TECHNICAL RESEARCH REPORT

Gradient Estimation of Two-Stage Continuous Transfer Lines Subject to Operation-Dependent Failures

by M. Fu, X. Xie

T.R. 98-57

Gradient estimation of two-stage continuous transfer lines subject to operation-dependent failures

Michael FU
Van Munching Hall
University of Maryland
College Park, MD 20742 USA
e-mail: mfu@umd5.umd.edu

Xiaolan XIE
INRIA/MACSI Team
ENIM-ILE DU SAULCY
57045 METZ CEDEX, FRANCE
e-mail: xie@loria.fr

October 1998

Abstract

This paper addresses the gradient estimation of transfer lines comprising two machines separated by a buffer of finite capacity. A continuous flow model is considered, where machines are subject to operation-dependent failures, i.e., a machine cannot fail when it is idle. Both repair times and failure times may be general, i.e., they need not be exponentially distributed. The system is hybrid in the sense that it has both continuous dynamics, as a result of continuous material flow, and discrete events: failures and repairs. The purpose of this paper is to estimate the gradient of the throughput rate with respect to the buffer capacity. Both IPA estimators and SPA estimators are derived. Simulation results show that IPA estimators do not work, contradicting the common belief that IPA always works for continuous flow models.

Keywords: Transfer lines, Continuous flow, Perturbation analysis, Operation-dependent failures

earlier version to appear in Proceedings of CDC (Fu and Xie 1998)

1 INTRODUCTION

A transfer line consists of a set of machines arranged in a serial configuration and separated by buffers. A part to be processed arrives to the first machine as raw material from outside the system. After being processed by the first machine, it queues in the first buffer, waiting to be processed by the second machine. It continues in this manner through all machines and reaches the inventory of finished products after being processed by the last machine. The rate at which a machine processes a part is called the machine's production rate.

The performance of a transfer line is adversely affected by machine failures. While a machine is being repaired, it is unable to process parts, thus disrupting the flow of the transfer line. During this down time, the level of the machine's downstream buffer decreases while the level of its upstream buffer increases. If the repair takes a long time, then the downstream buffer may empty out – starving the downstream machine, and/or the upstream buffer may fill to capacity – blocking the upstream machine. In either case, the affected machine is said to be forced down.

The machine failures may be either operation dependent or time dependent. Operation-dependent failures can only occur while a machine is processing a part, whereas time-dependent failures can occur even if it is forced down. Both types of failures have been considered in the literature. Operation-dependent failures are commonly considered in performance analysis of production lines, and time-dependent failures are usually assumed in the flow control of failure-prone manufacturing systems. Excellent literature surveys on performance evaluation of production lines can be found in Buzacott and Shanthikumar (1992) and Dallery and Gershwin (1992), though here only closely related work will be reviewed. We note here that for the most part little has been done for systems with generally distributed failure and repair times.

In this paper, we limit ourselves to transfer lines composed of two machines separated by a buffer of finite capacity. A continuous flow model is considered. The maximal production rates of the two machines are the same. Times to repair and times to failure of the machines are random variables with general distributions. Machines are subject to operation-dependent failures, and a machine cannot fail if it is forced down. Due to the generality of the failure/repair mechanisms, transfer lines considered in this paper are not analytically tractable, so that performance evaluation requires simulation. The purpose of this paper is to evaluate the gradient of the throughput rate with respect to the buffer capacity. Both infinitesimal perturbation analysis (IPA) estimators and smoothed perturbation analysis (SPA) estimators are derived. We show that the IPA estimators are biased, because certain event order changes cause significant jumps in the performance of the line. Thus, this is an example of a simple continuous flow model in which IPA does not apply, contradicting the common belief that IPA is always applicable to continuous flow models. It is worth noting, however, that IPA does work in the setting presented here if failures are instead time dependent (see Xie 1998).

Most related to this paper are works on performance evaluation and optimization of continuous production lines with operation-dependent failures (e.g., Plambeck et al. 1996, Shi et al. 1998, and Suri and Fu 1994). In particular, Suri and Fu (1994) proposed a GSMP model for representing the underlying stochastic process of a continuous production line, and perturbation analysis with respect to maximal production rates was considered in Shi et al. 1998.

The transfer line model considered in this paper is a piecewise deterministic control system (PDCS). Piecewise deterministic control systems have been addressed by many authors. Most of the existing works are motivated by the optimal control of manufacturing flow and consider time-dependent failures. Among them, convergence of stochastic approximation algorithms coupled with perturbation analysis was addressed in Haurie et al. 1994 under a fairly general framework. However, conditions of that paper are difficult to check and its results difficult to apply. Perturbation analysis was also applied in Caramanis and Liberopoulos (1992) to the flow controller design of manufacturing systems without internal buffers and with constant demand rates. A two-machine production line with constant demand rate was considered in Yan et al. (1994), where sample gradients of inventory cost with respect to control parameters were defined and proved to be strongly consistent. Perturbation analysis was also conducted in Brémaud et al. (1997) for a single-machine/single-item production system having multiple machine states. It is worth mentioning as well the work of Wardi and Melamed (1996), in which the gradient estimation for loss measures in continuous flow models of a single-queue system was addressed using IPA.

The rest of the paper is organized as follows. Notation and basic relations for continuous production lines are presented in Section 2. Section 3 presents the IPA estimators. Section 4 derives unbiased SPA estimators for the finite time horizon case, and Section 5 extends them to the long-run case. In Section 6, numerical results for two simple examples are presented. Section 7 concludes.

2 NOTATION AND BASIC RELATIONS

We consider a production line composed of two machines M_1 and M_2 separated by a buffer of capacity c. We assume the synchronous case where the maximal production rate of both machines is the same, and without loss of generality assumed to be 1. The following notation will be used throughout the paper (with $i = 1, 2$):

$$
\begin{aligned}
X_{ik} &= k\text{-th time to failure of machine } M_i, \\
Y_{ik} &= k\text{-th time to repair of machine } M_i, \\
F_i(\text{resp. } G_i) &= \text{distribution function of } X_{ik} \text{ (resp. } Y_{ik}), \\
f_i(\text{resp.} g_i) &= \text{density function of } X_{ik} \text{ (resp. } Y_{ik}), \\
\lambda_i, \mu_i &= \text{failure rate and repair rate, i.e., } \lambda_i = 1/E[X_{ik}] \text{ and } \mu_i = 1/E[Y_{ik}], \\
\alpha_{it} &= \text{state of machine } M_i \text{ at time } t\text{; 1 if up and 0 otherwise}, \\
r_{it} &= \text{remaining lifetime (until failure or repair) of machine } M_i \text{ in state } \alpha_{it} \text{ at time } t^+, \\
a_{it} &= \text{age (since last failure or repair) of machine } M_i \text{ in state } \alpha_{it} \text{ at time } t^+, \\
P_{it} &= \text{cumulative production of } M_i \text{ up to time } t, \\
x_t &= \text{buffer level at time } t, \\
e_k &= k\text{-th event} \in \{F_1, R_1, F_2, R_2, BF, BE\},
\end{aligned}
$$

$$\begin{aligned} t_k &= \text{epoch of event } e_k, \\ s_k &= \text{state of the system at time } t_k^+ = (a_{1k}, a_{2k}, r_{1k}, r_{2k}, x_k), \\ \tau_k &= \text{time from } e_{k-1} \text{ to } e_k = \tau_k - \tau_{k-1}, \\ \#(k,e) &= \text{number of occurrences of event } e \text{ in } e_1 e_2 ... e_k. \end{aligned}$$

In the above notation, for the sake of simplicity, x_k is used to denote x_{t_k}. This will not lead to confusion, since throughout the paper only x_t denotes the inventory level at time t and, in all other cases, $x_\bullet$ denotes in fact $x_{t_\bullet}$. The same abuse of notation is often followed for α_{ik}, r_{ik}, a_{ik}, τ_k, and s_k.

We take the initial condition $\alpha_0 = (1,1)$ and $x_0 = 0$, i.e., the system begins empty with both machines up. The performance measure considered in this paper is the throughput rate of the system defined as follows:

$$L = \lim_{t\to\infty} P_{2t}/t,$$

which is assumed to exist w.p. 1. Three finite-time estimators will be considered:

$$\begin{aligned} L_t &= P_{2t}/t, \\ L_n &= P_{2t_{w(n)}}/t_{w(n)}, \\ L_n &= P_{2t_n}/t_n, \end{aligned}$$

where $e_{w(n)}$ is the n-th repair of M_2.

The dynamics of the system can be characterized similar to a generalized semi-Markov process (GSMP) model: starting from s_0, the next event e_1 is determined and the state of the system is updated as to be described. Then the system evolves in the same way starting from its new state s_1.

Machine M_1 is said to be blocked in state $s_k = (\alpha_{1k}, \alpha_{2k}, r_{1k}, r_{2k}, x_k)$ if $\alpha_k = (1,0)$ and $x_k = c$. Machine M_2 is said to be starved in state s_k if $\alpha_k = (0,1)$ and $x_k = 0$. In either case, the machine is said to be *forced down*. When a machine M_i is forced down, it cannot produce, it cannot break down, and its remaining time to failure r_{ik} remains unchanged as long as it is forced down. Thus, in terms of the usual GSMP terminology, the noninterruption condition is violated; in this case, the corresponding events are not cancelled, but merely suspended.

