Large Size Asymptotics for Crossbar Switches with Input Queueing

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Abstract

With the advent of high–speed networks, various switch architectures have been proposed in order to meet the increasingly stringent performance requirements that are being placed on the underlying switching systems. In general, the performance analysis of such a switch architecture is a difficult task mainly due to the fact that a switch consists of a large number of queues which interact with each other in a fairly complicated manner. In this paper, we analyze a crossbar switch with input queueing in terms of maximum throughput, and formalize the phenomenon that virtual queues formed by the head–of–line cells become decoupled as the switch size grows unboundedly large. We also establish various properties of the limiting queue size processes so obtained.

Key words: Crossbar switches; Input queueing; Asymptotics; Maximum throughput.

1 Introduction

Rapid advances in all aspects of telecommunications, especially in the areas of transmission systems and fiber optics, have led to the introduction of new switching technologies for enabling the future B–ISDN (Broadband Integrated Services Digital Networks). In particular, space–division packet switching has been recognized

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as a key component in this ongoing evolution towards B–ISDN and multiprocessor interconnects. This is due to the high capacity, viz., in the range 10–100 Gps, that space–division packet switching can achieve through the use of a highly parallel switching fabric with simple per packet processing distributed among many high–speed VLSI circuits [8].

In non–blocking space–division packet switches, it is always possible to establish a connection between any idle input and output pair. However, output contention arises when more than one cell at different input ports demand to be routed to the same output. As the contending cells cannot be placed on the output port at the same time, a buffer has to be provided somewhere in the switch to store the cell(s) which cannot be served. This is typical for an ATM switch as it statistically multiplexes cells, and buffering must be provided to ensure that ATM cells destined to the same output port can be stored and are not dropped. Several buffering strategies have been reported in the literature to mitigate the effects of contention [3, 6, 7, 13], with proposed solutions depending on a variety of factors such as the speed of input and output lines relative to the cell transfer time across the switching fabric, and implementation complexity.

Noteworthy among proposed buffering strategies is input queueing whereby cells are enqueued in a buffer located at each input port [6, 8]. More specifically, consider a non–blocking crossbar switch with K input and K output ports. The switch operates in a synchronous mode with time divided into consecutive slots of equal duration; the length of a slot coincides with the transport time of a cell across the switching fabric. Each input port is equipped with a buffer of infinite capacity. At the beginning of each time slot, the switch controller mediates potential output contentions by randomly selecting one head–of–the–line (HOL) cell amongst the HOL cells which have the same output address. The HOL cells selected for transmission are then removed from their respective buffer and start being transmitted across the fabric; this transmission is completed by the end of the time slot. At the same time that transmission starts new cells which arrive into the system during a time slot are enqueued by the end of the slot. These steps are repeated from slot to slot.

Even under the most simplified set of assumptions, viz., traffic to the input ports is modeled by independent Bernoulli streams, and the output assignment is performed randomly and uniformly across ports, evaluating the switch performance in terms of delay and throughput is a fairly daunting task because of inter–queue correlations. Any direct solution to this problem, either analytic or computational, appears likely to meet with great difficulties due to the complexity of the resulting Markov chain model, and another approach is therefore required if progress is to be made. In [8] Karol, Hluchyj and Morgan provide such an approach by introducing the virtual queues which are formed by HOL cells (physically located in the input queues) with the same output destination . As these authors focus on the evaluation of the *maximum* system throughput (per port), they assume a *saturated* regime, i.e., the arrival rate per input port is 1, so that at the beginning of every slot there always exists a cell trying to reach the HOL at each input port. Under this "heavy–traffic" assumption, the corresponding Markov chain analysis is feasible only for small values of K, and the next key ingredient in the analysis of [8] is then to consider the asymptotic regime when the switch size grows unboundedly large. The discussion, with particular reference to Appendix A of [8], suggests that the K virtual queues asymptotically decouple as K grows unbounded, and that each such queue in the limit becomes a discrete–time $G|D|1$.

In this paper, we revisit the model of virtual queues in saturation that was introduced by Karol, Hluchyj and Morgan. By developing a suitable representation for the virtual queue size processes, we formalize the intuition used in [8] that the virtual queues become decoupled as the switch size grows unboundedly large. We then proceed to identify the limiting dynamics as a set of independent discrete–time queues characterized by a non–standard Lindley recursion where the statistics of the driving sequence are determined by the statistics of the recursion's output. Various properties of this limiting Lindley recursion are derived; in fact its stationary version coincides with the discrete–time $G|D|1$ of [8].

We note that related limiting results were derived in [4] for an extension of the model discussed here; however the approach used there should be contrasted with the rather elementary one used in this paper. The arguments we have given may in principle pave the way for obtaining convergence rates on the asymptotics; this ongoing work will be reported in [9, 10].

The paper is organized as follows: The model is presented in Section 2, and its asymptotic regime for large switch size is identified in Section 3. We discuss various properties of the limiting process in Section 4. The paper closes with some remarks on the maximal system throughput.

A few words on the notation used in this paper: We denote the set of non–negative integers by $\mathbb{N}$, and the set of all real (resp. non–negative real) numbers by $\mathbb{R}$ (resp. $\mathbb{R}_+$). Throughout the paper, K always denotes a positive integer. The k^{th}

component of any element $\boldsymbol{x}$ in $\mathbb{R}^K$ is denoted either by x^k or by x_k, $k = 1, \ldots, K$, so that $\boldsymbol{x} \equiv (x^1, \ldots, x^K)$ or $(x_1, \ldots, x_K)$. A similar convention is used for random variables (rvs). Finally two rvs X and Y are said to be *equal in law* if they have the same distribution and we denote it by $X =_{st} Y$. The Poisson distribution with parameter $\lambda > 0$ is denoted by $\mathcal{P}(\lambda)$. Weak convergence is denoted by $\Longrightarrow$.

2 The Model

The $K \times K$ switching fabric of interest is assumed saturated so that cells are always waiting in each input queue at the beginning of every time slot. Whenever a cell is transmitted through the switch, a new cell immediately moves to the head of the input queue. The output assignment is random and uniform across ports, while output contentions are resolved by randomly selecting one HOL cell amongst those HOL cells with the same output address.

