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Bonus systems in an open portfolio☆

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Abstract

In this paper, we study bonus systems in an open portfolio, i.e. we consider that a policyholder can transfer his policy to a different insurance company at any time. We make use of inhomogeneous Markov chains to model the system and show, under reasonable assumptions, that the stationary distribution is independent of the market shares, and is easily calculated.

Keywords: Bonus systems; Markov chains; Stationary distribution; Open model; Closed model

1. Introduction

Bonus systems can be developed using the theory of Markov chains. This has already been done by several authors such as Loimaranta (1972), Norberg (1976), Borgan et al. (1981), Gilde and Sundt (1989) and Lemaire (1995). However Loimaranta (1972) and Lemaire (1995) do not follow the view usually adopted in the theory of experience rating, and followed here.

In this paper, we follow closely the work by Norberg (1976), Borgan et al. (1981) and Gilde and Sundt (1989) and generalize it to cope with the Portuguese situation.

There is a big variety of bonus scales in Portugal. Each company can have its own tariff model, including experience rating systems. There are significant moves among different companies, explained not only by the aggressiveness of the market, but also by the lack of data disclosure among insurers. An individual can leave an insurer and buy another policy to a competitor simply declaring that this is the first insurance policy that he is getting. As a result, policyholders who are placed in a severe class will tend to leave the company. Hence, although there is a starting class for the drivers buying (or declaring to buy) an automobile policy for the first time, some policyholders are placed in some other class depending on the claims record reported by the former company or on the commercial aggressiveness policy of the insurance company.

Considering the Portuguese situation, which should be common to other countries, we tried to model the system to include transfers between an insurer and the rest of the market. We will name this model by "open model" as opposed to the model presented in Section 2, which will be named by "closed model". We assume that the structure of the transfers is the same along different periods.

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In Section 2, we summarize the results obtained by Norberg (1976), Borgan et al. (1981) and Gilde and Sundt (1989), in Section 3 we present the "open model" and in Section 4 we give a simplified example using both models.

2. Bonus systems and homogeneous Markov chains

The papers by Norberg (1976), Borgan et al. (1981) and Gilde and Sundt (1989) assume that a bonus system can be dealt in the framework of homogeneous Markov chains.

In their papers a bonus system is supposed to be an experience rating system such that:

1. The insurance periods have equal length (generally 1 year).
2. The policies in the portfolio are divided into a finite number of classes, numbered from 1 to K. A policy stays in the same class during each insurance period.
3. The premium per period of time is $\pi(j)$ for all the policies in class j in the portfolio. The vector $\boldsymbol{\pi} = (\pi(1), \ldots, \pi(K))$ is called the bonus scale.
4. All policies are placed in the same initial class, say k, in the first period.
5. The transition rules are such that the bonus class of a policyholder at any period can be determined as a function of the bonus class and the number of claims reported in the preceding period.

The transition rules are represented by a $K \times K$ matrix $\mathbf{T}$, whose entry $T(i, j)$ is the set of claim numbers leading from class i to class j. When the length period is given, the bonus system is defined by the triplet $S = (\mathbf{T}, \boldsymbol{\pi}, k)$.

It is assumed that $\mathbf{T}$ is such that the Markov chain is irreducible (i.e. all the classes communicate), and that there exists an "elite" class with the property that a policy in that class remains there after a claim free period, which implies that the Markov chain is aperiodic.

Let X_n be the total claim amount, in period n, for a single risk, chosen at random from an automobile portfolio of an insurance company. The accident proneness of the risk, which is represented by the risk parameter θ, is assumed to be picked out of the collective and is regarded as the outcome of a positive random variable Θ with distribution function $U(\cdot)$. It is assumed that, given θ, X_n are i.i.d. random variables and that each of them has a compound distribution, with claim numbers M_n and claim severities $\{Y_{nj}\}_{j=1}^{\infty}$, which are supposed to be i.i.d. and independent of (Θ, M_n).

Let $Z_{S,n}$ denote the bonus class in period n for a given policy. Then, for a given $\Theta = \theta$, $\{Z_{S,n}\}_{n=1}^{\infty}$ is a homogeneous Markov chain, with state space $\{1, 2, \ldots, K\}$, and one-step transition probability matrix $\mathbf{P}_{T,\theta} = [p_{T,\theta}(i, j)]$, with $p_{T,\theta}(i, j) = \Pr\{M_n \in T(i, j)\}$. The conditional distribution of $Z_{S,n}$ is

$$p_{S,\theta}^{(n)}(j) = \Pr_{\theta}(Z_{S,n} = j), \quad j = 1, 2, \ldots, K,$$

which depends on S through $\mathbf{T}$ and k. Since the Markov chain is finite, irreducible and aperiodic, it has a limiting distribution $\mathbf{p}_{T,\theta} = [p_{T,\theta}(j)]$ with

$$p_{T,\theta}(j) = \lim_{n\to\infty} p_{S,\theta}^{(n)}(j), \quad j = 1, \ldots, K, \tag{1}$$

which is the stationary distribution, i.e. the left normalized eigenvector associated to the unit eigenvalue of the matrix $\mathbf{P}_{T,\theta}$.

