ABSTRACT

Title of dissertation: AN INEXACT INTERIOR-POINT ALGORITHM
FOR CONIC CONVEX OPTIMIZATION PROBLEMS

Simon Peter Schurr, Doctor of Philosophy, 2006

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In this dissertation we study an algorithm for convex optimization problems in conic form. (Without loss of generality, any convex problem can be written in conic form.) Our algorithm belongs to the class of interior-point methods (IPMs), which have been associated with many recent theoretical and algorithmic advances in mathematical optimization. In an IPM one solves a family of slowly-varying optimization problems that converge in some sense to the original optimization problem. Each problem in the family depends on a so-called *barrier function* that is associated with the problem data. Typically IPMs require evaluation of the gradient and Hessian of a suitable ("self-concordant") barrier function. In some cases such evaluation is expensive; in other cases formulas in closed form for a suitable barrier function and its derivatives are unknown. We show that even if the gradient and Hessian of a suitable barrier function are computed *inexactly*, the resulting IPM can possess

the desirable properties of polynomial iteration complexity and global convergence to the optimal solution set.

In practice the best IPMs are primal-dual methods, in which a convex problem is solved together with its dual, which is another convex problem. One downside of existing primal-dual methods is their need for evaluation of a suitable barrier function, or its derivatives, for the *dual* problem. Such evaluation can be even more difficult than that required for the barrier function associated with the original problem. Our primal-dual IPM does not suffer from this drawback—it does not require exact evaluation, or even estimates, of a suitable barrier function for the dual problem.

Given any convex optimization problem, Nesterov and Nemirovski showed that there exists a suitable barrier function, which they called the *universal barrier function*. Since this function and its derivatives may not be available in closed form, we explain how a Monte Carlo method can be used to estimate the derivatives. We make probabilistic statements regarding the errors in these estimates, and give an upper bound on the minimum Monte Carlo sample size required to ensure that with high probability, our primal-dual IPM possesses polynomial iteration complexity and global convergence.

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by

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Dedication

This dissertation is dedicated to my loving and patient grandmother

Harmony Zoe Tootell,

who lived from September 8, 1917 to June 27, 2004.

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All Scripture is God-breathed and is useful for teaching, rebuking, correcting and training in righteousness, so that the man of God may be thoroughly equipped for every good work. (2 Timothy 3:16-17, NIV)

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A	Matrix or linear operator in the constraints of an optimization problem.
b	Right-hand side in the constraints of a "primal" optimization problem.
C	Multiplicative constant in the universal barrier function.
c	Objective function coefficient vector of a "primal" optimization problem.
$E_1(x)$	Absolute error in $F_1(x)$.
$E_2(x)$	Absolute error in $F_2(x)$.
$\mathbf{E}(f)$	Expected value of the random variable f.
$F(x)$	A self-concordant barrier function; sometimes denotes the universal barrier function.
$F_1(x)$	Estimate of the gradient of a self-concordant barrier function.
$F_2(x)$	Estimate of the Hessian of a self-concordant barrier function.
$F'(x), F''(x), F'''(x)$	Derivatives of F.
$F_*(s)$	Conjugate function (a.k.a. Legendre dual or Fenchel dual) of F.
K	A pointed closed convex cone having nonempty interior.
$L(\cdot)$	Lagrangian function associated with an optimization problem.
m	The number of linear equality constraints in a "primal" optimization problem.
$\tilde{N}, N, N_1, N_2, N_\xi, N_*$	The sizes of various Monte Carlo samples.
$\mathcal{N}(\theta)$	Neighborhood of the primal-dual central path.
$\mathcal{N}(\theta, \underline{\mu}, \bar{\mu})$	Set of points in $\mathcal{N}(\theta)$ whose duality measure lies in the interval $[\underline{\mu}, \bar{\mu}]$.
n	Number of variables in a "primal" optimization problem.
$\mathcal{O}(f)$	Order of the function f.
$\mathcal{R}^n$	n-dimensional Euclidean space.
$\mathcal{R}^n_+$	Nonnegative orthant of $\mathcal{R}^n$.
$\mathcal{R}^{m\times n}$	Set of real matrices having m rows and n columns.
S^{n-1}	$(n-1)$-dimensional unit sphere in $\mathcal{R}^n$.
$\mathcal{S}^n$	Vector space of real symmetric matrices of order n.
s	Vector of "dual" variables.
v_P	Optimal value of a "primal" optimization problem.
v_D	Optimal value of a "dual" optimization problem.
w	Vector of "dual" variables.
x	Vector of "primal" variables.
$\mathcal{X}(\theta)$	Primal components of triples lying in $\mathcal{N}(\theta)$.
$\mathcal{X}(\theta, \underline{\mu}, \bar{\mu})$	Primal components of triples lying in $\mathcal{N}(\theta, \underline{\mu}, \bar{\mu})$.
$y^1, y^2, \cdots$	Vectors in $\mathcal{R}^n$ generated by a Monte Carlo sample.
$\beta_0, \beta_1, \beta_2$	Quantities measuring the length of various Newton directions.

$\underline{\delta}, \bar{\delta}$	A lower and an upper bound, respectively, on the ratio of duality measures at successive iterations of an interior-point algorithm.
$\delta\varphi(x), \delta g(x), \delta H(x)$	Absolute errors in the estimates $\hat{\varphi}(x), \hat{g}(x)$, and $\hat{H}(x)$.
$\Delta x, \Delta w, \Delta s$	Newton directions corresponding to the variables x, w, and s.
ϵ_1	The largest allowable "relative error" in $F_1(x)$.
ϵ_2	The largest allowable "relative error" in $F_2(x)$.
ε	A bound on the distance from the final iterate of an interior-point algorithm to the optimal solution set.
θ	A parameter measuring the width of the neighborhood $\mathcal{N}(\theta)$ of the primal-dual central path.
κ	A parameter related to the centering parameter τ.
$\lambda_{\min}(M)$	Minimum eigenvalue of the symmetric matrix M.
$\lambda_{\max}(M)$	Maximum eigenvalue of the symmetric matrix M.
μ	Duality measure of a primal-dual point, or a barrier parameter.
$\nabla f, \nabla^2 f$	Gradient and Hessian of the function f.
ν, ψ	Complexity parameter of a self-concordant barrier function.
$\varphi(x), g(x), H(x)$	Characteristic function of a cone, its gradient, and its Hessian, respectively.
$\hat{\varphi}(x), \hat{g}(x), \hat{H}(x)$	Estimates of $\varphi(x), g(x)$, and $H(x)$, respectively.
$\rho(f)$	Third absolute moment of the random variable f.
$\sigma(f)$	Standard deviation of the random variable f.
τ	Centering parameter in an interior-point method.
ω	Newton decrement of a self-concordant barrier function.
inf	Infimum.
sup	Supremum.
log	Natural logarithm.
bnd(S)	Boundary of the set S.
cl(S)	Closure of the set S.
ri(S)	Relative interior of the set S.
int(S)	Interior of the set S.
S^*	Dual of the set S.
diag(x)	Diagonal matrix whose diagonal is the vector x.
:=, =:	"Is by definition".
$\equiv$	"Is equivalent to".
$\forall$	"For all".
$\prec$, $\preceq$, $\succ$, $\succeq$	Partial orderings with respect to the cone of positive semidefinite matrices.
$x \leq y$	Componentwise inequality between vectors x and y.
$\langle x, y \rangle$	Inner product between points x and y in a vector space.
$x \in S$	x is an element of the set S.
$\subset, \subseteq$	Strict and nonstrict set inclusion.
$\lfloor x \rfloor$	Largest integer less than or equal to x.
$\lceil x \rceil$	Smallest integer greater than or equal to x.
$\lvert x \rvert$	Absolute value if x is a scalar, modulus if x is a complex number, and componentwise absolute value if x is a vector or matrix.
$\lVert A \rVert_p, \lVert x \rVert_p$	p-norm of a matrix A or vector x; $p = 1, 2$, or ∞.
$\lVert h \rVert_{x,F}$, $\lVert h \rVert^*_{x,F}$	Local norms of a vector h induced by the Hessian of F evaluated at x.
A^{-1}	Inverse of the nonsingular matrix A.
$A^{1/2}$	Unique symmetric positive definite square root of the symmetric positive definite matrix A.

A^T	Transpose of the matrix A.
A^*	Adjoint of the linear operator A.
$\bar{X}$	Negation of the event X.

Chapter 1

Introduction and background

This dissertation studies a certain class of problems in the area of *optimization*, also known as *mathematical programming*. In a typical optimization problem, one minimizes or maximizes a given "objective function" of some unknown variables, subject to constraints on these variables. Each constraint can be considered as a restriction on the variables due to, say, the limited availability of a certain resource, or a physical law that is relevant to the particular problem at hand. Resources can be interpreted broadly, and may refer to raw materials, people, time, money, etc. In practical terms, by solving an optimization problem, one determines an allocation of resources that yields the highest return for their use. We shall study continuous optimization problems whose data is *convex*. That is, the finite-dimensional set S of admissible variable values is convex, and the objective function is a convex function on S. Convex optimization problems have a wide range of applications; such problems can be found in fields such as chemistry, engineering, and economics. As a concrete example of an application, consider that in the field of optimal control, one seeks the best way to externally control a dynamical system in such a way that the total energy of the system is minimized. (Minimization of energy corresponds to a stable state.) A condition for stability is given by the well-known Kalman-Yakubovich-Popov lemma, and can be verified by solving a certain type of convex problem called a semidefinite optimization problem.

Due to the wide range of applications of convex optimization problems, it is of interest to

study efficient and implementable algorithms for such problems. Many of the best practical algorithms require the accurate evaluation of special auxiliary functions associated with the problem data. In some cases such accurate evaluations may be difficult or even impossible to obtain. The effect of using approximate, or *inexact*, evaluations will be studied. It will be shown that in the presence of inexact evaluations—which in principle require less computational effort than exact evaluations—our algorithm inherits the desirable properties of algorithms that use exact evaluations.

Given an optimization problem, which is called the *primal* problem, one can formulate an associated problem known as the *dual* problem. The same set of problem data or "inputs" used to formulate the primal problem is used to describe the dual problem. The variables, constraints, and objective function of the dual are different from those of the primal, but the two problems are related in such a way that information about the solution to the primal can be used to obtain information about the solution to the dual, and vice versa. More specifically, the dual is constructed in such a way that the optimal values of the dual variables, which are sometimes called "Lagrange multipliers", represent *marginal prices* associated with the primal constraints. We can similarly think of the optimal primal variables as marginal prices associated with the dual constraints. The marginal price associated with a constraint on a certain resource is a measure of how valuable it would be to allow the use of an additional unit of that resource. Such information is very useful if the data defining the optimization problem is uncertain, or if, given the solution of an optimization problem, we wish to solve a problem with perturbed data.

We now describe our problem in more concrete terms. We study convex optimization problems of the form

$$v = \inf_x \{\langle c, x\rangle \mid Ax = b,\ x \in K\}, \tag{1.0.1}$$

where A is a linear operator between finite-dimensional real vector spaces, each of which is equipped with an inner product $\langle \cdot, \cdot \rangle$, b and c are vectors of the appropriate dimensions,

and K is a closed convex cone lying in a finite-dimensional real vector space. In words, the right-hand side of (1.0.1) is the infimum of the linear objective function $\langle c, x\rangle$ subject to the constraints $Ax = b$ and $x \in K$ on the components of the vector x. The set of permissible values of x, $\{x \mid Ax = b,\, x \in K\}$, is called the *feasible set* of (1.0.1). We seek an *optimal* vector x, i.e., a feasible x satisfying $v = \langle c, x\rangle$, if such an x exists. We also seek the *optimal value* v, an extended real number. We call (1.0.1) a *conic convex optimization problem* because its feasible set includes a constraint that x lies inside a convex cone. Since convexity of the cone will always be assumed in this work, we will refer to such problems simply as conic optimization problems. We see that the feasible set of (1.0.1) is the intersection of an affine subspace and a convex cone. Any convex set can be written as such an intersection, hence any convex optimization problem can be represented in the above conic form. Writing a problem in conic form does not make it intrinsically easier to solve, but can provide us with helpful theoretical insights, especially if the dual problem associated with (1.0.1) is also written as a conic optimization problem.

The study of problems that can be expressed in the form (1.0.1) began in the late 1940s. At that time, the constraints of (1.0.1) were expressed as linear equality constraints $Ax = b$ and linear inequality constraints, say, $Cx \geq d$, where C is a matrix and d is a vector. (The inequality between vectors is to be taken componentwise.) This amounts to K being a polyhedral cone. By a linear change of variables, one can rewrite the constraints so that K is the nonnegative orthant. Since the objective function is linear and the constraints are defined by linear functions of the variables, the resulting problems are called linear optimization or linear programming problems.[1] Ever since the 1940s, researchers have spent much effort studying theoretically and practically efficient algorithms for such problems. Extensions to *nonlinear* problems, i.e., instances of (1.0.1) for which K is *nonpolyhedral*, have also been the focus of much attention.

With regard to linear optimization problems, for several decades the simplex method

[1]The term linear programming was originally used to describe such problems, but in the last few decades the term "programming" has become synonymous with the unrelated field of computer programming. Therefore we will avoid this term.

of George Dantzig was the method of choice. Despite much effort, the simplex method has not been successfully generalized to solve nonlinear problems or linear problems over more general cones, and so the need arose for algorithms to solve such problems. The foundation for such algorithms was laid in the 1950s and 1960s with pioneering work on barrier methods and penalty methods. At that time, a theory of nonlinear optimization had developed to such an extent that in 1968 Fiacco and McCormick were able to publish a book [11] containing many foundational principles of optimization that are recognizable in the most efficient algorithms of today. However the gap between theory and practice was significant, and for various reasons—some of them related to poor computer implementations—such methods fell out of favor. Furthermore, despite the theoretical advances that were made in nonlinear optimization, from the standpoint of computational complexity most algorithms were deficient. In short, until the mid-1980s, all algorithms for solving linear and nonlinear optimization problems were deficient in one of two ways: either their worst-case computational complexity was exponential, or the running time required to solve even moderately-sized problem instances was large.

As for convex (linear and nonlinear) optimization problems, these deficiencies have in many respects been addressed in the last twenty years with the proliferation of *interior-point methods*, which are iterative methods producing iterates lying in the (relative) interior of the feasible set of an optimization problem. One of the main ingredients of interior-point methods is a so-called *barrier function* that forces iterates to stay away from the (relative) boundary of the feasible set, where the set of solutions to (1.0.1) lies. The barrier function can be considered as a regularization term that becomes prohibitively large at points close to the boundary of the feasible set. A positive "regularization parameter" μ times such a function is added to the objective function $\langle c, x \rangle$ in (1.0.1), resulting in a regularized function that is strictly convex on the interior of the feasible set. At each iteration of an interior-point method, an approximate minimizer of the regularized function over the feasible set is computed. From one iteration to the next the regularization parameter is reduced slowly in much the same way as one varies the homotopy parameter in a homotopy

method. As μ is decreased toward zero, the effect of the regularized function is gradually decreased, and under certain conditions the sequence of resulting approximate minimizers converges to an optimal solution of (1.0.1). So instead of solving (1.0.1) directly—which is difficult—one solves a sequence of optimization problems that converge in an appropriate sense to (1.0.1). We now address how to solve a particular optimization problem in the sequence.

The classical method for performing smooth minimization *without* constraints is Newton's method, but this method is local rather than global. Suppose that Newton's method is used to minimize a smooth nondegenerate convex function f of several variables. The well-known sufficient conditions for local convergence of Newton's method to a minimizer, as given by the classical Kantorovich theory, hold in a neighborhood (or "basin of attraction") of the minimizer, and quadratic convergence to the minimizer is guaranteed there.[2] However the form of the above-mentioned sufficient conditions is problematic if one uses the "standard" Kantorovich result rather than the "affine invariant" version, because these conditions are not affine invariant. Yet the Newton iteration *is* affine invariant with respect to the coordinate system, in the sense that under an affine change of coordinates the Newton direction will be unchanged. In addition, regardless of which form of Kantorovich's result is used, the sufficient conditions for local convergence involve norms of quantities depending on the gradient and Hessian of f. The choice of norm is arbitrary, and it was unclear which norm is to be preferred. As we have already noted, barrier functions give rise to a family of convex optimization problems that is parameterized by a regularization parameter μ, which is decreased to zero. Unfortunately as μ approaches zero, the volume of the basin of attraction of the minimizer shrinks to zero, hence the regularized problems become more difficult to solve accurately. As μ approaches zero, in order to obtain a unit increase in the accuracy, the amount of work required increases significantly.

To remedy this situation, Nesterov and Nemirovski in the late 1980s proposed that a

[2]For a modern survey on the Kantorovich theory and its connection to interior-point methods, see [41] and the references therein.

special norm, *dependent on* f, be used. They showed that if the Hessian of f is Lipschitz continuous *in the metric induced by itself* (see [33, p. 32]), hence the name *self-concordant function*, then progress towards the minimizer can be measured with respect to this same metric. Since we are interested in solving constrained problems such as (1.0.1), rather than unconstrained problems, a self-concordant function f used to solve (1.0.1) should be related to K in an appropriate way. For example, f should be a barrier function for K. The term *self-concordant barrier function* (for the cone K) is used by Nesterov and Nemirovski to describe a barrier function that can suitably regularize the original objective function $\langle c, x\rangle$ in the presence of the constraint $x \in K$. A "suitable" barrier function gives rise to interior-point methods having low worst-case iteration complexity.

Since interior-point methods applied to problems of the form (1.0.1) typically involve the application of Newton's method to the original objective function $\langle c, x\rangle$ plus a multiple μ of a (self-concordant) barrier function, in principle numerical values of the gradient and Hessian of this function are required. In this work, we consider the situation where evaluation of the gradient and Hessian is either impossible or expensive to perform exactly, even if rounding errors are ignored. For example, in semidefinite optimization, where the variables are not vectors, but symmetric matrices that are constrained to be positive semidefinite, it may be necessary to compute inverses or Cholesky decompositions of dense matrices in order to evaluate the gradient and Hessian of a suitable self-concordant barrier function. Since these linear algebra tasks are expensive, it may be preferable to instead compute approximate inverses or approximate Cholesky decompositions, and hence an approximate gradient and Hessian. In doing so, we choose to sacrifice accuracy in the computed gradient and Hessian at each iteration, at the cost of additional iterations. The question arises as to how much accuracy can be sacrificed at each iteration without destroying the *polynomial worst-case iteration complexity* of the interior-point method. Roughly speaking, an algorithm has polynomial worst-case iteration complexity if the number of iterations required to obtain a near-optimal solution is at most polynomial in the problem size and the number of digits of accuracy.