Fix $K = 1, 2, \ldots$ and $t = 0, 1, \ldots$. For each $k = 1, 2, \ldots, K$, we denote by B_t^k the number of HOL cells which are destined to output k at the beginning of time slot $[t, t+1)$ and by A_{t+1}^k the number of arrivals to the k^{th} virtual queue formed by the HOL cells waiting for transport to output port k during time slot $[t, t+1)$. The total number L_t of HOL cells present at the beginning of time slot $[t, t+1)$ is given by

$$L_t = \sum_{k=1}^{K} B_t^k, \tag{2.1}$$

so that there are $K - L_t$ empty HOL positions at the beginning of time slot $[t, t+1)$, or equivalently, $K - L_t$ fresh cells which move to the head of the line positions at that time, and which need to be addressed to one of K output ports.

In order to provide a precise description of the model, we introduce a collection $\{U_{t+1}^k,\ t = 0, 1, \ldots; k = 1, 2, \ldots\}$ of i.i.d. rvs which are uniformly distributed on $[0, 1]$. The process $\{(B_t^1, \ldots, B_t^K),\ t = 0, 1, \ldots\}$ evolves according to the following recursion: For each $k = 1, \ldots, K$,

$$B_0^k = 0, \quad B_{t+1}^k = \left[B_t^k - 1 + A_{t+1}^k\right]^+, \quad t = 0, 1, \ldots \tag{2.2}$$

with

$$A_{t+1}^k = \sum_{j=1}^{K-L_t} \mathbf{1}\left[\frac{k-1}{K} \le U_{t+1}^j < \frac{k}{K}\right], \quad t = 0, 1, \ldots. \tag{2.3}$$

The form of 2.3) reflects the fact that the output assignment is random and uniform across ports. We close this section with several relations which are found useful at a later stage of the discussion: First, we observe that

$$\begin{aligned} \sum_{k=1}^{K} A_{t+1}^{k} &= \sum_{k=1}^{K} \sum_{j=1}^{K-L_t} \mathbf{1}\left[\frac{k-1}{K} \leq U_{t+1}^{j} < \frac{k}{K}\right] \\ &= \sum_{j=1}^{K-L_t} \sum_{k=1}^{K} \mathbf{1}\left[\frac{k-1}{K} \leq U_{t+1}^{j} < \frac{k}{K}\right] \\ &= \sum_{j=1}^{K-L_t} 1 = K - L_t. \end{aligned} \tag{2.4}$$

Next, as the involved rvs are all integer–valued, we have

$$\left[B_t^k - 1 + A_{t+1}^k < 0\right] = \left[B_t^k = A_{t+1}^k = 0\right] \tag{2.5}$$

so that

$$\begin{aligned} B_{t+1}^k &= \left[B_t^k - 1 + A_{t+1}^k\right]^+ \\ &= \left[B_t^k - 1 + A_{t+1}^k\right] \mathbf{1}\left[B_t^k - 1 + A_{t+1}^k \geq 0\right] \\ &= B_t^k - 1 + A_{t+1}^k - \left[B_t^k - 1 + A_{t+1}^k\right] \mathbf{1}\left[B_t^k = A_{t+1}^k = 0\right] \\ &= B_t^k - 1 + A_{t+1}^k + \mathbf{1}\left[B_t^k = A_{t+1}^k = 0\right]. \end{aligned} \tag{2.6}$$

Adding these relations side by side for $k = 1, \ldots, K$, we conclude from (2.4)-(2.6) that

$$\begin{aligned} L_{t+1} &= L_t - K + \sum_{k=1}^{K} A_{t+1}^k + \sum_{k=1}^{K} \mathbf{1}\left[B_t^k = A_{t+1}^k = 0\right] \\ &= \sum_{k=1}^{K} \mathbf{1}\left[B_t^k = A_{t+1}^k = 0\right]. \end{aligned} \tag{2.7}$$

It is plain for (2.7) that the recursion (2.2)–(2.3) is indeed well defined.

3 Large K Asymptotics

Next we consider the asymptotic regime when the switch size K grows unboundedly large. In particular, we are interested in how some fixed number of input queues behave asymptotically as $K \to \infty$. The key convergence result for large K is contained in Proposition 3.1 below; its proof will be a simple consequence of the next four lemmas.

First, some notation: For the sake of clarity, we add a superscript K on all rvs associated with the K queue system. Let $\mathcal{I}$ denote a given set of input ports; there is no loss of generality in assuming that $\mathcal{I}$ is of the form $\mathcal{I} \equiv \{1, \ldots, I\}$ for some positive integer I and that $I \leq K$.

Fix $K = 1, 2, \ldots$. For each $k = 1, 2, \ldots, K$, we recursively define the rvs $\{Z_t^{K,k},\ t = 1, 2, \ldots\}$ by

$$Z_0^{K,k} \equiv \emptyset, \quad Z_{t+1}^{K,k} \equiv (Z_t^{K,k}, B_t^{K,k}, A_{t+1}^{K,k}), \quad t = 0, 1, \ldots \tag{3.1}$$

and we write

$$Z_t^{K,\mathcal{I}} \equiv (Z_t^{K,1}, \ldots, Z_t^{K,I}), \quad t = 0, 1, \ldots \tag{3.2}$$

For $t = 1, 2, \ldots$, we say that condition $(\mathcal{I}_t)$ holds if the convergence

$$Z_t^{K,\mathcal{I}} \Longrightarrow_K z_t^{\mathcal{I}} \equiv (z_t^1, \ldots, z_t^I) \tag{3.3}$$

takes place where the limiting rvs $z_t^1, \ldots, z_t^I$ are *i.i.d.* rvs. Obviously, under $(\mathcal{I}_t)$ we have $B_s^{K,i} \Longrightarrow_K b_s^i$ and $A_{s+1}^{K,i} \Longrightarrow_K a_{s+1}^i$, $i = 1, \ldots, I$, for all $s = 0, 1, \ldots, t-1$, and in analogy with (3.1) we have

$$z_0^i \equiv \emptyset, \quad z_{s+1}^i \equiv (z_s^i, b_s^i, a_{s+1}^i), \quad s = 0, 1, \ldots, t-1 \tag{3.4}$$

Finally, we say that for $t = 1, 2, \ldots$, condition $(\mathcal{L}_t)$ holds if

$$\frac{1}{K} L_{t-1}^K \xrightarrow{\mathbf{P}}_K \lambda_t \tag{3.5}$$

for some *nonrandom* λ_t.

