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Robust Control of Set-Valued Discrete Time Dynamical Systems

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Robust Control of Set-Valued Discrete Time Dynamical Systems*

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Abstract

This paper presents results obtained for the robust control of discrete time dynamical systems. The problem is formulated and solved using dynamic programming. Both necessary and sufficient conditions in terms of (stationary) dynamic programming equalities are presented. The output feedback problem is solved using the concept of an information state, where a decoupling between estimation and control is obtained.

1 Introduction

This paper is concerned with the robust control of systems modelled as inclusions. Systems of this type occur, for example in hybrid systems, where an upper logical level switches between different plant models depending on observed events [16],[9]. Stability results for such systems based on Lyapunov-like functions have been presented in [8],[16]. However, no concept of robust performance (in the sense of minimizing variations in the regulated outputs due to switching between different plant models and noise), particularly for the output feedback case, exists for such systems.

Another example of systems that can be modelled as inclusions are systems with bounded parameteric uncertainty. A number of results can be found in the literature concerned with stabilization and ultimate boundedness of such (linear) systems (e.g. [5],[20],[23],[17],[19],[6]). At the same time, it has been noted that with standard H_∞

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control methods, no robust behaviour on H_∞ performance along with stability can be guaranteed. This has led to optimal robust H_∞ control for linear systems [15] where one tries to obtain optimal guaranteeable H_∞ performance given uncertainty about the plant's state space parameters. What is not considered however, is the influence of the parameter variations themselves on the regulated outputs. Under the assumption that the noise is bounded, the above problem can be reduced to robust control of inclusions. In this context, both the influence of parameter variations, as well as of exogenous inputs on the regulated output can be considered in a unified context. Furthermore, we will show that ultimate boundedness of trajectories can also be established. It should be noted here that the concept of casting dynamical systems as inclusions is not new. However, previous work has been mainly concerned with the problem of control synthesis under input constraints [14].

Our work is motivated by recent results obtained in the nonlinear H_∞ context in [13]. We will use the dynamic game framework developed in [7],[13]. Furthermore, to establish the ultimate boundedness of trajectories, we will employ the theory of dissipative systems [22] to write down a version of the bounded real lemma. The latter is expressed in terms of a dissipation inequality, which has appeared repeatedly in papers dealing with nonlinear robust control (e.g.[3],[10],[11],[12],[13],[18],[21]). In the context of set-valued discrete time dynamical systems, in [1] the authors have employed dissipativity to identify conditions for the existence of fixed points of set-valued maps. For the output feedback problem, we will employ the concept of an information state. The exact form of the information state recursion was derived from an analogous set-valued stochastic control problem in [4]. Using the concept of an information state, we are able to obtain a separation between estimation and control.

Although we will be using concepts from dissipative systems, our final necessary and sufficient conditions for the existence of a solution to the robust control problem will be expressed in terms of (stationary) dynamic programming equalities. However, from the proof for the sufficient conditions, it will be clear that the necessary and sufficient conditions could also be expressed in terms of dissipation inequalities.

In section 2, we present the problem formulation. Section 3 deals with the state feedback case and section 4 with the output feedback problem. Finally an example is presented in section 5.

2 Problem Formulation

The system under consideration (Σ) is expressed as

$$\Sigma \begin{cases} x_{k+1} & \in & \mathcal{F}(x_k, u_k) \ , \quad x_0 \in X_0 \\ y_{k+1} & \in & \mathcal{G}(x_k, u_k) \\ z_{k+1} & = & l(x_{k+1}, u_k) \ , \quad k = 0, 1, \ldots \end{cases} \tag{1}$$

Here, $x_k \in \mathbf{R}^n$ are the states, $u_k \in U \subset \mathbf{R}^m$ are the control inputs, $y_k \in \mathbf{R}^t$ are the measured variables, and $z_k \in \mathbf{R}^q$ are the regulated outputs. The following assumptions are made on the system Σ

1. $0 \in X_0$.
2. $\mathcal{F}(x, u)$, $\mathcal{G}(x, u)$ are compact for all $x \in \mathbf{R}^n$ and $u \in U$.
3. The origin is an equilibrium point for $\mathcal{F}$, $\mathcal{G}$ and l. i.e

   $$\mathcal{F}(0,0) \ni 0 \ ; \quad \mathcal{G}(0,0) \ni 0 \ ; \quad l(0,0) = 0$$

4. $Int\mathcal{F} \neq \phi$ for all $x \in \mathbf{R}^n$, $u \in U$. i.e. there exists an $\bar{\epsilon} > 0$ such that for any $x \in \mathbf{R}^n$, $u \in U$, $B_{\bar{\epsilon}}(r) \in \mathcal{F}(x, u)$ for some $r \in \mathcal{F}(x, u)$. Here $B_{\bar{\epsilon}}(r)$ is the open ball of radius $\bar{\epsilon}$ centered at r.
5. $l(\cdot, u) \in C^1(\mathbf{R}^n)$ for all $u \in U$ and is such that, $\exists \gamma_{min} > 0$, such that

   $$\mathcal{L}^\gamma \triangleq \left\{ s \in \mathbf{R}^n | \exists u \in U \text{ s.t. } \left| \frac{\partial}{\partial x} l(s, u) \right| \leq \gamma \right\}$$

   is bounded and contains the origin $\forall \gamma \geq \gamma_{min}$.
6. $U \subset \mathbf{R}^m$ is compact.

Some of the notation employed in the paper will be as follows:
$|\cdot|$ denoted the Euclidean norm, $\|\cdot\|$ denotes the l^2 norm, , ${}^{u}_{0,k}(x)$ denotes the truncated forward cone of the point $x \in \mathbf{R}^n$ [1]. In particular

$$, {}^{u}_{0,k}(x) \triangleq \{x_{0,k} | x_{j+1} \in \mathcal{F}(x_j, u_j), j = 0, \ldots, k-1\}.$$

We furthermore define

$$, {}^{u}_{0,k}(x_0)\Big|_{l^2} \triangleq \{x^1_{0,k}, x^2_{0,k} \in \ , {}^{u}_{0,k}(x_0) | x^2 - x^1 \in l^2([0,k], \mathbf{R}^n)\}$$

and $X^u_k(x_0) \subset \mathbf{R}^n$ as the cross section of the forward cone of x_0 at time instant k.

The robust control problem can now be stated as:
Given $\gamma \geq \gamma_{min}$, find a controller u ($= u(x)$ or $u(y)$ depending on what is measured) such that the closed loop system Σ^u satisfies the following three conditions:

1. Σ^u is weakly asymptotically stable, in the sense that for each k, there exists an $\alpha_k \in \mathcal{F}(x_k, u_k)$ such that, the sequence $\alpha_k \longrightarrow 0$ as $k \longrightarrow \infty$.
2. Σ^u is ultimately bounded.
3. $$\sup_{r,s\in\Gamma^u(0)|_{l^2}, r\neq s} \frac{\|l(r,u)-l(s,u)\|}{\|r-s\|} \leq \gamma$$

3 State Feedback Case

In the state feedback case, the problem is to find a controller $u \in S$ i,e, $u_k = u(x_k)$, where $u : \mathbf{R}^n \longmapsto U$ such that the three conditions stated above are satisfied.

