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by R. Venkataraman, P.S. Krishnaprasad

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A model for a thin magnetostrictive actuator

R.Venkataraman, P.S. Krishnaprasad
Department of Electrical Engineering and Institute for Systems Research
University of Maryland, College Park, MD 20742
{*venkat,krishna*}@*isr.umd.edu* *

Abstract

In this paper, we propose a model for dynamic magnetostrictive hysteresis in a thin rod actuator. We derive two equations that represent magnetic and mechanical dynamic equilibrium. Our model results from an application of the energy balance principle. It is a dynamic model as it accounts for inertial effects and mechanical dissipation as the actuator deforms, and also eddy-current losses in the ferromagnetic material.

We also show rigorously that the model admits a periodic solution that is asymptotically stable when a periodic forcing function is applied.

1 Introduction

There is growing interest in the design and control of smart structures – systems with embedded sensors and actuators that provide enhanced ability to program a desired response from a system. Applications of interest include: (a) smart helicopter rotors with actuated flaps that alter the aerodynamic and vibrational properties of the rotor in conjunction with evolving flight conditions and aerodynamic loads; (b) smart fixed wings with actuators that alter airfoil shape to accomodate changing drag/lift conditions; (c) smart machine tools with actuators to compensate for structural vibrations under varying loads. In these and other examples, key technologies include actuators based on materials that respond to changing electric, magnetic, and thermal fields via piezoelectric, magnetostrictive and thermo-elasto-plastic interactions.

Typically such materials exhibit complex nonlinear and hysteretic responses (see Figure 1 for an example of a magnetostrictive material Terfenol-D used in a commercial actuator). Controlling such materials is thus a challenge. The present work is concerned with the development of a physics-based model for magnetostrictive material that captures hysteretic phenomena and can be subject to rigorous mathematical analysis towards control design.

In Section 2 we propose a model for magnetostriction that describes the dynamic behaviour of a thin rod actuator. It is a low (6) dimensional model with 10 parameters and hence is suitable for real-time control. The model incorporates features observed in a commercial actuator [1], like the hysteretic behaviour of magnetostriction as a function of the external field; dependence on the rate of the input, eddy current losses, inertial effects and mechanical damping effects. In Section 3 we analyze this model for periodic forcing functions. Using the Schauder Fixed Point Theorem, we prove that the solution is an asymptotically stable periodic orbit, when the parameters are subject to certain constraints.

2 Thin magnetostrictive actuator model

We are interested in developing a low dimensional model for a magnetostrictive actuator. The main motivation is to use it for control purposes. Therefore the starting point of our work is Jiles and Atherton's macroscopic model for hysteresis in a ferromagnetic rod [2]. In our model, we treat the actuator itself along with the associated prestress, magnetic path, to be a mass-spring system with magneto-elastic coupling. As we show later, our model is only technically valid when the input signal is periodic. However, this is the case in many applications where one obtains rectified linear or rotary motion by applying a periodic input at a high frequency to these actuators. For instance in our hybrid motor [3, 4], we produced a rotary motion using both piezoelectric and magnetostrictive motors in a mechanical clamp and push arrangement.

In an earlier work, we explored the connections between a bulk ferromagnetic hysteresis model and energy balance principles [5]. We present in this paper, an extension of this theory to include magnetostriction and eddy current losses. This is done by equating the work done by external sources (both magnetic and mechanical), with the change in the internal energy of the material, change in kinetic energy, and losses in the magnetization process and the mechanical deformation.

$$\delta W_{bat} + \delta W_{mech} = \underbrace{\delta W_{mag} + \delta W_{magel} + \delta W_{el}}_{Change\,in\,internal\,energy} + \underbrace{\delta L_{mag} + \delta L_{el}}_{losses} + \underbrace{\delta K}_{Change\,in\,kinetic\,energy} \quad (1)$$

In Equation 1, δK is the work done in changing the

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kinetic energy of the system consisting of the magnetoelastic rod, δW_{mag} is the change in the magnetic potential energy, δW_{magel} is the change in the magnetoelastic energy, δW_{el} is the change in the elastic energy, δL_{mag} are the losses due to the change in the magnetization, and δL_{el} are the losses due to the elastic deformation of the rod.