The unconditional distribution of $Z_{S,n}$ is

$$p_{S}^{(n)}(j) = \Pr\{Z_{S,n} = j\} = \int_0^{\infty} p_{S,\theta}^{(n)}(j)\,\mathrm{d}U(\theta), \quad j = 1, 2, \ldots, K. \tag{2}$$

If Z_T is a random variable with conditional distribution, for a given θ, equal to the limit distribution of $Z_{S,n}$, then the unconditional distribution of Z_T is

$$p_T(j) = \Pr\{Z_T = j\} = \int_0^{\infty} p_{T,\theta}(j)\,\mathrm{d}U(\theta), \quad j = 1, \ldots, K.$$

If we re-scale the claim severity to have mean 1, which means that its expected value is chosen as the monetary unit, and if the claim number distribution is parametrized to have expected value θ, then the value $\pi_T(j)$ that minimizes the mean square error $Q_0(S) = E\{[E(X_n|\theta) - \pi_T(Z_T)]^2\} = E\{[\theta - \pi_T(Z_T)]^2\}$ is $\pi_T(j) = \pi_T^A(j)$ such that

$$\pi_T^A(j) = E[\theta|Z_T = j] = \frac{\int_0^\infty \theta p_{T,\theta}(j)\,\mathrm{d}U(\theta)}{p_T(j)}, \quad j = 1, \dots, K. \tag{3}$$

Eq. (3) defines the optimal bonus scale for the specific transition rules **T**. It does not depend on the initial class k, because an asymptotic criterion was used. This is the bonus scale proposed by Norberg (1976).

Borgan et al. (1981) generalized Norberg (1976) work by introducing a set of non-negative weights w_n, $n = 0, 1, \dots$ and by measuring the performance of S by a weighted average of the form

$$Q(S) = \sum_{n=0}^{\infty} w_n Q_n(S) = \sum_{n=0}^{\infty} w_n \int_0^\infty \sum_{j=1}^{K} [\theta - \pi(j)]^2 p_{S,\theta}^{(n)}(j)\,\mathrm{d}U(\theta), \tag{4}$$

where w_n represents the weight given to period n ($n = 1, 2, \dots$) and w_0 is the weight to be given to the stationary distribution.

The minimizer of (4) is $\pi(j) = \pi_S^B(j)$,

$$\pi_S^B(j) = \frac{\sum_{n=0}^{\infty} w_n \int_0^\infty \theta p_{S,\theta}^{(n)}(j)\,\mathrm{d}U(\theta)}{p_S(j)} \tag{5}$$

with

$$p_S(j) = \sum_{n=0}^{\infty} w_n \int_0^\infty p_{S,\theta}^{(n)}(j)\,\mathrm{d}U(\theta). \tag{6}$$

The solution is now dependent on the starting class k, with exception of the solution when $w_0 > 0$ and $w_n = 0$ for $n > 0$, which corresponds to (3).

Both the former scales can lead to a situation where a more dangerous class of policyholders is paying less than a less dangerous one. Gilde and Sundt (1989) solved this problem by imposing a linear scale, i.e. by minimizing (4) subject to constraints of the form $\pi(j) = a + bj$, $j = 1, \dots, K$, and got that the optimal solution is $\pi_S^L(j) = a^L + b^L j$ with

$$b^L = \frac{\sum_{j=1}^{s} j b_S^B(j) p_S(j) - \sum_{j=1}^{s} j p_S(j) \sum_{j=1}^{s} \pi_S^B(j) p_S(j)}{\sum_{j=1}^{s} j^2 p_S(j) - \left(\sum_{j=1}^{s} j p_S(j)\right)^2}, \quad a^L = \sum_{j=1}^{s} \pi_S^B(j) p_S(j) - b_L \sum_{j=1}^{s} j p_S(j). \tag{7}$$

3. The open model

Let us consider that the collective is now the set of all drivers with an insurance policy in a given country and let us consider a given company. It is natural to assume that its bonus system attracts the policyholders differently, namely due to the existence of different risk proneness. Given θ, let $q_{T,\theta}^{(n)}$ with $0 < q_{T,\theta}^{(n)} < 1$, be the market share of the company at period n. The global market share of the company at period n, is given by $\int q_{T,\theta}^{(n)}\,\mathrm{d}U(\theta)$. In practice, future market shares will depend on the bonus system in force in the company as well as on the systems used by the competing companies, in the same way that they will depend on the a priori tariff structure of the company and its competitors. As such a market looks very difficult to model, we will assume that the market shares are exogenously given, which can be a questionable assumption. However, we will show that the long run distribution of the policyholders among the several classes of the bonus system, and hence the premium scale based on the long run distribution, are independent of these market shares. As a result we do not need to specify their values, unless we do not want to be confined to the long run solution.