3.1 Finite Time Case

For the finite time case, conditions 1 and 2 are not required. Condition 3 is equivalent to the existence of a finite $\beta^u_K(x_0)$, $\beta^u_K(0) = 0$ such that

$$\sum_{i=0}^{K-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 \leq \gamma^2 \sum_{i=0}^{K-1} \mid r_{i+1} - s_{i+1} \mid^2 + \beta^u_K(x_0),$$
$$\forall K \geq 1, \forall r, s \in , {}^{u}_{0,K}(x_0), \forall x_0 \in X_0 \tag{2}$$

3.2 Dynamic Game

Here, the robust control problem is converted into an equivalent dynamic game. For $u \in S_{k,K-1}$ and $\bar{x} \in X_k(x_0)$, where $X_k(x_0)$ is the set of states that the system can achieve at time k if it were started form x_0, define

$$J_{\bar{x},k}(u) = \sup_{r,s\in\Gamma^u_{k,K}(\bar{x})} \{\sum_{i=k}^{K-1} (\mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 - \gamma^2 \mid r_{i+1} - s_{i+1} \mid^2)\} \tag{3}$$

Clearly

$$J_{\bar{x},k}(u) \geq 0.$$

Now, the finite gain property can be expressed as below

Lemma 1 *$\Sigma_{\bar{x}}^u$ is finite gain on [k,K] if and only if there exists a finite $\beta_K^u(\bar{x})$, $\beta_K^u(0) = 0$ such that*

$$J_{\bar{x},j}(u) \leq \beta_K^u(\bar{x}), \quad j \in [k, K], \ \forall \bar{x} \in X_0 \tag{4}$$

The problem is hence reduced to finding a $u^* \in S_{k,K-1}$ which minimizes $J_{\bar{x},k}$.

3.3 Solution to the Finite Time State Feedback Robust Control Problem

We can solve the above using dynamic programming. Define

$$V_k(\bar{x}) = \inf_{u \in S_{k,K-1}} \sup_{r,s \in \Gamma_{k,K}^u(\bar{x})} \{ \sum_{i=k}^{K-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 - \gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \} \tag{5}$$

The corresponding dynamic programming equation is

$$\begin{array}{rcl} V_k(x) & = & \inf_{u \in U} \sup_{r,s \in \mathcal{F}(x,u)} \{ \mid l(r,u) - l(s,u) \mid^2 - \gamma^2 \mid r - s \mid^2 + V_{k+1}(r) \} \\ V_K(x) & = & 0 \end{array} \tag{6}$$

Note that we have abused notation, and here u is a vector instead of a function as in equation 5.

Theorem 1 *(Necessity) Assume that $u^* \in S_{0,K-1}$ solves the finite time state feedback robust control problem. Then, there exists a solution V to the dynamic programming equation (6) such that $V_k(x) \geq 0$, $V_k(0) = 0$, $k \in [0, K-1]$, $x \in X_0$.*

Proof:

For $x \in X_0$, $k \in [0, K-1]$ define

$$V_k(x) = \inf_{u \in S_{k,K-1}} J_{x,k}(u)$$

Then, we have

$$0 \leq V_k(x) \leq \beta_K^{u^*}(x), \quad k \in [0, K-1], \ x \in X_0$$

Thus, V_k is finite on X_0, and by dynamic programming, V satisfies equation 6. Also, since $\beta_K^{u^*}(0) = 0$, $V_k(0) = 0$.

□

Theorem 2 *(Sufficiency) Assume that there exists a solution V to the dynamic programming equation (6), such that $V_k(x) \geq 0$, $V_k(0) = 0$, $k \in [0, K-1]$, $x \in X_0$. Let $u^* \in S_{k,K-1}$ be a control policy such that u_k^* achieves the minimum in equation (6) for $k = 0, \ldots, K-1$. Then u^* solves the finite time state feedback robust control problem.*

Proof: Dynamic programming arguments imply that for a given $x \in X_0$

$$V_0(x) = J_{x,0}(u^*) = \inf_{u \in S_{0,K-1}} J_{x,0}(u)$$

Thus u^* is an optimal policy for the game and lemma 1 is satisfied with $u = u^*$, where we obtain $\beta_K^u(x) = V_0(x)$.

□

3.4 Infinite Time Case

Here, we are interested in the limit as $K \longrightarrow \infty$. Invoking stationarity equation (6) becomes

$$V(x) = \inf_{u \in U} \sup_{r,s \in \mathcal{F}(x,u)} \{V(s) + \mid l(r,u) - l(s,u) \mid^2 - \gamma^2 \mid r - s \mid^2\} \tag{7}$$

3.5 The Dissipation Inequality

We say that the system $\Sigma^{\bar{u}}$ is *finite gain dissipative* if there exists a function $V(x)$ (called the storage function), such that $V(x) \geq 0$, $V(0) = 0$, and it satisfies the dissipation inequality

$$V(x) \geq \sup_{r,s \in \mathcal{F}(x,\bar{u}(x))} \{V(s) - \gamma^2 \mid r - s \mid^2 + \mid l(r, \bar{u}(x)) - l(s, \bar{u}(x)) \mid^2\} \tag{8}$$

$$\forall x \in X_k^{\bar{u}}(x_0), \forall k \geq 0, \forall x_0 \in X_0$$

where $\bar{u}(x)$ is the control value for state x.

Theorem 3 *Let $u \in S$. The system Σ^u is finite gain if and only if it is finite gain dissipative.*

Proof:

(i) Assume Σ^u is finite gain dissipative. Then equation (8) implies

$$V(x_0) \geq V(r_k) - \gamma^2 \sum_{i=0}^{k-1} \mid r_{i+1} - s_{i+1} \mid^2 + \sum_{i=0}^{k-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2,$$

$$\forall k > 0; \quad \forall r, s \in \,,\, ^u(x_0)$$

This implies

$$V(x_0) + \gamma^2 \sum_{i=0}^{k-1} \mid r_{i+1} - s_{i+1} \mid^2 \geq V(r_k) + \sum_{i=0}^{k-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2$$

Since $V \geq 0$ for all $x \in X_k^u(x_0)$, this implies

$$\sum_{i=0}^{k-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 \leq \gamma^2 \sum_{i=0}^{k-1} \mid r_{i+1} - s_{i+1} \mid^2 + V(x_0)$$

Thus Σ^u is finite gain.

(ii) Assume Σ^u is finite gain. For any $x_0 \in X_0$ and $k \geq 0$, define for $x \in X_k^u(x_0)$

$$\tilde{V}_{k,j}^u(x, x_0) = \sup_{r,s \in \Gamma^u(x)} \{ \sum_{i=0}^{j-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 - \gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \}$$

Then we have for any $x \in X_k^u(x_0)$

$$0 \leq \tilde{V}_{k,j}^u(x, x_0) \leq \beta^u(x_0), \quad \forall j \geq 0$$

Furthermore

$$\tilde{V}_{k,j+1}^u(x, x_0) \geq \tilde{V}_{k,j}^u(x, x_0)\ , \ \forall x \in X_k^u(x_0)$$

Furthermore, note that by time invariance, $\tilde{V}_{k,j}^u(x, x_0)$ depends only on x and j. Thus if $x \in X_{k_1}^u(x_0^1) \bigcap X_{k_2}^u(x_0^2)$ then $\tilde{V}_{k_1,j}^u(x, x_0^1) \equiv \tilde{V}_{k_2,j}^u(x, x_0^2)$. Hence,

$$\tilde{V}_{k,j}^u(x, x_0) \longrightarrow V^u(x), \quad \text{as } k \longrightarrow \infty\ , \ \forall x \in x_k^u(x_0), k \geq 0, x_0 \in X_0$$

Also, we have

$$0 \leq V^u(x_0) \leq \beta^u(x_0)$$

Since

$$V(x) = \inf_{u \in S} V^u(x) = V^{\bar{u}}(x) \leq V^u(x)$$

dynamic programming implies that $V^u(x)$ solves the dissipation inequality (8) for all $x \in X_k^u(x_0)$, $k \geq 0$, $x_0 \in X_0$. Furthermore $V^u(x) \geq 0$ and $V^u(0) = 0$. Thus V^u is a storage function and hence Σ^u is finite gain dissipative.