Lemma 3.1 *If both conditions $(\mathcal{I}_t)$ and $(\mathcal{L}_t)$ hold for some $t = 1, 2, \ldots$, then $(\mathcal{L}_{t+1})$ also holds.*

Proof. First, from (2.7) and exchangeability, we get

$$\begin{aligned} \mathrm{var}\left[\frac{1}{K} L_t^K\right] &= \mathrm{var}\left[\frac{1}{K}\sum_{i=1}^K \mathbf{1}\left[B_{t-1}^{K,i} = A_t^{K,i} = 0\right]\right] \\ &= \frac{1}{K}\mathrm{var}\left[\mathbf{1}\left[B_{t-1}^{K,1} = A_t^{K,1} = 0\right]\right] \\ &\quad + \frac{K-1}{K}\mathrm{cov}\left[\mathbf{1}\left[B_{t-1}^{K,1} = A_t^{K,1} = 0\right], \mathbf{1}\left[B_{t-1}^{K,2} = A_t^{K,2} = 0\right]\right]. \end{aligned}$$

Under $(\mathcal{I}_t)$, $(Z_t^{K,1}, Z_t^{K,2}) \Longrightarrow_K (z_t^1, z_t^2)$ with z_t^1 and z_t^2 i.i.d. rvs, and we readily get

$$\begin{aligned} & \lim_{K\to\infty} \text{cov}\left[\mathbf{1}\left[B_{t-1}^{K,1} = A_t^{K,1} = 0\right],\ \mathbf{1}\left[B_{t-1}^{K,2} = A_t^{K,2} = 0\right]\right] \\ = & \text{ cov}\left[\mathbf{1}\left[b_{t-1}^1 = a_t^1 = 0\right],\ \mathbf{1}\left[b_{t-1}^2 = a_t^2 = 0\right]\right] \\ = & \ 0. \end{aligned} \tag{3.6}$$

by making use of the Bounded Convergence Theorem. Therefore, $\lim_{K\to\infty} \text{var}\left[\frac{1}{K}L_t^K\right] = 0$ and by Chebyshev's inequality we conclude that

$$\frac{1}{K}L_t^K - \mathbf{E}\left[\frac{1}{K}L_t^K\right] \xrightarrow{\mathbf{P}}_K 0. \tag{3.7}$$

Next, by exchangeability,

$$\begin{aligned} \mathbf{E}\left[\frac{1}{K}L_t^K\right] &= \mathbf{P}\left[B_{t-1}^{K,1} = 0,\ A_t^{K,1} = 0\right] \\ &= \mathbf{E}\left[(1-\frac{1}{K})^{K-L_{t-1}^K}\mathbf{1}\left[B_{t-1}^{K,1} = 0\right]\right]. \end{aligned} \tag{3.8}$$

Condition $(\mathcal{I}_t)$ guarantees $B_{t-1}^{K,1} \Longrightarrow_K b_{t-1}^1$, whereas $\frac{1}{K}L_{t-1}^K \xrightarrow{\mathbf{P}}_K \lambda_t$ by assumption. Together these two facts imply [1, Theorem 4.4, p. 27] that $(\frac{1}{K}L_{t-1}^K, B_{t-1}^{K,1}) \Longrightarrow_K (\lambda_t, b_{t-1}^1)$, whence

$$\lim_{K\to\infty} \mathbf{E}\left[\frac{1}{K}L_t^K\right] = e^{-(1-\lambda_t)}\mathbf{P}\left[b_{t-1}^1 = 0\right] \equiv \lambda_{t+1}, \tag{3.9}$$

and it is now plain from (3.7) and (3.9) that $(\mathcal{L}_{t+1})$ holds. ■

Lemma 3.2 *If $(\mathcal{I}_t)$ hold for some $t = 1, 2, \ldots$, then the convergence*

$$\left((Z_t^{K,1}, B_t^{K,1}), \ldots, (Z_t^{K,I}, B_t^{K,I})\right) \Longrightarrow \left((z_t^1, b_t^1), \ldots, (z_t^I, b_t^I)\right) \tag{3.10}$$

takes place where the limiting rvs $(z_t^1, b_t^1), \ldots, (z_t^I, b_t^I)$ are i.i.d. rvs.

Proof. For each $K = 1, 2, \ldots$, we note that

$$\begin{aligned} (Z_t^{K,i}, B_t^{K,i}) &= (Z_t^{K,i}, [B_{t-1}^{K,i} - 1 + A_t^{K,i}]^+) \\ &= F_t(Z_t^{K,i}), \quad i = 1, 2, \ldots, I \end{aligned} \tag{3.11}$$

for some mapping $F_t : \mathbb{N}^{2t} \to \mathbb{N}^{2t+1}$ which depends only on t and not on K or on i. This mapping is automatically continuous as it is defined on discrete spaces, and therefore preserves weak convergence [2]. Hence $(\mathcal{I}_t)$ immediately yields

$$\left((Z_t^{K,1}, B_t^{K,1}), \ldots, (Z_t^{K,I}, B_t^{K,I})\right) \Longrightarrow \left(F_t(z_t^1), \ldots, F_t(z_t^I)\right) \tag{3.12}$$

where the limiting rvs $F_t(z_t^1), \ldots, F_t(z_t^I)$ are obviously i.i.d. rvs. The conclusion (3.10) holds with $(z_t^i, b_t^i) = F_t(z_t^i)$, $i = 1, 2, \ldots, I$. ■

A little more can be extracted from the proof of Lemma 3.2. Indeed, the argument given above also shows that

$$b_t^i =_{st} [b_{t-1}^i - 1 + a_t^i]^+, \quad i = 1, \ldots, I. \tag{3.13}$$

Lemma 3.3 *If both conditions $(\mathcal{I}_t)$ and $(\mathcal{L}_t)$ hold for some $t = 1, 2, \ldots$, then $(\mathcal{I}_{t+1})$ also holds.*

Proof. Fix u_i and z_i, $i = 1, \ldots, I$, in the unit interval $[0, 1]$, and consider a bounded mapping $\varphi : \mathbb{N}^{2tI} \to \mathbb{R}$. With $\mathcal{F}_t$ denoting the σ-field generated by the rvs $\{U_s^j,\ j = 1, 2, \ldots;\ s = 1, 2, \ldots, t\}$, we observe that

$$\mathbf{E}\left[\prod_{i \in \mathcal{I}} z_i^{A_{t+1}^{K,i}} \mid \mathcal{F}_t\right] = \left(1 - \frac{1}{K}\sum\nolimits_{i \in \mathcal{I}}(1 - z_i)\right)^{K - L_t^K}. \tag{3.14}$$

