TECHNICAL RESEARCH REPORT

On Participation Factors for Linear Systems

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T.R. 99-41

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Revised Manuscript: June 1999

Abstract

Participation factors are nondimensional scalars that measure the interaction between the modes and the state variables of a linear system. Since their introduction by Verghese, Pérez-Arriaga and Schweppe, participation factors have been used for analysis, order reduction and controller design in a variety of fields. In this paper, participation factors are revisited, resulting in new definitions. The aim of these definitions is to achieve a conceptual framework that doesn't hinge on any particular choice of initial condition. The initial condition is modeled as an uncertain quantity, which can be viewed either in a set-valued or a probabilistic setting. If the initial condition uncertainty obeys a symmetry condition, the new definitions are found to reduce to the original definition of participation factors.

Keywords: participation factors, linear systems, modal analysis, stability, dynamics, probability

1 Introduction

Since its introduction by Verghese, Pérez-Arriaga and Schweppe [9],[5],[10], Selective Modal Analysis (SMA) has become a popular tool for system analysis, order reduction and actuator placement. In particular, this tool is extensively used in the electric power systems area [4].

Participation factors, a key element of SMA, provide a mechanism for assessing the level of interaction between system modes and system state variables.

In this paper, participation factors are revisited, resulting in new definitions. The purpose of these definitions is to achieve a conceptual framework that doesn't hinge on any particular choice of initial condition. The initial condition is modeled as an uncertain quantity, which can be viewed either in a set-valued or a probabilistic setting. If the initial condition uncertainty obeys a symmetry condition, the new definitions are found to reduce to the original definition of [9],[5],[10]. Since the initial condition is viewed as uncertain, the definitions model its effect on the level of mode-state interaction in an average sense. (For simplicity, the term "state" is used here interchangeably with "state variable.") This work provides a framework for a deeper appreciation of participation factors, as well as an opening to further useful generalizations of the concept in various directions. For instance, generalizations to other system types can be pursued. Also, definitions along the same lines could facilitate analytical treatment of the relation between control inputs and states/mode interaction, which in turn would have implications for actuator and sensor placement.

Fowllowing Verghese, Pérez-Arriaga and Schweppe [9],[5],[10], participation factors are considered in two basic senses. In the first sense, a participation factor measures the relative contribution of a mode to a state. In the second, a participation factor measures the relative contribution of a state to a mode. It isn't clear at the outset that these two senses should lead to identical formulas for participation factors. However, the precise definitions in [9],[5],[10] for these two senses of participation factors did indeed result in identical mathematical expressions. The same conclusion is found to apply in the present paper, under assumptions valid for a large class of problems.

It should be noted that there have been other interpretations of participation factors since the original work of [9],[5],[10]. For example, participation factors are often viewed as sensitivities of eigenvalues to changes in the diagonal entries of the state dynamics matrix (see, e.g., [8]). Interpretations in terms of modal energies are also common (see, e.g., [1]). Still another interpretation relates to eigenvalue mobility under state feedback [6].

The remainder of the paper is organized as follows. In Section 2, the original definition of participation factors [9],[5],[10] is recalled, and motivation for the work of this paper is given. In Section 3, the new definitions of this paper that address participation of *modes in states* are given. The relationship with the original definition is shown. Section 4 contains the new definitions that address participation of *states in modes* along with their relationship to the original definition in [9],[5],[10]. Concluding remarks are given in Section 5.

2 Background and Motivation

Participation factors were introduced by Verghese, Pérez-Arriaga and Schweppe [9],[5],[10] as a means for ranking the relative interactions between system modes and system states. The concept is one element of the Selective Modal Analysis (SMA) approach introduced by these authors, and its first applications were in the field of electric power systems. This section recalls the original definition of participation factors.