Practically speaking, the most efficient algorithms for conic optimization are *primal-dual* interior-point methods, so called because they solve a given primal convex optimization problem together with its dual, which is also a convex optimization problem. In this work we study how the errors in evaluating the gradient and Hessian of a self-concordant barrier for the underlying cone affect the convergence and the iteration complexity of a primal-dual interior-point algorithm. As our main contribution, we show that if the "relative errors" in our gradient and Hessian estimates are not too large, then such estimates can be used in a primal-dual "path-following" algorithm that is globally convergent and has polynomial worst-case iteration complexity.

To our knowledge, all primal-dual interior-point methods for conic optimization in the literature require the evaluation—or at least approximate evaluation—of a barrier function for the dual cone K^*, or the gradient and Hessian of such a function.[3] Another contribution of this work lies in the fact that our primal-dual interior-point method does not require the evaluation—or even approximate evaluation—of a dual barrier function, or its derivatives. This is desirable because such quantities are in general difficult to compute. While our algorithm avoids the evaluation—or approximate evaluation—of dual barrier information, it is not clear how one can obtain an estimate of the gradient and Hessian of a barrier function for K itself. (We have already given details as to how this might be done in the case of semidefinite optimization, but in doing so, we used the fact that simple explicit formulas for the gradient and Hessian of a suitable self-concordant barrier function are known. In general this is not the case.) Nesterov and Nemirovski showed in [33] that for every pointed closed convex cone K having nonempty interior, there exists a cone-dependent *universal barrier function* F, which is a self-concordant barrier for K. However F was written in terms of a multidimensional integral whose domain of integration depends on K^*. In [33, p. 50] Nesterov and Nemirovski write, "[T]he universal barrier usually is too complicated to be used in interior-point algorithms, so [the result proving that the universal barrier function is a self-concordant barrier, and gives rise to theoretically efficient

[3]For optimization problems in which $K = K^*$, a self-concordant barrier for K can also be used for K^*.

interior-point methods] should be regarded as nothing but an existence theorem." In other words, although it was shown that various classes of interior-point methods generate a near-optimal solution in a polynomial number of iterations, it is not known if the computational effort required *at each iteration* is polynomial—or even finite—since only for some classes of convex optimization problems has it been established that the gradient and Hessian of F can be evaluated using polynomial or finite computational effort. For example, in the case that K is the nonnegative orthant in $\mathcal{R}^n$, F takes a particularly simple form, and its gradient and Hessian can be computed in $\mathcal{O}(n)$ arithmetic operations.

Despite the difficulty mentioned by Nesterov and Nemirovski, we show in the second part of this work how to use their universal barrier function in an interior-point method, even when the multidimensional integral defining this function cannot be computed exactly. The basic idea is to use a Monte Carlo method to estimate this integral, and hence the barrier gradient and Hessian. We investigate the properties of such gradient and Hessian estimates, and give probabilistic error bounds for these estimates. We indicate how large the sample size should be in order that with "high probability" the errors in the gradient and Hessian estimates are small enough for the estimates to be used in our inexact interior-point algorithm.

This work is organized as follows. We first give notation and preliminaries in Chapter 2. In Chapter 3 we introduce and study properties of the class of self-concordant barrier functions. We also discuss an important class of self-concordant barrier functions for cones known as *logarithmically homogeneous* barrier functions, explaining the properties of these functions that play a key role in interior-point methods for conic optimization. The main contribution of this work is in Chapter 4. There we state and analyze a primal-dual interior-point method for conic problems, which uses inexact values of the gradient and Hessian of a self-concordant barrier for the underlying cone. We then show how accurate the gradient and Hessian estimates need to be in order for the algorithm to be globally convergent and to converge in a polynomial number of iterations. We study three types of perturbation (or "error") in the exact gradient and Hessian: unstructured perturbations, structured pertur-

bations (meaning that the errors in the gradient and Hessian are related in a certain way), and no perturbations (meaning that the exact gradient and Hessian are used). In each case our interior-point method solves the primal *and* dual conic optimization problems without needing to evaluate—or even approximately evaluate—a barrier function for the dual cone.

In Chapter 5 we study an application in which a Monte Carlo method is used to estimate the gradient and Hessian of the above-mentioned universal barrier function. These estimates may then be used in our inexact interior-point method. After making some introductory comments in Section 5.1, we study in Section 5.2 the universal barrier function of Nesterov and Nemirovski. We present several equivalent expressions for this function, and for several cones of interest, give "explicit" formulas for the universal barrier function, its gradient and its Hessian. In Section 5.3 some background to Monte Carlo methods is given in preparation for later analysis. In Section 5.4 we study an application of the structured perturbations discussed in Chapter 4. Specifically, we estimate the gradient and Hessian of the universal barrier function using a Monte Carlo method. In accordance with the standard theory of Monte Carlo error estimates, the expected error in our Monte Carlo estimates decreases as the sample size increases. We give a minimum sample size such that the approximate gradient and Hessian are suitable for use in our inexact interior-point algorithm, where "suitable" means that with high probability, the algorithm is globally convergent and has polynomial iteration complexity. Finally in Chapter 6 we present conclusions and give some directions for future research.

Chapter 2

Notation and Preliminaries

2.1 Linear algebra

Denote by $\mathcal{R}^n$ the n-dimensional real Euclidean vector space equipped with the inner product $\langle x, y\rangle = x^T y$, where x^T denotes the transpose of the vector x. This inner product induces the Euclidean norm (also called the vector 2-norm) $\|x\|_2 := (x^T x)^{1/2} = (\sum_j x_j^2)^{1/2}$, where x_j is the j-th element of the vector x. We will also refer to two other vector norms: $\|x\|_1 := \sum_j |x_j|$ and $\|x\|_\infty := \max_j |x_j|$. Let $\mathcal{R}^{m\times n}$ denote the set of real matrices of order m by n, and denote by I_n the identity matrix of order n. When the order of an identity matrix is clear from the context, we omit the subscript. The rank of a matrix is the number of linearly independent rows, which is equal to the number of linearly independent columns. The matrix A is said to have full row rank if all rows of A are linearly independent.

Let A be a real square matrix. If x is a nonzero (possibly complex-valued) vector satisfying $Ax = \lambda x$ for some (possibly complex) number λ, then x is said to be an *eigenvector* of A associated with the *eigenvalue* λ. A square matrix of order n has n eigenvalues, counting multiplicity. If A is a symmetric matrix, then all eigenvalues of A are real. A symmetric matrix A is said to be *positive semidefinite* if $y^T Ay \geq 0$ for every vector y of the appropriate dimension. Equivalently, all eigenvalues of A are nonnegative. If $y^T Ay > 0$ for every nonzero vector y, then A is said to be *positive definite*. Equivalently, all eigenvalues of A are positive. If A has some positive and some negative eigenvalues, then A is said to

be an *indefinite* matrix. The sum of positive semidefinite (respectively positive definite) matrices is also positive semidefinite (positive definite).

The matrix 2-norm of the real square matrix A is the matrix norm induced by the vector 2-norm:

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}. \tag{2.1.1}$$

Let $\lambda_1(A), \lambda_2(A), \cdots$ denote the distinct eigenvalues of the (not necessarily symmetric) matrix A, and denote the modulus of the complex number λ by $|\lambda|$. Let x^i be an eigenvector of A associated with the eigenvalue $\lambda_i(A)$. It follows from (2.1.1) that

$$\|A\|_2 \geq \max_i \frac{\|Ax^i\|_2}{\|x^i\|_2} = \max_i \frac{\|\lambda_i(A)x^i\|_2}{\|x^i\|_2} = \max_i |\lambda_i(A)|. \tag{2.1.2}$$

Now let λ be a eigenvalue of A^TA. The matrix A^TA is symmetric, and is also positive semidefinite since $y^T(A^TA)y = (Ay)^T(Ay) = \|Ay\|_2^2 \geq 0$ for every y of the appropriate dimension. Hence $\lambda \geq 0$. By definition, $A^TAx = \lambda x$ for some eigenvector x, so $\|Ax\|_2^2 = x^TA^TAx = x^T(\lambda x) = \lambda\|x\|_2^2$. It follows from (2.1.1) that $\|A\|_2^2 = \max_i \lambda_i(A^TA)$. This result does not assume A is symmetric, but when A is symmetric, the maximum eigenvalue of A^TA can be written as $\max_i \lambda_i(A^TA) = \max_i \lambda_i(A^2) = \max_i |\lambda_i(A)|^2$. So when A is symmetric, the inequality in (2.1.2) is tight:

$$\|A\|_2 = \max_i |\lambda_i(A)|. \tag{2.1.3}$$

We denote the minimum and maximum eigenvalues of a matrix A having all real eigenvalues by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ respectively. It can be shown that for every vector x of the appropriate dimension,

$$\lambda_{\min}(A)x^Tx \leq x^TAx \leq \lambda_{\max}(A)x^Tx.$$

The left-hand and right-hand inequalities are tight when x is an eigenvector corresponding to $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ respectively. Suppose now that A is nonsingular. Then its eigenvalues are nonzero, so $Ax = \lambda x$ implies $\lambda^{-1}x = A^{-1}x$. Hence

$$\lambda_{\max}(A^{-1}) = \frac{1}{\lambda_{\min}(A)}.$$

Given a symmetric positive definite matrix A, there exists a unique symmetric positive definite square root of A, which we denote by $A^{1/2}$: $A = A^{1/2}A^{1/2}$. We will also write $A^{-1/2}$ to denote the unique positive definite square root of A^{-1}. Let x be an eigenvector corresponding to an eigenvalue λ of $A^{1/2}$. Then $Ax = A^{1/2}(A^{1/2}x) = A^{1/2}(\lambda x) = \lambda^2 x$. Hence λ^2 is an eigenvalue of A. It follows that

$$\lambda_{\min}(A^{1/2}) = (\lambda_{\min}(A))^{1/2}.$$

We now give some relations between the 2-norms of various matrices.

Lemma 2.1.1. *Let M, P be symmetric matrices of the same order. We have:*

(i)

$$\|MP^2M\|_2 = \|MP\|_2^2,$$

(ii)

$$\|MPM\|_2 \leq \|M^2P\|_2.$$

(iii) *If M is also nonsingular,*

$$\|P\|_2 \leq \|M^{-1}PM\|_2.$$

Proof. If M and P are symmetric, then $MP^2M = (MP)(MP)^T$, which is symmetric and positive semidefinite. So

$$\|MP^2M\|_2 = \lambda_{\max}(MP^2M) = \lambda_{\max}((MP)(MP)^T) = \|MP\|_2^2,$$

proving (i). To prove (ii), note that the symmetry of M and P guarantees MPM is also symmetric. Assume that M is nonsingular. The eigenvalues of a matrix are unaffected by a similarity transformation, viz., $\lambda_i(MAM^{-1}) = \lambda_i(A)$ for any A of the appropriate dimensions. Using this fact as well as (2.1.3) and (2.1.2), we have

$$\|MPM\|_2 = \max_i |\lambda_i(MPM)| = \max_i |\lambda_i(MMPMM^{-1})| = \max_i |\lambda_i(M^2P)| \leq \|M^2P\|_2,$$

as required. If M is singular, then the result holds with M replaced by $M + \epsilon I$, for all sufficiently small positive ϵ. Taking the limit as $\epsilon \to 0$, we obtain the required result, since the maximum eigenvalue of a matrix is a continuous function of the matrix components. The result in (iii) follows from (ii) after replacing M and P in (ii) by M^{-1} and MPM respectively. □

The next result, which follows from, e.g., [18, Exercise (8.1.16), p. 491], is useful for bounding the norm of a matrix for which only bounds on the components are known. Given square matrices A and B of the same dimensions, we write $A \leq B$ to mean that $B - A$ is a nonnegative matrix, i.e., $A_{ij} \leq B_{ij}$ for all i and j. We also write $|A|$ to denote the componentwise absolute value of the matrix A.

Lemma 2.1.2. *Let $B \in \mathcal{R}^{n\times n}$ be a nonnegative matrix. Then*

$$\|B\|_2 = \max_{A\in\mathcal{R}^{n\times n}} \{\|A\|_2 \mid |A| \leq B\}.$$

Lemma 2.1.2 holds even when B is nonsquare, but we do not require this more general result.

2.2 Convex analysis and cones

2.2.1 Convexity

This work is heavily related to convex functions and convex sets, the definitions of which we review now. We use $\subset$ to denote strict set inclusion and $\subseteq$ to denote nonstrict inclusion. A set $S \subseteq \mathcal{R}^n$ is said to be a *convex set* if for any two points in S, the line segment connecting these points lies entirely in S. Mathematically, for all scalars $\beta \in [0, 1]$,

$$x, y \in S \implies \beta x + (1 - \beta)y \in S.$$

Given a convex set $S \subseteq \mathcal{R}^n$, a function $f : S \to \mathcal{R}$ is said to be a *convex function* on S if for all $\beta \in [0, 1]$,

$$x, y \in S \implies f(\beta x + (1 - \beta)y) \leq \beta f(x) + (1 - \beta)f(y).$$

Geometrically, f never lies above its secants. If f is twice continuously differentiable on S, then f is convex on S if and only if its *Hessian*, i.e., the matrix of mixed second partial derivatives of f, is positive semidefinite at all points in S whose neighborhood is contained in S. We denote the Hessian of f by f'' or $\nabla^2 f$. Let A be a symmetric matrix. Since the Hessian of the quadratic form $f(x) = x^T Ax$ is A, the function $f(x)$ is convex (everywhere) if A is positive semidefinite. The function f is said to be *strictly convex* on S if f lies strictly below all of its secants: for all $\beta \in (0, 1)$,

$$x, y \in S \implies f(\beta x + (1 - \beta)y) < \beta f(x) + (1 - \beta)f(y).$$

The vector of first partial derivatives of f is called the *gradient* of f, denoted by f' or ∇f. A characterization of convex functions in terms of the gradient of f is as follows. Suppose that f is continuously differentiable at all points in S whose neighborhood is contained in

S. Then f is convex on S if and only if

$$f(x+y) \geq f(x) + \langle y, \nabla f(x) \rangle$$

for all x and y such that a neighborhood of x is contained in S, and $x+y \in S$.

2.2.2 Elementary topology on sets

We now introduce some topological notions on sets. A nonempty set $\mathcal{L} \subseteq \mathcal{R}^n$ that is closed under addition and real scalar multiplication is said to be a *linear subspace* of $\mathcal{R}^n$. Geometrically, $\mathcal{L}$ is a flat manifold passing through the origin $0 \in \mathcal{R}^n$. Now let $S+T := \{s+t \mid s \in S, t \in T\}$ represent the Minkowski sum of the sets S and T. Given a vector $x \in \mathcal{R}^n$ and a linear subspace $\mathcal{L} \subseteq \mathcal{R}^n$, the set $x+\mathcal{L} = \{x+y \mid y \in \mathcal{L}\}$ is said to be an *affine subspace* of $\mathcal{R}^n$. The smallest affine subspace $x+\mathcal{L}$ containing a given set $S \subseteq \mathcal{R}^n$ is called the *affine hull* of S, denoted by $\mathrm{aff}(S)$. It can be shown that

$$\mathrm{aff}(S) = \left\{ \sum_i \alpha_i x^i \;\middle|\; \sum_i \alpha_i = 1,\ \alpha_i \in \mathcal{R},\ x^i \in S\ \forall\, i \right\}.$$

The *convex hull* of a set S, denoted by $\mathrm{conv}(S)$, is the smallest convex set containing S:

$$\mathrm{conv}(S) = \left\{ \sum_i \alpha_i x^i \;\middle|\; \sum_i \alpha_i = 1,\ \alpha_i \geq 0,\ x^i \in S\ \forall\, i \right\}.$$

It is the set of convex combinations of points in S. If $S \subseteq \mathcal{R}^n$, then the summations in the definition of $\mathrm{conv}(S)$ need not consist of more than $n+1$ points in order to generate the convex hull.

Let $B = \{x \in \mathcal{R}^n \mid \|x\| \leq 1\}$ be the unit ball in $\mathcal{R}^n$, where $\|\cdot\|$ is some norm on $\mathcal{R}^n$. The *interior* of S, denoted by $\mathrm{int}(S)$, is given by

$$\mathrm{int}(S) = \{x \in S \mid (x+\epsilon B) \subseteq S \text{ for some } \epsilon > 0\}.$$

In words, $\text{int}(S)$ is the set of points in S for which a neighborhood is contained in S. (We can think of $x + \epsilon B$ as being a neighborhood of x.) If $\text{int}(S) = S$, then S is said to be *open*. The *relative interior* of S, denoted by $\text{ri}(S)$, is the interior of S when S is regarded as a subset of its affine hull:

$$\text{ri}(S) = \{x \in \text{aff}(S) \mid (x + \epsilon B) \cap \text{aff}(S) \subseteq S \text{ for some } \epsilon > 0\}.$$

If S is a nonempty convex set, then $\text{ri}(S)$ and S have the same affine hull, so $\text{ri}(S)$ is nonempty [44, Theorem 6.2]. The *closure* of S, denoted by $\text{cl}(S)$, is given by

$$\text{cl}(S) = \{x \mid \text{there exists a sequence of points in } S \text{ converging to } x\}.$$

If $\text{cl}(S) = S$, then S is said to be *closed*. We will denote the *boundary* of S by $\text{bnd}(S)$. It is the set of points lying in the closure of S but not its interior. Similarly we define the *relative boundary* of S to be the set of points lying in $\text{cl}(S)$ but not in $\text{ri}(S)$.

