THESIS REPORT

Ph.D.

Robust Control of Set-Valued Discrete Time Dynamical Systems

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Ph.D. 95-8

ISR
INSTITUTE FOR SYSTEMS RESEARCH

ROBUST CONTROL OF SET-VALUED DISCRETE TIME DYNAMICAL SYSTEMS

by

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Dissertation submitted to the Faculty of the Graduate School
of The University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1995

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Abstract

Title of Dissertation: Robust Control of Set-Valued Discrete Time Dynamical
Systems.

Nital S. Patel, Doctor of Philosophy, 1995

Dissertation directed by: John S. Baras
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This thesis deals with the robust control of nonlinear systems subject to persistent

bounded non-additive disturbances. Such disturbances, could be due to exogenous

signals, or internal to the system as in the case of parametric uncertainty. The

problem solved could be viewed as an extension of l^1-optimal control to nonlinear

systems, however, now under very general non-additive disturbance assumptions.

We model such systems as inclusions, and set up an equivalent robust control

problem for the now set-valued dynamical system. Due to the fact, that inclusions

could arise from other considerations as well, we solve the control problem for

this general class of systems. The state feedback problem is solved via a game

theoretic approach, wherein the controller plays against the plant. For the output

feedback case, the concept of an information state is employed. The information

state dynamics define a new infinite dimensional system, and enables us to achieve

a separation between estimation and control. This concept is extended to the case of delayed measurements as well. For motivational purposes, we formally derive the information state from a risk-sensitive stochastic control problem via small noise limits. In general, the solution to the output feedback case involves solving an infinite dimensional dynamic programming equation. One way of avoiding this computation in practice is to consider certainty equivalence like controllers. This issue is considered, where we generalize the certainty equivalence controller to obtain other non-optimal, but dissipative output feedback policies. The approach followed yields both necessary and sufficient conditions for the solvability of the problem. We also present some applications of the theory developed.

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DEDICATION

To my parents

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude and appreciation to my advisor Dr. John S. Baras for his guidance and encouragement during my graduate research. His ideas and suggestions made my work all the more stimulating. I would also like to thank Dr. P. S. Krishnaprosad, Dr. Steven I. Marcus, Dr. W. P. Dayawansa, and Dr. Evanghelos Zafiriou for serving on my dissertation committee.

I am grateful to my colleagues in the Systems Integration Laboratory for making the lab a great place to work. Special thanks to Bernie Frankpitt for numerous discussions. Also, a special thank you to Tina Vigil.

Last, but most importantly, I thank Keya for being by my side.

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Chapter 1

Introduction

Robust control addresses the problem of designing high performance controllers when there is uncertainty in the system to be controlled. Various methodologies for robust control have evolved over the past decade. Linear systems have been the focus of attention, although research in nonlinear robust control is also gaining ground. We consider two distinct methodologies here.

The first one, H_∞ control evolved from a paper by Zames [58], where one is concerned with minimizing the H_∞ norm of the transfer function (T_{wz}) relating the exogenous inputs (w) to the regulated output (z), (see figure 1.1). In the time domain, the H_∞ norm of T_{wz} is nothing but the l^2 gain of the closed-loop system (from w to z). Motivated by the fact that such a definition (in terms of the l^2 gain) is independent of linearity, van der Schaft [53] postulated an equivalent performance problem for nonlinear systems in terms of minimizing the l^2 gain from w to z, i.e. given a nonlinear system (Σ), with zero initial conditions, find a controller
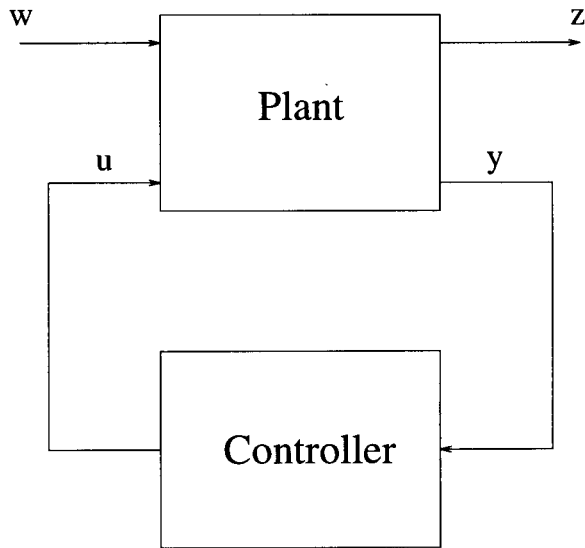
Figure 1.1: General plant-controller configuration.

u such that

$$\sup_{w\in l^2([0,\infty)),w\neq0} \frac{\|z\|}{\|w\|} \leq \gamma \qquad (1.1)$$

where $\gamma > 0$ is given. This soft constrained problem, also called nonlinear H_∞, has

a game theoretic interpretation, and a large body of literature has evolved around

it [6],[26], [28],[29],[31],[54]. However, certain key issues remain unresolved. The

primary one being, what interpretation does the l^2 norm attenuation have in terms

of performance specifications? Historically, H_∞ was postulated as a frequency

domain methodology for linear controller synthesis. For the linear case, the H_∞

technique can be employed to shape the maximum singular value plots (which

reduce to Bode plots for single input single output systems) of the sensitivity and

complimentary sensitivity functions. For the general nonlinear case, one does not

have such a nice interpretation. In [54] it was shown that the linearized nonlinear

state feedback H_∞ controller is identical to a linear H_∞ controller designed for the linearized plant with the same attenuation level γ. However, an analogous result for output feedback control has not been obtained. Also, notice that the supremum in equation (1.1) allows the noise to take on arbitrary large values. In the linear case, we can employ scaling to show that there is no conservativeness in this definition [24]. For the nonlinear case, however, this could yield extremely conservative policies. Finally, the case of parametric uncertainty, which is tackled in the linear case by representing it as a multiplicative perturbation (in the frequency domain) presents special difficulties in the nonlinear case.

The second methodology, based in the time domain, evolved from a paper by Vidyasagar [55]. It deals with minimizing the l^∞ norm of the regulated output (z) given persistent bounded exogenous inputs (w). For the linear case, this is called the l^1-optimal control problem. The solution to this problem was first obtained by [15], who employed Youla parameterization to parameterize all stabilizing compensators, and then solved for the optimal compensator via duality wherein the optimization can be recast as a linear programming problem. It has been noted that even the state feedback compensator could be dynamic [18], and this has led researchers to search for nonlinear static state feedback laws. A number of approaches based on identifying controlled-invariant sets have evolved [11],[50]. These sets, for the linear case are polytopes, and one again ends up with a linear programming problem. However, for the nonlinear case this niceness is lost.

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Moreover, these invariance based methods fail to extend to the output feedback case.

This dissertation is concerned with developing a framework for synthesizing control policies for general nonlinear systems subject to persistent bounded non-additive disturbances. The aim is to obtain ultimate boundedness controllers, where the bounding set is now specified by performance considerations. This problem can be viewed as an extension of l^1-optimal control to nonlinear systems. We employ a non-traditional approach to the synthesis problem, by first converting the system into an inclusion, where the right hand side is now given by a set-valued map.

Example 1: Let the system be given by

$$
\begin{aligned}
x_{k+1} &= f(x_k, u_k, w_k) \\
y_{k+1} &= g(x_k, u_k, v_k)
\end{aligned}
\tag{1.2}
$$

with $w_k \in \mathcal{W}$, $v_k \in \mathcal{V}$ for all k. Here, $\mathcal{W}$ and $\mathcal{V}$ are appropriate bounding sets for the noise inputs. We recast this system as an inclusion by defining set-valued maps $\mathcal{F}$, and $\mathcal{G}$ as follows:

$$
\begin{aligned}
\mathcal{F}(x, u) &\triangleq \bigcup_{w \in \mathcal{W}} f(x, u, w) \\
\mathcal{G}(x, u) &\triangleq \bigcup_{v \in \mathcal{V}} g(x, u, v)
\end{aligned}
$$

and rewrite the system (1.2) as

$$
\begin{aligned}
x_{k+1} &\in \mathcal{F}(x_k, u_k) \\
y_{k+1} &\in \mathcal{G}(x_k, u_k)
\end{aligned}
\tag{1.3}
$$

i.e. as an inclusion. This idea, of modeling dynamic systems as inclusions is not new. In fact, the study of such equations (albeit in continuous time) was initiated by Zaremba in 1934, and Marchaud in 1938. Interest was revived in the early sixties by the work of Fillipov and Ważewski. In this regard, we mention the pioneering work by Filippov [20] on discontinuous systems (e.g. systems subject to friction). In fact, it seems that set-valued maps have been carefully - maybe unconsciously - hidden in control theory. This is exemplified by the fact, that over the past several decades starting from the work of Schweppe [48], several researchers have developed (set-valued) ellipsoidal techniques for estimating the set of feasible states of linear systems subject to bounded noise [17],[21],[48],[34]. Striking similarities have emerged, between some of these algorithms, and well established techniques such as weighed recursive least squares, and the Kalman-Buchy solution to the Wiener filtering problem. However, these ellipsoidal techniques have not enjoyed much success as aiding in a synthesis theory. A notable exception is the work by Kurzhanski and Valyi [35] for the state feedback case, where the authors combined ellipsoidal techniques and viability theory [2].

Note that, description (1.3) is quite general, in the sense, that one could arrive at it by other considerations.

Example 2: Consider the model for a hybrid system where an upper logical level switches between different plants, depending on observed events [23],[42]. Here,

we assume that the system is subject to bounded additive noise.

$$x_{k+1} = f^i(x_k, u_k) + w_k, \quad x_0 = \bar{x} \tag{1.4}$$

$x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^m$, $i = 1, \ldots, N$, and $w_k \in \bar{\mathcal{B}}_\epsilon(0)$, where $\bar{\mathcal{B}}_\epsilon(0) \subset \mathbf{R}^n$ denotes the closed ball of radius ϵ centered at 0. We define

$$\mathcal{F}(x_k, u_k) \triangleq \bigcup_{i=1}^{N} f^i(x_k, u_k) + \bar{\mathcal{B}}_\epsilon(0)$$

Then, it can be argued that any trajectory generated by (1.4), can also be generated by the inclusion

$$x_{k+1} \in \mathcal{F}(x_k, u_k), \quad x_0 = \bar{x}$$

Stability results for hybrid systems (1.4) based on Lyapunov-like functions have been presented in [12],[42]. However, no concept of robust performance (in the sense of minimizing variations in the regulated outputs due to switching between different plant models and noise), particularly for the output feedback case exists for such systems.

Another common example is a special case of (1.2), where the noise contains components representing parametric uncertainty. Stabilization results for such (linear) systems (without exogenous noise) based on Lyapunov functions, and Ricatti equations can be found in [43],[46],[59],[33],[45]. In [7], Barmish *et al.* consider a similar problem, but now with bounded additive noise, and the aim is to obtain ultimately boundedness control under certain **matching** conditions. Linear inclusions also

occur in global linearization of nonlinear systems ([38],[37]). Furthermore, inclusions could arise naturally when the available information about the system is not sufficient to generate a reliable model.

The main contributions of this thesis are as follows: (i) We develop a rigorous framework for solving control problems for systems modeled as inclusions. Of particular interest here, is the fact that nonlinear systems under very general bounded noise structure can be represented as inclusions. Our approach yields both necessary and sufficient conditions for solvability. (ii) We give an example of a stochastic control problem, which is subject to both uniform and Gaussian noise. Under large deviations type limits on the Gaussian component, the problem tends to a deterministic robust control problem. This aids in motivating the problem formulation, and the information state employed to solve the output feedback problem. (iii) Some results are presented, which could enable one to view the information state as a weighed indicator function of the set of feasible states. (iv) We discuss certainty equivalence, and give a condition under which certainty equivalence holds. This condition maybe more tractable than the one currently existing in the literature. We also establish that the condition under which certainty equivalence holds is equivalent to the existence of a solution to a functional equation. (v) In doing so, we also give a class of controllers, which although non-optimal, guarantee that the closed-loop system satisfies a dissipation inequality. (vi) We explicitly treat the delayed measurement case. The main motivating factor for doing so was its

importance to industry[1].

To illustrate the use of this methodology, we present several examples. These are based on (i) an unstable nonlinear plant, (ii) a discontinuous system with parametric uncertainty, and (iii) run by run control, where we consider the problems of end-pointing and rate control in a low pressure chemical vapor deposition (LPCVD) reactor.

Some of the developments require the notions of continuity of set-valued maps and limits of sequences of sets. For a general background in set-valued calculus, we refer the reader to [5],[16],[3],[13].

1.1 Organization of the Thesis

We start by stating the problem in the next section. The motivation for the choice of cost function is delayed till chapter 3, where we consider a risk-sensitive stochastic control problem. It is observed that when we take the small noise limits of this problem, we can recover the deterministic cost function. However, the main aim of chapter 3 is to motivate the information state recursion employed to solve the output feedback problem in chapter 4. Chapter 3 is in that sense motivational, but serves two important purposes. Firstly, it links the problem under consideration to a stochastic control problem, and secondly it shows that the information

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[1]In this regard, we are grateful to Dr. Stephanie W. Butler of Texas Instruments for some interesting discussions.

state is a weighed indicator function of the set of feasible states. How one applies the cost function to the problem of controller synthesis is illustrated in chapter 6, which presents a design framework, and some examples. Chapters 2 and 4 deal with the state feedback and output feedback cases respectively. The approach is motivated by recent results obtained in the nonlinear H_∞ context by [31], which employs the dynamic game framework of [8]. In our case, we have an unusual game, in the sense that the controller is now playing against the system itself. For the infinite time case, we employ the theory of dissipative systems [57] to write down a version of the bounded real lemma. The latter is expressed in terms of a dissipation inequality, which has appeared repeatedly in literature dealing with nonlinear and linear robust control (e.g. [6],[26],[27],[28],[31],[44] ,[54],[24]). Chapter 4 also contains the case of delayed measurements, considering its importance to practical applications.

The results obtained in chapter 4, involve solving an infinite dimensional dynamic programming problem. A standard practice has been to employ a certainty equivalence controller [52]. In chapter 5, this issue is examined, and the notion of certainty equivalence is generalized to yield non-optimal dissipative policies. Finally, chapter 6 gives some applications of the methodology developed.

Before, concluding this section, we hint at the structure of the controller obtained. The controller is separated, in the sense it has two parts corresponding to estimation and control, and this is shown in figure 1.2.
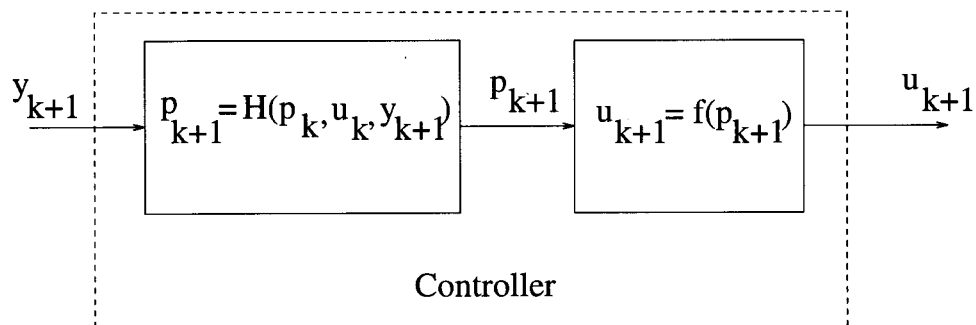
Figure 1.2: Structure of the controller.

1.2 Problem Formulation

The system under consideration (Σ), is expressed as

$$\Sigma \begin{cases} x_{k+1} & \in \quad \mathcal{F}(x_k, u_k) \ , \quad x_0 \in X_0 \\ y_{k+1} & \in \quad \mathcal{G}(x_k, u_k) \\ z_{k+1} & = \quad l(x_{k+1}, u_k) \ , \quad k = 0, 1, \ldots \end{cases} \tag{1.5}$$

Here, $x_k \in \mathbf{R}^n$ are the states, $u_k \in \mathbf{U} \subset \mathbf{R}^m$ are the control inputs, $y_k \in \mathbf{R}^t$ are the measured variables, and $z_k \in \mathbf{R}^q$ are the regulated outputs.

Remark 1.1 Note that the regulated output z_{k+1} is expressed in terms of x_{k+1} and u_k. The presence of u_k is incorporated for the sake of generality. In most situations of interest, u_k will not be present. This fact will be made explicit in chapter 6. This differs from the traditional linear quadratic type cost, which includes a term in the control. Linear quadratic control can qualitatively be viewed as "minimum effort" control. However, the justification for minimizing control effort has never been clarified. A common view due to Athans [1] is that minimizing control effort and state excursions reduces the likelihood that nonlinearities (e.g. saturation) will be encountered during a disturbance. For the case at hand, we could explicitly

specify the control bounds in the definition of U. Furthermore, as will be shown in chapter 6, the augmented plant could contain filters to further shape the control response. In that case, we could re-index the regulated output as $z_k = l(x_k)$. The (apparently) strange indexing of z is necessary to maintain indexing compatibility with the system dynamics.

1.2.1 Notation

Some of the notation employed in the thesis will be as follows:

$|\cdot|$ denotes any suitable norm.

$x_{i,j}$ denotes a sequence $\{x_i, x_{i+1}, \ldots, x_j\}$.

$\Gamma_{0,k}^u(x)$ denotes the truncated forward cone of the point $x \in R^n$ [4]. In particular

$$\Gamma_{0,k}^u(x) \triangleq \{x_{0,k} | x_{j+1} \in \mathcal{F}(x_j, u_j), j = 0, \ldots, k-1; x_0 = x\}.$$

i.e. $\Gamma_{0,k}^u(x)$ is the set of all possible state trajectories that the system can generate in the time interval $[0, k]$, given a control policy u, and initial condition x.

$X_k^u(x_0) \subset R^n$ as the cross section of the forward cone of x_0 at time instant k.

We furthermore write $r, s \in \Gamma_{0,k}^u(x)$ to denote the set of trajectories such that $r \in \Gamma_{0,k}^u(x)$ and $s_{i+1} \in \mathcal{F}(r_i, u_i)$ for $i = 0, \ldots, k-1$.

$\mathcal{B}_a(b)$ denotes an open ball of radius a centered at b, and $\bar{\mathcal{B}}_a(b)$ similarly denotes a closed ball.

S and O denote the space of static state and dynamic output feedback policies respectively. If $\bar{u} \in S$, then for any k, the control value $u_k = \bar{u}(x_k) \in U$. Furthermore, we write $S_{i,j}$ to denote policies defined only for the time interval $i, i+1, \ldots, j$. Similarly, if $\bar{u} \in O$, then for any k, $u_k = \bar{u}(y_{1,k}) \in U$. Furthermore, $O_{i,j}$ is defined in a similar manner as $S_{i,j}$.

$\delta_M : \boldsymbol{R}^n \to \boldsymbol{R}^*$ is defined by

$$\delta_M(x) \triangleq \begin{cases} 0 & \text{if } x \in M \\ -\infty & \text{else} \end{cases} \qquad (1.6)$$

For the output feedback case, we define

$$\Delta_{1,K}^u(x_0) = \{y_{1,K} \mid y_{k+1} \in \mathcal{G}(x_k, u_k), \ \forall x \in \Gamma_{0,K-1}^u(x_0)\}$$

$$\Gamma_{0,K}^{u,y}(x_0) = \{x_{0,K} \in \Gamma_{0,K}^u(x_0) \mid y_{k+1} \in \mathcal{G}(x_k, u_k), \ k = 0, \ldots, K-1\}$$

We will also write $r, s \in \Gamma_{0,K}^{u,y}(x)$ in a similar manner as for $\Gamma_{0,K}^u(x)$.

Finally, given any set-valued map $\Lambda(x)$, we occasionally write

$$\Lambda(M) = \bigcup_{x \in M} \Lambda(x)$$

Remark 1.2 Of particular interest here, is the case, when $| \cdot |$ denotes the ∞ norm. But since the results are norm-independent, we choose to pursue this level of generality.

1.2.2 Assumptions

The following assumptions are made on the system Σ

A1. $0 \in X_0$.

A2. $\mathcal{F}(x, u)$, $\mathcal{G}(x, u)$ are compact for all $x \in \boldsymbol{R}^n$ and $u \in \boldsymbol{U}$.

A3. The origin is an equilibrium point for $\mathcal{F}$, $\mathcal{G}$ and l. i.e.

$$\mathcal{F}(0,0) \ni 0 \; ; \quad \mathcal{G}(0,0) \ni 0 \; ; \quad l(0,0) = 0$$

A4. There exists an $\bar{\epsilon} > 0$, such that for all $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{U}$, $\bar{\mathcal{B}}_\epsilon(r) \subset \mathcal{F}(x, u)$ for

some $r \in \mathcal{F}(x, u)$, $\bar{\epsilon} > \epsilon > 0$.

A5. $l(\cdot, u) \in C^1(\boldsymbol{R}^n)$ for all $u \in \boldsymbol{U}$ and is such that, $\exists \gamma_{min} > 0$, such that

$$\mathcal{L}^\gamma \triangleq \left\{ s \in \boldsymbol{R}^n \mid \exists u \in \boldsymbol{U} \text{ s.t. } \left| \frac{\partial}{\partial x} l(s, u) \right| \leq \gamma \right\}$$

is compact and contains the origin $\forall \gamma \geq \gamma_{min}$.

A6. $\boldsymbol{U} \subset \boldsymbol{R}^m$ is compact.

Remark 1.3 The smoothness assumptions in **A5** can be relaxed. This is considered in Appendix A. Also, in **A5**, we can replace compactness by boundedness (see remark 2.14). Furthermore, in **A5**, we can get away with a subset of the states being bounded, provided we can use invariance to establish that the remaining

states are bounded as well. We can also relax assumption **A4**, to let $\mathcal{F}$ be locally connected. How one deals with these situations is very much problem dependent, and for clarity of exposition we assume that the above assumptions hold. Remark 6.2, on page 105 illustrates the idea via an example. To this end note that **A5** is not particularly restrictive, since a common cost (often encountered in practice), $l(x) = x^T Q x$, with $Q > 0$, trivially satisfies the assumption. Here, x^T denotes the vector transpose.

Remark 1.4 We have assumed that the initial states (x_0) belong to a set X_0, rather than all of $\boldsymbol{R}^n$. The reason being, we could have actuator limits (imposed by $\boldsymbol{U}$). In which case, we may want a bounded X_0. It is easy to come up with examples of systems, where allowing arbitrary x_0 results in unreasonable problems. As an example consider the following system

$$x_{k+1} \in [3,4] x_k + u_k + [-1,1]$$

with $\boldsymbol{U} = [-1,1]$. Let $x_0 = 10$. Clearly, one can check by direct computation that for any trajectory r of this system we have $r_k \to \infty$ as $k \to \infty$.

Remark 1.5 We could have also assumed that the set-valued maps have convex images. There is no loss of generality, considering that under mild assumptions the trajectories of the original system are dense in those generated by the convexified (relaxed) system (Filippov-Ważewski theorem [3]).

1.2.3 Statement of the Problem

The robust control problem can now be stated as:

Given $\gamma \geq \gamma_{min}$, find a controller u ($\in S$ or O depending on what is measured) such that the closed loop system Σ^u satisfies the following three conditions:

C1. Σ^u is weakly asymptotically stable, in the sense that for any trajectory x, and for each k, there exists an $\alpha_k \in \mathcal{F}(x_k, u_k)$ such that, the sequence $\alpha_k \longrightarrow 0$ as $k \longrightarrow \infty$. i.e. [5]

$$0 \in \liminf_{k \to \infty} \mathcal{F}(x_k, u_k)$$

C2. Σ^u is ultimately bounded.

C3. (Finite Gain) There exists a finite $\beta^u(x) \geq 0$, with $\beta^u(0) = 0$ such that

$$\sup_{r,s \in \Gamma^u(x_0), r \neq s} \sum_{i=0}^{\infty} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 -\gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \leq \beta^u(x_0), \quad (1.7)$$

$\forall x_0 \in X_0$. The above condition states a performance specification in terms of the (Lipschitz) induced norm. The reason for choosing this form of the cost function is elaborated upon in the next section. Define $\Gamma^u(0) \mid_{l^2} \triangleq \{r, s \in \Gamma^u(0) \mid r - s \in l^2\}$. Then, the above guarantees that for $r, s \in \Gamma^u(0) \mid_{l^2}$,

$$\sup_{r,s \in \Gamma^u(0)\mid_{l^2}, r \neq s} \frac{\|l(r, u) - l(s, u)\|_{l^2}}{\|r - s\|_{l^2}} \leq \gamma$$

provided, of course that $\mid \cdot \mid$ now denotes the Euclidean norm.

Although not explicitly stated, in what follows it is sufficient that $\beta^u(x)$ be defined only on X_0.

We call the closed-loop system **finite gain** if **C3** is satisfied.

Remark 1.6 Condition **C3** yields a soft-constrained problem, and typically one carries out γ-iterations to obtain the sub-optimal γ.

Remark 1.7 Assumptions **A4** and **A5** ensure non-triviality of **C3**. In particular, assumption **A4** precludes single-valued systems, for which **C3** is trivially satisfied for any value of γ. Also, assumption **A5**, precludes l from being uniformly Lipschitz continuous, since for such functions **C3** is satisfied by any control policy for γ large enough.

Remark 1.8 We can also generalize equation (1.7) as

$$\sup_{r,s \in \Gamma^u(x_0), r \neq s} \sum_{i=0}^{\infty} |l(r_{i+1}, u_i) - l(s_{i+1}, u_i)|^p - \gamma^p |r_{i+1} - s_{i+1}|^p \leq \beta^u(x_0)$$

for any $p \in [1, \infty)$. The results presented in this thesis for the deterministic problem remain unchanged. However, in order to maintain compatibility with the results from chapter 3 (where we consider a stochastic control problem), we explicitly set $p = 2$.

1.2.4 Motivating the Cost

In this section, we try to motivate the cost employed in condition **C3**. The aim of the robust control problem, is to attenuate the influence of the set-valued dynamics
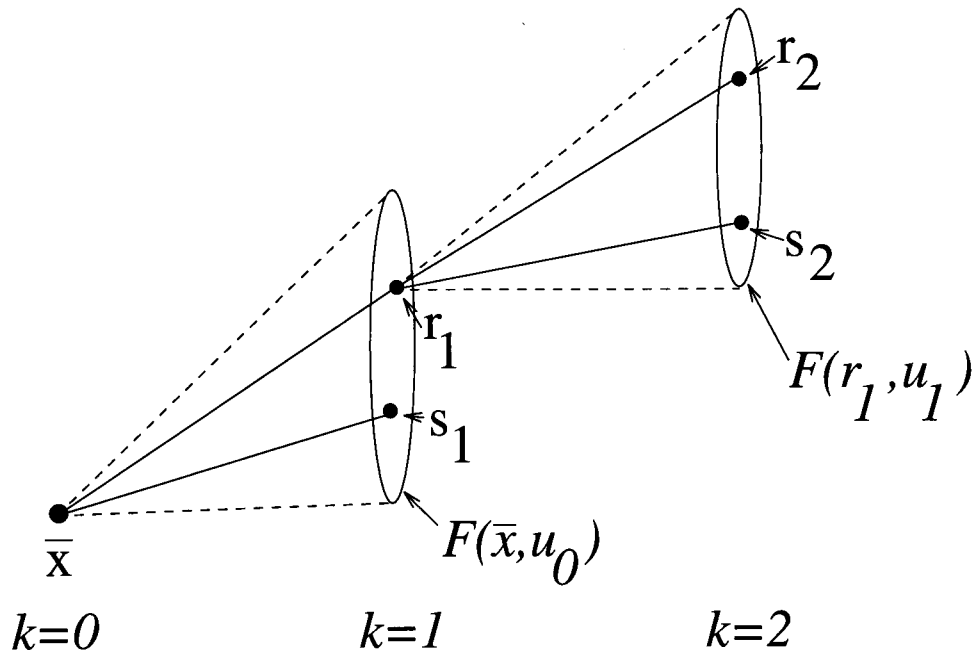
Figure 1.3: Set of possible state trajectories.

on the regulated output z. To this end consider a finite time problem, where the time horizon is 2. We are given a $\gamma > 0$, and an admissible control policy u. We denote the initial state value by $\bar{x}$.

Consider figure 1.3. From $\bar{x}$, we can go to any point in $\mathcal{F}(\bar{x}, u_0)$. Suppose that the next state the system goes to is r_1. Note that the system could have also gone to s_1. Now, from r_1 we can go to any arbitrary point in $\mathcal{F}(r_1, u_1)$, where u_1 is the control value at time $k = 1$. We again pick two points r_2, s_2 in $\mathcal{F}(r_1, u_1)$. The variation in the regulated output that could occur is therefore,

$$\left(|l(r_1, u_0) - l(s_1, u_0)|^2 + |l(r_2, u_1) - l(s_2, u_1)|^2 \right)^{\frac{1}{2}}$$

where we assume we are working with the Euclidean norm. We now normalize this

by the distance between r and s, i.e. by

$$\left(|r_1 - s_1|^2 + |r_2 - s_2|^2\right)^{\frac{1}{2}}$$

The reason for doing so is we are trying to attenuate the **influence** of the set-valued dynamics on the regulated output, and not the variation in the regulated output itself. We can write the worst-case normalized variation in the regulated output as

$$\sup_{r,s \in \Gamma_{0,2}^u(\bar{x})} \frac{\left(|l(r_1, u_0) - l(s_1, u_0)|^2 + |l(r_2, u_1) - l(s_2, u_1)|^2\right)^{\frac{1}{2}}}{\left(|r_1 - s_1|^2 + |r_2 - s_2|^2\right)^{\frac{1}{2}}}$$

If $\bar{x} = 0$, we now require that for the given γ, the control policy be such that this worst-case normalized variation is bounded by γ, or that

$$\sup_{r,s \in \Gamma_{0,2}^u(0)} \sum_{k=1}^{2} |l(r_k, u_{k-1}) - l(s_k, u_{k-1})|^2 - \gamma^2 |r_k - s_k|^2 \leq 0$$

Then, generalizing to arbitrary $\bar{x} \in X_0$, we require the existence of a finite $\beta^u(x) \geq 0$, $\beta^u(0) = 0$, such that

$$\sup_{r,s \in \Gamma_{0,2}^u(\bar{x})} \sum_{k=1}^{2} |l(r_k, u_{k-1}) - l(s_k, u_{k-1})|^2 - \gamma^2 |r_k - s_k|^2 \leq \beta^u(\bar{x})$$

for all $\bar{x} \in X_0$. This condition requires that the worst-case normalized variation of the regulated output be finite for all initial conditions. One now repeats the above process for an arbitrary large time horizon to obtain equation (1.7).