Consider a general continuous-time linear time-invariant system

$$\dot{x}(t) \;=\; Ax(t), \tag{1}$$

where $x \in \mathbb{R}^n$, and A is a real $n \times n$ matrix. Suppose for simplicity that A has a set of n distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Typically the evolution of each state variable is influenced by all the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. However, it is often desirable to quantify the participation of a particular mode (i.e., eigenmode) in a state variable. If the state vector is composed of physically meaningful variables, such a quantification leads to conclusions regarding the influence of system modes on physical components.

It is at first tempting to base the association of modes with state variables on the magnitudes of the entries in the right eigenvector associated with a mode. Let $(r^1, r^2, \ldots, r^n)$ be right eigenvectors of the matrix A associated with the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, respectively. Using this criterion, one would say that the mode associated with λ_i is significantly involved in the state x_k if r^i_k is large. As pointed out in [9],[5],[10], this approach has two main disadvantages. The first is that it requires a complete spectral analysis of the system, and is thus computationally expensive. The second, which is the more serious flaw, is that the numerical values of the entries of the eigenvectors depend on the choice of units for the corresponding state variables. This renders the criterion unreliable in providing a measure of the contribution of modes to state variables. This is true even if the variables are similar physically and are measured in the same units.

The foregoing was the motivation for the approach taken in SMA to quantify the relative contributions of modes to state variables. In SMA the entries of both the right and left eigenvectors are utilized to calculate "participation factors" that measure the level of participation of modes in states and the level of participation of states in modes. The participation factors defined in SMA are dimensionless quantities that are independent of the units in which state variables are measured.

Next, a brief summary of the original definition of participation factors in the sense of participation of *modes in states* is given. As noted earlier, the original definition also encompasses participation of *states in modes.*

Let $(l^1, l^2, \ldots,\ l^n)$ denote the left (row) eigenvectors of the matrix A associated with the eigenvalues $(\lambda_1, \lambda_2, \ldots,\ \lambda_n)$, respectively. The right and left eigenvectors are taken to satisfy the normalization [3, p.154]

$$l^i r^j = \delta_{ij}, \quad 1 \le i, j \le n, \tag{2}$$

where δ_{ij} is the Kronecker delta symbol:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In the remainder of the paper, the i-th mode refers to the mode associated with λ_i, indeed specifically to the $e^{\lambda_i t}$ terms in the state trajectory (and not, for simplicity, to the conjugates of these terms in case of a complex eigenvalue; see Remark 2 below).

In [9],[5],[10], the *participation factor* of the i-th mode in the k-th state x_k is *defined* to be the complex number

$$p_{ki} := l^i_k r^i_k. \tag{3}$$

The motivation for this definition given in [9],[5],[10] is as follows.

The solution of the dynamic system equation (1) satisfying the initial condition $x(0) = x^0$ is

$$x(t) \quad = \quad e^{At} x^0. \tag{4}$$

Since the eigenvalues of A are assumed distinct, A is similar to a diagonal matrix. Using this, Eq. (4) gives

$$x(t) = \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r^i. \tag{5}$$

This formula is alternatively (and conveniently) viewed as the superposition of solutions corresponding to initial conditions along the eigenvectors, where each such initial condition is the projection of x^0 along the eigenvector.

Now suppose the initial condition x^0 is e^k, the unit vector along the k-th coordinate axis. Then the evolution of the k-th state becomes

$$\begin{aligned} x_k(t) &= \sum_{i=1}^{n} (l_k^i r_k^i) e^{\lambda_i t} \\ &= \sum_{i=1}^{n} p_{ki} e^{\lambda_i t}. \end{aligned} \tag{6}$$

Eq. (6) indicates that p_{ki} can be viewed as the relative participation of the i-th mode in the k-th state at $t = 0$.