2.2.3 Cones

A *cone* $K \subseteq \mathcal{R}^n$ is a nonempty set such that $\alpha x \in K$ for all scalars $\alpha \geq 0$ and $x \in K$. Hence a cone always includes the origin.[1] If K is also a convex set, K is called a *convex cone*. A cone whose interior is nonempty is said to be *solid*. If K contains no lines, i.e., $K \cap -K = \{0\}$, then K is said to be *pointed*. A cone that is closed, convex, solid, and pointed, is said to be a *full cone*.[2]

Given two symmetric matrices M_1 and M_2, we will write $M_1 \preceq M_2$ or $M_2 \succeq M_1$ if and only if $M_2 - M_1$ is a positive semidefinite matrix. We will write $M_1 \prec M_2$ or $M_2 \succ M_1$ if and only if $M_2 - M_1$ is a positive definite matrix. These relations are *partial orderings* with respect to the cone of positive semidefinite matrices, which is a full cone.[3] That is,

[1] Some authors only require that $\alpha x \in K$ for all $\alpha > 0$, rather than for all $\alpha \geq 0$; they do not require that the origin lie inside a cone.

[2] Some authors call such a cone *proper* or *regular*.

[3] It is not difficult to verify that cone of positive semidefinite matrices is closed and convex. It is also solid

the relations are reflexive, transitive, and anti-symmetric.

The *dual* of a set $S \subseteq \mathcal{R}^n$ is the set of vectors making a nonnegative inner product with every vector in S:

$$S^* = \{y \mid \langle x, y\rangle \geq 0 \ \forall\, x \in S\}.$$

It can be easily verified that the dual of any set is a closed convex cone, so we may refer to S^* as the dual cone of S. Taking the dual of the dual does not always recover the original set. Let $S^{**} := (S^*)^*$ denote the double dual of S. It is shown in, e.g., [3, Theorem 2.2], that

$$S = S^{**} \iff S \text{ is a closed convex cone.} \tag{2.2.1}$$

If for some inner product $\langle \cdot, \cdot \rangle$, $K = K^*$, then the set K is said to be *self-dual*. It is not difficult to show that all self-dual sets are full cones. In the case that K is a pointed closed convex cone,

$$\text{int}(K^*) = \{y \mid \langle x, y\rangle > 0 \ \forall\, 0 \neq x \in K\},$$

so that K^* is solid, and it can be shown from the definitions that when K is a solid closed convex cone, K^* is pointed. It follows that if K is a full cone, then K^* is also. In Figure 2.1 we give two examples of sets in $\mathcal{R}^2$ with their dual cones.

In some contexts it will be convenient for us to consider matrices as linear operators. A full rank matrix $A \in \mathcal{R}^{m\times n}$ can be considered an *onto* linear operator $A : \mathcal{R}^n \to \mathcal{R}^m$, meaning that A is onto $\mathcal{R}^m$. The adjoint of a linear operator $A : \mathcal{R}^n \to \mathcal{R}^m$ is denoted by A^*, which maps $\mathcal{R}^m$ to $\mathcal{R}^n$, and is defined by $\langle x, A^*y\rangle = \langle Ax, y\rangle$ for all $x \in \mathcal{R}^n$ and $y \in \mathcal{R}^m$. It will be clear from the context whether * denotes the dual cone or an adjoint operator.

Let A be a linear operator and S a set lying in the domain of A. Define $A(S) = \{Ax \mid x \in$

because its interior, being the set of positive definite matrices, is nonempty. Finally, the cone is pointed because if M and $-M$ are both positive semidefinite matrices, then all eigenvalues of M and $-M$ are nonnegative. Hence they are all zero.

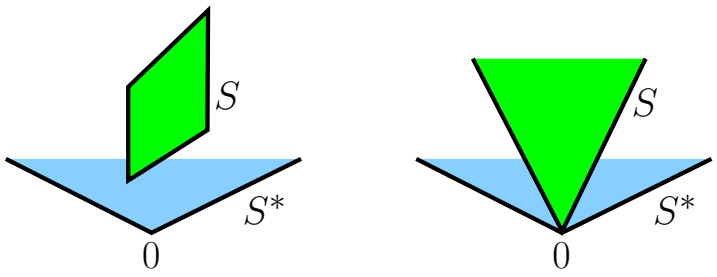

Figure 2.1: For two two-dimensional sets S, the dual S^* is shown. In the first figure only part of the (unbounded) dual cone is shown, and in the second figure only part of the cone S and its dual are shown.

$S\}$ to be the *linear image* of S (under A). The dual of the linear image $A(S)$ is given by

$$\begin{aligned} A(S)^* &= \{y \mid \langle y, z\rangle \geq 0 \ \forall\, z \in A(S)\} \\ &= \{y \mid \langle y, Ax\rangle \geq 0 \ \forall\, x \in S\} \\ &= \{y \mid \langle A^*y, x\rangle \geq 0 \ \forall\, x \in S\} \\ &= \{y \mid A^*y \in S^*\}. \end{aligned} \tag{2.2.2}$$

It is of interest in convex analysis and optimization to know when the linear image of a closed convex cone is closed. Related to this issue is the family of *theorems of the alternative*, which say that exactly one of two systems of linear or nonlinear inequalities or inclusions has a solution. These theorems come in many different forms; for a collection of theorems of the alternative involving only linear equalities and inequalities, see [27, p. 34]. Perhaps the most well known is Farkas' lemma [7], which we now give.

Lemma 2.2.1. *Let $A \in \mathcal{R}^{m\times n}$ and $b \in \mathcal{R}^m$. Exactly one of the following two systems of equalities and inequalities has a solution:*

(a) $Ax = b$ *and* $x \geq 0$*;*

(b) $A^Ty \geq 0$ *and* $b^Ty < 0$.

A cone is said to be *polyhedral* if it can written as the set of points satisfying a finite number of linear equalities and nonstrict inequalities. Hence a polyhedral cone can be written as

a linear image of the *nonnegative orthant* $\mathcal{R}^n_+ \equiv \{x \in \mathcal{R}^n \mid x \geq 0\}$. Farkas' lemma relies on the fact that any linear image of a polyhedral cone is closed. For *nonpolyhedral* closed convex cones it is not always the case that a linear image results in a closed set. An example of this is given by $A = \begin{bmatrix} \sqrt{2} & -2 & 0 \\ \sqrt{2} & 0 & -2 \end{bmatrix}$ and $K = \{x \in \mathcal{R}^3 \mid x_1 \geq (x_2^2 + x_3^2)^{1/2}\}$: it can be verified that $\begin{bmatrix} 1+\epsilon \\ -1 \end{bmatrix} \in A(K)$ for each $\epsilon > 0$, yet $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \notin A(K)$. We now generalize Lemma 2.2.1 to involve (possibly nonpolyhedral) convex cones; cf. [3, Theorem 3.1]. Note the assumption on the closedness of the linear image.

Lemma 2.2.2. *Let $A : \mathcal{R}^n \to \mathcal{R}^m$, $b \in \mathcal{R}^m$, and let $K \subseteq \mathcal{R}^n$ be a closed convex cone. Suppose that the convex cone $A(K)$ is closed. Then exactly one of the following two systems has a solution:*

(a) *$Ax = b$ and $x \in K$;*

(b) *$A^*y \in K^*$ and $\langle b, y\rangle < 0$.*

Proof. The statement in (a) is equivalent to $b \in A(K)$, which in view of the closedness of $A(K)$ and (2.2.1), is equivalent to $b \in \mathrm{cl}(A(K)) = A(K)^{**}$. By definition, this means that $\langle b, y\rangle \geq 0$ for all $y \in A(K)^*$, which in light of (2.2.2) is equivalent to $\langle b, y\rangle \geq 0$ for all y such that $A^*y \in K^*$. The last statement is the negation of (b). Since (a) and (b) are mutually exclusive alternatives, exactly one of (a) and (b) holds. □

Although the closedness condition in Lemma 2.2.2 may not necessarily hold if K is nonpolyhedral, an "asymptotic" Farkas Lemma that is slightly weaker than Lemma 2.2.2 always holds; see e.g., [43, Theorem 3.2.3]. Conditions under which the linear image of a closed convex set is closed can be found, e.g., in [38] and the references therein.

2.3 Convex optimization

In this section we explain some fundamentals of convex optimization. The problem data or "input" for a generic convex optimization problem consists of a closed convex set $S \subseteq \mathcal{R}^n$ and a convex function $f : \mathcal{R}^n \to \mathcal{R}$. We seek the minimum value of f over S, and a vector

x, if one exists, for which the minimum is attained. Mathematically we may write this as

$$v = \inf_{x \in \mathcal{R}^n} \{f(x) \mid x \in S\}. \tag{2.3.1}$$

The function f is known as the *objective function* and the set S is called the *feasible set.*[4] A feasible point/vector x for (2.3.1) is one satisfying $x \in S$; if $x \notin S$, then x is said to be *infeasible* for (2.3.1). The vector x contains n unknown variables. The infimum v is called the *optimal value* of (2.3.1). We seek feasible points x such that $v = f(x)$, i.e., the optimal value is attained. The set of such points is called the *optimal solution set* of (2.3.1). In the case that for any real number δ there exists an $x \in S$ with $f(x) < \delta$, we say that (2.3.1) is unbounded and set the optimal value to be $v = -\infty$. If the set S is empty, we adopt the convention that $v = +\infty$. There is no loss of generality in considering minimization problems as opposed to maximization problems, since $\sup\{f(x) \mid x \in S\} = -\inf\{-f(x) \mid x \in S\}$. If in (2.3.1) the "inf" is replaced by "sup", and the resulting problem is unbounded, we use the convention that $v = +\infty$. If instead S is empty, then we set $v = -\infty$.

It will be convenient to consider convex optimizations in a different form from that in (2.3.1)—one involving a convex cone. Firstly, we can assume without loss of generality that the objective function in (2.3.1) is *linear*. That is, $f(x) = \langle \alpha, x \rangle$ for some constant vector α. If f is not linear, we may add an additional constraint $f(x) \leq y$ to the existing constraint $x \in S$, where $y \in \mathcal{R}$ is a new variable, and minimize y instead of $f(x)$. So we have rewritten (2.3.1) as

$$v = \inf_{\substack{x \in \mathcal{R}^n \\ y \in \mathcal{R}}} \{y \mid f(x) \leq y,\ x \in S\}. \tag{2.3.2}$$

[4]Sometimes the feasible set is written as a finite number of inequality constraints $g_i(x) \leq 0$, $i = 1, \cdots, p$, where each $g_i : \mathcal{R}^n \to \mathcal{R}$ is a convex (possible nonlinear) function, and some linear equality constraints. There is no loss of generality in writing all constraints in $\leq$ form, since constraints of $\geq$ form can easily be converted to $\leq$ form. In fact the g_i need not be convex; what matters is that the points satisfying all constraints—however they are described—form a convex set.

The objective function of (2.3.2) is clearly convex, and the feasible set is convex since it is the intersection of two convex sets: the epigraph $\{(x, y) \mid f(x) \leq y\}$ of f, whose convexity follows from that of f, and S. Hence (2.3.2) is a convex optimization problem. In order for (x, y) to be feasible for (2.3.2), y can be decreased only as far as the infimum of $f(x)$ over $x \in S$. Therefore the optimal value and the x components of the optimal solution set for (2.3.2) are the same as those in (2.3.1).

It is preferable in some cases to write the feasible set in (2.3.2) as a "cone constraint", i.e., a constraint of the form $x \in K$ where K is a convex cone. This is done as follows. Given a closed convex set $\tilde{S} \subseteq \mathcal{R}^p$, define the *cone fitted to* $\tilde{S}$ by

$$K(\tilde{S}) = \mathrm{cl}\{(z, t) \mid z/t \in \tilde{S},\ t > 0\} \equiv \mathrm{cl}\{t(z, 1) \mid z \in \tilde{S},\ t > 0\} \subseteq \mathcal{R}^{p+1}. \tag{2.3.3}$$

The cone fitted to $\tilde{S}$ is an embedding of $\tilde{S}$ into $\mathcal{R}^{p+1}$, and $\tilde{S}$ is the intersection of $K(\tilde{S})$ with the affine subspace $\mathcal{R}^n \times \{t \mid t = 1\}$. It can be easily verified that $K(\tilde{S})$ is a closed convex cone, so we have written the convex feasible set of (2.3.2) as the intersection of a closed convex cone and an affine subspace. Hence any convex optimization problem can be written in the so-called *conic* form:

$$v_P = \inf_{x \in \mathcal{R}^n} \{\langle c, x \rangle \mid Ax = b,\ x \in K\}, \tag{2.3.4}$$

where $A : \mathcal{R}^n \to \mathcal{R}^m$, $b \in \mathcal{R}^m$, $c \in \mathcal{R}^n$, and $K \subseteq \mathcal{R}^n$ is a closed convex cone. We call this formulation a *conic (convex) optimization problem* because the feasible set involves a convex cone. We may assume without loss of generality that the operator A is onto. If this is not the case, then we can remove redundant constraints in the system $Ax = b$, reducing it to a smaller system whose linear operator is onto. We may also assume without loss of generality that the closed convex cone K is pointed and solid. If K is not pointed, i.e., contains a line, then it can be written as the (Minkowski) sum of a linear subspace and a pointed cone; see e.g., [44, p. 65]. As an example, the non-pointed closed convex cone $\{x \in \mathcal{R}^2 \mid x_2 \geq 0\}$ is the sum of the line $x_2 = 0$ and the nonnegative orthant $\mathcal{R}^2_+$, the latter being a full cone.

The existing linear equality constraints will be changed as a result of this reformulation. Now suppose $K \subseteq \mathcal{R}^n$ is not solid. Then K is solid as a subset of the span of the points in K, and this subset is a linear subspace having dimension, say, $k < n$. So (2.3.4) can be rewritten as a problem involving the cone constraint $x' \in K' \subseteq \mathcal{R}^k$, where K' is a solid cone. The existing linear equality constraints will be changed as a result of this change of variables.

The notion of *duality* is fundamental to the study of optimization problems. The basic idea is that for a given (not necessarily convex) optimization problem, which is called the *primal* problem, one can associate with it a *dual* optimization problem. Many types of dual problem exist. We will study perhaps the most widely known—the Lagrangian dual, but this dual is equivalent to others such as the Fenchel dual. A primal problem and its Lagrangian dual are linked through the Lagrangian function. The primal problem (2.3.4) can be written as the minimax problem

$$\inf_{x \in \mathcal{R}^n} \sup_{\substack{w \in \mathcal{R}^m \\ s \in K^*}} L(x, w, s), \tag{2.3.5}$$

where

$$L(x, w, s) \quad = \quad \langle c, x \rangle - \langle w, Ax - b \rangle - \langle s, x \rangle \tag{2.3.6}$$

is the Lagrangian associated with (2.3.4). Let us verify this assertion. For fixed $x \in K$,

$$\begin{aligned} \sup_{\substack{w \in \mathcal{R}^m \\ s \in K^*}} L(x, w, s) &= \langle c, x \rangle + \sup_{w \in \mathcal{R}^m} -\langle w, Ax - b \rangle + \sup_{s \in K^*} -\langle s, x \rangle \\ &= \langle c, x \rangle + \begin{cases} 0 & : \ Ax = b \\ +\infty & : \ \text{otherwise} \end{cases} + \begin{cases} 0 & : \ x \in X := \{x \mid \langle s, x \rangle \geq 0 \ \forall \, s \in K^*\} \\ +\infty & : \ \text{otherwise.} \end{cases} \end{aligned} \tag{2.3.7}$$

The set X is seen to be K^{**} by definition of a dual cone. From (2.2.1) we have $K^{**} = K$.

Hence

$$\sup_{\substack{w\in\mathcal{R}^m\\ s\in K^*}} L(x,w,s) = \begin{cases} \langle c,x\rangle & : \ Ax=b,\ x\in K \\ +\infty & : \ \text{otherwise,} \end{cases}$$

giving

$$\inf_{x\in\mathcal{R}^n} \sup_{\substack{w\in\mathcal{R}^m\\ s\in K^*}} L(x,w,s) = \inf_{x\in\mathcal{R}^n} \{\langle c,x\rangle \mid Ax=b,\ x\in K\},$$

which is (2.3.4). The Lagrangian dual of (2.3.4) is defined to be the associated maximin problem, i.e., (2.3.5) with the inf and sup reversed:

$$\begin{aligned} \sup_{\substack{w\in\mathcal{R}^m\\ s\in K^*}} \inf_{x\in\mathcal{R}^n} L(x,w,s) &= \sup_{\substack{w\in\mathcal{R}^m\\ s\in K^*}} \inf_{x\in\mathcal{R}^n} \langle c-A^*w-s,x\rangle + \langle b,w\rangle \\ &= \sup_{\substack{w\in\mathcal{R}^m\\ s\in K^*}} \begin{cases} \langle b,w\rangle & : \ c-A^*w-s=0 \\ -\infty & : \ \text{otherwise.} \end{cases} \end{aligned}$$

It can be verified in a similar way that the dual of the Lagrangian dual is the primal problem, so the following pair of problems are dual to each other:

$$v_P = \inf_x \{\langle c,x\rangle \mid Ax=b,\ x\in K\}, \tag{2.3.8}$$

$$v_D = \sup_{w,s} \{\langle b,w\rangle \mid A^*w+s=c,\ s\in K^*\}. \tag{2.3.9}$$

We now see the first advantage of using a conic formulation: symmetry between the primal problem, which involves optimization over the cone K, and the dual problem, which involves optimization over the dual cone K^*. Since K^* is a closed convex cone, (2.3.9) is also a convex optimization problem. The "structure" of the cones K and K^* play an important role in algorithms. We should point out that writing a pair of dual convex optimization problems in the conic form (2.3.8)–(2.3.9) does not make them intrinsically easier to solve.

However the symmetry between the two conic problems can assist us both theoretically and algorithmically.[5] We will refer to the constraints $Ax = b$ and $x \in K$ as the *primal constraints*, and the constraints $A^*w + s = c$ and $s \in K^*$ as the *dual constraints*. The components of the vector x will be called the *primal variables*, and the components of the vectors w and s will be called the *dual variables*.

We give three important special cases of the pair (2.3.8)–(2.3.9). In the case that $K = \mathcal{R}^n_+$ is the nonnegative orthant, (2.3.8) and (2.3.9) are called *linear optimization problems*. The nonnegative orthant is a self-dual cone (with respect to the Euclidean inner product), so the constraints $x \in K$ and $s \in K^*$ amount to $x \geq 0$ and $s \geq 0$ respectively.

Let $x_{1:n-1}$ be the vector $x \in \mathcal{R}^n$ with the last component removed, so $\|x_{1:n-1}\|_2 = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$. If K is the *second-order cone* $\{x \mid x_n \geq \|x_{1:n-1}\|_2\}$, then (2.3.8) and (2.3.9) are called *second-order cone optimization problems*.[6] It can be shown that the second-order cone is self-dual, hence it is a full cone. For a recent survey on second-order cone optimization problems, see [1].