□

We now have to show that the control policy $u \in S_{[0,\infty)}$ which renders Σ^u finite gain dissipative, also guarantees ultimate boundedness of trajectories, and furthermore under a certain *detectability* type assumption, the existence of a sequence $\alpha_n \in \mathcal{F}(x_n, u_n)$ such that $\lim_{n\to\infty} \alpha_n = 0$. The above can be also expressed as [2]

$$0 \in \liminf_{k\longrightarrow\infty} \mathcal{F}(x_k, u_k)$$

We, now study the convergence of

$$W_i^{\bar{u}} = \sup_{r\in\mathcal{F}(\bar{x}_i,\bar{u}_i)} (\mid l(r,\bar{u}_i) - l(\bar{x}_{i+1},\bar{u}_i) \mid^2 - \gamma^2 \mid r - \bar{x}_{i+1} \mid^2)$$

to zero, where $\bar{x}$ is a trajectory generated by the control $\bar{u}$.

Lemma 2 *If $W_k^{\bar{u}} \longrightarrow 0$, as $i \longrightarrow \infty$, then $\forall\epsilon > 0$, $\exists K$ such that $\forall k \geq K$, $\exists\delta$ such that*

$$\mid r - \bar{x}_{k+1} \mid < \delta \quad \Longrightarrow \quad \mid l(\bar{x}_{k+1},\bar{u}_k) - l(r,\bar{u}_k) \mid < \epsilon$$

Proof:
Suppose to the contrary. Then $\exists\epsilon > 0$ such that, $\forall K$, $\exists k \geq K$, such that $\forall\delta > 0$

$$\mid r - \bar{x}_{k+1} \mid < \delta \Longrightarrow \mid l(\bar{x}_{k+1},\bar{u}_k) - l(r,\bar{u}_k) \mid \geq \epsilon$$

Fix δ such that $0 < \delta < \bar{\epsilon}$ and $\delta < \sqrt{\epsilon}$. Then for any $s \in B_{\frac{\delta}{\gamma}}(\bar{x}_{k+1}) \bigcap \mathcal{F}(\bar{x}_k, u_k) \subset \mathcal{F}(\bar{x}_k, u_k)$

$$\mid l(\bar{x}_{k+1},u_k) - l(s,\bar{u}_k) \mid^2 - \gamma^2 \mid \bar{x}_{k+1} - s \mid^2 \geq \epsilon - \delta^2 = \eta$$

This contradicts the convergence of $W_k^{\bar{u}}$.

□

Remark: The above lemma gives a necessary condition for the sequence $W_k^{\bar{u}}$ to converge.

Lemma 3 *If $W_k^{\bar{u}} \longrightarrow 0$, then $\forall\epsilon$, $\hat{\epsilon} > 0$, $\bar{\epsilon} > \epsilon > 0$, $\exists K$ such that $\forall k \geq K$, $\exists r \in B_\epsilon(\bar{x}_{k+1}) \bigcap \mathcal{F}(\bar{x}_k,\bar{u}_k)$ with $r \neq \bar{x}_{k+1}$ and*

$$\frac{\mid l(r,\bar{u}_k) - l(\bar{x}_{k+1},\bar{u}_k) \mid}{\mid r - \bar{x}_{k+1} \mid} < \gamma + \hat{\epsilon} \tag{9}$$

Proof:
By contradiction. $\exists\hat{\epsilon}$, $\epsilon > 0$, $\bar{\epsilon} > \epsilon > 0$, such that $\forall K\ \exists k \geq K$ such that

$$\frac{| \, l(r, \bar{u}_k) - l(\bar{x}_{k+1}, \bar{u}_k) \, |}{| \, r - \bar{x}_{k+1} \, |} \geq \gamma + \hat{\epsilon}, \ \forall r \in B_\epsilon(\bar{x}_{k+1}) \bigcap \mathcal{F}(\bar{x}_k, \bar{u}_k), \ r \neq \bar{x}_{k+1}$$

Hence, $\exists\eta > 0$ such that

$$| \, l(r, \bar{u}_k) - l(\bar{x}_{k+1}, \bar{u}_k) \, |^2 - \gamma^2 \, | \, r - \bar{x}_{k+1} \, |^2 \geq \eta \, | \, r - \bar{x}_{k+1} \, |^2$$

Let $r \in B_\epsilon(\bar{x}_{k+1}) \bigcap \mathcal{F}(\bar{x}_k, \bar{u}_k)$ be such that $\epsilon > | \, r - \bar{x}_{k+1} \, | > \frac{\epsilon}{2}$. Thus,

$$| \, l(r, \bar{u}_k) - l(\bar{x}_{k+1}, \bar{u}_k) \, |^2 - \gamma^2 \, | \, r - \bar{x}_{k+1} \, |^2 \geq \eta \frac{\epsilon^2}{4} = \hat{\eta}$$

Hence, $\exists\hat{\eta} > 0$ such that $\forall K$, $\exists k \geq K$ such that

$$W_k^{\bar{u}} \geq \hat{\eta}$$

Hence, we get a contradiction.

□

Corollary 1 *If* $W_k^{\bar{u}} \longrightarrow 0$, *then*

$$\lim_{k \longrightarrow \infty} | \, \frac{\partial}{\partial x} l(\bar{x}_{k+1}, \bar{u}_k) \, | \leq \gamma$$

Proof:
Take the limit in equation (9) as ϵ, $\hat{\epsilon} \longrightarrow 0$.

□

Before we can prove weak asymptotic stability, we need the following additional assumption on the system Σ.

A: Assume that for a given $\gamma > 0$, the system $\Sigma^{\bar{u}}$ is such that

$$\lim_{k \longrightarrow \infty} | \, \frac{\partial}{\partial x} l(\bar{x}_{k+1}, \bar{u}_k) \, | \leq \gamma$$

implies $0 \in \liminf_{k \longrightarrow \infty} \mathcal{F}(\bar{x}_k, \bar{u}_k)$.

Remark: The assumption above, can be viewed to be analogous to the *detectability* assumption often encountered in H_∞ control literature e.g. [18],[11].

The following theorem gives a sufficient condition for weak asymptotic stability.