A simple conditioning argument then shows that

$$\begin{aligned}
&\mathbf{E}\left[\varphi(Z_t^{K,\mathcal{I}}) \prod_{i \in \mathcal{I}} u_i^{B_t^{K,i}} z_i^{A_{t+1}^{K,i}}\right] \\
&= \mathbf{E}\left[\varphi(Z_t^{K,\mathcal{I}}) \prod_{i \in \mathcal{I}} u_i^{B_t^{K,i}} \mathbf{E}\left[\prod_{i \in \mathcal{I}} z_i^{A_{t+1}^{K,i}} \mid \mathcal{F}_t\right]\right] \\
&= \mathbf{E}\left[\varphi(Z_t^{K,\mathcal{I}}) \prod_{i \in \mathcal{I}} u_i^{B_t^{K,i}} \left(1 - \frac{1}{K}\sum\nolimits_{i \in \mathcal{I}}(1 - z_i)\right)^{K - L_t^K}\right].
\end{aligned} \tag{3.15}$$

By Lemma 3.1, $\frac{1}{K}L_t^K \xrightarrow{\mathbf{P}}_K \lambda_{t+1}$ while $(\mathcal{I}_t)$ guarantees (3.10) by virtue of Lemma 3.2, and together these two facts imply [1, Theorem 4.4, p. 27]

$$\left(\frac{1}{K}L_t^K, (Z_t^{K,1}, B_t^{K,1}), \ldots, (Z_t^{K,I}, B_t^{K,I})\right) \Longrightarrow_K \left(\lambda_{t+1}, (z_t^1, b_t^1), \ldots, (z_t^I, b_t^I)\right). \tag{3.16}$$

Invoking the Bounded Convergence Theorem, we obtain

$$\begin{aligned}\lim_{K\to\infty} \mathbf{E}\left[\varphi(Z_t^{K,\mathcal{I}}) \prod_{i\in\mathcal{I}} u_i^{B_t^{K,i}} z_i^{A_{t+1}^{K,i}}\right] &= \mathbf{E}\left[\varphi(z_t^{\mathcal{I}}) \prod_{i\in\mathcal{I}} u_i^{b_t^i} \prod_{i\in\mathcal{I}} \exp(-(1-\lambda_{t+1})(1-z_i))\right] \\ &= \mathbf{E}\left[\varphi(z_t^{\mathcal{I}}) \prod_{i\in\mathcal{I}} u_i^{b_t^i}\right] \prod_{i\in\mathcal{I}} \exp(-(1-\lambda_{t+1})(1-z_i))\end{aligned}$$

and this establishes the convergence $(Z_{t+1}^{K,1}, \ldots, Z_{t+1}^{K,I}) \Longrightarrow (z_{t+1}^1, \ldots, z_{t+1}^I)$. In particular, it follows that $(A_{t+1}^{K,1}, \ldots, A_{t+1}^{K,I}) \Longrightarrow (a_{t+1}^1, \ldots, a_{t+1}^I)$, and we have

$$\mathbf{E}\left[\varphi(z_t^{\mathcal{I}}) \prod_{i\in\mathcal{I}} u_i^{b_t^i} \prod_{i\in\mathcal{I}} z_i^{a_{t+1}^i}\right] = \mathbf{E}\left[\varphi(z_t^{\mathcal{I}}) \prod_{i\in\mathcal{I}} u_i^{b_t^i}\right] \prod_{i\in\mathcal{I}} \exp(-(1-\lambda_{t+1})(1-z_i)) \quad (3.17)$$

Upon specializing $\varphi \equiv 1$ and $u_i = 1$, $i = 1, \ldots, I$, in (3.17), we readily get

$$\mathbf{E}\left[\prod_{i\in\mathcal{I}} z_i^{a_{t+1}^i}\right] = \prod_{i\in\mathcal{I}} \exp(-(1-\lambda_{t+1})(1-z_i)) = \prod_{i\in\mathcal{I}} \mathbf{E}\left[z_i^{a_{t+1}^i}\right] \quad (3.18)$$

and this last relation, in conjunction with (3.17) implies

$$\mathbf{E}\left[\varphi(z_t^{\mathcal{I}}) \prod_{i\in\mathcal{I}} u_i^{b_t^i} z_i^{a_{t+1}^i}\right] = \mathbf{E}\left[\varphi(z_t^{\mathcal{I}}) \prod_{i\in\mathcal{I}} u_i^{a_t^i}\right] \mathbf{E}\left[\prod_{i\in\mathcal{I}} z_i^{a_{t+1}^i}\right]. \quad (3.19)$$

It is now plain from (3.18) and (3.19) that the rvs $a_{t+1}^1, \ldots, a_{t+1}^I$ are i.i.d. rvs which are independent of the i.i.d. rvs $(z_t^1, b_t^1), \ldots, (z_t^I, b_t^I)$, and this completes the proof that $(\mathcal{I}_{t+1})$ holds. ■

Lemma 3.4 *Both conditions $(\mathcal{I}_t)$ and $(\mathcal{L}_t)$ hold for all $t = 1, 2, \ldots$.*

Proof. The proof proceeds by induction on t: Here, $B_0^{K,i} = 0$ so that $L_0^K \equiv 0$, and $(\mathcal{L}_t)$ trivially holds for $t = 1$ with $\lambda_1 = 0$. To establish the validity of $(\mathcal{I}_t)$ for $t = 1$ it suffices to show that $(A_1^{K,1}, \ldots, A_1^{K,I}) \Longrightarrow (a_1^1, \ldots, a_1^I)$ where the rvs $a_1^1, \ldots, a_1^I$ are i.i.d. To that end, for $z_1, \ldots, z_I$ in $[0,1]$, we have

$$\mathbf{E}\left[\prod_{i\in\mathcal{I}} z_i^{A_1^{K,i}}\right] = \left[1 - \frac{1}{K}\sum\nolimits_{i\in\mathcal{I}}(1-z_i)\right]^K \quad (3.20)$$

so that

$$\lim_{K\to\infty} \mathbf{E}\left[\prod_{i\in\mathcal{I}} z_i^{A_1^{K,i}}\right] = \prod_{i\in\mathcal{I}} \exp(-(1-z_i)). \quad (3.21)$$

and the desired conclusion follows.

Finally, the induction step is a direct consequence of Lemmas 3.1 – 3.3. ■

The proof of Lemma 3.3 and Lemma 3.4 also shows that for each $t = 0, 1, \ldots$, the i.i.d. rvs $a^1_{t+1}, \ldots, a^I_{t+1}$ are Poisson rvs with $\mathbf{E}\left[a^i_{t+1}\right] = 1 - \lambda_{t+1}$, $i = 1, \ldots, I$. In order to obtain an alternate expression for these constants, we need the following technical lemma.