For the two-machine systems, the following events are possible: the failure of M_1, the repair of M_1, the failure of M_2, the repair of M_2, buffer full, and buffer empty, denoted respectively by F_1, R_1, F_2, R_2, BF, BE. The determination of the next event e_{k+1} is as follows. The time to state change of machine M_i for $i = 1,2$, is as follows:

$$T_{ik} = \begin{cases} r_{ik}, & \text{if } M_i \text{ is not forced down in } s_k; \\ \infty, & \text{otherwise.} \end{cases}$$

The time to buffer full event is:

$$T_{Fk} = \begin{cases} c - x_k, & \text{if } \alpha_k = (1,0) \wedge x_k < c; \\ \infty, & \text{otherwise.} \end{cases}$$

The time to buffer empty event is:

$$T_{Ek} = \begin{cases} x_k, & \text{if } \alpha_k = (1,0) \wedge x_k \ > 0; \\ \infty, & \text{otherwise.} \end{cases}$$

As a result, the next event epoch $t_{k+1} = t_k + \tau_k$ with

$$\tau_k = \min\{T_{1k}, T_{2k}, T_{Fk}, T_{Ek}\}.$$

The next event is defined as follows:

$$e_k = \begin{cases} F_i, & \text{if } \tau_k = T_{ik} \wedge \alpha_{ik} = 1; \\ R_i, & \text{if } \tau_k = T_{ik} \wedge \alpha_{ik} = 0; \\ BF, & \text{if } \tau_k = T_{Fk}; \\ BE, & \text{if } \tau_k = T_{Ek}. \end{cases}$$

The next state can be updated as follows:

$$\begin{aligned}
x_{k+1} &= \begin{cases} x_k, & \text{if } M_1 \text{or } M_2 \text{ is forced down in } s_k, \\ x_k + (\alpha_{1k} - \alpha_{2k})\tau_k, & \text{otherwise;} \end{cases} \\
\alpha_{1k+1} &= \begin{cases} \alpha_{1k}, & \text{if } e_{k+1} \ \notin \{F_1, R_1\}; \\ 1 - \alpha_{1k}, & \text{otherwise;} \end{cases} \\
\alpha_{2k+1} &= \begin{cases} \alpha_{2k}, & \text{if } e_{k+1} \ \notin \{F_2, R_2\}; \\ 1 - \alpha_{2k}, & \text{otherwise;} \end{cases} \\
r_{k+1} &= \begin{cases} sample(Y_1), & \text{if } e_{k+1} = F_1; \\ sample(X_1), & \text{if } e_{k+1} = R_1; \\ r_{1k}, & \text{if } M_1 \text{ is blocked in } s_k; \\ r_{1k} - \tau_k, & \text{otherwise;} \end{cases} \\
r_{2k+1} &= \begin{cases} sample(Y_1), & \text{if } e_{k+1} = F_2; \\ sample(X_1), & \text{if } e_{k+1} = R_2; \\ r_{2k}, & \text{if } M_2 \text{ is starved in } s_k; \\ r_{2k} - \tau_k, & \text{otherwise.} \end{cases}
\end{aligned}$$

From the above equations, $e_{k+1} = R_2$ if M_1 is blocked, and $e_{k+1} = R_1$ if M_2 is starved.

3 IPA ANALYSIS

For the purpose of perturbation analysis, we compare the sample path of the system having buffer capacity c, called the nominal system, with that of the system having buffer capacity $c+\Delta$, $\Delta > 0$, called the perturbed system. As usual in IPA, we assume that the event sequence is identical for both system up to the k-th event, i.e.,

(A1) $e_1(c + \Delta) = e_1(c), ..., e_k(c + \Delta) = e_k(c)$.

Under this assumption, the following holds:

$$\begin{aligned} \alpha_k(c + \Delta) &= \alpha_k(c), \\ a_{ik}(c + \Delta) + r_{ik}(c + \Delta) &= a_{ik}(c) + r_{ik}(c), \forall i = 1, 2. \end{aligned} \tag{1}$$

Figure 1 HERE

Figure 1: A typical sample path

More relations between the nominal and perturbed systems can be obtained by detailed sample path analysis. For this purpose, consider the sample path of Figure 1 and let

$$\begin{aligned} H &= \min\{n \geq 1 : e_n = BF\}, \\ O &= \min\{n \geq H : e_n = BE\}. \end{aligned}$$

Clearly H and O define an operation cycle of the system. At event e_{O+1}, both machines are up and the buffer is empty, and the system starts a similar cycle as at time 0. Hence it is natural to first conduct the perturbation analysis for the first cycle, and then extend the results to the other cycles. However, we note that except for the case where times to failure of machine M_2 are exponential, the points are not in fact regenerative points. Five cases are considered for the first cycle.

Theorem 1. Under assumption (A1), the following hold:

- If $k < H : t_k(c+\Delta) = t_k(c), x_k(c+\Delta) = x_k(c), r_{ik}(c+\Delta) = r_{ik}(c), \forall i = 1, 2.$
- If $k = H : t_k(c+\Delta) = t_k(c) + \Delta, x_k(c+\Delta) = x_k(c) + \Delta, r_{ik}(c+\Delta) = r_{ik}(c) - \Delta, \forall i = 1, 2.$
- If $H < k < O$ and $e_k \in \{F_2, R_2\} : t_k(c+\Delta) = t_k(c), x_k(c+\Delta) = x_k(c) + \alpha_{1k}\Delta, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c).$
- If $H < k < O$ and $e_k = BF : t_k(c+\Delta) = t_k(c), x_k(c+\Delta) = x_k(c)+\Delta, r_{1k}(c+\Delta) = r_{1k}(c)-\Delta, r_{2k}(c+\Delta) = r_{2k}(c).$
- If $H < k < O$ and $e_k \in \{F_1, R_1\} : t_k(c+\Delta) = t_k(c) - \Delta, x_k(c+\Delta) = x_k(c) + \alpha_{2k}\Delta, r_{1k}(c+\Delta) = r_{1k}(c), r_{2k}(c+\Delta) = r_{2k}(c) + \Delta.$
- If $k = O : t_k(c+\Delta) = t_k(c), x_k(c+\Delta) = x_k(c) = 0, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c).$
- If $k = O+1 : t_k(c+\Delta) = t_k(c) - \Delta, x_k(c+\Delta) = x_k(c) = 0, r_{1k}(c+\Delta) = r_{1k}(c), r_{2k}(c+\Delta) = r_{2k}(c).$

Proof. Case $0 < k < H$ is obvious since, from $t = 0$ to t_H, the buffer level is always below H, so the sample paths of the nominal and perturbed systems are identical. Case $k = H$ is a trivial consequence of case $0 < k < H$. The results concerning cases $H < k < O$ will be proved later. Consider the case $k = O$. In this case, $\alpha_{k-1} = \alpha_k = (0, 1)$ and $e_{k-1} \in \{F_1, R_2\}$. Only the proof for $e_{k-1} = F_1$ is given, since that for $e_{k-1} = R_2$ is similar. If $e_{k-1} = F_1$, then $t_{k-1}(c+\Delta) = t_{k-1}(c) - \Delta$ and $x_{k-1}(c+\Delta) = x_{k-1}(c) + \Delta$. As a result, $t_k(c+\Delta) = t_{k-1}(c+\Delta) + x_{k-1}(c+\Delta) = t_{k-1}(c) + x_{k-1}(c) = t_k(c)$, i.e., $t_k(c+\Delta) - t_{k-1}(c+\Delta) = t_k(c) - t_{k-1}(c) + \Delta$. Hence, $r_{1k}(c+\Delta) = r_{1k}(c) - \Delta$, $r_{2k}(c+\Delta) = r_{2k}(c)$. Finally, the result for the case

$k = O + 1$ is a trivial consequence of the case $k = O$, since for $k = O + 1$, under assumption (A1), $e_k = R_1$, and machine M_2 is starved from e_{k-1} to e_k.

We now prove the cases $H < k < O$ by recursion on $k = O, O+1, ...$. Consider first the case $e_k \in \{F_2, R_2\}$. By definition of H and O, the buffer level is always positive from t_{H-1} to t_O. As a result of Case $0 < k < H$, machine M_2 fails and is repaired at the same time over period (t_{H-1}, t_O). As a result, $t_k(c + \Delta) = t_k(c)$. Consider first the case $e_k = F_2$. Hence, $r_{2k}(c + \Delta) = r_{2k}(c) = \mathit{sample}(Y_2)$, $\alpha_{k-1} = (\cdot, 1)$ and $e_{k-1} \in \{F_1, R_1, R_2\}$. For each possible e_{k-1}, by recursive assumption, $t_k(c + \Delta) = t_{k-1}(c + \Delta) + r_{2k-1}(c + \Delta) = t_{k-1}(c) + r_{2k-1}(c) = t_k(c)$, $x_k(c+\Delta) = x_{k-1}(c+\Delta) + (\alpha_{1k-1} - 1)(t_k(c+\Delta) - t_{k-1}(c+\Delta)) = x_k(c) + \alpha_{1k}\Delta$, $r_{1k}(c + \Delta) = r_{1k}(c) - \Delta$. The results for the case $e_k = R_2$, i.e. $\alpha_{k-1} = (\cdot, 0)$and $e_{k-1} \in \{F_1, R_1, R_2, BF\}$, can be established in a similar way by considering each possible event e_{k-1}.

For the case $e_k = BF, \alpha_{k-1} = \alpha_k = (1, 0), x_{k-1}(c+\Delta) < c+\Delta, x_{k-1}(c) < c$, and $e_{k-1} \in \{F_2, R_1\}$. Only the case $e_{k-1} = R_1$ is considered, since the proof is similar for $e_{k-1} = F_2$. Since $e_{k-1} = R_1$, $t_{k-1}(c + \Delta) = t_{k-1}(c) - \Delta$, $x_{k-1}(c + \Delta) = x_{k-1}(c)$, $r_{1k-1}(c + \Delta) = r_{1k-1}(c)$, $r_{2k-1}(c + \Delta) = r_{2k-1}(c) + \Delta$. Hence, $t_k(c+\Delta) = t_{k-1}(c+\Delta)+c+\Delta-x_{k-1}(c+\Delta) = t_{k-1}(c)+c-x_{k-1}(c) = t_k(c)$. Hence $t_k(c+\Delta)-t_{k-1}(c+\Delta) = t_k(c) - t_{k-1}(c) + \Delta$, $r_{1k}(c + \Delta) = r_{1k}(c) - \Delta, r_{2k}(c + \Delta) = r_{2k}(c)$. Finally, $x_k(c + \Delta) = c + \Delta = x_k(c)- \Delta$.