Theorem 4 *If for a given $\gamma > 0$, $\Sigma^{\bar{u}}$ is finite gain dissipative and satisfies assumption A, then $\Sigma^{\bar{u}}$ is weakly asymptotically stable.*

Proof:
From the dissipation inequality (equation 8), we obtain for any $x_0 \in X_0$

$$\sum_{i=0}^{K} \mid l(r_{i+1}, \bar{u}_i) - l(s_{i+1}, \bar{u}_i) \mid^2 - \gamma^2 \mid r_{i+1} - s_{i+1} \mid^2 \leq V(x_0), \quad \forall K;\ r, s \in , ^{\bar{u}}(x_0).$$

In particular for any $x \in , ^{\bar{u}}(x_0)$

$$\sum_{k=0}^{K} W_k^{\bar{u}} \leq V(x_0), \quad \forall K$$

We know that $W_k^{\bar{u}} \geq 0$, $\forall k$. This implies that

$$W_k^{\bar{u}} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty$$

Hence, by corollary 1 and assumption A, we obtain

$$0 \in \liminf_{k \longrightarrow \infty} \mathcal{F}(\bar{x}_k, \bar{u}_k)$$

This implies that $\exists \alpha_n \in \mathcal{F}(\bar{x}_n, \bar{u}_n)$ such that $\lim_{n \longrightarrow \infty} \alpha_n = 0$.
Hence, $\forall x \in , ^{\bar{u}}(x_0)$, $\exists \alpha_n \in \mathcal{F}(x_n, \bar{u}_n)$ such that $\lim_{n \longrightarrow \infty} \alpha_n = 0$.

□

Corollary 2 *If $\Sigma^{\bar{u}}$ is finite gain dissipative, then $\Sigma^{\bar{u}}$ is ultimately bounded.*

Proof: In the proof of theorem 4, we observe that if $\Sigma^{\bar{u}}$ is finite gain dissipative, then

$$W_k^{\bar{u}} \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

Hence, by corollary 1

$$\lim_{k \longrightarrow \infty} \left| \frac{\partial}{\partial x} l(x_{k+1}, \bar{u}_k) \right| \leq \gamma$$

Which implies that $\lim_{k \longrightarrow \infty} x_k \in \mathcal{L}^\gamma$, which is bounded by assumption 5.

□

Remark: Furthermore, if we impose sufficient smoothness assumptions on $\Sigma^{\bar{u}}$, such that V is continuous, then all trajectories generated by $\Sigma^{\bar{u}}$ are stable in the sense of lyapunov. In particular V then becomes a lyapunov function.

Remark: It is clear from above and from lemma 2, that we do need some form of continuity assumption on l as a necessary condition for the system to be finite gain dissipative.

3.6 Solution to the State Feedback Robust Control Problem

Although, the results above indicate that the controlled dissipation inequality is both a necessary and sufficient condition for the solvability of the state feedback robust control problem, we state the necessary and sufficient conditions in terms of dynamic programming equalities.

Theorem 5 *(Necessity) If a controller $\bar{u} \in S$ solves the state feedback robust control problem, then there exists a function $V(x)$ such that $V(x) \geq 0$, $V(0) = 0$ and V satisfies the following equation i.e.*

$$V(x) = \inf_{u \in U} \sup_{r,s \in \mathcal{F}(x,u)} \{|l(r,u) - l(s,u)|^2 - \gamma^2 |r - s|^2 + V(r)\} \tag{10}$$

$x \in X_k^{\bar{u}}(x_0),\ k \geq 0,\ x_0 \in X_0.$

Proof: Construct a sequence V_j, $j = 0, \dots$ of functions as follows

$$\begin{aligned} V_{j+1}(x) &= \inf_{u \in U} \sup_{r,s \in \mathcal{F}(x,u)} \{|l(r,u) - l(s,u)|^2 - \gamma^2 |r - s|^2 + V_j(r)\} \\ V_0(x) &= 0 \end{aligned}$$

Clearly,

$$V_j(x) \geq 0\ ,\ \forall x \in \mathbf{R}^n,\ \forall j \geq 0$$

and

$$V_{j+1}(x) \geq V_j(x)\ ,\ \forall x \in \mathbf{R}^n,\ j \geq 0$$

For any $x_0 \in X_0$ and $k \geq 0$, pick an $x \in X_k^{\bar{u}}(x_0)$. Then dynamic programming arguments imply that

$$0 \leq V_j(x) \leq \beta^{\bar{u}}(x_0)\ ,\quad \forall x \in X_k^{\bar{u}}(x_0)$$

Furthermore, note that $V_j(x)$ depends only on j and x. Hence,

$$V_j(x) \longrightarrow V(x) \text{ as } j \longrightarrow \infty\ ,\ \forall x \in X_k^{\bar{u}}(x_0),\ k \geq 0,\ x_0 \in X_0$$

and by definition, V satisfies equation (10). Furthermore, $V(x) \geq 0$ and $V(x_0) \leq \beta^{\bar{u}}(x_0)$. Hence, $V(0) = 0$.

□

Theorem 6 *(Sufficiency) Assume that there exists a solution V to the stationary dynamic programming equation (10) for all $x \in \mathbf{R}^n$, satisfying $V(x) \geq 0$ and $V(0) = 0$. Let $\bar{u}(x)$ be the control value which achieves the minimum in equation (10). Then $\bar{u} \in S$ solves the state feedback robust control problem provided that $\Sigma^{\bar{u}}$ satisfies assumption A.*

Proof: Since V satisfies equation (10), $\Sigma^{\bar{u}}$ satisfies equation (8) with equality. Hence, $\Sigma^{\bar{u}}$ is finite gain dissipative, and hence by theorem 3, $\Sigma^{\bar{u}}$ is finite gain. Furthermore, by theorem 4 $\Sigma^{\bar{u}}$ is weakly asymptotically stable and by corollary 2 $\Sigma^{\bar{u}}$ is ultimately bounded.

□

Corollary 3 *If $X_0 = \mathbf{R}^n$, then the existence of a solution to the stationary dynamic programming equation (10) for all $x \in \mathbf{R}^n$, is both a necessary and sufficient condition for the existence of a solution to the state feedback robust control problem.*

Remark: It can be seen from the statement of theorem 5 and the proof of theorem 6, that we could have expressed the necessary and sufficient conditions for the solvability of the state feedback robust control problem in terms of dissipation inequalities.

4 Output Feedback Case

We now consider the output feedback robust control problem. We denote the set of control policies as O. Hence, if $u \in O$, then $u_k = f(y_{1,k}, u_{0,k-1})$.

4.1 Finite Time

Given $\gamma > 0$, and a finite time interval $[0, K]$, find a control policy $u \in O_{0,K-1}$, such that there exists a finite quantity $\beta_K^u(x)$ with $\beta_K^u(0) = 0$ and

$$\sum_{i=0}^{K-1} \mid l(r_{i+1}, u_i) - l(s_{i+1}, u_i) \mid^2 \leq \gamma^2 \sum_{i=0}^{K-1} \mid r_{i+1} - s_{i+1} \mid^2 + \beta_K^u(x_0),$$

$$\forall r, s \in \, {}^{u}_{0,K}(x_0), \ \forall x_0 \in X_0$$

We introduce for convenience the following notation.

$$\begin{aligned} \Delta^{u}_{1,K}(x_0) &= \{y_{1,K} \mid y_{k+1} \in \mathcal{G}(x_k, u_k),\ \forall x \in ,\ {}^{u}_{0,K-1}(x_0)\} \\ ,\ {}^{u,y}_{0,K}(x_0) &= \{x_{0,K} \in ,\ {}^{u}_{0,K}(x_0) \mid y_{k+1} \in \mathcal{G}(x_k, u_k),\ k = 0, \ldots, K-1\} \end{aligned}$$

4.2 Dynamic Game

In this section, we transform the output feedback robust control problem to a dynamic game. We introduce the function space

$$\mathcal{E} = \{p : \mathbf{R}^n \longrightarrow \mathbf{R}^*\}$$

and define for each $x \in \mathbf{R}^n$ a function $\delta_x : \mathbf{R}^n \longrightarrow \mathbf{R}^*$ by

$$\delta_x(\xi) \triangleq \begin{cases} 0 & \text{if } \xi = x \\ -\infty & \text{if } \xi \neq x \end{cases}$$

For $u \in O_{0,K-1}$, and $p \in \mathcal{E}$ define a functional $J_{p,k}(u)$ by

$$J_{p,k}(u) \triangleq \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^u(x_0)} \{p(x_0) + \sum_{i=1}^{k} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 - \gamma^2 \mid s_i - r_i \mid^2\} \quad (11)$$

for $k = 0, \ldots, K$.

