-\rho}}\mathbf{P}[\sigma^\star > x]. \tag{15}$$

Proof. Combining Theorem 1 with Proposition 2, we already get that $B^\star \in \mathcal{S}$ if and only $U \in \mathcal{S}$, in which case

$$\mathbf{P}[B^\star > x] \sim (1-K)^{-1}\mathbf{P}[U > x]. \tag{16}$$

Next, with (8) in mind, we observe that $\lim_{x\to\infty}\mathbf{P}[\sigma^\star > x] = 0$, so that

$$e^{\rho\mathbf{P}[\sigma^\star > x]} = 1 + \rho\mathbf{P}[\sigma^\star > x] + o(\mathbf{P}[\sigma^\star > x]).$$

Hence, from (8) we get

$$\mathbf{P}[U > x] = \frac{e^{-\rho}}{K}\left(\rho\mathbf{P}[\sigma^\star > x] + o(\mathbf{P}[\sigma^\star > x])\right),$$

and the conclusion

$$\mathbf{P}[U > x] \sim \frac{\rho e^{-\rho}}{K} \mathbf{P}[\sigma^\star > x] \tag{17}$$

follows. Therefore, since $\mathcal{S}$ is closed under tail-equivalence [3, Lemma A3.15, p. 572], we get $U \in \mathcal{S}$ if and only if $\sigma^\star \in \mathcal{S}$, and we complete the proof of (15) by injecting this last fact with (17) into the equivalence (16). ■

4 Orderings in the $M|G|\infty$ queue

For $\mathbb{R}$–valued random variables X and Y, we say that X is smaller than Y in the strong stochastic (resp. convex, increasing convex) ordering if

$$\mathbf{E}[\varphi(X)] \leq \mathbf{E}[\varphi(Y)] \tag{18}$$

for all mappings $\varphi : \mathbb{R} \to \mathbb{R}$ which are monotone increasing (resp. convex, increasing and convex) provided the expectations in (18) exist. In that case we write $X \leq_{st} Y$ (resp. $X \leq_{cx} Y$, $X \leq_{icx} Y$). Additional material on these orderings can be found in the monographs [11, 13, 14]. The following result is well known [14, Prop. 2.2.5, p. 45].

Proposition 4 *Let the $\mathbb{N}$–valued rv N be independent of the sequences of i.i.d. rvs $\{X, X_n,\ n = 1, 2, \ldots\}$ and $\{Y, Y_n,\ n = 1, 2, \ldots\}$. If $X \leq_{st} Y$, then it holds that*

$$X_1 + \ldots + X_N \leq_{st} Y_1 + \ldots + Y_N \tag{19}$$

Results similar to (19) hold *mutatis mutandis* under the weaker assumptions $X \leq_{cx} Y$ and $X \leq_{icx} Y$. The main result of this section is the following stochastic comparison.

Proposition 5 *Consider two $M|G|\infty$ queues (λ, σ_1) and (λ, σ_2) with generic busy period rvs B_1 and B_2, respectively. If $\sigma_1 \leq_{cx} \sigma_2$, then $B_1 \leq_{cx} B_2$.*

Throughout, for each $i = 1, 2$, quantities associated with the $M|G|\infty$ queues (λ, σ_i) are subscripted by i.

Proof. Under the condition $\sigma_1 \leq_{cx} \sigma_2$, $\mathbf{E}[\sigma_1] = \mathbf{E}[\sigma_2]$, whence $\rho_1 = \rho_2$

and $K_1 = K_2$, and the inequalities $\mathbf{E}[(\sigma_1 - x)^+] \leq \mathbf{E}[(\sigma_2 - x)^+]$ hold for all $x \geq 0$. Invoking (9), we immediately conclude that $U_1 \leq_{st} U_2$. Next, applying Proposition 4 to the random sum representation (6), we see that $B_1^\star \leq_{st} B_2^\star$, namely

$$\frac{1}{\mathbf{E}[B_1]} \int_x^\infty \mathbf{P}[B_1 > t]\, dt \leq \frac{1}{\mathbf{E}[B_2]} \int_x^\infty \mathbf{P}[B_2 > t]\, dt, \quad x \geq 0. \tag{20}$$

This is equivalent to

$$\frac{\mathbf{E}[(B_1 - x)^+]}{\mathbf{E}[B_1]} \leq \frac{\mathbf{E}[(B_2 - x)^+]}{\mathbf{E}[B_2]}, \quad x \geq 0. \tag{21}$$

Finally, observe from (7) and from the observations made above that $\mathbf{E}[B_1] = \mathbf{E}[B_2]$, so that $\mathbf{E}[(B_1 - x)^+] \leq \mathbf{E}[(B_2 - x)^+]$ for all $x \geq 0$, and the desired conclusion readily follows [14, Thm. 1.3.1, p. 9]. ■

The literature contains few stochastic comparison results for infinite server queues; they deal mostly with the number of customers, e.g., [14, Prop. 7.1.1, p. 127], [14, Table 7.2, p. 147]. However, a simple coupling argument readily leads to the following comparison:

Proposition 6 *Consider two $M|G|\infty$ queues (λ, σ_1) and (λ, σ_2) with generic busy period rv B_1 and B_2, respectively. If $\sigma_1 \leq_{st} \sigma_2$, then $B_1 \leq_{st} B_2$.*

Finally, we combine Propositions 5 and 6 with the characterization of the convex increasing ordering provided in [9]:

Proposition 7 *Consider two $M|G|\infty$ queues (λ, σ_1) and (λ, σ_2) with generic busy period rv B_1 and B_2, respectively. If $\sigma_1 \leq_{icx} \sigma_2$, then $B_1 \leq_{icx} B_2$.*

5 A Proof of Theorem 1

Consider the process of particle counting as described in Chapter 6 of the monograph by Takács [15, p. 205]. Type II counters are equivalent to infinite server queues if particles are interpreted as customers. So-called "registered" particles [15, p. 205] are those particles which arrive at an instant when there is no other particle present; in the infinite server queue context, such a registered customer is a customer that initiates a busy period. Let the rv R denote the length of time that elpases between the arrival epochs of two consecutive registered particles, or equivalently, in the infinite server queue, the time duration between the start of two consecutive busy periods.