For the case $e_k \in \{F_1, R_1\}$, $e_{k-1} \neq BF$ and neither M_1 nor M_2 is forced down from e_{k-1} to e_k. If $e_{k-1} \in \{F_1, R_1\}$, then $t_{k-1}(c + \Delta) = t_{k-1}(c)- \Delta$, $x_{k-1}(c + \Delta) = x_{k-1}(c) + \alpha_{2k-1}\Delta$, $r_{1k-1}(c + \Delta) = r_{1k-1}(c)$, $r_{2k-1}(c+\Delta) = r_{2k-1}(c)- \Delta$. Further $\alpha_{2k} = \alpha_{2k-1}$, $t_k = t_{k-1}+r_{1k-1}$ and $x_k = x_{k-1}+(\alpha_{1k-1}-\alpha_{2k-1})r_{1k-1}$. As a result, $t_k(c+\Delta) = t_k(c)-\Delta$, $x_k(c+\Delta) = x_k(c)+\alpha_{2k}\Delta$, $r_{1k}(c+\Delta) = r_{1k}(c)$, $r_{2k}(c+\Delta) = r_{2k}(c)+\Delta$. If $e_{k-1} \in \{F_2, R_2\}$, then $t_{k-1}(c+\Delta) = t_{k-1}(c)$, $x_{k-1}(c+\Delta) = x_{k-1}(c)+\alpha_{1k-1}\Delta$, $r_{1k-1}(c+\Delta) = r_{1k-1}(c)-\Delta$, $r_{2k-1}(c + \Delta) = r_{2k-1}(c)$. Further $\alpha_{1k} = \alpha_{1k-1}$, $t_k = t_{k-1} + r_{2k-1}$and $x_k = x_{k-1} + (\alpha_{1k-1} - \alpha_{2k-1})r_{2k-1}$. The results can then be established. □

It should be noted that at the occurrence of e_{O+1}, the sample path of the perturbed system can be derived from the one of the nominal system as above and the only difference is that the event epochs $t_k(c + \Delta)$ is further shifted leftward by Δ. To generalize the results, we decompose the sample path into cycles as follows:

$$\begin{aligned} H_m &= \min\{n = O_{m-1} : e_n = BF\}, \\ O_m &= \min\{n = H_m : e_n = BE\}, \end{aligned}$$

where $O_0 = 0$. Clearly, $H_m > O_{m-1} + 1$ and $O_m > H_m + 1$.

Theorem 2: Assume that assumption (A1) holds and that e_k is an event of cycle $m + 1$ with $m \geq 0$, i.e. $O_m + 1 < k \leq O_{m+1} + 1$. Then the following hold:

- If $k < H_{m+1} : t_k(c + \Delta) = t_k(c) - m\Delta, x_k(c + \Delta) = x_k(c), r_{ik}(c + \Delta) = r_{ik}(c), \forall i = 1, 2.$
- If $k = H_{m+1}$ and $e_k = BF : t_k(c + \Delta) = t_k(c) - (m - 1)\Delta, x_k(c + \Delta) = x_k(c) + \Delta, r_{ik}(c + \Delta) = r_{ik}(c) - \Delta, \forall i = 1, 2.$

- If $H_{m+1} < k < O_{m+1}$ and $e_k \in \{F_2, R_2\} : t_k(c+\Delta) = t_k(c) - m\Delta, x_k(c+\Delta) = x_k(c) + \alpha_{1k}\Delta, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c)$.

- If $H_{m+1} < k < O_{m+1}$ and $e_k = BF : t_k(c+\Delta) = t_k(c) - m\Delta, x_k(c+\Delta) = x_k(c) + \Delta, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c)$,

- If $H_{m+1} < k < O_{m+1}$ and $e_k \in \{F_1, R_1\} : t_k(c+\Delta) = t_k(c) - (m+1)\Delta, x_k(c+\Delta) = x_k(c) + \alpha_{2k}\Delta, r_{1k}(c+\Delta) = r_{1k}(c), r_{2k}(c+\Delta) = r_{2k}(c) + \Delta$.

- If $k = O_{m+1}$ and $e_k = BE : t_k(c+\Delta) = t_k(c) - m\Delta, x_k(c+\Delta) = x_k(c) = 0, r_{1k}(c+\Delta) = r_{1k}(c) - \Delta, r_{2k}(c+\Delta) = r_{2k}(c)$.

- If $k = O_{m+1} + 1$ and $e_k = R_1 : t_k(c+\Delta) = t_k(c) - (m+1)\Delta, x_k(c+\Delta) = x_k(c) = 0, r_{1k}(c+\Delta) = r_{1k}(c), r_{2k}(c+\Delta) = r_{2k}(c)$.

From the above results, we can derive the sensitivities of the performance measures.

Theorem 3: Consider the estimator $L_n = P_{2t_{w(n)}}/t_{w(n)}$. Assume that assumption (A1) holds $1 \leq k \leq w(n)$ where $w(n)$ is the n-th occurrence of R_2. Then,

$$L_n(c+\Delta) - L_n(c) = \frac{L_n(c)\,\Delta}{t_{w(n)}(c)/Q_{w(n)} - \Delta},$$

where Q_k is the number of cycles defined above completed by the $k-$th event.

Proof. From the definition of $w(n)$, we have:

$$P_{2t_{w(n)}}(c+\Delta) = P_{2t_{w(n)}}(c) = \sum_{k=1}^{n} X_{2k},$$

where X_{2k} for $1 \leq k \leq n$ are times to failure of machine M_2. From Theorem 2,

$$t_{w(n)}(c+\Delta) = t_{w(n)}(c) - Q_n\Delta.$$

The combination of the two relations gives:

$$L_n(c+\Delta) - L_n(c) = \frac{P_{2t_{w(n)}}(c)}{t_{w(n)}(c) - Q_n\Delta} - \frac{P_{2t_{w(n)}}(c)}{t_{w(n)}(c)} = \frac{P_{2t_{w(n)}}(c)Q_n\Delta}{t_{w(n)}(c)(t_{w(n)}(c) - Q_n\Delta)} = \frac{L_n(c)\Delta}{t_{w(n)}(c)/Q_{w(n)} - \Delta}.$$

□

Theorem 4: Consider the estimator $L_t = P_{2t}/t$. Assume that assumption (A1) holds $1 \leq k \leq N(t)+1$, where $N(t)$ is the number of events up to time t, i.e. $N(t) = \inf\{k : t_k \leq t\}$. Further, assume that $N(t)$ is the same for both perturbed and nominal systems. Then,

$$L_t(c+\Delta) - L_t(c) = \frac{\alpha_{2t}\gamma_{2t}}{t/Q_{N(t)}}\Delta + \frac{v_t}{t}\Delta,$$

where $\gamma_{2t} = 1$ if machine M_2 is not starved at time t and $\gamma_{2t} = 0$ otherwise, and v_t is a random variable such that $-2 \leq v_t \leq 2$.

Proof. Clearly,

$$P_{2t} = \sum_{k=1}^{\#(N(t),F_2)} X_{2k} + \alpha_{2t}(a_{2N(t)} + \gamma_{2t}(t - t_{N(t)})).$$

We note that γ_{2t} can be derived from the event sequence up to $e_{N(t)}$ as follows:

$$\gamma_{2t=} \begin{cases} \alpha_{1k}, & \text{if } \exists n \leq N(t)/e_n = BE \wedge e_k \in \{F_1, R_1\}, \forall n < k \leq N(t); \\ 1 & \text{otherwise;} \end{cases}$$

As a result, under the condition of the theorem, $\alpha_{2t}(c + \Delta) = \alpha_{2t}(c)$ and $\gamma_{2t}(c + \Delta) = \gamma_{2t}(c)$. Therefore,

$$P_{2t}(c + \Delta) - P_{2t}(c) = \alpha_{2t}(a_{2N(t)}(c + \Delta) - a_{2N(t)}(c)) - \alpha_{2t}\gamma_{2t}(t_{N(t)}(c + \Delta) - t_{N(t)}(c)).$$

From relation (1), $a_{2N(t)} = X_{2,\#(N(t),F_2)+1} - r_{2N(t)}$. This relation together with Theorem 2 leads to $-\Delta \leq a_{2N(t)}(c + \Delta) - a_{2N(t)}(c) \leq \Delta$. Furthermore, from Theorem 2, $-(Q_{N(t)} + 1)\Delta \leq t_{N(t)}(c + \Delta) - t_{N(t)}(c) \leq -(Q_{N(t)} - 1)\Delta$. Since α_{2t}, $\gamma_{2t} \in \{0, 1\}$, it holds that $-2 \leq v_t \leq 2$ with

$$v_t = \frac{P_{2t}(c + \Delta) - P_{2t}(c) - \alpha_{2t}\gamma_{2t}Q_{N(t)}\Delta}{\Delta}.$$

As a result,

$$L_t(c + \Delta) - L_t\,(c) = \frac{\alpha_{2t}\gamma_{2t}}{t/Q_{N(t)}}\Delta + \frac{v_t}{t}\Delta.$$ □

Similarly, it can be proved that:

Theorem 5: Consider the estimator $L_n = P_{2t_n}/t_n$. Assume that assumption (A1) holds $1 \leq k \leq n$. Then,

$$L_n(c + \Delta) - L_n\,(c) = \frac{L_n(c)Q_n + v_t}{t_n(c + \Delta)}\Delta,$$

where $t_n(c+\Delta) = t_n(c) - (Q_n + u_n)\Delta$, v_n and u_n are random variables such that $-2 \leq v_n \leq 2, -1 \leq u_n \leq 1$.