Remark 1. The quantities defined in Eq. (3) are also used in [9],[5],[10] to measure relative participation of *states in modes*. The motivation for this interpretation given in [5] is somewhat similar to the above and need not be reproduced here. ■

As noted above, any notion of participation factors is useful only if it results in quantities that do not depend on the units in which state variables are measured. In this remark, we record for later use the effect of changes in units on right and left eigenvectors. A general change in units is represented by a transformation $\tilde{x}_j = \alpha_j x_j$, $j = 1, \ldots, n$, where the α_j are positive constants. Let r^i and l^i be right and left eigenvectors, respectively, corresponding to eigenvalue λ_i, in the original units. It is straightforward to show that these eigenvectors become, in the new units, $\tilde{r}^i$ and $\tilde{l}^i$ with components $\tilde{r}_j^i = \alpha_j r_j^i$ and $\tilde{l}_j^i = \alpha_j^{-1} l_j^i$. In particular, it is clear that the quantities p_{ki} defined in (3) above are independent of units.

The approach of the present paper to defining participation factors builds on the original work in [9],[5],[10]. The focus is on extending the participation factors concept of [9],[5],[10] to explicitly incorporate the effect of uncertainty in the initial condition x^0.

The reconsideration of modal participation in this light is motivated by the following simple observations for the linear system (1) and its solution (5). If the initial condition x^0 lies along the i-th eigenvector, then the only mode that participates in the evolution of any state is the i-th mode (i.e., the $e^{\lambda_i t}$ mode). On the other hand, if the initial condition lies along the k-th coordinate axis, then the evolution of the k-th state involves all system modes according to Eq. (6) above. Clearly, then, the degree to which a mode participates in a state depends on the initial condition. Similar considerations can be given regarding participation of states in modes.

In the next section, a new approach to defining participation factors is given for the case of participation of modes in states. Section 4 gives the corresponding results for participation of states in modes. Both sections rely on explicitly incorporating the effect of uncertainty in the system initial condition x^0 through an averaging operation.

3 Participation Factors: Modes in States

The linear system (1) usually represents the small perturbation dynamics of a nonlinear system in the neighborhood of an equilibrium. The initial condition for such a perturbation is naturally viewed as being an uncertain vector of small norm.

There are several ways in which the foregoing comments can be implemented to result in definitions of participation factors. For example, one can suppose that the initial condition is known to lie in a specified symmetric set in $\mathbb{R}^n$ and calculate "average participations" of modes in state variables by averaging over the set. This is done in Subsection 3.1. If the initial uncertainty set is not symmetric according to Definition 1 below, then the set-theoretic approach of Subsection 3.1 does not lead to a useful notion of participation factors. In some applications, such as ecological modeling, it is natural to restrict the initial condition to lie in a sector and not in a full neighborhood of the equilibrium. A formalism that allows such settings is given in Subsection 3.2. The approach of Subsection 3.2 entails modeling the initial condition uncertainty probabilistically. This technique is used again in Section 4 to streamline the analysis of participation of states in modes.

3.1 Set-Valued Initial Condition Uncertainty

In this subsection, the initial condition x^0 is taken to lie in a connected set $\mathcal{S}$ containing the origin:

$$x^0 \in \mathcal{S}. \tag{7}$$

In fact, the case of greatest interest is when $\mathcal{S} = \mathbb{R}^n$.

Sets $\mathcal{S}$ that are symmetric in the sense of the next definition are of particular significance.

Definition 1. The set $\mathcal{S}$ is symmetric with respect to each of the hyperplanes $x_k = 0$, $k = 1, \ldots, n$. That is, for any $k \in \{1, \ldots, n\}$ and $z = (z_1, \ldots, z_k, \ldots, z_n) \in \mathbb{R}^n$, $z \in \mathcal{S}$ implies that $(z_1, \ldots, -z_k, \ldots, z_n) \in \mathcal{S}$. ■

Although probabilistic arguments aren't used until the next subsection, intuitively, Definition 2 below is consistent with assuming a uniform density for the initial condition and computing the *expectation* of the *relative contribution* of the i-th mode in the k-th state at time $t = 0$. Note that the *average contribution* at time $t = 0$ of the i-th mode to state x_k vanishes and so is not useful as a notion of participation factor:

$$\underset{x^0 \in \mathcal{S}}{\text{avg}} \quad (l^i x^0) r^i_k \quad = \quad 0. \tag{8}$$