The system can now be treated by means of a Markov chain with $K+1$ states, where state labeled $K+1$ refers to the outside world. The first K states correspond to the bonus classes.

We assume that, given θ, the policies when entering the portfolio are placed in the different bonus classes according to the row vector $\mathbf{v}_\theta = [v_\theta(i)]$, where $v_\theta(i)$ is the probability that a new policy is placed in class i, $i = 1, 2, \dots, K$, once it enters the system. In practice, the entry of new policies is made by means of a set of well defined rules, which applies to all policyholders. The probability of a new policy, with risk parameter θ, to be placed in class i depends on θ, because the starting class depends on the past risk performance. In our model the vectors v_θ are given. Their estimation should be done using the past information and the assignment made, according to those rules, to the different bonus classes. However, some difficulties in the estimation process can arise. For example, let us suppose that only the policies with past claim record (in other companies) of no claims for the last 10 years are assigned to a given class i^*. Given θ, and assuming a Poisson process for the number of claims, $v_\theta(i^*)$ is the product of $\mathrm{e}^{-10\theta}$ by the probability that a new policy provides past record information for at least the last 10 years. This last probability can be difficult to estimate. If we assume it to be independent of θ, we will be considering that the policies will be equally likely to provide information when they change companies. This is not certainly the case in Portugal, where it can happen, due to the lack of data disclosure among insurers, that a driver changes company, but declares that he was never insured. In a practical application we might be led to consider that the initial distribution is a vector $\mathbf{v}$ independent of θ (which would be the expected value of $\mathbf{v}_\theta$, calculated over $\theta \in \Theta$), which can easily be estimated.

For each $i = 1, \dots, K$, let $d_{T,m,\theta}(i)$, $0 \leqslant d_{T,m,\theta}(i) < 1$, be the conditional probability, given θ, that a policyholder, who was in class i in the preceding period and declares m accidents in the period, exits the company (we assume that all the exits are made at the end of the period). Let $d_{S,\theta}(i)$ be the conditional probability, given θ, that a policyholder exits the company having been in class i in the preceding period. Again, we assume that these probabilities do not depend on $\boldsymbol{\pi}$. Let $p_\theta(m)$ be the probability that a policyholder with risk parameter θ declares m accidents in a period of time. Then

$$d_{S,\theta}(i) = \sum_{m=0}^{\infty} p_\theta(m) d_{T,m,\theta}(i). \tag{8}$$

The bonus scale $\boldsymbol{\pi}$ has to reflect now the transition rules $\mathbf{T}$ and, for the different values of θ, the vector of entrances $\mathbf{v}_\theta$, the probabilities $d_{T,m,\theta}(i)$, and the given market shares.

Let $\mathbf{P}_{T,\theta} = [p_{T,\theta}(i, j)]$ be the $K \times K$ transition probability matrix in the "closed model" (as in Section 2) and let $\mathbf{A}_{S,\theta}^{(n,n+1)} = [a_{S,\theta}^{(n,n+1)}(i, j)]$ be the $(K+1) \times (K+1)$ transition probability matrix from period n to $n+1$, in the open model.

It is straightforward to see that

$$a_{S,\theta}^{(n,n+1)}(i, j) = \sum_{l \in T(i,j)} p_\theta(l)(1 - d_{T,l,\theta}(i)), \quad i, j = 1, \dots, K. \tag{9}$$

As $a_{S,\theta}^{(n,n+1)}(i, j)$ does not depend on n whenever $i, j = 1, \dots, K$, we will drop the superscript $(n, n+1)$ and write just $a_{S,\theta}(i, j)$ for $i, j = 1, \dots, K$. Let $\mathbf{A}_{S,\theta}^* = [a_{S,\theta}(i, j)]_{i,j=1,\dots,K}$. The elements $a_{S,\theta}^{(n,n+1)}(i, K+1)$ are equal to $d_{S,\theta}(i)$ for $i = 1, \dots, K$.