By taking the limit of the above gradient estimators, we obtain:

$$\begin{aligned} IPA1 &= \lim_{n\to\infty}\lim_{\Delta\to 0}\frac{L_n\,(c + \Delta) - L_n(c)}{\Delta} = \frac{L_\infty(c)}{C} \text{ w.p.1,} \\ IPA2 &= \lim_{t\to\infty}\lim_{\Delta\to 0}\frac{L_t\,(c + \Delta) - L_t(c)}{\Delta} = \frac{\alpha_{2\infty}\gamma_{2\infty}}{C} \text{ w.p.1,} \\ IPA3 &= \lim_{n\to\infty}\lim_{\Delta\to 0}\frac{L_n\,(c + \Delta) - L_n(c)}{\Delta} = \frac{L_\infty(c)}{C} \text{ w.p.1,} \end{aligned}$$

where C is the average length of the cycles. Note that $IPA1$ and $IPA3$ are constants, whereas $IPA2$ is a random variable. Since $P(\alpha_{2\infty}\gamma_{2\infty} = 0) > 0$, if a long simulation is performed to estimate $IPA2$, we can expect that $IPA2$ is sometime equal to 0 and sometime equal to $1/C$ and the related estimator does not converge. The relationship between the estimators can be established as follows. Since $\alpha_{2t}\gamma_{2t} = 1$ if and only if machine M_2 is producing, $P(\alpha_{2\infty}\gamma_{2\infty} = 1)$ is equal to the throughput rate of the system. As a result,

$$E\,[IPA2] = \frac{P(\alpha_{2\infty}\gamma_{2\infty} = 1)}{C} = \frac{L}{C} = IPA1 = IPA3.$$

Unfortunately, because event changes that occur when assumption (A1) is violated can lead to large (non-infinitesimal) jumps in the performance measure (as confirmed by the numerical simulation experiments described in Section 6, $IPA1$ and $IPA3$ are not strongly consistent, i.e., $IPA1 = IPA3 \neq dL/dc$. This means that event changes need to be considered in the derivation of gradient estimators, which leads to the SPA analysis considered in the following section.

4 SPA ANALYSIS: FINITE HORIZON

The purpose of this section is to derive SPA estimators following the approach proposed by Fu and Hu (1997). Only the performance measure $L_t = P_{2t}/t$ is considered here. Under the framework of Fu and Hu (1997), the sample path space is partitioned into sets (probability events) $\mathcal{A}(\Delta)$ and $\mathcal{A}^c(\Delta)$, where $\mathcal{A}(\Delta)$ contains the sample paths that experience no event changes due to a perturbation of size Δ, and the complement set $\mathcal{A}^c$ contains the sample paths on which event changes do occur as a result of the perturbation. Using this partition,

$$\begin{aligned}
\frac{dE\left[L_t\right]}{dc} &= \lim_{\Delta\to 0} E\left[\frac{L_t(c+\Delta)-L_t(c)}{\Delta}|\mathcal{A}\right]P(\mathcal{A}) + \lim_{\Delta\to 0} E\left[\frac{L_t(c+\Delta)-L_t(c)}{\Delta}|\mathcal{A}^c\right]P(\mathcal{A}^c) \\
&= E\left[\frac{dL_t}{dc}\right] + E\left[\lim_{\Delta\to 0} E\left[(L_t(c+\Delta)-L_t(c))|z,\mathcal{A}^c\right]\lim_{\Delta\to 0}\frac{P(\mathcal{A}^c|z)}{\Delta}\right] \\
&= E\left[\frac{dL_t}{dc}\right] + E\left[\left(E_z\left[L^{PP}\right]-E_z\left[L^{DNP}\right]\right)\frac{dP_z}{dc}\right], \\
\text{where } E_z\left[L^{PP}\right] &= \lim_{\Delta\to 0} E\left[L_t\left(c+\Delta\right)|z,\mathcal{A}^c\right] \\
E_z\left[L^{DNP}\right] &= \lim_{\Delta\to 0} E\left[L_t\left(c\right)|z,\mathcal{A}^c\right] \\
\frac{dP_z}{dc} &= \lim_{\Delta\to 0}\frac{P(\mathcal{A}^c|z)}{\Delta},
\end{aligned}$$

where dP_z/dc is the probability rate of event changes, z is a set of sample path quantities selected to smooth the effect of event changes, L^{DNP} is the performance measure of a so-called degenerated nominal path denoted by DNP, L^{PP} is the performance measure of a so-called perturbed path, denoted by PP. PP is identical to DNP up to time, say t_k, where an event change occurs. At this instant, an event occurs on both nominal and perturbed paths; however, the event on the nominal path differs from the one on the perturbed path.

In the following, we examine the possible event changes, select the characterization and determine the effect of event changes, i.e., $L^{PP} - L^{DNP}$. Assume that the k-th event changes, i.e.,

$$e_1(c+\Delta) = e_1(c), ..., e_{k-1}(c+\Delta) = e_{k-1}(c), e_k(c+\Delta) \neq e_k(c).$$

Clearly the above assumption implies that the state of the machines α_{k-1} is identical for both nominal and perturbed systems. Consider as well the following indicator of the buffer state $\beta_k = (\beta_{1k}, \beta_{2k})$ with $\beta_{1k} = 1\{x_k = c\}$ and $\beta_{2k} = 1\{x_k = 0\}$. β_{k-1} is also identical for both nominal and perturbed systems since it is totally determined by the sequence of events up to e_{k-1} as follows:

$$\beta_{1k-1} = \begin{cases} 1, & \text{if } \exists n \leq k-1/e_n = BF \wedge e_i \in \{F_2, R_2\}, \forall n < k \leq k-1; \\ 0 & \text{otherwise;} \end{cases}$$

$$\beta_{2k-1} = \begin{cases} 1, & \text{if } \exists n \leq k-1/e_n = BE \wedge e_i \in \{F_1, R_1\}, \forall n < k \leq k-1; \\ 0 & \text{otherwise.} \end{cases}$$

A change of the k-th event with the next event is not possible if machine M_1 is blocked in state s_{k-1} or M_2 is starved in state s_{k-1}, i.e., $\alpha_{k-1} = (1,0) \wedge \beta_{k-1} = (1,0)$ or $\alpha_{k-1} = (0,1) \wedge \beta_{k-1} = (0,1)$, because only one event (R_2 in the first case and R_1 in the second case) is feasible. For a third case with $\alpha_{k-1} = (1,1)$ and $\beta_{k-1} = (0,1)$, an event change is not possible, because there are only two competing events, and the relative remaining lifetimes of both are unchanged by a change in c.

Case B: $\alpha_{k-1} = (1,1) \wedge \beta_{k-1} = (0,0)$. According to Theorem 2, an event change is possible only if $H_{m+1} < k-1 < O_{m+1}$ and $e_{k-1} = R_2$ or $e_{k-1} = R_1$. As a result, either $r_{1k-1}(c+\Delta) = r_{1k-1}(c) - \Delta \wedge r_{2k-1}(c+\Delta) = r_{2k-1}(c)$ or $r_{1k-1}(c+\Delta) = r_{1k-1}(c) \wedge r_{2k-1}(c+\Delta) = r_{2k-1}(c) + \Delta$. The only event change is: $e_k(c+\Delta) = F_1 \wedge e_k(c) = F_2$ under conditions $r_{1k}(c) - \Delta < r_{2k}(c) < r_{1k}(c)$ or $r_{2k}(c) < r_{1k}(c) < r_{2k}(c) + \Delta$. Since $\beta_{k-1} = (0,0)$, i.e. $0 < x_k < c$ and, for small enough Δ, $e_{k+1}(c+\Delta) = F_2 \wedge e_{k+1}(c) = F_1$. In the limiting case, i.e. $\Delta \to 0$, $r_{ik-1}(c+\Delta) = r_{ik-1}(c)$ and the two sample paths PP and DNP become identical everywhere except at time t_k where F_1F_2 occurs in PP and F_2F_1 occurs in DNP. As a result, $L^{PP} - L^{DNP} = 0$.

By similar reasoning, results for the remaining cases can be established, and we summarize all the cases here:

- Case A: $\alpha_{k-1} \wedge \beta_{k-1} = (1,0) \wedge (1,0), (0,1) \wedge (0,1)$:

  No event change possible.

- Case B: $\alpha_{k-1} = (1,1) \wedge \beta_{k-1} = (0,0)$

  $e_k(c+\Delta) = F_1 \wedge e_k(c) = F_2$ and $L^{PP} - L^{DNP} = 0$

- Case C: $\alpha_{k-1} = (1,1) \wedge \beta_{k-1} = (1,0)$:

  $e_k(c+\Delta) = F_1 \wedge e_k(c) = F_2$ and $L^{PP} - L^{DNP} \neq 0$

- Case D: $\alpha_{k-1} = (0,0) \wedge \beta_{k-1} = (x,x), x \in \{0,1\}$:

  $e_k(c+\Delta) = R_1 \wedge e_k(c) = R_2$ and $L^{PP} - L^{DNP} = 0$

- Case E: $\alpha_{k-1} = (0,1) \wedge \beta_{k-1} = (0,0)$ or $\alpha_{k-1} = (0,1) \wedge \beta_{k-1} = (1,0)$:

  (i) $e_k(c+\Delta) = R_1 \wedge e_k(c) = F_2$ and $L^{PP} - L^{DNP} = 0$;

  (ii) $e_k(c+\Delta) = R_1 \wedge e_k(c) = BE$ and $L^{PP} - L^{DNP} = 0$;

- Case F: $\alpha_{k-1} = (1,0) \wedge \beta_{k-1} = (0,x), x \in \{0,1\}$:

  (i) $e_k(c+\Delta) = R_2 \wedge e_k(c) = BF$ and $L^{PP} - L^{DNP} = 0$;

  (ii) If $x = 0$, $e_k(c+\Delta) = F_1 \wedge e_k(c) = R_2$ and $L^{PP} - L^{DNP} = 0$ (only $x = 0$ is possible);

  (iii) $e_k(c+\Delta) = F_1 \wedge e_k(c) = BF$ and $L^{PP} - L^{DNP} \neq 0$;

Figure 2 HERE

Figure 2: A critical event change

- Case G: Interchange of the last event and the end of horizon:

  $e_{N(t)}(c+\Delta) > t \wedge e_{N(t)}(c) < t$ and $L^{PP} - L^{DNP} = 0$.