Definition 2. The participation factor for the mode associated with λ_i in state x_k with respect to an uncertainty set $\mathcal{S}$ symmetric according to Definition 1 is

$$p_{ki} \quad := \quad \underset{x^0 \in \mathcal{S}}{\text{avg}} \quad \frac{(l^i x^0) r^i_k}{x^0_k} \tag{9}$$

whenever this quantity exists. Here, "$\text{avg}_{x^0 \in \mathcal{S}}$" is an operator that computes the average of a function over the set $\mathcal{S}$. (In computing the implied multidimensional volume integral, the argument function is undefined for $x^0_k = 0$, and the Cauchy principal value [7] of the integral is to be used. For example, if the uncertainty set is $\mathbb{R}^n$, then a limit is taken over a symmetric set $\mathcal{S}$ that tends to $\mathbb{R}^n$. The limit is taken in a symmetric way as the states tend to ∞ at the upper integration limits and to 0 at the lower integration limits.) ■

This quantity measures the *average relative contribution* at time $t = 0$ of the i-th mode to state x_k. In the definition, the i-th mode is interpreted as the $e^{\lambda_i t}$ term in (5). Also, the denominator on the right side of (9) is simply the sum of the contributions from all modes to $x_k(t)$ at $t = 0$:

$$x^0_k \quad = \quad \sum_{j=1}^{n} (l^j x^0) r^j_k. \tag{10}$$

Remark 2. When λ_i is complex its conjugate is also an eigenvalue, and one naturally views the mode as consisting of both the $e^{\lambda_i t}$ term and the $e^{\bar{\lambda}_i t}$ term in (5). Because of this, it would also be reasonable to define the participation factor for complex modes as the twice the real part of the quantity defined in (9), i.e., $2\text{Re}(l_k^i r_k^i)$. This is not done here for two reasons. The first is to simplify the presentation. The second, which is the more important of the two, is to ensure that the new definitions not differ from the original one merely because of a convention such as whether a mode includes just one exponential term or also includes its conjugate. ■

Next, the expression on the right side of Eq. (9) is simplified for the case of a symmetric set $\mathcal{S}$ according to Definition 1. Let

$$\text{Vol}(\mathcal{S}) := \int_{x^0 \in \mathcal{S}} dx^0 \tag{11}$$

denote the volume of the set $\mathcal{S}$. If the set $\mathcal{S}$ has infinite volume, then the construction below is performed for a finite symmetric subset, and then a limit is taken as discussed in Definition 2. From Eq. (9), p_{ki} is given by

$$\begin{aligned} p_{ki} &= \underset{x^0 \in \mathcal{S}}{\text{avg}} \frac{(l_k^i x_k^0) r_k^i}{x_k^0} + \underset{x^0 \in \mathcal{S}}{\text{avg}} \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{(l_j^i x_j^0) r_k^i}{x_k^0} \\ &= l_k^i r_k^i + \int_{x^0 \in \mathcal{S}} \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{(l_j^i x_j^0) r_k^i}{x_k^0} dx^0 / \text{Vol}(\mathcal{S}) \\ &= l_k^i r_k^i + \sum_{\substack{j=1 \\ j \neq k}}^{n} l_j^i r_k^i \int_{x^0 \in \mathcal{S}} \frac{x_j^0}{x_k^0} dx^0 / \text{Vol}(\mathcal{S}) \\ &= l_k^i r_k^i. \end{aligned} \tag{12}$$

The last step follows from the observation that, because $\mathcal{S}$ is symmetric according to Definition 1,

$$\int_{x^0 \in \mathcal{S}} \frac{x_j^0}{x_k^0} dx^0 = 0 \tag{13}$$

for any $j \neq k$, where the integral is interpreted in the sense of the Cauchy principal value. Summarizing this result, it has been found that for a set-valued uncertainty for the initial condition with a symmetric uncertainty set $\mathcal{S}$, Definition 2 for p_{ki} reduces to

$$p_{ki} \;\; := \;\; l_k^i r_k^i. \tag{14}$$

This *result* for p_{ki} agrees with the original *definition* of Verghese, Pérez-Arriaga and Schweppe [9],[5],[10]. However, Definition 2 is based on the symmetry condition as given in Definition 1. Without this symmetry condition, the integral implied in the definition (9) becomes cumbersome to evaluate, and its Cauchy principal value may fail to exist.