Let $\mathbf{e}_{S,\theta}^{(n)} = [e_{S,\theta}^{(n)}(i)]$ be a row vector whose ith entry denotes the probability that, in period n, a policyholder with risk parameter θ is in state i, for $i = 1, \dots, K+1$ (we consider, for the moment, that $\mathbf{e}_{S,\theta}^{(0)}$ is known). As $e_{S,\theta}^{(n+1)}(K+1) = 1 - q_{T,\theta}^{(n+1)}$, we get, by conditioning on the state of the system at period n, for $n = 0, 1, \dots,$

$$1 - q_{T,\theta}^{(n+1)} = a_{S,\theta}^{(n,n+1)}(K+1, K+1)(1 - q_{T,\theta}^{(n)}) + \sum_{j=1}^{K} e_{S,\theta}^{(n)}(j) d_{S,\theta}(j) \tag{10}$$

from which it follows that

$$a_{S,\theta}^{(n,n+1)}(K+1,K+1)=\frac{1-q_{T,\theta}^{(n+1)}-\sum_{j=1}^{K}e_{S,\theta}^{(n)}(j)d_{S,\theta}(j)}{1-q_{T,\theta}^{(n)}}. \tag{11}$$

Then, the other entries $a_{S,\theta}^{(n,n+1)}(K+1,j)$, $j=1,\dots,K$, are easily calculated according to

$$a_{S,\theta}^{(n,n+1)}(K+1,j)=(1-a_{S,\theta}^{(n,n+1)}(K+1,K+1))v_\theta(j),\quad j=1,\dots,K. \tag{12}$$

Note that when writing (10), we are assuming that the set of all policyholders (in and out of the company) is the same for all time periods. Eqs. (11) and (12) also reflect the hypothesis made that the market shares, the vector of entrances and the vectors of exits are, for each θ, independent of the premium scale and exogenously given.

The matrix $\mathbf{A}_{S,\theta}^{(n,n+1)}$ can then be written by blocks as

$$\mathbf{A}_{S,\theta}^{(n,n+1)}=\begin{bmatrix}\mathbf{A}_{S,\theta}^{*} & \mathbf{d}'_{S,\theta}\\ (1-a_{S,\theta}^{(n,n+1)}(K+1,K+1))\mathbf{v}_\theta & a_{S,\theta}^{(n,n+1)}(K+1,K+1)\end{bmatrix},$$

where $\mathbf{d}'_{S,\theta}$ is a column vector with $\mathbf{d}'_{S,\theta}=[d_{S,\theta}(i)]_{i=1,\dots,K}$. Note that the underlying Markov chain is now inhomogeneous, because the right-hand side of (11) and (12) depend on the time n.

Lemma 1. *Let*

$$\mathbf{P}_{S,\theta}^{*(n,n+1)}=\begin{bmatrix}\tilde{\mathbf{P}}_{S,\theta} & \mathbf{0}\\ (1-l_{S,\theta}^{(n,n+1)})\mathbf{v}_\theta & l_{S,\theta}^{(n,n+1)}\end{bmatrix}, \tag{13}$$

with $\tilde{\mathbf{P}}_{S,\theta}=[\tilde{\mathbf{p}}_{S,\theta}(i,j)]_{i,j=1,\dots,K}$, *where*

$$\tilde{\mathbf{p}}_{S,\theta}(i,j)=a_{S,\theta}(i,j)+d_{S,\theta}(i)v_\theta(j),\quad i,j=1,\dots,K, \tag{14}$$

and

$$l_{S,\theta}^{(n,n+1)}=\frac{1-q_{T,\theta}^{(n+1)}}{1-q_{T,\theta}^{(n)}}, \tag{15}$$

then the transition probability matrix $\mathbf{A}_{S,\theta}^{(n,n+1)}$ *is equivalent to the matrix* $\mathbf{P}_{S,\theta}^{*(n,n+1)}$, *in the sense that*

$$\mathbf{e}_{S,\theta}^{(n+1)}=\mathbf{e}_{S,\theta}^{(n)}\mathbf{A}_{S,\theta}^{(n,n+1)}=\mathbf{e}_{S,\theta}^{(n)}\mathbf{P}_{S,\theta}^{*(n,n+1)},\quad n=0,1,\dots. \tag{16}$$

