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STATIONARY PROCESSES THAT LOOK LIKE RANDOM WALKS— THE BOUNDED RANDOM WALK PROCESS IN DISCRETE AND CONTINUOUS TIME

João Nicolau
Universidade Técnica de Lisboa

Several economic and financial time series are bounded by an upper and lower finite limit (e.g., interest rates). It is not possible to say that these time series are random walks because random walks are limitless with probability one (as time goes to infinity). Yet, some of these time series behave just like random walks. In this paper we propose a new approach that takes into account these ideas. We propose a discrete-time and a continuous-time process (diffusion process) that generate bounded random walks. These paths are almost indistinguishable from random walks, although they are stochastically bounded by an upper and lower finite limit. We derive for both cases the ergodic conditions, and for the diffusion process we present a closed expression for the stationary distribution. This approach suggests that many time series with random walk behavior can in fact be stationarity processes.

1. INTRODUCTION

The study of stationary versus nonstationary time series has became a key issue in both time series and econometrics analysis. Their implications for economic theory are extremely important. Some time series seems to be nonstationary, such as industrial production, consumer prices, and stock prices, among others (see Kwiatkowski, Phillips, Schmidt, and Shin, 1992). However there are others where there seems to be no consensus. For example, Perron (1989) can not reject the unit root hypothesis for the nominal interest rate. In the same direction, Chan, Karolyi, Longstaff, and Sanders (1992) point out that the mean reversion for the U.S. interest rate is very weak, which is a sign of unit root possibility. However, Dahlquist (1996) finds some mean (linear) reversion effects for interest rates in Denmark, Germany, Sweden, and the UK. On the other hand, Aït-Sahalia (1996) concluds, that the mean linear reversion

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is not adequate for the 7-day Eurodollar deposit rate. He finds that the drift (infinitesimal coefficient of the diffusion process) of the spot rate process is essentially zero as long as the rate is between 4 and 17% but pulls it strongly toward this middle region whenever it escapes. Thus, in the interval from 4 to 17% the process behaves like a random walk (RW) process (as the drift is zero) but is not a true RW as the process shows reversion effects whenever some high or low value is reached. Aït-Sahalia's interpretation seems to solve the puzzle: the interest rate behaves like a RW—so the usual test of stationarity is not able to reject the unit root—but the process is obviously bounded with reversion effects at high and low levels, which eventually leads to stationarity and a mean reversion.

Regarding the exchange rate, there is now a considerable amount of evidence that real exchange rates do not have unit roots (see Rogoff, 1996; Rose, 1996) despite initial views (see Roll, 1979; Adler and Lehman, 1983). Therefore it is expected, in general, that real exchange rates are bounded in probability (i.e., do not diverge to $\pm\infty$) and they have long-run values to converge to. With regard to nominal exchange rates it can be argued from economic and statistical considerations that some nominal exchange rates should be bounded or be in some kind of implicit target zone regime. Nicolau (1999) argues that the DEM/USD exchange rate is not a RW (at least in the last 15 years) despite the conclusions of the Dickey–Fuller test. That is, the DEM/USD behaves like a RW but cannot be a true RW as there is some evidence that this exchange rate is bounded[1] (we emphasize that the unit root process goes to $+\infty$ or $-\infty$ with probability one as time goes to $+\infty$ so a bounded process can not have a unit root).

These ideas suggest that some economic and financial time series can behave just like a RW (with some volatility pattern) but because of some economic reasons they are bounded processes (e.g., in probability) and even stationary processes. To build a model with such features it is necessary to allow RW behavior most of the time but force mean reversions whenever the processes try to escape from some interval. There is some evidence that the usual Dickey–Fuller test under general specification of alternative hypothesis has low power to detect stationary processes. Nicolau (1999) shows that the power of the Dickey–Fuller test is extremely low when the alternative hypothesis is a stationary bounded RW process (we present this model next; see also Kwiatkowski et al., 1992).

In this paper our aim is to present a new model in discrete and in continuous time that can generate paths with the following features: as long as the process is in the interval of moderate values, the process basically looks like a RW, but there are reversion effects toward the interval of moderate values whenever the process reaches some high or low values. As we will see, these processes can admit—relying on the parameters—stationary (or ergodic) distributions, so we will come to the following interesting conclusion: processes that are almost

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indistinguishable from the RW process can be, in effect, stationary with ergodic distributions.

The paper consists of two main sections. In Section 2 we introduce and discuss the main properties of the bounded random walk (BRW) process in discrete time. We do the same in Section 3 for a continuous-time version of the BRW. In the discrete-time version we admit a GARCH representation for the volatility, whereas in the continuous-time case we discuss an exponential form that could be appropriate for modeling a "smile" curve for volatility (Krugman and Miller, 1992). In all cases we state the conditions under which the processes are ergodic.[2] We present examples to make these models clear.

2. THE BOUNDED RANDOM WALK IN DISCRETE TIME

The previous section shows that a BRW model can be appropriate for modeling some economic and financial time series. In this section we start to discuss some proprieties that a BRW model should satisfy.

2.1. Some Properties

If a process is a RW, the function $E[\Delta X_t | X_{t-1} = x]$ (where $\Delta X_t = X_t - X_{t-1}$) must be zero (for all x). On the other hand, if a process is bounded (in probability) and mean-reverting to τ (say), the function $E[\Delta X_t | X_{t-1} = x]$ must be positive if x is below τ and negative if x is above τ.

Now consider a process that is bounded but behaves like a RW. What kind of function should $E[\Delta X_t | X_{t-1} = x]$ be? As the process behaves like a RW, (i) it must be zero in some interval and, because the process is bounded, (ii) it must be positive (negative) when x is "low" ("high"). Moreover we expect that (iii) $E[\Delta X_t | X_{t-1} = x]$ is a monotonic function, which, associated with (ii), means that the reversion effect should be strong if x is far from the interval of reversion and should be weak in the opposite case; (iv) $E[\Delta X_t | X_{t-1} = x]$ is differentiable (on the state space of X) to assure a smooth effect of reversion. This kind of behavior, for instance, with regard to interest and exchange rates, is implicit in Aït-Sahalia (1996), Stanton (1997), and Nicolau (1999) through the nonparametric estimation of $E[\Delta X_t | X_{t-1} = x]$. A possible representation for $E[\Delta X_t | X_{t-1} = x]$ will be provided in Section 2.3.