Let $\mathcal{S}^{\hat{n}}$ be the vector space of symmetric matrices of order $\hat{n}$ equipped with the trace inner product $\langle X_1, X_2 \rangle = \text{trace}(X_1 X_2)$ for $X_1, X_2 \in \mathcal{S}^{\hat{n}}$. If K is the *positive semidefinite cone* $\{X \in \mathcal{S}^{\hat{n}} \mid X \succeq 0\}$, and $n = \frac{\hat{n}(\hat{n}+1)}{2}$, then (2.3.8) and (2.3.9) are called *semidefinite optimization problems*. The vectors x and s can be considered as matrices in $\mathcal{S}^{\hat{n}}$ or vectors in $\mathcal{R}^n$. The positive semidefinite cone is self-dual under the trace inner product. For recent surveys on semidefinite optimization problems, see [57, 52].

We have explained notions of feasibility and optimality for the generic convex optimization problem (2.3.1), but we now formally define these and other concepts for the pair (2.3.8)–(2.3.9) since we will focus our attention on this pair.

Definition 2.3.1 (Feasibility, strong feasibility[7]). *For the primal-dual pair of conic*

[5] In contrast to (2.3.9), the Lagrangian dual of a convex problem whose feasible set is written as a finite number of linear equality and nonlinear inequality constraints sometimes cannot be written down in a simple form. The reason is that in the maximin formulation of the dual problem, it is sometimes not possible to find the optimal value of the inner optimization problem $\inf_x L(x, w, s)$ explicitly in terms of w and s.

[6] The second-order cone is also known as the Lorentz cone or ice-cream cone.

[7] Some authors refer to strong feasibility as *strict* feasibility.

problems (2.3.8)–(2.3.9)*:*

(a) *The point* x *is said to be* feasible *for* (2.3.8) *if* $Ax = b$ *and* $x \in K$*;*

(b) *The pair* (w, s) *is said to be* feasible *for* (2.3.9) *if* $A^*w + s = c$ *and* $s \in K^*$*;*

(c) *The point* x *is said to be* strongly feasible *for* (2.3.8) *if* $Ax = b$ *and* $x \in \text{int}(K)$*;*

(d) *The pair* (w, s) *is said to be* strongly feasible *for* (2.3.9) *if* $A^*w + s = c$ *and* $s \in \text{int}(K^*)$*;*

(e) *A triple* (x, w, s) *is said to be a* (strongly) feasible primal-dual point *if* x *is (strongly) feasible for* (2.3.8) *and* (w, s) *is (strongly) feasible for* (2.3.9)*;*

(f) *If there exists a (strongly) feasible point for* (2.3.8)*,* (2.3.8) *is said to be a* (strongly) feasible problem*. Similarly for* (2.3.9)*.*

It can be shown that for any sets X and Y and any function $F : X \times Y \to \mathcal{R}$, the inequality $\inf_{x\in X} \sup_{y\in Y} F(x, y) \geq \sup_{y\in Y} \inf_{x\in X} F(x, y)$ holds. It follows that the optimal values of (2.3.8) and (2.3.9) are related by $v_P \geq v_D$. This result is known as *weak duality.* Recall that the optimal values are extended real numbers, i.e., they can take on the values $\pm\infty$. The (possibly infinite) quantity $v_P - v_D$ is called the *duality gap* of the pair (2.3.8)–(2.3.9). It is common to also refer to the duality gap associated with a primal-dual triple (x, w, s) as the difference between the primal and dual objective function values, $\langle c, x\rangle - \langle b, w\rangle$. If (x, w, s) is a feasible primal-dual point, then the (nonnegative) duality gap associated with (x, w, s) is given by

$$\begin{aligned}
\langle c, x\rangle - \langle b, w\rangle &= \langle A^*w + s, x\rangle - \langle Ax, w\rangle \\
&= \langle A^*w, x\rangle + \langle s, x\rangle - \langle Ax, w\rangle \\
&= \langle A^*w, x\rangle + \langle s, x\rangle - \langle x, A^*w\rangle \\
&= \langle s, x\rangle.
\end{aligned}$$

Definition 2.3.2 (Optimality). (a) *A feasible point* x *for* (2.3.8) *is said to be* optimal *for* (2.3.8) *if* $v_P = \langle c, x\rangle$*.*

(b) *A feasible pair* (w, s) *for* (2.3.9) *is said to be* optimal *for* (2.3.9) *if* $v_D = \langle b, w\rangle$*.*

(c) *A triple* (x, w, s) *is said to be an* optimal primal-dual solution *if* x *is optimal for* (2.3.8)

and (w, s) *is optimal for* (2.3.9).

Although we might in an application be interested in solving only one conic optimization problem, say (2.3.8), the most efficient practical algorithms solve (2.3.8) together with (2.3.9). The information gained by solving (2.3.9) is used in helping solve (2.3.8), and vice versa.

When solving optimization problems, one usually desires *global* optimal solutions, i.e., solutions for which the objective function is the best over all possible feasible points. Sometimes this is too ambitious, and instead one has to settle for *local* optimal solutions that give the best objective function value only locally. For convex optimization problems the set of locally optimal solutions is a (possibly empty) convex set, and all local optimal solutions are global optimal solutions.

Although all pairs of dual optimization problems satisfy weak duality ($v_P \geq v_D$), only some satisfy a stronger property known as *strong duality*.

Definition 2.3.3 (Strong duality). *If the problems* (2.3.8) *and* (2.3.9) *are such that* $v_P = v_D$*, and both optimal values are attained, then* strong duality *is said to hold for* (2.3.8)–(2.3.9).

It can be shown using Farkas' lemma (Lemma 2.2.1) that when K is the nonnegative orthant, if either (2.3.8) or (2.3.9) is feasible, then strong duality holds. This is not necessarily the case if K is nonpolyhedral. In fact it was shown in [49] that if K is any nonpolyhedral closed convex cone, then there exists a triple (A, b, c) in (2.3.8)–(2.3.9) such that either (2.3.8) or (2.3.9) is feasible, yet the duality gap is nonzero. Many examples of such problems can be found in the literature. See e.g., [26, Section 6.1], [43, Section 3.2], and [52, Section 4].

If the duality gap $v_P - v_D$ is zero, then the optimal values of the dual variables—whenever they exist—measure the sensitivity of the primal objective function $\langle c, x \rangle$ to changes in the primal constraint data. For this reason, among others, it is desirable for a pair of dual convex optimization problems to have a zero duality gap. It would also be desirable to know whether this is the case *before* solving (2.3.8)–(2.3.9). In order to ensure a priori

that a zero duality gap occurs, we usually must assume that the data (A, b, c, K) defining these problems satisfies a regularity condition called a *constraint qualification*. Roughly speaking a constraint qualification for a particular optimization problem is a condition on the constraints that ensures the feasible set is regular is a certain sense. Many types of constraint qualification are known; see e.g., [2]. We will use only one, which we now define formally.

Definition 2.3.4 (Generalized Slater constraint qualification). *The* generalized Slater constraint qualification (GSCQ) *is said to hold for* (2.3.8) *if this problem is strongly feasible. Similarly for* (2.3.9).

A sufficient condition for strong duality of (2.3.8)–(2.3.9) is that both problems satisfy the GSCQ. Moreover, under this condition, the optimal primal-dual solution set is nonempty. This set is also bounded under the further assumption that A is onto; for a proof, see e.g., [30, Theorem 1]. If the GSCQ holds for only one of the two problems, then a zero duality gap still results, but the optimal values need not both be attained, although at least one value is attained. Given a full cone K, if A is onto and the GSCQ holds for both (2.3.8) and (2.3.9), then the optimal primal-dual solution set is stable with respect to perturbations in A, b, and c. In particular, if the perturbations in A, b, and c are sufficiently small, then the optimal primal-dual solution set remains nonempty and bounded, and the primal and dual optimal values are continuous functions of A, b, and c.

Of interest are conditions that guarantee a zero duality gap *regardless* of the perturbations in the primal and dual objective function and right-hand side, b and c. Such a notion was called *universal duality* in [48]. The sufficient conditions (on A and K) for universal duality were also shown in [48] to be necessary. Furthermore, for fixed K, such conditions were shown to hold generically in both a metric and a topological sense.

2.4 Miscellaneous preliminaries

In this section we state other preliminary results that will be used elsewhere in this work. First we give the Cauchy-Schwarz inequality on the inner product space of real square integrable functions, and also a discretized version of this inequality.

Lemma 2.4.1. (a) *Let $a_1 : Y \to \mathcal{R}$ and $a_2 : Y \to \mathcal{R}$ be real square integrable functions, i.e., $\int_Y a_i(y)^2 < \infty$ for each i. Then*

$$\left(\int_Y |a_1(y)\, a_2(y)| \, \mathrm{d}y \right)^2 \leq \left(\int_Y a_1(y)^2 \, \mathrm{d}y \right) \left(\int_Y a_2(y)^2 \, \mathrm{d}y \right) .$$

(b) *Let $\{a_k\}$ and $\{b_k\}$ be square summable sequences, i.e., $\sum_k a_k^2 < \infty$ and $\sum_k b_k^2 < \infty$. Then*

$$\left(\sum_k |a_k b_k| \right)^2 \leq \sum_k a_k^2 \, \sum_k b_k^2 \, .$$

The following lemma follows from [59, Theorem 3.2]. It will be used in the analysis of our conic optimization algorithm. For completeness, we provide a proof.

Lemma 2.4.2. *Let $\varepsilon \in (0,1)$ and $\psi > 1$, and suppose that $\mu_0, \mu_1, \cdots$ is a sequence of positive numbers satisfying $\mu_{k+1} \leq (1 - \frac{1}{\psi})\mu_k$ for $k = 0, 1, \cdots$. Then*

$$k \geq \left\lceil \psi \log \left(\frac{\mu_0}{\varepsilon} \right) \right\rceil \implies \mu_k \leq \varepsilon.$$

Proof. For any nonnegative integer k we have $\mu_k \leq (1 - \frac{1}{\psi})^k \mu_0$, so $\log(\mu_k) \leq k \log(1 - \frac{1}{\psi}) + \log(\mu_0)$. Therefore $\mu_k \leq \varepsilon$ is implied by $k \log(1 - \frac{1}{\psi}) + \log(\mu_0) \leq \log(\varepsilon)$. The latter inequality is in turn implied by $-\frac{k}{\psi} + \log(\mu_0) \leq \log(\varepsilon)$, in view of the inequality $\log(1 - x) \leq -x$ for $x < 1$. Rearranging this gives the required result. □

Finally, given a set $S \subseteq \mathcal{R}^n$ and functions $f, g : S \to \mathcal{R}$, we write $f(x) = \mathcal{O}(g(x))$ to mean that there exists a positive constant C independent of x such that $|f(x)| \leq C|g(x)|$ for all $x \in S$.

Chapter 3

Self-concordant barrier functions

3.1 Introduction

In this chapter we discuss a key ingredient in the design of efficient algorithms for convex optimization problems, especially those in conic form. Given a full cone K, we seek a smooth convex function F that is "compatible" with the cone K in the conic problem (2.3.8). The properties of F that make for compatibility will be explained in this chapter. Basically, F must be capable of regularizing (2.3.8) in such a way that when Newton's method is applied to the regularized problem, fast convergence ensues. Once we have characterized the class of functions that help realize this goal, we will be in a position to present an algorithm for (2.3.8)–(2.3.9) based upon Newton's method. The algorithm will generate a near-optimal solution of (2.3.8)–(2.3.9) in an efficient way, where efficiency is measured in terms of a worst-case bound on the number of iterations of the algorithm.

3.2 Properties and characterizations of self-concordant barrier functions

Given a set $S \subseteq \mathcal{R}^n$ and a function $F : S \to \mathcal{R}$, we will denote the *k-th directional derivative* of F at the point $x \in \mathrm{int}(S)$ in the directions $h_1, \cdots, h_k$, by

$$F^{(k)}(x)[h_1, \cdots, h_k] = \frac{\partial^k}{\partial t_1 \cdots \partial t_k} F(x + t_1 h_1 + \cdots + t_k h_k)\big|_{t_1 = \cdots = t_k = 0} ,$$

assuming that F is k times continuously differentiable in a neighborhood of x. It follows from the smoothness of F that the differentiation operators commute, i.e., the directional derivatives are symmetric with respect to the collection of directions $h_1, \cdots, h_k$; see e.g., [25, Chapter XVII, Theorem 6.2]. If F is three times continuously differentiable on $\mathrm{int}(S)$, then for all $h_1, h_2, h_3 \in \mathcal{R}^n$, and all $x \in \mathrm{int}(S)$, we have $F'(x)[h_1] = F'(x)^T h_1$, $F''(x)[h_1, h_2] = h_1^T F''(x) h_2$, and $F'''(x)[h_1, h_2, h_3] = \frac{\mathrm{d}}{\mathrm{d}\alpha} h_1^T F''(x + \alpha h_3) h_2\big|_{\alpha=0}$. Define $F'''(x)[h_1, h_2]$ to be the vector satisfying

$$F'''(x)[h_1, h_2, h_3] = h_3^T F'''(x)[h_1, h_2]. \tag{3.2.1}$$

Both modern and classical barrier-type methods for convex optimization problems involve functions that become prohibitively large as the relative boundary of the feasible set is approached. This is referred to as the *barrier property*.

Definition 3.2.1 (Barrier property). *Let $S \subset \mathcal{R}^n$ be a closed convex set with nonempty interior. The function $F : \mathrm{int}(S) \to \mathcal{R}$ is said to satisfy the* barrier property *if for every sequence $\{x_i\} \subset \mathrm{int}(S)$ converging to a boundary point of S, $F(x_i) \to \infty$. Such an F is called a* barrier function *for S.*

It will be helpful to have a special designation for functions whose Hessian is positive definite on its entire domain.

Definition 3.2.2 (Nondegenerate convex function). *Let $S \subset \mathcal{R}^n$ be a closed convex set*

with nonempty interior. Let $F : \mathrm{int}(S) \to \mathcal{R}$ *be a twice continuously differentiable convex function. If* $F''(x)$ *is positive definite (implying that* F *is strictly convex) on* $\mathrm{int}(S)$*, then* F *is a* nondegenerate convex *function.*

A key concept in the study of barrier functions for interior-point methods is that of self-concordancy, introduced in [33].

Definition 3.2.3 (Complexity parameter, Newton decrement, self-concordant barrier function). *Let* $S \subset \mathcal{R}^n$ *be a closed convex set with nonempty interior. Suppose that the function* $F : \mathrm{int}(S) \to \mathcal{R}$ *satisfies the following properties:*

(a) F *is convex and three times continuously differentiable;*

(b)

$$|F'''(x)[h,h,h]| \le 2(F''(x)[h,h])^{3/2} \qquad \forall\, x \in \mathrm{int}(S),\ h \in \mathcal{R}^n; \tag{3.2.2}$$

(c)

$$\psi := \sup_{x \in \mathrm{int}(S)} \omega(F,x)^2 < \infty, \tag{3.2.3a}$$

$$\textit{where} \qquad \omega(F,x) := \inf_t \{t \mid |F'(x)[h]| \le t(F''(x)[h,h])^{1/2}\ \forall\, h \in \mathcal{R}^n\}. \tag{3.2.3b}$$

The quantity ψ *is called the* complexity parameter *of* F*, and* $\omega(F,x)$ *is called the* Newton decrement *of* F *at* x*;*

(d) F *satisfies the barrier property.*

Then F *is said to be a* ψ-self-concordant barrier function *for* S*.*

The following result from [33, Proposition 2.3.1] allows us to easily generate new self-concordant barrier functions from existing functions.

Lemma 3.2.4 (Properties of self-concordant barrier functions). *Let* $S_1 \subset \mathcal{R}^m$ *and* $S_2 \subset \mathcal{R}^n$ *be closed convex sets having nonempty interior, let* $F_1(x)$ *be a* ν_1*-self-concordant barrier for* S_1*, and let* $F_2(y)$ *be a* ν_2*-self-concordant barrier for* S_2*. Then:*

(a) $F_1(x) + F_2(y)$ *is a* $(\nu_1 + \nu_2)$*-self-concordant barrier for* $S_1 \times S_2 \subseteq \mathcal{R}^m \times \mathcal{R}^n$*;*

(b) *If the image of the affine map* $Az - b : \mathcal{R}^n \to \mathcal{R}^m$ *intersects* $\mathrm{int}(S_1)$*, then the restriction* $F(z) := F_1(Az - b)$ *is a* ν_1*-self concordant barrier for the closed convex set* $S := \{z \in \mathcal{R}^n \mid Az - b \in S_1\}$*.*

We now give two examples of self-concordant barrier functions.

Example 3.2.5. Let $a_i \in \mathcal{R}^n$ and $b_i \in \mathcal{R}$ for $i = 1, \cdots, m$ be such that the set

$$S_0 = \{x \mid a_i^T x > b_i,\ i = 1, \cdots, m\} \subset \mathcal{R}^n$$

is nonempty. It can then be verified that S_0 is the interior of the (closed convex) polyhedron

$$S = \{x \mid a_i^T x \geq b_i,\ i = 1, \cdots, m\},$$

and the function

$$F(x) = -\sum_{i=1}^{m} \log(a_i^T x - b_i)$$

is a nondegenerate m-self-concordant barrier for S.[1] We can verify this directly, i.e., from Definition 3.2.3, or more simply, we may build up F from one-dimensional barrier functions. We first verify that $F(x) = -\log(x)$ is a nondegenerate 1-self-concordant barrier function for the nonnegative half line $\mathcal{R}_+$. Clearly F is smooth on the positive half line, which is the interior of $\mathcal{R}_+$. Now for all $h \in \mathcal{R}$ and $x > 0$, the first three directional derivatives of F are

$$\begin{aligned} F'(x)[h] &= \frac{-h}{x}, \\ F''(x)[h,h] &= \frac{h^2}{x^2}, \\ F'''(x)[h,h,h] &= \frac{-2h^3}{x^3}. \end{aligned}$$

[1] To be more precise, according to Definition 3.2.2, F is a nondegenerate *convex* self-concordant barrier. However convexity is implied by self-concordancy, so it is redundant to describe such a barrier as convex.