The expression for δW_{mag} will be given shortly. The elastic energy is given by $W_{el} = \frac{1}{2} l\, x^2$, where x is the total strain multiplied the length of the actuator. Chikazumi [6] derives an expression for the magnetoelastic energy density of a three dimensional crystal. It is of the form where the strain components multiply the square of the magnetization components. In our one dimensional case, we can similarly write down the following expression for the magnetoelastic energy W_{magel}.

$$W_{magel} = b\, M^2\, x\, \mathcal{V}$$

where b is the magneto-elastic coupling constant and $\mathcal{V}$ is the volume of the magnetostrictive rod. M is the average magnetic moment of the rod. The expression for the magnetic hysteresis losses δL_{mag} is due to Jiles and Atherton. The motivation for this term is the observation that the hysteresis losses are due to irreversible domain wall motions in a ferromagnetic solid. They arise from various defects in the solids and are discussed in detail by Jiles and Atherton [2]. The mathematical consequences of this hypothesis is discussed in detail in our earlier work [5].

$$\delta L_{mag} = \oint \mathcal{V}\, k\, sign(\dot{H})\, (1-c)\, dM_{irr}$$

The line integral implies that the integration is carried out over one full cycle of the input voltage/current which is assumed to be periodic. The reason for this will be discussed shortly. The losses due to mechanical damping are assumed to be $\delta L_{el} = \oint c_1\, \dot{x}\, dx$. The change in the kinetic energy $\delta K = \oint m_{eff}\, \ddot{x}\, dx$. Therefore,

$$\begin{aligned} \delta W_{bat} + \delta W_{mech} = \delta W_{mag} + \underbrace{\oint m_{eff}\, \ddot{x}\, dx}_{\delta K} \;\ldots \\ + \underbrace{\mathcal{V} \oint b\, M^2\, dx + \mathcal{V} \oint 2\, b\, M\, x\, dM}_{\delta W_{magel}} + \underbrace{\oint l\, x\, dx}_{\delta W_{el}} \\ + \underbrace{\mathcal{V} \oint k\, sign(\dot{H})\, (1-c)\, dM_{irr}}_{\delta L_{mag}} + \underbrace{\oint c_1\, \dot{x}\, dx}_{\delta L_{el}} \end{aligned} \tag{2}$$

Now we obtain expressions for the left hand side of the above equation. For a thin cylindrical magnetostrictive actuator, with an average magnetic moment M, and an uniform magnetic field in the x direction H, the work done by the battery in changing the magnetization per unit volume, in one cycle, is given by

$$\delta W_{bat} = \oint \mu_0 H\; dM$$

Let an external force F in the x direction produce a uniform compressive stress in the x direction σ within the actuator. The total displacement of the edge of the actuator rod be x. Thus the mechanical work done by the external force in a cycle of magnetization is given by [7],

$$\delta W_{mech} = \oint F\; dx$$

The total work done by the battery and the external force is

$$\delta W_{bat} + \delta W_{mech} = \mathcal{V} \oint \mu_0 H\; dM + \oint F\; dx$$

We see that adding the integral of any perfect differential over a cycle does not change the value on the left hand side. Therefore,

$$\delta W_{bat} + \delta W_{mech} = \mathcal{V} \left(\oint \mu_0 H\; dM + \oint \alpha M\; dM \right) + \oint F\; dx \tag{3}$$

Equations 2 and 3 give,

$$\begin{aligned} \mathcal{V}\, \mu_0 \oint (H + \alpha\, M - \tfrac{2\, b\, M\, x}{\mu_0})\, dM \quad \cdots \\ + \oint (F - lx - c_1 \dot{x} - m_{eff} \ddot{x} - \mathcal{V} b M^2)\, dx \;=\; \delta W_{mag} \\ + \mathcal{V} \oint k\, sign(\dot{H})\, (1-c)\, dM_{irr} \end{aligned} \tag{4}$$