To satisfy (i)–(iv) we assume $E[\Delta X_t | X_{t-1} = x] = e^k(e^{-\alpha_1(x-\tau)} - e^{\alpha_2(x-\tau)})$ with $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $k < 0$. Let us fix $a(x) = e^k(e^{-\alpha_1(x-\tau)} - e^{\alpha_2(x-\tau)})$. As we will see, this function is sufficiently general and flexible in the sense that it can generate a vast range of BRWs in a stationary (or nonstationary) framework (because stationary processes are bounded in probability, it is quite natural to expect that BRWs are stationary). In addition, this function has good properties, allowing the extension of our discrete-time model to a continuous-time framework. There may be other functions similar to $a(x)$ that can generate BRWs, so the $a(x)$ function is not unique. As far as we know, the only alternative models, in the literature, are the SETAR(3,1,1) (self-exciting threshold autoregressive; Tong, 1990) and the regime-switching model (Hamilton, 1994), see however equation (11). In Section 2.5, we show that our model has several important advantages over these two alternative models.

With our assumption about $E[\Delta X_t | X_{t-1} = x]$ we propose, therefore, the BRW in discrete time:

$$X_t = X_{t-1} + e^k(e^{-\alpha_1(X_{t-1}-\tau)} - e^{\alpha_2(X_{t-1}-\tau)}) + \sigma_t \varepsilon_t \qquad (1)$$

($\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $k < 0$) where $\{\varepsilon_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $E[\varepsilon_t] = 0$ and $\text{Var}[\varepsilon_t] = 1$ and σ_t (volatility) belongs to the information set $\mathcal{F}_{t-1} = \sigma(X_\tau : \tau \leq t-1)$, for example, σ_t^2 can have a GARCH representation (Bollerslev, 1986).

2.2. The Function $a(x)$

We now analyze the function $a(x) = e^k(e^{-\alpha_1(x-\tau)} - e^{\alpha_2(x-\tau)})$. It is evident that the case $\alpha_1 + \alpha_2 = 0$ leads to the unit root (because it implies $a(x) = 0, \forall x$). It is still obvious (in more general cases, e.g., $\alpha_1 > 0$ and $\alpha_2 > 0$) that $a(\tau) = 0$, so X_t must behave just like a RW whenever $X_{t-1} = \tau$.

On the other hand, we can select the parameters k, α_1, and α_2 such that $a(x) \approx 0$ whenever X_{t-1} is in the neighborhood of τ. The range of the interval where $a(x)$ is approximately null depends on the parameters. Typically, the case $\alpha_1 > 0$, $\alpha_2 > 0$, $k < 0$, and $|k|$ is high with regard to α_1 and α_2 entails $a(x) \approx 0$ over a large interval centered on τ. Suppose that X_{t-1} moves significantly away from τ, for example, $X_{t-1} > \tau$. Then, $a(x)$ turns out to be negative, and the probability that X_t decreases will be high. Thus, if X_{t-1} is "high" and far away from τ there will be reversion effects that pull it toward lower values. However, near τ these reversion effects will be almost null.

Let us see, in more detail, the meaning of the parameters α_1, α_2, τ, and k. The parameter k controls the range of the interval under which the process behaves like a RW. When $k < 0$ and $|k|$ is high (low) the range tends to be high (low). The τ parameter is a central measure of the process because, as we have seen, $a(\tau) = 0$. We should expect τ to be the mean of the process under the hypothesis $\alpha_1 = \alpha_2$ (we will turn again to this issue in Section 3.3). However, if $a(x)$ is approximately zero over a large interval (centered in τ) we should expect a reversion effect toward a neighborhood of τ (and not exactly to τ). Finally, the $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ parameters measure the reversion effect of the process whenever it escapes from the interval where the function $a(x)$ is approximately zero. High values of these parameters imply a strong reversion effect. It is easy to see that the α_1 (α_2) parameter is linked to the reversion effect when the process is low (high). We notice that the case $\alpha_1 \neq \alpha_2$ leads to asymmetrical effects. For instance, according to Gray (1996), among others, the interest rate process behaves typically like a RW when the process is moderate or low, showing, however, a strong reversion effect when it is very high. Therefore, we expect $\alpha_2 > \alpha_1$ (what prevents the process from reaching the zero state is actually the very low volatility—see our discussion at the end of Section 3.3). Another example of the case $\alpha_1 \neq \alpha_2$ is analyzed in Nicolau (1999) in the study of the DEM/USD exchange rate.

2.3. An Example

In Figure 1 we draw $a(x) = e^k(e^{-\alpha_1(x-\tau)} - e^{\alpha_2(x-\tau)})$ for the following values: $k = -15$, $\alpha_1 = \alpha_2 = 3$, and $\tau = 100$. We see that in the interval $I = (95,105)$ the function $a(x)$ is (approximately) zero, so in this interval X behaves like a RW. Outside of I there are reversions toward I.

In Figure 2, we simulated two paths ($t = 1,2,\ldots,1{,}000$): a BRW path from (1), using the values presented previously and $\sigma_t = \sigma = .4$, and a RW path from $X_t = X_{t-1} + .4\varepsilon_t$. In both processes we use the same values of ε_t (we assume that $\{\varepsilon_t\}$ is a sequence of i.i.d. random variables with $N(0,1)$ distribution). We see that both trajectories are almost indistinguishable until $t = 140$ (approximately). Near the value 105 there are reversion effects only in the BRW,

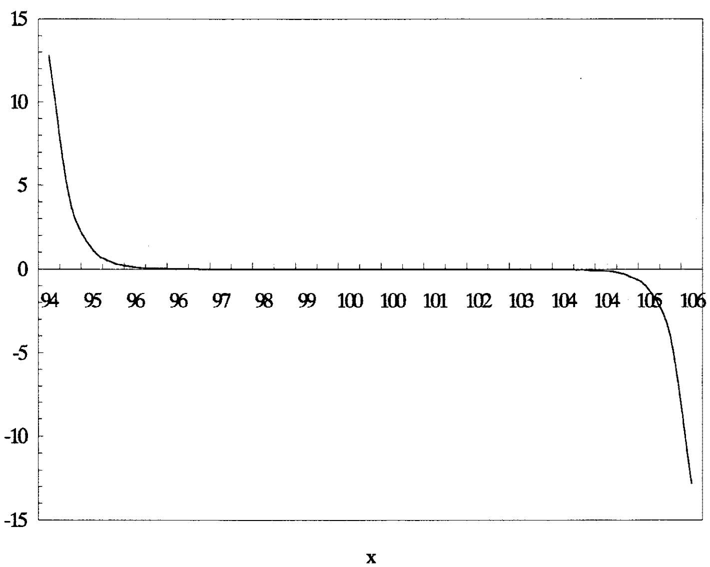

FIGURE 1. Curve $a(x)$.