Hence

$$|F'''(x)[h,h,h]| = 2\frac{|h|^3}{x^3} = 2\left(\frac{h^2}{x^2}\right)^{3/2} = 2(F''(x)[h,h])^{3/2},$$
$$\omega(F,x) = \inf_t \left\{ t \;\middle|\; \frac{|h|}{x} \le t\left(\frac{h^2}{x^2}\right)^{1/2} \;\forall\, h \right\} = 1,$$
$$\psi = \sup_{x\in \mathrm{int}(S)} \omega(F,x)^2 = 1,$$

verifying properties (b) and (c) of Definition 3.2.3. Clearly $F(x) \to \infty$ as $x \to 0^+$, so F also satisfies the barrier property. Finally $F''(x)[h,h] > 0$ for all nonzero h and for all $x > 0$, so F is nondegenerate. This verifies that F is a nondegenerate 1-self-concordant barrier function for $\mathcal{R}_+$. Of course $\mathcal{R}_+^m$ is the direct product of m nonnegative half lines, so from Lemma 3.2.4(a), $-\sum_{i=1}^m \log(x_i)$ is an m-self-concordant barrier function for $\mathcal{R}_+^m$, where x_i is the i-th component of the vector x. The function $-\sum_{i=1}^m \log(x_i)$ is called the *logarithmic barrier function* for the nonnegative orthant. Now apply Lemma 3.2.4(b), where A is the matrix whose rows are a_i^T and b is the vector whose i-th component is b_i. We conclude that F as given above is a nondegenerate m-self-concordant barrier for S.

Example 3.2.6. The function F given by $F(x) = -\log(1 - \|x\|_2^2)$ is a nondegenerate 1-self-concordant barrier for the unit ball $S = \{x \mid \|x\|_2 \le 1\} \subset \mathcal{R}^n$. Clearly F is smooth on $\mathrm{int}(S) = \{x \mid \|x\|_2 < 1\}$. Now for all $h \in \mathcal{R}^n$ and $x \in \mathrm{int}(S)$, the first three directional derivatives of F are

$$F'(x)[h] = \frac{2h^T x}{1 - \|x\|_2^2},$$
$$F''(x)[h,h] = \frac{2\|h\|_2^2(1 - \|x\|_2^2) + 4(h^T x)^2}{(1 - \|x\|_2^2)^2},$$
$$F'''(x)[h,h,h] = \frac{12h^T x\|h\|_2^2(1 - \|x\|_2^2) + 16(h^T x)^3}{(1 - \|x\|_2^2)^3}.$$

Using these, we can verify that properties (b) and (c) of Definition 3.2.3 hold with $\psi = 1$. It is easily seen that F also satisfies the barrier property, and since $F''(x)[h,h] > 0$ for all nonzero directions $h \in \mathcal{R}^n$ and all $x \in \mathrm{int}(S)$, F is nondegenerate. This shows that F is a

nondegenerate 1-self-concordant barrier for S.

Note that in the differential inequality in property (b) of Definition 3.2.3, the exponent $3/2$ ensures invariance with respect to a linear scaling of the direction h. The constant 2 is somewhat arbitrary, but was chosen so that the logarithmic barrier function $-\sum_{i=1}^{n} \log(x_i)$ is a self-concordant barrier for the nonnegative orthant in $\mathcal{R}^n$. We now discuss the complexity parameter and Newton decrement defined in Definition 3.2.3. The Newton decrement is so called because it measures the difference between the minimum of F and the minimum of the quadratic approximation of F about x; the location of the latter is the objective of Newton's method. The name "complexity parameter" for ψ was coined by Renegar [43], and is preferable to the term "parameter of the barrier" used in [33], since the latter may be confused with the term "barrier parameter" that dates back to classical barrier methods. The complexity parameter plays a central role in the theoretical development of interior-point methods for convex optimization problems. As the name suggests, it appears in the worst-case computational complexity estimates of interior-point methods. Given a closed convex set S having nonempty interior, it is desirable from a theoretical viewpoint to find self-concordant barriers for S for which ψ is as small as possible. It was shown in [33, Corollary 2.3.3] that if $S \subset \mathcal{R}^n$ is a closed convex set with nonempty interior and F is a ψ-self-concordant barrier for S, then $\psi \geq 1$. This bound is tight, as evidenced by the 1-self-concordant barrier given in Example 3.2.6.

Let F be a nondegenerate ψ-self-concordant barrier for $S \subset \mathcal{R}^n$, and let $x \in \mathrm{int}(S)$. Then $F''(x)$ and $F''(x)^{-1}$ are positive definite matrices for each $x \in \mathrm{int}(K)$. Therefore the second differential of F and the inverse of this second differential induce the following dual "local" norms on $\mathcal{R}^n$:

$$\|h\|_{x,F} = (h^T F''(x) h)^{1/2}, \qquad \|h\|_{x,F}^* = (h^T F''(x)^{-1} h)^{1/2}, \quad h \in \mathcal{R}^n. \tag{3.2.4}$$

The norms are called local because they depend on x. That the norms are dual follows from

$$\max_y \{h^T y \mid \|y\|_{x,F} \leq 1\} = (h^T F''(x)^{-1} h)^{1/2} \equiv \|h\|^*_{x,F} \ .$$

Given $r > 0$, the ball $\{y \mid \|x - y\|_{x,F} \leq r\}$ is called is called the *Dikin ellipsoid* of radius r centered at x. The elongation of the Dikin ellipsoid depends on x. It is readily seen from property (b) of Definition 3.2.3 that for any $x \in \text{int}(S)$, a self-concordant barrier satisfies $|F'''(x)[h, h, h]| \leq 2$ inside the unit ball $\|h\|_{x,F} \leq 1$. Therefore, inside this ball the Hessian of F is (locally) Lipschitz continuous with respect to the metric induced by the local norm $\|\cdot\|_{x,F}$. In other words, a quadratic approximation to F at x is reasonably reliable within a unit ball centered at x. This is important in order that Newton's method be successfully applied to the problem of minimizing F, or of more importance to us, the primal objective function $\langle c, x\rangle$ plus a positive multiple of F. So on one hand, the differential inequality in (3.2.2) implies that the Hessian of F—and by extension, F itself—does not grow too fast. In fact Renegar showed the following result regarding the rate of increase of F:

Lemma 3.2.7 ([43, Theorem 2.3.8]). *Let F be a nondegenerate ψ-self-concordant barrier for the closed convex set S having nonempty interior, and let $x \in \text{int}(S)$ and $y \in \text{cl}(S)$. For all $t \in (0, 1]$,*

$$F(y + t(x - y)) \leq F(x) - \psi \log(t).$$

On the other hand, property (3.2.3a) in Definition 3.2.3 shows that the Hessian of F does not grow too slowly (relative to F'). The following characterization of the Newton decrement was mentioned in [33, Section 2.2.1].

Lemma 3.2.8. *Let F be a nondegenerate convex function on a closed convex set S having nonempty interior. For all $x \in \text{int}(S)$, the Newton decrement of F is given by*

$$\omega(F, x) = \|F'(x)\|^*_{x,F} \ .$$

Proof. From the definition of the Newton decrement in (3.2.3b), we have

$$\begin{aligned}
\omega(F,x) &:= \inf_t\{t \mid |F'(x)[h]| \le t\|h\|_{x,F} \ \forall\, h \in \mathcal{R}^n\} \\
&= \inf_t\{t \mid F'(x)[h] \le t\|h\|_{x,F} \ \forall\, h \in \mathcal{R}^n\} \\
&= \inf_t\{t \mid F'(x)[h] \le t\|h\|_{x,F} \ \forall\, \|h\|_{x,F} = 1\} \\
&= \inf_t\{t \mid F'(x)[h] \le t \ \forall\, \|h\|_{x,F} = 1\} \\
&= \max_h \{F'(x)[h] \mid \|h\|_{x,F} = 1\} \\
&= \max_h \{F'(x)[h] \mid \|h\|_{x,F} \le 1\} \\
&= \|F'(x)\|^*_{x,F} \ ,
\end{aligned}$$

where the penultimate equality follows from the fact that the maximum of a linear function over a compact set exists and lies on the boundary. □

We see from Lemma 3.2.8 that the Newton decrement of F at x can be interpreted as the dual local norm of the gradient of F at x. The following result—which builds upon [33, Theorem 2.1.1]—characterizes the local behavior of the Hessian of a self-concordant barrier.

Lemma 3.2.9. *Let F be a self-concordant barrier for the closed convex set $S \subset \mathcal{R}^n$ having nonempty interior.*

(a) *If $x, y \in \mathrm{int}(S)$ are such that $r := \|x - y\|_{x,F} < 1$, then for all $h \in \mathcal{R}^n$,*

$$(1-r)\|h\|_{x,F} \le \|h\|_{y,F} \le \frac{1}{1-r}\|h\|_{x,F}, \tag{3.2.5}$$

and if F is also nondegenerate, then for all $h \in \mathcal{R}^n$,

$$(1-r)\|h\|^*_{x,F} \le \|h\|^*_{y,F} \le \frac{1}{1-r}\|h\|^*_{x,F}, \tag{3.2.6}$$

$$1 - r \le \|F''(x)^{-1/2}F''(y)^{1/2}\|_2 \le \frac{1}{1-r}, \tag{3.2.7}$$

$$1 - r \le \|F''(y)^{-1/2}F''(x)^{1/2}\|_2 \le \frac{1}{1-r}. \tag{3.2.8}$$

(b) *For every* $x \in \text{int}(S)$, $\|x-y\|_{x,F} < 1$ *implies that* $y \in \text{int}(S)$.

Proof. The inequalities in (3.2.5) are from [33, Theorem 2.1.1]. We now prove the inequalities in (3.2.6), (3.2.7), and (3.2.8) under the nondegeneracy assumption on F. To prove (3.2.7), observe that (3.2.5) is equivalent to $(1-r)^2F''(x) \preceq F''(y) \preceq \frac{1}{(1-r)^2}F''(x)$, which due to the nondegeneracy of F, is equivalent to $(1-r)^2 I \preceq F''(x)^{-1/2}F''(y)F''(x)^{-1/2} \preceq \frac{1}{(1-r)^2}I$. It follows that $(1-r)^2 \leq \|F''(x)^{-1/2}F''(y)F''(x)^{-1/2}\|_2 \leq \frac{1}{(1-r)^2}$. Since $\|F''(x)^{-1/2}F''(y)F''(x)^{-1/2}\|_2 = \|F''(x)^{-1/2}F''(y)^{1/2}\|_2^2$ by Lemma 2.1.1(a)(i), we have proven (3.2.7). The proof of (3.2.8) is similar, beginning from $(1-r)^2F''(y) \preceq F''(x) \preceq \frac{1}{(1-r)^2}F''(y)$, which is equivalent to (3.2.5). We now prove (3.2.6). Letting $D = F''(x)^{-1/2}F''(y)^{1/2}$ and using (3.2.7) and (3.2.8), we have

$$\|h\|_{x,F}^* = \|F''(x)^{-1/2}h\|_2 = \|DF''(y)^{-1/2}h\|_2 \leq \|D\|_2 \, \|h\|_{y,F}^* \leq \frac{1}{1-r}\|h\|_{y,F}^*,$$
$$\|h\|_{y,F}^* = \|F''(y)^{-1/2}h\|_2 = \|D^{-1}F''(x)^{-1/2}h\|_2 \leq \|D^{-1}\|_2 \, \|h\|_{x,F}^* \leq \frac{1}{1-r}\|h\|_{x,F}^*.$$

The result in (b) was proven in [33, Theorem 2.1.1]. □

Given a self-concordant barrier F for the closed convex set $S \subset \mathcal{R}^n$ having nonempty interior, the set $R(F) = \{h \in \mathcal{R}^n \mid \|h\|_{x,F} = 0\}$ is called the *recessive subspace* of F at $x \in \text{int}(S)$. It is a linear subspace, and is independent of x (see [33, Corollary 2.1.1]).

Remark 3.2.10. *It follows from Lemma 3.2.9 that a self-concordant barrier F for a full cone K is necessarily nondegenerate. To see why, suppose that F is degenerate, i.e., $\|h\|_{x,F} = 0$ for some $x \in \text{int}(K)$ and nonzero $h \in \mathcal{R}^n$. By Lemma 3.2.9(b), it follows that $x + h \in \text{int}(K)$ for any $h \in R(F)$. But this is impossible in light of the pointedness of K. So F must be nondegenerate.*

We will primarily be interested in self-concordant barriers for full cones. Although such barriers are necessarily nondegenerate, we will still call them nondegenerate self-concordant barriers for the sake of clarity. Another consequence of Lemma 3.2.9(b) is that for any

$x \in \text{int}(S)$, the Dikin ellipsoid of radius 1 centered at x will be contained in $\text{int}(S)$. However as x gets closer to the boundary of S, this ellipsoid becomes more elongated.

The following result follows from Definition 3.2.3 and [33, Proposition 9.1.1].

Lemma 3.2.11. *Let F be a self-concordant barrier for the closed convex set $S \subset \mathcal{R}^n$ having nonempty interior. We have*

$$|F'''(x)[h_1, h_2, h_3]| \leq 2\|h_1\|_{x,F}\|h_2\|_{x,F}\|h_3\|_{x,F} \qquad \forall\, x \in \text{int}(S),\ h_1, h_2, h_3 \in \mathcal{R}^n.$$

The following technical result will be used in the analysis of our interior-point method.

Lemma 3.2.12. *Let F be a nondegenerate self-concordant barrier for the full cone $K \subset \mathcal{R}^n$. Let $x \in \text{int}(K)$ and $h \in \mathcal{R}^n$ be such that $\beta := \|h\|_{x,F} < 1$, and let $\alpha \in [0, 1]$. We have*

$$\|F'''(x + \alpha h)[h, h]\|^*_{x+h,F} \leq \frac{2\beta^2}{(1-\beta)(1-\alpha\beta)}.$$

Proof. For all $\alpha \in [0, 1]$, $\|x - (x + \alpha h)\|_{x,F} = \alpha\beta < 1$. It follows from Lemma 3.2.9(b) that for all $\alpha \in [0, 1]$, $x + \alpha h \in \text{int}(K)$, hence the third directional derivatives of F at $x + \alpha h$ are well defined, as is the matrix $F''(x + \alpha h)^{-1/2}$. The result in (3.2.9a) follows from the definition of the $\|\cdot\|^*_{\cdot,F}$ norm in (3.2.4), and (3.2.9b) follows from the relation $\|h\|_2 = \max_{\|y\|=1} |y^T h|$. The result in (3.2.9c) follows from the definition of F''' in (3.2.1), (3.2.9d) follows from Lemma 3.2.11, and (3.2.9e) follows from the definition of the $\|\cdot\|_{\cdot,F}$

norm in (3.2.4).

$$\begin{aligned}
\|F'''(x+\alpha h)[h,h]\|^*_{x+\alpha h,F} &= \|F''(x+\alpha h)^{-1/2}F'''(x+\alpha h)[h,h]\|_2 && \text{(3.2.9a)}\\
&= \max_{y:\|y\|_2=1} |y^T F''(x+\alpha h)^{-1/2}F'''(x+\alpha h)[h,h]| && \text{(3.2.9b)}\\
&= \max_{y:\|y\|_2=1} |F'''(x+\alpha h)[h,h,F''(x+\alpha h)^{-1/2}y]| && \text{(3.2.9c)}\\
&\le \max_{y:\|y\|_2=1} 2\|h\|^2_{x+\alpha h,F}\|F''(x+\alpha h)^{-1/2}y\|_{x+\alpha h,F} && \text{(3.2.9d)}\\
&= 2\|h\|^2_{x+\alpha h,F} \max_{y:\|y\|_2=1} \|y\|_2 && \text{(3.2.9e)}\\
&= 2\|h\|^2_{x+\alpha h,F}.
\end{aligned}$$

Now from (3.2.5) (with $y = x+\alpha h$ so that $r = \alpha\beta$) we have $\|h\|_{x+\alpha h,F} \le \frac{\beta}{1-\alpha\beta}$. Using this and (3.2.6) (with y and x replaced by $x+h$ and $x+\alpha h$ respectively), we get

$$\begin{aligned}
\|F'''(x+\alpha h)[h,h]\|^*_{x+h,F} &\le \frac{1}{1-\|(1-\alpha)h\|_{x+\alpha h,F}}\ \|F'''(x+\alpha h)[h,h]\|^*_{x+\alpha h,F}\\
&\le \frac{1}{1-(1-\alpha)\|h\|_{x+\alpha h,F}}\ 2\|h\|^2_{x+\alpha h,F}\\
&\le \frac{1}{1-(1-\alpha)\frac{\beta}{1-\alpha\beta}}\ 2\left(\frac{\beta}{1-\alpha\beta}\right)^2\\
&= \frac{2\beta^2}{(1-\beta)(1-\alpha\beta)}.
\end{aligned}$$

□

3.3 Logarithmic homogeneity

In this section we study a special class of self-concordant barriers for convex cones called *logarithmic homogeneous barriers*, first defined in [33, Definition 2.3.2].

Definition 3.3.1 (Logarithmically homogeneous barrier, normal barrier). *Let K be a full cone. A function $F : \mathrm{int}(K) \to \mathcal{R}$ is said to be a* ν-logarithmically homogeneous barrier *for K if $\nu \ge 1$ and the following properties hold:*

(a) *F is convex and twice continuously differentiable;*

(b) *The barrier property holds for F on* $\mathrm{int}(K)$*;*

(c) *The* logarithmic-homogeneity relation *holds:*

$$F(tx) = F(x) - \nu \log(t) \qquad \forall\ x \in \mathrm{int}(K),\ t > 0. \tag{3.3.1}$$

If F also satisfies properties (a) *and* (b) *of Definition 3.2.3, then F is said to be a* ν-normal barrier.

We now give some important properties of logarithmically homogeneous barriers. Most of them are true for any twice continuously differentiable functions satisfying the logarithmic-homogeneity relation: it is not necessary that these functions be convex or satisfy the barrier property. Most of the results in the following lemma are taken from [33, Proposition 2.3.4]. It is shown in, e.g., [43, Theorem 3.3.1] that F' maps $\mathrm{int}(K)$ to $-\mathrm{int}(K^*)$.