Define the effective field to be,

$$H_e = H + \alpha\, M - \frac{2\, b\, M\, x}{\mu_0}$$

As the integration is over one cycle of magnetization, we have

$$\oint H_e\, dM = -\oint M\; dH_e$$

It was observed in [5], that if M is a function of H_e then there are no losses in one cycle. This is the situation for a paramagnetic material where $M = M_{an}$ is given by Langévin's expression as a function of H_e. Hence for the lossless case, the magnetic potential energy is given by,

$$\delta W_{mag} = -\mathcal{V} \oint M_{an}\, dH_e$$

Thus Equation 4 can be rewritten as

$$\begin{aligned} \mathcal{V}\, \mu_0 \oint (M_{an} - M - \tfrac{k\, sign(\dot{H})\, (1-c)}{\mu_0}\, \tfrac{dM_{irr}}{dH_e})\, dH_e \\ + \oint (F - d\, x - c_1\, \dot{x} - m_{eff}\, \ddot{x} - b\, M^2\, \mathcal{V})\, dx = 0 \end{aligned}$$

Note that the above equation is valid only if H, M, x, $\dot{x}$ are periodic functions of time. In other words, the trajectory is a periodic orbit. We now make the *hypothesis* that the following equation is valid when we go from one point to another point on this periodic orbit.

$$\mathcal{V}\,\mu_0 \int (M_{an} - M - \tfrac{k\,sign(\dot{H})\,(1-c)}{\mu_0}\,\tfrac{dM_{irr}}{dH_e})\,dH_e + \int (F - d\,x - c_1\,\dot{x} - m_{eff}\,\ddot{x} - b\,M^2\,\mathcal{V})\,dx = 0$$

The above equation is assumed to hold only on the periodic orbit. Since dx and dH_e are independent variations arising from independent control of the external prestress and applied magnetic field respectively, the integrands must be equal to zero.

$$M_{an} - M - \frac{k\,sign(\dot{H})\,(1-c)}{\mu_0}\,\frac{dM_{irr}}{dH_e} = 0 \quad (5)$$

$$m_{eff}\,\ddot{x} + c_1\,\dot{x} + d\,x + b\,M^2\,\mathcal{V} = F \quad (6)$$

Jiles and Atherton relate the irreversible and the reversible magnetizations as follows [2],

$$\begin{aligned} M &= M_{rev} + M_{irr}. \\ M_{rev} &= c\,(M_{an} - M_{irr}). \\ \frac{dM}{dH} &= \delta_M\,(1-c)\,\frac{dM_{irr}}{dH} + c\,\frac{dM_{an}}{dH} \end{aligned} \quad (7)$$

where δ_M is defined by,

$$\delta_M = \begin{cases} 0 & : \ \dot{H} < 0 \ \ and \ \ M_{an}(H_e) - M(H) > 0 \\ 0 & : \ \dot{H} > 0 \ \ and \ \ M_{an}(H_e) - M(H) < 0 \\ 1 & : \ otherwise. \end{cases} \quad (8)$$

Finally after some algebraic manipulations, the equations for the magnetostriction model are given by

$$\frac{dM}{dt} = \frac{\frac{k\,\delta}{\mu_0}\,c\,\frac{dM_{an}}{dH_e} + \delta_M\,(M_{an} - M)}{\frac{k\,\delta}{\mu_0} - \left(\delta_M\,(M_{an} - M) + \frac{k\,\delta}{\mu_0}\,c\,\frac{dM_{an}}{dH_e}\right)\left(\alpha - \frac{2\,b\,x}{\mu_0}\right)}\,\frac{dH}{dt} \quad (9)$$

$$m_{eff}\,\ddot{x} + c_1\,\dot{x} + d\,x + b\,M^2\,\mathcal{V} = F \quad (10)$$

The inputs to the above set of equations are $\frac{dH}{dt}$ and F, while x is the mechanical displacement.

A magnetostrictive material has finite resistivity, and therefore there are eddy currents circulating within the rod. Using Maxwell's equations, we can derive the following simple expression for the power losses due to eddy currents [3].

$$P_{eddy} = \frac{V^2\,l_m}{N^2\,8\pi\rho}\,\frac{B^2 + A^2}{B^2 - A^2}$$

where A, B are the inner and the outer radii of the rod, l_m is its length, N is the number of turns of coil on the rod, and V is the voltage across the coil of the inductor. Hence the eddy current losses can be represented equivalently as a resistor in parallel with the hysteretic inductor. This idea is quite well known and a discussion can be found in [6] or [3]. From the above expression for the power lost, the value of the resistor is,

$$R_{ed} = \frac{N^2\,8\pi\rho}{l_m}\,\frac{B^2 - A^2}{B^2 + A^2}$$

The actual work done by the battery in changing the magnetization and to replenish the losses due to the eddy currents in one cycle is now given by

$$\begin{aligned} \delta\bar{W}_{bat} &= \delta W_{bat} + \oint P_{eddy}\,dt \\ &= -\mathcal{V}\oint \mu_0 M\,dH_e + \oint P_{eddy}\,dt \end{aligned}$$

Figure 2 shows a schematic of the full model. The hysteretic inductor stands for the magnetostrictive actuator model.