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Chapter 2

The State Feedback Case

In the state feedback case, the problem is to find a controller $u \in S$, i.e. $u_k = u(x_k)$, with $u_k \in U(x_k)$, where $U : R^n \longmapsto R^m$, such that the three conditions (**C1-C3**) stated on page 15 are satisfied. We pursue this level of generality here, due to the importance of this problem to the case when one has state constrained controls [2]. The special case of this is clearly $U(x) \equiv U$. We also assume that the set-valued map $U(x)$ assumes compact values for all $x \in R^n$.

2.1 Finite Time Case

For the finite time case, conditions **C1** and **C2** of section 1.2.3 are not required. From condition **C3** of section 1.2.3 we require the existence of a finite $\beta_K^u(x_0)$, $\beta_K^u(0) = 0$ such that

$$\sum_{i=0}^{K-1} \left(\mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 -\gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \right) \leq \beta_K^u(x_0),$$

$$(2.1)$$

$$\forall K \geq 1, \forall r, s \in \Gamma_{0,K}^u(x_0), \forall x_0 \in X_0$$

2.1.1 Dynamic Game

Here, the robust control problem is converted into an equivalent dynamic game. For $u \in S_{k,K-1}$ and $\bar{x} \in X_k(x_0)$, where $X_k(x_0)$ is the set of states that the system can achieve at time k if it were started from x_0, define

$$J_{\bar{x},k}(u) = \sup_{r,s \in \Gamma_{k,K}^u(\bar{x})} \{ \sum_{i=k}^{K-1} (\mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 - \gamma^2 \mid r_{i+1} - s_{i+1} \mid^2) \} \quad (2.2)$$

Clearly

$$J_{\bar{x},k}(u) \geq 0.$$

Now, the finite gain property can be expressed as below

Lemma 2.1 $\Sigma_{\bar{x}}^u$ is finite gain on [k,K] if and only if there exists a finite $\beta_K^u(\bar{x})$, $\beta_K^u(0) = 0$ such that

$$J_{\bar{x},j}(u) \leq \beta_K^u(\bar{x}), \quad j \in [k, K], \ \forall \bar{x} \in X_0 \quad (2.3)$$

$\square$

The problem is hence reduced to finding a $u^* \in S_{k,K-1}$ which minimizes $J_{\bar{x},k}$.

2.1.2 Solution to the Finite Time State Feedback Robust Control Problem

We can solve the above using dynamic programming. Define

$$V_k(\bar{x}) = \inf_{u \in S_{k,K-1}} \sup_{r,s \in \Gamma_{k,K}^u(\bar{x})} \{ \sum_{i=k}^{K-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 -\gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \} \quad (2.4)$$

The corresponding dynamic programming equation is

$$\begin{aligned} V_k(x) &= \inf_{u \in U(x)} \sup_{r,s \in \mathcal{F}(x,u)} \{ \mid l(r, u) - l(s, u) \mid^2 -\gamma^2 \mid r - s \mid^2 +V_{k+1}(r) \} \\ V_K(x) &= 0 \end{aligned}$$

$$(2.5)$$

Note that we have abused notation, and here u is a vector instead of a function as in equation (2.4).

Theorem 2.2 (Necessity) Assume that $u^* \in S_{0,K-1}$ solves the finite time state feedback robust control problem. Then, there exists a solution V to the dynamic programming equation (2.5) such that $V_k(x) \geq 0$, $V_k(0) = 0$, $k \in [0, K-1]$, $x \in X_0$.

Proof:

For $x \in X_0$, $k \in [0, K-1]$ define

$$V_k(x) = \inf_{u \in S_{k,K-1}} J_{x,k}(u)$$

Then, we have

$$0 \leq V_k(x) \leq \beta_K^{u^*}(x), \quad k \in [0, K-1], \ x \in X_0$$

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Thus, V_k is finite on X_0, and by dynamic programming, V satisfies equation (2.5).

Also, since $\beta_K^{u^*}(0) = 0$, $V_k(0) = 0$.

□

Theorem 2.3 (Sufficiency) Assume that there exists a solution V to the dynamic programming equation (2.5), such that $V_k(x) \geq 0$, $V_k(0) = 0$, $k \in [0, K-1]$, $x \in X_0$. Let $u^* \in S_{k,K-1}$ be a control policy such that u_k^* achieves the minimum in equation (2.5) for $k = 0, \ldots, K-1$. Then u^* solves the finite time state feedback robust control problem.

Proof: Dynamic programming arguments imply that for a given $x \in X_0$

$$V_0(x) = J_{x,0}(u^*) = \inf_{u \in S_{0,K-1}} J_{x,0}(u)$$

Thus u^* is an optimal policy for the game and lemma 2.1 is satisfied with $u = u^*$, where we obtain $\beta_K^u(x) = V_0(x)$.

□

2.2 Infinite Time Case

Here, we are interested in the limit as $K \longrightarrow \infty$. Invoking stationarity equation (2.5) becomes

$$V(x) = \inf_{u \in \mathbf{U}(x)} \sup_{r,s \in \mathcal{F}(x,u)} \left\{ V(s) + \mid l(r,u) - l(s,u) \mid^2 - \gamma^2 \mid r - s \mid^2 \right\} \qquad (2.6)$$

2.2.1 The Dissipation Inequality

We say that the system $\Sigma^{\bar{u}}$ is **finite gain dissipative** if there exists a function $V(x)$ (called the storage function), such that $V(x) \geq 0$, $V(0) = 0$, and it satisfies the dissipation inequality

$$V(x) \geq \sup_{r,s \in \mathcal{F}(x,\bar{u}(x))} \left\{ V(s) - \gamma^2 \mid r - s \mid^2 + \mid l(r, \bar{u}(x)) - l(s, \bar{u}(x)) \mid^2 \right\} \qquad (2.7)$$

$$\forall x \in X_k^{\bar{u}}(x_0), \forall k \geq 0, \forall x_0 \in X_0$$

where $\bar{u}(x)$ is the control value for state x.

Theorem 2.4 Let $u \in S$. The system Σ^u is finite gain if and only if it is finite gain dissipative.

Proof:

(i) Assume Σ^u is finite gain dissipative. Then equation (2.7) implies

$$V(x_0) \geq V(r_k) - \gamma^2 \sum_{i=0}^{k-1} \mid r_{i+1} - s_{i+1} \mid^2 + \sum_{i=0}^{k-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2,$$

$$\forall k > 0; \quad \forall r, s \in \Gamma^u(x_0)$$

Since $V \geq 0$ for all $r, s \in \Gamma_{0,k}^u(x_0)$, this implies

$$\sum_{i=0}^{k-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 - \gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \leq V(x_0)$$

Thus Σ^u is finite gain.

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(ii) Assume Σ^u is finite gain. For any $x_0 \in X_0$ and $k \geq 0$, define for $x \in X_k^u(x_0)$

$$\tilde{V}_{k,j}^u(x, x_0) = \sup_{r,s \in \Gamma^u(x)} \left\{ \sum_{i=0}^{j-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 -\gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \right\}$$

Then we have for any $x \in X_k^u(x_0)$

$$0 \leq \tilde{V}_{k,j}^u(x, x_0) \leq \beta^u(x_0), \quad \forall j \geq 0$$

Furthermore

$$\tilde{V}_{k,j+1}^u(x, x_0) \geq \tilde{V}_{k,j}^u(x, x_0) , \quad \forall x \in X_k^u(x_0)$$

Furthermore, note that by time invariance, $\tilde{V}_{k,j}^u(x, x_0)$ depends only on x and j.

Thus if $x \in X_{k_1}^u(x_0^1) \bigcap X_{k_2}^u(x_0^2)$ then $\tilde{V}_{k_1,j}^u(x, x_0^1) \equiv \tilde{V}_{k_2,j}^u(x, x_0^2)$. Hence,

$$\tilde{V}_{k,j}^u(x, x_0) \longrightarrow V^u(x), \quad \text{as } k \longrightarrow \infty , \quad \forall x \in X_k^u(x_0), k \geq 0, x_0 \in X_0$$

Also, we have

$$0 \leq V^u(x_0) \leq \beta^u(x_0)$$

Since

$$V(x) = \inf_{u \in S} V^u(x) = V^{\bar{u}}(x) \leq V^u(x)$$

dynamic programming implies that $V^u(x)$ solves the dissipation inequality 2.7 for all $x \in X_k^u(x_0)$, $k \geq 0$, $x_0 \in X_0$. Furthermore $V^u(x) \geq 0$ and $V^u(0) = 0$. Thus V^u is a storage function and hence Σ^u is finite gain dissipative.

$\square$

We now have to show that the control policy $u \in S_{[0,\infty)}$ which renders Σ^u finite gain dissipative, also guarantees ultimate boundedness of trajectories, and furthermore under a certain **detectability** type assumption, the existence of a sequence $\alpha_n \in \mathcal{F}(x_n, u_n)$ such that $\lim_{n\to\infty} \alpha_n = 0$. The above can be also expressed as [5]

$$0 \in \liminf_{k \to \infty} \mathcal{F}(x_k, u_k)$$

We, first study the convergence of

$$W_i^{\bar{u}} = \sup_{r \in \mathcal{F}(\bar{x}_i, \bar{u}_i)} (\mid l(r, \bar{u}_i) - l(\bar{x}_{i+1}, \bar{u}_i) \mid^2 - \gamma^2 \mid r - \bar{x}_{i+1} \mid^2)$$

to zero, where $\bar{x}$ is a trajectory generated by the control $\bar{u}$.

Lemma 2.5 If $W_k^{\bar{u}} \longrightarrow 0$, as $k \longrightarrow \infty$, then $\forall \epsilon > 0$, $\exists K$ such that $\forall k \geq K$, $\exists \delta$ such that

$$\mid r - \bar{x}_{k+1} \mid < \delta \implies \mid l(\bar{x}_{k+1}, \bar{u}_k) - l(r, \bar{u}_k) \mid < \epsilon$$

Proof:

Suppose to the contrary. Then $\exists \epsilon > 0$ such that, $\forall K$, $\exists k \geq K$, such that $\forall \delta > 0$

$$\mid r - \bar{x}_{k+1} \mid < \delta \implies \mid l(\bar{x}_{k+1}, \bar{u}_k) - l(r, \bar{u}_k) \mid \geq \epsilon$$

Fix δ such that $0 < \delta < \bar{\epsilon}$ and $\delta < \sqrt{\epsilon}$. Then for any $s \in \mathcal{B}_{\frac{\delta}{\gamma}}(\bar{x}_{k+1}) \bigcap \mathcal{F}(\bar{x}_k, u_k)$

$\subset \mathcal{F}(\bar{x}_k, u_k)$

$$\mid l(\bar{x}_{k+1}, u_k) - l(s, \bar{u}_k) \mid^2 - \gamma^2 \mid \bar{x}_{k+1} - s \mid^2 \geq \epsilon - \delta^2 = \eta$$

This contradicts the convergence of $W_k^{\bar{u}}$.

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$\square$

Remark 2.6 The above lemma gives a necessary condition for the sequence $W_k^{\bar{u}}$ to converge.

Lemma 2.7 If $W_k^{\bar{u}} \longrightarrow 0$, as $k \longrightarrow \infty$, then $\forall \epsilon, \hat{\epsilon} > 0, \bar{\epsilon} > \epsilon > 0, \exists K$ such that $\forall k \geq K, \exists r \in \mathcal{B}_\epsilon(\bar{x}_{k+1}) \bigcap \mathcal{F}(\bar{x}_k, \bar{u}_k)$ with $r \neq \bar{x}_{k+1}$ and

$$\frac{\mid l(r, \bar{u}_k) - l(\bar{x}_{k+1}, \bar{u}_k) \mid}{\mid r - \bar{x}_{k+1} \mid} < \gamma + \hat{\epsilon} \tag{2.8}$$

Proof:

By contradiction. $\exists \hat{\epsilon}, \epsilon > 0, \bar{\epsilon} > \epsilon > 0$, such that $\forall K \exists k \geq K$ such that

$$\frac{\mid l(r, \bar{u}_k) - l(\bar{x}_{k+1}, \bar{u}_k) \mid}{\mid r - \bar{x}_{k+1} \mid} \geq \gamma + \hat{\epsilon}, \ \forall r \in \mathcal{B}_\epsilon(\bar{x}_{k+1}) \bigcap \mathcal{F}(\bar{x}_k, \bar{u}_k), \ r \neq \bar{x}_{k+1}$$

Hence, $\exists \eta > 0$ such that

$$\mid l(r, \bar{u}_k) - l(\bar{x}_{k+1}, \bar{u}_k) \mid^2 - \gamma^2 \mid r - \bar{x}_{k+1} \mid^2 \geq \eta \mid r - \bar{x}_{k+1} \mid^2$$

Let $r \in \mathcal{B}_\epsilon(\bar{x}_{k+1}) \bigcap \mathcal{F}(\bar{x}_k, \bar{u}_k)$ be such that $\epsilon > \mid r - \bar{x}_{k+1} \mid > \frac{\epsilon}{2}$. Thus,

$$\mid l(r, \bar{u}_k) - l(\bar{x}_{k+1}, \bar{u}_k) \mid^2 - \gamma^2 \mid r - \bar{x}_{k+1} \mid^2 \geq \eta \frac{\epsilon^2}{4} = \hat{\eta}$$

Hence, $\exists \hat{\eta} > 0$ such that $\forall K, \exists k \geq K$ such that

$$W_k^{\bar{u}} \geq \hat{\eta}$$

Hence, we get a contradiction.

$\square$

Corollary 2.8 If $W_k^{\bar{u}} \longrightarrow 0$, as $k \longrightarrow \infty$, then

$$\limsup_{k \to \infty} \mid \frac{\partial}{\partial x} l(\bar{x}_{k+1}, \bar{u}_k) \mid \leq \gamma$$

Proof:

Take the limit in equation (2.8) as ϵ, $\hat{\epsilon} \longrightarrow 0$, and use Assumption **A4**.

$\square$

Before we can prove weak asymptotic stability, we need the following additional assumption on the system Σ.

A7. Assume that for a given $\gamma > 0$, the system $\Sigma^{\bar{u}}$ is such that

$$\limsup_{k \to \infty} \mid \frac{\partial}{\partial x} l(\bar{x}_{k+1}, \bar{u}_k) \mid \leq \gamma$$

implies $0 \in \liminf_{k \to \infty} \mathcal{F}(\bar{x}_k, \bar{u}_k)$.

Remark 2.9 The assumption above, can be viewed to be analogous to the detectability assumption often encountered in H_∞ control literature e.g. [44],[27].

The following theorem gives a sufficient condition for weak asymptotic stability.

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Theorem 2.10 If for a given $\gamma > 0$, $\Sigma^{\bar{u}}$ is finite gain dissipative and satisfies assumption **A7**, then $\Sigma^{\bar{u}}$ is weakly asymptotically stable.

Proof:

From the dissipation inequality (2.7), we obtain for any $x_0 \in X_0$

$$\sum_{i=0}^{K} \mid l(r_{i+1}, \bar{u}_i) - l(s_{i+1}, \bar{u}_i) \mid^2 - \gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \leq V(x_0), \quad \forall K; \; r, s \in \Gamma^{\bar{u}}(x_0).$$

In particular for any $x \in \Gamma^{\bar{u}}(x_0)$

$$\sum_{k=0}^{K} W_k^{\bar{u}} \leq V(x_0), \quad \forall K$$

We know that $W_k^{\bar{u}} \geq 0$, $\forall k$. This implies that

$$W_k^{\bar{u}} \longrightarrow 0 \;\; \text{as} \; k \longrightarrow \infty$$

Hence, by corollary 2.8 and assumption **A7**, we obtain

$$0 \in \liminf_{k \longrightarrow \infty} \mathcal{F}(\bar{x}_k, \bar{u}_k)$$

This implies that $\exists \alpha_n \in \mathcal{F}(\bar{x}_n, \bar{u}_n)$ such that $\lim_{n \longrightarrow \infty} \alpha_n = 0$.

Hence, $\forall x \in \Gamma^{\bar{u}}(x_0)$, $\exists \alpha_n \in \mathcal{F}(x_n, \bar{u}_n)$ such that $\lim_{n \longrightarrow \infty} \alpha_n = 0$.

$\square$

Corollary 2.11 If $\Sigma^{\bar{u}}$ is finite gain dissipative, then $\Sigma^{\bar{u}}$ is ultimately bounded.

Proof: In the proof of theorem 2.10 , we observe that if $\Sigma^{\bar{u}}$ is finite gain dissipative, then

$$W_k^{\bar{u}} \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

Hence, by corollary 2.8

$$\limsup_{k \to \infty} \left| \frac{\partial}{\partial x} l(x_{k+1}, \bar{u}_k) \right| \leq \gamma$$

Which implies that $x_k \to \mathcal{L}^\gamma$, as $k \to \infty$, which is bounded by assumption **A5**, section 1.2.2. Since, suppose not. Then there exists an $\epsilon > 0$, s.t. $\forall K \geq 0$, $\exists k \geq K$, such that $\mathcal{B}_\epsilon(x_k) \cap \mathcal{L}^\gamma = \phi$. This implies that there exists an $\hat{\epsilon} > 0$, such that $\forall K \geq 0$, $\exists k \geq K$, such that $|\frac{\partial}{\partial x} l(x_{k+1}, \bar{u}_k)| > \gamma + \hat{\epsilon}$. Which implies that

$$\limsup_{k \to \infty} \left| \frac{\partial}{\partial x} l(x_{k+1}, \bar{u}_k) \right| > \gamma$$

A contradiction.

$\square$

Remark 2.12 Furthermore, if we impose sufficient smoothness assumptions on $\Sigma^{\bar{u}}$, such that V is continuous, then all trajectories generated by $\Sigma^{\bar{u}}$ are stable in the sense of Lyapunov. In particular, V then becomes a Lyapunov function. To this effect, the recent work by Blanchini [10] is similar in spirit. He constructs state feedback compensators for discrete time linear systems to achieve ultimate boundedness control via set-induced Lyapunov functions. This procedure was then applied to the state feedback l^1-optimal control problem [11].

Remark 2.13 It is clear from above and from lemma 2.5, that we do need some form of continuity assumption on l as a necessary condition for the system to be finite gain dissipative.

Remark 2.14 We can replace the compactness assumption (**A5**, page 13) on $\mathcal{L}^\gamma$ by boundedness, in which case we would have $x_k \to \bar{\mathcal{L}}^\gamma$, as $k \to \infty$, in corollary 2.11.

2.3   Solution to the State Feedback Problem

Although, the results above indicate that the controlled dissipation inequality is both a necessary and sufficient condition for the solvability of the state feedback robust control problem, we state the necessary and sufficient conditions in terms of dynamic programming equalities.

Theorem 2.15 (Necessity) If a controller $\bar{u} \in S$ solves the state feedback robust control problem, then there exists a function $V(x)$ such that $V(x) \geq 0$, $V(0) = 0$ and V satisfies the following equation i.e.

$$V(x) = \inf_{u \in U(x)} \sup_{r,s \in \mathcal{F}(x,u)} \{|l(r,u) - l(s,u)|^2 - \gamma^2 |r - s|^2 + V(r)\} \qquad (2.9)$$

$x \in X_k^{\bar{u}}(x_0)$, $k \geq 0$, $x_0 \in X_0$.

Proof: Construct a sequence V_j, $j = 0, \ldots$ of functions as follows

$$
\begin{aligned}
V_{j+1}(x) &= \inf_{u \in \boldsymbol{U}(x)} \sup_{r,s \in \mathcal{F}(x,u)} \{ |l(r, u) - l(s, u)|^2 - \gamma^2 |r - s|^2 + V_j(r) \} \\
V_0(x) &= 0
\end{aligned}
$$

Clearly,

$$
V_j(x) \geq 0 \ , \ \forall x \in \boldsymbol{R}^n, \ \forall j \geq 0
$$

and

$$
V_{j+1}(x) \geq V_j(x) \ , \ \forall x \in \boldsymbol{R}^n, \ j \geq 0
$$

For any $x_0 \in X_0$ and $k \geq 0$, pick an $x \in X_k^{\bar{u}}(x_0)$. Then dynamic programming

arguments imply that

$$
0 \leq V_j(x) \leq \beta^{\bar{u}}(x_0) \ , \quad \forall x \in X_k^{\bar{u}}(x_0)
$$

Furthermore, note that $V_j(x)$ depends only on j and x. Hence,

$$
V_j(x) \longrightarrow V(x) \text{ as } j \longrightarrow \infty \ , \ \forall x \in X_k^{\bar{u}}(x_0), \ k \geq 0, \ x_0 \in X_0
$$

and by definition, V satisfies equation (2.9). Furthermore, $V(x) \geq 0$ and $V(x_0) \leq$

$\beta^{\bar{u}}(x_0)$. Hence, $V(0) = 0$.

$\square$

Theorem 2.16 (Sufficiency) Assume that there exists a solution V to the station-

ary dynamic programming equation (2.9) for all $x \in \boldsymbol{R}^n$, satisfying $V(x) \geq 0$ and

$V(0) = 0$. Let $\bar{u}(x)$ be the control value which achieves the minimum in equation

(2.9). Then $\bar{u} \in S$ solves the state feedback robust control problem provided that $\Sigma^{\bar{u}}$ satisfies assumption **A7**.

Proof: Since V satisfies equation (2.9), $\Sigma^{\bar{u}}$ satisfies equation (2.7) with equality. Hence, $\Sigma^{\bar{u}}$ is finite gain dissipative, and hence by theorem 2.4, $\Sigma^{\bar{u}}$ is finite gain. Furthermore, by theorem 2.10 $\Sigma^{\bar{u}}$ is weakly asymptotically stable and by corollary 2.11 $\Sigma^{\bar{u}}$ is ultimately bounded.

$\square$

Remark 2.17 The condition that a solution exist for all $x \in \mathbf{R}^n$ is quite strong. It is sufficient that a solution exist on a set $\mathcal{D} \subset \mathbf{R}^n$, with a corresponding control policy $\bar{u}$, provided $\bigcup_{x_0 \in X_0} \Gamma^{\bar{u}}(x_0) \subset \mathcal{D}$. This may however be difficult to check in practice.

Corollary 2.18 If $X_0 = \mathbf{R}^n$, then the existence of a solution to the stationary dynamic programming equation (2.9) for all $x \in \mathbf{R}^n$, is both a necessary and sufficient condition for the existence of a solution to the state feedback robust control problem.

$\square$

Remark 2.19 It can be seen from the statement of theorem 2.15 and the proof of theorem 2.16, that we could have expressed the necessary and sufficient condi-

tions for the solvability of the state feedback robust control problem in terms of dissipation inequalities.

Chapter 3

The Information State

This chapter is aimed at motivating the cost definition used so far, as well as to obtain an information state recursion for the output feedback problem. A key idea used is that if we start out with a risk-sensitive stochastic control problem, and take the small noise limits, we obtain the formulation for a robust control problem for a deterministic system.

The main steps are as follows. In section 3.1, we consider a risk-sensitive stochastic control problem. We employ the idea in [31], where the small noise limit of a risk-sensitive stochastic control problem is taken to formally obtain an information state solution to the deterministic nonlinear H_∞ control problem. In [31] an exponential cost function motivated from [9] was used, and small noise limits taken. We use the information state recursion derived from the stochastic control problem as the basis to an information state controller for the deterministic problem. From our viewpoint, the stochastic control problem is entirely motivational, and we drop

most of the assumptions associated with the small noise limit derivation when considering the deterministic problem in section 3.2.

For the remainder of this chapter, we assume that $|\cdot|$ denotes the Euclidean norm.

3.1 The Stochastic Control Problem

We will consider a special case of the risk sensitive stochastic control problem. On a probability space $(\Omega, \mathcal{S}, \mathbf{P}^u)$ consider the stochastic control problem

$$
\begin{aligned}
x^\varepsilon_{k+1} &= \xi^\varepsilon_k + w^\varepsilon_{k+1}, \ \xi^\varepsilon_k \in \mathcal{F}(x^\varepsilon_k, u_k) \\
y^\varepsilon_{k+1} &= \nu^\varepsilon_k + v^\varepsilon_{k+1}, \ \nu^\varepsilon_k \in \mathcal{G}(x^\varepsilon_k)
\end{aligned}
$$

on the finite time interval $k = 0, \ldots, K - 1$. The process $y^\varepsilon \in \mathbf{R}$ is measured, and is called the observation process. $x^\varepsilon \in \mathbf{R}^n$ represent the states. For convenience, we will write the dynamics as

$$
\begin{aligned}
x^\varepsilon_{k+1} &\in \mathcal{F}(x^\varepsilon_k, u_k) + w^\varepsilon_{k+1} \\
y^\varepsilon_{k+1} &\in \mathcal{G}(x^\varepsilon_k, u_k) + v^\varepsilon_{k+1}
\end{aligned}
$$

Let $\mathcal{W}_k, \mathcal{Y}_k$ denote the complete filtrations generated by $(x^\varepsilon_{0,k}, y^\varepsilon_{0,k})$ and $y^\varepsilon_{0,k}$ respectively. We assume

B1. $y^\varepsilon_0 = 0$

B2. $\{w^\varepsilon_k\}$ is an $\mathbf{R}^n$-valued i.i.d. noise sequence with density

$$
\psi^\varepsilon(w) = (2\pi\varepsilon)^{-n/2} \exp(-\tfrac{1}{2\varepsilon} \mid w \mid^2).
$$

B3. $\{v_k^\varepsilon\}$ is a real-valued i.i.d. noise sequence with density

$\phi^\varepsilon(v) = (2\pi\varepsilon)^{-1/2} \exp(-\frac{1}{2\varepsilon} \mid v \mid^2)$, independent of $\{w_k^\varepsilon\}$.

B4. $\{\xi_k^\varepsilon\}$ is an $\boldsymbol{R}^n$-valued independent noise sequence with $\xi_k^\varepsilon \in \mathcal{F}(x_k^\varepsilon, u_k)$, having

a uniform density $\chi(x_k^\varepsilon, u_k) = (\int_{\xi \in \mathcal{F}(x_k^\varepsilon, u_k)} d\xi)^{-1}$. Furthermore, for each k, ξ_k^ε

is independent of w_l^ε and v_l^ε, $l = k + 1, \ldots, K$. Similarly, $\{\nu_k^\varepsilon\}$ is a $\boldsymbol{R}$-

valued independent noise sequence with $\nu_k^\varepsilon \in \mathcal{G}(x_k^\varepsilon)$ having a uniform density

$\theta(x_k^\varepsilon) = (\int_{\nu \in \mathcal{G}(x_k^\varepsilon)} d\nu)^{-1}$. Furthermore, for each k, ν_k^ε is independent of ξ_l^ε,

w_{l+1}^ε, v_{l+1}^ε for $l = k, \ldots, K - 1$.

B5. The controls u_k take values in $\boldsymbol{U} \subset \boldsymbol{R}^m$ assumed compact and are $\mathcal{Y}_k$ measurable.

B6. $\mathcal{F}$ is a set-valued map from $\boldsymbol{R}^n \times \boldsymbol{R}^m$ to $\boldsymbol{R}^n$, uniformly continuous in x, uniformly in $u \in \boldsymbol{U}$. $\mathcal{G}$ is a set-valued map from $\boldsymbol{R}^n$ to $\boldsymbol{R}$, satisfying the same assumptions as $\mathcal{F}$.

B7. Furthermore, $\mathcal{F}$, $\mathcal{G}$ assume convex compact values and have a non-empty interior for all x and u. χ, θ are uniformly continuous in x, uniformly in $u \in \boldsymbol{U}$, and are bounded.

B8. x_0^ε has density $\rho = (2\pi)^{-n/2} \exp(\frac{-1}{2}|x|^2)$.

Note that assumption **B7** places restrictions on $\mathcal{F}$, $\mathcal{G}$. An example of $\mathcal{F}$ which satisfies these assumptions is

$$\mathcal{F}(x, u) = Ax + Bu + \bar{\mathcal{B}}_r(0)$$

where, A, and B are matrices of appropriate dimensions, and $\bar{\mathcal{B}}_r(0)$ is the closed ball of radius r, centered at 0. At time k, let $U(k)$ denote the set of control functions $\bar{u}_k$ which satisfy **B5**, i.e. $\bar{u}_k$ take values in $\boldsymbol{U}$, and are a function of $y_{0,k}^{\varepsilon}$. Note that ξ_k^{ε}, ν_k^{ε} will in general depend on all the past values of w^{ε}, and v^{ε}, through the state x_{k-1}^{ε}, and control u_{k-1}. For $l \geq 0$, we write $U_{k,k+l} = U(k) \bigcup U(k+1) \cdots \bigcup U(k+l)$. The cost function is defined for admissible $u \in U_{0,K-1}$, $\mu > 0$ by

$$J^{\mu,\varepsilon}(\rho, u) = \mathbf{E}^u \left[\exp \frac{\mu}{\varepsilon} \left(\sum_{k=1}^{K} L(x_k^{\varepsilon}, w_k^{\varepsilon}, u_{k-1}) \right) \right]$$

and the partially observed risk-sensitive stochastic control problem is to find $u^* \in U_{0,K-1}$ such that

$$J^{\mu,\varepsilon}(\rho, u^*) = \inf_{u \in U_{0,K-1}} J^{\mu,\varepsilon}(\rho, u)$$

We further assume that

B9. $L \in C(\boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^m)$ is single-valued, nonnegative, bounded and uniformly continuous.