Lemma 3.5 *For each* $t = 0, 1, \ldots$, *the rvs* $\{B^{K,k}_t,\ k = 1, 2, \ldots, K;\ K = 1, 2, \ldots\}$ *are uniformly integrable.*

Proof. Fix $K = 1, 2, \ldots$ and $t = 0, 1, \ldots$. For each $k = 1, \ldots, K$, the bound

$$A^{K,k}_{t+1} \leq \sum_{j=1}^{K} \mathbf{1}\left[\frac{k-1}{K} \leq U^j_{t+1} < \frac{k}{K}\right] \tag{3.22}$$

and standard properties of binomial distributions readily yield

$$\mathbf{E}\left[\left|A^{K,k}_{t+1}\right|^2\right] \leq \frac{2K-1}{K} \leq 2. \tag{3.23}$$

¿From the obvious inequality $B^k_{t+1} \leq B^k_t + A^k_{t+1}$, we get

$$\mathbf{E}\left[\left|B^k_{t+1}\right|^2\right] \leq 2\mathbf{E}\left[\left|B^k_t\right|^2\right] + 2\mathbf{E}\left[\left|A^k_{t+1}\right|^2\right], \quad t = 0, 1, \ldots \tag{3.24}$$

and an argument by induction now show that $\sup_{K,k} \mathbf{E}\left[\left|B^k_t\right|^2\right] < \infty$, thereby implying the stated uniform integrability. ■

By exchangeability, for each $t = 0, 1, \ldots$, we have

$$\mathbf{E}\left[\frac{1}{K} L^K_t\right] = \mathbf{E}\left[B^{K,1}_t\right], \quad K = 1, 2, \ldots \tag{3.25}$$

so that

$$\begin{aligned} \lambda_{t+1} &= \lim_{K\to\infty} \mathbf{E}\left[\frac{1}{K} L^K_t\right] \\ &= \lim_{K\to\infty} \mathbf{E}\left[B^{K,1}_t\right] = \mathbf{E}\left[b^1_t\right] \end{aligned} \tag{3.26}$$

where the last step made use of the convergence $B_t^{K,1} \Longrightarrow_K b_t^1$ and of the uniform integrability stated in Lemma 3.5.

Combining Lemma 3.4 with the remark following its proof and with (3.13), we conclude that for large values of K, the processes $\{B_t^{K,i},\ t=0,1,\ldots\}$, $i=1,\ldots,I$, are asymptotically independent, and that each one is well approximated by a process generated through a Lindley type recursion driven by a sequence of *time-varying* independent Poisson rvs. Formally, we have

Proposition 3.1 *For any finite set $\mathcal{I} \equiv \{1,\ldots,I\}$ and $t=0,1,\ldots$, we have the convergence*

$$\left((B_s^{K,i}, A_{s+1}^{K,i}),\ i \in \mathcal{I},\ s=0,1,\ldots,t\right) \Longrightarrow_K \left((b_s^k, a_{s+1}^k),\ i \in \mathcal{I},\ s=0,1,\ldots,t\right)$$

with the following properties for the limiting processes: The processes $\{(b_s^i, a_{s+1}^i),\ s=0,1,\ldots,t\}$, $i=1,\ldots,I$, are mutually independent, and for each $i=1,\ldots,I$, the rvs $\{b_t^i,\ t=0,1,\ldots\}$ satisfy the Lindley recursion

$$b_0^i = 0; \quad b_{t+1}^i = \left[b_t^i - 1 + a_{t+1}^i\right]^+, \quad t=0,1,\ldots. \tag{3.27}$$

where the rvs $\{a_{t+1}^i,\ t=0,1,\ldots)$ are independent Poisson rvs with $\mathbf{E}\left[a_{t+1}^i\right] = 1-\mathbf{E}\left[b_t^i\right]$ for all $t=0,1,\ldots$.

4 Properties of the Process $\{b_t,\ t=0,1,\ldots\}$

We now analyze the limiting recursion (3.27) which we find useful to generalize somewhat: Let $\mathcal{B}$ denote the collection of all $\mathbb{N}$–valued rvs b (or equivalently, their distributions) such that $0 \le \mathbf{E}[b] \le 1$. The process $\{b_t,\ t=0,1,\ldots\}$ of interest is generated through the recursion

$$b_0 = b; \quad b_{t+1} = \left[b_t - 1 + a_{t+1}\right]^+, \quad t=0,1,\ldots \tag{4.1}$$

where we assume that (i) the initial condition b is an element of $\mathcal{B}$; (ii) the rvs $\{a_{t+1},\ t=0,1,\ldots\}$ are independent Poisson rvs with $\mathbf{E}[a_{t+1}] = 1-\mathbf{E}[b_t]$ for each $t=0,1,\ldots$, and (iii) the initial condition b is independent of the rvs $\{a_{t+1},\ t=0,1,\ldots\}$.

This recursion (4.1) differs from the standard Lindley recursion for $GI/GI/1$ queues in that here the statistics of the driving sequence $\{a_{t+1},\ t=0,1,\ldots\}$ are determined recursively from the statistics of the recursion's output.

As was the case with the recursion (3.27), because all involved rvs are $\mathbb{N}$–valued, we have

$$[b_t - 1 + a_{t+1} < 0] = [b_t = a_{t+1} = 0] \tag{4.2}$$

so that

$$b_{t+1} = b_t - 1 + a_{t+1} + \mathbf{1}\,[b_t = a_{t+1} = 0]\,. \tag{4.3}$$

This relation readily leads to the following fact.

Proposition 4.1 *If the initial condition b is an element of $\mathcal{B}$, then the rv b_t is also an element of $\mathcal{B}$ for all $t = 1, 2, \ldots$, with*

$$\mathbf{E}\,[b_{t+1}] = \mathbf{P}\,[b_t = 0]\exp\left(-(1 - \mathbf{E}\,[b_t])\right) \tag{4.4}$$

and the recursion (4.1) is therefore well defined.