The symmetry assumption on $\mathcal{S}$ is reasonable for typical engineering system models. However, there are important examples of system models for which the state vector is restricted. For example, in population dynamics and chemical process dynamics, state variables are often restricted to be nonnegative. If the origin is an equilibrium for such a system, then initial conditions near the origin must be restricted to have nonnegative components. In part to accommodate such settings, a probabilistic framework to studying participation of modes in states is pursued in the next subsection. This probabilistic framework will be found in Section 4 to greatly facilitate the study of participation of states in modes. Indeed, the set-valued approach was found to be cumbersome for the study of participation of states in modes, and will not be discussed in that context.

3.2 Random Initial Condition Uncertainty

Next, a probabilistic definition of participation factors for participation of modes in states is given. This is achieved by letting the initial condition x^0 be a random vector and then taking the expectation of the relative contribution of a mode to a state. The initial condition is assumed to be a random vector satisfying the following assumption of independence.

Assumption 1. The components x_j^0, $j = 1, \ldots, n$, of the initial condition vector x^0 are independent random variables with probability density function $f_{X_j^0}(x_j^0)$. ■

The probabilistic notion of participation factors for participation of modes in states follows. Denote by E the expectation operator.

Definition 3. Suppose Assumption 1 holds. Define p_{ki}, the participation at time $t = 0$ of the mode λ_i in state x_k, as the expectation

$$p_{ki} \;\; := \;\; E\,\{\frac{(l^i x^0) r_k^i}{x_k^0}\} \tag{15}$$

whenever this expectation exists. ■

In applications of this definition to specific problems, it is useful to rewrite the defining formula (15) as follows:

$$
\begin{aligned}
p_{ki} &= E\,\{\frac{(l^i x^0) r^i_k}{x^0_k}\} \\
&= E\,\{\sum_{j=1}^{n} \frac{(l^i_j x^0_j) r^i_k}{x^0_k}\} \\
&= E\,\{\frac{(l^i_k x^0_k) r^i_k}{x^0_k}\} + E\,\{\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{(l^i_j x^0_j) r^i_k}{x^0_k}\} \\
&= l^i_k r^i_k + \sum_{\substack{j=1\\ j\neq k}}^{n} l^i_j r^i_k E\,\{\frac{x^0_j}{x^0_k}\}.
\end{aligned} \tag{16}
$$

To verify that the quantities p_{ki} of Definition 3 do not depend on the units of the state variables x_j, $j = 1, \ldots, n$, invoke Assumption 1 in Eq. (15) and expand the dot product in the numerator, to get

$$
\begin{aligned}
p_{ki} &= E\,\{\frac{(l^i x^0) r^i_k}{x^0_k}\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{X^0_1}(x^0_1) f_{X^0_2}(x^0_2)\cdots f_{X^0_n}(x^0_n) (\sum_{j=1}^{n} \frac{(l^i_j x^0_j) r^i_k}{x^0_k}) dx^0_1 \cdots dx^0_{n-1} dx^0_n.
\end{aligned} \tag{17}
$$

As in Section 2 (comments following Remark 1), consider a change of variables $\tilde{x}_j = \alpha_j x_j$, $j = 1, \ldots, n$. In the new coordinates, the right and left eigenvectors corresponding to eigenvalue λ_i have components $\tilde{r}^i_j = \alpha_j r^i_j$ and $\tilde{l}^i_j = {\alpha_j}^{-1} l^i_j$. Also, clearly, the initial condition components become $\tilde{x}^0_j = \alpha_j x^0_j$, $j = 1, \ldots, n$. The density functions for the initial condition components are given in the new coordinates by

$$
f_{\tilde{X}^0_j}(\tilde{x}^0_j) = \frac{1}{\alpha_j} f_{X^0_j}(x^0_j). \tag{18}
$$