If $0<q_{T,\theta}^{(n)}<1$, $n=0,1,\dots$, then $\mathbf{P}_{S,\theta}^{*(n,n+1)}$ is a Markov matrix provided that $q_{T,\theta}^{(n+1)}\geqslant q_{T,\theta}^{(n)}$. If $q_{T,\theta}^{(n+1)}<q_{T,\theta}^{(n)}$, then $l_{S,\theta}^{(n,n+1)}>1$, but (16) still holds. This result allows us to say that

$$e_{S,\theta}^{(n)}=\mathbf{e}_{S,\theta}^{(0)}\mathbf{A}_{S,\theta}^{(0,n)}=\mathbf{e}_{S,\theta}^{(0)}\mathbf{P}_{S,\theta}^{*(0,n)},\quad n=1,2,\dots \tag{17}$$

and then

$$\lim_{n\to\infty}e_{S,\theta}^{(n)}=\mathbf{e}_{S,\theta}^{(0)}\lim_{n\to\infty}\mathbf{A}_{S,\theta}^{(0,n)}=\mathbf{e}_{S,\theta}^{(0)}\lim_{n\to\infty}\mathbf{P}_{S,\theta}^{*(0,n)}. \tag{18}$$

The matrix $\mathbf{P}_{S,\theta}^{*(0,n)}$ is easily calculated and is

$$\mathbf{P}_{S,\theta}^{*(0,n)}=\begin{bmatrix}\tilde{\mathbf{P}}_{S,\theta}^{n} & \mathbf{0}\\ \mathbf{x}_{S,\theta}^{(0,n)} & l_{S,\theta}^{(0,n)}\end{bmatrix} \tag{19}$$

with

$$\mathbf{x}_{S,\theta}^{(0,n)}=\frac{1}{1-q_{T,\theta}^{(0)}}\mathbf{v}_\theta\sum_{j=1}^{n}\Delta q_{T,\theta}^{(j)}\tilde{\mathbf{P}}_{S,\theta}^{n-j}, \tag{20}$$

where

$$\Delta q_{T,\theta}^{(j)}=q_{T,\theta}^{(j)}-q_{T,\theta}^{(j-1)} \tag{21}$$

and

$$l_{S,\theta}^{(0,n)}=\frac{1-q_{T,\theta}^{(n)}}{1-q_{T,\theta}^{(0)}}. \tag{22}$$

Theorem 1. *Given θ and provided that $\sum_{k=1}^{\infty}\Delta q_{T,\theta}^{(n)}$ with $0<q_{T,\theta}^{(n)}<1$, $\forall n$ is absolutely convergent, then the long run distribution of the policyholders among the K classes of the bonus system is independent of the market shares and is given by the stationary distribution of the matrix $\tilde{\mathbf{P}}_{S,\theta}$, i.e. is given by the vector $\tilde{\mathbf{p}}_{S,\theta}=[\tilde{p}_{S,\theta}(i)]_{i=1,\dots,K}$ satisfying the system of equations*

$$\tilde{\mathbf{p}}_{S,\theta}=\tilde{\mathbf{p}}_{S,\theta}\tilde{\mathbf{P}}_{S,\theta},\qquad \sum_{i=1}^{K}\tilde{p}_{S,\theta}(i)=1, \tag{23}$$

which is to say, it is given by the normalised left eigenvector of the matrix $\tilde{\mathbf{P}}_{S,\theta}$ associated to the eigenvalue 1.

Proof. Note that $\tilde{\mathbf{P}}_{S,\theta}$, whose entries are given by (14), is calculated as a function of the transition probability matrix in the closed model, $\mathbf{P}_{T,\theta}$, of the values $d_{T,m,\theta}(i)$ and of the vector $\mathbf{v}_\theta$. Hence, $\tilde{\mathbf{P}}_{S,\theta}$ and its eigenvectors depend on all those quantities. Note, however, that an entry of $\tilde{\mathbf{P}}_{S,\theta}$ is zero only if the respective entry of $\mathbf{P}_{T,\theta}$ is zero, and the element on the diagonal of $\tilde{\mathbf{P}}_{S,\theta}$ corresponding to the elite class will be positive. Then, if the rules are such that the Markov chain in the closed model is finite, irreducible and aperiodic, the same happens to the Markov matrix $\tilde{\mathbf{P}}_{S,\theta}$, and we can guarantee using the Jordan canonical form, see for instance Cox and Miller (1965), that there exists a matrix $\mathbf{B}_{S,\theta}$ such that $\tilde{\mathbf{P}}_{S,\theta}=\mathbf{B}_{S,\theta}^{-1}\mathbf{J}_{S,\theta}\mathbf{B}_{S,\theta}$, where $\mathbf{B}_{S,\theta}$ is the matrix of the left eigenvectors of $\tilde{\mathbf{P}}_{S,\theta}$ and $\mathbf{J}_{S,\theta}$ is the Jordan matrix

$$\mathbf{J}_{S,\theta}=\begin{bmatrix}\mathbf{J}_1(1)=1 & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_{m_2}(\lambda_2) & \cdots & \mathbf{0}\\ \cdots & \cdots & \cdots & \cdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{m_l}(\lambda_l)\end{bmatrix}, \tag{24}$$