Remark: The functional $p \in \mathcal{E}$ in equation (11) can be chosen to reflect any *a priori* knowledge concerning the initial state x_0 of Σ^u.

The finite gain property of Σ^u can now be expressed in terms of J as follows.

Lemma 4 *$\Sigma^u_{x_0}$ is finite gain on [0,K] if and only if there exists a finite quantity $\beta^u_K(x_0)$, $\beta^u_K(0) = 0$, such that*

$$J_{\delta_{x_0},k}(u) \leq \beta^u_K(x_0), \quad k = 0, \ldots, K$$

For notational convenience, we introduce the following pairing

$$(p, q) = \sup_{x \in \mathbf{R}^n} \{p(x) + q(x)\}$$

and a restricted version

$$(p, q \mid X) = \sup_{x \in X} \{p(x) + q(x)\}$$

Lemma 5 *If each map $\Sigma^u_{x_0}$ is finite gain on $[0, K]$, then*

$$(p, 0 \mid X_0) \leq J_{p,K}(u) \leq (p, \beta^u_K \mid X_0)$$

Proof:
Set $r = s \in$, $^u(x_0)$ in equation (11). Then clearly

$$(p, 0 \mid X_0) \leq J_{p,K}(u)$$

Since, $\Sigma^u_{x_0}$ is finite gain on $[0, K]$ for all $x_0 \in X_0$, this implies that for any $x_0 \in X_0$

$$p(x_0) + \sum_{i=1}^{K} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 - \gamma^2 \mid s_i - r_i \mid^2 \leq p(x_0) + \beta^u_K(x_0) \leq (p, \beta^u_K \mid X_0)$$

Hence, $J_{p,K}(u) \leq (p, \beta^u_K \mid X_0)$.

□

Thus, we can define

$$dom\ J_{p,K}(u) = \{p \in \mathcal{E} : (p, 0 \mid X_0), (p, \beta^u_K \mid X_0) \text{ is finite } \}$$

The finite time output feedback dynamic game is to find a control policy $u \in O_{0,K-1}$, which minimizes each functional $J_{\delta_{x_0},K}$.

4.3 Information State Formulation

Motivated by results obtained in the set-valued stochastic control problem [4], for a fixed $y_{1,k} \in \Delta^u_{1,k}(X_0)$, and $u_{1,k-1}$, we define the *information state* $p_k \in \mathcal{E}$ by

$$p_k(x) \triangleq \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^{u,y}_{0,k}(x_0)} \{p_0(x_0) + \sum_{i=1}^{k} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 - \gamma^2 \mid r_i - s_i \mid^2 \mid r_k = x\} \tag{12}$$

Here, the convention is that the supremum over an empty set is $-\infty$. Furthermore, for convenience we redefine p_0 as

$$p_0(x) = \begin{cases} p_0(x) & \text{, if } x \in X_0 \\ -\infty & \text{, else} \end{cases}$$

Clearly, if Σ^u is finite gain, then

$$-\infty \leq p_k(x) \leq (p_0, \beta^u_K) < +\infty$$

and a finite lower bound for $p_k(x)$ is obtained for all *feasible* $x \in \mathbf{R}^n$.

Now, define $H(p,u,y) \in \mathcal{E}$ by

$$H(p,u,y)(x) \triangleq \sup_{\xi \in \mathbf{R}^n} \{p(\xi) + B(\xi,x,u,y)\}$$

where the function B is defined by

$$B(\xi,x,v,y) \triangleq \begin{cases} \sup_{s \in \mathcal{F}(\xi,v)} \{\mid l(x,v) - l(s,v) \mid^2 - \gamma^2 \mid x-s \mid^2\} & \text{if } \begin{cases} x & \in & \mathcal{F}(\xi,v) \\ y & \in & \mathcal{G}(\xi,v) \end{cases} \\ -\infty & \text{else} \end{cases}$$

Lemma 6 *The information state is the solution of the following recursion*

$$\begin{cases} p_{k+1} & = & H(p_k,u_k,y_{k+1}), \ \ k = 0,\ldots,K-1 \\ p_0 & \in & \mathcal{E} \end{cases} \tag{13}$$

Proof:
We use induction. Assume that (13) is true for $0,\ldots,k$; we must show that p_{k+1} defined by (12) equals $H(p_k,u_k,y_{k+1})$. Now

$$\begin{aligned} H(p_k,u_k,y_{k+1})(x) &= \sup_{\xi \in \mathbf{R}^n} \{p_k(\xi) + B(\xi,x,u_k,y_{k+1})\} \\ &= \sup_{\xi \in \mathbf{R}^n} \{p_k(\xi) + \sup_{s \in \mathcal{F}(\xi,u_k)} (\mid l(x,u_k) - l(s,u_k) \mid^2 - \\ &\quad \gamma^2 \mid x - s \mid^2 \mid \ \ y_{k+1} \in \mathcal{G}(\xi,u_k), x \in \mathcal{F}(\xi,u_k)\} \\ &= p_{k+1}(x) \end{aligned}$$

by the definition (12) for p_k, and p_{k+1}.

□

Remark: Note that we can write

$$p_k(x) = \sup_{\xi \in \Gamma_{0,k}^{u,y}(X_0)} \{p_0(\xi_0) + \sum_{i=0}^{k-1} B(\xi_i,\xi_{i+1},u_i,y_{i+1}) \ \mid \ \xi_k = x\}$$

for $k = 1,\ldots,K$.

Remark: The relationship between the information state and the indicator function of the *feasible sets* was established in [4]. In particular, it was established that if $p_0 = \delta_{x_0}$, then $p_k(x) \geq 0$ if and only if $x \in \mathcal{X}_k^{y,u}(x_0)$, where $\mathcal{X}_k^{y,u}(x_0)$ is the set of feasible states at time k, given $u_{0,k-1}$ and $y_{1,k}$.

Theorem 7 *For $u \in O_{0,k-1}$, $p \in \mathcal{E}$, such that $J_{p,k}(u)$ is finite, we have*

$$J_{p,k}(u) = \sup_{y_{1,k} \in \Delta^u(X_0)} \{(p_k, 0) \mid p_0 = p\}, \;\; k \in [0, K] \tag{14}$$

Proof:
We have

$$\begin{aligned}
&\sup_{y_{1,k} \in \Delta^u(X_0)} \{(p_k, 0) \mid p_0 = p\} \\
&\quad = \sup_{y_{1,k} \in \Delta^u(X_0)} \sup_{\xi \in \Gamma^{u,y}_{0,k}(X_0)} \{p(\xi_0) + \sum_{i=0}^{k-1} B(\xi_i, \xi_{i+1}, u_i, y_{i+1})\} \\
&\quad = \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^u_{0,k}(x_0)} \{p(x_0) + \sum_{i=1}^{k} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 \\
&\qquad\qquad - \gamma^2 \mid r_i - s_i \mid^2\} \\
&\quad = J_{p,k}(u)
\end{aligned}$$

□

Remark: This representation theorem is actually a separation principle.