3 Qualitative analysis of the magnetostrictive actuator model

It is very important to note that the model equations (9 – 10) are only valid when all the M, H are periodic in time. What we mean, is that solution of the equations represent the physics of the system under these conditions. But, usually in practice we do not know apriori what state the system is in. Then can we use the above model? The answer in the affirmative is provided in this section. We show analytically that even if we start at the origin in the $M - H$ plane (which is usually not on the hysteresis loop), and apply a periodic input $\dot{H}$, we *tend* asymptotically towards this periodic solution.

It was shown in an earlier work [5], that Equation (9) has an orbitally asymptotically stable limit cycle when $b = 0$ (no coupling) and the input is co-sinusoidal. The situation is much more complicated when the coupling is non-zero. It remains to be shown that there exists an orbitally asymptotically stable limit cycle for co-sinusoidal inputs $\dot{H}$, with $b \neq 0$.

3.1 The uncoupled model with periodic perturbation

Before studying the full coupled system, we consider the effect of periodic perturbations on the uncoupled models. Define state variables,

$$\begin{aligned} x_1 &= H \\ x_2 &= M \\ y_1 &= x \\ y_2 &= \dot{x} \end{aligned}$$

Let,

$$z = \frac{x_1 + (\alpha - \frac{2\,b\,g(t)}{\mu_0})\,x_2}{a}.$$

Then the state equations are:

$$\dot{x_1} = u. \quad (11)$$

$$\dot{x_2} = f_2(x_1, x_2, x_3, x_4, g(t))\, u \quad (12)$$

where the function $f_2(\cdot)$ is obtained by substituting the state variables in Equation (9) and $g(\cdot)$ for $x(\cdot)$.

$$x_3 = sign(u). \quad (13)$$

$$x_4 = \begin{cases} 0 & : \quad x_3 < 0 \quad and \quad \coth(z) - \frac{1}{z} - \frac{x_2}{M_s} > 0 \\ 0 & : \quad x_3 > 0 \quad and \quad \coth(z) - \frac{1}{z} - \frac{x_2}{M_s} < 0 \\ 1 & : \quad otherwise. \end{cases} \quad (14)$$

$$\dot{y} = A\, y + \frac{b\, \mathcal{V}}{m_{eff}}\, h^2(t) \quad (15)$$

where $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$; $A = \begin{bmatrix} 0 & 1 \\ -\frac{d}{m_{eff}} & -\frac{c_1}{m_{eff}} \end{bmatrix}$. $g(\cdot)$, $h(\cdot)$ are $\frac{2\pi}{\omega}$ periodic functions. The input is given by,

$$u(t) = U\, \cos(\omega\, t). \quad (16)$$

3.1.1 Analysis of the uncoupled magnetic system

The proof of existence and uniqueness of trajectories for the system (11 - 14) is exactly as in the earlier paper [5], with the some modifications.

Theorem 1 *Consider the system of equations (11 - 14) with $b \neq 0$. Let the input be given by Equation (16). Suppose*

$$|g| \leq G \quad (17)$$

and $\tilde{\alpha} = \alpha - \frac{2\, b\, G}{\mu_0}$ satisfies

$$\frac{c\, \tilde{\alpha}\, M_s}{3\, a} < 1 \quad (18)$$

$$\frac{k}{\mu_0}\left(1 - \frac{\tilde{\alpha}\, c\, M_s}{3a}\right) - 2\, \tilde{\alpha}\, M_s > 0. \quad (19)$$

Also

$$0 < c < 1. \quad (20)$$

Then the exists a solution to the system with initial condition $x(0) = (0, 0)$. Moreover this solution is unique for all time $t \geq 0$ and lies in the compact set $[-\frac{U}{\omega}, \frac{U}{\omega}] \times [-M_s, M_s]$.

Proof The proof is very similar to that of the system with $b = 0$ [5].
□.