Proof. Consider $t = 0, 1, \ldots$ such that $0 \leq \mathbf{E}\,[b_t] \leq 1$ – note that this property holds for $t = 0$. Taking expectations on both sides of (4.3), we get

$$\mathbf{E}\,[b_{t+1}] = \mathbf{P}\,[b_t = a_{t+1} = 0] = \mathbf{P}\,[b_t = 0]\,\mathbf{P}\,[a_{t+1} = 0] \tag{4.5}$$

where we have used the independence of the rvs b_t and a_{t+1}, and the fact $\mathbf{E}\,[a_{t+1}] = 1 - \mathbf{E}\,[b_t]$. The expression (4.4) is a simple consequence of the Poisson character of the rv a_{t+1}, and the stated conclusions readily follow by induction. ■

In the case of a Lindley recursion driven by an i.i.d. sequence, or even a stationary ergodic one, it is well known [11] under what conditions the output sequence converges weakly to a honest rv – the so–called stable case. As these classical results do not apply here, we embark now on establishing a similar convergence result for the recursion (4.1). We begin by investigating the existence of stationary solutions to the recursion (4.1). This is best achieved by considering the probability generating function of the distributions of interest. For $t = 0, 1, \ldots$ and z in $[0, 1]$, we set

$$A_{t+1}(z) = \mathbf{E}\,[z^{a_{t+1}}] = \exp\left(-(1 - \mathbf{E}\,[b_t])(1 - z)\right) \tag{4.6}$$

and

$$B_t(z) = \mathbf{E}\left[z^{b_t}\right]. \tag{4.7}$$

Fix $t = 0, 1, \ldots$ and z in the interval $(0, 1)$:. Using (4.2) we see that

$$\begin{aligned} z^{b_{t+1}} &= \mathbf{1}\left[b_t - 1 + a_{t+1} \geq 0\right] z^{b_t - 1 + a_{t+1}} + \mathbf{1}\left[b_t = a_{t+1} = 0\right] \\ &= z^{b_t - 1 + a_{t+1}} + \mathbf{1}\left[b_t = a_{t+1} = 0\right](1 - z^{-1}). \end{aligned} \tag{4.8}$$

Taking expectations on both sides of this last relation and using the independence of the rvs b_t and a_{t+1}, we get

$$B_{t+1}(z) = z^{-1} B_t(z) A_{t+1}(z) + \left(1 - z^{-1}\right) \mathbf{P}\left[b_t = 0\right] \mathbf{P}\left[a_{t+1} = 0\right]. \tag{4.9}$$

An $I\!N$–valued rv b in $\mathcal{B}$ is said to be a *stationary* solution to the recursion (4.1) if the rvs $\{b_t,\ t = 0, 1, \ldots\}$ generated by the recursion with $b_0 = b$ form a stationary sequence; of course in that case we have $b_t =_{st} b$ for all $t = 0, 1, \ldots$.

Proposition 4.2 *The recursion (4.1) admits exactly one stationary solution b in $\mathcal{B}$; its probability generating function B is given by*

$$B(z) = \frac{\lambda(1 - z)}{\exp(-(1 - \lambda)(1 - z)) - z}, \quad z \in (0, 1) \tag{4.10}$$

with

$$\lambda = \sqrt{2} - 1 = \mathbf{E}\left[b\right]. \tag{4.11}$$

Proof. Let b be an element in $\mathcal{B}$, and let a denote a Poisson rv with parameter $1 - \mathbf{E}\left[b\right]$; its probability generating function A is given by

$$A(z) = \mathbf{E}\left[z^a\right] = \exp\left(-(1 - \mathbf{E}\left[b\right])(1 - z)\right), \quad z \in [0, 1]. \tag{4.12}$$

Fix $t = 0, 1, \ldots$ and z in the interval $(0, 1)$. If b is a stationary solution to the recursion (4.1), then $b_t =_{st} b$, so that $\mathbf{E}\left[b_t\right] = \mathbf{E}\left[b\right]$ and $B_t(z) = B(z)$, with B denoting the probability generating function of b. Reporting this information into (4.9), we obtain the relation

$$B(z) = z^{-1} B(z) A(z) + \left(1 - z^{-1}\right) \mathbf{P}\left[b = 0\right] \mathbf{P}\left[a = 0\right]. \tag{4.13}$$

Rearranging terms, we get

$$\left(1 - z^{-1} A(z)\right) B(z) = \left(1 - z^{-1}\right) \mathbf{P}\left[b = 0\right] \mathbf{P}\left[a = 0\right] \tag{4.14}$$

or equivalently,

$$B(z) = \frac{z - 1}{z - A(z)} \mathbf{P}\left[b = 0\right] \mathbf{P}\left[a = 0\right]. \tag{4.15}$$

Upon noting from (4.4) that $\mathbf{E}[b] = \mathbf{P}[b=0]\,\mathbf{P}[a=0]$, we conclude from (4.15) that

$$B(z) = \frac{z-1}{z-A(z)}\mathbf{E}[b]. \tag{4.16}$$

To complete the determination of $B(z)$ we need to evaluate $\mathbf{E}[b]$: We first differentiate both sides of (4.16) with respect to z and find that

$$\dot{B}(z) = \mathbf{E}[b]\,\frac{1+(z-1)\dot{A}(z)-A(z)}{(z-A(z))^2}. \tag{4.17}$$

Next, letting $z \uparrow 1$ in this last relation and using l' Hospital's rule twice, we obtain

$$\begin{aligned}
\mathbf{E}[b] &= \lim_{z\uparrow 1}\dot{B}(z) \\
&= \mathbf{E}[b]\lim_{z\uparrow 1}\left(\frac{1+(z-1)\dot{A}(z)-A(z)}{(z-A(z))^2}\right) \\
&= \mathbf{E}[b]\,\frac{\ddot{A}(1)}{2(1-\dot{A}(1))^2}.
\end{aligned} \tag{4.18}$$

It is plain from (4.12) that $\dot{A}(1) = 1-\mathbf{E}[b]$ and $\ddot{A}(1) = (1-\mathbf{E}[b])^2$, and substituting these expressions into (4.18) leads to the relation $2\mathbf{E}[b]^2 = (1-\mathbf{E}[b])^2$. The only positive solution to this quadratic equation is given by (4.11), and the relation (4.10) is now an immediate consequence of (4.16). ■

Proposition 4.3 *For any initial condition b in $\mathcal{B}$, if $b_t \Longrightarrow_t b_\infty$, then the limiting rv b_∞ is the unique stationary solution (4.10)–(4.11) to (4.1) in $\mathcal{B}$.*

The next three lemmas prepare the proof of Proposition 4.3.