The following corollary enables us to express the finite gain property of Σ^u in terms of the information state p.

Corollary 4 *For any output feedback controller $u \in O_{0,K-1}$, the closed loop system Σ^u is finite gain on $[0, K]$ if and only if the information state p_k satisfies*

$$\sup_{y_{1,k} \in \Delta^u(X_0)} \{(p_k, 0) \mid p_0 = \delta_{x_0}\} \leq \beta^u_K(x_0), \;\; \forall k \in [0, K]$$

for some finite $\beta^u_K(x_0)$, with $\beta^u_K(0) = 0$.

□

Remark: Thus the name *information state* for p is justified, since p_k contains all the information relevant to the finite gain property of Σ^u that is available in the observations $y_{1,k}$.

The information state dynamics (13) may be regarded as a new (infinite dimensional) control system Ξ, with control u and uncertainty parameterized by y. The state p_k, and the disturbance y_k are available to the controller, so the original output feedback dynamic game is equivalent to a new game with *full information.* The cost is now given by (14).

We now need an appropriate class $I_{i,K}$ of controllers, which feedback this new state variable. A control u belongs to $I_{i,K}$, if for each $k \in [i, K]$, there exists a map $\bar{u}_k$ from a subset of $\mathcal{E}^{k-i+1}$ (sequences $p_{i,k}$) into U, such that $u_k = \bar{u}(p_{i,k})$. Note that since p_k depends on the observable information $y_{i,k}$, $I_{0,k-1} \subset O_{0,k-1}$.

4.4 Solution to the Finite Time Output Feedback Robust Control Problem

We use dynamic programming to solve the game. Define the value function by

$$M_k(p) = \inf_{u \in O_{0,k-1}} \sup_{y \in \Delta^u_{1,k}(X_0)} \{(p_k, 0) \mid p_0 = p\} \tag{15}$$

for $k \in [0, K]$, and the corresponding dynamic programming equation is

$$M_k(p) = \inf_{u \in U} \sup_{y \in \mathbf{R}^t} \{M_{k-1}(H(p, u, y))\},\ k \in [1, K] \tag{16}$$

with the initial condition

$$M_0(p) = (p, 0)$$

Remark: In the above equations, we have inverted the time index to enable ease of exposition when dealing with the infinite time case. Since, the system is assumed to be time invariant, it does not matter if we write the equations as above, or as

$$\tilde{M}_k(p) = \inf_{u \in U} \sup_{y \in \mathbf{R}^t} \{\tilde{M}_{k+1}(H(p, u, y))\},\ k \in [0, K-1]$$

with the initial condition

$$\tilde{M}_K(p) = (p, 0)$$

as far as we invert the index of the control policy obtained by solving equation(16).

Define for a function $M : \mathcal{E} \longrightarrow \mathbf{R}^*$,

$$dom\ M = \{p \in \mathcal{E} \mid M(p)\ finite\}$$

Theorem 8 *(Necessity) Assume that $\bar{u} \in O_{0,K-1}$ solves the finite time output feedback robust control problem. Then there exists a solution M to the dynamic programming equation (16) such that dom $J_{p,K}(\bar{u}) \subset$ dom M_k, $M_k(\delta_0) = 0$, $M_k(p) \geq (p, 0)$, $k \in [0, K]$.*

Proof:
For $p \in dom\ J_{p,K}(\bar{u})$, define $M_k(p)$ by (15). Then

$$M_k(p) = \inf_{u \in O_{0,k-1}} J_{p,k}(u)$$

Now, we also have

$$M_k(p) = \inf_{u \in O_{0,k-1}} \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^u_{0,k}(x_0)} \{p(x_0) + \sum_{i=1}^{k} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 - \gamma^2 \mid s_i - r_i \mid^2\}$$

For $u = \bar{u}$, by using the finite gain property for $\Sigma^{\bar{u}}$ we get

$$M_k(p) \leq \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^{\bar{u}}_{0,k}(x_0)} \{p(x_0) + \sum_{i=1}^{k} \mid l(s_i, \bar{u}_{i-1}) - l(r_i, \bar{u}_{i-1}) \mid^2 - \gamma^2 \mid s_i - r_i \mid^2\} \leq (p, \beta^{\bar{u}}_K)$$

Thus, $dom\ J_{p,K}(\bar{u}) \subset dom\ M_k$. Also

$$M_k(p) \geq (p, 0)$$

Since, $\beta^{\bar{u}}_K(0) = 0$, and $(\delta_0, 0) = 0$, we have that $M_k(\delta_0) = 0$.

□

Theorem 9 *(Sufficiency) Assume there exists a solution M to the dynamic programming equation (16) such that $\delta_x \in dom\ M_k$ for all $x \in X_0$, $M_k(\delta_0) = 0$, $M_k(p) \geq (p, 0)$, $k \in [0, K]$. Let $\bar{u}^* \in I_{0,K-1}$ be a policy such that $u^*_k = \bar{u}^*_{K-k}(p_k)$, where $\bar{u}^*_k(p)$ achieves the minimum in (16); $k = 0, \ldots, K-1$. Then u^* solves the finite time output feedback robust control problem.*

Proof:
We see that

$$M_K(p) = J_{p,K}(u^*) \leq J_{p,K}(u)$$

for all $u \in O_{0,K-1}$, $p \in dom\ M_K$. Now

$$\sup_{y \in \Delta^{u^*}_{1,K}(X_0)} \{(p_K, 0) \mid p_0 = \delta_{x_0}\} \leq M_K(\delta_{x_0})$$

which implies by corollary 4 that Σ^{u^*} is finite gain with $\beta^{u^*}_K(x_0) = M_K(\delta_{x_0})$, and hence u^* solves the finite time output feedback robust control problem.

□

Corollary 5 *If the finite time output feedback robust control problem is solvable by an output feedback controller $\bar{u} \in O_{0,K-1}$, then it is also solvable by an information state feedback controller $u^* \in I_{0,K-1}$.*

□

4.5 Infinite Time Case

We pass to the limit as $K \longrightarrow \infty$ in the dynamic programming equation (16).

$$\lim_{k \longrightarrow \infty} M_k(p) = M(p)$$

where $M_k(p)$ is defined by (15), to obtain a stationary version of equation (16)

$$M(p) = \inf_{u \in U} \sup_{y \in \mathbf{R}^t} \{M(H(p,u,y))\} \tag{17}$$

4.6 Dissipation Inequality

The following lemma is a consequence of corollary 4.

Lemma 7 *For any $u \in O$, the closed loop system Σ^u is finite gain if and only if the information state satisfies*

$$\sup_{k \geq 1} \sup_{y \in \Delta^u_{1,k}(x_0)} \{(p_k, 0) \mid p_0 = \delta_{x_0}\} \leq \beta^u(x_0) \tag{18}$$

for some finite $\beta^u(x_0)$, with $\beta^u(0) = 0$.

□

By using lemma 7 we say that the information state system Ξ^u ((13) with information state feedback $u \in I$) is finite gain if and only if the information state p_k satisfies (18) for some finite $\beta^u(x_0)$, with $\beta^u(0) = 0$. If Σ^u is finite gain, we write

$$dom\ J_p(u) = \{p \in \mathcal{E} \mid (p,0), (p, \beta^u) \text{ finite}\}$$

where $J_p(u) = \sup_{k \geq 0} J_{p,k}(u)$.