The differentials scale as

$$
d\tilde{x}^0_j = \alpha_j dx^0_j, \quad j = 1, \ldots, n. \tag{19}
$$

Denote by $\tilde{p}_{ki}$ the participation factors computed in the new coordinates using Eq. (17). Using the facts noted in the preceding paragraph, it follows immediately that $\tilde{p}_{ki} = p_{ki}$, i.e., the participation factors defined in Definition 2 are indeed independent of the choice of units for the system state variables.

The following well-known fact from probability theory will be useful in the examples below. It will also be employed in Section 4 for the study of participation of states in modes.

Lemma 1. (See, e.g., [2, pp. 163-164].) Let X and Y be random variables and let $g(X,Y)$ be a function of X and Y. Then

$$E\ \{g(X,Y)\} = E_Y\{E_X\{g(X,Y)|Y\}\ \}, \tag{20}$$

where the notation E_X and E_Y is used to emphasize that the inner expectation is conditioned on Y and taken with respect to X, and the outer expectation is unconditional and taken with respect to Y. ■

For the purposes of this paper, the following observation, based on Lemma 1, is particularly valuable.

Remark 3. If X and Y are independent random variables and the probability density of at least one of X or Y is symmetric with respect to the origin, then:

$$\begin{aligned} E\ \{XY\} &= 0, \quad \text{and} && (21)\\ E\ \{\frac{X}{Y}\} &= 0. && (22) \end{aligned}$$

■

Two examples of the use of Definition 3 are given next.

Example 1. Suppose that the marginal densities $f_{X_j^0}(x_j^0)$ are symmetric with respect to $x_j^0 = 0$, i.e., that they are even functions of x_j^0, for $j = 1, \ldots, n$. With this assumption, Eq. (16) gives

$$\begin{aligned} p_{ki} &= l_k^i r_k^i + \sum_{\substack{j=1\\ j\neq k}}^{n} l_j^i r_k^i E\ \{\frac{x_j^0}{x_k^0}\} \\ &= l_k^i r_k^i \end{aligned} \tag{23}$$

by Remark 3, Eq. (22). Thus, in this setting, the participation factors (15) of Definition 3 reduce to those of the original definition (3) of [9],[5],[10]. It is also interesting to note that the calculations above are still valid for a particular state x_k for which the corresponding initial value x_k^0 is known to be distributed symmetrically, even if the other initial conditions are not distributed symmetrically. ■

Example 2. Suppose that the state variables are restricted to be nonnegative, and, more specifically, that the density functions $f_{X_j^0}(x_j^0)$ are Rayleigh:

$$f_{X_j^0}(x_j^0) = \begin{cases} \frac{x_j^0}{b_j} e^{-\frac{(x_j^0)^2}{2b_j}} & \text{if } x_j^0 \geq 0, \\ 0 & \text{if } x_j^0 < 0, \end{cases} \tag{24}$$

for $j = 1, \ldots, n$. Here the b_j are positive parameters of the individual Rayleigh densities. The mean of X_j^0 is $E\left\{X_j^0\right\} = \sqrt{\frac{\pi b_j}{2}}$, $j = 1, \ldots, n$. The participation factors p_{ki} of Definition 3 are evaluated using (16) as follows (Lemma 1 is not used in the following calculation, though it is easy to check that it would give the same result):