From now on until the end of this section, it is always assumed that the parameters satisfy conditions (17-20).

Theorem 2 *Consider the system given by Equations (11–14), with input given by Equation (16) and $b \neq 0$. If $(x_1, x_2)(0) = (0, 0)$, then the Ω-limit set of the system is an asymptotically orbitally stable periodic orbit.*

Proof

The proof is identical to the one with $b = 0$ [5].
□

Denote the periodic solution of the perturbed magnetic system (11 - 14) with perturbation $g(\cdot)$ and input $u(\cdot)$, as $\bar{x}(\cdot)$. It is a two dimensional vector and a $T = \frac{2\pi}{\omega}$ periodic function. Define the sets $B = \{\phi \in \mathcal{C}([0,T], \mathcal{R}) \; : \; |\phi| \leq \beta_1; \; |\phi(t) - \phi(\bar{t})| \leq M_1\, |t - \bar{t}| \;\; \forall\, t, \bar{t} \in [0,T]\}$, $D = \{\psi \in \mathcal{C}([0,T], \mathcal{R}) \; : \; |\psi| \leq \beta_2; \; |\psi(t) - \psi(\bar{t})| \leq M_2\, |t - \bar{t}| \;\; \forall\, t, \bar{t} \in [0,T]\}$, where $\beta_1, \beta_2, M_1, M_2$ are positive constants. Let $\mathcal{P}_1$, $\mathcal{P}_2 : \mathcal{C}([0,T], \mathcal{R}^2) \to \mathcal{C}([0,T], \mathcal{R})$ denote the projection operators defined by $\mathcal{P}_1(f,g) = f$ and $\mathcal{P}_2(f,g) = g$.
Consider the mappings $\mathcal{G} : B \to \mathcal{C}([0,T], \mathcal{R}^2)$; $g(\cdot) \mapsto \bar{x}(\cdot)$ and $\mathcal{H} : D \to \mathcal{C}([0,T], \mathcal{R}^2)$; $h(\cdot) \mapsto \bar{y}(\cdot)$. We first show $\mathcal{G}$ to be continuous.

Theorem 3 *$\mathcal{G}$ is a continuous map.*

Proof Let the system (11 - 14) be represented by

$$\dot{x} = f(t, x, \tilde{\alpha})\; ; \quad (t, x) \in D \subset \mathcal{R}^3$$

where $\tilde{\alpha} = \alpha - \frac{2\, b\, g(t)}{\mu_0}$, and D is an open set. The state x is 2-dimensional because the discrete states x_3 and x_4 are functions of x_1, x_2 and u. Let the initial condition be $(x_1, x_2)(0) = (0, 0)$.
If $g_n \to g$ in the uniform norm over $[0, T]$ where T is the period of f, then $\tilde{\alpha}_n \to \tilde{\alpha}$. Consider the sequence of systems $\dot{x} = f_n(t, x) = f(t, x, \tilde{\alpha}_n)$. As f is continuous in $\tilde{\alpha}$, $f_n \to f$ in the uniform norm if $\tilde{\alpha}_n \to \tilde{\alpha}$ (Theorem 8). The solutions of each of the systems $\{f_n\}$ and f exist and is unique for $t \in [0, T]$. Then by Theorem 9, the solutions $\phi_n(t)$ of $\dot{x} = f_n(t, x)$ converge uniformly to $\phi(t)$ the solution of $\dot{x} = f(t, x, \tilde{\alpha})$ for $t \in [0, T]$.
Consider the time interval $[T, 2T]$. We have shown that $\phi_n(T) \to \phi(T)$. Then again by Theorem 9, $\phi_n(t) \to \phi(t)$ for $t \in [T, 2T]$. Thus we can keep extending the solutions $\phi_n(t)$ and $\phi(t)$ and obtain uniform convergence over any interval $[mT, (m+1)T]$ where $m > 0$. Therefore, for each m and $\epsilon > 0$, there exists $N(m) > 0$ such that $|\phi_n - \phi| < \frac{\epsilon}{3} \;\; \forall\; n \geq N(m)$.
By Theorem 2 there exist asymptotically orbitally stable periodic orbits $\bar{x}_n$ of the systems $\dot{x} = f_n(t, x)$ and $\bar{x}$ of the system $\dot{x} = f(t, x, \tilde{\alpha})$. Hence for each $\epsilon > 0$, there exists $M \geq 0$ such that $|\bar{x}_n - \phi_n| < \frac{\epsilon}{3}$ and $|\bar{x} - \phi| < \frac{\epsilon}{3} \;\; \forall\; m \geq M$ and $t \in [mT, (m+1)T]$.
Hence for all $n \geq N(M)$ and $t \in [mT, (m+1)T]$ where $m \geq M$, we have $|\bar{x}_n - \bar{x}| \leq |\bar{x}_n - \phi_n| + |\phi_n - \phi| + |\bar{x} - \phi| < \epsilon$. Hence $\mathcal{G}$ is a continuous map.
□