Theorem 1 in [15, p. 210] provides a closed form expression for the Laplace–Stieltjes transform of the rv R when customers arriva according to a Poisson process: For the $M|G|\infty$ queue (λ, σ), it holds that

$$\mathbf{E}\left[e^{-\theta R}\right] = 1 - \frac{1}{\lambda+\theta} \cdot \frac{1}{T(\theta)}, \quad \theta \geq 0 \tag{22}$$

with

$$T(\theta) := \int_0^\infty \exp\left(-\theta t - \lambda \int_0^t \mathbf{P}\left[\sigma > x\right] dx\right) dt. \tag{23}$$

Lemma 1 *With the notation (4)–(5), it holds that*

$$T(\theta) = \frac{1}{\theta}\left(1 - K\mathbf{E}\left[e^{-\theta U}\right]\right), \quad \theta > 0. \tag{24}$$

Proof. Fix $\theta \geq 0$. From (5) we note that

$$e^{-\lambda \mathbf{E}[\sigma]\mathbf{P}[\sigma^\star \leq t]} = 1 - K\mathbf{P}\left[U \leq t\right], \quad t \geq 0. \tag{25}$$

Making use of this fact in the definition (23), we find

$$\begin{aligned} T(\theta) &= \int_0^\infty e^{-\theta t} e^{-\lambda \mathbf{E}[\sigma]\mathbf{P}[\sigma^\star \leq t]} dt \\ &= \int_0^\infty e^{-\theta t}\left(1 - K\mathbf{P}\left[U \leq t\right]\right) dt \\ &= \frac{1}{\theta} - K \int_0^\infty e^{-\theta t}\mathbf{P}\left[U \leq t\right] dt \\ &= \frac{1}{\theta} - K\left(\left[\frac{e^{-\theta t}}{-\theta}\mathbf{P}\left[U \leq t\right]\right]_0^\infty - \int_0^\infty \frac{e^{-\theta t}}{-\theta}\frac{d}{dt}\mathbf{P}\left[U \leq t\right] dt\right) \end{aligned}$$

and the desired conclusion (24) follows from the fact $\mathbf{P}\left[U \leq 0\right] = 0$. ■

Fix $\theta > 0$. Thus, using the expression from Lemma 1 in (22), we conclude that

$$\mathbf{E}\left[e^{-\theta R}\right] = 1 - \frac{1}{\lambda+\theta} \cdot \frac{\theta}{1 - K\mathbf{E}\left[e^{-\theta U}\right]}. \tag{26}$$

However, for the $M|G|\infty$ queue (λ, σ), it is plain that $R =_{st} B + I$ with (i) B and I independent; (ii) the rv B is distributed according to a busy cycle length; and (iii) the rv I is distributed according to an idle period, thus is an exponential rv with parameter λ so that

$$\mathbf{E}\left[e^{-\theta I}\right] = \frac{\lambda}{\lambda+\theta}. \tag{27}$$

Consequently,

$$\mathbf{E}\left[e^{-\theta B}\right] = \frac{\mathbf{E}\left[e^{-\theta R}\right]}{\mathbf{E}\left[e^{-\theta I}\right]} = \frac{\lambda+\theta}{\lambda} - \frac{\theta}{\lambda(1 - K\mathbf{E}\left[e^{-\theta U}\right])} \tag{28}$$

Hence,

$$\begin{aligned}
\frac{1 - \mathbf{E}\left[e^{-\theta B}\right]}{\theta} &= \frac{1}{\lambda} \cdot \frac{K\mathbf{E}\left[e^{-\theta U}\right]}{1 - K\mathbf{E}\left[e^{-\theta U}\right]} \\
&= \frac{1}{\lambda} \cdot \sum_{\ell=1}^{\infty} (K\mathbf{E}\left[e^{-\theta U}\right])^{\ell} \\
&= \frac{K}{\lambda(1-K)} \cdot \sum_{\ell=1}^{\infty} (1-K)K^{\ell-1}(\mathbf{E}\left[e^{-\theta U}\right])^{\ell} \\
&= \frac{K}{\lambda(1-K)} \mathbf{E}\left[e^{-\theta(U_1+\ldots+U_\nu)}\right] \qquad (29)
\end{aligned}$$

where ν is the $\mathbb{N}$–valued rv with geometric distribution (4). Letting θ go to zero in (29) we conclude that the rv B is integrable with mean given by (7). This allows a rewriting of (29) as

$$\frac{1 - \mathbf{E}\left[e^{-\theta B}\right]}{\theta \mathbf{E}\left[B\right]} = \mathbf{E}\left[e^{-\theta(U_1+\ldots+U_\nu)}\right], \tag{30}$$

whence

$$\mathbf{E}\left[e^{-\theta B^\star}\right] = \mathbf{E}\left[e^{-\theta(U_1+\ldots+U_\nu)}\right] \tag{31}$$

This completes the proof of Theorem 1.

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