3.1.1 Change of Measure

Using an idea from [19], suppose there exists a reference measure $\mathbf{P}^\dagger$ such that under $\mathbf{P}^\dagger$, $\{y_k^\varepsilon\}$ is i.i.d. with density ϕ^ε, independent of $\{x_k^\varepsilon\}$ where x^ε satisfies

$$x_{k+1}^\varepsilon \in \mathcal{F}(x_k^\varepsilon, u_k) + w_{k+1}^\varepsilon$$

Define

$$\Lambda_k^\varepsilon = \Pi_{l=1}^k \left(\theta(x_{l-1}^\varepsilon) \frac{\int_{\mathcal{G}(x_{l-1}^\varepsilon)} \phi^\varepsilon(v_l^\varepsilon + \xi) d\xi}{\phi^\varepsilon(v_l^\varepsilon)} \right)$$

and define $\mathbf{P}^\dagger$ by setting

$$\left. \frac{d\mathbf{P}^\dagger}{d\mathbf{P}^u} \right|_{\mathcal{W}_k} = \Lambda_k^\varepsilon$$

i.e. by setting the Radon-Nykodym derivative, restricted to $\mathcal{W}_k$ to equal Λ_k^ε. Note that in general, $\mathbf{P}^\dagger$ at k, may depend on the states $x_{0,k-1}^\varepsilon$ (but not on x_k^ε), however we hide this to prevent notational clutter. Then

Lemma 3.1 Under $\mathbf{P}^\dagger$, the random variables $\{y_l^\varepsilon\}$ are i.i.d. with density function ϕ^ε.

Proof: Let $t \in R$, and consider

$$
\begin{aligned}
\mathbf{P}^\dagger(y_k^\varepsilon \leq t | \mathcal{W}_{k-1}) &= \mathbf{E}^\dagger[I(y_k^\varepsilon \leq t) | \mathcal{W}_{k-1}] \\
&= \frac{\mathbf{E}^u[\Lambda_k^\varepsilon I(y_k^\varepsilon \leq t) | \mathcal{W}_{k-1}]}{\mathbf{E}^u[\Lambda_k^\varepsilon | \mathcal{W}_{k-1}]}
\end{aligned}
$$

Now

$$\mathbf{E}^u[\Lambda_k^\varepsilon | \mathcal{W}_{k-1}] = \Lambda_{k-1}^\varepsilon \theta(x_{k-1}^\varepsilon) \int_R \int_{\mathcal{G}(x_{k-1}^\varepsilon)} \phi^\varepsilon(v_k^\varepsilon + \xi) d\xi dv_k^\varepsilon$$

$$= \Lambda_{k-1}^{\varepsilon} \theta(x_{k-1}^{\varepsilon}) \int_{\mathcal{G}(x_{k-1}^{\varepsilon})} \int_{\mathbf{R}} \phi^{\varepsilon}(y_k^{\varepsilon}) dy_k^{\varepsilon} d\xi$$

by changing the order of integration and by

a change of variables.

$$= \Lambda_{k-1}^{\varepsilon}$$

and

$$\mathbf{E}^u[\Lambda_k^{\varepsilon} I(y_k^{\varepsilon} \le t)|\mathcal{W}_{k-1}] = \Lambda_{k-1}^{\varepsilon} \theta(x_{k-1}^{\varepsilon}) \int_{\mathbf{R}} \int_{\mathcal{G}(x_{k-1}^{\varepsilon})} I(y_k^{\varepsilon} \le t) \phi^{\varepsilon}(v_k^{\varepsilon} + \xi) d\xi dv_k^{\varepsilon}$$

$$= \Lambda_{k-1}^{\varepsilon} \theta(x_{k-1}^{\varepsilon}) \int_{\mathcal{G}(x_{k-1}^{\varepsilon})} \int_{\mathbf{R}} I(y_k^{\varepsilon} \le t) \phi^{\varepsilon}(y_k^{\varepsilon}) dy_k^{\varepsilon} d\xi$$

by changing the order of integration and by

a change of variables.

$$= \Lambda_{k-1}^{\varepsilon} \int_{-\infty}^{t} \phi^{\varepsilon}(y_k^{\varepsilon}) dy_k^{\varepsilon}$$

The result follows.

$\square$

It is clear that under $\mathbf{P}^{\dagger}$, y_l^{ε}, and x_l^{ε} are independent. Furthermore, the existence

of $\mathbf{P}^{\dagger}$ is guaranteed by Kolmogorov's extension theorem. In a similar manner, we

define the inverse transformation relating $\mathbf{P}^u$ to $\mathbf{P}^{\dagger}$ as follows.

$$\frac{d\mathbf{P}^u}{d\mathbf{P}^{\dagger}}\bigg|_{\mathcal{W}_k} = \mathcal{Z}_k^{\varepsilon} = \Pi_{l=1}^{k} \Psi^{\varepsilon}(x_{l-1}^{\varepsilon}, y_l^{\varepsilon})$$

where

$$\Psi^{\varepsilon}(x_{l-1}^{\varepsilon}, y_l^{\varepsilon}) \triangleq \theta(x_{l-1}^{\varepsilon}) \frac{\int_{\mathcal{G}(x_{l-1}^{\varepsilon})} \phi^{\varepsilon}(y_l^{\varepsilon} - \xi) d\xi}{\phi^{\varepsilon}(y_l^{\varepsilon})}$$

$$= \theta(x_{l-1}^{\varepsilon}) \int_{\mathcal{G}(x_{l-1}^{\varepsilon})} \exp(-\frac{1}{\varepsilon}(\frac{1}{2}|\xi|^2 - y_l^{\varepsilon}\xi)) d\xi$$

3.1.2　Information State

We consider the space $L^\infty(\mathbf{R}^n)$ and its dual $L^{\infty*}(\mathbf{R}^n)$. We will denote the natural bilinear pairing between $L^\infty(\mathbf{R}^n)$ and $L^{\infty*}(\mathbf{R}^n)$ by $< \tau, \eta >$ for $\tau \in L^{\infty*}(\mathbf{R}^n)$, $\eta \in L^\infty(\mathbf{R}^n)$.

We define the information state process $\sigma_k^{\mu,\varepsilon} \in L^{\infty*}(\mathbf{R}^n)$ by

$$< \sigma_k^{\mu,\varepsilon}, \eta > = \mathbf{E}^\dagger \left[\eta(x_k^\varepsilon) \exp \left(\frac{\mu}{\varepsilon} \sum_{l=1}^k L(x_l^\varepsilon, w_l^\varepsilon, u_{l-1}) \right) \mathcal{Z}_k^\varepsilon \mid \mathcal{Y}_k \right]$$

for all test functions $\eta \in L^\infty(\mathbf{R}^n)$, for $k = 1, \dots, K$, with $\sigma_0^{\mu,\varepsilon} = \rho \in L^1(\mathbf{R}^n)$. We introduce the bounded linear operator $\Sigma^{\mu,\varepsilon} : L^\infty(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)$ defined by

$$\Sigma^{\mu,\varepsilon}(u, y)\eta(\xi) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} \int_{\mathcal{F}(\xi,u)} \psi^\varepsilon(z-r) \exp \left(\frac{\mu}{\varepsilon} L(z, z-r, u) \right) \eta(z) dr dz \Psi^\varepsilon(\xi, y)\chi(\xi, u)$$

and its adjoint $\Sigma^{\mu,\varepsilon*} : L^{\infty*}(\mathbf{R}^n) \to L^{\infty*}(\mathbf{R}^n)$ defined by

$$\Sigma^{\mu,\varepsilon*}(u, y)\sigma(z) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} \int_{\mathcal{F}(\xi,u)} \chi(\xi, u)\psi^\varepsilon(z-r) \exp \left(\frac{\mu}{\varepsilon} L(z, z-r, u) \right) \Psi^\varepsilon(\xi, y)\sigma(\xi) dr d\xi$$

$$(3.1)$$

Lemma 3.2 The information state $\sigma_k^{\mu,\varepsilon}$ satisfies

$$\begin{cases} \sigma_k^{\mu,\varepsilon} &= \Sigma^{\mu,\varepsilon*}(u_{k-1}, y_k^\varepsilon)\sigma_{k-1}^{\mu,\varepsilon} \\ \sigma_0^{\mu,\varepsilon} &= \rho \end{cases} \qquad (3.2)$$

Proof:

$$< \sigma_k^{\mu,\varepsilon}, \eta > \quad =$$

$$
\begin{aligned}
&= \mathbf{E}^{\dagger} \left[\eta(x_k^{\varepsilon}) \exp \left(\frac{\mu}{\varepsilon} \sum_{l=1}^{k} L(x_l^{\varepsilon}, w_l^{\varepsilon}, u_{l-1}) \right) \mathcal{Z}_k^{\varepsilon} \, | \mathcal{Y}_k \right] \\
&= \mathbf{E}^{\dagger} \left[\eta(x_k^{\varepsilon}) \exp \left(\frac{\mu}{\varepsilon} L(x_k^{\varepsilon}, w_k^{\varepsilon}, u_{k-1}) \right) \Psi^{\varepsilon}(x_{k-1}^{\varepsilon}, y_k^{\varepsilon}) \cdot \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. \exp \left(\frac{\mu}{\varepsilon} \sum_{l=1}^{k-1} L(x_l^{\varepsilon}, w_l^{\varepsilon}, u_{l-1}) \right) \mathcal{Z}_{k-1}^{\varepsilon} \, | \mathcal{Y}_k \right] \\
&= \mathbf{E}^{\dagger} \left[\int_{\boldsymbol{R}^n} \int_{\mathcal{F}(x_{k-1}, u_{k-1})} \eta(z) \exp \left(\frac{\mu}{\varepsilon} L(z, z - r, u_{k-1}) \right) \Psi^{\varepsilon}(x_{k-1}^{\varepsilon}, y_k^{\varepsilon}) \cdot \right. \\
&\qquad \left. \chi(x_{k-1}, u_{k-1}) \exp \left(\frac{\mu}{\varepsilon} \sum_{l=1}^{k-1} L(x_l^{\varepsilon}, w_l^{\varepsilon}, u_{l-1}) \right) \mathcal{Z}_{k-1}^{\varepsilon} \psi^{\varepsilon}(z - r) dr dz \, | \mathcal{Y}_k \right] \\[2em]
&= \ < \sigma_{k-1}^{\varepsilon}, \int_{\boldsymbol{R}^n} \int_{\mathcal{F}(\cdot, u_{k-1})} \eta(z) \exp \left(\frac{\mu}{\varepsilon} L(z, z - r, u_{k-1}) \right) \cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad \chi(\cdot, u_{k-1}) \Psi^{\varepsilon}(\cdot, y_k^{\varepsilon}) \psi^{\varepsilon}(z - r) dr dz > \\[1em]
&= \ < \sigma_{k-1}^{\varepsilon}, \Sigma^{\mu,\varepsilon}(u_{k-1}, y_k^{\varepsilon}) \eta > \\[1em]
&= \ < \Sigma^{\mu,\varepsilon *}(u_{k-1}, y_k^{\varepsilon}) \sigma_{k-1}^{\varepsilon}, \eta >
\end{aligned}
$$

for any $\eta \in L^{\infty}(\boldsymbol{R}^n)$.

$\square$

Observe that for all $u \in U_{0,K-1}$, we have

$$
\begin{aligned}
\mathbf{E}^{\dagger} \left[< \sigma_K^{\mu,\varepsilon}, 1 > \right] &= \mathbf{E}^{\dagger} \left[\mathbf{E}^{\dagger} \left[\exp \left(\frac{\mu}{\varepsilon} \sum_{l=1}^{K} L(x_l^{\varepsilon}, w_l^{\varepsilon}, u_{l-1}) \right) \mathcal{Z}_K^{\varepsilon} \, | \mathcal{Y}_K \right] \right] \\
&= \mathbf{E}^{\dagger} \left[\exp \left(\frac{\mu}{\varepsilon} \sum_{l=1}^{K} L(x_l^{\varepsilon}, w_l^{\varepsilon}, u_{l-1}) \right) \mathcal{Z}_K^{\varepsilon} \right] \\
&= J^{\mu,\varepsilon}(\rho, u)
\end{aligned}
$$

Thus, the cost can be expressed as a function of $\sigma_K^{\mu,\varepsilon}$ alone, and hence the name

information state for $\sigma_k^{\mu,\varepsilon}$ is justified.

3.1.3   Small Noise Limit

We first define some spaces following [31]. For $\gamma \in M \triangleq \{\gamma \in \boldsymbol{R}^2 \mid \gamma_1 > 0, \gamma_2 \geq 0\}$

define

$$\mathcal{D}^\gamma \triangleq \{\tilde{p} \in C(\boldsymbol{R}^n) \mid \tilde{p}(x) \leq -\gamma_1 \mid x \mid^2 + \gamma_2\}$$

$$\mathcal{D} \triangleq \{\tilde{p} \in C(\boldsymbol{R}^n) \mid \tilde{p}(x) \leq -\gamma_1 \mid x \mid^2 + \gamma_2 \text{ for some } \gamma \in M\}$$

We equip these spaces with the topology of uniform convergence on compact subsets. Define $\Lambda^{\mu*} : \mathcal{D} \to \mathcal{D}$ by

$$\Lambda^{\mu*}(u,y)\tilde{p}(z) \triangleq \sup_{\xi \in \boldsymbol{R}^n} \left\{ \tilde{p}(\xi) + \sup_{r \in \mathcal{F}(\xi,u)} \left(L(z, z - r, u) - \frac{1}{2\mu} \mid z - r \mid^2 \right) - \right.$$

$$\left. \frac{1}{\mu} \inf_{s \in \mathcal{G}(\xi)} \left(\frac{1}{2} \mid s \mid^2 - sy \right) \right\}$$

for $\tilde{p} \in \mathcal{D}$.

Then we have

Theorem 3.3

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\mu} \log \Sigma^{\mu,\varepsilon*}(u,y) e^{\frac{\mu}{\varepsilon}\tilde{p}} = \Lambda^{\mu*}(u,y)\tilde{p}$$

in $\mathcal{D}$ uniformly on compact subsets of $\boldsymbol{U} \times \boldsymbol{R} \times \mathcal{D}^\gamma$ for each $\gamma \in M$.

Proof:

From (3.1) we have

$$\frac{\varepsilon}{\mu} \log \Sigma^{\mu,\varepsilon*}(u,y) e^{\frac{\mu}{\varepsilon}\tilde{p}}(z) =$$

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$$\frac{\varepsilon}{\mu} \log \int_{\boldsymbol{R}^n} \int_{\mathcal{F}(\xi,u)} \int_{s \in \mathcal{G}(\xi)} \exp \frac{\mu}{\varepsilon} \left(\frac{-1}{2\mu} \mid z - r \mid^2 - \frac{n\varepsilon}{2\mu} \log(2\pi\varepsilon) + L(z, z - r, u) + \right.$$

$$\left. \tilde{p}(\xi) + \frac{\varepsilon}{\mu} \log \chi(\xi, u) + \frac{\varepsilon}{\mu} \log \theta(\xi) - \frac{1}{\mu} \left[\frac{1}{2} \mid s \mid^2 - sy \right] \right) dr d\xi$$

Under the assumptions made on the system, a straightforward application of the

Varadhan-Laplace lemma (Appendix B) yields the result.

$\square$

Remark 3.4 In particular, setting $\sigma^{\mu,\varepsilon} = e^{\frac{\mu}{\varepsilon}\tilde{p}}$ in equation (3.2), and employing

the result of theorem 3.3, we obtain

$$\tilde{p}_{k+1}(z) = \sup_{\xi \in \boldsymbol{R}^n} \left\{ \tilde{p}_k(\xi) + \sup_{r \in \mathcal{F}(\xi,u_k)} \left(L(z, z - r, u_k) - \frac{1}{2\mu} \mid z - r \mid^2 \right) - \right.$$

$$\left. \frac{1}{\mu} \inf_{s \in \mathcal{G}(\xi)} \left(\frac{1}{2} \mid s \mid^2 - sy_{k+1} \right) \right\} \quad (3.3)$$

for $k = 0, \ldots, K - 1$.

3.2   The Deterministic Case

We now consider the deterministic problem. The system is defined as

$$\Sigma \left\{ \begin{array}{rcl} x_{k+1} & \in & \mathcal{F}(x_k, u_k) \\ y_{k+1} & \in & \mathcal{G}(x_k) \end{array} \right. \quad (3.4)$$

for $k = 0, \ldots, K - 1$. We assume that the system (3.4) satisfies the relevant

assumptions of section 3.1. Namely, that $\mathcal{F}$, $\mathcal{G}$ take on compact values with non-

empty interior, and $u_k \in \boldsymbol{U}$, with $\boldsymbol{U}$ compact. We first simplify the information

state recursion (3.3) for this case. Here, it is assumed that we have access to the function L, which is tied to the particular kind of robust control problem being considered. More will be said about this in the next subsection.

We carry out the following change of variables in equation (3.3)

$$p_0(x) \stackrel{\triangle}{=} \tilde{p}_0(x)$$

$$p_k(x) \stackrel{\triangle}{=} \tilde{p}_k(x) - \frac{1}{2\mu} \sum_{j=0}^{k-1} | y_{j+1} |^2 \, , \ k = 1, \ldots, K.$$

Then equation (3.3) can be written as

$$p_{k+1}(x) = \sup_{\xi \in \mathbf{R}^n} \left\{ p_k(\xi) + \sup_{r \in \mathcal{F}(\xi, u_k)} \left(L(x, x - r, u_k) - \frac{1}{2\mu} | x - r |^2 \right) - \frac{1}{2\mu} \inf_{s \in \mathcal{G}(\xi)} (s - y_{k+1})^2 \right\} \tag{3.5}$$

Using the convention that the supremum over an empty set is $-\infty$, we can place a natural restriction on ξ. Define

$$\Omega(x, y, u) \stackrel{\triangle}{=} \{ \xi \in \mathbf{R}^n \mid x \in \mathcal{F}(\xi, u) \text{ and } y \in \mathcal{G}(\xi) \}$$

This just ensures that the values of ξ are compatible with x, u, and y, given the dynamics (3.4). Then equation (3.5) can be written as

$$p_{k+1}(x) = \sup_{\xi \in \Omega(x, y_{k+1}, u_k)} \left\{ p_k(\xi) + \sup_{r \in \mathcal{F}(\xi, u_k)} \left(L(x, x - r, u_k) - \frac{1}{2\mu} | x - r |^2 \right) \right\} \tag{3.6}$$

or

$$\begin{aligned} p_{k+1} &= H(p_k, y_{k+1}, u_k) \\ p_0 &= \bar{p} \end{aligned}$$

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Remark 3.5 Here, $\bar{p}$ is a function which weighs the initial states $x_0 \in \mathbf{R}^n$. In particular, it could incorporate any *a priori* information we have about x_0.

3.2.1 Motivating the Robust Control Problem

Consider the information state recursion (3.6), with $x_0 = 0$. Then the appropriate choice for $\bar{p}$ is $\bar{p} = \delta_{\{0\}}$, where δ is as defined in equation (1.6). By inspection, one obtains

$$p_k(x) = \sup_{r,s \in \Gamma^{u,y}_{0,k}(0)} \{ \sum_{l=0}^{k-1} L(r_{l+1}, r_{l+1} - s_{l+1}, u_l) - \frac{1}{2\mu} |r_{l+1} - s_{l+1}|^2 | \ r_k = x \} \quad (3.7)$$

where $\Gamma^{u,y}_{0,k}(0)$ is the set of all state trajectories $x_{0,k}$ that can be generated by the closed loop system Σ^u, which are compatible with the measurements $y_{1,k}$ (see section 1.2.1). We now consider the following control problem for the system (3.4). Find a control policy $u \in O_{0,K-1}$, such that

$$\sum_{k=0}^{K-1} \{ L(r_{k+1}, r_{k+1} - s_{k+1}, u_k) - \frac{1}{2\mu} |r_{k+1} - s_{k+1}|^2 \} \leq 0 \quad (3.8)$$

over all trajectories $r, s \in \Gamma^u_{0,K}(0)$. Note that, if $r - s \in l^2([0,K])$, then the above guarantees that

$$\frac{(\sum_{k=0}^{K-1} L(r_{k+1}, r_{k+1} - s_{k+1}, u_k))^{\frac{1}{2}}}{\|r - s\|_{l^2}} \leq \frac{1}{\sqrt{2\mu}}$$

This immediately yields a method to set up robust control problems for the system Σ. Consider for example the following regulated output.

$$z_{k+1} = h(x_{k+1}, u_k)$$

where x_k evolve via the dynamics (3.4). One could now consider attenuating the (Lipschitz) induced norm of z, (provided of course that h is not, say, uniformly Lipschitz continuous in x) by defining L as

$$L(r, w, u) = |h(r, u) - h(r - w, u)|^2$$

Then, we obtain the cost considered in in this thesis. Note that alternate costs are possible by defining L differently.

3.2.2 Information State and Feasible States

For the remainder of the chapter, we let $x_0 \in \mathbf{R}^n$ be arbitrary, but known. An interesting property of the information state is that it also acts as an indicator function for feasible states.

Definition: For a given initial state x_0, an output trajectory $y_{1,k+1}$, and a control trajectory $u_{0,k}$, a state $\bar{x}_{k+1}$ is called **feasible** at time $k + 1$ if there exists a state trajectory $x_{0,k+1}$ with $x_{k+1} = \bar{x}_{k+1}$ such that $x_{j+1} \in \mathcal{F}(x_j, u_j)$ and $y_{j+1} \in \mathcal{G}(x_j)$ for $j = 0, \ldots, k$.

Remark 3.6 Hence, one could think of feasible states at a given time instant, to be those values which the system states could assume given the available information up to that time.

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Consider now the following recursion

$$
\begin{aligned}
\mathcal{X}_{k+1}^{y,u}(x_0) &= \mathcal{F}(\mathcal{G}^{-1}(y_{k+1}) \cap \mathcal{X}_k^{y,u}(x_0), u_k), \quad k = 0, \ldots, K-1 \\
\mathcal{X}_0^{y,u}(x_0) &= \{x_0\}
\end{aligned}
$$

where $\mathcal{G}^{-1}(y_{k+1}) = \{x \in \mathbf{R}^n \mid y_{k+1} \in \mathcal{G}(x)\}$, and for a set $M \subset \mathbf{R}^n$, we define

$\mathcal{F}(M, u) = \bigcup_{x \in M} \mathcal{F}(x, u)$.

Lemma 3.7 $\bar{x} \in \mathcal{X}_{k+1}^{y,u}(x_0)$ if and only if $\bar{x}$ is feasible at time $k + 1$.

Proof:

We show this by induction on k. Clearly for $k = 0$ the assertion holds. Assume it

to be true for k. Now consider instance $k + 1$.

(i) Suppose $\bar{x} \in \mathcal{X}_{k+1}^{y,u}(x_0)$. Then there exists a ξ such that $\bar{x} \in \mathcal{F}(\xi, u_k)$ and

$$
\xi \in \mathcal{G}^{-1}(y_{k+1}) \cap \mathcal{X}_k^{y,u}(x_0).
$$

Hence, ξ is feasible at time k, and is such that $y_{k+1} \in \mathcal{G}(\xi)$. Hence, by definition

$\bar{x} \in \mathcal{F}(\xi, u_k)$ is feasible at time $k + 1$.

(ii) Suppose $\bar{x}$ is feasible at time $k + 1$. Then by definition there exists a trajectory

$x_{0,k+1}$ such that $x_k \in \mathcal{X}_k^{y,u}(x_0)$, and $y_{k+1} \in \mathcal{G}(x_k)$. Hence,

$$
x_k \in \mathcal{G}^{-1}(y_{k+1}) \cap \mathcal{X}_k^{y,u}(x_0)
$$

Thus,

$$
\bar{x} = x_{k+1} \in \mathcal{F}(\mathcal{G}^{-1}(y_{k+1}) \cap \mathcal{X}_k^{y,u}(x_0), u_k) = \mathcal{X}_{k+1}^{y,u}(x_0)
$$

□

Define the following limiter function $\tau(\cdot) : \boldsymbol{R}_+ \to \boldsymbol{R}^-$ as

$$\tau(x) \overset{\triangle}{=} \begin{cases} 0 & \text{if } x \geq 0 \\ x & \text{else} \end{cases}$$

where $\boldsymbol{R}_+$ denotes the extended real line and $\boldsymbol{R}^-$ denotes $\{x \in \boldsymbol{R}_+ \mid x \leq 0\}$. Also,

recall the definition of $\delta_M : \boldsymbol{R}^n \to \boldsymbol{R}^*$ from equation (1.6)

Theorem 3.8 Suppose $p_0 = \delta_{\{x_0\}}$ then

$$\tau(p_k(x)) = \delta_{\mathcal{X}_k^{y,u}(x_0)}(x), \forall x \in \boldsymbol{R}^n$$

Proof:

First notice that if $p_0 = \delta_{\{x_0\}}$, then for a given $x \in \boldsymbol{R}^n$ either $p_k(x) \geq 0$ or

$p_k(x) = -\infty$. Now

$$\tau(p_0(x)) = 0 \text{ iff } x = x_0$$

Assume true for $p_k(x)$ and let $x \in \mathcal{X}_{k+1}^{y,u}(x_0)$. Then there exists a $\xi \in \boldsymbol{R}^n$ such that

$x \in \mathcal{F}(\xi, u_k)$, $y \in \mathcal{G}(\xi)$ and $\xi \in \mathcal{X}_k^{y,u}(x_0)$. Hence $p_k(\xi) \geq 0$ and

$$\sup_{r \in \mathcal{F}(\xi, u_k)} L(x, x - r, u_k) - \frac{1}{2\mu} \mid x - r \mid^2 \geq 0$$

Hence

$$p_{k+1}(x) \geq p_k(\xi) + \sup_{r \in \mathcal{F}(\xi, u_k)} L(x, x - r, u_k) - \frac{1}{2\mu} \mid x - r \mid^2 \geq 0$$

Hence, $\tau(p_{k+1}(x)) = 0$.

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Now let $\tau(p_{k+1}(x)) = 0$. i.e. $p_{k+1}(x) \geq 0$. This implies that x is such that

$x \in \mathcal{F}(\xi, u_k)$, $y_{k+1} \in \mathcal{G}(\xi)$ and $p_k(\xi) \neq -\infty$ for some $\xi \in \mathbf{R}^n$. Hence, $p_k(\xi) \geq 0$

and thus $\xi \in \mathcal{X}_k^{y,u}(x_0)$ by assumption. Hence,

$$\xi \in \mathcal{X}_k^{y,u}(x_0) \bigcap \mathcal{G}^{-1}(y_{k+1})$$

Therefore,

$$\begin{aligned} x \in \mathcal{F}(\xi, u_k) &\subset \mathcal{F}(\mathcal{X}_k^{y,u}(x_0) \bigcap \mathcal{G}^{-1}(y_{k+1}), u_k) \\ &= \mathcal{X}_{k+1}^{y,u}(x_0) \end{aligned}$$

Hence, $x \in \mathcal{X}_{k+1}^{y,u}(x_0)$.

$\square$

Remark 3.9 Thus, we see that the information state can be transformed by a simple limiting operation to the indicator function of the set of feasible states.

This has consequences on the solvability of the problem. In particular, (skipping ahead for the moment),

Remark 3.10 Suppose the system starts from rest (i.e. $x_0 = 0$). Then, from corollary 4.9, if we start with $x_0 = 0$ with $p_0 = \delta_{\{0\}}$, then the information state is always non-positive. Hence, it is zero on feasible states, and $-\infty$ elsewhere. Thus, instead of computing the information state via (3.6), one could consider propagating the set of feasible states (the so called problem of **guaranteed estimation** [49],[34]).

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Chapter 4

Output Feedback Case

We now consider the output feedback robust control problem. We denote the set of control policies as O. Hence, if $u \in O$, then $u_k = f(y_{1,k})$. Although, one could extend the developments in chapter 3, we choose to develop the solution independently. Thus, this chapter is independent of chapter 3, and the role of chapter 3 is mainly motivational.

4.1 Finite Time

For the finite time case, we are only interested in the satisfaction of condition **C3** of section 1.2.3, page 15. Hence, the problem is, given $\gamma \geq \gamma_{min}$, and a finite time interval $[0, K]$, find a control policy $u \in O_{0,K-1}$, such that there exists a finite quantity $\beta_K^u(x)$ with $\beta_K^u(0) = 0$ and

$$\sum_{i=0}^{K-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 - \gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \leq \beta_K^u(x_0),$$

$$\forall r, s \in \Gamma_{0,K}^u(x_0), \ \forall x_0 \in X_0$$

4.1.1 Dynamic Game

In this section, we transform the output feedback robust control problem to a dynamic game. We introduce the function space.

$$\mathcal{E} = \{p : \mathbf{R}^n \longrightarrow \mathbf{R}^*\}$$

For $u \in O_{0,K-1}$, and $p \in \mathcal{E}$ define a functional $J_{p,k}(u)$ by

$$J_{p,k}(u) \triangleq \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^u(x_0)} \{p(x_0) + \sum_{i=1}^{k} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 - \gamma^2 \mid s_i - r_i \mid^2\} \quad (4.1)$$

for $k = 0, \ldots, K$.

Remark 4.1 The functional $p \in \mathcal{E}$ in equation (4.1) can be chosen to reflect any *a priori* knowledge concerning the initial state x_0 of Σ^u.

The finite gain property of Σ^u can now be expressed in terms of J as follows.

Lemma 4.2 (i) $\Sigma_{x_0}^u$ is finite gain on [0,K] if and only if there exists a finite quantity $\beta_K^u(x_0)$, $\beta_K^u(0) = 0$, such that

$$J_{\delta_{\{x_0\}},k}(u) \leq \beta_K^u(x_0), \quad k = 0, \ldots, K$$

(ii) Σ^u is finite gain on $[0, K]$, if and only if there exists a finite $\beta_K^u \geq 0$ on X_0, with $\beta_K^u(0) = 0$, such that

$$J_{-\beta_K^u,k}(u) \leq 0, \quad k = 0, \ldots, K$$

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where $\delta_{\{x\}}$ is defined in equation 1.6.