where $\lambda_1=1,\lambda_2,\dots,\lambda_l$, are the eigenvalues of $\tilde{\mathbf{P}}_{S,\theta}$ with multiplicity $1,m_2,\dots,m_l$, respectively, which are such that $|\lambda_i|<1$ for $i=2,\dots,l$ and with

$$\mathbf{J}_{m_i}(\lambda_i)=\begin{bmatrix}\lambda_i & 1 & 0 & 0 & 0\\ 0 & \lambda_i & 1 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & \lambda_i\end{bmatrix}=\lambda_i\mathbf{I}_{m_i}+\mathbf{M}_{m_i}, \tag{25}$$

where $\mathbf{I}_{m_i}$ is the identity matrix $(m_i \times m_i)$ and $\mathbf{M}_{m_i}$ is a $(m_i \times m_i)$ matrix with

$$\mathbf{M}_{m_i} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{26}$$

Then

$$\mathbf{x}_{S,\theta}^{(0,n)} = \frac{1}{1 - q_{T,\theta}^{(0)}} \mathbf{v}_\theta \sum_{j=1}^{n} \Delta q_{T,\theta}^{(j)} \tilde{\mathbf{P}}_{S,\theta}^{n-j} = \frac{1}{1 - q_{T,\theta}^{(0)}} \mathbf{v}_\theta \mathbf{B}_{S,\theta}^{-1} \left(\sum_{j=1}^{n} \Delta q_{T,\theta}^{(j)} \mathbf{J}_{S,\theta}^{n-j} \right) \mathbf{B}_{S,\theta}. \tag{27}$$

We will prove now that, under our assumptions on the market shares,

$$\lim_{n\to\infty} \left(\sum_{j=1}^{n} \Delta q_{T,\theta}^{(j)} \mathbf{J}_{S,\theta}^{n-j} \right) = \sum_{j=1}^{n} \Delta q_{T,\theta}^{(j)} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{28}$$

We start by noting that

$$\mathbf{J}_{m_i}^{n-j}(\lambda_i) = \sum_{l=0}^{n-j} \binom{n-j}{l} \lambda_i^{n-j-l} \mathbf{M}_{m_i}^{l} = \sum_{l=0}^{\min(m_i-1,n-j)} \binom{n-j}{l} \lambda_i^{n-j-l} \mathbf{M}_{m_i}^{l} \tag{29}$$

since $\mathbf{M}_{m_i}^{l} = \mathbf{0}$ for $l \geqslant m_i$. Then

$$\sum_{j=1}^{n} \Delta q_{T,\theta}^{(j)} \mathbf{J}_{S,\theta}^{n-j} = \sum_{j=1}^{n} \Delta q_{T,\theta}^{(j)} \sum_{l=0}^{\min(m_i-1,n-j)} \binom{n-j}{l} \lambda_i^{n-j-l} \mathbf{M}_{m_i}^{l} = \sum_{j+k=n} \Delta q_{T,\theta}^{(j)} \sum_{l=0}^{\min(m_i-1,k)} \binom{k}{l} \lambda_i^{k-l} \mathbf{M}_{m_i}^{l}. \tag{30}$$

This last expression can be regarded as the general term of the Cauchy product of the series $\sum_{j=1}^{\infty} \Delta q_{T,\theta}^{(j)}$, which is absolutely convergent by hypothesis, and the series

$$\sum_{k=1}^{\infty} \sum_{l=0}^{\min(m_i-1,k)} \binom{k}{l} \lambda_i^{k-l} \mathbf{M}_{m_i}^{l} = \sum_{k=1}^{m_i-1} \sum_{l=0}^{k} \binom{k}{l} \lambda_i^{k-l} \mathbf{M}_{m_i}^{l} + \sum_{k=m_i}^{\infty} \sum_{l=0}^{m_i-1} \binom{k}{l} \lambda_i^{k-l} \mathbf{M}_{m_i}^{l}. \tag{31}$$

The last term is an absolutely convergent series because

$$\sum_{k=m_i}^{\infty} \binom{k}{l} |\lambda_i|^{k-l}, \quad l = 0, \ldots, m_i - 1 \tag{32}$$

is convergent for $|\lambda_i| < 1$.

As the Cauchy product of two absolutely convergent series is still convergent, then its general term has to vanish when n goes to ∞ and (28) is proved.