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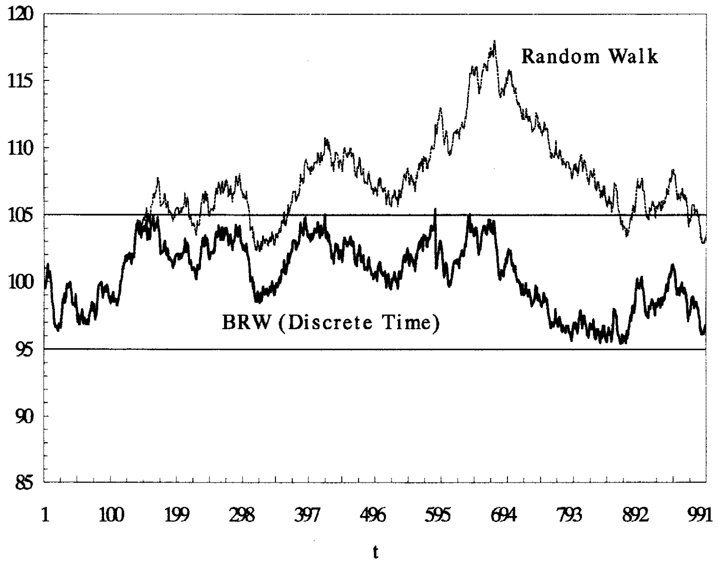

FIGURE 2. Bounded random walk vs. random walk.

that is, $a(x) < 0$ (see Figure 1), which prevent the process from increasing beyond 105. After that the processes follow different trajectories.

2.4. Stationarity

First, consider a homogeneous stochastic discrete process $X = \{X_t; t = 0,1,2,\ldots\}$ with initial value $X_0 = x_0$ (possibly random). Let us assume that X is a Markov process governed by the function

$$X_t = f(X_{t-1}, \varepsilon_t), \tag{2}$$

where

$$f(X_{t-1}, \varepsilon_t) = \mu(X_{t-1}) + \sigma(X_{t-1})\varepsilon_t,$$

$$\mu(X_{t-1}) = X_{t-1} + a(X_{t-1}),$$

$a: \mathbf{R} \to \mathbf{R}$ is a nonlinear function of X_{t-1}, and $\{\varepsilon_t; t \geq 1\}$ is a stochastic process. There are several approaches to check stationarity and ergodicity (see Borovkov, 1998). For instance, consider Proposition 1, which follows (adapted from Chang, Appendix 1, in Tong, 1990, pp. 448–466).

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PROPOSITION 1. *Assume that f is continuous everywhere and continuously differentiable in a neighborhood of the origin. Suppose that conditions A1–A7 (see Appendix A) hold. Then* $X = f(X_{t-1}, \varepsilon_t)$ *is geometrically ergodic.*

Proof. See Tong (1990, pp. 448–466).

For nonlinear time series, it is very difficult to check the (very important) A2 condition (see Appendix A). This is also true in our model. Basically, A2 indicates (in state space $\mathbf{R}^1$) that $|\mu_t(x_0)| \le Ke^{-ct}|x_0|$ where $K > 0$, $c > 0$, and $\mu_t(x_0)$ is the tth iteration $\mu_t(x_0) = \mu(\mu(\ldots\mu(x_0)))$ given the initial value x_0. The problem is that it is generally impossible, for nonlinear time series, to get $\mu_t(x_0) = \mu(\mu(\ldots\mu(x_0)))$ as a function of the initial value. However, in the case under analysis, with $\mu(x) = x + a(x)$, we consider the following conditions,[3] which are easy to check.

H1 $a(x) > 0$ if $x < 0$ and $a(x) < 0$ if $x > 0$.

H2 $a(x) < -2x$ if $x < 0$ and $a(x) > -2x$ if $x > 0$.

PROPOSITION 2. *The conditions H1 and H2 imply A2.*

Proof. See Appendix B.

H1 indicates that when the process is, for example, above its equilibrium value $\tau = 0$, that is, $X_{t-1} > 0$, there will be adjustments forcing X_t to decrease, so we must observe $E[\Delta X_t | X_{t-1}] = a(X_{t-1}) < 0$. The H2 condition indicates that these adjustments must be moderate and progressive to preclude explosions.

Consider now the BRW process (1). Without any loss of generality we fix $\tau = 0$ (this can be achieved by the transformation $Y_t = X_t - \tau$). As we have seen, the BRW satisfies condition A1. However, there is a potential problem: some adjustments can be explosive in the sense that every attempt to correct the path (when X_{t-1} is too "low" or too "high," beyond some level) can be excessive and turn out to be explosive. The hypothesis H2, if satisfied, certainly avoids explosive behavior. Unfortunately our function $a(x)$ does not satisfy H2 for all x. In practice this is not a major problem—in practical applications we never expect that the case $|a(x)| \ge 2|x|$ will happen.

Nevertheless, we force H2 by coupling (1) with the additional regularity condition (in the case $\tau = 0$)

$$X_t = \phi X_{t-1} + \sigma_t \varepsilon_t, \qquad 0 < \phi < 1 \quad \text{if } |a(X_{t-1})| \ge 2|X_{t-1}| \tag{3}$$

(we note that this condition is not necessary in the continuous-time case to analyze stationarity). We assume that the set $\{x : a(x) < -2x \text{ if } x < 0 \text{ and } a(x) > -2x \text{ if } x > 0\}$ has a positive Lebesgue measure. We now state the following proposition.

PROPOSITION 3. *Suppose that* $\alpha_1 > 0$, $\alpha_2 > 0$, $\sigma_t > 0$ $(\forall t)$, ε_t *satisfy A4 (in Appendix A) and* $\{u_t = \sigma_t \varepsilon_t; t = 1,2,\dots\}$ *is covariance stationary. Then X (see equation (1)) with the condition (3) is geometrically ergodic.*

Proof. It is easy to see that the conditions H1 and H2 and also A1–A7 (in Appendix A) are satisfied. ■

What kind of stationary distributions can we expect from the BRW model? Because X behaves like a RW most of the time, say, in the interval I, we must expect a flat distribution in the center. On the other hand, outside of I there are strong reversions, so the tails of distribution must not be heavy. This is what we have observed in some financial time series in levels. Notice that in the first difference sequence we observe a completely different pattern (heavy tails, peak distribution at the center, high kurtosis).