$\square$

For notational convenience, we introduce the following pairing

$$(p, q) = \sup_{x \in \mathbf{R}^n} \{p(x) + q(x)\}$$

and a restricted version

$$(p, q \mid X) = \sup_{x \in X} \{p(x) + q(x)\}$$

Lemma 4.3 If each map $\Sigma_{x_0}^u$ is finite gain on $[0, K]$, then

$$(p, 0 \mid X_0) \leq J_{p,K}(u) \leq (p, \beta_K^u \mid X_0)$$

Proof:

Set $r = s \in \Gamma^u(x_0)$ in equation (4.1). Then clearly

$$(p, 0 \mid X_0) \leq J_{p,K}(u)$$

Since, $\Sigma_{x_0}^u$ is finite gain on $[0, K]$ for all $x_0 \in X_0$, this implies that for any $x_0 \in X_0$

$$p(x_0) + \sum_{i=1}^{K} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 - \gamma^2 \mid s_i - r_i \mid^2 \leq p(x_0) + \beta_K^u(x_0) \leq (p, \beta_K^u \mid X_0)$$

Hence, $J_{p,K}(u) \leq (p, \beta_K^u \mid X_0)$.

$\square$

Thus, we can define

$$dom \ J_{p,K}(u) = \{p \in \mathcal{E} : (p, 0 \mid X_0), (p, \beta_K^u \mid X_0) \text{ is finite }\}$$

The finite time output feedback dynamic game is to find a control policy $u \in O_{0,K-1}$, which minimizes $J_{p,K}$.

4.1.2 Information State Formulation

Motivated by the results obtained in chapter 3, for a fixed $y_{1,k} \in \Delta_{1,k}^u(X_0)$, and $u_{1,k-1}$, we define the **information state** $p_k \in \mathcal{E}$ by

$$p_k(x) \overset{\triangle}{=} \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma_{0,k}^{u,y}(x_0)} \{p_0(x_0) + \sum_{i=1}^{k} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 -$$

$$\gamma^2 \mid r_i - s_i \mid^2 \mid r_k = x\} \tag{4.2}$$

Here, the convention is that the supremum over an empty set is $-\infty$. Furthermore, for convenience we redefine p_0 as

$$p_0(x) = \begin{cases} p_0(x) & \text{, if } x \in X_0 \\ -\infty & \text{, else} \end{cases}$$

Clearly, if Σ^u is finite gain, then

$$-\infty \leq p_k(x) \leq (p_0, \beta_K^u) < +\infty$$

and a finite lower bound for $p_k(x)$ is obtained for all feasible $x \in \mathbf{R}^n$.

Now, define $H(p, u, y) \in \mathcal{E}$ by

$$H(p, u, y)(x) \overset{\triangle}{=} \sup_{\xi \in \mathbf{R}^n} \{p(\xi) + B(\xi, x, u, y)\} \tag{4.3}$$

where the function B is defined by

$$
B(\xi, x, v, y) \triangleq \begin{cases} \sup_{s \in \mathcal{F}(\xi,v)} \{| \ l(x,v) - l(s,v) \ |^2 \\ \qquad\qquad -\gamma^2 \ | \ x - s \ |^2 \} & \text{if} \begin{cases} x & \in \quad \mathcal{F}(\xi, v) \\ y & \in \quad \mathcal{G}(\xi, v) \end{cases} \\ -\infty & \text{else} \end{cases}
$$

Lemma 4.4 The information state is the solution of the following recursion

$$
\begin{cases} p_{k+1} & = \quad H(p_k, u_k, y_{k+1}), \ k = 0, \dots, K-1 \\ p_0 & \in \quad \mathcal{E} \end{cases} \tag{4.4}
$$

Proof:

We use induction. Assume that (4.4) is true for $0, \dots, k$; we must show that p_{k+1}

defined by (4.2) equals $H(p_k, u_k, y_{k+1})$. Now

$$
\begin{aligned}
H(p_k, u_k, y_{k+1})(x) &= \sup_{\xi \in \mathbf{R}^n} \{p_k(\xi) + B(\xi, x, u_k, y_{k+1})\} \\
&= \sup_{\xi \in \mathbf{R}^n} \{p_k(\xi) + \sup_{s \in \mathcal{F}(\xi, u_k)} (| \ l(x, u_k) - l(s, u_k) \ |^2 - \\
&\qquad \gamma^2 \ | \ x - s \ |^2 \ | \ \ y_{k+1} \in \mathcal{G}(\xi, u_k), x \in \mathcal{F}(\xi, u_k)\} \\
&= p_{k+1}(x)
\end{aligned}
$$

by the definition (4.2) for p_k, and p_{k+1}.

$\square$

Remark 4.5 Note that we can write

$$
p_k(x) = \sup_{\xi \in \Gamma_{0,k}^{u,y}(X_0)} \{p_0(\xi_0) + \sum_{i=0}^{k-1} B(\xi_i, \xi_{i+1}, u_i, y_{i+1}) \ | \ \xi_k = x\}
$$

for $k = 1, \dots, K$.

Remark 4.6 The relationship between the information state and the indicator function of the feasible sets was established in chapter 3. In particular, it was established that if $p_0 = \delta_{\{x_0\}}$, then $p_k(x) \geq 0$ if and only if $x \in \mathcal{X}_k^{y,u}(x_0)$, where $\mathcal{X}_k^{y,u}(x_0)$ is the set of feasible states at time k, given $u_{0,k-1}$ and $y_{1,k}$.

Theorem 4.7 For $u \in O_{0,k-1}$, $p \in \mathcal{E}$, such that $J_{p,k}(u)$ is finite, we have

$$J_{p,k}(u) = \sup_{y_{1,k} \in \Delta^u(X_0)} \{(p_k, 0) \mid p_0 = p\}, \ k \in [0, K] \qquad (4.5)$$

Proof:

We have

$$\sup_{y_{1,k} \in \Delta^u(X_0)} \{(p_k, 0) \mid p_0 = p\}$$

$$= \sup_{y_{1,k} \in \Delta^u(X_0)} \sup_{\xi \in \Gamma_{0,k}^{u,y}(X_0)} \{p(\xi_0) + \sum_{i=0}^{k-1} B(\xi_i, \xi_{i+1}, u_i, y_{i+1})\}$$

$$= \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma_{0,k}^u(x_0)} \{p(x_0) + \sum_{i=1}^{k} \mid l(s_i, u_{i-1})$$

$$-l(r_i, u_{i-1}) \mid^2 -\gamma^2 \mid r_i - s_i \mid^2\}$$

$$= J_{p,k}(u)$$

$\square$

Remark 4.8 This representation theorem is actually a separation principle.

The following corollary enables us to express the finite gain property of Σ^u in terms of the information state p.

Corollary 4.9 For any output feedback controller $u \in O_{0,K-1}$, the closed loop system Σ^u is finite gain on $[0, K]$ if and only if the information state p_k satisfies

$$\sup_{y_{1,k} \in \Delta^u(X_0)} \{(p_k, 0) \mid p_0 = -\beta_K^u\} \leq 0, \ \forall k \in [0, K]$$

for some finite $\beta_K^u(x_0) \geq 0$, $x_0 \in X_0$, with $\beta_K^u(0) = 0$.

$\square$

Remark 4.10 Thus the name **information state** for p is justified, since p_k contains all the information relevant to the finite gain property of Σ^u that is available in the observations $y_{1,k}$.

The information state dynamics (4.4) may be regarded as a new (infinite dimensional) control system Ξ, with control u and uncertainty parameterized by y. The state p_k, and the disturbance y_k are available to the controller, so the original output feedback dynamic game is equivalent to a new game with **full information**. The cost is now given by (4.5). Note that, now the control will depend only on the information state. Hence, the controller has a separated structure.

We now need an appropriate class $I_{i,K-1}$ of controllers, which feedback this new state variable. A control u belongs to $I_{i,K-1}$, if for each $k \in [i, K-1]$, there exists

a map $\bar{u}_k$ from a subset of $\mathcal{E}^{k-i+1}$ (sequences $p_{i,k}$) into $\boldsymbol{U}$, such that $u_k = \bar{u}(p_{i,k})$.

Note that since p_k depends on the observable information $y_{i,k}$, $I_{0,k-1} \subset O_{0,k-1}$, for

$k = 1, \ldots, K$.

4.1.3   Solution to the Finite Time Output Feedback Robust Control Problem

We use dynamic programming to solve the game. Define the value function by

$$M_k(p) = \inf_{u \in O_{0,k-1}} \sup_{y \in \Delta_{1,k}^u(X_0)} \{(p_k, 0) \mid p_0 = p\} \tag{4.6}$$

for $k \in [0, K]$, and the corresponding dynamic programming equation is

$$M_k(p) = \inf_{u \in \boldsymbol{U}} \sup_{y \in \boldsymbol{R}^t} \{M_{k-1}(H(p, u, y))\}, \ k \in [1, K] \tag{4.7}$$

with the initial condition

$$M_0(p) = (p, 0)$$

Remark 4.11 In the above equations, we have inverted the time index to enable ease of exposition when dealing with the infinite time case. Since, the system is assumed to be time invariant, it does not matter if we write the equations as above, or as

$$\tilde{M}_k(p) = \inf_{u \in \boldsymbol{U}} \sup_{y \in \boldsymbol{R}^t} \{\tilde{M}_{k+1}(H(p, u, y))\}, \ k \in [0, K-1]$$

with the initial condition

$$\tilde{M}_K(p) = (p, 0)$$

as far as we invert the index of the control policy obtained by solving equation (4.7).

Define for a function $M : \mathcal{E} \longrightarrow \mathbf{R}^*$,

$$\text{dom } M = \{p \in \mathcal{E} \mid M(p) \text{ finite}\}$$

Theorem 4.12 (Necessity) Assume that $\bar{u} \in O_{0,K-1}$ solves the finite time output feedback robust control problem. Then there exists a solution M to the dynamic programming equation (4.7) such that $\text{dom } J_{p,K}(\bar{u}) \subset \text{dom } M_k$, $M_k(-\beta_K^{\bar{u}}) = 0$, $M_k(p) \geq (p, 0)$, $p \in \text{dom } M_k$, $k \in [0, K]$.

Proof:

For $p \in \text{dom } J_{p,K}(\bar{u})$, define $M_k(p)$ by (4.6). Then

$$M_k(p) = \inf_{u \in O_{0,k-1}} J_{p,k}(u)$$

Now, we also have

$$M_k(p) = \inf_{u \in O_{0,k-1}} \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma_{0,k}^u(x_0)} \left\{ p(x_0) + \sum_{i=1}^{k} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 \right.$$
$$\left. -\gamma^2 \mid s_i - r_i \mid^2 \right\}$$

For $u = \bar{u}$, by using the finite gain property for $\Sigma^{\bar{u}}$ we get

$$M_k(p) \leq \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma_{0,k}^{\bar{u}}(x_0)} \{ p(x_0) + \sum_{i=1}^{k} \mid l(s_i, \bar{u}_{i-1}) - l(r_i, \bar{u}_{i-1}) \mid^2$$
$$-\gamma^2 \mid s_i - r_i \mid^2 \}$$
$$\leq (p, \beta_K^{\bar{u}})$$

Thus, $dom\ J_{p,K}(\bar{u}) \subset dom\ M_k$. Also

$$M_k(p) \geq (p, 0)$$

Also $-\beta_K^{\bar{u}} \leq 0$, with $\beta_K^{\bar{u}}(0) = 0$, and hence $M_k(-\beta_K^{\bar{u}}) = 0$.

$\square$

Theorem 4.13 (Sufficiency) Assume there exists a solution M to the dynamic programming equation (4.7) on some non-empty domain $dom\ M_k$, such that $-\beta \in dom\ M_k$ ($\beta(x) \geq 0$ and finite for all $x \in X_0$, with $\beta(0) = 0$), $M_k(-\beta) = 0$, $M_k(p) \geq (p, 0)$, $k \in [0, K]$. Let $u^* \in I_{0,K-1}$ be a policy such that $u_k^* = \bar{u}_{K-k}^*(p_k)$, $k = 0, \ldots, K-1$; where $\bar{u}_k^*(p)$ achieves the minimum in (4.7), $p_0 = -\beta$, and let p be the corresponding information state trajectory with $p_k \in dom\ M_{K-k}$, $k = 1, \ldots, K$. Then u^* solves the finite time output feedback robust control problem.

Proof:

We see that

$$M_K(p) = J_{p,K}(u^*) \leq J_{p,K}(u)$$

for all $u \in O_{0,K-1}$, $p \in dom\ M_K$. Now

$$\sup_{y \in \Delta_{1,K}^{u^*}(X_0)} \{(p_K, 0) \mid p_0 = -\beta\} \leq M_K(-\beta) = 0$$

which implies by corollary 4.9 that Σ^{u^*} is finite gain, and hence u^* solves the finite time output feedback robust control problem.

□

Corollary 4.14 If the finite time output feedback robust control problem is solvable by an output feedback controller $\bar{u} \in O_{0,K-1}$, then it is also solvable by an information state feedback controller $u^* \in I_{0,K-1}$.

□

4.2 Infinite Time Case

For the infinite time case, we need to satisfy the conditions **C1-C3** stated in section 1.2.3, page 15. We pass to the limit as $K \longrightarrow \infty$ in the dynamic programming equation (4.7).

$$\lim_{k \longrightarrow \infty} M_k(p) = M(p)$$

where $M_k(p)$ is defined by (4.6), to obtain a stationary version of equation (4.7)

$$M(p) = \inf_{u \in U} \sup_{y \in \mathbf{R}^t} \{M(H(p, u, y))\} \qquad (4.8)$$

4.2.1 Dissipation Inequality

The following lemma is a consequence of corollary 4.9.

Lemma 4.15 For any $u \in O$, the closed loop system Σ^u is finite gain if and only if the information state satisfies

$$\sup_{k \geq 1} \sup_{y \in \Delta^u_{1,k}(X_0)} \left\{ (p_k, 0) \mid p_0 = -\beta^u \right\} \leq 0 \qquad (4.9)$$

for some finite $\beta^u(x_0) \geq 0$ on X_0, with $\beta^u(0) = 0$.

$\square$

By using lemma 4.15 we say that the information state system Ξ^u ((4.4) with information state feedback $u \in I$) is finite gain if and only if the information state p_k satisfies equation (4.9) for some finite $\beta^u(x_0)$, with $\beta^u(0) = 0$. If Σ^u is finite gain, we write

$$dom \ J_p(u) = \{ p \in \mathcal{E} \mid (p, 0), (p, \beta^u) \ \text{finite} \}$$

where $J_p(u) = \sup_{k \geq 0} J_{p,k}(u)$.

We say that the information state system $\Xi^{\bar{u}}$ is **finite gain dissipative** if there exists a function (storage function) $M(p)$, such that $dom \ M$ contains $-\beta$ ($\beta \geq 0$ and finite for all $x \in X_0$, with $\beta(0) = 0$), $M(p) \geq (p, 0)$, $M(-\beta) = 0$, and satisfies the following dissipation inequality

$$M(p) \geq \sup_{y \in \mathbf{R}^t} \left\{ M(H(p, \bar{u}(p), y)) \right\}, \ \forall p \in dom \ M \qquad (4.10)$$

Note that if $\Xi^{\bar{u}}$ is finite gain dissipative, and $p \in dom \ M$, then $H(p, \bar{u}(p), y) \in dom \ M$ for all $y \in \mathbf{R}^t$. Consequently, $p_0 \in dom \ M$, implies $p_k \in dom \ M$, $\forall k > 0$.

Lemma 4.16 M_k is monotone non-decreasing. i.e.

$$M_{k-1}(p) \leq M_k(p)$$

Proof:

Note that

$$M_k(p) = \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^u_{0,k}(x_0)} \{ p(x_0) + \sum_{i=1}^{k} \mid l(r_i, u_{i-1}) - l(s_i, u_{i-1}) \mid^2 - \gamma^2 \mid s_i - r_i \mid^2 \}$$

Then for any $\epsilon > 0$, choose $x_0' \in X_0$, and $r', s' \in \Gamma^u_{0,k-1}(x_0')$ such that

$$M_{k-1}(p) \leq p(x_0') + \sum_{i=1}^{k-1} \mid l(r_i', u_{i-1}) - l(s_i', u_{i-1}) \mid^2 - \gamma^2 \mid r_i' - s_i' \mid^2 + \epsilon$$

Let $x_0 = x_0'$, and define $r, s \in \Gamma^u_{0,k}(x_0)$ by $r = r'$, $s = s'$ on $[0, k-1]$, and $r_k = s_k$.

Then

$$
\begin{aligned}
M_k(p) &\geq p(x_0) + \sum_{i=1}^{k} \mid l(r_i, u_{i-1}) - l(s_i, u_{i-1}) \mid^2 - \gamma^2 \mid r_i - s_i \mid^2 \\
&\geq p(x_0') + \sum_{i=1}^{k-1} \mid l(r_i', u_{i-1}) - l(s_i', u_{i-1}) \mid^2 - \gamma^2 \mid r_i' - s_i' \mid^2 + \\
&\qquad\qquad\qquad\qquad\qquad \mid l(r_k, u_{k-1}) - l(s_k, u_{k-1}) \mid^2 \\
&\geq M_{k-1}(p) - \epsilon
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, letting $\epsilon \longrightarrow 0+$ gives

$$M_k(p) \geq M_{k-1}(p)$$

$\square$

We are now in a position to prove a version of the bounded real lemma for the information state system Ξ.

Theorem 4.17 Let $u \in I$. Then the information state system Ξ^u is finite gain if and only if it is finite gain dissipative.

Proof:

(i)

Assume that Ξ^u is finite gain dissipative. Then by the dissipation inequality (4.10)

$$M(p_k) \le M(p_0), \ \forall k > 0, \ \forall y \in \Delta_{1,k}^u(X_0)$$

Setting $p_0 = -\beta$, and using the fact that $M(p) \ge (p, 0)$, we get

$$(p_k, 0) \le M(-\beta) = 0, \ \forall k > 0, \forall y \in \Delta_{1,k}^u(x_0)$$

Therefore Ξ^u is finite gain, with $p_0 = -\beta$.

(ii)

Assume Ξ^u is finite gain. Then

$$(p, 0) \le J_{p,k}(u) \le (p, \beta^u), \ \forall k \ge 0, p \in dom \ J_p(u)$$

Writing $M_k(p) = J_{p,k}(u)$, so that

$$(p, 0) \le M_k(p) \le (p, \beta^u), \ k \ge 0, p \in dom \ J_p(u)$$

By lemma 4.16, M_k is monotone non-decreasing. Therefore

$$M_a(p) = \lim_{k \longrightarrow \infty} M_k(p)$$

exists, and is finite on *dom* M_a, which contains *dom* $J_p(u)$.

To show that M_a satisfies the dissipation inequality (4.10), fix $p \in dom\ M_a$, $y \in \mathbf{R}^t$, and $\epsilon > 0$. Select $k > 0$, and $\tilde{y}_{1,k-1}$ such that

$$M_a(H(p, u(p), y)) \leq (\tilde{p}_{k-1}, 0) + \epsilon$$

where, $\tilde{p}_j$, $j = 0, \ldots, k-1$ is the information state trajectory generated by $\tilde{y}$, with $\tilde{p}_0 = H(p, u(p), y)$.

Define

$$y_i = \left\{ \begin{array}{ll} y & \text{, if } i = 1 \\ \tilde{y}_{i-1} & \text{, if } i = 2, \ldots, k \end{array} \right.$$

and let p_j, $j = 0, \ldots, k$ denote the corresponding information state trajectory with $p_0 = p$. Then

$$
\begin{aligned}
M_a(p) &\geq (p_k, 0) \\
&= (\tilde{p}_{k-1}, 0) \\
&\geq M_a(H(p, u(p), y)) - \epsilon
\end{aligned}
$$

Since, y and ϵ are arbitrary, we have

$$M_a(p) \geq \sup_{y \in \mathbf{R}^t} M_a(H(p, u(p), y))$$

Hence, M_a solves the dissipation inequality. Also, by definition $(p, 0) \leq M_a(p) \leq (p, \beta^u)$. This implies that $M_a(-\beta^u) = 0$. Thus, Ξ^u is finite gain dissipative.

$\square$

We, now again assume that Σ^u satisfies assumption **A7** on page 27.

Theorem 4.18 Let $u \in I$. If Ξ^u is finite gain dissipative and Σ^u satisfies assumption **A7**, then Σ^u is weakly asymptotically stable.

Proof:

Inequality (4.10) implies

$$\sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^u_{0,k}(x_0)} \{p(x_0) + \sum_{i=1}^{k} \mid l(r_i, u_{i-1}) - l(s_i, u_{i-1}) \mid^2 -\gamma^2 \mid s_i - r_i \mid^2\} \leq M(p)$$

for all $k \geq 1$. Let $x_0 \in X_0$, and let $p = -\beta^u$. Then the above gives

$$\sup_{r,s \in \Gamma^u_{0,k}(x_0)} \{\sum_{i=1}^{k} \mid l(r_i, u_{i-1}) - l(s_i, u_{i-1}) \mid^2 -\gamma^2 \mid s_i - r_i \mid^2\} \leq M(p) + \beta^u(x_0) < +\infty$$

For any $\bar{r} \in \Gamma^u_{0,k}(x_0)$, there is a sequence

$$
\begin{aligned}
W^u_k &= \sup_{g,h \in \mathcal{F}(\bar{r}_k, u_k)} \{\mid l(g, u_k) - l(h, u_k) \mid^2 -\gamma^2 \mid g - h \mid^2\} \\
&\geq 0
\end{aligned}
$$

Also, from above we obtain that

$$\sum_{i=0}^{k} W^u_k < +\infty, \; \forall k \geq 0$$

Hence, $W_k^u \longrightarrow 0$, as $k \longrightarrow \infty$ and by corollary 2.8 and assumption **A7**

$$0 \in \liminf_{k \longrightarrow \infty} \mathcal{F}(\bar{r}_k, u_k)$$

Hence, Σ^u is weakly asymptotically stable.

$\square$

Corollary 4.19 If Ξ^u is finite gain dissipative, then Σ^u is ultimately bounded.

Proof: Similar to that of corollary 2.11.

$\square$

We also need to show that the information state system Ξ^u is stable.

Theorem 4.20 Let $u \in I$. If Ξ^u is finite gain dissipative, then Ξ^u is stable on all feasible $x \in \mathbf{R}^n$.

Proof:

The dissipation inequality (4.10) implies that

$$p_k(x) \leq (p_k, 0) \leq M(p_0) < +\infty$$

for all $p_0 \in dom \ M$, and for all $k \geq 0$. For the lower bound, note that by definition (4.2)

$$p_k(x) = \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma_{0,k}^u(x_0)} \{ p_0(x_0) + \sum_{i=1}^{k} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 - \gamma^2 \mid r_i - s_i \mid^2 \}$$

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For any $x_0 \in X_0$, this implies that for any feasible $x \in \mathbf{R}^n$

$$p_k(x) \geq p_0(x_0) > -\infty, \ \forall k \geq 0$$

Therefore, Ξ^u is stable.

$\square$

4.3 Solution to the Output Feedback Robust Control Problem

As in the state feedback case, it can be inferred from the previous results, that the controlled dissipation inequality (4.10) is both a necessary and sufficient condition for the solvability of the output feedback robust control problem.

However, we now state necessary and sufficient conditions for the solvability of the output feedback robust control problem in terms of dynamic programming equalities.

Theorem 4.21 (Necessity) Assume that there exists a controller $\bar{u} \in O$ which solves the output feedback robust control problem. Then there exists a function $M(p)$, such that $dom\ J_p(\bar{u}) \subset dom\ M(p)$, $M(p) \geq (p, 0)$, $M(-\beta^{\bar{u}}) = 0$ and M solves the stationary dynamic programming equation

$$M(p) = \inf_{u \in U} \sup_{y \in \mathbf{R}^t} \{M(H(p, u, y))\} \tag{4.11}$$

for all $p \in dom\ J_p(\bar{u})$.

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Proof: For $p \in dom\ J_p(\bar{u})$, define $M_k(p)$, $k = 0, \ldots$ as follows

$$M_k(p) = \inf_{u \in U} \sup_{y \in R^t} M_{k-1}(H(p, u, y))$$
$$M_0(p) = (p, 0)$$

Clearly

$$(p, 0) \leq M_k(p) \leq (p, \beta^{\bar{u}}) < +\infty, \ \forall p \in dom\ J_p(\bar{u})$$

Furthermore, a modification of lemma 4.16 establishes that

$$M_{k+1}(p) \geq M_k(p), \quad \forall p \in dom\ J_p(\bar{u})$$

Hence,

$$M_k(p) \longrightarrow M(p) \text{ as } k \longrightarrow \infty$$

and $M(p)$ satisfies equation (4.11) for all $p \in dom\ J_p(\bar{u})$.

Furthermore, $dom\ J_p(\bar{u}) \subset dom\ M(p)$ and $(p, 0) \leq M(p) \leq (p, \beta^{\bar{u}})$. Thus, since $-\beta^{\bar{u}} \leq 0$, $\beta^{\bar{u}}(0) = 0$, $M(-\beta^{\bar{u}}) = 0$.

$\square$

Theorem 4.22 (Sufficiency) Assume that there exists a solution M to the stationary dynamic programming equation (4.11) on some non-empty domain $dom\ M$, such that $-\beta \in dom\ M$, $(\beta(x) \geq 0$, and finite $\forall x \in X_0$, with $\beta(0) = 0)$, $M(-\beta) = 0$, and $M(p) \geq (p, 0)$. Let $\bar{u} \in I$ be a policy such that $\bar{u}(p)$ achieves the minimum in (4.11). Let $p_0 = -\beta$, and let p_k be the corresponding information state trajectory satisfying $p_k \in dom\ M$, $k = 0, 1, \ldots$. Then, $\bar{u} \in I$ solves the

information state feedback robust control problem if the closed loop system $\Sigma^{\bar{u}}$ satisfies assumption **A7**.

Proof: Since M satisfies equation (4.11), $\Sigma^{\bar{u}}$ satisfies equation (4.10) with equality. Hence, $\Xi^{\bar{u}}$ is finite gain dissipative and by theorem 4.17, $\Xi^{\bar{u}}$ is finite gain. Furthermore, theorem 4.18 establishes that $\Sigma^{\bar{u}}$ is weakly asymptotically stable, and by corollary 4.19 $\Sigma^{\bar{u}}$ is ultimately bounded. Also by theorem 4.20, $\Xi^{\bar{u}}$ is stable for all feasible $x \in \mathbf{R}^n$.

$\square$

Remark 4.23 As in the state feedback case, we can from the statement of theorem 4.21 and the proof of theorem 4.22, obtain necessary and sufficient conditions for the solvability of the robust control problem in terms of dissipation inequalities.

4.4 Delayed Measurements

In this section, we present some results for the case of delayed measurements. We will restrict ourselves to the finite time case. Hence, we need only satisfy condition **C3**, of section 1.2.3, page 15. The infinite time case can be tackled as in the case of no delay, by writing down a dissipation inequality based on the stationary dynamic programming equation. It is interesting to observe, that for the delayed measurement case, the information state is no longer just the **cost to**

come function. Furthermore, we will see that the we now have to also solve an open-loop dynamic programming equation, in addition to solving the filter equation for the information state, and the dynamic programming equation to obtain the control.

The system under consideration is

$$\Sigma \begin{cases} x_{k+1} & \in \quad \mathcal{F}(x_k, u_k) \\ y_{k+1} & \in \quad \mathcal{G}(x_{k-\tau}, u_{k-\tau}) \\ z_{k+1} & = \quad l(x_{k+1}, u_k). \end{cases} \tag{4.12}$$

Here, $x_k \in \boldsymbol{R}^n$ are the states, $y_k \in \boldsymbol{R}^t$ are the measurements. $u_k \in \boldsymbol{U} \subset \boldsymbol{R}^m$ are the control inputs, and $z_k \in \boldsymbol{R}^q$ are the regulated outputs. Here, $\tau \geq 0$ is assumed to be fixed. It is clear that if $k \leq \tau$, then no measurements y_k are available.

We denote the space of output feedback policies as O. Hence, if $u \in O$ then $u_k = u(y_{\tau+1,k}, u_{0,k-1})$, where in general $s_{i,j}$ is the vector $[s_i \; s_{i+1} \ldots s_j]$.

Before proceeding further, we introduce the spaces

$$\mathcal{E} \triangleq \{p : \boldsymbol{R}^n \to \boldsymbol{R}^*\}$$

and

$$\mathcal{U}^k \triangleq \{u : u = u_{i,j}, u_t \in \boldsymbol{U}, i \leq t \leq j, 0 \leq j - i \leq k, \text{or } u = \phi\}.$$

Now consider their direct sum

$$\mathcal{D} \triangleq \mathcal{E} \oplus \mathcal{U}^{\tau-1} = \left\{ \begin{bmatrix} p \\ u \end{bmatrix} : p \in \mathcal{E}, u \in \mathcal{U}^{\tau-1} \right\}$$

and define the operators $\pi_1 : \mathcal{D} \to \mathcal{E}$, and $\pi_2 : \mathcal{D} \to \mathcal{U}^{\tau-1}$ as

$$\pi_1 \left(\begin{bmatrix} p \\ u \end{bmatrix} \right) = p \text{ and, } \pi_2 \left(\begin{bmatrix} p \\ u \end{bmatrix} \right) = u.$$

Also, we associate with a sequence $u_{i,j}$ its length given by $\sigma(u) = j - i + 1$. Here, we use the convention that $\sigma(\phi) = 0$. We now consider the functional (similar to the delay free case)

$$L_{p,k}(u) \triangleq \sup_{x_0 \in \mathbf{R}^n} \sup_{r,s \in \Gamma_{0,k}^u(x_0)} \left\{ p(x_0) + \sum_{i=1}^{k-1} |l(r_{i+1}, u_i) - l(s_{i+1}, u_i)|^2 - \gamma^2 |r_{i+1} - s_{i+1}|^2 \right\}$$

Then, we have

Lemma 4.24 (i) Σ^u is finite gain on $[0, K]$ if and only if there exists a finite quantity $\beta_K^u(\cdot) \geq 0$, such that

$$L_{-\beta_K^u, K}(u) \leq 0.$$

(ii) If each map $\Sigma_{x_0}^u$ is finite gain on $[0, K]$, then

$$(p, 0) \leq L_{p,K}(u) \leq (p, \beta_K^u).$$

$\square$

The robust control problem can now be expressed as, find $u^* \in O_{0,k-1}$, such that

$$L_{p,k}(u^*) = \inf_{u \in O_{0,k-1}} L_{p,k}(u)$$

4.4.1 Information State

For a fixed $u_{0,k-1}$, $y_{\tau+1,k} \in \Delta_{\tau+1,k}^u(X_0)$, we define the **cost to come** function $p_k \in \mathcal{E}$ as

$$p_k(x) \quad \triangleq \quad \sup_{x_0 \in \mathbf{R}^n} \sup_{r,s \in \Gamma_{0,k}^u(x_0)} \left\{ p_0(x_0) + \sum_{i=0}^{k-1} |l(r_{i+1}, u_i) - l(s_{i+1}, u_i)|^2 - \quad (4.13) \right.$$

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$$\gamma^2 |r_{i+1} - s_{i+1}|^2 : r_k = x, y_{i+1} \in \mathcal{G}(r_{i-\tau}, u_{i-\tau}), \tau \le i \le k - 1\}.$$

We would like to express p_k as a dynamical equation. For this purpose, define $H(p, u, y) \in \mathcal{E}$ as done in the delay free case (c.f. equation 4.3).