Thus, only event changes of Case C and Case F(iii) need to be further considered, i.e., these correspond to the critical event changes. These event changes are important, since machine M_1 is blocked in state s_k, and the event F_1 is suspended until the repair of M_2 in DNP, whereas it is under repair in PP (see Figure 2).

The characterization for smoothing the discrete changes is the set of all random variables except X_{1k*}:

$$z_k = \{X_{in}, \forall i = 1, 2 \wedge n \geq 1\} \backslash \{X_{1k*}\},$$

where X_{1k*} is the newest time to failure of M_1 sampled prior to e_k. As a result, the probability rate for event change can be determined as follows:

Case C: $e_k(c+\Delta) = F_1 \wedge e_k(c) = F_2$ implies that $r_{1k-1}(c+\Delta) = r_{2k-1}(c+\Delta)$ and $r_{2k-1}(c) < r_{1k-1}(c)$. From Theorem 2, $e_k(c+\Delta) = F_1 \Leftrightarrow r_{1k-1}(c) - \Delta \leq r_{2k-1}(c)$ and $e_k(c) = F_2 \Leftrightarrow r_{2k-1}(c) < r_{1k-1}(c)$. Since $X_{ik} = a_{ik} + r_{ik}$, $e_k(c+\Delta) = F_1 \Leftrightarrow X_{1k-1}(c) = a_{1k-1}(c) + r_{2k-1}(c) + \Delta$ and $e_k(c) = F_2 \Leftrightarrow X_{1k}(c) > a_{1k-1}(c) + r_{2k-1}(c)$. As a result, the rate of event change is

$$\begin{aligned}\frac{dP_{z_k}}{dc} &= \lim_{\Delta \to 0} \frac{P(X_{1k} \leq a_{1k-1} + r_{2k-1} + \Delta | X_{1k} > a_{1k-1} + r_{2k-1})}{\Delta} \\ &= \frac{f_1(a_{1k-1} + r_{2k-1})}{1 - F_1(a_{1k-1} + r_{2k-1})}.\end{aligned}$$

Case F(iii): By similar arguments as in case C,

$$\begin{aligned}\frac{dP_{z_k}}{dc} &= \lim_{\Delta \to 0} \frac{P(X_{1k} \leq a_{1k-1} + +c + \Delta - x_{k-1} | X_{1k} > a_{1k-1} + c + x_{k-1})}{\Delta} \\ &= \frac{f_1(a_{1k-1} + c - x_{k-1})}{1 - F_1(a_{1k-1} + c - x_{k-1})}.\end{aligned}$$

Since NP and DNP_k are identical up to t_k with $e_k = BF$ for case F(iii) and $e_k = F_2$ for case C, then it holds for DNP_k and NP that $a_{1k} = a_{1k-1} + c - x_{k-1}$ in case F(iii) and $a_{1k} = a_{1k-1} + r_{2k-1}$ in case C. As a result, it holds in both cases that:

$$\frac{dP_{z_k}}{dc} = \frac{f_1(a_{1k})}{1 - F_1(a_{1k})}.$$

Combining the above results, we obtain the following gradient estimator:

$$\frac{dL_t(c)}{dc} + \sum_{k \in BFS} \left(L^{PP,k} - L^{DNP,k}\right) \frac{f_1(a_{1k})}{1 - F_1(a_{1k})}. \tag{2}$$

where $BFS = \{k | \alpha_k = (1,0) \wedge x_k = c\}$ is the set of blocking states.

In order to establish unbiasedness of the estimator we define some sets of sample paths that are characterized by their behavior when a perturbation of size Δ is introduced. Denote a sample path by ω. For $k = 1, 2, ..., n = N(t)$, let

$$\begin{aligned}
\mathcal{U}_k(\Delta) &= \{\omega : e_1(c+\Delta) = e_1(c), \cdots, e_k(c+\Delta) = e_k(c)\},\\
\mathcal{V}_k(\Delta) &= \{\omega : e_1(c+\Delta) = e_1(c), \cdots, e_k(c+\Delta) \neq e_k(c)\}.
\end{aligned}$$

By definition, $\mathcal{V}_k = \mathcal{U}_{k-1} \backslash \mathcal{U}_k$. The set $\mathcal{U}_k$ contains sample paths that experience no change in their event sequence through the kth transition, due to the introduction of a perturbation of size Δ. In particular, the set $\mathcal{U}_n$ contains sample paths that experience no change in their *entire* event sequence due to the introduction of a perturbation of size Δ. On the other hand, the set $\mathcal{V}_k$ contains sample paths in which the first change occurs in the event sequence at the kth transition. Since ω is usually understood, its explicit display will henceforth be omitted except when used in defining new sets of sample paths. Also, the dependence of $\mathcal{U}_k$ and $\mathcal{V}_k$ on Δ is usually omitted for notational brevity. To take into account the time horizon, we further partition the set of sample paths as follows:

$$\begin{aligned}
\mathcal{A} &= \{\omega \in \mathcal{U}_{N(t,c)} : N(t, c+\Delta) = N(t, c)\},\\
\mathcal{B}_k &= \{\omega \in \mathcal{V}_k : k \leq N(t, c)\},\\
\mathcal{C} &= \{\omega \in \mathcal{U}_{N(t,c)} : N(t, c+\Delta) > N(t, c)\}.
\end{aligned}$$

Since $\mathcal{A}$, $\mathcal{B}_k$, $k = 1, 2, ...,$ and $\mathcal{C}$ partition the set of possible sample paths, by conditioning on whether or not Δ causes a change in the event sequence, we write

$$\begin{aligned}
E[L_t(c+\Delta)] - E[L_t(c)] &= E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathcal{A})] + E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathcal{C})]\\
&\quad + \sum_{k=1}^{\infty}(E[L_t(c+\Delta)\mathbf{1}(\mathcal{B}_k)] - E[L_t(c)\mathbf{1}(\mathcal{B}_k)]).
\end{aligned}$$

Dividing by Δ and then taking the limit, we have

$$\begin{aligned}
\frac{dE[L_t(c)]}{dc} &= \lim_{\Delta\to 0} \frac{E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathcal{A})]}{\Delta}\\
&\quad + \lim_{\Delta\to 0} \frac{E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathcal{C})]}{\Delta}\\
&\quad + \sum_{k=1}^{\infty} \lim_{\Delta\to 0} \frac{E[L_t(c+\Delta)\mathbf{1}(\mathcal{B}_k)] - E[L_t(c)\mathbf{1}(\mathcal{B}_k)]}{\Delta}.
\end{aligned}$$

We now establish

Theorem 6. If $F_1(\cdot)$ is Lipschitz continuous with Lipschitz constant K and density $f_1(\cdot)$, then the estimator given by (2) is an unbiased estimator for $dE[L_t(c)]/dc$.

Proof. We can further partition $\mathcal{B}_k$ into the various cases to consider, i.e., $\mathcal{B}_k^\alpha$, where α corresponds to a particular case. We need to show the following:

(a) $\lim_{\Delta\to 0} \frac{E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathcal{A})]}{\Delta} = E\left[\frac{dL_t(c)}{dc}\right];$

(b) $\lim_{\Delta\to 0} \frac{E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathcal{C})]}{\Delta} = 0;$

(c) $\lim_{\Delta\to 0} \frac{E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathcal{B}_k^{\alpha})]}{\Delta} = E\left[\frac{dP_{z_k}}{dc}(L_t^{PP,k(\alpha)} - L_t^{DNP,k(\alpha)})\right],$

where the (α) indicates the dependence on the case, $z_k = \{X_{in}, Y_{in}, i = 1, 2, n \geq 1\}\backslash\{X_{1k*}\}$, and X_{1k*} is the value of X_{1j} active at e_k. All three parts are established via the dominated convergence theorem.

A bound for part (a), the IPA portion of the estimator, can be established using Theorem 4, since $|v_t| \leq 2, \alpha_{2t}\gamma_{2t} \leq 1$, and from the definition of a cycle, it takes at least c time units to go from empty to full and c time units to go from full to empty, so $Q_{N(t)}/t \leq 1/2c$, which leads to

$$E\left[\sup_{c\in(c_{min},c_{max})}\left|\frac{L_t(c+\Delta) - L_t(c)}{\Delta}\mathbf{1}(\mathcal{A})\right|\right] \leq \frac{1}{2c_{min}} + \frac{2}{t},$$

where c_{min} and c_{max} denote the lower and upper bounds, respectively, of the buffer capacity. Applying the dominated convergence theorem completes the proof of part (a).