$$\begin{aligned}
p_{ki} &= l_k^i r_k^i + \sum_{\substack{j=1 \\ j \neq k}}^{n} l_j^i r_k^i \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{x_j^0}{x_k^0} \frac{x_1^0 x_2^0 \cdots x_n^0}{b_1 b_2 \cdots b_n} e^{-\frac{(x_1^0)^2}{2b_1} - \frac{(x_2^0)^2}{2b_2} - \cdots - \frac{(x_n^0)^2}{2b_n}} dx_1^0 \cdots dx_{n-1}^0 dx_n^0 \\
&= l_k^i r_k^i + \sum_{\substack{j=1 \\ j \neq k}}^{n} l_j^i r_k^i \int_0^\infty \int_0^\infty \frac{(x_j^0)^2}{b_j b_k} e^{-\frac{(x_j^0)^2}{2b_j} - \frac{(x_k^0)^2}{2b_k}} dx_j^0 dx_k^0 \\
&= l_k^i r_k^i + \sum_{\substack{j=1 \\ j \neq k}}^{n} l_j^i r_k^i \int_0^\infty \frac{e^{-\frac{(x_k^0)^2}{2b_k}}}{b_k} \int_0^\infty \frac{(x_j^0)^2 e^{-\frac{(x_j^0)^2}{2b_j}}}{b_j} dx_j^0 dx_k^0 \\
&= l_k^i r_k^i + \sum_{\substack{j=1 \\ j \neq k}}^{n} l_j^i r_k^i \int_0^\infty \frac{e^{-\frac{(x_k^0)^2}{2b_k}}}{b_k} E\left\{X_j^0\right\} dx_k^0
\end{aligned}$$

$$= \quad l_k^i r_k^i + \sum_{\substack{j=1 \\ j \neq k}}^{n} l_j^i r_k^i \int_0^\infty \frac{e^{-\frac{(x_k^0)^2}{2b_k}}}{b_k} \left(\sqrt{\frac{\pi b_j}{2}}\right) dx_k^0$$

$$= \quad l_k^i r_k^i + \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{\pi}{2} \sqrt{\frac{b_j}{b_k}}\, l_j^i r_k^i. \tag{25}$$

Thus, the participation factors obtained for this example differ from those of the original definition of [9],[5],[10]. This illustrates the importance of initial condition uncertainty models in the study of mode-state interaction. ■

4 Participation Factors: States in Modes

In the foregoing section, the emphasis was on participation of modes in states. In the present section, attention is focused on assessing the other aspect of mode-state interaction, namely participation of states in modes. The development in this section continues to address the effect of initial condition uncertainty on mode-state interaction. The uncertainty is modeled probabilistically rather than set-theoretically, since the authors have found the probabilistic approach to lead to simpler analysis.

Consider again Eq. (1), repeated here for convenience:

$$\dot{x}(t) \quad = \quad Ax(t). \tag{26}$$

Denote by V the matrix of right eigenvectors of A:

$$V = [r^1 \; r^2 \; \cdots \; r^n]. \tag{27}$$

Recall that the corresponding left eigenvectors are row vectors denoted as $l^1, l^2, \ldots, l^n$. From the normalization (2) and the orthogonality of left and right eigenvectors corresponding to distinct eigenvalues, it follows that

$$V^{-1} = \begin{pmatrix} l^1 \\ l^2 \\ \vdots \\ l^n \end{pmatrix}. \tag{28}$$

Perform the change of variables

$$z := V^{-1}x. \tag{29}$$

Then z follows the dynamics

$$\begin{aligned} \dot{z}(t) &= V^{-1}AVz(t) \\ &= \Lambda z, \end{aligned} \tag{30}$$

where $\Lambda := \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. This implies that the new states z_i, $i = 1, \dots, n$ evolve according to

$$z_i(t) = e^{\lambda_i t} z_i^0, \tag{31}$$

where $z^0 = (z_1^0, \dots, z_n^0)^T := z(0)$. It is clear that $z_i(t)$ represents the evolution of the i-th mode. To define the participation of the original states x_k, $k = 1, \dots, n$ in z_i, z_i^0 is written in terms of x^0 and then an appropriate expectation is taken. This is done next.

From Eqs. (29) and (28), $z_i(t)$ is given by

$$\begin{aligned} z_i(t) &= e^{\lambda_i t} l^i x^0 \\ &= e^{\lambda_i t} \sum_{j=1}^{n} (l_j^i x_j^0). \end{aligned} \tag{32}$$

This equation, which shows the contribution of each component of the initial state x_j^0, $j = 1, \dots, n$, to the i-th mode, motivates the following definition for the participation factor governing participation of states in modes.