Let $\hat{p} \in \mathcal{D}$, and define the shift/pad operation $\eta : \mathcal{U}^{\tau-1} \times \boldsymbol{U} \to \mathcal{U}^{\tau-1}$ by

$$\eta(u_{i,j}, u_{j+1}) \triangleq \begin{cases} u_{i,j+1} & \text{if } j - i < \tau - 1 \\ u_{i+1,j+1} & \text{else} \end{cases}$$

and the functional $J : \mathcal{D} \to \boldsymbol{R}^*$ by

$$J(\hat{p}_k)(x) \triangleq \sup_{x_0 \in \boldsymbol{R}^n} \sup_{r,s \in \Gamma_{0,\sigma(\pi_2(\hat{p}_k))}^{\pi_2(\hat{p}_k)}} \{ \pi_1(\hat{p}_k)(x_0) + \sum_{i=0}^{\sigma(\pi_2(\hat{p}_k))-1} |l(r_{i+1}, \pi_2(\hat{p}_k)_i) - l(s_{i+1}, \pi_2(\hat{p}_k)_i)|^2 - \gamma^2 |r_{i+1} - s_{i+1}|^2 : r_{\sigma(\pi_2(\hat{p}_k))} = x \} \tag{4.14}$$

where $\pi_2(\hat{p}_k)_i$ denotes the ith element of $\pi_2(\hat{p}_k)$, assuming that the indexing starts from 0. In particular, if $\pi_2(\hat{p}_k) = \phi$, then $J(\hat{p}_k)(x) = \pi_1(\hat{p}_k)(x)$. We now define the functional $F \in \mathcal{D}$ by

$$F(\hat{p}_k, u_k, y_{k+1})(x) \triangleq \begin{bmatrix} H(\pi_1(\hat{p}_k), \pi_2(\hat{p}_k)_0, y_{k+1})(x) \\ \eta(\pi_2(\hat{p}_k), u_k) \end{bmatrix}.$$

Then we have the following result:

Lemma 4.25 The cost to come function (p_k) is the solution to the following recursion

$$\Xi \begin{cases} \hat{p}_{k+1} = F(\hat{p}_k, u_k, y_{k+1}), k \in [0, K-1] \\ p_{k+1} = J(\hat{p}_{k+1}) \end{cases} \tag{4.15}$$

for any $\hat{p}_0 \in \mathcal{D}$ of the form $\begin{bmatrix} p_0 \\ \phi \end{bmatrix}$, with $p_0 \in \mathcal{E}$.

Proof:

Given the initial condition of the form $\hat{p}_0 = \begin{bmatrix} p_0 \\ \phi \end{bmatrix}$, with $p_0 \in \mathcal{E}$, we have

$$\hat{p}_{k+1}(x) = \begin{bmatrix} \pi_1(\hat{p}_{k+1})(x) \\ \pi_2(\hat{p}_{k+1}) \end{bmatrix} = \begin{bmatrix} H(\pi_1(\hat{p}_k), \pi_2(\hat{p}_k)_0, y_{k+1})(x) \\ \eta(\pi_2(\hat{p}_k), u_k) \end{bmatrix}.$$

By the definition of η it is clear that

$$\pi_2(\hat{p}_{k+1}) = \begin{cases} u_{0,k} & \text{if } k < \tau \\ u_{k-\tau+1,k} & \text{if } k \geq \tau. \end{cases}$$

Also, by definition,

$$H(\pi_1(\hat{p}_k), \pi_2(\hat{p}_k)_0, y_{k+1})(x) = p_0(x) \text{ if } k < \tau$$

else, if $k \geq \tau$, we have

$$\begin{aligned} H(\pi_1(\hat{p}_k), \pi_2(\hat{p}_k)_0, y_{k+1})(x) &= H(\pi_1(\hat{p}_k), u_{k-\tau}, y_{k+1})(x) \\ \\ &= \sup_{\xi \in \mathbf{R}^n} \{\pi_1(\hat{p}_k)(\xi) + B(\xi, x, u_{k-\tau}, y_{k+1})\} \\ \\ &= \sup_{\xi \in \mathbf{R}^n} \{\pi_1(\hat{p}_k)(\xi) + \sup_{s \in \mathcal{F}(\xi, u_{k-\tau})} |l(x, u_{k-\tau}) - \\ & \quad l(s, u_{k-\tau})|^2 - \gamma^2 |x - s|^2 : x \in \mathcal{F}(\xi, u_{k-\tau}), \\ \\ & \quad y_{k+1} \in \mathcal{G}(\xi, u_{k-\tau})\}. \end{aligned}$$

Which implies that

$$\begin{aligned} \pi_1(\hat{p}_{k+1})(x) = \ & \sup_{x_0 \in \mathbf{R}^n} \sup_{r,s \in \Gamma^u_{0,k-\tau+1}(x_0)} \{p_0(x_0) + \sum_{i=0}^{k-\tau} |l(r_{i+1}, u_i) - \\ & l(s_{i+1}, u_i)|^2 - \gamma^2 |r_{i+1} - s_{i+1}|^2 : \quad\quad (4.16) \\ & r_{k-\tau+1} = x, y_{i+1+\tau} \in \mathcal{G}(x_i, u_i), 0 \leq i \leq k - \tau\} \end{aligned}$$

for $k \geq \tau$.

Recalling the definition of J (equation (4.14)), and assuming $k < \tau$, we have

$$J(\hat{p}_{k+1})(x) = \sup_{\xi \in \mathbf{R}^n} \sup_{r,s \in \Gamma^{\pi_2(\hat{p}_{k+1})}_{0,k+1}(\xi)} \{p_0(\xi) + \sum_{i=0}^{k} |l(s_{i+1}, \pi_2(\hat{p}_{k+1})_i) - l(r_{i+1}, \pi_2(\hat{p}_{k+1})_i)|^2 - \gamma^2 |r_{i+1} - s_{i+1}|^2 : r_{k+1} = x\}$$

which equals p_{k+1} by definition.

Now, assuming that $k \geq \tau$, we obtain

$$J(\hat{p}_{k+1})(x) = \sup_{\xi \in \mathbf{R}^n} \sup_{r,s \in \Gamma^{\pi_2(\hat{p}_{k+1})}_{k-\tau+1,k+1}(\xi)} \{\pi_1(\hat{p}_{k+1})(\xi) + \sum_{i=k-\tau+1}^{k} |l(r_{i+1}, \pi_2(\hat{p}_{k+1})_{i+\tau-1-k})$$
$$- l(s_{i+1}, \pi_2(\hat{p}_{k+1})_{i+\tau-1-k})|^2 - \gamma^2 |r_{i+1} - s_{i+1}|^2 : r_{k+1} = x\}$$

$$= p_{k+1} \text{ by substituting } \pi_1(\hat{p}_{k+1}) \text{ from equation (4.16).}$$

$\square$

Similar to the case of no delay, we get the following theorem which expresses the L in terms of p.

Theorem 4.26 For $u \in O_{0,k-1}$, $p \in \mathcal{E}$, such that $L_{p,k}(u)$ is finite, we have

$$L_{p,k}(u) = \begin{cases} \sup_{y \in \Delta^u_{\tau+1,k}(X_0)} \{(p_k, 0) : p_0 = p\}, & k \in [\tau+1, K] \\ \{(p_k, 0) : p_0 = p\}, & k \leq \tau. \end{cases}$$

Proof:

Similar to Theorem 4.7.

$\square$

This immediately yields the following corollary.

Corollary 4.27 For any output feedback controller $u \in O_{0,K-1}$, the closed-loop system Σ^u is finite gain if and only if $\hat{p}_k$ satisfies

$$0 \geq \begin{cases} \sup_{y \in \Delta^u_{\tau+1,k}(X_0)} \left\{ (J(\hat{p}_k), 0) : \hat{p}_0 = \begin{bmatrix} -\beta^u_K \\ \phi \end{bmatrix} \right\}, & \forall k \in [\tau+1, K] \\ \left\{ (J(\hat{p}_k), 0) : \hat{p}_0 = \begin{bmatrix} -\beta^u_K \\ \phi \end{bmatrix} \right\}, & \forall k \in [0, \tau] \end{cases} \tag{4.17}$$

for some finite $\beta^u_K(x) \geq 0$, $\beta^u_K(0) = 0$.

$\square$

In fact, the above result yields a separation principle, in the sense that $\hat{p}_k \in \mathcal{D}$ contains all the relevant information required to solve the problem. This justifies naming $\hat{p}_k \in \mathcal{D}$ obtained via dynamics (4.15), with initial conditions of the form $\begin{bmatrix} p_0 \\ \phi \end{bmatrix}$, $p_0 \in \mathcal{E}$ the **information state**. In particular, we have transformed the problem into one with full information, with a new (infinite dimensional) system Ξ, whose states are $\hat{p}_k$, and the disturbance are the measurements y_k. The cost is now given by (4.17).

Remark 4.28 Note that the **information state** is no longer the cost to come, as it was in the case of no measurement delay. However, in the case, we have $\tau = 0$, the two definitions coincide.

Remark 4.29 Furthermore, note that we could have taken the supremum in equation (4.17) over $y \in \Delta^u_{1,k}(X_0)$, since the cost is independent of y_k, for $k \in [0, \tau]$.

4.4.2 Solution to the Delayed Measurement Problem

We employ dynamic programming to solve the problem. Define

$$M_k(\hat{p}) \triangleq \inf_{u \in O_{0,k-1}} \sup_{y \in \Delta_{1,k}^u(X_0)} \{(J(\hat{p}_k), 0) : \hat{p}_0 = \hat{p}\}. \tag{4.18}$$

For a function $M : \mathcal{D} \to \boldsymbol{R}^*$, we write

$$dom\ M = \{\hat{p} \in \mathcal{D} : M(\hat{p}) \text{ is finite}\}$$

and, we also write

$$dom\ L_{.,k}(u) = \{p \in \mathcal{E} : L_{p,k} \text{ is finite}\}.$$

Now consider the following dynamic programming equation.

$$
\begin{aligned}
W_k(\hat{p}) &= \inf_{u \in \boldsymbol{U}} \sup_{y \in \boldsymbol{R}^t} \{W_{k-1}(F(\hat{p}, u, y))\} \\
&\quad \hat{p} \in dom\ W_k, \ k \in [1, K] \\
W_0(\hat{p}) &= (\pi_1(\hat{p}), Q_0^{\pi_2(\hat{p})})
\end{aligned}
\tag{4.19}
$$

where $Q_0^{\pi_2(\hat{p})}$ is obtained via the following open-loop dynamic programming equation

$$
\begin{aligned}
Q_k^{\pi_2(\hat{p})}(x) &= \sup_{s,r \in \mathcal{F}(x, \pi_2(\hat{p})_k)} \{|l(r, \pi_2(\hat{p})_k) - l(s, \pi_2(\hat{p})_k)|^2 - \gamma^2 |r - s|^2 + \\
&\quad Q_{k+1}^{\pi_2(\hat{p})}(r)\}, \ x \in \boldsymbol{R}^n, \ k = 0, \ldots, \sigma(\pi_2(\hat{p})) - 1 \\
Q_{\sigma(\pi_2(\hat{p}))}^{\pi_2(\hat{p})}(x) &= 0.
\end{aligned}
\tag{4.20}
$$

Lemma 4.30 Let $\hat{p} \in \mathcal{D}$, and let $Q_0^{\pi_2(\hat{p})}$ be obtained as a solution to the open-loop dynamic programming equation (4.20). Then

$$(J(\hat{p}), 0) = (\pi_1(\hat{p}), Q_0^{\pi_2(\hat{p})}).$$

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Proof:

Dynamic programming arguments imply that

$$Q_0^{\pi_2(\hat{p})}(x) = \sup_{r,s \in \Gamma_{0,\sigma(\pi_2(\hat{p}))}^{\pi_2(\hat{p})}(x)} \{ \sum_{i=0}^{\sigma(\pi_2(\hat{p}))-1} |l(r_{i+1}, \pi_2(\hat{p})_i) - l(s_{i+1}, \pi_2(\hat{p})_i)|^2$$

$$-\gamma^2 |r_{i+1} - s_{i+1}|^2 \}.$$

Which in turn implies that

$$
\begin{aligned}
(\pi_1(\hat{p}), Q_0^{\pi_2(\hat{p})}) &= \sup_{\xi \in \boldsymbol{R}^n} \{ \pi_1(\hat{p})(\xi) + \\
&\quad \sup_{\Gamma_{0,\sigma(\pi_2(\hat{p}))}^{\pi_2(\hat{p})}(\xi)} \sum_{i=0}^{\sigma(\pi_2(\hat{p}))-1} |l(r_{i+1}, \pi_2(\hat{p})_i) \\
&\quad - l(s_{i+1}, \pi_2(\hat{p})_i)|^2 - \gamma^2 |r_{i+1} - s_{i+1}|^2 \} \\
&= \sup_{x \in \boldsymbol{R}^n} J(\hat{p})(x) \\
&= (J(\hat{p}), 0).
\end{aligned}
$$

$\square$

Theorem 4.31 Let W be the solution of the dynamic programming equation (4.19), initialized via (4.20). Then $W = M$.

Proof:

Note that $M_0(\hat{p}) = (J(\hat{p}), 0) = W_0(\hat{p})$. We now establish that M satisfies (4.19).

We use induction. Let this be true for k. Then we have

$$
\begin{aligned}
M_{k+1}(\hat{p}) &= \inf_{u \in O_{0,k}} \sup_{y \in \Delta_{1,k+1}^u(X_0)} \{ (J(\hat{p}_{k+1}), 0) : \hat{p}_0 = \hat{p} \} \\
&= \inf_{u_0 \in U} \sup_{y_1 \in \boldsymbol{R}^t} \inf_{u \in O_{1,k}} \sup_{y \in \Delta_{2,k+1}^u(X_0)} \{ (J(\hat{p}_{k+1}), 0) : \\
&\qquad\qquad\qquad\qquad \hat{p}_1 = H(\hat{p}, u, y) \}
\end{aligned}
$$

(where we interchange the minimization over $u_{1,k}$ and maximization over y_1, since $u_{1,k}$ is a function of y_1.)

$$
\begin{aligned}
&= \inf_{u_0 \in U} \sup_{y_1 \in \boldsymbol{R}^t} \inf_{u \in O_{0,k-1}} \sup_{y \in \Delta_{1,k}^u(\mathcal{F}(X_0, u_0))} \{ (J(\hat{p}_k), 0) : \\
&\qquad\qquad\qquad\qquad \hat{p}_0 = H(\hat{p}, u, y) \}
\end{aligned}
$$

(due to time invariance.)

$$= \inf_{u \in U} \sup_{y \in \boldsymbol{R}^t} M_k(H(\hat{p}, u, y)).$$

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Hence, since $M_0 = W_0$, an induction argument also establishes that $M_k = W_k$, $k \in [0, K]$.

$\square$

We now state the necessary and sufficient conditions for the solvability of the finite time robust control problem.

Theorem 4.32 (Necessity) Assume that $u^o \in O_{0,K-1}$ solves the finite time output feedback problem for the system subject to a constant measurement delay of $\tau \geq 0$. Then there exists a solution M to the dynamic programming equation (4.19), such that $dom\ L_{.,K}(u^o) \subset \pi_1(dom\ M_k)$, $M_k\left(\begin{bmatrix} -\beta_K^{u^o} \\ \phi \end{bmatrix} \right) = 0$, $M_k(\hat{p}) \geq (J(\hat{p}), 0)$, $\hat{p} \in dom\ M_k$, $k \in [0, K]$.

Proof: We first establish that $M_k(\hat{p}) \geq (J(\hat{p}), 0)$. Let $\hat{p} \in dom\ M_k$. We can write $M_k(\hat{p})$ as

$$
\begin{aligned}
M_k(\hat{p}) &= \inf_{u \in O_{0,k-1}} \sup_{x_0 \in \mathbf{R}^n} \sup_{r,s \in \Gamma_{0,k}^u(x_0)} \{ J(\hat{p})(x_0) + \sum_{i=0}^{k-1} |l(r_{i+1}, u_i) \\
&\qquad\qquad l(s_{i+1}, u_i)|^2 - \gamma^2 |r_{i+1} - s_{i+1}|^2 \} \\
&\geq (J(\hat{p}), 0).
\end{aligned}
$$

Let $p \in dom\ L_{.,K}(u^o)$, and set $\hat{p} = \begin{bmatrix} p \\ \phi \end{bmatrix}$. Now by (4.18)

$$
\begin{aligned}
M_k\left(\begin{bmatrix} p \\ \phi \end{bmatrix} \right) &= \inf_{u \in O_{0,k-1}} L_{p,k}(u) \\
&\leq L_{p,k}(u^o) \\
&\leq (p, \beta_K^{u^o}).
\end{aligned}
$$

Thus, $dom\ L_{.,K}(u^o) \subset dom\ M_k$. Since, $\beta_K^{u^o}(x) \geq 0$, $\beta_K^{u^o}(0) = 0$, we have

$$J\left(\begin{bmatrix} -\beta_K^{u^o} \\ \phi \end{bmatrix}\right) = (-\beta_K^{u^o}, 0) = 0$$

This implies that $M_k\left(\begin{bmatrix} -\beta_K^{u^o} \\ \phi \end{bmatrix}\right) = 0$. Also Theorem 2 establishes that M is the

unique solution to the dynamic programming equation (4.19).

$\square$

Theorem 4.33 (Sufficiency) Assume there exists a solution M to the dynamic

programming equation (4.19) on some non-empty domain $dom\ M_k$, such that

$\begin{bmatrix} -\beta \\ \phi \end{bmatrix} \in dom\ M_k$, $M_k\left(\begin{bmatrix} -\beta \\ \phi \end{bmatrix}\right) = 0$, for some $\beta \geq 0$, $\beta(0) = 0$ and also

that $M_k(\hat{p}) \geq (J(\hat{p}), 0)$, for all $\hat{p} \in dom\ M_k$, $k \in [0, K]$. Let $\bar{u}_k(\hat{p})$ achieve the

minimum in (4.19) for each $\hat{p} \in dom\ M_k$, $k \in [1, K]$. Let u^* be a policy such that

$u_k^* = \bar{u}_{K-k}(\hat{p}_k)$, where $\hat{p}_k$ is the corresponding trajectory with initial conditions

$\hat{p}_0 = \begin{bmatrix} -\beta \\ \phi \end{bmatrix}$, assuming $\hat{p}_k \in dom\ M_{K-k}$, $k \in [0, K]$. Then u^* solves the finite-time

output feedback problem for the system subject to a constant delay of $\tau \geq 0$.

Proof: Observe that

$$M_k\left(\begin{bmatrix} p \\ \phi \end{bmatrix}\right) = L_{p,k}(u^*) \leq L_{p,k}(u)$$

for all $u \in O_{0,k-1}$, $\begin{bmatrix} p \\ \phi \end{bmatrix} \in dom\ M_k$. Hence,

$$\sup_{y \in \Delta_{1,k}(X_0)} \left\{ (J(\hat{p}_k), 0) : \hat{p}_0 = \begin{bmatrix} -\beta \\ \phi \end{bmatrix}, u = u^* \right\} \leq M_k(-\beta) = 0$$

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which implies by corollary 4.27 that Σ^{u^*} is finite gain, and thus u^* solves the finite time output feedback problem.

$$\square$$

Remark 4.34 Note the strong resemblance of the necessary and sufficient conditions for solvability of the delayed measurement problem to those when we have no delay. In fact, our conditions collapse to those of the delay free case, when $\tau = 0$.

Remark 4.35 We see that solvability of the delayed measurement case requires: (i) existence of a solution $\hat{p}_k$ to (4.15), (ii) existence of a solution $Q^{\pi_2(\hat{p})}$ to (4.20), (iii) existence of a solution M to (4.19), and (iv) a coupling condition, viz. $\hat{p}_k \in$ *dom* M_{K-k}.

Chapter 5

Reducing Controller Complexity

This chapter will consider some of the issues arising in trying to reduce the controller complexity. In general solving the output feedback dynamic programming equation

$$M(p) = \inf_{u \in U} \sup_{y \in R^t} M(H(p, u, y)) \tag{5.1}$$

is computationally hard. We will first consider the certainty equivalence controller, and then generalize the notion of certainty equivalence. A sufficiency condition exists for a certainty equivalence controller to be optimal. This condition deals with the ability to rewrite the upper value function of the output feedback game in terms of the upper value function of the state feedback game. We will give another condition which is implied by the former, which no longer includes the upper value function of the output feedback game (something which we are trying to avoid having to compute in the first place!). We will also establish that the upper value function of the state feedback game is the unique solution (if the

solution exists at all) of the equation

$$M(p) = (p, U), \forall p \in \mathcal{E}$$

where $U : \mathbf{R}^n \to \mathbf{R}$.

In this chapter we will be interested in the infinite time case, and in establishing conditions for the output feedback policy to be dissipative. A standing assumption throughout this chapter will be that the domain of the upper value function V of the state feedback game is large enough to encompass all points where the information state p_k is not equal to $-\infty$, for all k. i.e. let $dom \ V \triangleq \{x \in \mathbf{R}^n | V(x) < +\infty\}$, and let $dom \ p_k \triangleq \{x \in \mathbf{R}^n | p_k \text{ is finite}\}$. Then

$$dom \ p_k \subset dom \ V, \ \forall k \geq 0$$

5.1 Certainty Equivalence

Since, Whittle [56] first postulated the minimum stress estimate for the solution of a risk-sensitive stochastic optimal control problem, it has evolved into the certainty equivalence principle. The latter states that under appropriate conditions, an optimal output feedback controller can be obtained by inserting an estimate of the state into the corresponding state feedback law. In general, however the controller so obtained is non-optimal. The certainty equivalence property is known to hold for linear systems with a quadratic cost [8]. The recent interest in nonlinear H_∞ control has led researchers to examine whether, certainty equivalence could be

carried over to nonlinear systems. If certainty equivalence were to hold, it would result in a tremendous reduction in the complexity of the problem. In a recent paper [32], sufficient conditions were given for certainty equivalence to hold in terms of a saddle point condition. Also, in [30], a simple example is given to demonstrate the non-optimal nature of the certainty equivalence controller. An implementation of the certainty equivalence controller can be found in [52].

The certainty equivalence controller is constructed as follows. We identify p_k as the **past stress**, and V as the **future stress**, and then compute

$$\hat{x}_k \in \arg \max_{x \in \mathbf{R}^n} \{p_k(x) + V(x)\} \tag{5.2}$$

and use the feedback policy $u_k(p_k) = u_F(\hat{x}_k)$, where u_F is the optimal state feedback policy. In [30] a sufficient condition is stated for certainty equivalence to hold. This condition is

Theorem 5.1 (Certainty Equivalence) Let M be the upper value function for the output feedback game. Let V be the upper value function of the state feedback game. Then the certainty equivalence controller is optimal, provided

$$M(p_k) = (p_k, V) \tag{5.3}$$

for all k, where p_k is the information state trajectory generated by the system.

□

Unfortunately, the result above involves the quantity M, which we are trying to avoid computing. In addition, since the information state trajectory is not known *a priori*, we may need to actually check for all $p \in \mathcal{E}$.

For the delayed measurement case, the results remain identical with p_k being replaced by $J(\hat{p}_k)$ where $J(\hat{p}_k)$ is as defined in equation (4.1). In particular, the certainty equivalence controller can now be constructed as follows. Given the information state $\hat{p}_k$ at time k, and V the upper value function of the state feedback game, we compute

$$\hat{x}_k \in \arg \max_{x \in \boldsymbol{R}^n} \{J(\hat{p}_k)(x) + V(x)\} \tag{5.4}$$

and implement $u_k(\hat{p}_k) = u_F(\hat{x}_k)$, where u_F is the optimal state feedback policy.

We now turn to generalizations of certainty equivalence, and some of the bearings these have on the certainty equivalence controller.

5.2 Reduced Complexity Controllers

In this section, we present conditions for a reduced complexity controller to exist. These conditions apply for both optimal and non-optimal policies. In general, conditions for an optimal solution may not be satisfied. Hence, one may be satisfied with a reduced complexity non-optimal policy, which guarantees dissipativity of the closed-loop system. In the special case, we show that the policies so obtained are certainty equivalence policies. Furthermore, in doing so, we will be able to give

an equivalent sufficiency condition for certainty equivalence which may be more tractable than the one given above.

The dynamic programming equation (5.1), is infinite dimensional in general. Hence, this motivates us to search for reduced complexity control policies, which preserve the stability properties of the closed-loop system. We again introduce the function $\delta_{\{x\}} \in \mathcal{E}$ as defined by equation (1.6)

For a given $x \in \mathbf{R}^n$, and $u \in \mathbf{U}$, we define

$$\Omega(x, u) \stackrel{\triangle}{=} \{\xi \in \mathbf{R}^n \mid x \in \mathcal{F}(\xi, u)\}.$$

Then we have the following basic result.

Lemma 5.2 Let $h : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$. Then for any $u \in \mathbf{U}$, we have

$$\sup_{x \in \mathbf{R}^n} \sup_{\xi \in \Omega(x,u)} h(x, \xi) = \sup_{\xi \in \mathbf{R}^n} \sup_{r \in \mathcal{F}(\xi,u)} h(r, u)$$

Proof:

For any $\epsilon > 0$ there exists a $x^\epsilon \in \mathbf{R}^n$, and a $\xi^\epsilon \in \Omega(x^\epsilon, u)$, such that

$$
\begin{aligned}
\sup_{x \in \mathbf{R}^n} \sup_{\xi \in \Omega(x,u)} h(x, \xi) \ &< \ h(x^\epsilon, \xi^\epsilon) + \epsilon \\
&\leq \ \sup_{r \in \mathcal{F}(\xi^\epsilon, u)} h(r, \xi^\epsilon) + \epsilon \\
&\leq \ \sup_{\xi \in \mathbf{R}^n} \sup_{r \in \mathcal{F}(\xi,u)} h(r, \xi) + \epsilon
\end{aligned}
$$

Since ϵ is arbitrary, the result follows.

The reverse inequality can be shown as follows.

$$\sup_{x \in \mathbf{R}^n} \sup_{\xi \in \Omega(x,u)} h(x,\xi) \geq h(x,\xi), \; x \in \mathcal{F}(\xi,u)$$

$$\geq \sup_{r \in \mathcal{F}(\xi,u)} h(r,\xi)$$

$$\geq \sup_{\xi \in \mathbf{R}^n} \sup_{r \in \mathcal{F}(\xi,u)} h(r,\xi)$$

□

Define, $J_U^p : \mathbf{R}^n \times \mathbf{U} \to \mathbf{R}$ as

$$J_U^p(x,u) \triangleq \left\{ p(x) + \sup_{r,s \in \mathcal{F}(x,u)} \{| \, l(r,u) - l(s,u) \, |^2 - \gamma^2 \, | \, r - s \, |^2 + U(r) \} \right\}$$

We now state a fundamental result, which will be used repeatedly.

Lemma 5.3 For any $u \in \mathbf{U}$, and $U : \mathbf{R}^n \to \mathbf{R}$, and $p_k \in \mathcal{E}$,

$$\sup_{x \in \mathbf{R}^n} J_U^{p_k}(x,u) \geq \sup_{y \in \mathbf{R}^t} (H(p_k,u,y),U).$$

Proof:

$$\sup_{y \in \mathbf{R}^t} (p_{k+1},U) = \sup_{y \in \mathbf{R}^t} \sup_{x \in \mathbf{R}^n} \sup_{\xi \in \mathbf{R}^n} \{ p_k(\xi) + \sup_{r \in \mathcal{F}(\xi,u)} (| \, l(x,u) - l(r,u) \, |^2 -$$

$$\gamma^2 \, | \, x - r \, |^2 | \, x \in \mathcal{F}(\xi,u), y \in \mathcal{G}(\xi,u)) + U(x) \}$$

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$$
\leq \sup_{x \in \mathbf{R}^n} \sup_{\xi \in \Omega(x,u)} \{ p_k(\xi) + \sup_{r \in \mathcal{F}(\xi,u)} (\mid l(x,u) - l(r,u) \mid^2 -
$$

$$
\gamma^2 \mid x - r \mid^2) + U(x) \}
$$

$$
= \sup_{\xi \in \mathbf{R}^n} \{ p_k(\xi) + \sup_{r,x \in \mathcal{F}(\xi,u)} \mid l(x,u) - l(r,u) \mid^2 - \gamma^2 \mid x - r \mid^2 +
$$

$$
U(x) \}
$$

$$
= \sup_{\xi \in \mathbf{R}^n} J_U^p(\xi, u)
$$

$\square$

We now state the main theorem, which gives a sufficient condition for the existence of dissipative reduced complexity policies.

Theorem 5.4 Given $U : \mathbf{R}^n \to \mathbf{R}$, $U \geq 0$, and $U(0) = 0$. If for all $p_k \in \mathcal{E}$

$$
(p_k, U) \geq \inf_{u \in U} \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, u)
$$

then $\hat{u}(p_k) \in \arg\min_{u \in U} \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, u)$, solves the output feedback problem, and the associated storage function is $W(p_k) = (p_k, U)$.