For part (b), let us first notice that $t_{k+1} = t_k + \tau(r_k, x_k, \alpha_k, \gamma_k)$, where τ is a continuous function of r_k and x_k. For all $\omega \in \mathcal{U}_k$, Theorem 2 implies that $t_k(c+\Delta) \to t_k(c)$, $r_k(c+\Delta) \to r_k(c)$, $x_k(c+\Delta) \to x_k(c)$. Further, $\omega \in \mathcal{U}_k$ implies that $\alpha_k(c+\Delta) = \alpha_k(c)$ and $\gamma_k(c+\Delta) = \gamma_k(c)$. As a result, for all $\omega \in \mathcal{C}$, $t_{N(t,c)+1}(c+\Delta) \to t_{N(t,c)+1}(c)$. By definition, $t_{N(t,c)+1}(c) > t$, which implies that $t_{N(t,c)+1}(c+\Delta) > t$ for Δ small enough. Therefore,

$$E\left[\lim_{\Delta\to 0} \frac{(L_t(c+\Delta) - L_t(c))\,\mathbf{1}(C)}{\Delta}\right] = 0.$$

The dominated convergence theorem will be used to conclude part (b). To this end, from Theorem 2, we have $t_k(c+\Delta) \geq t_k(c) - (Q_k+1)\Delta$, $r_{ik}(c+\Delta) = r_{ik}(c) + u_{ik}\Delta$, and $x_k(c+\Delta) = x_k(c) + u_{0k}\Delta$, with $u_{ik} \in \{-1, 0, 1\}$ and $k = N(t, c)$. Hence,

$$P_{2k}(c+\Delta) = P_{2k}(c) - u_{2k}\Delta.$$

Furthermore, $\omega \in \mathcal{U}_{N(t,c)}$ implies that $\alpha_t(c+\Delta)$ and $\gamma_t(c+\Delta)$ for $t \in (t_k(c+\Delta), t_k(c+\Delta)+t-t_k(c)-\Delta)$ are equal to $\alpha_t(c)$ and $\gamma_t(c)$ for $t \in (t_k(c), t)$. As a result,

$$\begin{aligned} P_{2t}(c+\Delta) &\leq P_{2k}(c+\Delta) + \alpha_{2k}\gamma_{2k}(t - t_{N(t,c)} - \Delta) + (Q_k+1)\Delta + \Delta \\ &= P_{2t}(c) - u_{2k}\Delta - \alpha_{2k}\gamma_{2k}\Delta + (Q_k+1)\Delta + \Delta \\ &\leq P_{2t}(c) + (Q_k+3)\Delta. \end{aligned}$$

Therefore,

$$0 \leq \frac{(L_t(c+\Delta) - L_t(c))\,\mathbf{1}(C)}{\Delta} \leq \frac{Q_t+3}{t} \leq \frac{1}{2c} + \frac{3}{t}.$$

The proof of part (b) is then established by the dominated convergence theorem.

For part (c), we will just consider one α here, Case F(iii), which we will denote by $\alpha = F3$:

$$\mathcal{B}_k^{F3} = \{\omega \in \mathcal{B}_k : e_k = BF, a_{1k-1} + c - x_{k-1} < X_{1k*} \leq a_{1k-1} + c + \Delta - x_{k-1}\},$$

where X_{1k*} corresponds to the time to failure at M1 sampled prior to e_k. We note that $e_k = BF$ implies $\alpha_{k-1} = (1, 0), \beta_{k-1} = (0, 0)$. The other cases are analogous.

Specifically, we need to show

$$\lim_{\Delta \to 0} \frac{E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathcal{B}_k^{F3})]}{\Delta}$$
$$= E\left[\frac{f_1(a_{1k} + c - x_{k-1})}{1 - F_1(a_{1k} + c - x_{k-1})}\mathbf{1}\{X_{1k*} > a_{1k-1} + c - x_{k-1}\}(L_t^{PP,k(F3)} - L_t^{DNP,k(F3)})\right],$$

where

$$\begin{aligned} L_t^{PP,k(F3)} &= \lim_{\Delta \to 0} E[L_t(c+\Delta)\mathbf{1}\{e_k = BF\}|z_k, a_{1k-1} + c - x_{k-1} < X_{1k*} \leq a_{1k-1} + c + \Delta - x_{k-1}]\mathbf{1}(\mathcal{U}_{k-1}), \\ L_t^{DNP,k(F3)} &= \lim_{\Delta \to 0} E[L_t(c)\mathbf{1}\{e_k = BF\}|z_k, a_{1k-1} + c - x_{k-1} < X_{1k*} \leq a_{1k-1} + c + \Delta - x_{k-1}]\mathbf{1}(\mathcal{U}_{k-1}). \end{aligned}$$

We consider $E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathcal{B}_k^{F3})], k = 1, \ldots, n$. First we rewrite it as

$$E[E[(L_t(c+\Delta) - L_t(c))\mathbf{1}(\mathcal{B}_k^{F3})|z_k]],$$

and consider the inner conditional expectation term.

$$\begin{aligned} &|E[L_t(c+\Delta)\mathbf{1}(\mathcal{B}_k^{F3})|z_k]| \\ &\quad = |E[L_t(c+\Delta)\mathbf{1}\{e_k = BF\}|z_k, a_{1k-1} + c - x_{k-1} < X_{1k*} \leq a_{1k-1} + c + \Delta - x_{k-1}] \\ &\qquad \mathbf{1}(\mathcal{U}_{k-1})P(a_{1k-1} + c - x_{k-1} < X_{1k*} \leq a_{1k-1} + c + \Delta - x_{k-1})| \\ &\quad = |E[L_t(c+\Delta)\mathbf{1}\{e_k = BF\}|z_k, a_{1k-1} + c - x_{k-1} < X_{1k*} \leq a_{1k-1} + c + \Delta - x_{k-1})] \\ &\qquad \mathbf{1}(\mathcal{U}_{k-1})(F_1(a_{1k-1} + c + \Delta - x_{k-1})) - F_1(a_{1k-1} + c - x_{k-1}))| \\ &\quad \leq K\Delta E[L_t(c+\Delta)|z_k, a_{1k-1} + c - x_{k-1} < X_{1k*} \leq a_{1k-1} + c + \Delta - x_{k-1}]. \end{aligned}$$

Also, since $P_{2t} \leq t$, we have

$$E\left[\sup_c E[L_t(c+\Delta)|z_k, a_{1k-1} + c - x_{k-1} < X_{1k*} \leq a_{1k-1} + c + \Delta - x_{k-1}]\right] \leq 1,$$

so once again invoking the dominated convergence theorem, we have

$$\begin{aligned} &\lim_{\Delta \to 0} \frac{E[L_t(c+\Delta)\mathbf{1}(\mathcal{B}_k^{F3})]}{\Delta} \\ &= E\left[\lim_{\Delta \to 0} \frac{E[L_t(c+\Delta)\mathbf{1}(\mathcal{B}_k^{F3})|z_k]}{\Delta}\right] \\ &= E\left[\lim_{\Delta \to 0} \frac{(F_1(a_{1k-1} + c + \Delta - x_{k-1}) - F_1(a_{1k-1} + c - x_{k-1}))}{\Delta}\right. \\ &\qquad \left.\times \lim_{\Delta \to 0} E\left[L_t(c+\Delta)\mathbf{1}\{e_k = BF\}|z_k, a_{1k-1} + c - x_{k-1} < X_{1k*} \leq a_{1k-1} + c + \Delta - x_{k-1}\right]\mathbf{1}(\mathcal{U}_{k-1})\right] \\ &= E\left[f_1(a_{1k-1} + c - x_{k-1})L_t^{PP,k(F3)}\right] \\ &= E\left[\frac{f_1(a_{1k-1} + c - x_{k-1})}{1 - F_1(a_{1k-1} + c - x_{k-1})}\mathbf{1}\{X_{1k*} > a_{1k-1} + c - x_{k-1}\}L_t^{PP,k(F3)}\right]. \end{aligned}$$

Similarly, we can show

$$\lim_{\Delta c \to 0} \frac{E[L_t(c)\mathbf{1}(\mathcal{B}_k^{F3})]}{\Delta} = E\left[\frac{f_1(a_{1k-1}+c-x_{k-1})}{1-F_1(a_{1k-1}+c-x_{k-1})}\mathbf{1}\{X_{1k*} > a_{1k-1}+c-x_{k-1}\}L_t^{DNP,k(F3)}\right].$$

□

5 SPA ANALYSIS: INFINITE HORIZON CASE

5.1 Regenerative case

From Theorem 4, the IPA term in (2) is given as follows:

$$\frac{dL_t(c)}{dc} = \frac{\alpha_{2t}\gamma_{2t}}{t/Q_{N(t)}} + \frac{v_t}{t}, \tag{3}$$

where γ_{2t} and v_t are defined in Theorem 4. Note that $t/Q_{N(t)}$ is the average length of an operation cycle defined in Section 3, and $\alpha_{2t}\gamma_{2t} = 1$ if and only if machine M_2 is producing at time t. We assume that the steady state exists; more precisely, we make the following assumptions:

(A2) There exists a finite $C > 0$ such that $\lim_{t\to\infty} t/Q_{N(t)} = C$ w.p. 1.

(A3) $\lim_{t\to\infty} E[\alpha_{2t}\gamma_{2t}] = L$ with $L = \lim_{t\to\infty} P_{2t}/t$ w.p. 1.

Lemma 1. Under assumptions (A2)-(A3), $\lim_{t\to\infty} E[dL_t(c)/dc] = L/C$.

Proof. From relation (3),

$$E\left[\frac{dL_t(c)}{dc}\right] = \frac{E\left[\alpha_{2t}\gamma_{2t}\right]}{C} + E\left[\alpha_{2t}\gamma_{2t}\left(\frac{Q_{N(t)}}{t} - \frac{1}{C}\right)\right] + E\left[\frac{v_t}{t}\right].$$

Since $|v_t| \le 2, E[v_t/t] \to 0$. From assumption (A2), $w_t = \alpha_{2t}\gamma_{2t}(Q_{N(t)}/t - 1/C) \to 0$ with probability 1. Since $\alpha_{2t}\gamma_{2t} \le 1$, $|w_t| \le Q_{N(t)}/t + 1/C$. From the definition of a cycle, it takes at least c time units to go from empty to full and c time units to go from full to empty, $Q_{N(t)}/t \le 1/2c$, which leads to $|w_t| \le 1/2c + 1/C$. Applying the dominated convergence theorem yields $E[w_t] \to 0$. The lemma is then established by combining the above results and assumption (A3). □

From this result, $L_t/(t/Q_{N(t)})$ is a strongly consistent estimator of $\lim_{t\to\infty} E[dL_t(c)/dc]$. This leads to the following strongly consistent estimator of $dL(c)/dc$:

$$\frac{L_t}{t/Q_{N(t)}} + \sum_{k\in BFS}\left(L^{PP,k} - L^{DNP,k}\right)\frac{f_1(a_{1k})}{1-F_1(a_{1k})}. \tag{4}$$

A rigorous proof of strong consistency, though not carried out explicitly here, can be established under the regenerative assumption below along the lines used in Fu and Hu (1997), since we will show that

$$\lim_{t\to\infty}\left(L^{PP,k} - L^{DNP,k}\right)$$

is a function of r_{2k} only, so the estimator (asymptotically) depends only on a_{1k} and r_{2k} within a regenerative cycle. This regenerative property of the estimator, along with some technical assumptions, are the main conditions required in such a proof.