We say that the information state system $\Xi^{\bar{u}}$ is *finite gain dissipative* if there exists a function (storage function) $M(p)$, such that $dom\ M$ contains δ_x for all $x \in X_0$, $M(p) \geq (p, 0)$, $M(\delta_0) = 0$, and satisfies the following dissipation inequality

$$M(p) \geq \sup_{y \in \mathbf{R}^t} \{M(H(p, \bar{u}(p), y))\} \tag{19}$$

Note that if $\Xi^{\bar{u}}$ is finite gain dissipative, and $p \in dom\ M$, then $H(p, \bar{u}(p), y) \in dom\ M$ for all $y \in \mathbf{R}^t$. Consequently, $p_0 \in dom\ M$, implies $p_k \in dom\ M$, $\forall k > 0$.

Lemma 8 *M_k is monotone non-decreasing. i.e.*

$$M_{k-1}(p) \leq M_k(p)$$

Proof:
Note that

$$M_k(p) = \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^u_{0,k}(x_0)} \{p(x_0) + \sum_{i=1}^{k} \mid l(r_i, u_{i-1}) - l(s_i, u_{i-1}) \mid^2 - \gamma^2 \mid s_i - r_i \mid^2\}$$

Then for any $\epsilon > 0$, choose $x_0^{'} \in X_0$, and $r^{'}, s^{'} \in \Gamma^u_{0,k-1}(x_0^{'})$ such that

$$M_{k-1}(p) \leq p(x_0^{'}) + \sum_{i=1}^{k-1} \mid l(r_i^{'}, u_{i-1}) - l(s_i^{'}, u_{i-1}) \mid^2 - \gamma^2 \mid r_i^{'} - s_i^{'} \mid^2 + \epsilon$$

Let $x_0 = x_0^{'}$, and define $r, s \in \Gamma^u_{0,k}(x_0)$ by $r = r^{'}$, $s = s^{'}$ on $[0, k-1]$, and $r_k = s_k$. Then

$$\begin{aligned} M_k(p) &\geq p(x_0) + \sum_{i=1}^{k} \mid l(r_i, u_{i-1}) - l(s_i, u_{i-1}) \mid^2 - \gamma^2 \mid r_i - s_i \mid^2 \\ &\geq p(x_0^{'}) + \sum_{i=1}^{k-1} \mid l(r_i^{'}, u_{i-1}) - l(s_i^{'}, u_{i-1}) \mid^2 - \gamma^2 \mid r_i^{'} - s_i^{'} \mid^2 + \\ &\qquad \mid l(r_k, u_{k-1}) - l(s_k, u_{k-1}) \mid^2 \\ &\geq M_{k-1}(p) - \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, letting $\epsilon \longrightarrow 0+$ gives

$$M_k(p) \geq M_{k-1}(p)$$

□

We are now in a position to prove a version of the bounded real lemma for the information state system Ξ.

Theorem 10 *Let $u \in I$. Then the information state system Ξ^u is finite gain if and only if it is finite gain dissipative.*

Proof:
(i)
Assume that Ξ^u is finite gain dissipative. Then by the dissipation inequality (19)

$$M(p_k) \leq M(p_0), \ \forall k > 0, \ \forall y \in \Delta^u_{1,k}(X_0)$$

Setting $p_0 = \delta_{x_0}$, and using the fact that $M(p) \geq (p, 0)$, we get

$$(p_k, 0) \leq M(\delta_{x_0}),\ \forall k > 0, \forall y \in \Delta^u_{1,k}(x_0)$$

Therefore Ξ^u is finite gain, with $\beta^u(x_0) \stackrel{\triangle}{=} M(\delta_{x_0})$.

(ii)
Assume Ξ^u is finite gain. Then

$$(p, 0) \leq J_{p,k}(u) \leq (p, \beta^u),\ \forall k \geq 0, p \in dom\ J_p(u)$$

Writing $M_k(p) = J_{p,k}(u)$, so that

$$(p, 0) \leq M_k(p) \leq (p, \beta^u),\ k \geq 0, p \in dom\ J_p(u)$$

By lemma 8, M_k is monotone non-decreasing. Therefore

$$M_a(p) = \lim_{k \longrightarrow \infty} M_k(p)$$

exists, and is finite on $dom\ M_a$, which contains $dom\ J_p(u)$.

To show that M_a satisfies the dissipation inequality (18), fix $p \in dom\ M_a$, $y \in \mathbf{R}^t$, and $\epsilon > 0$. Select $k > 0$, and $\tilde{y}_{1,k-1}$ such that

$$M_a(H(p, u(p), y)) \leq (\tilde{p}_{k-1}, 0) + \epsilon$$

where, $\tilde{p}_j$, $j = 0, \ldots, k-1$ is the information state trajectory generated by $\tilde{y}$, with $\tilde{p}_0 = H(p, u(p), y)$.

Define

$$y_i = \begin{cases} y & ,\text{if } i = 1 \\ \tilde{y}_{i-1} & ,\text{if } i = 2, \ldots, k \end{cases}$$

and let p_j, $j = 0, \ldots, k$ denote the corresponding information state trajectory with $p_0 = p$. Then

$$\begin{aligned} M_a(p) &\geq (p_k, 0) \\ &= (\tilde{p}_{k-1}, 0) \\ &\geq M_a(H(p, u(p), y)) - \epsilon \end{aligned}$$

Since, y and ϵ are arbitrary, we have

$$M_a(p) \geq \sup_{y \in \mathbf{R}^t} M_a(H(p, u(p), y))$$

Hence, M_a solves the dissipation inequality. Also, by definition $(p, 0) \leq M_a(p)$. This and (18) imply that $M_a(\delta_0) = 0$. Thus, Ξ^u is finite gain dissipative.

□

We, now again assume that Σ^u satisfies assumption A.

Theorem 11 *Let $u \in I$. If Ξ^u is finite gain dissipative and Σ^u satisfies assumption A, then Σ^u is weakly asymptotically stable.*

Proof:
Inequality (19) implies

$$\sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^u_{0,k}(x_0)} \{p(x_0) + \sum_{i=1}^{k} \mid l(r_i, u_{i-1}) - l(s_i, u_{i-1}) \mid^2 - \gamma^2 \mid s_i - r_i \mid^2\} \leq M(p)$$

for all $k \geq 1$. Let $x_0 \in X_0$, and let $p = \delta_{x_0}$. Then the above gives

$$\sup_{r,s \in \Gamma^u_{0,k}(x_0)} \{\sum_{i=1}^{k} \mid l(r_i, u_{i-1}) - l(s_i, u_{i-1}) \mid^2 - \gamma^2 \mid s_i - r_i \mid^2\} \leq M(p)$$

For any $\bar{r} \in , {}^u_{0,k}(x_0)$, there is a sequence

$$\begin{aligned} W^u_k &= \sup_{g,h \in \mathcal{F}(\bar{r}_k, u_k)} \{\mid l(g, u_k) - l(h, u_k) \mid^2 - \gamma^2 \mid g - h \mid^2\} \\ &\geq 0 \end{aligned}$$

Also, from above we obtain that

$$\sum_{i=0}^{k} W^u_k \leq M(p), \ \forall k \geq 0$$

Hence, $W^u_k \longrightarrow 0$, as $k \longrightarrow \infty$ and by corollary 1 and assumption A

$$0 \in \liminf_{k \longrightarrow \infty} \mathcal{F}(\bar{r}_k, u_k)$$

Hence, Σ^u is weakly asymptotically stable.