Proof:

$$
\begin{aligned}
(p_k, U) & \geq \inf_{u \in U} \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, u) \\
& = \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, \hat{u}(p_k)) \\
& \geq \sup_{y \in \mathbf{R}^t} (H(p_k, \hat{u}(p_k), y), U)
\end{aligned}
$$

Furthermore, $(p_k, U) \geq (p_k, 0)$, and $(-U, U) = 0$. Hence, (p_k, U) is a storage function, and $\hat{u}$ is a (non-optimal) solution to the output feedback problem, with the information state trajectory initialized by $p_0 = -U$.

$\square$

Remark 5.5 We could have considered any $\hat{u}_k$ such that

$$\sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, u(x)) \geq \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, \hat{u}_k).$$

where U is a storage function for the state feedback game, and $u(\cdot)$ is the corresponding state feedback policy.

Corollary 5.6 (Certainty Equivalence) Given $U \equiv V$, the upper value function of the state feedback game, and the optimal state feedback policy u_F. If for all $p_k \in \mathcal{E}$

$$(p_k, V) = \inf_{u \in U} \sup_{x \in \mathbf{R}^n} J_V^{p_k}(x, u) \tag{5.5}$$

then $u(p_k) = u_F(\hat{x})$, where $\hat{x} \in \arg \max_{x \in \mathbf{R}^n} \{p_k(x) + V(x)\}$, is an optimal control policy for the output feedback problem.

Proof:

Clearly (5.5) implies that

$$\sup_{x \in \mathbf{R}^n} J_V^{p_k}(x, u_F(x)) = \sup_{x \in \mathbf{R}^n} \inf_{u \in U} J_V^{p_k}(x, u) = \inf_{u \in U} \sup_{x \in \mathbf{R}^n} J_V^{p_k}(x, u)$$

Hence, a saddle point exists, and so for any $\hat{x} \in \arg\max_{x \in \mathbf{R}^n} (p_k(x) + V(x))$, and

$\hat{u} = u_F(\hat{x})$,

$$(p_k, V) = J_V^{p_k}(\hat{x}, \hat{u}) = \sup_{x \in \mathbf{R}^n} J_V^{p_k}(x, \hat{u}) \geq \sup_{y \in \mathbf{R}^t} (H(p_k, \hat{u}, y), V)$$

Hence, $W(p_k) = (p_k, V)$ is a storage function, and $W(\delta_{\{x\}}) = V(x)$, the optimal cost of the state feedback game. Hence, the policy is optimal for the output feedback game.

$\square$

Remark 5.7 It is sufficient that the conditions in theorem 5.4 and corollary 5.6 hold only for all p_k, $k = 0, 1, \ldots$. If this is the case, then U need not be a storage function for the state feedback problem. It is only when we need the conditions to hold for $p_k \in \{\delta_{\{x\}} \mid x \in dom\ U\}$ that U is forced to be a storage function.

In general, conditions for the optimal policy maybe difficult to establish. However, there may exist non-optimal state feedback policies such that their storage functions satisfy the conditions of theorem 5.4. In that case, using such non-optimal policies will guarantee that the system is dissipative.

We now return to equation (5.3) which characterizes certainty equivalence in terms of the upper value function of the output feedback game.

Lemma 5.8 Let $\bar{u} \in \mathcal{I}$ be , with W its storage function. Then

$$W(p_k) \geq \inf_{u \in U} \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, u), \quad k = 0, 1, \ldots$$

where, $U(x) \overset{\triangle}{=} W(\delta_{\{x\}})$.

Proof:

$$
\begin{aligned}
W(p_k) \ &\geq\ \sup_{x \in \mathbf{R}^n} \{ p_k(x) + \sup_{r,s \in \Gamma_{[k,\infty)}^{\bar{u}}(x)} \sum_{i=k}^{\infty} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 - \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \} \\[6pt]
&=\ \sup_{x \in \mathbf{R}^n} \{ p_k(x) + \sup_{r,s \in \mathcal{F}(x, \bar{u}(p_k))} (\mid l(r, \bar{u}(p_k)) - l(s, \bar{u}(p_k)) \mid^2 - \gamma^2 \mid r - s \mid^2 + \\
&\qquad\qquad \sup_{r',s' \in \Gamma_{[k+1,\infty)}^{\bar{u}}(r)} \sum_{i=k+1}^{\infty} \mid l(r'_{i+1}, u_i) - l(s'_{i+1}, u_i) \mid^2 - \gamma^2 \mid r'_{i+1} - s'_{i+1} \mid^2) \} \\[6pt]
&=\ \sup_{x \in \mathbf{R}^n} \{ p_k(x) + \sup_{r,s \in \mathcal{F}(x, \bar{u}(p_k))} (\mid l(r, \bar{u}(p_k)) - l(s, \bar{u}(p_k)) \mid^2 - \gamma^2 \mid r - s \mid^2 + \\
&\qquad \sup_{\xi \in \mathbf{R}^n} \{ \delta_{\{r\}}(\xi) + \sup_{r',s' \in \Gamma_{[k+1,\infty)}^{\bar{u}}(\xi)} \sum_{i=k+1}^{\infty} \mid l(r'_{i+1}, u_i) - l(s'_{i+1}, u_i) \mid^2 - \\
&\qquad\qquad \gamma^2 \mid r'_{i+1} - s'_{i+1} \mid^2 \}) \} \\[6pt]
&\geq\ \inf_{u \in U} \sup_{x \in \mathbf{R}^n} \{ p_k(x) + \sup_{r,s \in \mathcal{F}(x,u)} (\mid l(r, u) - l(s, u) \mid^2 - \gamma^2 \mid r - s \mid^2 + U(r)) \} \\[6pt]
&=\ \inf_{u \in U} \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, u)
\end{aligned}
$$

$\square$

Theorem 5.9 (Unicity) Let M be the upper value function of the output feedback game. If there exists a function $U : \mathbf{R}^n \to \mathbf{R}$, such that $M(p_k) = (p_k, U)$, for all $p_k \in \mathcal{E}$, then $U \equiv V$, the upper value function of the state feedback game.

Proof:

We have

$$(p_k, U) = M(p_k) \geq \inf_{u \in \mathbf{U}} \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, u).$$

Let $\hat{u}(p_k) \in \arg\min_{u \in \mathbf{U}} \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, u)$. Then

$$
\begin{aligned}
(p_k, U) &\geq \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, \hat{u}(p_k)) \\
&\geq \sup_{y \in \mathbf{R}^t} (H(p_k, \hat{u}(p_k), y), U) \\
&= \sup_{y \in \mathbf{R}^t} M(H(p_k, \hat{u}(p_k), y))
\end{aligned}
$$

Hence, $\hat{u}$ is an optimal policy since $(p_0, U) = M(p_0), \forall\, p_o \in \mathcal{E}$. Thus,

$$M(p_k) = \sup_{y \in \mathbf{R}^t} M(H(p_k, \hat{u}(p_k), y))$$

which implies that

$$(p_k, U) = \inf_{u \in \mathbf{U}} \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, u).$$

Setting, $p_k = \delta_{\{x\}}$, we obtain

$$U(x) = \inf_{u \in \mathbf{U}} \sup_{r, s \in \mathcal{F}(x, u)} \{| \, l(r, u) - l(s, u) \,|^2 - \gamma^2 \,| \, r - s \,|^2 + U(r)\}.$$

Hence, $U \equiv V$.

$\square$

Corollary 5.10 If there exists a p_k such that $M(p_k) \neq (p_k, V)$, then there exists no function $Y : \mathbf{R}^n \to \mathbf{R}$, such that $M(p) = (p, Y)$ for all $p \in \mathcal{E}$.

□

Corollary 5.11 Let W be a storage function for an (non-optimal) information state feedback policy $\bar{u} \in \mathcal{I}$, and let $W(p_k) = (p_k, U)$, $k \geq 0$. Then $\hat{u}(p_k) \in \arg\min_{u \in U} \sup_{x \in \mathbf{R}^n} J_U^{p_k}(x, u)$ solves the output feedback problem with the storage function $W(p)$. Furthermore, is we insist that $W(\delta_{\{x\}}) = (\delta_{\{x\}}, U)$, $\forall \ x \in \ dom \ U$, then U is a storage function for a (non-optimal) state feedback policy. Also, if $W \equiv M$, the upper value function of the output feedback game, then the controller is a certainty equivalence controller.

□

Remark 5.12 It is clear from the proof of theorem 5.9 , that if (5.3) holds, then so does (5.5). However, (5.5) is a more tractable condition, since it does not involve the upper value function M, which is what we are trying to avoid having to compute in the first place.

In particular, we also have the following result, which states a sufficient condition for certainty equivalence to hold, in terms of solvability of a functional equation.

Corollary 5.13 (Certainty Equivalence) Let M be the upper value function of the output feedback game. Then certainty equivalence holds if the equation

$$M(p) = (p, U), \ \text{for all } p \in \mathcal{E}$$

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has a solution $U : \boldsymbol{R}^n \to \boldsymbol{R}$.

$\Box$

Chapter 6

Applications

We first discuss some consequences of the methodology developed, before treating the general design problem. Let us start by recalling the definition of controlled-invariant sets [50].

Definition: Let $\mathcal{F} : \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^n$ be a set-valued map. Let $\mathcal{U} : \boldsymbol{R}^n \to \boldsymbol{U} \subset \boldsymbol{R}^m$ be a set-valued map. The set $\mathcal{K} \subset \boldsymbol{R}^n$ is called **controlled-invariant** under $(\mathcal{F}, \mathcal{U})$ if for every $x \in \mathcal{K}$, there is a $u \in \mathcal{U}(x)$ such that $\mathcal{F}(x, u) \subset \mathcal{K}$.

This definition can be interpreted as saying that there exists a control policy u^* such that under u^*, the state trajectory of the system $(\mathcal{F}, \mathcal{U})$ is **viable** in $\mathcal{K}$, i.e. $x_k \in \mathcal{K}$ for all $k \geq 0$ [2].

The notion of controlled-invariance plays a fundamental role in several methodologies of controller design. The most prominent amongst these being l^1-optimal control for linear systems, where one is concerned with attenuating the influence of

persistent bounded additive noise [55]. The solution to this problem was first obtained by [15]. It has been noted, that for such problems, the linear state feedback compensator could be dynamic [18]. Recently, researchers have tried to obtain static nonlinear compensators [11],[50]. The key ideas here are the construction of an appropriate controlled-invariant set. However, it is not clear how the methodology will extend to the measurement feedback case. In passing, we note that these problems are related to the targeting problems considered by [35]. In particular, given a set $\mathcal{M}$, find a state feedback policy u, such that if $x_0 \in \mathcal{M}$, then $x_k \in \mathcal{M}$ for all $k \geq 0$, else if $x_0 \notin \mathcal{M}$, then $x_k \to \mathcal{M}$ as $k \to \infty$. In the l^1 case, the requirements are weaker, since we require that if $x_0 = 0$, then $x_k \in \mathcal{M}$ for all k (assuming that $0 \in \mathcal{M}$) and if $x_0 \neq 0$ then $x_k \to \mathcal{M}$, as $k \to \infty$. Here, the set $\mathcal{M}$ is defined in a suitable fashion. The set $\mathcal{M}$ thus includes the positive limit set of trajectories generated by the controlled system, with 0 in the positive limit set.

Now let $\mathcal{K}^u$ be the positive limit set of any trajectory $\{x_k\}$ generated by the set-valued dynamical system

$$
\begin{aligned}
x_{k+1} &\in \mathcal{F}(x_k, u_k), \ x_0 \in X_0 \\
y_{k+1} &\in \mathcal{G}(x_k, u_k)
\end{aligned}
$$

via a (state or output feedback) control policy u. Let $\gamma > 0$ be given, and let $l : \mathbf{R}^n \times \mathbf{U} \to \mathbf{R}^q$ be a C^1 function such that

$$
\mathcal{L}^\gamma \triangleq \left\{ s \in \mathbf{R}^n \middle| \exists u \in \mathbf{U} \text{ s.t. } \left| \frac{\partial}{\partial x} l(s, u) \right| \leq \gamma \right\}
$$

is compact and contains the origin. Then we have

Theorem 6.1 Suppose there exists a solution to the robust control problem, with an admissible control policy u^*. Then for any trajectory $x \in \Gamma^{u^*}(X_0)$,

$$\mathcal{K}^{u^*} \subset \mathcal{L}^\gamma$$

Furthermore, if $x_0 = 0$, then $x_k \in \mathcal{L}^\gamma$ for all $k \geq 0$.

Proof:

The proof follows from corollary 2.11 for the state feedback case, and corollary 4.19 for the output feedback case.

□

Thus, we see that the choice of the regulated output yields a bounding set around the positive limit set of the controlled inclusion. Hence, given a regulated output $h(x)$, and a desired γ, we construct $l(x) = \int_0^x h(s)ds$, and solve the robust control problem. If the problem has a solution then we obtain $|h(x_k)| \leq \gamma$, for all k, given $x_0 = 0$. Else if $x_0 \neq 0$, then $\limsup_{k \to \infty} |h(x_k)| \leq \gamma$. This yields a method of constructing output feedback ultimate boundedness controllers for nonlinear systems subject to persistent bounded excitation. Before considering this, we present an example based on targeting of inclusions. The problem of synthesizing state feedback policies (in the linear case) was considered by [35].

Example: Targeting Inclusions. Given an inclusion

$$
\begin{aligned}
x_{k+1} &\in \mathcal{F}(x_k, u_k) \\
y_{k+1} &\in \mathcal{G}(x_k, u_k)
\end{aligned}
$$

$x_0 \in X_0$. Consider the problem of synthesizing an output feedback policy such that

$$
x_k \to \mathcal{E}(0, Q)
$$

where $\mathcal{E}(0, Q)$ represents an ellipsoid given by

$$
\mathcal{E}(0, Q) \triangleq \left\{ x \in \boldsymbol{R}^n \mid x^T Q x \leq 1 \right\}
$$

where x^T denotes the transpose of the vector x. Here, $Q > 0$ is a positive definite symmetric matrix. Now define

$$
z_k = \frac{1}{2} x_k^T Q^{\frac{1}{2}} x_k
$$

where $Q^{\frac{1}{2}} > 0$ is the positive definite square root of Q. Let $\gamma = 1$, and solve the output feedback robust control problem (with $|\cdot|$ representing the Euclidean norm). Suppose, a solution exists with u^* the corresponding policy. Then we have

$$
\begin{aligned}
\limsup_{k \to \infty} |Q^{\frac{1}{2}} x_k| &\leq \gamma \\
\limsup_{k \to \infty} x_k^T Q x_k &\leq \gamma^2 = 1
\end{aligned}
$$

which implies that

and hence, $x_k \to \mathcal{E}(0, Q)$ as $k \to \infty$.

6.1   Ultimate Boundedness Control

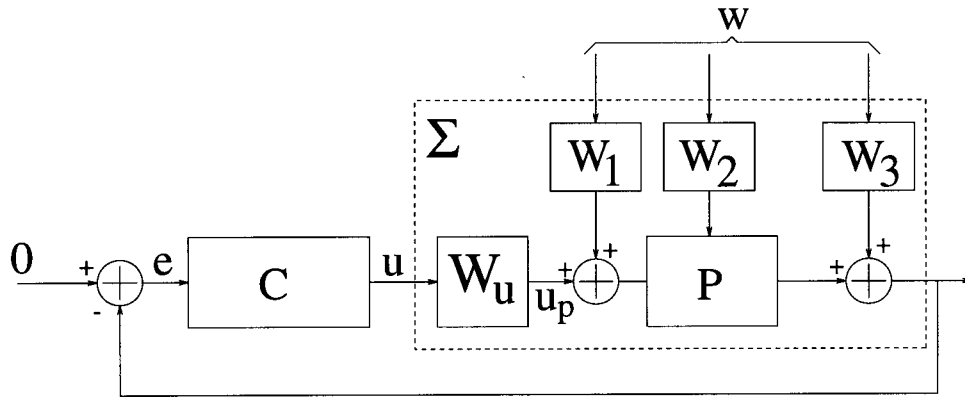Consider the system in figure 6.1. Here C represents the controller section to

Figure 6.1: A typical feedback configuration.

be synthesized, P the (nonlinear) plant and w are the persistent bounded noise signals. The aim is to maintain the error, e as close to 0 as possible. u is an intermediate control signal with $u \in U$, where U could represent the actuator limits. Here, Σ represents the augmented plant. W_u represents the (stable linear) filter incorporated to shape the controller output. Hence, u_p is the control signal seen by the plant, and the implemented controller will consist of both C and W_u. It is assumed that there is no direct feed-through of the signal u to e. One way of ensuring this is to force W_u to be strictly proper. This is a mild requirement, since in most practical situations high frequencies in the control action (chatter) are undesirable. W_1, W_2, W_3 represent (stable) filters which weigh the noise signal w. Note that W_1 and W_3 are shown for purposes of clarity only, since they could be incorporated in W_2. We assume that there is no direct feed-through of the noise w to e. This is mainly a technical assumption, since we have been working with single-valued regulated outputs. This assumption can be dropped, if we consider,

for example the Chebyshev center of the (now set-valued) regulated output $\mathcal{E}_k$. The Chebyshev center e_k^* of the set $\mathcal{E}_k$ is defined through the relation

$$\min_z \max_{\xi \in \mathcal{E}_k} |z - \xi| = \max_{\xi \in \mathcal{Z}_k} |e_k^* - \xi|$$

and is obviously the center of the smallest ball that includes the set $\mathcal{E}_k$. Furthermore, if $\mathcal{E}_k$ is convex, then $e_k^* \in \mathcal{E}_k$. There is no loss of generality in doing so, since if the noise is bounded, then the error (e_k) is also bounded around the Chebyshev center (e_k^*), and one can explicitly obtain this bound. The system Σ is converted to its state space form, with the states represented by x. We also assume that $x = 0$, $u = 0$ is an equilibrium point for the system Σ, with $w = 0$. The system can now be re-written as an inclusion.

Now define $z_k = s(e_k) = l(x_k)$, where l is such that $l(0) = 0$, and $\mathcal{L}^\gamma$ is compact and contains the origin (this condition is in general stronger than necessary, and one may be able to invoke invariance arguments to relax it, in particular, one may be able to show that if the states which appear in l are ultimately bounded, then all the other states are as well). Let $\mathcal{K}^u$ represent the positive limit set of any state trajectory. The synthesis problem is, given $\gamma > 0$, $w_k \in \mathcal{W}$, find a controller C^* (corresponding to a policy u^*), such that $\mathcal{K}^{u^*} \subset \mathcal{L}^\gamma$, with $0 \in \mathcal{K}^{u^*}$. Clearly, by appropriately defining l, one should be able to induce various properties on the closed-loop system. This has still to be looked into. For example, if $e_k = h(x_k)$, then by defining $l(x) = \int_0^x h(\xi)d\xi$, we can bound the magnitude of the error e_k, in the sense, if $x_0 = 0$, then $|e_k| \leq \gamma$ for all k, else if $x_0 \neq 0$, then $\limsup_{k\to\infty} |e_k| \leq \gamma$.

We now consider some examples. Due to the computational complexity associated with dynamic programming, the systems and control objectives considered have been kept as simple as possible. In each case, we carry out γ iterations similar to those in linear H_∞ control [14],[24], and implement the controller which corresponds to the smallest value of γ obtained.

6.2   Unstable Nonlinear Plant

This example is taken from [22], and concerns the stabilization and disturbance rejection for a single state unstable nonlinear plant. The system is given by

$$\begin{aligned} \dot{x} &= u - x - \beta \frac{1.0}{1.0 + K_1/x + x/K_2} + w \\ y &= x \end{aligned}$$

Here, y is the reactant concentration, t is the dimensionless time, u is the feed reactant concentration, K_1, K_2 are kinetic constants, β is a constant, and w represents the input disturbance, and is assumed to belong to $[-0.05, 0.05]$. The values of the parameters are chosen as in [22] to be $\beta = 2$, $K_1 = 0.01$, and $K_2 = 0.1$. Furthermore, we assume that $U = [0, 5]$. The model for single enzyme-catalyzed reaction with substrate-inhibited kinetics [41] as well as the model for the ethylene hydrogenation in an isothermal CSTR [39] are of the above form. The model has two stable and one unstable steady state points. The unstable steady state is at $\bar{x} = 0.125$, and $\bar{u} = 0.9834$.

We discretize the system via the Euler transform. Recently, [40] showed that under

very mild assumptions, trajectories of such a discretization converge to those of the original inclusion, as the time step tends to zero. The sampling interval chosen is 0.1. We set

$$z_{k+1} = 1000(x_{k+1} - \bar{x})^2$$

and carry out dynamic programming. We choose $\Delta x = 0.001$, and $\Delta u = 0.05$. The dynamic programming equation converges after 10 iterations, and optimal value of γ lies between 11.518 and 11.508. The controller implemented corresponds to $\gamma = 11.518$. We plot the control values versus the state in figure 6.2. Observe that the controller can be very well approximated by a linear one, and we obtain the following linear control policy

$$u_k = -12.29(x_k - 0.125) + 0.9834$$

This policy is then used instead of carrying out a table look-up.

For the simulations, we employ a zero-order hold with a sampling time of 0.1. The initial state is chosen as $x_0 = 0.14$. We induce a pulse disturbance of magnitude 0.05 from $t = 10$ to $t = 30$. The pulse train has a period of 3, and a pulse width of 1.5. This is followed by a 0.2Hz sinusoid with an amplitude of 0.05. The state trajectory is shown in figure 6.3.

We observe that the controller stabilizes the system in the absence of noise, and when the noise is introduced, the state gets perturbed by an amount equal to
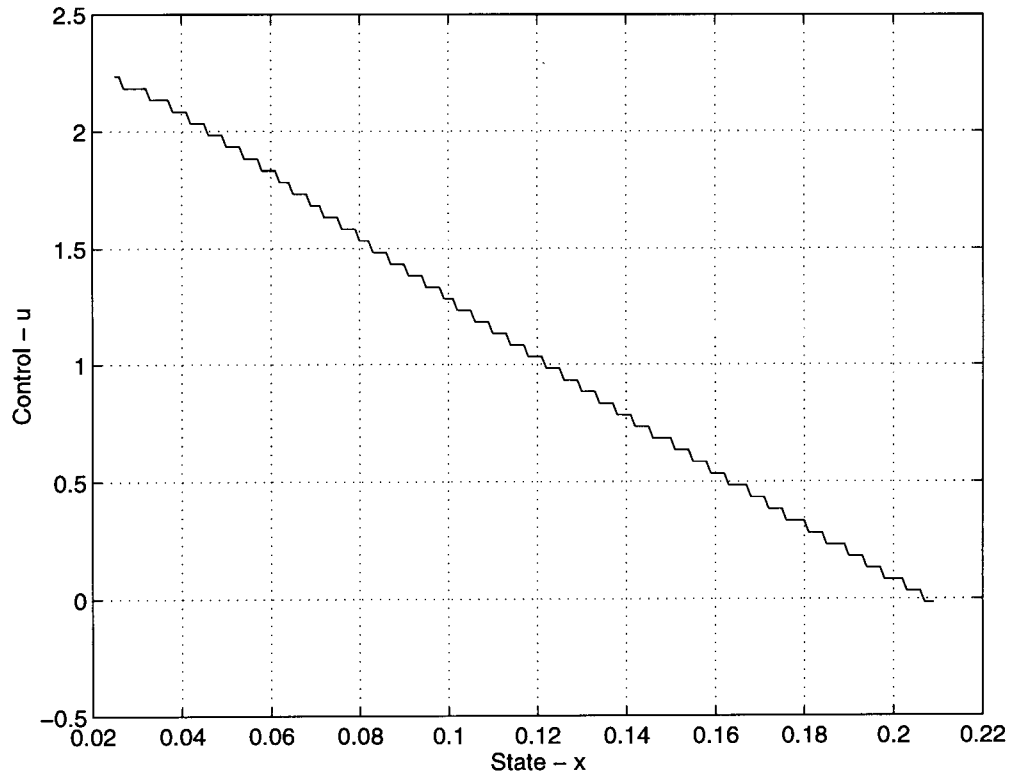
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Figure 6.2: Control values vs. states.

0.0057. Note that this is consistent, since, we have

$$\frac{\partial}{\partial x}|l(x_k)| \leq \gamma$$

which implies that

$$
\begin{aligned}
2000|x_k - 0.125| &\leq 11.518 \\
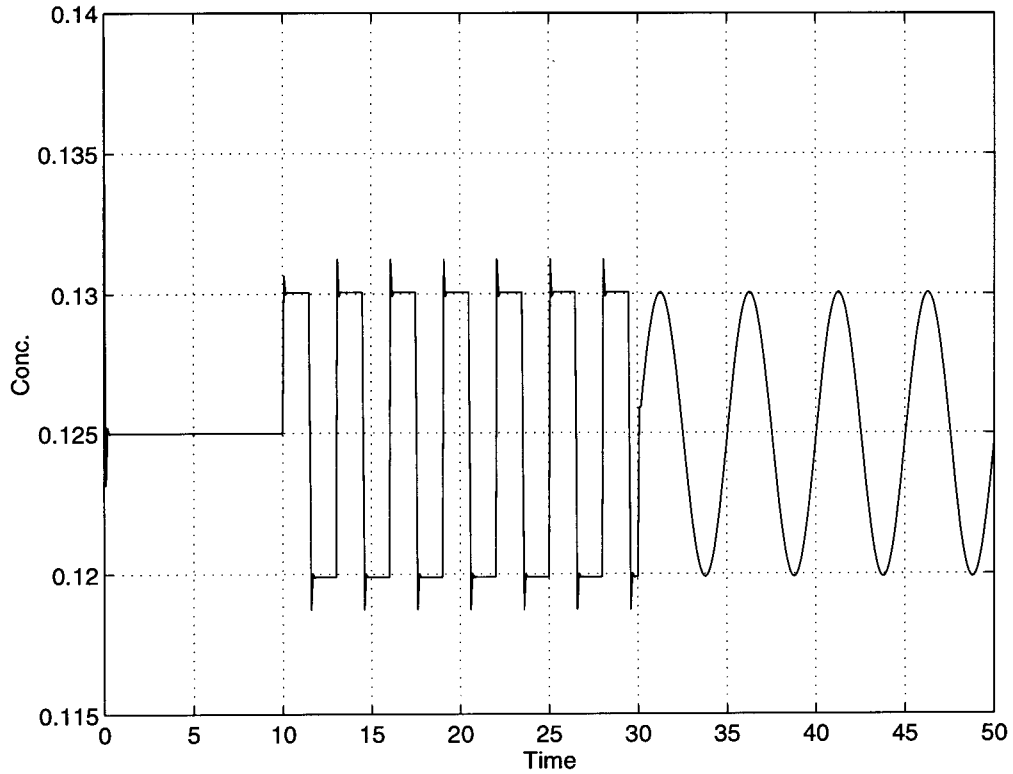|x_k - 0.125| &\leq 0.0058
\end{aligned}
$$

Figure 6.3: State trajectory with system subject to a pulse/sinusoidal disturbance.

6.3 Discontinuous System with Parametric Uncertainty

We consider a discontinuous system given by

$$\theta[k+2] + (c_1 - 2)\theta[k+1] + (1 - c_1)\theta[k] + c_2 f \operatorname{sgn}(\theta[k+1] - \theta[k]) = c_3 u[k] \quad (6.1)$$

with $\theta[1] = \theta[0] = 0$. Here θ is the position in radians, and the nominal values of the parameters are: $c_1^0 = 0.162$, $c_2^0 = 0.1$, $c_3^0 = 6.43 \times 10^{-4}$, and $f^0 = 7.2 \times 10^{-3}$. Furthermore, $-10 \leq u[k] \leq 10$, and we assume that the actual values of the parameters are within $\pm 10\%$ of the nominal. The above system corresponds to

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a sampling time of 0.01s. The controller is supposed to reject output additive disturbances (i.e. $y[k] = \theta[k] - r[k]$) of magnitude ± 0.25 radians, with a cutoff at 0.5 Hz. We first carry out the H_∞ design.

6.3.1 H_∞ Design

We, first smoothen and approximately linearize the system (6.1) via dithering and nonlinear $(\tanh(\theta[k] - \theta[k-1]))$ feedback, to obtain $G^0(z) = \frac{c_3^0}{(z-1)(z-1+c_1^0)}$ as the nominal plant. Due to the pole at $z = 1$, it is not possible to carry out an H_∞ design on this nominal plant. So, we apply a unity feedback, and shift the pole away from 1. Thus, we work with $P^0(z) = \frac{G^0(z)}{1+G^0(z)}$. Representing the parameter variations as a multiplicative perturbation ($\Delta_m(z)$), we obtain $\|\Delta_m\|_\infty < 0.25$. Transforming the discrete time plant into continuous time, using the inverse Tustin transform (which preserves the H_∞ norm), the problem can be stated as: Given $P^0(s)$, $W_1(s)$, $W_2(s)$, and $W_3(s)$, maximize ρ, while ensuring that $\left\| \begin{array}{c} \rho W_1 T_{w,e} \\ W_2 T_{w,u} \\ W_3 T_{w,y} \end{array} \right\|_\infty < 1$, where $T_{w,e}$ is the sensitivity function, $T_{w,y}$ is the complementary sensitivity function, and $T_{w,u}$ is the transfer function from the disturbance to the controller output. Here, $W_3(s) = 0.25$, $W_2(s) = 10^{-2}\frac{s+10}{s+350}$, and $W_1(s) = \frac{\pi}{s+\pi}$. The solution is obtained via [14]. We get $\rho = 8.625$, and the corresponding controller $C_P(z)$ is

$$\frac{81.83z^4 - 137.86z^3 - 25.37z^2 + 137.86z - 56.45}{z^4 - 2.41z^3 + 1.45z^2 + 0.36z - 0.39}$$

which can now be transformed to correspond to the original plant $G(z)$ by $C_G(z) = C_P(z) + 1$.