Lemma 3.3.2. *Let K be a full cone, and $F : \mathrm{int}(K) \to \mathcal{R}$ be a twice continuously differentiable function satisfying the logarithmic-homogeneity relation* (3.3.1)*. Then for all $x \in \mathrm{int}(K)$ and $t > 0$:*

(a) $F'(tx) = \frac{1}{t}F'(x)$ *and* $F''(tx) = \frac{1}{t^2}F''(x)$*;*

(b) $F'(x)^T x = -\nu$*;*

(c) $F''(x)x = -F'(x)$*;*

(d) $\|x\|^2_{x,F} \equiv x^T F''(x)x = \nu$*;*

(e) *If F is also nondegenerate, then* $\|F'(x)\|^*_{x,F} \equiv \left(F'(x)^T F''(x)^{-1} F'(x)\right)^{1/2} = \nu^{1/2}$.

Proof. (a) Differentiating (3.3.1) with respect to x gives $tF'(tx) = F'(x)$. Differentiating again with respect to x gives $t^2F''(tx) = F''(x)$.

(b) Differentiating (3.3.1) with respect to t gives $F'(tx)^T x = -\nu/t$. Setting $t = 1$ gives the required result.

(c) Differentiating the relation in (b) with respect to x gives $F''(x)x + F'(x) = 0$.

(d) Take the inner product of the relation in (c) with x, and then use (b) to get the required result.

(e) Since $F''(x)$ is now assumed positive definite, $F''(x)^{-1}$ is well defined, and it follows

from (c) that $F''(x)^{-1}F'(x) = -x$. Taking the inner product of this equation with $F'(x)$ and using (b) gives the required result. □

The following corollary shows that if a nondegenerate convex function satisfies the logarithmic-homogeneity relation, then its gradient cannot grow too fast relative to its Hessian.

Corollary 3.3.3. *Let K be a full cone, and $F : \mathrm{int}(K) \to \mathcal{R}$ be a nondegenerate convex function satisfying the logarithmic-homogeneity relation with parameter ν. (That is, F is nondegenerate and satisfies properties (a) and (c) of Definition 3.3.1.) Then F satisfies* (3.2.3a) *with $\psi = \nu$.*

Proof. Since F is a nondegenerate convex function, it follows from Lemma 3.2.8 that the square of the Newton decrement is given by $(\|F'(x)\|^*_{x,F})^2$, and the latter quantity is ν in light of Lemma 3.3.2(e). □

Corollary 3.3.4—which is suggested by Corollary 3.3.3—combines results from [33, Corollary 2.3.2] and [33, Proposition 2.3.5], and gives a connection between logarithmically homogeneous barriers and self-concordant barriers.

Corollary 3.3.4. *If F is a ν-normal barrier for the full cone K, then F is also a nondegenerate ν-self-concordant barrier for K.*

Proof. From the definition, a ν-normal barrier satisfies properties (a), (b), and (d) of Definition 3.2.3. It follows from Corollary 3.3.3 that property (c) also holds with $\psi = \nu$. Hence F is a ν-self-concordant barrier for K. The nondegeneracy of F was established in Remark 3.2.10. □

Given a full cone K and a function $F : \mathrm{int}(K) \to \mathcal{R}$, we will make use of the *conjugate*

function of F, $F_* : \mathrm{int}(K^*) \to \mathcal{R}$, which is given by[2]

$$F_*(s) := \sup_{x \in \mathrm{int}(K)} \{-x^T s - F(x)\}, \quad s \in \mathrm{int}(K^*). \tag{3.3.2}$$

Since $F_*(s)$ is the pointwise supremum of linear functions of s, F_* is convex on $\mathrm{int}(K^*)$. We gather some properties of self-concordant barrier functions and their conjugates.

Lemma 3.3.5. *Let K be a full cone.*

(a) *If F is a nondegenerate self-concordant barrier for K, then F_* is a nondegenerate self-concordant barrier for K^*.*

(b) *If F is a ν-normal barrier for K, then F_* is a ν-normal barrier for K^*.*

(c) *Let F be a ν-normal barrier for K. For any scalars $\alpha > 0$ and β, and $s \in \mathrm{int}(K^*)$, we have $(\alpha F + \beta)_*(s) = \alpha F_*(s) + \alpha\nu \log(\alpha) - \beta$.*

Proof. Parts (a) and (b) are contained in [43, Theorem 3.3.1]. We now prove (c). From (3.3.2)

$$\begin{aligned}
(\alpha F + \beta)_*(s) &= \sup_{x \in \mathrm{int}(K)} \{-x^T s - \alpha F(x) - \beta\} \\
&= \alpha \sup_{x \in \mathrm{int}(K)} \{-x^T (s/\alpha) - F(x)\} - \beta \\
&= \alpha F_*(s/\alpha) - \beta \\
&= \alpha\big(F_*(s) - \nu \log(1/\alpha)\big) - \beta,
\end{aligned}$$

where the last equality follows from (b) and (3.3.1). □

In light of Lemma 3.3.5(a), the next result follows immediately from Lemma 3.2.9.

Lemma 3.3.6. *Let F be a nondegenerate self-concordant barrier for the full cone K.*

[2]Strictly speaking, in the definition of a conjugate function, the domain of F_* is not restricted to $\mathrm{int}(K^*)$. We include such a restriction here so that F_* is finite, which is the only case of interest to us. The definition of a conjugate function used here is found in, e.g., [35, 36, 43]. It is slightly different from that found elsewhere in the literature, including [33]. The difference is the minus sign in front of the $x^T s$ term, which turns the domain of F_* from $\mathrm{int}(-K^*)$ into $\mathrm{int}(K^*)$.

(a) *If* $t, s \in \text{int}(K^*)$ *are such that* $r := \|t - s\|_{t,F_*} < 1$*, then for all* $h \in \mathcal{R}^n$*,*

$$(1-r)\|h\|_{t,F_*} \leq \|h\|_{s,F_*} \leq \frac{1}{1-r}\|h\|_{t,F_*}.$$

(b) *For every* $s \in \text{int}(K^*)$*,* $\|s - t\|_{s,F_*} < 1$ *implies that* $t \in \text{int}(K^*)$*.*

The following results relate the derivatives of F to those of F_*. They use the fact that F' maps $\text{int}(K)$ to $-\text{int}(K^*)$, which was noted earlier, and furthermore, F'_* maps $\text{int}(K^*)$ to $-\text{int}(K)$, which follows from [43, Theorem 3.3.4].

Lemma 3.3.7. *Let* F *be a nondegenerate self-concordant barrier function for the full cone* K*. For every* $x \in \text{int}(K)$ *and* $s \in \text{int}(K^*)$ *we have:*

(a) $F'_*(-F'(x)) = -x$*;*

(b) $F'(-F'_*(s)) = -s$*;*

(c) $F''_*(-F'(x)) = F''(x)^{-1}$*;*

(d) $F''(-F'_*(s)) = F''_*(s)^{-1}$*;*

(e) *If* F *is also logarithmically homogeneous, then* $\|h\|_{-\mu F'(x),F_*} = \frac{1}{\mu}\|h\|^*_{x,F}$ *for all* $\mu > 0$ *and* $h \in \mathcal{R}^n$*.*

Proof. See [43, Theorem 3.3.4] for a proof of (a) and (c). (These results are proven for a larger class of functions than nondegenerate self-concordant barrier functions.) See [35, Section 2] for (b) and (d). We now prove (e). It follows from Lemma 3.3.5(b) that for some ν, F_* is a ν-normal barrier for K^*. Using Lemma 3.3.2(a), and then (c) above, we obtain for all $x \in \text{int}(K)$ and $\mu > 0$,

$$F''_*(-\mu F'(x)) = \frac{1}{\mu^2}F''_*(-F'(x)) = \frac{1}{\mu^2}F''(x)^{-1}.$$

So for all $h \in \mathcal{R}^n$, $\|h\|_{-\mu F'(x),F_*} = \|h\|_{x,F^{-1}/\mu^2} = \frac{1}{\mu}\|h\|_{x,F^{-1}} = \frac{1}{\mu}\|h\|^*_{x,F}$. □

To end this section we briefly discuss an important class of full cones and normal barrier functions that have been used successfully in practical algorithms. The following is taken from [35, 36].

Definition 3.3.8 (Self-scaled barrier, self-scaled cone). *Let F be a ν-normal barrier for the full cone K. F is said to be a* self-scaled barrier *for K if for all $x, w \in \operatorname{int}(K)$,*

(a) $F''(w)x \in \operatorname{int}(K^*)$,

(b) $F_*(F''(w)x) = F(x) - 2F(w) - \nu$.

If K admits a self-scaled barrier, then K is said to be a self-scaled cone.

The conditions in Definition 3.3.8 are symmetric in that if F is a self-scaled barrier for K, then F_* is a self-scaled barrier for K^*: for any $t, s \in \operatorname{int}(K^*)$, we have $F_*''(t)s \in \operatorname{int}(K)$, and $F(F_*''(t)s) = F_*(s) - 2F_*(t) - \nu$; see [35, Proposition 3.1].

The name "self-scaled" is due to property (a) in Definition 3.3.8: points can be mapped from $\operatorname{int}(K)$ to $\operatorname{int}(K^*)$ via the linear operator $F''(w)$, which is the Hessian of F evaluated at a point in K itself. It was shown that for any $x \in \operatorname{int}(K)$ and $s \in \operatorname{int}(K^*)$, the *scaling point* $w \in \operatorname{int}(K)$ such that $F''(w)x = s$, is unique. Another important property of self-scaled cones is that they are self-dual, i.e., $K = K^*$.

The class of self-scaled cones has been completely classified. In fact self-scaled cones are precisely the *homogeneous self-dual* or *symmetric* cones, as was noted in [16]. This class of cones includes the (real symmetric) positive semidefinite cone, the second-order cone, the cones of positive semidefinite Hermitian complex matrices and positive semidefinite quaternion matrices, and a certain 27-dimensional cone. Direct products of these cones are also self-scaled; see [35, Theorem 2.1]. The nonnegative orthant is also a self-scaled cone. This can be proven directly, or using the facts that (i) the real symmetric positive semidefinite cone is self-scaled, and (ii) the restriction of a self-concordant barrier function to a linear subspace is also a self-concordant barrier function (Lemma 3.2.4(b)). These facts are relevant because the nonnegative orthant is the restriction of the positive semidefinite cone to the subspace of diagonal matrices.

Chapter 4

An inexact primal-dual interior-point method for conic optimization

Contents

4.1 Introduction

We will study an algorithm to solve the conic convex optimization problem

$$v_P = \inf_x \{c^T x \mid Ax = b,\ x \in K\}, \tag{4.1.1}$$

where $A \in \mathcal{R}^{m \times n}$, $b \in \mathcal{R}^m$, $c \in \mathcal{R}^n$, and $K \subseteq \mathcal{R}^n$ is a closed convex cone. It was shown in Section 2.3 that any convex optimization problem can be recast in the form (4.1.1), where in Section 2.3 a general inner product $\langle \cdot, \cdot \rangle$ was used. There is essentially no loss of generality

in using the Euclidean inner product here. The Lagrangian dual of (4.1.1) is

$$v_D = \sup_{w,s} \{b^T w \mid A^T w + s = c,\ s \in K^*\}. \tag{4.1.2}$$

The following assumptions on the problem data will be made.

Assumption 4.1.1. (a) *A has full row rank.*

(b) *The equality constraints $Ax = b$ in* (4.1.1) *are nonvacuous.*

(c) *K is a full cone, i.e., a pointed closed convex cone having nonempty interior.*

As shown in Section 2.2, it follows from Assumption 4.1.1(c) that K^* is also a full cone. Nesterov and Nemirovski showed the following theoretically important result in [33, Theorem 2.5.1].

Lemma 4.1.2. *There exists a constant $C > 0$ independent of n such that for any full cone $K \subset \mathcal{R}^n$, there exists a ν-normal barrier F for K with $\nu = Cn$.*

Remark 4.1.3. *Our interest is in algorithms for which strong feasibility of* (4.1.1) *is maintained at each iteration. Therefore what is important is the behavior of F not on the whole of K, but on the restriction of K to the affine subspace $\{x \mid Ax = b\}$. The restriction of a self-concordant barrier to a linear subspace is also a self-concordant barrier (Lemma 3.2.4(b)). So the presence of linear constraints $Ax = b$ in* (4.1.1) *is of no concern to us as we study self-concordant barriers for K. The complexity parameter ν for the set $\{x \mid Ax = b,\ x \in K\}$ will be less than or equal to that for the cone K, since $\{x \mid Ax = b,\ x \in K\} \subset K$.*

The proof of Lemma 4.1.2 is constructive in the sense that a specific F, called the *universal barrier function*, was given in [33]. We discuss this function further in Section 5.2. For now we may be content knowing that for any convex optimization problem there exists a suitable barrier function. In this section we explain how a barrier function for the cone K may be used in an interior-point method to solve the conic optimization problems (4.1.1) and (4.1.2). First we give a brief history of some discoveries in the field of interior-point methods. We note that while some properties of self-concordant barrier functions such as

(b) and (c) in Definition 3.2.3 were not developed until the late 1980s, other properties such as the barrier property date back to much earlier work on interior-point methods for nonlinear optimization; see e.g., [11, Chapter 3].

It was not until the work of Khachiyan in [20, 21], which later became known as the ellipsoid algorithm, that a polynomial time bound on the worst-case computational complexity of a linear optimization algorithm was proven. (We can think of such a complexity bound as being a worst-case estimate on the amount of work required to obtain a near-optimal solution.) Khachiyan's algorithm was based on earlier algorithms for convex optimization by Yudin and Nemirovski [60], and independently, Shor [51]. Although a theoretical milestone had been reached, these algorithms proved to be extremely slow in practice. The next major development came in 1984 when Karmarkar's seminal paper [19] appeared. In it a polynomial-time algorithm for linear programming, or in our nomenclature, linear optimization, problems was presented. Karmarkar's algorithm was also novel in several respects, and contained some underlying ideas that would find their way into more recent algorithms, and that can be motivated in an easier way. One such idea was that of staying away from the (relative) boundary of the feasible set of the problem. By approaching the solution—which lies on the boundary—from the "center" of the feasible set, one can make faster progress. Progress was measured by Karmarkar in the form of a so-called "potential function", which combined a measure of the distance from the current objective function value to the optimal value, with a function measuring proximity to the "center" of the feasible set. (The potential function can be interpreted in various other ways.) More recently, variants of Karmarkar's algorithm called *potential reduction* methods have been developed to solve not just linear, but also classes of nonlinear convex problems quite efficiently in practice, while maintaining polynomial worst-case complexity. For surveys on potential reduction methods for linear optimization problems, see e.g., [53, Sections 2-6] and [59, Chapter 4], and the references therein. Potential reduction methods have also been developed for general conic problems (of the form (4.1.1)) in [33, Chapter 4], [30], and [53, Section 8]. Another major class of algorithms known as *path-following* algorithms has been developed to solve conic

optimization problems. In this chapter we will present an algorithm belonging to this class. Path-following algorithms for linear optimization can be found in e.g., [15] and [59, Chapters 5-6]. Primal algorithms for general conic problems are studied in e.g., [33, Chapter 3] and [43, Section 2.4].

We now explain the interior-point framework we will use to solve (4.1.1)–(4.1.2). Consider the *barrier problem* associated with (4.1.1):

$$v_P(\mu) = \inf_x \{c^T x + \mu F(x) \mid Ax = b\}. \tag{4.1.3}$$

where $\mu > 0$ is called the *barrier parameter*, and the function F in (4.1.3) is defined on the *interior* of the cone K. The following will be a standing assumption on F throughout the remainder of this chapter.

Assumption 4.1.4. *The function F is a ν-normal barrier, i.e., a ν-logarithmically homogeneous self-concordant barrier, for K.*

It follows from Assumption 4.1.4 that (4.1.3) is a convex optimization problem having a strictly convex objective function on the interior of K. The constraint $x \in K$ from (4.1.1) is enforced implicitly here by restricting the domain of F to $\text{int}(K)$. The properties of a ν-normal barrier are what make F "compatible" with K; the precise sense in which F is compatible with K will be explained in various ways throughout the rest of this chapter. Observe that the optimal solutions of (4.1.1) and (4.1.2) occur when the infimum or supremum of a linear function over a closed convex set is attained. Therefore these optimal solutions—if they exist—lie on the boundary of the convex (feasible) set. Since the primal and dual feasible sets are the intersection of an affine subspace and a closed convex cone, the primal and dual optimal solutions lie on the boundary of the cone in question (K for (4.1.1) and K^* for (4.1.2)).

We can consider (4.1.3) as a *family* of convex optimization problems parameterized by μ. Although the feasible set $\{x \mid Ax = b,\ x \in \text{int}(K)\}$ of (4.1.3) excludes points on the boundary of K, one can show that under certain conditions, (4.1.3) has a unique

minimizer, and as μ is decreased to zero, the sequence of minimizers of (4.1.3) converges to the minimizer of (4.1.1), which lies on the boundary of K. Furthermore, the corresponding sequence of optimal values $v_P(\mu)$ converges to v_P. So the classical idea was to solve a sequence of barrier problems for a sequence of μ values decreasing to zero. One can think of μ as being a weighting, used to balance the original objective function $c^T x$ and the barrier function F.

We see that the only (explicit) constraints in (4.1.3) are of linear equality form, and these cause us little difficulty as far as solving (4.1.3) is concerned. We will make the following assumption on the pair (4.1.1)–(4.1.2).[1]

Assumption 4.1.5. *The dual optimization problems* (4.1.1) *and* (4.1.2) *each satisfy the generalized Slater constraint qualification (Definition 2.3.4). That is, there exists an* $x \in \text{int}(K)$ *such that* $Ax = b$*, and a pair* $(w, s) \in \mathcal{R}^m \times \text{int}(K^*)$ *such that* $A^T w + s = c$*.*

As was noted in Section 2.3, under Assumptions 4.1.5 and 4.1.1(a) the optimal primal and dual solution sets are nonempty and bounded, and the duality gap is zero, i.e., $v_P = v_D$. Since the sequence of minimizers of (4.1.3) ideally converges to a solution of (4.1.1), the set of such minimizers has special significance. It was shown in [30, Lemma 1] that under Assumption 4.1.5, (4.1.3) has a unique minimizer for each $\mu > 0$.