Therefore, under some conditions, including $\alpha_1 > 0$ and $\alpha_2 > 0$, the BRW process is geometrically ergodic, is bounded in probability, and is not persistent in the sense that shocks do not have permanent effects on the process. The implications for economic theory and policy actions are extremely important and are well described in the literature. For instance, policy actions are not so important in stationary models because shocks only have a transitory effect. Nicolau (1999) found some evidence that the DEM/USD is a BRW (using daily observations from the last 14 years). Thus, there must be an implicit but effective target zone regime that limits the size of exchange rate fluctuations. This conclusion, applicable to the EURO/USD, is relevant for financial markets and central banks.

2.5. Alternative Models

We now address the issue of alternative models. As far as we know, only the SETAR(3,1,1) (Tong, 1990) and regime-switching model (with two regimes) (Hamilton, 1994) can generate similar (but not identical) behavior to the BRW. In the SETAR model the $E[\Delta X_t | X_{t-1} = x]$ function is not differentiable at the threshold parameters and implies sudden transition of regimes as soon as some threshold parameter is crossed by the process. In the regime-switching model the Markov chain must depend on the past information of X (in the sense that, if the process is in the RW regime and if it crosses some high or low value, the probability of entering in the stationary regime must be high). We think that our model has some advantages over the two previously mentioned alternative models in modeling bounded processes that behave like a RW. It is a simpler model based on a single regime (without suddenly switching regimes) and is easier to estimate, for instance, through pseudo- (or quasi-) maximum likelihood. Furthermore, under some weak conditions, it converges weakly to a continuous-time process (more precisely to a diffusion process) as the interval of time between observations goes to zero.

We notice that the pseudo maximum likelihood (or even the maximum likelihood if we want to assume a distribution for the innovations) is immediately applicable to the BRW (with white noise or GARCH innovations) in a stationary framework. However, we should point out that the model is not identifiable in the case $\alpha_1 + \alpha_2 = 0$ (it can be shown that the Fisher information matrix is singular). Actually, the case $\alpha_1 + \alpha_2 = 0$ leads to the RW model. Therefore, in the estimation procedure it is suitable to restrict the parameters α_1 and α_2 to the stationarity region, that is, imposing $\alpha_1 > 0$ and $\alpha_2 > 0$.

3. THE BOUNDED RANDOM WALK DIFFUSION MODEL

3.1. A Convergence Result

In the previous section we assumed that the interval between observations was fixed and equal to one (say, $\Delta = 1$). Now we consider the case where X is a continuous-time process. There are some advantages to assuming this hypothesis. For instance, continuous-time processes are frequently preferred in finance theory. Moreover, in a continuous-time framework it is, in general, easier to find limit properties such as stationary moments and distributions and in general laws of probability governing the process than in the discrete-time version. On the other hand, continuous-time processes are more difficult to estimate when the observations are discrete, but even in this situation (and this is always the case) it can be argued that continuous-time formulation is closer to the way that the data are actually generated: "the economy does not move in regular discrete jumps corresponding to the observations—it is adjusting in between observations and it can change at any point of time" (Bergstrom, 1993).

How can we define a BRW in continuous time? One way consists of analyzing the limit process of the stochastic difference equation (1) as the length of the discrete-time intervals between observations goes to zero. So, let us consider (1) in a more convenient notation, assuming, for simplicity that σ_t is constant:

$$X_{t_i} = X_{t_{i-1}} + e^{k_\Delta}(e^{-\alpha_1(X_{t_{i-1}}-\tau)} - e^{\alpha_2(X_{t_{i-1}}-\tau)}) + \sigma_\Delta \varepsilon_{t_i}, \qquad X_0 = c. \tag{4}$$

Here t_i are the instances at which the process is observed, $(0 \le t_0 \le t_1 \le \cdots \le T)$, Δ is the interval between observations, $\Delta = t_i - t_{i-1}$, k_Δ and σ_Δ are parameters depending on Δ (if $\Delta \equiv 1$, we consider $k_\Delta = k$ and $\sigma_\Delta = \sigma$), and $\{\varepsilon_{t_i}, i = 1,2,\dots\}$ is a sequence of i.i.d. random variables with $E[\varepsilon_{t_i}] = 0$ and $\mathrm{Var}[\varepsilon_{t_i}] = 1$. We notice that when Δ is changing some parameters in (4) must change accordingly. We are concerned with the following problem: which process must (4) converge to when $\Delta \downarrow 0$? We are actually concerned with the convergence of the sequence $\{X_t^\Delta\}$ formed as a step function from X_{t_i}, that is, $X_t^\Delta = X_{t_i}$ if $t_i \le t < t_{i+1}$. It can be proved, under the conditions $k_\Delta = k +$

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$\log \Delta$ and $\sigma_\Delta = \sigma\sqrt{\Delta}$, that the sequence $\{X_t^\Delta\}$ converges weakly (i.e., in distribution) as $\Delta \downarrow 0$ to the X_t process defined by the stochastic integral equation

$$X_t = c + \int_0^t e^k (e^{-\alpha_1(X_s-\tau)} - e^{\alpha_2(X_s-\tau)})\, ds + \int_0^t \sigma dW_s, \tag{5}$$

where W is a standard Brownian motion, independent of c.[4]

3.2. Two Diffusion Models

First, we suppose that X is governed by the following stochastic differential equations (SDE):

$$dX_t = e^k (e^{-\alpha_1(X_t-\tau)} - e^{\alpha_2(X_t-\tau)})\, dt + \sigma dW_t, \qquad X_{t_0} = c, \tag{6}$$

where c is a constant and W is a standard Wiener process ($t \geq t_0$).[5]

As in the discrete version, it is evident that the case $a(x) = 0$ (for all x) leads to the Wiener process (which can be understood as the RW process in continuous time but having some special features—see Arnold, 1974, Ch. 3). It is still obvious that $a(\tau) = 0$, so X_t must behave just like a Wiener process when X_t crosses τ. However, it is possible, by selecting adequate values for k, α_1, and α_2, to have a Wiener process behavior over a large interval centered on τ (i.e., such that $a(x) \approx 0$ over a large interval centered on τ). Nevertheless, whenever X_t escapes from some levels there will always be reversion effects toward the τ.