6.3.2   Set-valued Design

As in the H_∞ case, we pick $W_1(s) = \frac{\pi}{s+\pi}$. We discretize it using the Euler transform, which is known to preserve the H_∞ and l^1 norms [51]. Weighing the disturbance with W_1, we obtain $x_3[k+1] = 0.9686x_3[k] + 0.0314r$ with $r \in [-0.25, 0.25]$. Furthermore transforming (6.1) into its state space form, and allowing for parameter variations, we obtain

$$
\begin{aligned}
x_1[k+1] &\in \mathcal{A}x_1[k] + \mathcal{E}u[k] + \mathcal{C} \\[4pt]
x_2[k+1] &= x_2[k] + 0.01x_1[k] \\[4pt]
x_3[k+1] &\in 0.9686x_3[k] + 0.0314[-0.25, 0.25] \\[4pt]
y[k+1] &= x_2[k] - x_3[k] \\[4pt]
z[k+1] &= 10(x_2[k+1] - x_3[k+1])^2
\end{aligned}
$$

with $x_1[0] = x_2[0] = x_3[0] = 0$. Here, $\mathcal{A} = [0.8212, 0.8542]$, $\mathcal{E} = [0.05787, 0.07073]$, and $\mathcal{C} = [-0.08712, 0.08712]$. Note that $\theta[k] = x_2[k]$, and the error $e[k] = y[k]$. To avoid the infinite time dynamic programming, in practice we implement a certainty equivalence controller (equation (5.2)).

Remark 6.2 The system above clearly violates assumptions **A4**, **A5** of section 1.2.2, page 13. However, we can indirectly establish ultimate boundedness. Note that **A4** and **A5** are employed to establish properties of the state trajectory provided that the robust control problem has a solution. However, violation of

these assumptions does not violate the solvability of the problem. Let $l(x_2, x_3) = 10(x_2 - x_3)^2$. Suppose, that the problem has a solution. Then we obtain

$$\limsup_{k \to \infty} \left| \frac{\partial}{\partial x_3} l(x_2[k+1], x_3[k+1]) \right| \leq \gamma$$

Note that the derivative is taken with respect to x_3 alone, since $x_2[k+1]$ evolves through single-valued dynamics. This implies that for any arbitrary $\epsilon > 0$, $\exists K \geq 0$, s.t.

$$20|x_2[k] - x_3[k]| < \gamma + \epsilon \text{ for all } k \geq K$$

Now, x_3 is always bounded. Hence, x_2 is ultimately bounded, and for this to be true, from the dynamics of x_2 we obtain that x_1 is ultimately bounded as well.

We solve the state feedback problem with $x_1 \in [-2.5, 2.5]$, $\Delta x_1 = 0.1$, $x_2 \in [-0.3, 0.3]$, $\Delta x_2 = 0.01$, $x_3 \in [-0.35, 0.35]$, $\Delta x_3 = 0.05$, and we use $\Delta u = 0.5$. For this discretization, the optimal value of γ lies between 0.12 and 0.14. We pick $\gamma = 0.14$. The dynamic programming equation converges after 14 iterations. For the certainty equivalence controller, since the system is initially at rest, we pick $p_0 = \delta_{\{0\}}$ as the initial value for the information state recursion. While carrying out the simulations, the plant parameters c_1, c_2, c_3, and f are sinusoidally varied between their nominal values at frequencies of 0.1, 0.2, 0.25, and 0.4 Hz respectively. Figure 6.4 shows the error incurred due to a sinusoidal disturbance of frequency 0.5Hz, and having a magnitude 0.05 radians. The dashed line represent the error response of the H_∞ controller, the solid line represents that due to the set-valued design,
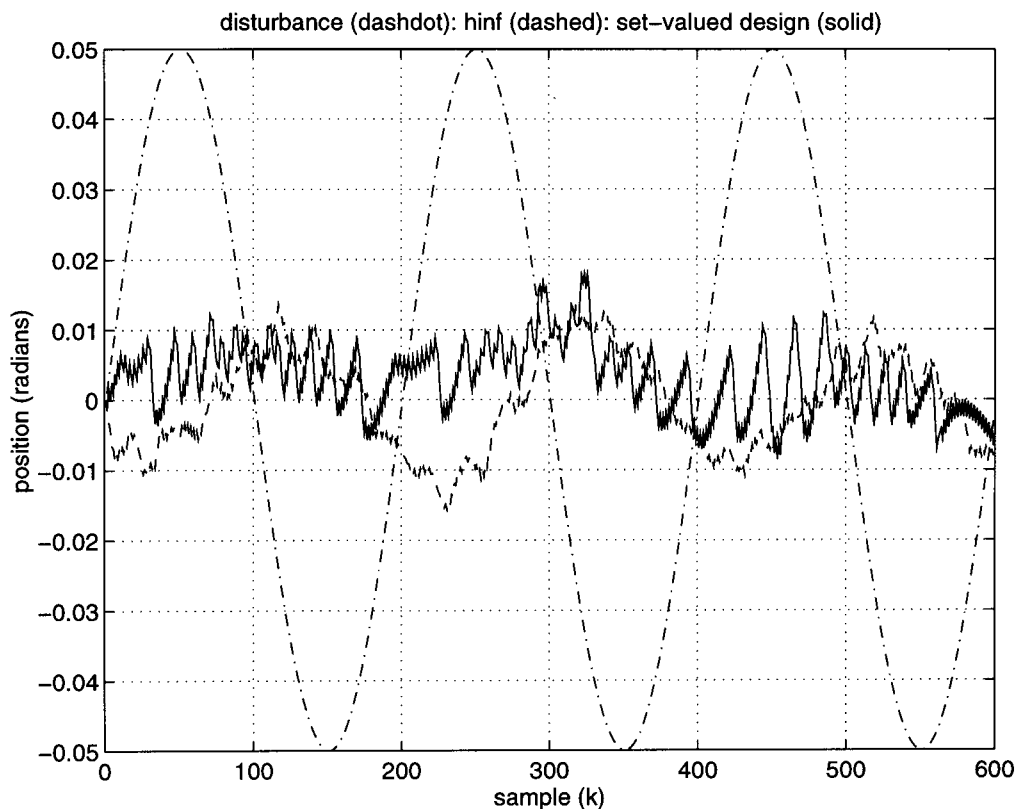
Figure 6.4: Error incurred under sinusoidal disturbance of 0.05 radians.

and the dash-dot line shows the disturbance. Clearly, the performance for both the designs is comparable.

However, if we increase the amplitude of the disturbance to 0.15 radians, the H_∞ controller becomes unstable due to saturation of the input. This is shown in figure 6.5. Moreover, using a more stringent $W_2(s)$ during the H_∞ design process, yields a controller that is ineffective in rejecting disturbances (due to extremely small values of ρ). Figure 6.6 shows the response of the set-valued design to a sinusoidal disturbance having an amplitude of 0.25 radians. Finally, figure 6.7 shows the response of both the H_∞ and the set-valued design to a step disturbance.
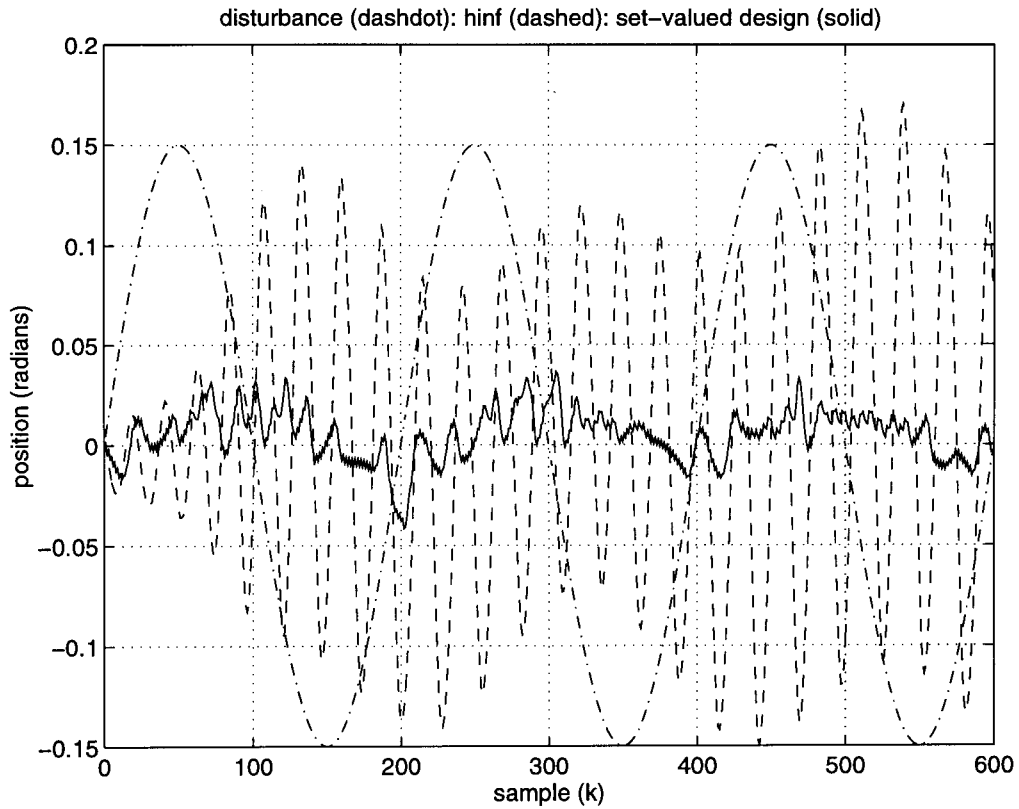
Figure 6.5: Error incurred under sinusoidal disturbance of 0.15 radians.

6.4 Run by Run Control

This section develops an application of the set-valued design to the problem of run by run (RbR) control. We consider the problems of end-pointing and LPCVD rate control.

6.4.1 End-Pointing

The scenario is as follows. Lots, consisting of 24 wafers are processed through a single wafer reactor. Here, we assume that the process under consideration is
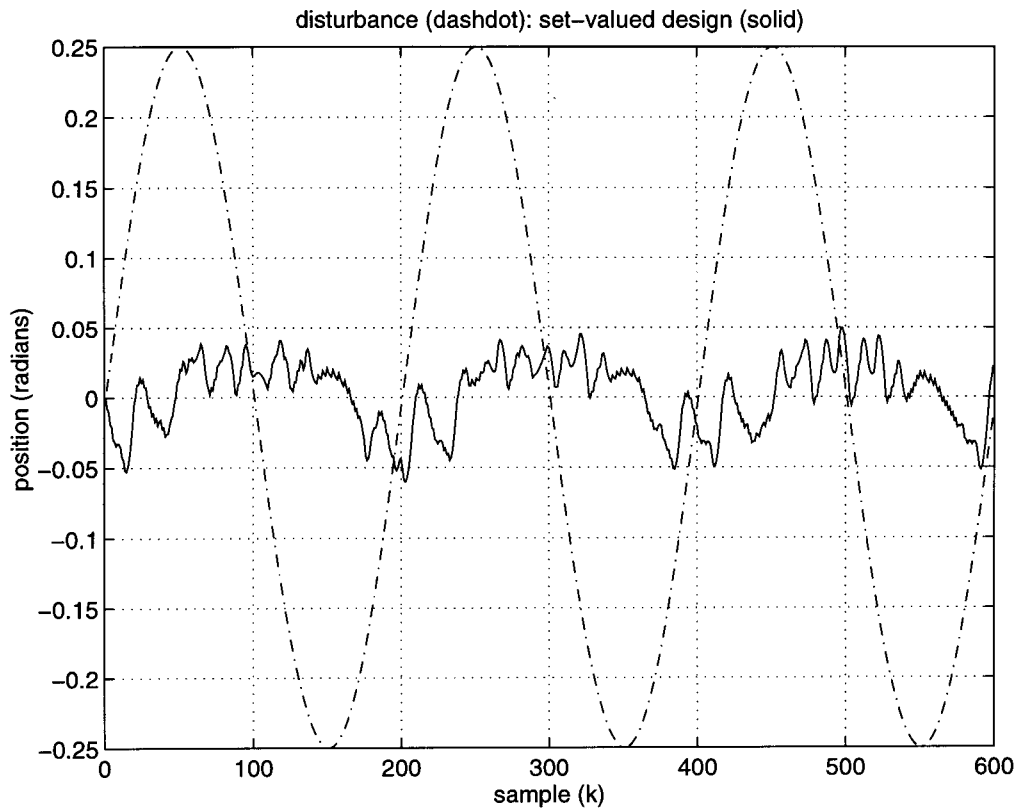
Figure 6.6: Error incurred under sinusoidal disturbance of 0.25 radians.

deposition. Measurements are carried out on the last wafer of each lot. The aim is

to determine the processing time, so as to achieve a given target thickness. Here,

it is assumed that the processing time per wafer is constant for all wafers across a

lot. We assume that the process is subject to three kinds of noise: (i) variation in

the average deposition rate at the test wafer from lot to lot, (ii) variation in the

instantaneous rate from test wafer to test wafer, due to changes in both the wafer

surface, and deposition conditions, and (iii) measurement noise, either due to finite

resolution of the measurement apparatus, or due to experimental error. Here, the

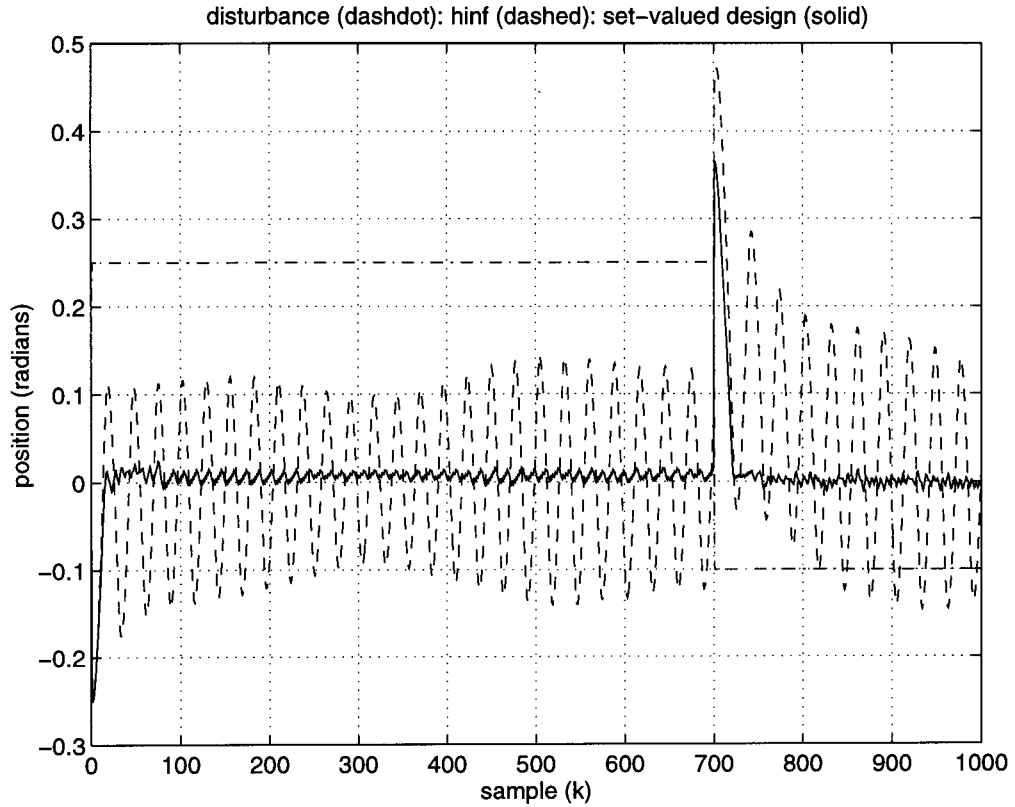basic process can be modeled as (assuming for now that we have no measurement

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Figure 6.7: Error incurred under step disturbances.

delay)

$$
\begin{aligned}
\hat{r}_{k+1} &= \hat{r}_k + v_k \\
\hat{\xi}_{k+1} &= (\hat{r}_k + w_k)\hat{t}_k \\
\hat{y}_{k+1} &= (\hat{r}_k + w_k)\hat{t}_k + m_k \\
\hat{R}_{k+1} &= \hat{T}_k
\end{aligned}
$$

where $\hat{r}_k$ is the average deposition rate for the test wafer for lot k, $\hat{\xi}_{k+1}$ is the

actual deposition thickness on the test wafer for lot k for a deposition time $\hat{t}_k$, and

$\hat{y}_{k+1}$ is the measured thickness. Also, $\hat{T}_k$ is the target thickness for lot k. Here,

v_k is the noise used to model the variation in the average deposition rate, w_k is

the noise used to model the rate variation per wafer, and m_k is the noise modeling

the measurement error. It is assumed that the controller knows $\hat{T}_k$ before the

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processing time for lot k is computed.

We now give some (fictitious) numbers to enable simulations. It is assumed that the nominal (or average) thickness required is $\bar{T} = \bar{\xi} = 1500\mathring{A}$. Furthermore, the nominal deposition rate of the apparatus is assumed to be $\bar{r} = 300\mathring{A}/min$, and hence the nominal deposition time is $\bar{t} = 5$ min. Hence, we can now express the system in terms of deviations from the nominal, i.e. as

$$
\begin{aligned}
r_{k+1} &= r_k + v_k \\
\xi_{k+1} &= r_k t_k + \bar{r} t_k + \bar{t} r_k + (\bar{t} + t_k) w_k \\
R_{k+1} &= T_k \\
y_{k+1} &= r_k t_k + \bar{r} t_k + \bar{t} r_k + (\bar{t} + t_k) w_k + m_k
\end{aligned}
$$

here, $r_k = \hat{r}_k - \bar{r}$, $t_k = \hat{t}_k - \bar{t}$, etc. We can now interpret $[r_k \; \xi_k \; T_k]$ as the **states**, and $[R_k \; y_k]$ as the **measurements**. Also, note that before the deposition time for lot k is computed, we also know T_k. Hence, what is not known on the onset of run k are ξ_k, i.e. the actual deposition for lot $k-1$, and r_k, i.e. the deposition rate for lot k. The deposition time is fixed to belong to the set $t_k \in [-4.8, 20]$, where this restriction could be obtained via scheduling constraints. Furthermore, we assume that $r_k \in \mathcal{R}$, where $\mathcal{R}$ denotes the operating range of the equipment. Here, it is assumed that $\mathcal{R} = [-125, 300]$. The operating range denotes the range of parameter values over which the equipment is supposed to be operating. If the rate exceeds this range, it is assumed that a maintainence call will be placed. The controller can raise a maintainence alarm by checking the information state, since during normal operation we should have $p_k(r, \xi, T) = -\infty$ for all $r \notin \mathcal{R}$. This follows from subsection 3.2.2, and the fact that $\mathcal{R}^c$ is an **infeasible** set. Hence, if

$p_k(r, \xi, T) \neq -\infty$ for some $r \notin \mathcal{R}$, a maintainence alarm is issued.

Now let v_k, and w_k be zero mean, Gaussian, with standard deviations $\sigma_v = 4$, and $\sigma_w = 1$ respectively. Also, the measurement error is modeled as being uniformly distributed between $-10\mathring{A}$ and $10\mathring{A}$. Such a distribution models the resolution limit of the measurement apparatus. Considering the $\pm 3\sigma_v$, and $\pm 3\sigma_w$ limits on v_k, and w_k, we obtain the following set-valued dynamical system

$$
\begin{array}{rcl}
r_{k+1} & \in & r_k + [-12, 12] \\
\xi_{k+1} & \in & r_k t_k + \bar{r}t_k + \bar{t}r_k + (\bar{t} + t_k)[-3, 3] \\
R_{k+1} & = & T_k \\
y_{k+1} & \in & r_k t_k + \bar{r}t_k + \bar{t}r_k + (\bar{t} + t_k)[-3, 3] + [-10, 10]
\end{array}
$$

Furthermore, we define the regulated output z_k as

$$
z_{k+1} = 0.1(\xi_{k+1} - R_{k+1})^2
$$

where we assume that there is no cost associated with the control action. Note that since we have no information on the power spectrum of the noise, we try to attenuate the influence of the noise on the regulated output over all frequencies. Now, assuming that $r_k \in \mathcal{R}$, we solve the state feedback problem assuming that $V(r, \xi, T) = 0$ for all $r \notin \mathcal{R}$. We iteratively test different values of γ, and the smallest value of γ which yields $V(0, 0, 0) = 0$ is found to be 6.5. Hence, this is the guaranteed value of which is true for the nominal deposition over the entire operating range $\mathcal{R}$. It is clear that the value of γ depends on the operating range $\mathcal{R}$. In fact, the smaller the range $\mathcal{R}$ the smaller the value of γ. We assume that the initial rate is within $\pm 20\mathring{A}/\text{min}$ of the nominal value $\bar{r}$, and hence initialize the

information state as $p_0(r, \xi, T) = 0$ if $r \in [-20, 20]$, or $p_0(r, \xi, T) = -\infty$ else. The certainty equivalence controller (5.2) is then implemented.

We also consider the case having a one lot delay, where the measurement is now given by

$$y_{k+1} \in r_{k-1}t_{k-1} + \bar{r}t_{k-1} + \bar{t}r_{k-1} + (\bar{t} + t_{k-1})[-3, 3] + [-10, 10]$$

and we again implement the certainty equivalence controller (5.4). We compare the performance of the robust controller designed, to a simple controller obtained via an EWMA estimate of the rate. Here, we assume that the initial rate is perfectly known. This controller is implemented as:

$$
\begin{aligned}
a_k &= \lambda \left(\frac{\hat{y}_{k-\tau}}{\hat{t}_{k-1-\tau}} \right) + (1 - \lambda)a_{k-1} \\
\hat{t}_k &= \frac{\hat{T}_k}{a_k}
\end{aligned}
$$

with a_0 equal to the initial rate. Here, we choose $\lambda = 0.1$ (a typical value for the EWMA weight [25]), and $\lambda = 0.8$. Also, $\tau = 0$ for the case with no delay, or $\tau = 1$ if we have a one lot delay. Note that the above implementation is simplistic, in that it assumes that the rate remains more or less constant over runs.

Figure 6.8 illustrates the behavior of the controllers when the equipment has a steady rate drift of $8\mathring{A}/\text{min}$ between lots with the target held at $1500\mathring{A}$. As expected, the EWMA controller with $\lambda = 0.1$ (even with no measurement delay) is unable to compensate for the drift, and yields a large offset. The performance is definitely better for $\lambda = 0.8$. Furthermore, we observe that the robust controller compensates for both the cases i.e. with no delay, and with one lot delay, although
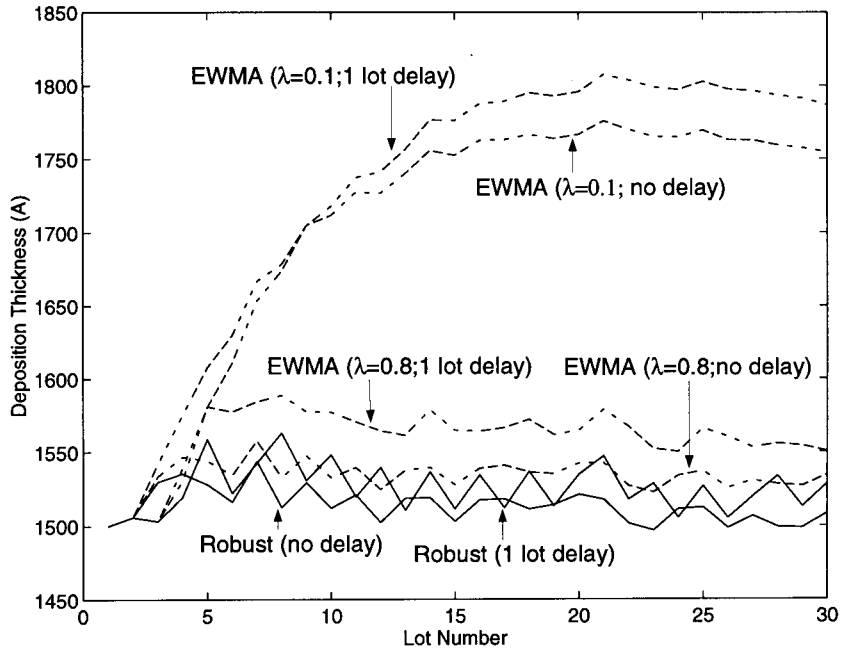
Figure 6.8: End-Pointing: Process under a steady rate drift of $8\mathring{A}/\text{min}$ between lots.

the performance is better when we have no delay. Here, it should be noted that unless explicitly stated, in all figures the plots show the actual deposition (ξ), and not the measured one (y).

The above is an extreme situation, and we now let all the noises entering the system be random. So v_k, and w_k are zero mean, Gaussian, with standard deviations $\sigma_v = 4$, $\sigma_w = 1$ respectively, and m_k is uniformly distributed between -10 and 10. However, we execute a sequence of target changes. The case having no measurement delay is shown in figure 6.9. We see that the robust controller is able to track the target much more tightly than the EWMA controller, which tends to drift around the target. Figure 6.10 shows what happens when we have a one lot delay
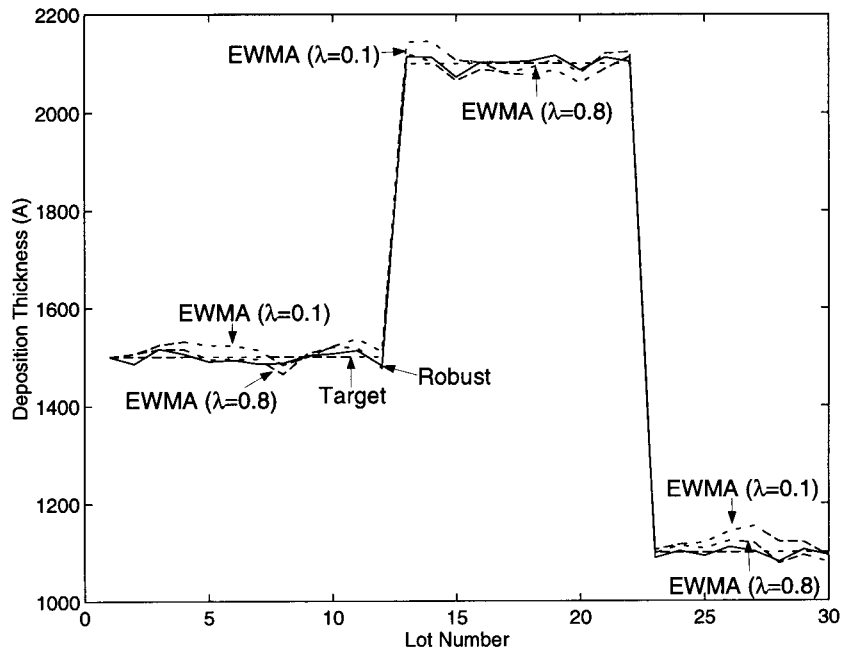
Figure 6.9: End-Pointing: Process subject to random noise and no measurement delay.

in the measurements. The performance of both controllers degrade, however, the robust controller tracks the target much more tightly than the EWMA controller.

Table 6.1, gives the mean and standard deviation (STD) of the errors for the cases discussed above. Here the error is given by

$$\text{error} = \text{actual deposition} - \text{target}$$

So far, we have assumed that the system conforms to the model assumptions. One would however, like a course of action in case the system does in fact violate the model assumptions. If the model has been correctly defined, the probability of this occurring should be negligible (i.e. $\ll 1\%$), and we may never observe a
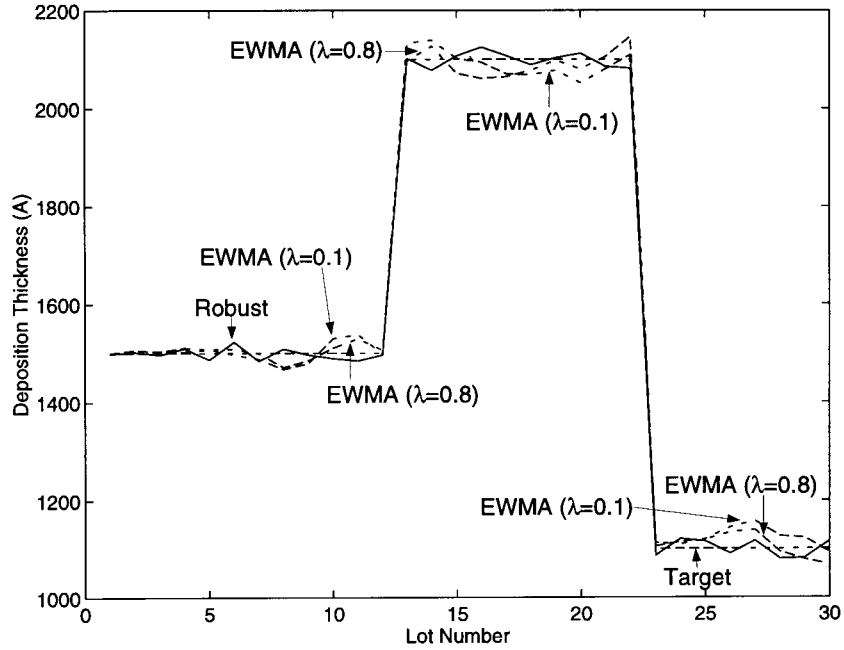
Figure 6.10: End-Pointing: Process subject to random noise and one lot measurement delay.

violation in practice. However, if the model is incorrectly defined, the chances of violation increases. Here, the violation is caused by either, **bad** data points, i.e. violating the bounds on the measurement error, or an **exceptional** process shift. This would result in the information state being set to $-\infty$ for all the states, since the observations would no longer conform to the system dynamics. This is a

		EWMA				Robust	
		$\lambda = 0.1$		$\lambda = 0.8$			
Drift		Mean	STD	Mean	STD	Mean	STD
Steady	No delay	209.60	83.43	33.57	11.34	13.73	11.75
	1 lot delay	223.72	102.80	59.53	22.74	25.66	16.54
Random	No delay	11.67	22.14	0.19	17.65	-0.89	11.87
	1 lot delay	4.75	23.96	-0.22	23.82	-1.03	14.12

Table 6.1: End-Pointing: Error statistics of the controllers.

potentially hazardous situation since the controller will stall.