3.1.2 Analysis of the uncoupled mechanical system

In this subsection, we consider the mechanical system with periodic perturbation given by Equation (15). We assume the homogenous system (that is, (15) with $h(t) = 0$) to be asymptotically stable. The relevant results are collected in the appendix.

Theorem 4 *Consider the system (15). If the eigenvalues of A have negative real parts and $h(\cdot)$ is an $\frac{2\pi}{\omega}$ periodic function, then (15) has an $\frac{2\pi}{\omega}$ periodic solution that is asymptotically orbitally stable.*

Proof This follows from Lemma 10 and Theorem 11 in the appendix.
□

Theorem 5 *If the eigenvalues of A have negative real parts, then $\mathcal{H}$ is a continuous map.*

Proof This again follows from Lemma 10 and Theorem 11.
□

3.1.3 Analysis of the coupled magnetostriction model

In this section, we prove the existence of an orbitally asymptotically stable periodic orbit for the magnetostriction model.
Let D_1 denote the range of $\mathcal{P}_2 \circ \mathcal{G}$ and B_1 denote the range of $\mathcal{P}_1 \circ \mathcal{H}$. Thus $\mathcal{P}_2 \circ \mathcal{G} : B \mapsto D_1$ and $\mathcal{P}_1 \circ \mathcal{H} : D \mapsto B_1$.

Theorem 6 *There exists a $\bar{b} > 0$ such that if $|b| \leq \bar{b}$ then $\mathcal{P}_2 \circ \mathcal{G} : B_1 \mapsto D_1$ and $\mathcal{P}_1 \circ \mathcal{H} : D_1 \mapsto B_1$.*

Proof First we show that the sets B_1 and D_1 have the same structure as that of B and D respectively. Then we choose $\bar{b}$ so that the domains and ranges of $\mathcal{G}$ and $\mathcal{H}$ are suitably adjusted. Choose $\beta_1 = M_s$ and $M_1 = \frac{M_s}{3a} U$ in the definition of the set D.

By Theorem 1, the elements of D_1 are uniformly bounded by M_s. Let $\bar{x} = \mathcal{G}\, g$. Therefore $\mathcal{P}_2 \circ \mathcal{G}\, g = \bar{x}_2$. Now $\bar{x}_2(t_2) - \bar{x}_2(t_1) = \int_0^1 \dot{\bar{x}}_2(t_1 + s\,(t_2 - t_1))\,(t_2 - t_1)\, ds$ by the Mean Value Theorem. As the parameters of the system (11 - 14) satisfy the conditions (17 - 20), the vector field $f(t,x)\,u(t)$ is uniformly bounded. Therefore $|\bar{x}_2(t_1) - \bar{x}_2(t_2)| \leq M_1\,|t_2 - t_1|$. Thus D_1 has the same structure of D.

Let $\bar{y} = \mathcal{H}\, h$. Therefore $\bar{y}_1 = \mathcal{P}_1 \circ \mathcal{H}\, h$. The elements of B_1 are uniformly bounded because $\mathcal{H}$ is linear in h^2 and the functions $h \in D$ are uniformly bounded. $|\bar{y}_1| \leq |\bar{y}| \leq |\mathcal{P}_1 \circ \mathcal{H}|\, M_s^2 = \beta_2$. We need to choose $\bar{b}$ so that $\tilde{\alpha} = \alpha - \frac{2\,b\,G}{\mu_0}$ defined in Theorem 1 satisfies Conditions (18) and (19). Such a non-zero $\bar{b}$ obviously exists. Now $\bar{y}_1(t_2) - \bar{y}_1(t_1) = \int_0^1 \dot{\bar{y}}_1(t_1 + s\,(t_2 - t_1))\,(t_2 - t_1)\, ds$ by the Mean Value Theorem. $|\dot{\bar{y}}| \leq |A|\,\beta_2 + b\,\mathcal{V}\,\beta_1^2 = M_2$. Therefore $|\bar{y}_1(t_2) - \bar{y}_1(t_1)| \leq M_2\,|t_2 - t_1|$. Thus B_1 has the same structure of B.