Hence, if we take limits to both sides of (27) and use (28), we get that

$$\lim_{n\to\infty} \mathbf{x}_{S,\theta}^{(0,n)} = \frac{1}{1 - q_{T,\theta}^{(0)}} \mathbf{v}_\theta \sum_{j=1}^{\infty} \Delta q_{T,\theta}^{(j)} \tilde{\mathbf{P}}_{S,\theta}^{\infty}, \tag{33}$$

where $\tilde{\mathbf{P}}^{\infty}_{S,\theta}$ is the limit of the Markov matrix $\tilde{\mathbf{P}}^{n}_{S,\theta}$, and has all its rows equal to the stationary distribution $\tilde{\mathbf{p}}_{S,\theta}$. Hence (33) is equivalent to

$$\lim_{n\to\infty} \mathbf{x}^{(0,n)}_{S,\theta} = \frac{1}{1-q^{(0)}_{T,\theta}} \sum_{j=1}^{\infty} \Delta q^{(j)}_{T,\theta} \tilde{\mathbf{p}}_{S,\theta} = \frac{1}{1-q^{(0)}_{T,\theta}} (q^{(\infty)}_{T,\theta} - q^{(0)}_{T,\theta}) \tilde{\mathbf{p}}_{S,\theta}, \tag{34}$$

where $q^{(\infty)}_{T,\theta}$ represents the long run market share of the company.

Then

$$\lim_{n\to\infty} e^{(n)}_{S,\theta}(i) = q^{(\infty)}_{T,\theta} \tilde{p}_{S,\theta}(i), \quad i = 1, \dots, K, \tag{35}$$

and hence the probability that θ is in class i of the bonus system provided that it is in the company is $\tilde{p}_{S,\theta}(i)$. □

Hence, if we wish to calculate the optimal bonus scale in the open model, when Norberg's asymptotic criterion is used, we only have to calculate, for each θ, the matrix $\tilde{\mathbf{P}}_{S,\theta}$, according to (14) and use (3) (or (7) with $w_0 > 0$ and $w_n = 0$ for $n \geqslant 1$, if we wish a linear scale). As it is known, the limit distribution is independent of the initial distribution. This implies that the same happens for scales based on the asymptotic criteria.

If we wish to calculate the optimal bonus scale in the open model, using the generalization proposed in Borgan et al. (1981), we need to foresee the market shares of the company in the future, to calculate $e^{(n)}_{S,\theta}$ according to (17) and to use (5) with $p^{(n)}_{T,\theta}(j)$ substituted by $e^{(n)}_{S,\theta}(j)/q^{(n)}_{T,\theta}$. The same substitution is made in (7) when we wish to apply the linear scale for some positive w_n for $n \geqslant 1$. In both cases, the optimal scale does not depend on the initial market share $q^{(0)}_{T,\theta}$ but it will depend on the initial percentage of policies in each bonus class, i.e. on $e^{0}_{S,\theta}(j)/q^{(0)}_{T,\theta}$.

4. An example

Let us consider the Swiss bonus system, which is a system with 22 classes, numbered in an increasing order of dangerousness, with entry in class 10 (for a first automobile insurance policy), and with the following transition rules for a policy in class i: for each claim free year the policy goes to class $\max(i-1, 1)$ and with m claims it goes to $\min(i+4m, 22)$.

We assume that the number of claims M_n, conditional on $\Theta = \theta$, is Poisson distributed with parameter θ, and that $\Theta = \theta$ is distributed according to the discrete structure function in Table 1. To keep the example as simple as possible we considered both $\mathbf{v}_\theta$ and $d_{m,\theta}(i)$ independent of θ, and the last also independent of m. Those figures are presented in Table 2.

Table 3 gives the results for the open and closed models. The first comments suggested by Table 3 is that the two long run distributions are quite different. Taking the open model as closer to reality, we can say that the closed model overevaluates the probabilities of the extreme classes. As a result, Norberg's asymptotic criterion, when applied to

Table 1
The structure function

θ	0.0050	0.0165	0.0310	0.0485	0.0690	0.0925	0.1190	0.1485	0.1810	0.2165	0.2550	0.2965	0.3410	0.3885	0.4390	0.4925	0.5490	0.6105	0.6845	0.8000
$u(\theta)$	0.2142	0.1368	0.1185	0.1039	0.0898	0.0761	0.0630	0.0509	0.0401	0.0307	0.0231	0.0169	0.0120	0.0084	0.0057	0.0038	0.0024	0.0016	0.0011	0.0010

Table 2
Vectors **v** and **d**

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$v_\theta(i)$	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.79	0.005	0.005	0.005	0.005	0.002	0.002	0.002	0.002	0.001	0.001	0	0
$d_{m,\theta}(i)$	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.095	0.11	0.125	0.14	0.155	0.17	0.185	0.2	0.22	0.24	0.26	0.28	0.30	0.32	0.34