A possible drawback of model (6) is that the diffusion coefficient is constant. In the exchange rate framework and under a target zone regime, we should observe a volatility of shape $\cap$ with respect to x (maximum volatility at the central rate) (see Krugman and Miller, 1992). On the other hand, under a free floating regime, it is common to observe a "smile" volatility (see Krugman and Miller, 1992). For both possibilities, we allow the volatility to be of shape $\cap$ or $\cup$ by assuming a specification such as $\exp\{\sigma + \beta(x-\mu)^2\}$. Depending on the β we will have volatility of $\cap$ or $\cup$ form. Naturally, $\beta = 0$ leads to constant volatility. This specification, with $\beta > 0$, can also be appropriate for interest rates (see Nicolau, 1998; Gray, 1996). We propose, therefore,

$$dX_t = e^k (e^{-\alpha_1(X_t-\tau)} - e^{\alpha_2(X_t-\tau)})\, dt + e^{\sigma/2+\beta/2(X_t-\mu)^2} dW_t, \qquad X_{t_0} = c. \tag{7}$$

3.3. Properties of the Models

Let us first consider the constant volatility model (6). The first question to ask is whether this process has exactly one continuous global solution over the entire interval $[t_0, \infty)$. According to Arnold (1974, Theorem 6.37, p. 114), if the infinitesimal coefficients a (drift) and b (diffusion) are continuously differentiable then X has a unique local solution that is defined up to a random explo-

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sion time η in the interval $t_0 < \eta \leq \infty$. Therefore, the main question is if $P[\eta = \infty] = 1$. We now state the following proposition.

PROPOSITION 4. *The solution of the SDE (6) with* $\alpha_1 = 0$, $\alpha_2 > 0$ *or* $\alpha_1 > 0$, $\alpha_2 = 0$ *or* $\alpha_1 > 0$, $\alpha_2 > 0$ *verifies* $P[\eta = \infty] = 1$.

Proof. See Appendix B.

So, (6) has exactly one continuous global solution over the entire interval $[t_0, \infty)$.

PROPOSITION 5. *The solution of the SDE (6) with* $\alpha_1 = 0$, $\alpha_2 > 0$ *or* $\alpha_1 > 0$, $\alpha_2 = 0$ *or* $\alpha_1 > 0$, $\alpha_2 > 0$ *is ergodic and has a stationary density of the form*

$$\bar{p}(x) \propto \sigma^{-2} \exp\left\{ -\frac{2e^k}{\sigma^2} \left(\frac{e^{-\alpha_1(x-\tau)}}{\alpha_1} + \frac{e^{\alpha_2(x-\tau)}}{\alpha_2} \right) \right\}. \tag{8}$$

Proof. See Appendix B.

The boundaries $l = -\infty$ and $r = \infty$ are natural so they are not attracting and cannot be attained (see Karlin and Taylor, 1981) starting the process at $X_0 = x$ where $l < x < r$. This is an important difference from the discrete version of model (1). In effect, whereas the discrete-time version recursion of (1) cannot start from an arbitrary value in $\mathbf{R}$, the continuous-time version does not have such a restriction (we only avoid having the initial points be $l = -\infty$ or $r = \infty$). Intuitively, if X is continuous, an oscillation explosive behavior in the bounded RW is precluded. For example, if the initial value $X_0 = x$ is far from the equilibrium point, there will be strong reversion effects toward τ. As soon as X_t starts to approximate τ the reversion effect will decrease and eventually will stop when τ is reached. In discrete time, if $X_0 = x$ is far from the equilibrium and if the condition (3) is not used, any tendency to return to equilibrium is made by explosive oscillations that are further and further from τ.

To exemplify this model we consider the case $k = -2$, $\alpha_1 = \alpha_2 = 2$, $\tau = 100$, and $\sigma = 4$. In the neighborhood of $\tau = 100$ the function $a(x)$ is (approximately) zero, so X behaves as a Wiener process (or a RW in continuous time). In effect, if $a(x) = 0$, we have $dX_t = \sigma dW_t$ (or $X_t = X_0 + \sigma W_t$). In Figure 3 we simulate two trajectories in the period $t \in [0, 20]$, with $X_0 = 100$. We compare the Wiener process ("unbounded RW") with the BRW, solution of (6), which was simulated by the Platen and Wagner discretization method (or scheme of order 1.5; see Kloeden and Platen, 1992). We draw two arbitrary lines to show that the BRW almost never crosses these lines. On the other hand, within the bands the BRW behaves like a pure RW.

Density can be very flat near τ as Figure 4 shows. As we have already pointed out, it is necessary to distinguish the distribution in levels from the distribution in the first differences sequence. This latter is usually leptokurtic. Financial time in levels is usually integrated or near integrated; therefore, over a large

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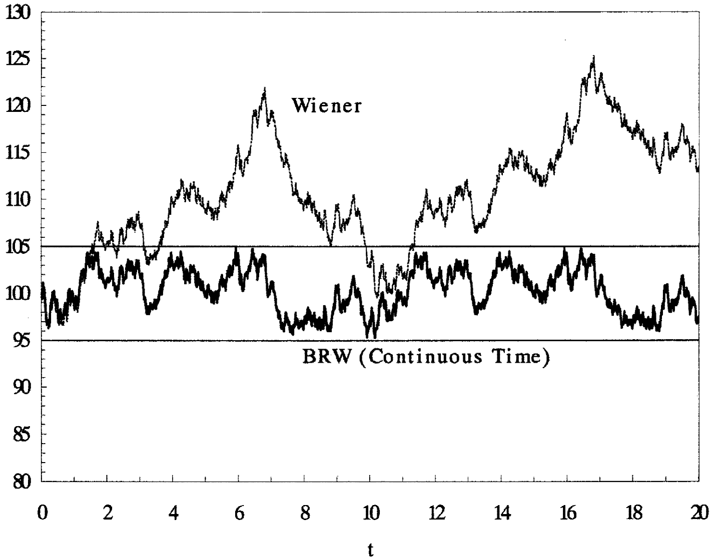

FIGURE 3. Bounded random walk vs. Wiener process.

interval of the state space, we do not expect to find values more likely than others, so actually the distribution in levels (if existing)[6] must be flat at the center.

In the preceding example, we have $d\bar{p}(\tau)/dx = 0$, so τ is a central measure. In the case $\alpha_1 = \alpha_2$ it seems clear that τ must be the stationary mean (notice that the stationary density is symmetrical around τ and the tails of distributions fall abruptly on the x axis, which must assure the existence of several moments—actually, this can be proved).

Let us now consider the exponential RW model. We now state the following proposition about (7).

PROPOSITION 6. *If $\beta > 0$ then $P[\eta = \infty] = 1$. If $\beta < 0$ and $\alpha_1 + \alpha_2 > 0$ then $P[\eta = \infty] = 1$.*

Proof. See Appendix B.

So, under the conditions of the previous proposition, (7) has exactly one continuous global solution on the entire $[t_0, \infty)$.