Our choice of $\bar{b} > 0$ ensures that if $|b| \leq \bar{b}$ then $\mathcal{P}_2 \circ \mathcal{G} : B_1 \mapsto D_1$ and $\mathcal{P}_1 \circ \mathcal{H} : D_1 \mapsto B_1$.
□

We now return to the dynamic model of magnetostriction (9,10) and prove the main theorem of this paper.

Theorem 7 *Consider the dynamic model for magnetostriction given by Equations (11 - 16). Suppose the matrix A has eigenvalues with negative real parts and the parameters satisfy conditions (18-20) with the magnetostriction constant $b \leq \bar{b}$ defined in the statement of Theorem 6. Then there exists an orbitally asymptotically stable periodic orbit of the system.*

Proof The sets B_1 and D_1 are compact and convex by Theorem 12. Then $B_1 \times D_1$ is compact in the uniform product norm by Theorem 13. Obviously it is also convex.

Let Ψ be defined as, $\Psi : B_1 \times D_1 \rightarrow B_1 \times D_1; \Psi\ (x_2, y_1) = (\mathcal{P}_1 \circ \mathcal{H}(x_2), \mathcal{P}_2 \circ \mathcal{G}(y_1))$. Then Ψ is continuous because $\mathcal{P}_2 \circ \mathcal{G}$ and $\mathcal{P}_1 \circ \mathcal{H}$ are continuous by Theorems 3 and 5, and the continuity of the projection operator.

Then by the Schauder Fixed Point Theorem (Theorem 14), there exists a limit point of the mapping Ψ in the set $B_1 \times D_1$. This gives us the periodicity of the two state variables x_2 and y_1. In general, the fixed point may not be unique, but when the initial state is the origin, the Ω limit set is unique by the uniqueness of solutions. Now, $(y_1, y_2) = \mathcal{G}\, x_2$ and by Theorem 4, (y_1, y_2) is an asymptotically stable periodic orbit. Also $(x_1, x_2) = \mathcal{H}\, y_1$ and by Theorem 2, (x_1, x_2) is an asymptotically stable periodic orbit. The other state variables (x_3, x_4) are periodic because they are determined by x_1, x_2 and u.
□

A Mathematical Preliminaries

Theorem 8 *If X and Y are normed linear spaces and f is a mapping from X to Y, then f is continuous at x if and only if for each sequence $\{x_n\}$ in X converging to x we have $\{f(x_n)\}$ converging to $f(x)$ in Y.*

Theorem 9 *Suppose $\{f_n\}$, $n = 1, 2, \cdots$, is a sequence of uniformly bounded functions defined and satisfying the Carathéodory conditions on an open set D in $\mathcal{R}^{n+1}$ with $\lim_{n\to\infty} f_n = f_0$ uniformly on compact subsets of D. Suppose (t_n, x_n) is a sequence of points in D converging to (t_0, x_0) in D as $n \to \infty$ and let $\phi_n(t)$, $n = 1, 2, \cdots$, be a solution of the equation $\dot{x} = f_n(t, x)$ passing through the point (t_n, x_n). If $\phi_0(t)$ is defined on $[a, b]$ and is unique, then there is an integer n_0 such that each $\phi_n(t)$, $n \geq_0$, can be defined on $[a, b]$ and converges uniformly to $\phi_0(t)$ uniformly on $[a, b]$.*

Consider the homogenous linear periodic system

$$\dot{x} = A(t)x \tag{21}$$

and the non-homogenous system

$$\dot{x} = A(t)x + f(t) \tag{22}$$

where $A(t+T) = A(t)$, $T > 0$ and $A(t)$ is a continuous $n \times n$ real or complex matrix function of t.