Definition 4.1.6 (Primal central path). *Let* $x(\mu)$ *be the (unique) minimizer of* (4.1.3)*. The set*

$$\{x(\mu) \mid \mu > 0\} \subset \text{int}(K)$$

is called the primal central path*.*

In Chapter 3 we defined the conjugate function F_* associated with a function F. Lemma 3.3.5(b) shows that F_* is a ν-normal barrier for the full cone K^*. Hence F_* is a suitable barrier for

[1] This assumption can be made without loss of generality, since if it fails to hold, one can embed (4.1.1)–(4.1.2) in a higher-dimensional conic problem for which the assumption does hold. See e.g., [34].

the dual problem (4.1.2). The resulting dual barrier problem is

$$v_D(\mu) = \sup_{w,s}\{b^T w - \mu F_*(s) \mid A^T w + s = c\}. \tag{4.1.4}$$

We may define a concept analogous to the primal central path for the dual barrier problem. It was shown in [30, Lemma 2] that under Assumption 4.1.5, (4.1.4), like (4.1.3), has a unique minimizer for each $\mu > 0$.

Definition 4.1.7 (Dual central path). *Let $(w(\mu), s(\mu))$ be the (unique) minimizer of* (4.1.4)*. The set*

$$\{(w(\mu), s(\mu)) \mid \mu > 0\} \subset \mathcal{R}^m \times \mathrm{int}(K^*)$$

is called the dual central path*.*

Of special importance in primal-dual algorithms is the set of triples (x, w, s) such that for some $\mu > 0$, x lies on the primal central path and (w, s) lies on the dual central path. The resulting set—which is a curve—is important since it leads to the optimal primal-dual solution set, in which we are ultimately interested.

Definition 4.1.8 (Primal-dual central path). *Let $x(\mu)$ be the minimizer of* (4.1.3) *and let $(w(\mu), s(\mu))$ be the minimizer of* (4.1.4)*. The set*

$$\{(x(\mu), w(\mu), s(\mu)) \mid \mu > 0\} \subset \mathrm{int}(K) \times \mathcal{R}^m \times \mathrm{int}(K^*)$$

is called the primal-dual central path*.*

Figure 4.1 illustrates a possible primal central path and its connection to the primal optimal solution set.[2]

To exploit the fact that the primal-dual central path is a curve culminating at a point in the primal-dual optimal solution set, we will design an iterative algorithm whose iterates

[2]The figure could also describe a dual central path or a primal-dual central path projected onto two-dimensional space.

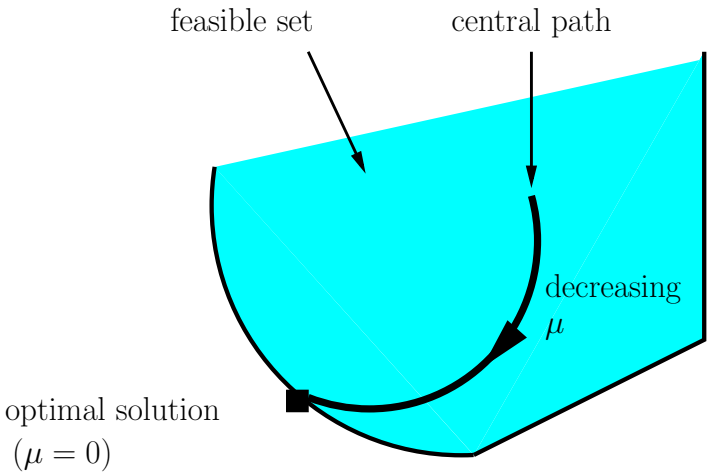

Figure 4.1: The shaded feasible set is a portion of the convex feasible set in (4.1.1). The primal central path is traced by varying $\mu \in (0, \infty)$.

stay close to the primal-dual central path while also converging to the primal-dual optimal solution set. To this end, we now characterize more explicitly the points lying on the primal-dual central path. In other words, we characterize the optimal solutions of the primal and dual barrier problems.

The study of necessary and sufficient optimality conditions for general nonlinear (including *nonconvex*) continuous optimization problems received much attention during the second half of the twentieth century. Some of the main results in this area can be found in, e.g., [11, 27, 2]. We will refer only to the results from this theory that are relevant to our context. Let $L(x, w)$ be the Lagrangian function associated with the problem (4.1.3):

$$L(x, w) = c^T x + \mu F(x) - w^T (Ax - b),$$

where w is the vector of dual variables associated with the m constraints represented by $Ax = b$. Solutions to (4.1.3) are related to stationary points of the Lagrangian function,

and the latter occur when

$$\begin{bmatrix} \nabla_x L \\ \nabla_w L \end{bmatrix} = \begin{bmatrix} c + \mu F'(x) - A^T w \\ -(Ax - b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.1.5}$$

The theory of necessary and sufficient optimality conditions for smooth nonlinear optimization problems, i.e., those problems whose objective and constraint functions are smooth functions of the variables, shows that the conditions in (4.1.5) are *necessary* in the presence of a constraint qualification for (4.1.3), such as that in Assumption 4.1.5. (The generalized Slater constraint qualification for (4.1.3) is the same as that for (4.1.1).) Under this constraint qualification, the conditions in (4.1.5) are necessary in the sense that in order for x to solve (4.1.3), it is necessary that there exists a w such that (x, w) solves (4.1.5); see e.g., [11, Theorem 21]. On the other hand, the conditions in (4.1.5) are *sufficient* for optimality of x in light of the convexity of the objective function and feasible set of (4.1.3); see e.g., [11, Theorem 20].

It was noted in Chapter 3 that for every $x \in \text{int}(K)$ we have $-F'(x) \in \text{int}(K^*)$. It follows from the first equation in (4.1.5) that if (x, w) solves (4.1.5), then $s := c - A^T w \in \text{int}(K^*)$. So for a fixed $\mu > 0$, the points x on the primal central path satisfy the following system of equations and inclusions for some w and s:

$$\begin{aligned} Ax &= b \\ A^T w + s &= c \\ \mu F'(x) + s &= 0 \\ x \in \text{int}(K), &\quad s \in \text{int}(K^*). \end{aligned} \tag{4.1.6}$$

Let us identify the vectors w and s in (4.1.6) as the dual vectors in (4.1.2). Then the conditions in (4.1.6) are just strong feasibility of (4.1.1) and (4.1.2), in addition to the nonlinear system of equations $\mu F'(x) + s = 0$ that links the primal and dual variables.

Now using the constraint qualification for (4.1.2) in Assumption 4.1.5, we can similarly

show that for a fixed μ the points (w, s) on the *dual* central path satisfy the following system of equations and inclusions for some x:

$$\begin{aligned} Ax &= b \\ A^T w + s &= c \\ \mu F_*'(s) + x &= 0 \\ x \in \operatorname{int}(K), &\quad s \in \operatorname{int}(K^*). \end{aligned} \tag{4.1.7}$$

For any $x \in \operatorname{int}(K)$ and $s \in \operatorname{int}(K^*)$, $\mu F'(x) + s = 0$ if and only if $\mu F_*'(s) + x = 0$, i.e., the third equation in (4.1.6) is equivalent to that in (4.1.7): $\mu F'(x) + s = 0$ is equivalent to $s = -\mu F'(x)$, which implies

$$\begin{aligned} \mu F_*'(s) &= \mu F_*'(-\mu F'(x)) \\ &= \mu F_*'(-F'(x/\mu)) && \text{(4.1.8a)} \\ &= \mu(-x/\mu) && \text{(4.1.8b)} \\ &= -x. \end{aligned}$$

Here (4.1.8a) follows from Lemma 3.3.2(a) and (4.1.8b) follows from Lemma 3.3.7(a). The reverse implication is proved similarly, using Lemmas 3.3.2(a) and 3.3.7(b). Therefore the primal and dual central paths are the same when embedded in (x, w, s) space. The equivalent relations $\mu F'(x) + s = 0$ and $\mu F_*'(s) + x = 0$ allow us to generate the dual central path from the primal central path and vice versa. Furthermore, the primal-dual central path is nothing but the set of triples (x, w, s) satisfying (4.1.6). The vectors w and s in the definition of the primal central path in fact belong to the dual central path, and the vector x in the definition of the dual central path belongs to the primal central path.

On the primal-dual central path there exists a simple relation between x and s:

$$x^T s = x^T(-\mu F'(x)) = -\mu x^T F'(x) = \nu\mu,$$

where the last equality follows from Lemma 3.3.2(b). Therefore to follow the primal-dual central path to the optimal primal-dual solution set, we define what is known as the *duality measure*,

$$\mu = \frac{x^T s}{\nu}, \tag{4.1.9}$$

and decrease μ to zero. The duality measure of a triple (x, w, s) may be thought of as the normalized duality gap; the duality gap associated with (x, w, s), $x^T s$, was defined in Section 2.3, and is nonnegative for any feasible (x, w, s). Throughout this chapter the parameter μ will be defined according to (4.1.9).

One might wonder whether it is better to replace the central path equation $\mu F'(x)+s = 0$ by the equation $-\mathrm{diag}(F'(x))^{-1}s + \mu e = 0$, where e is the vector of ones and $\mathrm{diag}(F'(x))$ is the matrix whose diagonal is the vector $F'(x)$. If $F(x) = -\sum_i \log(x_i)$ is the standard logarithmic barrier function for the nonnegative orthant, then $-\mathrm{diag}(F'(x))^{-1}s + \mu e = 0$ becomes the familiar $Xs = \mu e$, which is symmetric in x and s. Here X denotes the matrix whose diagonal is x. However there is no guarantee that for a given cone K, $\mathrm{diag}(F'(x))$ is nonsingular for every strongly feasible x.[3] Applying Newton's method to the nonlinear system of equations in (4.1.6), we obtain the linear system

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ \mu F''(x) & 0 & I \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta w \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T w - s \\ -\mu F'(x) - s \end{bmatrix}. \tag{4.1.10}$$

The primal Newton direction is Δx. Similarly we may apply Newton's method to the

[3]Consider, for example, the case that $K = \{x \in \mathcal{R}^2 \mid x_2 \geq |x_1|\}$ is the two-dimensional second-order cone. The self-concordant barrier function $F(x) = -\log(x_2^2 - x_1^2)$ for K is such that $\mathrm{diag}(F'(x))$ is singular along the strongly feasible ray $\{x \mid x_1 = 0,\ x_2 > 0\}$.

nonlinear system of equations in (4.1.7) to obtain the linear system

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ I & 0 & \mu F_*''(s) \end{bmatrix} \begin{bmatrix} \Delta x_* \\ \Delta w_* \\ \Delta s_* \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T w - s \\ -\mu F_*'(s) - x \end{bmatrix}. \tag{4.1.11}$$

The dual Newton direction is $(\Delta w_*, \Delta s_*)$. We now discuss the relationship between the solutions of (4.1.10) and (4.1.11).

Note that (4.1.10) and (4.1.11) do not generally have the same solution. However they do for self-scaled cones (Definition 3.3.8) due to the additional structure possessed by these cones and their associated self-concordant barriers. In this case $(\Delta x, \Delta w, \Delta s)$ can be considered a true primal-dual Newton direction. It is further noted in [35] that for the most widely studied self-scaled cones—the nonnegative orthant, the positive semidefinite cone, and the second-order cone—the optimal normal barrier F is such that $F(x)$ and $F_*(s)$ have exactly the same form up to an additive constant. The formulas for F, F_*, and the gradient and Hessian of F for such cones are given in Section 5.2.3. In [43, Section 3.5.1] the connection between self-scaled cones and the relation "$F(s) - F_*(s) = \text{constant}$" for $s \in \text{int}(K^*)$ is explained. (Note that $F(s)$ is well defined for a self-scaled cone, since such cones are self-dual ($K = K^*$), implying that $s \in \text{int}(K^*)$ lies in the domain of F.) A detailed discussion of self-scaled cones and self-scaled barrier functions associated with these cones can be found in [35, 36].

We will present a *short-step* algorithm to solve (4.1.1) and (4.1.2). Short-step algorithms date back to an important paper of Renegar [42], in which a polynomial-time primal algorithm for linear optimization was given. The name "short-step" arises from that fact that this class of algorithms generates at each iteration Newton steps that are "short" enough to be feasible. That is, no line search is required. This is a definite plus, since line searches may be expensive and difficult for many classes of cones K. The major downside is that such Newton steps are usually too conservative; in practice larger steps may be possible, leading to a faster reduction in the duality measure (to zero), and hence faster convergence

to the set of optimal primal-dual solutions. Before giving the algorithm and analyzing it, we first discuss some preliminary issues. We have noted that the primal-dual central path converges to the optimal primal-dual solution set, so it seems advantageous to design an algorithm that stays close to this path and makes progress towards the optimal primal-dual solution set. We now quantify what it means for a point to lie close to the central path. Given $\theta \in (0,1)$, define the $\mathcal{N}(\theta)$ *neighborhood* of the primal-dual central path by

$$\begin{aligned}\mathcal{N}(\theta) := \Big\{ & (x,w,s) \mid (x,w,s) \text{ is strongly feasible for (4.1.1)–(4.1.2)}, \\ & \|s + \mu F'(x)\|^*_{x,F} \leq \theta\mu,\ \mu = \frac{x^T s}{\nu} \Big\}.\end{aligned} \tag{4.1.12}$$

The neighborhood $\mathcal{N}(\theta)$ defined in (4.1.12) was used in [36, Section 6] for optimization over self-scaled cones. In the case that K is the nonnegative orthant and $F(x) = -\sum_i \log(x_i)$ is the standard logarithmic barrier function, we have $\|s + \mu F'(x)\|^*_{x,F} = \|Xs - \mu e\|_2$, so $\mathcal{N}(\theta)$ is the familiar $\mathcal{N}_2$ neighborhood used in linear optimization; see e.g., [59, p. 9]. Note that points in the set $\mathcal{N}(\theta)$ satisfy all conditions in (4.1.6) with the possible exception of the system of equations $s + \mu F'(x) = 0$, whose residual is sufficiently small (as measured in the $\|\cdot\|^*_{x,F}$ norm). Larger values of θ correspond to a wider neighborhood of the primal-dual central path, since θ dictates the extent to which points in the neighborhood fail to satisfy $s + \mu F'(x) = 0$, where μ is the duality measure. In fact, if for some $x \in \text{int}(K)$, $s \in \mathcal{R}^n$, and scalar $\tilde{\mu} > 0$, the quantity $\|s + \tilde{\mu} F'(x)\|^*_{x,F}$ is small, then $\tilde{\mu}$ is close to the duality measure of (x,w,s). The proof is as follows, where (4.1.13a) follows from Lemma 3.3.2(b) and (4.1.13b) follows from Lemma 3.3.2(d).

$$\begin{aligned}
|x^T s - \nu\tilde{\mu}| &= |x^T s + \tilde{\mu} x^T F'(x)| && \text{(4.1.13a)} \\
&= |x^T (s + \tilde{\mu} F'(x))| \\
&= |\big(F''(x)^{1/2} x\big)^T F''(x)^{-1/2} (s + \tilde{\mu} F'(x))| \\
&\leq \|F''(x)^{1/2} x\|_2 \ \|F''(x)^{-1/2} (s + \tilde{\mu} F'(x))\|_2 \\
&= \nu^{1/2} \|s + \tilde{\mu} F'(x)\|^*_{x,F}. && \text{(4.1.13b)}
\end{aligned}$$

We now give a key relation between the norms induced by the Hessians of F and F_*. To our knowledge it is new. It shows that in a neighborhood of the central path, the dual norm induced by the Hessian of F is approximately proportional to the norm induced by the Hessian of F_*.

Lemma 4.1.9. *Let F be a ν-normal barrier for the full cone K for some ν. Let $\theta \in (0,1)$ and $(x,w,s) \in \mathcal{N}(\theta)$. For any vector $h \in \mathcal{R}^n$,*

$$(1-\theta)\frac{1}{\mu}\|h\|^*_{x,F} \leq \|h\|_{s,F_*} \leq \frac{1}{(1-\theta)\mu}\|h\|^*_{x,F} \ .$$

Proof. Let $t = -\mu F'(x)$. As noted in Chapter 3, $F'(x) \in -\mathrm{int}(K^*)$ for all $x \in \mathrm{int}(K)$, so $t \in \mathrm{int}(K^*)$. Since $x \in \mathrm{int}(K)$ and $\mu > 0$, we may invoke Lemma 3.3.7(e). Using this lemma and $(x,w,s) \in \mathcal{N}(\theta)$, we have

$$r := \|t-s\|_{t,F_*} = \|\mu F'(x) + s\|_{-\mu F'(x),F_*} = \frac{1}{\mu}\|\mu F'(x) + s\|^*_{x,F} \leq \theta < 1.$$

Therefore the hypotheses of Lemma 3.3.6(a) hold for t and s as defined above. Applying this result and using $r \leq \theta$, we obtain

$$(1-\theta)\|h\|_{-\mu F'(x),F_*} \leq \|h\|_{s,F_*} \leq \frac{1}{1-\theta}\|h\|_{-\mu F'(x),F_*}. \tag{4.1.14}$$

But by Lemma 3.3.7(e) again, we have

$$\|h\|_{-\mu F'(x),F_*} = \frac{1}{\mu}\|h\|^*_{x,F} \ . \tag{4.1.15}$$

Combining (4.1.14) and (4.1.15) gives the required result. □

Given $\theta \in (0,1)$ and $0 < \underline{\mu} \leq \bar{\mu} \leq \infty$, we will find it convenient to define the truncated neighborhood

$$\mathcal{N}(\theta, \underline{\mu}, \bar{\mu}) := \big\{(x,w,s) \in \mathcal{N}(\theta) \mid \underline{\mu} \leq \frac{x^T s}{\nu} \leq \bar{\mu}\big\}. \tag{4.1.16}$$

We have already noted that under Assumptions 4.1.1 and 4.1.5, the primal and dual optimal solution sets are nonempty and bounded. It follows that the level sets of the duality measure must also be bounded in (x, w, s) space; see [30, Theorem 1]. Hence $\mathcal{N}(\theta, \underline{\mu}, \bar{\mu})$ is bounded for fixed $\theta, \underline{\mu}$, and $\bar{\mu}$.