PROPOSITION 7. *If $\beta > 0$ the process (7) is ergodic and has a stationary density of the form*

$$\bar{p}(x) \propto \exp\left\{-\beta(x-\mu)^2 - \sigma + \frac{A_1[A_2 f_1(x) - A_3 f_2(x)]}{\sqrt{\beta}}\right\}, \quad \textbf{(9)}$$

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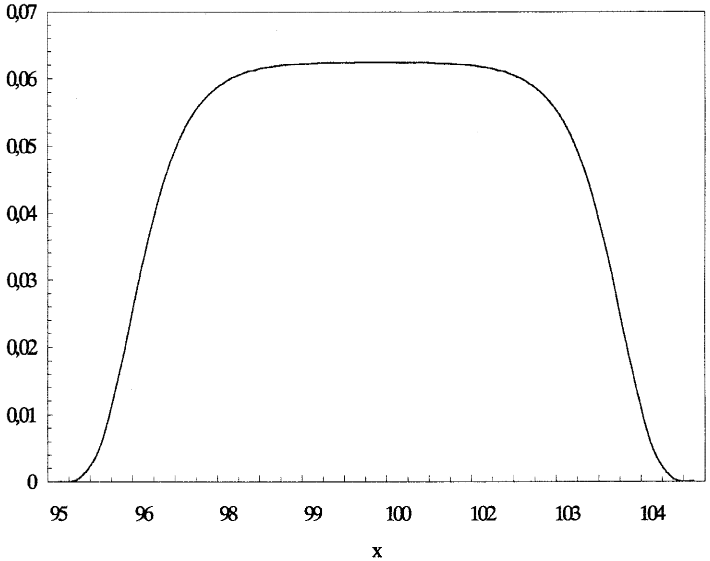

FIGURE 4. Stationary density.

where

$$A_1 = \sqrt{\pi} e^{k-\sigma-\mu\alpha_1-\tau\alpha_2},$$

$$A_2 = e^{\tau(\alpha_1+\alpha_2)+(\alpha_1^2/4\beta)},$$

$$A_3 = e^{\mu(\alpha_1+\alpha_2)+(\alpha_2^2/4\beta)},$$

$$f_1(x) = \text{erf}\left(\frac{2\beta(x-\mu)+\alpha_1}{2\sqrt{\beta}}\right),$$

$$f_2(x) = \text{erf}\left(\frac{2\beta(x-\mu)-\alpha_2}{2\sqrt{\beta}}\right),$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du.$$

Let $\beta^+ = |\beta|$. *If* $\beta < 0$ *and* $\alpha_1 + \alpha_2 > 0$ *the process (7) is ergodic and has a stationary density of the form*

$$\bar{p}(x) \propto \exp\left\{\beta^+(x-\mu)^2 - \sigma - \frac{A_4[-A_5 f_3(x) + A_6 f_4(x)]}{\sqrt{\beta^+}}\right\}, \quad \textbf{(10)}$$

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where

$$A_4 = \sqrt{\pi} e^{k-\sigma},$$

$$A_5 = e^{-(\alpha_1(4\beta^+(-\tau+\mu)+\alpha_1)/4\beta^+)},$$

$$A_6 = e^{-(\alpha_2(4\beta^+(\tau-\mu)+\alpha_2)/4\beta^+)},$$

$$f_3(x) = \text{erfi}\left(\frac{2\beta^+(x-\mu)-\alpha_1}{2\sqrt{\beta^+}}\right),$$

$$f_4(x) = \text{erfi}\left(\frac{2\beta^+(x-\mu)+\alpha_2}{2\sqrt{\beta^+}}\right),$$

$$\text{erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{u^2}du.$$

Proof. See Appendix B.

Finally, we point out a new form of RW behavior in a stationary framework. From Proposition 7 we see that the solution of the particular case

$$dX_t = e^{\sigma/2+\beta/2(X_t-\mu)^2} dW_t, \qquad \beta > 0 \tag{11}$$

turns out to be stationary, despite the drift's nullity. At first sight, as the drift is null, the process should not have any attraction toward a stable point. However, the process drifts to μ as $t \to \infty$ (it can be proved that μ is the stationary mean). Moreover, $E[X_t|X_s] = X_s$ $(t \geq s)$, but the process is not integrated in the usual econometric sense because integrated processes diverge (almost surely). Intuitively we can explain it as follows: when the process is near μ the instantaneous volatility, $b(x) = \exp\{\sigma/2 + \beta/2(x-\mu)^2\}$, is low, and the process tends to remain near μ. If X drifts away from μ, volatility increases. Now X is much more irregular, so there is a positive probability that the process crosses μ again. It is the volatility that pushes the process toward a steady point.[7] On the other hand, in this model, "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes." In effect, when X is near μ small changes tend to be followed by small changes as volatility is low; when X is far from μ large changes tend to be followed by large changes as volatility is high.

Therefore, roughly speaking, it is also possible to have RW behavior (with a null drift) in a stationary framework as long as the volatility pushes the process toward a steady point as described in the last paragraph. This, ultimately, makes the process bounded (see an application for interest rates in Nicolau, 1998); also, in this paper, we address the estimation issue in a continuous-time model).

NOTES

1. There are two major reasons why the DEM/USD exchange rate should not reach any arbitrary large value. First, the inflation rate and the gross national product (GNP) growth of the two economies have not shown strong and persistent differences for, at least, the last 15 years. Second, the G-7 council of economic ministers agreed on a set of policies with regard to exchange rates. In effect, by the end of 1986, the G-7 council considered that the USD had depreciated "too much." The Plaza and Louvre Accord (1985 and 1987) agreed to stabilize exchange rates. This meant limiting the size of exchange rate fluctuations with the use of coordinated central bank intervention. So in practice it is possible that the DEM/USD has been in an implicit target zone regime.

2. If X is geometrically ergodic then X is asymptotically stationary exponentially fast (despite the initial value). Furthermore, if X_0 (initial value) has stationary distribution π, X is strictly stationary and covariance stationary if $\int x^2\pi(dx) < \infty$.

3. It would be possible to replace **H1** by the following condition: there exist a $K > 0$ and a $M > 0$ such that $a(x) > K$ if $x < -M$ and $a(x) < -K$ if $x > M$.