Definition 1 *If $A(t)$ is an $n \times n$ continuous matrix function on $(-\infty, \infty)$ and $\mathcal{D}$ is a given class of functions which contains the zero function, the homogenous system $\dot{x} = A(t)x$ is said to be noncritical with respect to $\mathcal{D}$ if the only solution of Equation (21) which belongs to $\mathcal{D}$ is the solution $x = 0$. Otherwise, system (21) is said to be critical with respect to $\mathcal{D}$.*

The set $\mathcal{P}_T$ denoting the set of T-periodic continuous functions is a Banach space with the sup-norm. That is, $|f| = \sup_{-\infty<t<\infty} |f(t)|; \quad f \in \mathcal{P}_T$. Let $\mathcal{B}$ denote the set of continuous bounded functions from $\mathcal{R}$ to $\mathcal{R}^n$.

Theorem 10 *[8] (a) System (21) with $A(t) \in \mathcal{P}_T$ is noncritical with respect to $\mathcal{B}$ if and only if the characteristic exponents of (21) have nonzero real parts.*

(b) System (21) with $A \in \mathcal{P}_T$ is noncritical with respect to $\mathcal{P}_T$ if and only if $I - X(T)$ is nonsingular, when $X(t)$, $X(0) = I$, is a fundamental matrix solution of (21).

Theorem 11 *Suppose A is in $\mathcal{P}_T$. Then the nonhomogenous equation (22) has a solution $\mathcal{K}f$ in $\mathcal{P}_T$, if and only if system (21) is noncritical with respect to $\mathcal{P}_T$. Furthermore, if system (21) is noncritical with respect to $\mathcal{P}_T$, then $\mathcal{K}f$ is the only solution of (22) in $\mathcal{P}_T$ and is linear and continuous in f.*

Theorem 12 *Suppose D is a compact subset of $\mathcal{R}^m$; M, β are positive constants and $\mathcal{A}$ is the subset of $\mathcal{C}(D, \mathcal{R}^n)$ such that $\phi \in \mathcal{A}$ implies $|\phi| \leq \beta$; $|\phi(t) - \phi(\bar{t})| \leq M\,|t - \bar{t}|$ for $t, \bar{t} \in D$. Then the set $\mathcal{A}$ is convex and compact.*

Theorem 13 *[9] Let A, B be compact subsets of $X, |\cdot|_1)$ and $(Y, |\cdot|_2)$ respectively. Then $A \times B$ is compact (under either of the standard metrics $\|(x,y),(\acute{x},\acute{y})\|_1 = |x-\acute{x}|_1 + |y-\acute{y}|_2$, $\|(x,y),(\acute{x},\acute{y})\|_2 = \max(|x-\acute{x}|_1, |y-\acute{y}|_2)$).*

Theorem 14 *(Schauder) [8] If $\mathcal{A}$ is a convex, compact subset of a Banach space $\mathcal{X}$ and $f : \mathcal{A} \to \mathcal{A}$ is continuous, then f has a fixed point in $\mathcal{A}$.*

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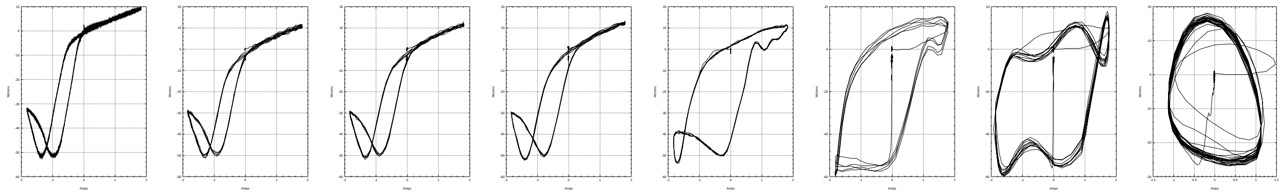

(a) 0.5 Hz (b) 1 Hz (c) 5 Hz (d) 10 Hz (e) 50 Hz (f) 100 Hz (g) 200 Hz (h) 500 Hz

Figure 1: ETREMA MP 50/6 Actuator displacement (Microns) vs current (Amps) characteristic at different driving frequencies .

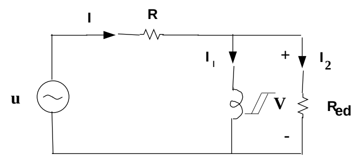

Figure 2: A thin magnetostrictive actuator in a resistive circuit.