Figure 3 HERE

Figure 3: Possible regeneration points

Figure 4 HERE

Figure 4: M_1 still fails at t_{k+1} in PP

The main difficulty in estimating (4) using simulation is the computation of $L^{PP,k} - L^{DNP,k}$. Its evaluation for the general case is still an open issue. In the following we consider the regenerative case and make the following assumption:

(A4) The underlying stochastic process of the two-machine line has a regeneration point **sr**.

The points A, B, C and D of Figure 3 are possible regeneration points depending on the distribution of times to failure. If the time to failure X_1 of machine M_1 has a phase-type distribution, the points A and D can be used to define regeneration points by extending the definition of a machine state to include the phase. Similarly, if the time to failure X_2 of machine M_2 has a phase-type distribution, the points B and C can be used to defined regeneration points. Of course, A and D are regeneration points if X_1 is exponentially distributed, and B and C are regeneration points if X_2 is exponentially distributed.

The estimation of the gradient estimator requires the estimation of $L^{PP,k} - L^{DNP,k}$. We recall the relationship between PP_k and DNP_k. PP_k is identical to DNP_k up to time t_k- where an event change occurs. At this instant, machine M_1 is blocked in the nominal system, whereas it breaks down in the perturbed system. Clearly, M_1 immediately fails in DNP_k following the repair of M_2 at time t_{k+1}. Furthermore, $x_t = c$ at time $t = t_{k+1}$ for both PP_k and DNP_k. Two cases are possible concerning the state of machine M_1 in PP at t_{k+1} (from DNP): (i) M_1 still failed at t_{k+1} or (ii) it is repaired at t_{k+1} (see Figures 4 and 5). Case (i) occurs w.p. $\overline{G_1}(r_{2k})$ and case (ii) occurs w.p. $G_1(r_{2k})$. Finally, since $e_k = BF$ in both DNP_k and NP, then $e_{k+1} = R_2$ in both DNP_k and NP. As a result, t_{k+1} is identical for both DNP_k and NP, hence r_{2k} can be taken form NP.

To summarize, the inventory trajectory and the cumulative production are identical for both PP_k and DNP_k up to time $t = t_{k+1}$ (from NP). At this point, $x_t = c$ and M_2 is just repaired in both PP_k and DNP_k. M_1 breaks down in DNP_k while it is under repair with age r_{2k} with probability $\overline{G_1}(r_{2k})$ and is just repaired with probability $G_1(r_{2k})$ in PP_k.

In order to estimate $L^{PP,k} - L^{DNP,k}$, let us consider the following construction of PP_k and DNP_k (see Figure 6). For DNP_k, a piece of sample path I is inserted at time t_{k+1} by starting with appropriate initial state defined above and this piece of sample path stops when the regeneration point **sr** is met. Similarly, for PP_k, a piece of sample path II is inserted at time t_{k+1} until the regeneration point **sr** is met. From there

Figure 5 HERE

Figure 5: M_1 is repaired at t_{k+1} in PP

Figure 6 HERE

Figure 6: Construction of PP and DNP

on, both DNP_k and PP_k pursue the same sample path III.

Let

$t_I^{PP,k}, P_I^{PP,k}$: length and cumulative production of sample path I,

$t_{II}^{DNP,k}, P_{II}^{DNP,k}$: length and cumulative production of sample path II,

$t_{III} = t - t_{k+1}$, P_{III}: length and cumulative production of sample path III.

Under this construction, we have for large t,

$$L_t^{PP,k} \approx L_{t+t_{II}^{PP,k}}^{PP,k} = \frac{P_{2,t+t_{II}^{PP,k}}^{PP,k}}{t + t_{II}^{PP,k}} = \frac{P_{2,t_{k+1}} + P_{II}^{PP,k} + P_{III}}{t + t_{II}^{PP,k}},$$

$$L_t^{DNP,k} \approx L_{t+t_I^{DNP,k}}^{DNP,k} = \frac{P_{2,t+t_I^{DNP,k}}^{DNP,k}}{t + t_I^{DNP,k}} = \frac{P_{2,t_{k+1}} + P_I^{DNP,k} + P_{III}}{t + t_I^{DNP,k}},$$

leading to

$$\begin{aligned} L_t^{PP,k} - L_t^{DNP,k} &\approx \frac{t_I^{DNP,k} P_{2,t+t_{II}^{PP,k}}^{PP,k} - t_{II}^{PP,k} P_{2,t+t_I^{DNP,k}}^{DNP,k} + t\left(P_{II}^{PP,k} - P_I^{DNP,k}\right)}{\left(t + t_{II}^{PP,k}\right)\left(t + t_I^{DNP,k}\right)} \\ &= \frac{t_I^{DNP,k} L_{t+t_{II}^{PP,k}}^{PP,k}}{t + t_I^{DNP,k}} - \frac{t_{II}^{PP,k} L_{t+t_I^{DNP,k}}^{DNP,k}}{t + t_{II}^{PP,k}} + t \frac{P_{II}^{PP,k} - P_I^{DNP,k}}{\left(t + t_{II}^{PP,k}\right)\left(t + t_I^{DNP,k}\right)} \\ &\approx \frac{L_t}{t}(t_I^{DNP,k} + t_{II}^{PP,k}) + \frac{1}{t}\left(P_{II}^{PP,k} - P_I^{DNP,k}\right). \end{aligned}$$

Therefore, we take the following as our long-run gradient estimator:

$$\begin{aligned} \text{SPA0} = \frac{L_t}{t/Q_{N(t)}} + \frac{L_t}{t} \sum_{k \in BFS} \frac{f_1(a_{1k})}{1 - F_1(a_{1k})} \left(t_I^{DNP,k} - t_{II}^{PP,k}\right) \\ + \frac{1}{t} \sum_{k \in BFS} \frac{f_1(a_{1k})}{1 - F_1(a_{1k})} \left(P_{II}^{PP,k} - P_I^{DNP,k}\right). \end{aligned} \tag{5}$$

When computing the above estimator using simulation, all terms except the two summations can be evaluated easily. Whenever an event e_k leading to a blocking state, extra simulation is performed to construct sample paths I and II as described above. We then update these two summations. The estimator (5) can be obtained at the end of the simulation accordingly.

5.2 A particular case

In this subsection, we assume that Y_{1k} and X_{2k} are exponentially distributed and derive strongly consistent gradient estimators computable without extra simulation. As shown above, the estimation of $L^{PP,k} - L^{DNP,k}$ requires the consideration of two cases : (i) M_1 still fails at t_{k+1} or (ii) it is repaired at t_{k+1}. Case (i) occurs w.p. $\overline{G_1}(r_{2k})$ and case (ii) occurs w.p. $G_1(r_{2k})$. By using different construction of sample paths, we prove in the following that

$$L^{PP,k} - L^{DNP,k} = 0$$

for case (i) and

$$L^{PP,k} - L^{DNP,k} \approx \frac{1 - L_t/E_2}{\lambda_1 t}$$

for case (ii), where E_2 is the isolated average throughput rate of M_2 given by $E_2 = \mu_2/(\lambda_2 + \mu_2)$. Our final gradient estimator for dL/dc is the following:

$$\text{SPA1} = \frac{L_t}{t/Q_{N(t)}} + \frac{1 - L_t/E_2}{\lambda_1 t} \sum_{k \in BFS} G_1(r_{2k}) \frac{f_1(a_{1k})}{1 - F_1(a_{1k})}. \tag{6}$$

Of course, if X_{1k} are exponentially distributed as well, then $f_1(x)/(1 - F_1(x)) = \lambda_1$ for all $x \geq 0$, and the gradient estimator becomes:

$$\frac{L_t}{t/Q_{N(t)}} + \frac{1 - L_t/E_2}{t} \sum_{k \in BFS} G_1(r_{2k}).$$

If all random variables are exponentially distributed, r_{2k} are exponentially distributed with mean equal to $E[Y_{2k}]$. Hence, case (ii) occurs w.p. $\mu_1/(\mu_1+\mu_2)$ and the gradient estimator becomes:

$$\text{SPA2} = \frac{L_t}{t/Q_{N(t)}} + (1 - L_t/E_2) \frac{\mu_1}{\mu_1 + \mu_2} \frac{\#(t, BF)}{t}, \tag{7}$$

where $\#(t, BF)$ denotes the number of blocking states up to time t.

Let us consider now the estimation of $L^{PP,k} - L^{DNP,k}$. Appropriate sample path constructions will be used, and Figures 4 and 5 are helpful for understanding what follows. If M_1 still fails at t_{k+1} in PP_k, the states of the machines and the buffer level are the same at t_{k+1} in both DNP_k and PP_k. We construct the portion of sample path following t_{k+1} as follows. At time t_{k+1}, new samples of Y_{1k} and X_{2k} is generated for setting the time to repair of M_1 and the time to failure of M_2 in DNP_k. At this point, we also use the same samples to reset the time to repair of M_1 and to set the time to failure of M_2 in PP_k. As a result, the state of the system at time t_{k+1}^+ is the same in PP_k and DNP_k. Let us notice that resetting the time to repair of M_1 in PP_k is possible due to the exponential distribution of Y_{1k}. As a result of above construction, we have: $L^{PP,k} - L^{DNP,k} = 0$.