4. We apply Theorem 2.2 in Nelson (1990).

5. Obviously, (6) is equivalent to (5).

6. We are talking about stationary distributions, so these distributions only make sense when the process is stationary.

7. Technically, when the process is in its natural scale (see the following notation in Appendix B), that is, $s(x) = 1$, the quantity $m(x)\varepsilon^2$ is of the order of the expected time the process spends in the interval $(x - \varepsilon, x + \varepsilon)$ given $X_0 = x$ before departure thereof (see Karlin and Taylor, 1981, pp. 197–198; in effect $E[T_{x-\varepsilon,x+\varepsilon}|X_0 = x] = m(x)\varepsilon^2$ where $T_{a,b} = \min\{T_a, T_b\}$ and T_a is hitting time of a, so $T_{a,b}$ is the first time the process reaches either a or b). It can be proved that $m(x) = \exp\{-\beta(x - \mu)^2\}$, so $m(x)$ is maximum when $x = \mu$. That is, the process spends more time in the interval $(\mu - \varepsilon, \mu + \varepsilon)$ than in any other interval (with fixed ε).

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APPENDIX A: THE CONDITIONS A1–A7

A1. $0 \in \mathbf{R}$ is an equilibrium state in the sense that $\mu(0) = 0$.

A2. $0 \in \mathbf{R}$ is exponentially asymptotic stable; that is, $\exists K, c > 0$ such that $\forall t \geq 0$, and $x_0 \in \mathbf{R}$ $\|\mu_t(x_0)\| \leq Ke^{-ct}\|x_0\|$, where $\|x\|$ is the Euclidean norm of x and $\mu_t(x_0)$ is the tth iteration $\mu_t(x_0) = \mu(\mu(\ldots\mu(x_0)))$ given the initial value x_0 (observe that $\mu_t(x_0) = \mu_k(\mu_s(x_{t-s-k}))$, $t > k > s$).

A3. $\forall x \in \mathbf{R}$, and for all neighborhoods V of $0 \in \mathbf{R}$ there is a nonnull conditional probability of $\sigma(X_{t-1})\varepsilon_t$ being in V given $X_{t-1} = x$.

A4. The distribution of ε_t has an absolutely continuous component (with respect to the Lebesgue measure) with positive probability density function over some open interval $(-\delta, \delta)$.

A5. $\partial f(0,0)/\partial x \neq 0$.

A6. μ is Lipschitz continuous over $\mathbf{R}$; that is, $\exists M > 0$ such that $\forall x, y \in \mathbf{R}$ $\|\mu(x) - \mu(y)\| \leq M\|x - y\|$.

A7. $\forall x \in \mathbf{R}$, $0 < E[\|\sigma(X_{t-1})\varepsilon_t\| | X_{t-1} = x] < \infty$.

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APPENDIX B: PROOFS

Proof of Proposition 2. The joint condition of H1 and H2 is $0 < a(x) < -2x$ if $x < 0$ and $-2x < a(x) < 0$ if $x > 0$. Therefore, we have, in both cases,

$$-2x < a(x) < 0 \Leftrightarrow |x + a(x)| < |x|,$$

$$0 < a(x) < -2x \Leftrightarrow |x + a(x)| < |x|,$$

so we can write $|\mu(x)| = |x + a(x)| < |x|$ or $|\mu(x)| \leq \phi|x|$ where $0 < \phi < 1$. Now

$$|\mu(x_0)| \leq \phi|x_0|,$$

$$|\mu_2(x_0)| = |\mu(\mu(x_0))| \leq \phi|\mu(x_0)| \leq \phi^2|x_0|,$$

$$\dots$$

$$|\mu_t(x_0)| \leq \phi^t|x_0|,$$

so A2 holds with $c = -\log\phi > 0$. ■

The remaining proofs are based on the following concepts.

Let $dX_t = a(X_t)\,dt + b(X_t)\,dW_t$, $t > t_0$, be a diffusion process, $I = (l, r)$ the state of space of X process, $s(z) = \exp\{-\int_{z_0}^{z} 2a(u)/b^2(u)\,du\}$ the scale density function, where z_0 is an arbitrary point inside I, and $m(u) = (b^2(u)s(u))^{-1}$ the speed density (see Karlin and Taylor, 1981). Let $S(l, x] = \lim_{x_1 \to l} \int_{x_1}^{x} s(u)\,du$ and $S[x, r) = \lim_{x_2 \to r} \int_{x}^{x_2} s(u)\,du$ where $l < x_1 < x < x_2 < r$. According to Arnold (1974, p. 114), if the infinitesimal coefficients a and b have continuous derivatives with respect to x, then there exists a unique continuous process defined until the random moment explosion η in the interval $t_0 < \eta \leq \infty$. Ikeda and Watanabe (1981, pp. 362–363) have proved that if $S(l, x] = S[x, r) = \infty$ then $P[\eta = \infty | X_0 = x] = 1$.

Now, it is known that if $S(l, x] = S[x, r) = \infty$ and $\int_l^r m(x)\,dx < \infty$ then X is ergodic and the invariant distribution P^0 has density $\bar{p}(x) = m(x)/\int_l^r m(u)\,du$ with respect to the Lebesgue measure (see Skorokhod, 1989, Theorem 16).

Proof of Proposition 4. We must prove $S(l, x] = S[x, r) = \infty$ where $l = -\infty$ and $r = \infty$ and $a(x) = e^k(e^{-\alpha_1(x-\tau)} - e^{\alpha_2(x-\tau)})$, $b(x) = \sigma$. First, consider the case $\alpha_1 > 0$, $\alpha_2 > 0$. We have

$$s(x) = \exp\left\{\frac{2e^k}{\sigma^2}\left(\frac{e^{-\alpha_1(x-\tau)}}{\alpha_1} + \frac{e^{\alpha_2(x-\tau)}}{\alpha_2}\right)\right\}.$$

Now, if $\alpha_1 > 0$, $\alpha_2 > 0$ then $s(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$. Then, $S(l, x] = S[x, r) = \infty$. In the case $\alpha_1 > 0$, $\alpha_2 = 0$ (the analysis of $\alpha_1 = 0$, $\alpha_2 > 0$ is similar) we have

$$s(x) = \exp\left\{\frac{2e^k}{\sigma^2}\left(\frac{e^{-\alpha_1(x-\tau)}}{\alpha_1} + x\right)\right\},$$

and so $s(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$. Then, $S(l, x] = S[x, r) = \infty$. ■

Proof of Proposition 5. We must prove $\int_l^r m(x)\,dx < \infty$. First, consider the case $\alpha_1 > 0$, $\alpha_2 > 0$. We have

$$m(x) = \sigma^{-2} \exp\left\{-\frac{2e^k}{\sigma^2}\left(\frac{e^{-\alpha_1(x-\tau)}}{\alpha_1} + \frac{e^{\alpha_2(x-\tau)}}{\alpha_2}\right)\right\}.$$