We can check for violation in the following manner. Let the measurement delay be τ. Assume we have just finished with run k. Then the controller flags a violation if given the current information state $p_{k-\tau}$, the recipe $u_{k-\tau}$, and the measurement y_{k+1},

$$H(p_{k-\tau}, u_{k-\tau}, y_{k+1}) = -\infty, \text{ for all } x \in \boldsymbol{R}^n$$

To recover from this situation, we reinitialize the information state. At this time, all we can say is that the process has either shifted by an exceptional amount, or that y_{k+1} is an exceptionally bad data point. Note that we are using the term **exceptional** to classify this situation, since the model already incorporates both expected measurement errors, and process shifts. To reinitialize the information state sequence, we do the following:

1. Define $\hat{p}_{k-\tau}$ in the following manner

$$\hat{p}_{k-\tau}(x) \triangleq \begin{cases} p_{k-\tau}(x) & \text{if } p_{k-\tau}(x) \neq -\infty \\ 0 & \text{if } p_{k-\tau}(x) = -\infty \text{ and } y_{k+1} \in \mathcal{G}(x, u_{k-\tau}) \end{cases}$$

2. Now define $p_{k+1-\tau}$ as

$$p_{k+1-\tau}(x) \triangleq \sup_{\xi \in \boldsymbol{R}^n} \{ \hat{p}_{k-\tau}(\xi) + \hat{B}(\xi, x, u_{k-\tau}) \}$$

where $\hat{B}$ is defined as

$$\hat{B}(\xi, x, u) \triangleq \sup_{s \in \mathcal{F}(\xi, u)} \{ |l(x, u) - l(s, u)|^2 - \gamma^2 |x - s|^2 \}$$

if $x \in \mathcal{F}(\xi, u)$, or else is equal to $-\infty$.

The information state is now propagated as before from the next run onwards, unless another violation is flagged. Note, that this also gives an indication about changes in the model assumptions. In particular, very frequent violations would imply that the model was grossly inaccurate to start with.

We now present a simulation of the above case for end-pointing. To simulate this case, we set the target as $1500\overset{\circ}{A}$. All the noise entering the system is now random, except for lot 5, and lot 14. During lot 5, we set $v_5 = 120$. This corresponds to an **exceptional process shift**. During lot 14, we set $m_{14} = 130$ and $w_{14} = 4$. This corresponds to a **bad data point**. Large values for the shift, and measurement error have been chosen to amplify their effects. Figure 6.11 shows the actual and measured deposition trajectories for both the cases, i.e. when we have no measurement delay (figure 6.11 (top)), and the case when we have a one lot delay (figure 6.11 (bottom)). The actual deposition is shown by a solid line, and the measured deposition by a dashed line. We also plot the target, which is a straight line corresponding to a thickness of $1500\overset{\circ}{A}$. For the case when we have no delay, the controller recovers immediately from the shift. In the case when we have a lot delay, the effects are more pronounced. In the case of exceptional shift the controller can only bring the process back to target after two runs. This is as expected, since the measurement is available to the controller only after the run following the shift, i.e. after run 6. The effect of the bad data point does not seem to be significant, although the controller does flag a violation.
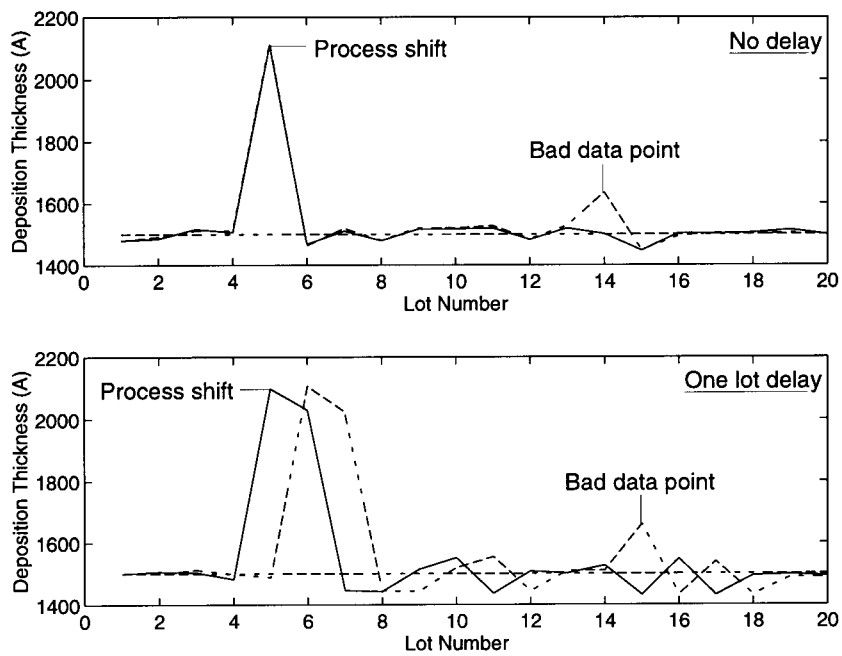
Figure 6.11: End-Pointing: Controller response to exceptional disturbances: No measurement delay (top); one lot measurement delay (bottom). Actual deposition thickness (solid); measured deposition thickness (dashed).

6.4.2 LPCVD Rate Control

In this section, we will briefly consider the inverse problem, i.e. of controlling the rates in an LPCVD reactor. The model we work with is an experimentally determined one, and is presented in [36]. Here, we limit our attention to the deposition on the first and last wafer. We augment the models with drift terms. The models express the deposition rates in terms of deposition temperature T, deposition pressure P, and the silane flow rate Q. They are given by

$$
\begin{aligned}
\hat{R}_1 &= \exp(c_1 + c_2 \ln P + c_3 T^{-1} + c_4 Q^{-1}) + d_1 \\
\hat{R}_2 &= \hat{R}_1 \left[\frac{1 - S' C_{gs} \hat{R}_1 Q^{-1}}{1 + S' C_{gs} \hat{R}_1 Q^{-1}} \right] + d_2
\end{aligned}
\tag{6.2}
$$

with the rates expressed in Å/min, P in mtorr, T in K, and Q in sccm. The parameters are given [36] to be $c_1 = 20.65$, $c_2 = 0.29$, $c_3 = -15189.21$, $c_4 = -47.97$, $S' = 4777.8$, and $C_{gs} = 1.85 \times 10^{-5}$, where we have dropped the units for convenience. d_1, and d_2 represent the drift terms. The actual rates (R_1 and R_2) are obtained from the above model by adding a zero mean noise to $\hat{R}_1$ and $\hat{R}_2$. The noise is assumed to be Gaussian with a variance of 9. Furthermore, we assume that the maximum drift expected between runs is 0.3. This actually represents a shift of σ in 10 runs, and may be too large to be true in practice. However, we choose this value, since it enables us to see the corrective action of the RbR controller in a fewer number of runs. The targets T_1 for R_1, and T_2 for R_2 are fixed at 169.75 Å/min and 141.7 Å/min respectively. It is also assumed that the other parameters of the model do not undergo changes from run to run. We also assume that what

is to be controlled are the measured rates (R_1 and R_2), since we do not have any information of the actual rates. We can express the system in a set-valued form as done in the previous example. The exact equations are

$$
\begin{aligned}
c_1[k+1] &= c_1[k] \\
c_2[k+1] &= c_2[k] \\
c_3[k+1] &= c_3[k] \\
c_4[k+1] &= c_4[k] \\
S'C_{gs}[k+1] &= S'C_{gs}[k] \\
d_1[k+1] &\in d_1[k] + [-0.3, 0.3] \\
d_2[k+1] &\in d_2[k] + [-0.3, 0.3] \\
R_1[k+1] &\in f(P[k], T[k], Q[k]) + [-9, 9] \\
R_2[k+1] &\in f(P[k], T[k], Q[k])g(Q[k]) + d_2[k] + [-9, 9] \\
y_1[k+1] &\in f(P[k-\tau], T[k-\tau], Q[k-\tau]) + [-9, 9] \\
y_2[k+1] &\in f(P[k-\tau], T[k-\tau], Q[k-\tau])g(Q[k-\tau]) + d_2[k-\tau] + [-9, 9]
\end{aligned}
$$

where

$$
f(P[k], T[k], Q[k]) = \exp(c_1[k] + c_2[k]\ln P[k] + c_3[k]T[k]^{-1} + c_4[k]Q[k]^{-1}) + d_1[k]
$$

and

$$
g(Q[k]) = \frac{1 - S'C_{gs}[k]f(P[k], T[k], Q[k])Q[k]^{-1}}{1 + S'C_{gs}[k]f(P[k], T[k], Q[k])Q[k]^{-1}}
$$

where we have not shown the dependence of f and g on the parameters (c_1, c_2, etc.) for convenience. Here $[c_1 \ c_2 \ c_3 \ c_4 \ S'C_{gs} \ d_1 \ d_2 \ R_1 \ R_2]$ represent the **states**, and $[y_1 \ y_2]$ represent the **measurements**. Also $\tau \geq 0$ represents the measurement delay. We assume that the operating region of the equipment $\mathcal{R}$ is $\pm 20\%$ around the parameters (i.e. c_1, c_2 etc.), and with the drifts d_1, and d_2 restricted to $[-30, 30]$. The regulated output is given by

$$
z[k+1] = (R_1[k+1] - T_1)^2 + (R_2[k+1] - T_2)^2
$$

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Solving the state feedback problem (after centering the system around the origin), and forcing the control inputs (i.e. P, T, and Q) to lie in the experimental design space [36], we obtain the value of γ as 30. We then implement the certainty equivalence controllers given by equation (5.2) for the case of no delay, and equation (5.4) for the case of one lot delay. Here, we assume that the system starts from 0, and hence we set $p_0 = \delta_{\{0\}}$.

For the purposes of simulation, we allow the drift to be zero mean, Gaussian with a standard deviation of 0.1. We force a process shift during run 4. This shift corresponds to a change in c_3 of 580, c_4 of -8, C_{gs} of 1×10^{-6}, d_1 of 2.5, and d_3 of 11. After this, we force bad data points during run 10, by forcing $n_1 = 20$, and $n_2 = -20$. Finally, we subject the process to a steady drift of 0.3 for both R_1 and R_2 between runs 15 and 30. Figure 6.12 (top) illustrates the controller performance for R_1, and figure 6.12 (bottom) for R_2. The horizontal lines show the target, the solid line is the case when we have no measurement delay, and the dotted line is the case when we have a one lot delay. It is observed that the controller effectively compensates against these disturbances. We again observe that the shift is compensated for in the very next run in case of no delayed measurements, and after two runs in the case of one run measurement delay.
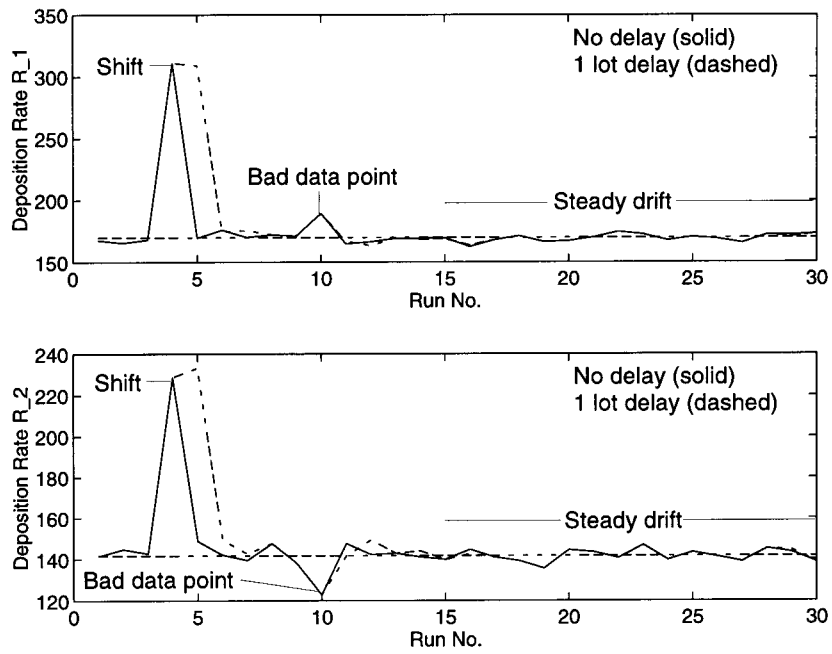
Figure 6.12: LPCVD Rate Control: Process subject to shift, bad data points, and steady drift.

Chapter 7

Concluding Remarks

In this thesis, we have considered the problem of constructing robust controllers for set-valued discrete time dynamical systems. An application of the techniques developed to the problem of rejecting persistent bounded (non-additive) disturbances for nonlinear systems was presented. The methodology developed could be viewed as being analogous to the l^1-optimal control problem for linear systems.

A number of problems remain open. Clearly, the most pressing one is the development of good approximations to the problem in order to avoid the infinite dimensional dynamic programming encountered during output feedback. Generalization of certainty equivalence is one way of attacking this problem, although, the issue has yet to be satisfactorily resolved. An alternative is to obtain finite dimensional approximations to the information state. In either case, one still needs to solve a finite dimensional dynamic programming equation. Here, due to the "curse of dimensionality" practical applications are somewhat limited. In this area, the

issues are similar to solutions to other optimal, and stochastic control problems solved via dynamic programming. Progress in addressing this issue is being made by a number of researchers, with an excellent, and extremely readable treatment in [47]. We note here, that the optimization problem to be solved at each time step is simpler in our case (when compared to say other nonlinear dynamic games), since by converting the system to a set-valued form, we have in a sense "smoothened" out the exact dependence on the disturbances. Also, more applications need to be developed. There is also a need to investigate, whether (if at all), a closed form solution exists for special cases. Finally, one would like an extension of this theory to continuous time. In this regard, one would still have to discretize the problem in order to obtain computational solutions. To this end, the Euler transform employed in the examples could play a fundamental role [40].

Appendix A

Smoothness of the Cost Function

The assumption, that $l(\cdot, u) \in C^1(\boldsymbol{R}^n)$, for all $u \in U$ can be dropped, if one considers the perturbed system

$$x_{k+1} \in \mathcal{F}(x_k, u_k) + \bar{\mathcal{B}}_\epsilon(0) = \mathcal{F}^\epsilon(x_k, u_k) \qquad (\text{A.1})$$

for some arbitrary $\epsilon > 0$. Let the set of trajectories corresponding to $\mathcal{F}$ be denoted by Γ^u, and those corresponding to $\mathcal{F}^\epsilon$ be $\Gamma^{u,\epsilon}$. Then clearly

$$\Gamma^u(x_0) \subset Int \ \Gamma^{u,\epsilon}(x_0)$$

for all $x_0 \in X_0$.

Assumption **A4** on page 13 may now be dropped, and assumption **A5** maybe replaced by the following.

A5'. $l(\cdot, u) : \boldsymbol{R}^n \to \boldsymbol{R}$, is continuous for all $u \in U$, and is such that, its **directional derivatives** with respect to the first argument exist for all x, and

$u \in \boldsymbol{U}$. Assume that there exists a $\gamma_{min} > 0$ such that

$$\mathcal{L}^{\gamma} \triangleq \{s \in \boldsymbol{R}^n \mid \exists u \in \boldsymbol{U} \text{ s.t. } |D_x l(s, u)| \leq \gamma\}$$

is bounded, and contains the origin for all $\gamma \geq \gamma_{min}$. Here, $|D_x l(s, u)|$ is

defined by

$$|D_x l(s, u)| = \sup_{|h|=1} |D_x^h l(s, u)|$$

where $D_x^h l(s, u)$ is the **directional derivative** of $l(\cdot, u)$ with respect to its

first argument at the point s in the direction h. i.e.

$$D_x^h l(s, u) = \lim_{t \to 0+} \frac{l(s + ht, u) - l(s, u)}{t}$$

Consider a sequence $\{x_k\}_{k=0}^{\infty}$ such that

$$x_{k+1} \in Int \; \mathcal{F}^{\epsilon}(x_k, u_k)$$

for all k. Now construct the sequence $\{W_k^u\}_{k=0}^{\infty}$ as

$$W_k^u = \sup_{r \in \mathcal{F}^{\epsilon}(x_k, u_k)} \left(|l(r, u_k) - l(x_{k+1}, u_k)|^2 - \gamma^2 |r - x_{k+1}|^2 \right)$$

Then we have

Lemma A.1 If $W_k^u \to 0$, as $k \to \infty$, then for all h, $|h| = 1$, for all $\epsilon_1, \epsilon_2 > 0$, $\exists K$,

such that $\forall k \geq K$, and for all

$$r = x_{k+1} + \epsilon_1 h \in \mathcal{F}^{\epsilon}(x_k, u_k)$$

we have

$$\frac{|l(x_{k+1} + \epsilon_1 h, u_k) - l(x_{k+1}, u_k)|}{\epsilon_1} < \gamma + \epsilon_2 \tag{A.2}$$

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Proof:

Suppose not, i.e. there exists h, $|h| = 1$, and there exist $\epsilon_1, \epsilon_2 > 0$, such that for all K, $\exists k \geq K$ such that $x_{k+1} + \epsilon_1 h \in \mathcal{F}^\epsilon(x_k, u_k)$ and

$$\frac{|l(x_{k+1} + \epsilon_1 h, u_k) - l(x_{k+1}, u_k)|}{\epsilon_1} \geq \gamma + \epsilon_2$$

Which implies that there exists an $\eta > 0$ such that

$$|l(x_{k+1} + \epsilon_1 h, u_k) - l(x_{k+1}, u_k)|^2 - \gamma^2 \epsilon_1^2 \geq \eta > 0$$

This implies that there exists an $s \in \mathcal{F}^\epsilon(x_k, u_k)$ such that

$$|l(s, u_k) - l(x_{k+1}, u_k)|^2 - \gamma^2 |s - x_{k+1}|^2 \geq \eta > 0$$

Which means $W_k^u \geq \eta$, a contradiction.

$\square$

This immediately yields the following corollary.

Corollary A.2 Let W_k^u be constructed as above. If $W_k^u \to 0$, as $k \to \infty$, then

$$\limsup_{k \to \infty} |D_x l(x_{k+1}, u_k)| \leq \gamma$$

Proof:

Take the limit in equation (A.2), as $\epsilon_1 \to 0$ to obtain

$$\limsup_{k \to \infty} \sup_{|h|=1} |D_x^h l(x_{k+1}, u_k)| < \gamma + \epsilon_2$$

Since ϵ_2 is arbitrary, the result follows.

$\square$

We can now obtain a version of the bounded real lemma . Only the state feedback case is presented here, and is similar to theorem 2.4.

Theorem A.3 If for a given $\gamma > 0$, $\Sigma_{\mathcal{F}\epsilon}^u$ is finite gain dissipative, then $\Sigma_{\mathcal{F}}^u$ is ultimately bounded.

Proof:

From the dissipation inequality (2.7), we obtain for any $x_0 \in X_0$

$$\sum_{i=0}^{k} |l(r_{i+1}, u_i) - l(s_{i+1}, u_i)|^2 - \gamma^2 |r_{i+1} - s_{i+1}|^2 \leq V(x_0)$$

for all k, $r, s \in \Gamma^{u,\epsilon}(x_0)$.

In particular, for any trajectory $x \in \Gamma^u(x_0) \subset Int\ \Gamma^{u,\epsilon}(x_0)$, we have

$$\sum_{i=0}^{k} W_i^u \leq V(x_0),\ \forall k$$

Which implies that

$$W_i^u \to 0 \text{ as } i \to \infty$$

and hence $x_k \to \mathcal{L}^{\gamma+\epsilon_2}$, as $k \to \infty$, for any arbitrary $\epsilon_2 > 0$.

Since, suppose not. Then, there exits an ϵ_2, $\epsilon_3 > 0$, such that for all K, there exists a $k \geq K$ such that

$$\mathcal{B}_{\epsilon_3}(x_{k+1}) \bigcap \mathcal{L}^{\gamma+\epsilon_2} = \phi$$

Which implies, that there exists an $\epsilon_2 > 0$, such that for all K, $\exists k \geq K$, such that

$$|D_x l(x_{k+1}, u_k)| > \gamma + \epsilon_2$$

This contradicts (via corollary A.2) the fact that $W_k^u \to 0$ as $k \to \infty$.

$\square$

Appendix B

The Varadhan-Laplace Lemma

Here, we give an extension of the Varadhan-Laplace lemma presented in [32]. Below ρ denotes a metric on $C(\boldsymbol{R}^n \times \boldsymbol{R}^p)$ corresponding to uniform convergence on compact subsets. $\mathcal{B}_r(x)$ denotes the open ball centered at x of radius r. $L_G^a(\cdot) : \boldsymbol{R}^n \to \boldsymbol{R}^p$ defined as Furthermore, it is assumed that $| \cdot |$ denotes the Euclidean norm. In what follows, F_a^ε, F_a denote single-valued maps, and G^a is a set-valued map. We also define, $L_G^a(M)$ as

$$L_G^a(M) \triangleq \bigcup_{x \in M} G^a(x)$$

Lemma B.1 Let A be a compact space, F_a^ε, $F_a \in C(\boldsymbol{R}^n \times \boldsymbol{R}^p)$ and assume

i. $\lim_{\varepsilon \to 0+} \sup_{a \in A} \rho(F_a^\varepsilon, F_a) = 0$

ii. The function F_a is uniformly continuous in each argument on each set $\mathcal{B}_R(0) \times \mathcal{B}_{\hat{R}}(0)$; $R, \hat{R} > 0$, uniformly in $a \in A$.

iii. $\exists \gamma_1 > 0$, $\gamma_2 \geq 0$ such that

$$F_a^\varepsilon(x, w), F_a(x, w) \leq -\gamma_1 \left(\mid x \mid^2 + \mid w \mid^2 \right) + \gamma_2$$

$\forall x \in \mathbf{R}^n$, $\forall w \in \mathbf{R}^m$, $\forall a \in A$, $\forall \varepsilon > 0$.

iv. $G^a : \mathbf{R}^n \to \mathbf{R}^m$ is a set-valued map, uniformly continuous with convex compact values on each set $\mathcal{B}_R(0)$, uniformly in $a \in A$.

v. $IntG^a \neq \phi$, $\forall x \in \mathbf{R}^n$, $\forall a \in A$.

Then

$$\lim_{\varepsilon \to 0+} \sup_{a \in A} \left| \varepsilon \log \int_{\mathbf{R}^n} \int_{G^a(x)} e^{F_a^\varepsilon(x,w)/\varepsilon} dw dx - \sup_{x \in \mathbf{R}^n} \sup_{w \in G^a(x)} F_a(x, w) \right| = 0$$

Proof:

Write

$$\bar{F}_a^\varepsilon = \sup_{x \in \mathbf{R}^n} \sup_{w \in G^a(x)} F_a^\varepsilon(x, w)$$

$$\bar{F}_a = \sup_{x \in \mathbf{R}^n} \sup_{w \in G^a(x)} F_a(x, w)$$

Then our assumptions ensure that

$$\lim_{\varepsilon \to 0+} \sup_{a \in A} \left| \bar{F}_a^\varepsilon - \bar{F}_a \right| = 0$$

For a given $\delta > 0$, define

$$\mathcal{B}_\delta^{a,\varepsilon} = \left\{ x \in \mathbf{R}^n \mid \exists w \in G^a(x) \text{ s.t. } F_a^\varepsilon(x, w) > \bar{F}_a^\varepsilon - \delta \right\}$$

Then assumption (**iii**) ensures that there exists $R > 0$ s.t.

$$\mathcal{B}_\delta^{a,\varepsilon} \subset \mathcal{B}_R(0).$$

Furthermore, by Berge's theorem [13], $L_G^a(\bar{\mathcal{B}}_R(0))$ is compact. Hence, $\exists \hat{R} > 0$ such that $L_G^a(\mathcal{B}_R(0)) \subset \mathcal{B}_{\hat{R}}(0)$.

By hypothesis (**iii**) on $\mathcal{B}_R(0) \times \mathcal{B}_{\hat{R}}(0)$ and using the uniform convergence of F_a^ε to F_a on $\bar{\mathcal{B}}_R(0) \times \bar{\mathcal{B}}_{\hat{R}}(0)$, $\exists r > 0$ such that

$$|x - x'| < \frac{r}{2} \text{ implies } |F_a^\varepsilon(x, w) - F_a^\varepsilon(x', w)| < \frac{\delta}{2}$$

for all $w \in \mathcal{B}_{\hat{R}(0)}$, $\forall x, x' \in \mathcal{B}_R(0)$, $a \in A$ and $\varepsilon > 0$ sufficiently small. And

$$|w - w'| < \frac{r}{2} \text{ implies } |F_a^\varepsilon(x, w) - F_a^\varepsilon(x, w')| < \frac{\delta}{2} \qquad .$$

$\forall x \in \mathcal{B}_R(0)$, $\forall w, w' \in \mathcal{B}_{\hat{R}}(0)$, $a \in A$ and $\varepsilon > 0$ sufficiently small.

Pick a

$$x_a^\varepsilon \in \arg\max_{\mathbf{R}^n} \sup_{w \in G^a(x)} F_a^\varepsilon(x, w) \subset \mathcal{B}_R(0)$$

and

$$w_a^\varepsilon \in \arg\max_{w \in G^a(x_a^\varepsilon)} F_a^\varepsilon(x_a^\varepsilon, w) \subset \mathcal{B}_{\hat{R}}(0)$$

By compactness, $w_a^\varepsilon \in G'^a(x_a^\varepsilon)$.

Now, let $\hat{\epsilon}$ be such that $0 < \hat{\epsilon} < \frac{r}{2}$, and define

$$\mathcal{W} \triangleq \{ w \mid \ | w - w_a^\varepsilon | < \frac{r}{2} - \hat{\epsilon} \}$$

Then, by the uniform continuity of G^a on $\mathcal{B}_{\hat{R}}(0)$, $\exists \hat{r}$ such that $\forall x$ with $| x_a^\varepsilon - x | < \hat{r}$

$$G^a(x) \bigcap \mathcal{W} \neq \phi$$

$\forall a \in A.$

Let $\bar{r} = \min\{\frac{r}{2}, \hat{r}\}$. Then, $\forall x \in \mathcal{B}_{\bar{r}}(x_a^\varepsilon)$ and for any $\bar{w}_x^a \in G^a(x) \bigcap \mathcal{W}$

$$\mathcal{B}_\varepsilon(\bar{w}_x^a) \bigcap G^a(x) \neq \phi$$

and for any $w \in \mathcal{B}_{\hat{\varepsilon}}(\bar{w}_x^a) \bigcap G^a(x)$

$$\mid w - w_a^\varepsilon \mid < \frac{r}{2}$$

Hence,

$$\mathcal{B}_{\bar{r}}(x_a^\varepsilon) \subset \mathcal{B}_\delta^{\alpha,\varepsilon}, \quad \forall a \in A, \varepsilon > 0 \text{ sufficiently small.}$$

Now, let

$$\alpha_a^\varepsilon \triangleq \int_{\mathbf{R}^n} \int_{w \in G^a(x)} \exp(F_a^\varepsilon(x, w)/\varepsilon) dw dx$$

For each $x \in \mathcal{B}_{\bar{r}}(x_a^\varepsilon)$ pick a $w_a^\varepsilon(x) \in G^a(x) \bigcap \mathcal{W}$. Then

$$
\begin{aligned}
\alpha_a^\varepsilon &\geq \int_{\mathcal{B}_{\bar{r}}(x_a^\varepsilon)} \int_{G^a(x) \bigcap \mathcal{B}_\varepsilon(w_a^\varepsilon(x))} \exp(F_a^\varepsilon(x, w)/\varepsilon) dw dx \\
&\geq \int_{\mathcal{B}_{\bar{r}}(x_a^\varepsilon)} \int_{G^a(x) \bigcap \mathcal{B}_\varepsilon(w_a^\varepsilon(x))} \exp\left(\frac{\bar{F}_a^\varepsilon - \delta}{\varepsilon}\right) dw dx \\
&\geq C_m \bar{r}^m a_{\hat{\varepsilon}}^R \exp\left(\frac{\bar{F}_a^\varepsilon - \delta}{\varepsilon}\right)
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
\varepsilon \log \alpha_a^\varepsilon &\geq \varepsilon \log C^m \bar{r}^m a_{\hat{\varepsilon}}^R + \bar{F}_a^\varepsilon - \delta \\
&\geq \bar{F}_a - 3\delta
\end{aligned}
$$

for all $\varepsilon > 0$ sufficiently small and for all $a \in A$.

Next, for $R > 0$ write

$$\alpha_a^\varepsilon \overset{\triangle}{=} \int_{|x| \leq R} \int_{w \in G^a(x)} \exp\left(F_a^\varepsilon(x, w)/\varepsilon\right) dw dx +$$

$$\int_{|x| \geq R} \int_{w \in G^a(x)} \exp\left(F_a^\varepsilon(x, w)/\varepsilon\right) dw dx$$

$$= J + K$$

Note that

$$\varepsilon \log \alpha_a^\varepsilon = \varepsilon \log J + O(K/J)$$

for R sufficiently large.

Let

$$z \overset{\triangle}{=} \begin{bmatrix} x \\ w \end{bmatrix}$$

Then

$$\begin{aligned}
K &\leq \int_{|z| \geq R} \exp\left(\frac{-\gamma_1 |z|^2 + \gamma_2}{\varepsilon}\right) dz \\
&\leq C_R \exp\left(\frac{C_1 - C_2 R^2}{\varepsilon}\right) \\
&\leq C_R \exp\left(-C'/\varepsilon\right)
\end{aligned}$$

where $C_R, C_1, C_2 > 0$, and $C' > 0$ for R large enough.

Increase R if necessary to ensure that

$$\arg \max_{x \in \mathbf{R}^n} \sup_{w \in G^a(x)} F_a^\varepsilon(x, w) \subset \mathcal{B}_R(0)$$

Then

$$\varepsilon \log J \leq \varepsilon \log \int_{|x| \leq R} \int_{w \in G^a(x)} \exp\left(\bar{F}_a^\varepsilon/\varepsilon\right) dw dx$$

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$$\le \quad \varepsilon \log M_R \int_{|x| \le R} \exp\left(\bar{F}_a^\varepsilon / \varepsilon\right) dx$$

$$\le \quad \varepsilon \log C_n R^n M_R + \bar{F}_a^\varepsilon$$

Thus

$$\varepsilon \log \alpha_a^\varepsilon \le \bar{F}_a + 3\delta$$

for all $\varepsilon > 0$ sufficiently small and for all $a \in A$. Thus

$$\sup_{a \in A} \left| \varepsilon \log \alpha_a^\varepsilon - \bar{F}_a \right| < 3\delta$$

for all $\varepsilon > 0$ sufficiently small.

$\square$

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