If M_1 is repaired at t_{k+1} in PP_k, then an independent portion of sample path is inserted in PP_k until machine M_1 fails. At this point, the portion of DNP_k following t_{k+1}^+ is added to construct PP_k. Clearly the correctness of this construction is due to the exponential distribution of X_{2k}. For the inserted portion, let $\tilde{X}_1$ be time to failure of M_1, N be the number of failures of M_2 and $\tilde{Y}_{2k}$ the related times to repair. Let T be the length of the inserted portion. Since in the inserted portion, M_1 is blocked whenever M_2 fails. Thus,

$$T = \widetilde{X}_1 + \sum_{k=1}^{N} \widetilde{Y}_{2k},$$

and the production of M_2 during the inserted portion is equal to $\tilde{X}_1$. As a result, for large t,

$$L_t^{PP} - L_t^{DNP} \approx L_{t+T}^{PP} - L_t^{DNP} = \frac{P_{2t}^{DNP} + \widetilde{X}_1}{t+T} - \frac{P_{2t}^{DNP}}{t} = \frac{\widetilde{X}_1 - TL_t^{DNP}}{t+T} \approx \frac{\widetilde{X}_1 - TL_t}{t}.$$

By taking expectation with respect to random variables of the inserted portion, we have

$$E_{\widetilde{X},\widetilde{Y}}\left[L_t^{PP}-L_t^{DNP}\right]\approx\frac{E[\widetilde{X}_1]-E[T]L_t}{t}.$$

Notice that $\{\widetilde{Y}_{2k}\}$ are independent of N. Hence,

$$E\left[T\right]=E\left[\widetilde{X}_1\right]+E\left[\sum_{k=1}^{N}\widetilde{Y}_{2k}\right]=E\left[\widetilde{X}_1\right]+E\left[N\right]E\left[\widetilde{Y}_{2k}\right].$$

Since $\{X_{2k}\}$ are exponentially distributed, N follows a Poisson distribution, so:

$$E\left[N\right]=E\left[E\left[N|\widetilde{X}_1\right]\right]=E\left[\lambda_2\widetilde{X}_1\right]=\lambda_2E\left[\widetilde{X}_1\right].$$

The combination of the above results gives:

$$E_{\widetilde{X},\widetilde{Y}}\left[L_t^{PP}-L_t^{DNP}\right]\approx\frac{1-L_t/E_2}{\lambda_1 t}.$$

6 NUMERICAL RESULTS

We compared the numerical properties of the various estimators by performing simulation experiments on two examples. The biased IPA estimator and the three SPA estimators – SPA0 given by (5), SPA1 given by (6), SPA2 given by (7) – are compared with symmetric difference (SD) estimates using common random numbers and $\Delta c = 0.05$. In all three cases, sample means and 95% confidence half-widths based on 20 independent replications are calculated.

Example 1: All random variables are exponentially distributed with $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$. Analytical results are available, which can be used to assess the convergence of the various estimators. The throughput rate can be found in Dallery and Gershwin (1992) and is equal to $L = \left(2+(1+c)^{-1}\right)^{-1}$, leading to $dL/dc = (3+2c)^{-2}$. The simulation results are summarized in Table 1, and they confirm that the IPA estimator is biased, whereas all three SPA estimators converge to the correct value. SPA2 and SPA1 seem to have a similar convergence rate that is substantially faster than SPA0, the estimator based on regenerative analysis. The SD estimates fare worse than SPA1/SPA2, but slightly better than SPA0.

Example 2: In this example, Y_{1k} and X_{2k} are exponentially distributed, X_{1k} and Y_{2k} have two-stage Erlang distributions. All random variables have mean equal to one, i.e., $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$. Only the case $c = 1$ is considered, and the results are shown in Table 2. For this example, an analytical solution is not available. SPA0 and SPA1 appear to converge to the same limiting value that is consistent with the SD estimates, whereas the results indicate that SPA2 is biased for this example, which is not surprising, since not all the random variables are exponentially distributed. Again, the convergence rate of SPA0 appears to be substantially slower than that of SPA1, and the SD estimates fall between the two.

7 CONCLUSION

In this paper, we have considered a continuous production line with two machines subject to operation-dependent failures, where failure and repair times have general probability distributions. Estimation of the derivative of the throughput rate with respect to the buffer capacity has been addressed. Both IPA estimators and SPA estimators were proposed. Simulation results confirm that IPA does not work, which contradicts a common belief that IPA always works for continuous flow models.

c	t	L_t	L	dL/dc	IPA	SPA2	SPA1	SPA0	SD
.5	1000	.3722	.375	.0625	.0304	.0623	.0625	.0658	.0573
		(.0045)			(.0010)	(.0017)	(.0018)	(.0071)	(.0075)
	10000	.3740	.375	.0625	.031	.0624	.0625	.0638	.0625
		(.0014)			(.0003)	(.0004)	(.0005)	(.0034)	(.0028)
	100000	.3749	.375	.0625	.0312	.0625	.0626	.0629	.0620
		(.0005)			(.0001)	(.0001)	(.0001)	(.0009)	(.0010)
	1000000	.3751	.375	.0625	.0313	.0625	.0625	.0624	.0625
		(.0001)			(.00003)	(.00005)	(.00004)	(.0003)	(.0002)
1	1000	.4000	.4	.04	.0197	.0390	.0393	.0432	.0366
		(.0050)			(.0008)	(.0015)	(.0015)	(.0105)	(.0059)
	10000	.4001	.4	.04	.0200	.0399	.0399	.0387	.0386
		(.0017)			(.0003)	(.0004)	(.0004)	(.0033)	(.0024)
	100000	.4002	.4	.04	.0200	.0399	.0399	.0402	.0404
		(.0005)			(.00006)	(.0002)	(.00017)	(.0008)	(.0006)
	1000000	.3999	.4	.04	.0200	.0400	.0400	.0399	.0400
		(.0001)			(.00003)	(.00004)	(.00004)	(.0002)	(.0002)
2	1000	.4276	.4286	.02041	.01028	.02081	.02090	.02582	.02114
		(.0048)			(.00064)	(.00110)	(.00117)	(.01366)	(.00581)
	10000	.4274	.4286	.02041	.01022	.02076	.02079	.01980	.02030
		(.0016)			(.00020)	(.00041)	(.00043)	(.00309)	(.00153)
	100000	.4284	.4286	.02041	.01018	.02040	.02039	.02016	.02031
		(.0005)			(.00005)	(.00010)	(.00010)	(.00124)	(.00042)
	1000000	.4285	.4286	.02041	.01021	.02043	.02043	.02063	.02038
		(.0002)			(.00001)	(.00003)	(.00003)	(.00047)	(.00019)

Table 1: Simulation results for Example 1 (95% confidence half-widths in parentheses)

t	L_t	IPA	SPA2	SPA1	SPA0	SD
1000	.4090	.02120	.04047	.04308	.04748	.04046
	(.0040)	(.00076)	(.00119)	(.00146)	(.01112)	(.00512)
10000	.4104	.02148	.04021	.04295	.04228	.04447
	(.0010)	(.00015)	(.00026)	(.00028)	(.00271)	(.00200)
100000	.4101	.02155	.04041	.04308	.04290	.04318
	(.0004)	(.00008)	(.00008)	(.00009)	(.00092)	(.00048)
1000000	.4100	.02153	.04036	.04304	.04319	.04303
	(.0001)	(.00003)	(.00005)	(.00006)	(.00030)	(.00017)

Table 2: Simulation results for Example 2 (95% confidence half-widths in parentheses)

ACKNOWLEDGMENTS

Michael C. Fu was supported in part by the National Science Foundation under Grant INT-9402580.

References

[1] P. Brémaud, R.P. Malhamé, and L. Massoulié, "A manufacturing system with general stationary failure process: stability and IPA of hedging control policies," *IEEE Trans. Automatic Control*, vol. 42, pp. 155-170, 1997.

[2] J.A. Buzacott and J.G. Shanthikumar, *Stochastic Models of Manufacturing Systems*, Prentice-Hall, Englewood Cliff, NJ, 1992.

[3] M. Caramanis and G. Liberopoulos, "Perturbation analysis for the design of flexible manufacturing system flow controllers," *Operations Research*, vol. 40, pp. 1107-1125, 1992.

[4] Y. Dallery and S.B. Gershwin, "Manufacturing flow line systems: a review of models and analytical results," *Queueing Systems,* vol. 12, pp. 3 - 94, 1992.

[5] M.C. Fu and J.Q. Hu, *Conditional Monte Carlo: Gradient Estimation and Optimization Applications,* Kluwer Academic Publishers, Boston, 1997.

[6] M.C. Fu and X. Xie, "Perturbation Analysis of Two-Stage Continuous Transfer Lines Subject to Operation-Dependent Failures," *Proceedings of the Conference on Decision and Control*, 1998.

[7] A. Haurie, P. L'Ecuyer, and Ch. Van Delft, "Convergence of stochastic approximation coupled with perturbation analysis in a class of manufacturing flow control models," *Discrete Event Dynamic Systems: Theory and Applications,* vol. 4, pp. 87-111, 1994.

[8] E.L. Plambeck, B.-R. Fu, S.M. Robinson, and R. Suri, "Sample-path optimization of convex stochastic performance functions," *Mathematical Programming,* vol. 75, pp. 137-176, 1996.

[9] L. Shi, B.-R. Fu, and R. Suri, "Sample path analysis for continuous tandem production lines," *Discrete Event Dynamic Systems: Theory and Applications,* submitted, 1998.

[10] R. Suri and B.-R. Fu, "On using continuous flow lines to model discrete production lines," *Discrete Event Dynamic Systems: Theory and Applications,* vol. 4, pp. 127-169, 1994.

[11] Y. Wardi and B. Melamed, "IPA gradient estimation for loss measures in continuous flow models," submitted, 1996.

[12] X.L. Xie, under preparation.

[13] H. Yan, G. Yin, and S.X.C. Lou, "Using stochastic optimization to determine threshold values for the control of unreliable manufacturing systems," *J. Optimization Theory and Applications,* vol. 83, pp. 511-539, 1994.