Lemma 4.1 *For any initial condition b in $\mathcal{B}$, if $b_t \Longrightarrow_t b_\infty$, then $\mathbf{P}[b_\infty = 0] > 0$ and $\underline{\lim}_{t\to\infty}\mathbf{E}[b_t] > 0$.*

Proof. We begin by noting the easy relations

$$\mathbf{P}[b_t \neq 0] \le \mathbf{E}[b_t] \quad \text{and} \quad \mathbf{E}[b_{t+1}] \le \mathbf{P}[b_t = 0], \quad t = 0,1,\ldots. \tag{4.19}$$

If $\lim_t \mathbf{P}[b_t = 0] = 0$, then the first inequality implies $\lim_t \mathbf{E}[b_t] = 1$ whereas the second inequality yields $\lim_t \mathbf{E}[b_t] = 0$, a clear contradiction and we must have

$\lim_t \mathbf{P}[b_t = 0] > 0$. The second part of the statement is now immediate from the bound

$$\mathbf{P}[b_t = 0] \leq e\mathbf{E}[b_{t+1}], \quad t = 0, 1, \ldots. \tag{4.20}$$

■

Lemma 4.2 *Consider the discrete–time Markov chain $\{\beta_{s+1},\ s = 0, 1, \ldots\}$ governed by the Lindley recursion*

$$\beta_0 = \beta; \quad \beta_{s+1} = [\beta_s - 1 + \alpha_{s+1}]^+, \quad s = 0, 1, \ldots \tag{4.21}$$

where the rvs $\{\alpha_{s+1},\ s = 0, 1, \ldots\}$ are i.i.d. rvs which are $\mathcal{P}(\gamma)$–distributed, and independent of the initial condition β. If $\gamma < 1$ and β is integrable, then the rvs $\{\beta_s,\ s = 0, 1, \ldots\}$ are uniformly integrable.

Proof. If $\beta = 0$, then it is well known [12] that $\beta_t \leq_{st} \beta_{t+1}$ for all $t = 0, 1, \ldots$. Hence the rvs $\{\beta_s,\ s = 0, 1, \ldots\}$ converge weakly, say to some rv β_∞, and by making use of the Monotone Convergence Theorem, we can easily conclude that $\lim_{s\to\infty} \mathbf{E}[\beta_s] = \mathbf{E}[\beta_\infty]$. When $\gamma < 1$, standard z–transform arguments readily yield $\mathbf{E}[\beta_\infty] < \infty$ and the uniform integrability of $\{\beta_s,\ s = 0, 1, \ldots\}$ follows [5].

We now consider the case when β is arbitrary. Let $\{\beta_s^0,\ s = 0, 1, \ldots\}$ denote the output to the recursion (4.21) when the initial condition is zero. It is easy to check by induction that

$$\beta_t \leq |\beta| + \beta_t^0, \quad t = 0, 1, \ldots \tag{4.22}$$

and the uniform integrability of $\{\beta_s,\ s = 0, 1, \ldots\}$ is implied by that of the sequence $\{\beta_s^0,\ s = 0, 1, \ldots\}$ and by the integrability of β. ■

Lemma 4.3 *For any initial condition b in $\mathcal{B}$, if $b_t \Longrightarrow_t b_\infty$, then the rvs $\{b_t,\ t = 0, 1, \ldots\}$ are uniformly integrable, and we have*

$$\lim_{t\to\infty} \mathbf{E}[b_t] = \mathbf{E}[b_\infty]. \tag{4.23}$$

Proof. By Lemma 4.1, for some $\varepsilon > 0$, there exists an integer t_ε such that

$\mathbf{E}[b_t] > \varepsilon$ for all $t \geq t_\varepsilon$. Next we consider the recursion (4.21) with $\beta_0 =_{st} b_{t_\varepsilon}$ and $\{\alpha_{s+1}, s = 0, 1, \ldots\}$ i.i.d. rvs which are $\mathcal{P}(1-\varepsilon)$–distributed. Noting [12] that $a_{t_\varepsilon+s+1} \leq_{st} \alpha_{s+1}$ for all $s = 0, 1, \ldots$, we readily conclude by an induction argument that

$$b_{t_\varepsilon+s+1} \leq_{st} \beta_{s+1}, \quad s = 0, 1, \ldots. \tag{4.24}$$

The desired result is then an easy consequence of these bounds and of the fact that the rvs $\{\beta_s, s = 0, 1, \ldots\}$ are uniformly integrable as established in Lemma 4.2. The convergence (4.23) is a well–known consequence of the uniform integrability of the rvs $\{b_t,\ t = 0, 1, \ldots\}$ [2, Theorem 16.13, p. 220]. ■

Proof. Our point of departure is the relation (4.9): The convergence $b_t \Longrightarrow b_\infty$ already implies $\lim_t \mathbf{P}[b_t = 0] = \mathbf{P}[b_\infty = 0]$ and with an obvious notation, $\lim_t B_t(z) = B_\infty(z)$ for all z in $[0, 1]$. We let t go to ∞ in (4.9), and using (4.23) we conclude that

$$B_\infty(z) = z^{-1}B_\infty(z)A_\infty(z) + \left(1 - z^{-1}\right)\mathbf{P}[b_\infty = 0]\,\mathbf{P}[a_\infty = 0] \tag{4.25}$$

where a_∞ is a Poisson rv with $\mathbf{E}[a_\infty] = 1 - \mathbf{E}[b_\infty]$, and probability generating function A_∞. It is plain from (4.4) and (4.23) that $\mathbf{E}[b_\infty] = \mathbf{P}[b_\infty = 0]\,\mathbf{P}[a_\infty = 0]$. After substitution of this fact into (4.25), we get the functional equation

$$B_\infty(z) = \frac{z-1}{z - A_\infty(z)}\mathbf{E}[b_\infty], \quad z \in (0, 1). \tag{4.26}$$

This equation has the same form as (4.16), whose unique solution is the unique stationary solution (4.10) of the recursion (4.1), whence b_∞ is indeed the unique stationary solution to the recursion (4.1). ■

As was the case for the classical Lindley recursion, we can also establish the following convergence result whose proof is available in [9, 10]; the convergence can also be extracted from the results of [4]:

Proposition 4.4 *Under the foregoing assumptions, for any initial condition b in $\mathcal{B}$, we have the convergence $b_t \Longrightarrow_t b_\infty$.*

5 Steady State Results

For each $K = 1, 2, \ldots$, under the enforced assumptions the rvs $\{(B_t^1, \ldots, B_t^K),\ t = 0, 1, \ldots\}$ form a discrete–time Markov chain with finite state space $\mathcal{S}_K$ given by

$$\mathcal{S}_K \equiv \{(x_1, \ldots, x_K) \in \{0, 1, \ldots, K\}^K : \sum_{k=1}^{K} \mathbf{1}[x_k > 0](x_k + 1) \leq K\}. \tag{5.1}$$

This chain is irreducible and aperiodic, hence ergodic and we have the convergence

$$(B_t^{K,1}, \ldots, B_t^{K,K}) \Longrightarrow_t (B_\infty^{K,1}, \ldots, B_\infty^{K,K}). \tag{5.2}$$

As in Section 3, we are interested in the asymptotic behavior of the system in steady–state for large K. In view of Proposition 3.1, we would expect the following result to hold.