It is easy to conclude that there are some positive constants k_1, k_2 such that

$$m(x) \le f(x) = \exp\{k_1 - k_2 x^2\}, \quad \forall x \in I.$$

Now $\int_{\mathbf{R}} f(x) < \infty \Rightarrow \int_{\mathbf{R}} m(x)\,dx < \infty$. In the case $\alpha_1 > 0$, $\alpha_2 = 0$ (the analysis of $\alpha_1 = 0$, $\alpha_2 > 0$ is similar) we have

$$m(x) = \sigma^{-2} \exp\left\{-\frac{2e^k}{\sigma^2}\left(\frac{e^{-\alpha_1(x-\tau)}}{\alpha_1} + x\right)\right\}.$$

As

$$\begin{aligned}\int_{\mathbf{R}} m(x)\,dx &= \int_{-\infty}^0 m(x)\,dx + \int_0^\infty m(x)\,dx \\ &\le \int_{-\infty}^0 \exp\{k_1 - k_2 x^2\}dx + \int_0^\infty \exp\{-k_3 x\}dx < \infty\end{aligned}$$

then $\int_l^r m(x)\,dx < \infty$ (for $k_1, k_2, k_3 > 0$). ■

Proof of Proposition 6. We must prove $S(l, x] = S[x, r) = \infty$ where $l = -\infty$ and $r = \infty$ and $a(x) = e^k(e^{-\alpha_1(x-\tau)} - e^{\alpha_2(x-\tau)})$, $b(x) = e^{\sigma/2+\beta/2(x-\mu)^2}$. Consider the case $\beta > 0$. We have

$$s(x) = \exp\left\{\frac{-A_1[A_2 f_1(x) + A_3 f_2(x)]}{\sqrt{\beta}}\right\},$$

where

$$A_1 = \sqrt{\pi} e^{k-\sigma-\mu\alpha_1-\tau\alpha_2},$$

$$A_2 = e^{\tau(\alpha_1+\alpha_2)+(\alpha_1^2/4\beta)},$$

$$A_3 = e^{\mu(\alpha_1+\alpha_2)+(\alpha_2^2/4\beta)},$$

$$f_1(x) = \operatorname{erf}\left(\frac{2\beta(x-\mu)+\alpha_1}{2\sqrt{\beta}}\right),$$

$$f_2(x) = \operatorname{erf}\left(\frac{2\beta(x-\mu)-\alpha_2}{2\sqrt{\beta}}\right),$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

It is not possible to determine $S(x) = \int^x s(u)\,du$, but it can be proved that $S(x)$ is not integrable. The function $s(x)$ is of type

$$\exp\{c_1 \operatorname{erf}(c_2 + c_3 x) + c_4 \operatorname{erf}(c_5 + c_6 x)\},$$

where c_i are parameters. As $-1 \le \operatorname{erf}(x) \le 1$ it follows that $s(x)$ does not tend to zero as $|x| \to \infty$. Therefore, $S(l, x] = S[x, r) = \infty$. Now consider the case $\beta < 0$ and let $\beta^+ = |\beta|$. We have

$$s(x) = \exp\left\{-\frac{A_4[-A_5 f_3(x) + A_6 f_4(x)]}{\sqrt{\beta^+}}\right\},$$

where

$$A_4 = \sqrt{\pi} e^{k-\sigma},$$

$$A_5 = e^{-(\alpha_1(4\beta^+(-\tau+\mu)+\alpha_1)/4\beta^+)},$$

$$A_6 = e^{-(\alpha_2(4\beta^+(\tau-\mu)+\alpha_2)/4\beta^+)},$$

$$f_3(x) = \operatorname{erfi}\left(\frac{2\beta^+(x-\mu) - \alpha_1}{2\sqrt{\beta^+}}\right),$$

$$f_4(x) = \operatorname{erfi}\left(\frac{2\beta^+(x-\mu) + \alpha_2}{2\sqrt{\beta^+}}\right),$$

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{u^2} du.$$

The function $s(x)$ is of type

$$\exp\left\{-c_1 \operatorname{erfi}\left(c_2 x + c_3 - \frac{\alpha_1}{\sqrt{\beta^+}}\right) + c_4 \operatorname{erfi}\left(c_2 x + c_3 + \frac{\alpha_2}{\sqrt{\beta^+}}\right)\right\}$$

where $c_1, c_2, c_4 > 0$ and $c_3 \in \mathbf{R}$. As $\operatorname{erfi}(x)$ is not bounded, this case needs to be analyzed carefully. Let us study the function

$$\phi(x) = -c_1 \operatorname{erfi}(x + k_1) + c_4 \operatorname{erfi}(x + k_2),$$

$c_1, c_2 > 0$. The derivative of ϕ is

$$\phi'(x) = \frac{2}{\sqrt{\pi}}(-c_1 e^{(x+k_1)^2} + c_4 e^{(x+k_2)^2}).$$

If $k_2 > k_1$ then

$$\phi'(x) = \begin{cases} >0 & x > x_0 \\ <0 & x < x_0 \end{cases}$$

for some x_0, and, under these conditions, $\phi(x)$ increases as $|x|$ increases (to any arbitrary large value). Without any loss of generality, let us take $s(x) = e^{\phi(x)}$ where $k_2 = \alpha_2$

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and $k_1 = -\alpha_1$. Therefore the condition $\alpha_2 + \alpha_2 > 0$ ensures that $s(x) \to \infty$, and so $S(l, x] = S[x, r) = \infty$. ■

Proof of Proposition 7. We must prove $\int_l^r m(x)\,dx < \infty$. First, consider the case $\beta > 0$. We have

$$m(x) \propto \exp\left\{-\beta(x-\mu)^2 - \sigma + \frac{A_1[A_2 f_1(x) - A_3 f_2(x)]}{\sqrt{\beta}}\right\},$$

where A_1, A_2, A_3, f_1, and f_2 are those given in the proof of Proposition 6. The function $m(x)$ is of type

$$\exp\{c_1 + c_2 x - c_3 x^2 + f(x)\},$$

where $c_2, c_3 > 0$, and $f(x)$ satisfies $|f(x)| < L < \infty$ (notice that $f(x)$ depends on constants and on the erf function). Therefore it is possible to find $k_1, k_2, k_3 > 0$ such that $m(x) \leq \exp\{k_1 + k_2 x - k_3 x^2\}$, so $\int_l^r m(x)\,dx \leq \infty$. In the case $\beta < 0$ and using the ideas of the proof of Proposition 6 it can be shown that $\log m(x)$ tends quickly to $-\infty$ as $|x| \to \infty$. So $m(x)$ is integrable. ■