Table 3
Long run distributions and optimal scales

Class j	Closed model			Open model		
	$p_T(j)$	$\pi_T^A(j)$	$\pi_T^L(j)$	$p_S(j)$	$\pi_S^A(j)$	$\pi_S^L(j)$
1	0.6901	0.0395	0.0413	0.5573	0.0418	0.0426
2	0.0284	0.0852	0.0558	0.0355	0.0828	0.0561
3	0.0310	0.0884	0.0703	0.0391	0.0871	0.0695
4	0.0339	0.0916	0.0848	0.0437	0.0922	0.0830
5	0.0373	0.0951	0.0993	0.0499	0.0983	0.0964
6	0.0138	0.1283	0.1138	0.0336	0.1083	0.1099
7	0.0133	0.1343	0.1283	0.0365	0.1144	0.1233
8	0.0125	0.1415	0.1429	0.0405	0.1221	0.1368
9	0.0113	0.1507	0.1574	0.0461	0.1322	0.1502
10	0.0085	0.1699	0.1719	0.0526	0.1448	0.1637
11	0.0082	0.1789	0.1864	0.0114	0.1870	0.1771
12	0.0079	0.1894	0.2009	0.0112	0.2007	0.1906
13	0.0076	0.2016	0.2154	0.0104	0.2169	0.2040
14	0.0073	0.2159	0.2300	0.0084	0.2349	0.2175
15	0.0075	0.2284	0.2445	0.0043	0.2416	0.2309
16	0.0078	0.2424	0.2590	0.0041	0.2580	0.2444
17	0.0084	0.2580	0.2735	0.0036	0.2766	0.2578
18	0.0092	0.2753	0.2880	0.0029	0.2949	0.2713
19	0.0104	0.2941	0.3025	0.0018	0.2976	0.2847
20	0.0122	0.3156	0.3171	0.0019	0.3254	0.2982
21	0.0148	0.3401	0.3316	0.0021	0.3636	0.3116
22	0.0187	0.3682	0.3461	0.0029	0.4040	0.3251

Table 4
Optimal scales given by (5) and (7)

Class j	Closed model		Open model	
	$\pi_T^B(j)$	$\pi_T^L(j)$	$\pi_T^B(j)$	$\pi_T^L(j)$
1	0.0448	0.0338	0.0513	0.0407
2	0.0572	0.0444	0.0602	0.0484
3	0.0607	0.0551	0.0625	0.0562
4	0.0647	0.0657	0.0656	0.0640
5	0.0696	0.0763	0.0699	0.0718
6	0.0716	0.0869	0.0695	0.0795
7	0.0758	0.0975	0.0731	0.0873
8	0.0807	0.1081	0.0782	0.0951
9	0.0865	0.1187	0.0850	0.1029
10	0.1304	0.1293	0.1014	0.1106
11	0.1403	0.1399	0.1380	0.1184
12	0.1503	0.1505	0.1496	0.1262
13	0.1617	0.1611	0.1635	0.1340
14	0.1793	0.1717	0.1832	0.1417
15	0.1905	0.1823	0.1810	0.1495
16	0.2038	0.1929	0.1955	0.1573
17	0.2186	0.2035	0.2114	0.1651
18	0.2353	0.2141	0.2242	0.1728
19	0.2524	0.2247	0.1970	0.1806
20	0.2811	0.2354	0.2221	0.1884
21	0.3149	0.2460	0.2591	0.1962
22	0.3565	0.2566	0.3101	0.2039

the closed model, under prices the extremes (both the “elite” and the worst classes). When the linear scale is used, we obtain a smaller premium in all the classes for the open model, exception to classes 1 and 2.

To apply the generalization proposed by Borgan et al. (1981) we have used the following scenario:

- Time horizon: 20 years with weights such that $w_i = w_{i-1}/1.05$ and $\sum_{i=1}^{20} w_i = 1$.
- Initial distribution, $\mathbf{c} = \mathbf{e}_{T,\theta}^{(0)}/q_{T,\theta}^{(0)}$, independent of θ, given by $c(1) = 0.55$, $c(2) = \cdots = c(5) = 0.04$, $c(6) = \cdots = c(9) = 0.03$, $c(10) = \cdots = c(13) = 0.02$, $c(14) = \cdots = c(22) = 0.01$.
- Market shares independent of θ, and such that $\Delta q_{T,\theta}^{(1)}/q_{T,\theta}^{(0)} = 0.05$ and $\Delta q_{T,\theta}^{(n+1)}/q_{T,\theta}^{(n)} = 0.95 \Delta q_{T,\theta}^{(n)}/q_{T,\theta}^{(n-1)}$ for $n = 1, 2, \ldots$.

Table 4 shows the results using (5) and (7) for both the closed and open models. The conclusions are quite similar to those established for the asymptotic criterion.

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