Proposition 5.1 *For any finite set $\mathcal{I} \equiv \{1, \ldots, I\}$, we have*

$$(B_\infty^{K,1}, \ldots, B_\infty^{K,I}) \Longrightarrow_K (b_\infty^1, \ldots, b_\infty^I) \tag{5.3}$$

where $b_\infty^1, \ldots, b_\infty^I$ denote i.i.d. rvs, each distributed like the unique stationary point in $\mathcal{B}$ of the recursion (4.1).

We begin with a useful technical fact:

Lemma 5.1 *The rvs $\left\{(B_t^{K,k}, \frac{1}{K}L_t^K),\ K = 1, 2, \ldots;\ k = 1, \ldots, K;\ t = 0, 1, \ldots\right\}$ constitute a tight collection of rvs.*

Proof. Fix $t = 0, 1, \ldots$ and $K = 1, 2, \ldots$. The rvs $B_t^{K,1}, \ldots, B_t^{K,K}$ are obviously exchangeable, whence

$$\mathbf{E}\left[B_t^{K,1}\right] = \cdots = \mathbf{E}\left[B_t^{K,K}\right] = \mathbf{E}\left[\frac{1}{K}L_t^K\right] \leq 1 \tag{5.4}$$

where in the last step we have used the fact $0 \leq L_t^K \leq K$. Tightness is now immediate as we note that these bounds are uniform in $t = 0, 1, \ldots$, $K = 1, 2, \ldots$ and $k = 1, 2, \ldots$. ■

6 System Throughput

In this section we are concerned with the (maximal) system throughput per port which can be defined as the average number of cells coming out of the system per time slot normalized with respect to switch size K. We denote by $TH(K)$ (resp. TH) the system throughput when the switch size is K (resp. ∞). Accordingly, $TH(K)$ can be expressed by

$$\begin{aligned}
TH(K) &\equiv \frac{1}{K}\mathbf{E}\left[\sum_{k=1}^{K}\mathbf{1}\left[B_\infty^{K,k}+A_\infty^{K,k}>0\right]\right] \\
&= \mathbf{P}\left[B_\infty^{K,1}+A_\infty^{K,1}>0\right] \\
&= 1-\mathbf{P}\left[B_\infty^{K,1}=A_\infty^{K,1}=0\right] \\
&= 1-\mathbf{E}\left[\mathbf{1}\left[B_\infty^{K,1}=0\right](1-\frac{1}{K})^{K-L_\infty^K}\right] \qquad (6.1)
\end{aligned}$$

while

$$\begin{aligned}
TH &= \mathbf{P}\left[b_\infty+a_\infty>0\right] \\
&= 1-\mathbf{P}\left[b_\infty=a_\infty=0\right] \\
&= 1-\mathbf{P}\left[b_\infty=0\right]\exp(-(1-\mathbf{E}\left[b_\infty\right])). \qquad (6.2)
\end{aligned}$$

We note that

$$\begin{aligned}
&|TH(K)-TH| \\
&= \left|\mathbf{E}\left[\mathbf{1}\left[B_\infty^{K,1}=0\right](1-\frac{1}{K})^{K-L_\infty^K}\right]-\mathbf{P}\left[b_\infty=0\right]\exp(-(1-\mathbf{E}\left[b_\infty\right]))\right| \\
&\le \left|\mathbf{E}\left[\mathbf{1}\left[B_\infty^{K,1}=0\right]\left\{(1-\frac{1}{K})^{K-L_\infty^K}-\exp(-(1-\mathbf{E}\left[b_\infty\right]))\right\}\right]\right| \\
&\quad +\left|\mathbf{P}\left[B_\infty^{K,1}=0\right]-\mathbf{P}\left[b_\infty=0\right]\right|\exp(-(1-\mathbf{E}\left[b_\infty\right])) \\
&\le \mathbf{E}\left[\left|(1-\frac{1}{K})^{K-L_\infty^K}-\exp(-(1-\mathbf{E}\left[b_\infty\right]))\right|\right]+\left|\mathbf{P}\left[B_\infty^{K,1}=0\right]-\mathbf{P}\left[b_\infty=0\right]\right|.
\end{aligned}$$

Next we observe that

$$\begin{aligned}
&\mathbf{E}\left[\left|(1-\frac{1}{K})^{K-L_\infty^K}-\exp(-(1-\mathbf{E}\left[b_\infty\right]))\right|\right] \\
&= \mathbf{E}\left[\left|\exp(-(K-L_\infty^K)\ln(\frac{K}{K-1}))-\exp(-(1-\mathbf{E}\left[b_\infty\right]))\right|\right] \\
&\le \mathbf{E}\left[\left|(K-L_\infty^K)\ln(\frac{K}{K-1}))-(1-\mathbf{E}\left[b_\infty\right])\right|\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}\left[\left|(1-\mathbf{E}[b_\infty]) - K(1-\frac{L_\infty^K}{K})\ln(\frac{K}{K-1})\right|\right] \\
&= \mathbf{E}\left[\left|(1-\mathbf{E}[b_\infty]) + (1-\frac{L_\infty^K}{K})\ln(1-\frac{1}{K})^K\right|\right] \\
&\leq (1-\mathbf{E}[b_\infty])\left(1+\ln(1-\frac{1}{K})^K\right) + \left|\ln(1-\frac{1}{K})^K\right|\mathbf{E}\left[\left|\mathbf{E}[b_\infty] - \frac{L_\infty^K}{K}\right|\right].
\end{aligned}
$$

It is easy to see that $1-\frac{1}{K} \leq e^{-K}$ for all $K = 1, 2, \ldots$, and that

$$
\lim_{K\to\infty}\left(1+\ln(1-\frac{1}{K})^K\right) = 0 \quad \text{with} \quad 1+\ln(1-\frac{1}{K})^K \sim -\frac{1}{2K} \tag{6.3}
$$

Hence, $TH(K)$ converges to TH at a rate which is determined by that of the convergences $\lim_{K\to\infty}\mathbf{P}\left[B_\infty^{K,1}=0\right] = \mathbf{P}[b_\infty = 0]$ and $\lim_{K\to\infty}\mathbf{E}\left[\left|\mathbf{E}[b_\infty] - \frac{L_\infty^K}{K}\right|\right] = 0$. Work is in progress on these rates of convergence will be reported in [9, 10].

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