□

Corollary 6 *If Σ^u is finite gain dissipative, then Σ^u is ultimately bounded.*

Proof: Similar to that of corollary 2.

□

We also need to show that the information state system Ξ^u is stable.

Theorem 12 *Let $u \in I$. If Ξ^u is finite gain dissipative, then Ξ^u is stable on all feasible $x \in \mathbf{R}^n$.*

Proof:
The dissipation inequality (19) implies that

$$p_k(x) \leq (p_k, 0) \leq M(p_0) < +\infty$$

for all $p_0 \in dom\ M$, and for all $k \geq 0$. For the lower bound, note that by definition (12)

$$p_k(x) = \sup_{x_0 \in X_0} \sup_{r,s \in \Gamma^u_{0,k}(x_0)} \{p_0(x_0) + \sum_{i=1}^{k} \mid l(s_i, u_{i-1}) - l(r_i, u_{i-1}) \mid^2 - \gamma^2 \mid r_i - s_i \mid^2\}$$

For any $x_0 \in X_0$, this implies that for any feasible $x \in \mathbf{R}^n$

$$p_k(x) \geq p_0(x_0) > -\infty,\ \forall k \geq 0$$

Therefore, Ξ^u is stable.

□

4.7 Solution to the Output Feedback Robust Control Problem

As in the state feedback case, it can be inferred from the previous results, that the controlled dissipation inequality (8) is both a necessary and sufficient condition for the solvability of the output feedback robust control problem.

However, we now state necessary and sufficient conditions for the solvability of the output feedback robust control problem in terms of dynamic programming equalities.

Theorem 13 *(Necessity) Assume that there exists a controller $\bar{u} \in O$ which solves the output feedback robust control problem. Then there exists a function $M(p)$, such* *that dom* $J_p(\bar{u}) \subset$ *dom* $M(p)$, $M(p) \geq (p, 0)$, $M(\delta_0) = 0$ *and* M *solves the stationary dynamic programming equation*

$$M(p) = \inf_{u \in U} \sup_{y \in \mathbf{R}^t} \{M(H(p, u, y))\} \tag{20}$$

for all $p \in$ *dom* $J_p(\bar{u})$.

Proof: For $p \in$ *dom* $J_p(\bar{u})$, define $M_k(p)$, $k = 0, \ldots$ as follows

$$\begin{aligned} M_k(p) &= \inf_{u \in U} \sup_{y \in \mathbf{R}^t} M_{k-1}(H(p, u, y)) \\ M_0(p) &= (p, 0) \end{aligned}$$

Clearly

$$(p, 0) \leq M_k(p) \leq (p, \beta^{\bar{u}}) < +\infty, \ \forall p \in dom \ J_p(\bar{u})$$

Furthermore, a modification of lemma 8 establishes that

$$M_{k+1}(p) \geq M_k(p), \quad \forall p \in dom \ J_p(\bar{u})$$

Hence,

$$M_k(p) \longrightarrow M(p) \text{ as } k \longrightarrow \infty$$

and $M(p)$ satisfies equation (20) for all $p \in$ *dom* $J_p(\bar{u})$. Furthermore, *dom* $J_p(\bar{u}) \subset$ *dom* $M(p)$ and $(p, 0) \leq M(p) \leq (p, \beta^{\bar{u}})$. Thus, since $(\delta_0, \beta^{\bar{u}}) = 0$, $M(\delta_0) = 0$.

□

Theorem 14 *(Sufficiency) Assume that there exists a solution* M *to the stationary dynamic programming equation (20) such that* $\delta_x \in$ *dom* M, $\forall x \in X_0$, $M(\delta_0) = 0$, *and* $M(p) \geq (p, 0)$. *Let* $\bar{u} \in I$ *be a policy such that* $\bar{u}(p)$ *achieves the minimum in (20). Then,* $\bar{u} \in I$ *solves the information state feedback robust control problem if the closed loop system* $\Sigma^{\bar{u}}$ *satisfies assumption A.*

Proof: Since M satisfies equation (20), $\Sigma^{\bar{u}}$ satisfies equation (19) with equality. Hence, $\Xi^{\bar{u}}$ is finite gain dissipative and by theorem 10, $\Xi^{\bar{u}}$ is finite gain. Furthermore, theorem 11 establishes that $\Sigma^{\bar{u}}$ is weakly asymptotically stable, and by corollary 6 $\Sigma^{\bar{u}}$ is ultimately bounded. Also by theorem 12, $\Xi^{\bar{u}}$ is stable for all feasible $x \in \mathbf{R}^n$.

□

Remark: As in the state feedback case, we can from the statement of theorem 13 and the proof of theorem 14, obtain necessary and sufficient conditions for the solvability of the robust control problem in terms of dissipation inequalities.

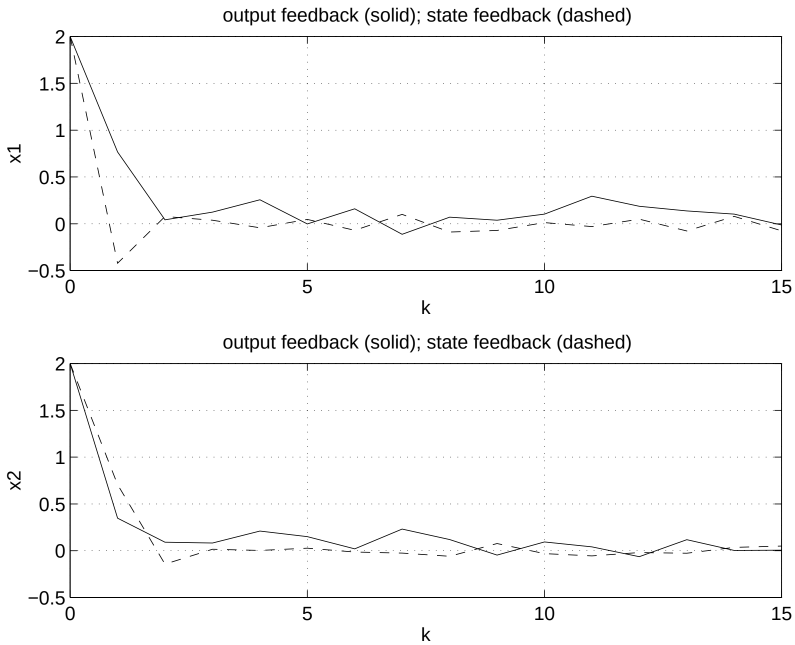

Figure 1: State Trajectories: x^1 (top) and x^2 (bottom)

5 Example

In this section we present a simple example. The system being considered is

$$\begin{aligned} x^1_{k+1} &\in [2,3]x^1_k + x^2_k + 7\sin(u^1_k) + [-0.1, 0.1] \\ x^2_{k+1} &\in 1.5x^1_k + [1,2]x^2_k + 6\sin(u^2_k) + [-0.1, 0.1] \end{aligned}$$

with the measurement equations

$$\begin{aligned} y^1_{k+1} &\in x^1_k + [-0.05, 0.05] \\ y^2_{k+1} &\in [1, 1.1]x^2_k + [-0.05, 0.05] \end{aligned}$$

and the regulated output

$$z_k = | x_k |^2$$

The value of γ is set to 0.8, and the initial state was set to $(2, 2)$.

Figure 1 gives the state trajectories for the output feedback case. The trajectories of the optimal state feedback case are also presented for comparison.

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