Definition 4. Suppose Assumption 1 holds. Define p_{ki}, the participation at time $t = 0$ of the state x_k in the mode λ_i, as the expectation

$$p_{ki} := E\,\{\frac{l_k^i x_k^0}{z_i^0}\} \tag{33}$$

whenever this expectation exists. ■

Eq. (33) can be rewritten in a form that can be directly applied depending on the probability distribution of the initial condition, much in the same way as was done for Eq.

(15) of Definition 3. To wit, rewrite x_k^0 in terms of the components of z^0. From Eq. (29), it follows that

$$x^0 = Vz^0. \tag{34}$$

Thus,

$$x_k^0 = \sum_{j=1}^{n} r_k^j z_j^0. \tag{35}$$

Using Eq. (35) in Definition 4 gives the following result, analogous to Eq. (16) for the case of participation of modes in states:

$$\begin{aligned} p_{ki} &= E\,\{\frac{l_k^i x_k^0}{z_i^0}\} \\ &= E\,\{\frac{l_k^i \sum_{j=1}^{n} r_k^j z_j^0}{z_i^0}\} \\ &= E\,\{\frac{l_k^i r_k^i z_i^0}{z_i^0}\} + \sum_{\substack{j=1 \\ j \neq i}}^{n} l_k^i r_k^j E\,\{\frac{z_j^0}{z_i^0}\} \\ &= l_k^i r_k^i + \sum_{\substack{j=1 \\ j \neq i}}^{n} l_k^i r_k^j E\,\{\frac{z_j^0}{z_i^0}\}. \end{aligned} \tag{36}$$

The general result (36) along with Lemma 1 imply that Definition 4 reduces to the original definition of participation factors (3) under Assumption 1 and under the assumption of symmetric probability densities for the initial condition components x_k^0. This is illustrated in the following example, which is analogous to Example 1. If the densities for the initial condition components x_k^0 are not symmetric, then the corresponding densities for the z_i^0 would need to be calculated and used in the formula (36).

Example 3. Suppose that the marginal densities $f_{X_j^0}(x_j^0)$ are symmetric with respect to $x_j^0 = 0$, i.e., that they are even functions of x_j^0, for $j = 1, \ldots, n$. Since z^0 is given by a linear transformation of x^0, $E\,\{x^0\} = 0$ implies that $E\,\{z^0\} = 0$ also. Therefore, under the symmetry assumption, Eq. (36) along with Remark 3, Eq. (22) give

$$
\begin{aligned}
p_{ki} &= l_k^i r_k^i + \sum_{\substack{j=1 \\ j \neq i}}^{n} l_k^i r_k^j E\left\{\frac{z_j^0}{z_i^0}\right\} \\
&= l_k^i r_k^i. \qquad (37)
\end{aligned}
$$

5 Conclusions

A new approach to the definition of participation factors has been given. The approach does not assume any particular choice for the system initial condition. Rather, the initial condition is taken to be an uncertain quantity and the participation factor is defined as an average contribution of a mode to a state, or of a state to a mode. The averaging is taken over the set of possible initial conditions, in either a set-theoretic or a probabilistic framework. The results of this paper were shown to generalize the original definition of participation factors as given in references [9],[5],[10]. This work can lead to further useful generalizations of the participation factors concept in various directions. For instance, generalizations to other system types can be pursued. Also, definitions along the same lines could facilitate analytical treatment of the relation between control inputs and states/mode interaction, which in turn would have implications for actuator and sensor placement. The authors are pursuing such generalizations.

Acknowledgments

This research has been supported by the U.S. Air Force Office of Scientific Research under Grant F49620-96-1-0161, the Office of Naval Research under Grant N00014-96-1-1123, the Electric Power Research Institute, Palo Alto, CA, under Contract RP8050-5, and by a Senior Fulbright Scholar Award.

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