4.2 Statement of our interior-point method

We now present an iterative algorithm to solve (4.1.1)–(4.1.2). This algorithm is a *primal-dual feasible-point* algorithm, meaning that at each iteration the iterates are feasible with respect to the primal and dual constraints. This is achieved by using a feasible starting triple (x^0, w^0, s^0), giving $b - Ax^0 = 0$ and $c - A^T w^0 - s^0 = 0$ in the right-hand side of (4.1.10). It involves the application of Newton's method to the system of equations in (4.1.6) for a sequence of μ values converging to zero. That is, we solve a sequence of linear systems of equations, each having a form similar to (4.1.10). The algorithm uses two parameters. One is $\theta \in (0, 1)$, which stipulates the width of the neighborhood $\mathcal{N}(\theta)$ inside which the initial iterate is constrained to lie. To explain the need for the second parameter, let us observe that the solution of (4.1.6) for a fixed $\mu > 0$ is the triple $(x(\mu), w(\mu), s(\mu))$ on the primal-dual central path. It is desirable to find this point since following the central path from this point leads to an optimal solution. However in order to guarantee progress at each iteration toward the optimal solution set, we need to not only make progress towards the central path, but also toward the optimal solution itself. In order to balance these two objectives, we multiply μ in the right-hand side of (4.1.10) by a so-called *centering parameter* $\tau \in (0, 1)$. (This idea is not new; see e.g., [59, p. 8], where the parameter is denoted by σ.) Now suppose we can find a triple $(x^0, w^0, s^0) \in \mathcal{N}(\theta)$ to use as a starting point in an iterative algorithm that solves (4.1.1)–(4.1.2).[4] By choosing τ and θ appropriately, we will ensure

[4]Finding such an initial point can be as difficult as solving (4.1.1) and (4.1.2). However such a point can be found by solving a so-called *homogeneous self-dual* optimization problem. This class of optimization problems is not to be confused with the class of homogeneous self-dual cones, mentioned in Section 3. A homogeneous self-dual optimization problem can be formulated regardless of the convex cone. This procedure was first developed for linear optimization problems. The basic idea is to embed (4.1.1) and (4.1.2) in a higher-dimensional conic optimization problem that possesses certain symmetries, and furthermore, an obvious

that all iterates of our algorithm stay in the neighborhood $\mathcal{N}(\theta)$. Moreover, the iterates converge to an optimal solution in a reasonable number of iterations by guaranteeing at each iteration a geometric reduction in the duality measure from its initial value of $(x^0)^T s^0/\nu$.

In practice, computations are performed in real arithmetic as opposed to rational arithmetic, in which all quantities are expressed as rational numbers. Therefore we cannot realistically expect to compute an exact optimal solution. So our goal will be to compute feasible primal and dual points that are within some prescribed distance of the set of optimal primal-dual solutions. One measure of the closeness of a feasible primal-dual point to the optimal solution set is the duality measure $x^T s/\nu$, which due to Assumption 4.1.5 is zero at an optimal primal-dual solution.

Definition 4.2.1 (ε-optimal solution). *Given $\varepsilon \in (0,1)$, an* ε-optimal solution *of* (4.1.1)–(4.1.2) *is a feasible primal-dual point whose duality measure is no greater than ε.*

So when we refer to convergence of our algorithm, we mean that for some prescribed ε, an ε-optimal solution of (4.1.1)–(4.1.2) has been generated.

We now explain the sense in which our interior-point method uses *inexact barrier function evaluations*. We will consider three cases, each of which involves a different assumption on the errors in $F'(x)$ and $F''(x)$ at a given $x \in \text{int}(K)$, due to computing these quantities inexactly. We first analyze the case where the errors or "perturbations" in the gradient and Hessian of F are completely unstructured. We then study the case that there are no perturbations, i.e., F' and F'' are evaluated exactly. We finally consider the case where the perturbations are structured, i.e., the errors in our estimates of F' and F'' are related. We stress that our primal-dual interior-point method, unlike others in the literature, does not require the evaluation—or an estimate—of a barrier function or its derivatives for the dual problem (4.1.2). Only in the analysis do we make use of a suitable barrier function (F_*) for the dual problem.

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strongly feasible point. For the details of how this can be extended to convex optimization problems, see [34], where it is assumed that exact barrier and Hessian information is known. Since we are not making such an assumption, we need to *assume* that an initial point can still be obtained in polynomial time.

Let $x \in \text{int}(K)$. Suppose estimates of the gradient and Hessian of F at x are known. Denote these estimates by $F_1(x)$ and $F_2(x)$ respectively. For the sake of discussion, in this section only we will make the following assumption on F_2. (In the analysis of our algorithm (Section 4.3) we will make a stronger assumption on $F_2(x)$.)

Assumption 4.2.2. *For all $x \in \text{int}(K)$, $F_2(x)$ is a positive definite matrix.*

Assumption 4.2.2 is sensible because from Assumption 4.1.4 the exact Hessian is positive definite for all $x \in \text{int}(K)$. Our short-step interior-point algorithm is presented below as Algorithm short_step.[5]

Algorithm short_step

Let $\theta, \tau \in (0,1)$ and $(x^0, w^0, s^0) \in \mathcal{N}(\theta)$.

For $k = 0, 1, \cdots$ until convergence:

(1) Let $\mu_k = (x^k)^T s^k / \nu$ and solve the linear system

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ \mu_k F_2(x^k) & 0 & I \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta w^k \\ \Delta s^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\tau\mu_k F_1(x^k) - s^k \end{bmatrix}. \tag{4.2.1}$$

(2) Set $(x^{k+1}, w^{k+1}, s^{k+1}) = (x^k, w^k, s^k) + (\Delta x^k, \Delta w^k, \Delta s^k)$.

end

Note the appearance of the centering parameter τ, and compare (4.2.1) with (4.1.10).

Before analyzing Algorithm short_step, we first explain the different ways in which the linear system (4.2.1) can be solved. We also give an overview of the literature for exact and inexact interior-point methods. (Here the term "exact" refers to the exact evaluation of the gradient and Hessian of F.)

[5]The algorithm is given in what we may call a "generic" form, since in order to prove convergence and polynomial iteration complexity, it is necessary to choose θ and τ appropriately, and to choose appropriate bounds on the errors in the estimates F_1 and F_2.

The linear system (4.2.1) is equivalent to

$$\begin{aligned} AF_2(x^k)^{-1}A^T\Delta w^k &= AF_2(x^k)^{-1}(\tau\mu_k F_1(x^k) + s^k), \\ \Delta s^k &= -A^T\Delta w^k, \\ \mu_k F_2(x^k)\Delta x^k &= -\tau\mu_k F_1(x^k) - s^k - \Delta s^k. \end{aligned} \tag{4.2.2}$$

Since A has full row rank (Assumption 4.1.1) and $F_2(x^k)$ is positive definite (Assumption 4.2.2), $F_2(x^k)^{-1}$ is defined and the system (4.2.2), which is called the *normal equations* system, has a unique solution. Hence the solution of (4.2.1) is also unique. We can use the positive definiteness of F_2 to solve the normal equations system by finding a Cholesky factorization of $F_2(x^k)$, say $F_2(x^k) = LL^T$. We then solve $LG = A^T$ for $G \in \mathcal{R}^{n\times m}$ by solving m triangular systems—one for each column of G. We then compute the vector $u = F_2(x^k)^{-1}(\tau\mu_k F_1(x^k) + s^k)$ by solving $LL^T u = \tau\mu_k F_1(x^k) + s^k$ using forward then backward substitution. Since $AF_2(x^k)^{-1}A^T = G^T G$ is positive definite, the vector Δw^k may be computed by solving $G^T G\Delta w^k = Au$ via a Cholesky decomposition of $G^T G$. The computation of Δs^k is straightforward, and the Cholesky factorization of $F_2(x^k)$ is reused to compute Δx^k. In addition to the cost of computing $F_1(x^k)$ and $F_2(x^k)$, the cost of forming and solving (4.2.2) can be estimated as follows:

- $\mathcal{O}(n^3)$ arithmetic operations to find the Cholesky factorization of $F_2(x^k)$ if $F_2(x^k)$ is dense.
- $\mathcal{O}(mn^2)$ arithmetic operations to solve $LG = A^T$ for $G \in \mathcal{R}^{n\times m}$.
- $\mathcal{O}(n^2)$ arithmetic operations to solve $LL^T u = \tau\mu_k F_1(x^k) + s^k$ for u.
- $\mathcal{O}(m^2 n)$ arithmetic operations to form $G^T G$.
- $\mathcal{O}(m^3)$ arithmetic operations to solve $G^T G\Delta w^k = Au$ for Δw^k.
- $\mathcal{O}(mn)$ arithmetic operations to compute Δs^k.
- $\mathcal{O}(n^2)$ arithmetic operations to compute Δx^k.

Since $m \leq n$ the total cost is $\mathcal{O}(n^3)$ arithmetic operations. An advantage of solving (4.2.2) is that the size of each of the three systems of equations is no greater than n, whereas the size of (4.2.1) is $m + 2n$, so the cost of blindly solving (4.2.1) without regard to the structure apparent in the matrix is greater. Solving (4.2.2) does have its disadvantages however. Unless the matrices A and $F_2(x^k)^{-1}$ are sparse, it is unlikely that $AF_2(x^k)^{-1}A^T$ will be sparse, so (4.2.2) involves two linear systems whose matrices are dense. Furthermore, $AF_2(x^k)^{-1}A^T$ becomes increasing ill-conditioned as x^k approaches the optimal solution set of (4.1.1). To compensate for this a preconditioner is sometimes used. Sometimes it may be better to solve the so-called *augmented system*, which results from eliminating only the variables Δs^k from (4.2.1):

$$\begin{aligned} \begin{bmatrix} -\mu_k F_2(x^k) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta w^k \end{bmatrix} &= \begin{bmatrix} \tau\mu_k F_1(x^k) + s^k \\ 0 \end{bmatrix}, \\ \Delta s^k &= -\tau\mu_k F_1(x^k) - s^k - \mu_k F_2(x^k)\Delta x^k. \end{aligned} \tag{4.2.3}$$

Although the linear system in (4.2.3) is larger than those in (4.2.2), if A is sparse, the cost of solving these systems may be comparable. Note however that the linear system (4.2.3) involves a symmetric indefinite matrix. The ill-conditioning in the augmented system is generally not as severe as for the normal equations. Still, in some practical algorithms an indefinite preconditioner is used. See e.g., [12], where an algorithm for linear optimization is presented.

There is an extensive literature on inexact interior-point methods for linear optimization problems, and a smaller body of work on inexact methods for nonlinear convex optimization problems. We note the interesting paper [56], in which various types of primal-dual potential reduction methods are given for conic optimization. It is shown that the assumptions made on the availability of a self-concordant barrier function and its derivatives affect the convergence and worst-case complexity properties of the potential reduction algorithm. Specifically, three cases are studied. In the first case, it is supposed that one cannot evaluate a self-concordant barrier F or its derivatives for the cone K in question, nor is it possible to

evaluate the conjugate barrier or its derivatives. Due to the absence of derivative information, the resulting algorithm is called a "zeroth order algorithm". Then it is supposed that we can evaluate the first derivatives of F and its conjugate, giving a "first order algorithm". Finally, it is supposed that we can evaluate both the first and second derivatives of F and its conjugate, giving a "second-order algorithm". In each case the inverse Hessian of a self-concordant barrier function is approximated by a matrix that may be updated according to a Quasi-Newton method.

In the interior-point method literature, the term "inexact algorithms" typically refers to algorithms in which approximate right-hand sides of say, (4.2.1), (4.2.2), or (4.2.3), are used. In some schemes approximate coefficient matrices are also used. Our inexact algorithm can also be considered this way, although our primary concern is to estimate F' and F'' rather than the coefficient matrix and right-hand-side of a linear system per se. In [28, 13, 24] infeasible-point methods for linear optimization are presented in which (4.2.1) is solved approximately. In [4] an inexact algorithm is presented for monotone horizontal linear complementarity problems. (This class of complementarity problems includes as special cases linear optimization problems and convex quadratic optimization problems; the latter problems have linear constraints and a convex quadratic objective function.) Inexact methods for other classes of nonlinear convex problems have also been studied. In [61] an inexact primal-dual method was presented for semidefinite optimization. This method generated a near-optimal solution in polynomial time under certain conditions on the inexactness. We also mention the paper [55] in which an inexact primal-dual path-following algorithm is given for a class of convex quadratic semidefinite optimization problems.

Only the system of equations resulting from the linearization of the complementarity condition is perturbed, but a polynomial worst-case complexity result is proven. In the seminal interior-point method paper of Karmarkar [19], a primal (rather than primal-dual) method was presented for linear optimization. At each iteration of Karmarkar's method, instead of computing the matrix in the linear system directly, Karmarkar used an update from the matrix used in the previous iteration. This method of updating, while it resulted

in inexact linear systems at each iteration, led to a decrease in the worst-case complexity of his algorithm. If an iterative method such as the (preconditioned) conjugate gradient method is used to solve the linear system of equations that occurs at each iteration, one can control the error in the estimated solution by varying the number of conjugate gradient iterations. For previous work on the use of the conjugate gradient method in interior-point methods for linear optimization, see e.g., [58] and the references therein.

To conclude this section, we make a remark about the uniqueness of the Newton direction in Algorithm short_step. It is known that for semidefinite and second-order cone optimization, the linearization of some formulations of the central path equations does not necessarily yield a well-defined Newton direction, even when A has full row rank; see [54, p. 778] and [39, Section 3.1] for examples. In other words, alternate forms of (4.1.10) may not possess a unique solution even if F_1 and F_2 are exact. This can occur, for example, if the equation $\mu F'(x) + s = 0$ is rewritten to be symmetric in x and s.[6] In the case that K is a self-scaled cone, this difficulty may be circumvented by instead computing the Nesterov-Todd direction, as explained above. However by directly linearizing $\mu F'(x) + s = 0$ instead, as we have done, the resulting linear system (4.1.10) always has a unique solution. By extension, the same is true for (4.2.1), where the Hessian of F is evaluated inexactly.

4.3 Unstructured perturbations in the gradient and Hessian of F

Given an $x \in \text{int}(K)$, suppose that the exact gradient and Hessian of $F(x)$ are unknown or too expensive to compute "exactly", but we can compute estimates, which we shall call $F_1(x)$ and $F_2(x)$ respectively. Since the Hessian of F is symmetric, we will assume that

[6] In semidefinite optimization, for example, one can use the n-normal barrier function $F(X) = -\log(\det(X))$, where $X \in \mathcal{R}^{n\times n}$ is the symmetric positive semidefinite matrix of primal variables, and $\det(X)$ denotes the determinant of X; see Section 5.2.3. If $S \in \mathcal{R}^{n\times n}$ denotes the symmetric matrix of dual variables analogous to s in (4.1.1), the central path equation $\mu F'(x) + s = 0$ becomes $-\mu X^{-1} + S = 0$, i.e., $XS = \mu I$. The last equation is symmetric in X and S. Now μI is a symmetric matrix, whereas the product XS need not be. So it is possible that when the linearization of $XS = \mu I$ is incorporated into (4.1.10), the resulting system fails to have a unique solution.

$F_2(x)$ is also symmetric. Denote the errors in the gradient and Hessian of $F(x)$ by

$$E_1(x) = F'(x) - F_1(x), \qquad E_2(x) = F''(x) - F_2(x).$$

By definition $E_2(x)$ is a symmetric matrix. Throughout this section, we will assume that the errors $E_1(x)$ and $E_2(x)$ are "small enough". We now quantify this. First let us define for $\theta \in (0, 1)$

$$\mathcal{X}(\theta) := \{x \mid (x, w, s) \in \mathcal{N}(\theta) \text{ for some } w, s\}.$$

Assumption 4.3.1. *The neighborhood parameter θ satisfies $\theta \in (0, 1)$. For some $\epsilon_1, \epsilon_2 > 0$, the absolute errors $E_1(x)$ and $E_2(x)$ satisfy the following relations:*

$$x \in \mathcal{X}(\theta) \quad \Longrightarrow \quad \|E_1(x)\|^*_{x,F} \equiv \|F''(x)^{-1/2} E_1(x)\|_2 \leq \epsilon_1 < 1, \tag{4.3.1}$$

$$x \in \mathcal{X}(\theta) \quad \Longrightarrow \quad \|F''(x)^{-1/2} E_2(x) F''(x)^{-1/2}\|_2 \leq \epsilon_2 < 1. \tag{4.3.2}$$

Loosely speaking, we will refer to the quantities $\|E_1(x)\|^*_{x,F}$ and $\|F''(x)^{-1/2} E_2(x) F''(x)^{-1/2}\|_2$ as the "relative errors" in $F_1(x)$ and $F_2(x)$. (More accurately, they measure the absolute errors relative to the Hessian of F. Note also that $\|E_1(x)\|^*_{x,F}$ is not invariant under a positive scaling of F.) These "relative errors" are in fact upper bounds on the true relative errors in $F_1(x)$ and $F_2(x)$, measured in appropriate norms: the relative error in $F_1(x)$, measured in the $\|\cdot\|^*_{x,F}$ norm, is

$$\frac{\|E_1(x)\|^*_{x,F}}{\|F'(x)\|^*_{x,F}} = \frac{\|E_1(x)\|^*_{x,F}}{\nu^{1/2}} \leq \|E_1(x)\|^*_{x,F},$$

where we have used Lemma 3.3.2(e) and $\nu \geq 1$. The relative error in $F_2(x)$, measured in

the matrix 2-norm, is (using Lemma 2.1.1(a)(iii))

$$\begin{aligned}
\frac{\|E_2(x)\|_2}{\|F''(x)\|_2} &\leq \frac{\|F''(x)^{-1/2}E_2(x)F''(x)^{1/2}\|_2}{\|F''(x)\|_2} \\
&= \frac{\|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}F''(x)\|_2}{\|F''(x)\|_2} \\
&\leq \frac{\|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2 \ \|F''(x)\|_2}{\|F''(x)\|_2} \\
&= \|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2.
\end{aligned}$$

So we may think of ϵ_1 and ϵ_2 in Assumption 4.3.1 as upper bounds on the maximum allowable relative errors in our estimates of F' and F''. The assumption in (4.3.2) implies that the eigenvalues of $F_2(x)$ are close to those of $F''(x)$ when $x \in \mathcal{X}(\theta)$:

Lemma 4.3.2. *The Hessian estimate $F_2(x)$ satisfies $(1-\epsilon_2)F''(x) \preceq F_2(x) \preceq (1+\epsilon_2)F''(x)$. Moreover $F_2(x)$ is positive definite.*

Proof. The nonstrict inequality in (4.3.2) implies that

$$-\epsilon_2 I \preceq F''(x)^{-1/2}E_2(x)F''(x)^{-1/2} \preceq \epsilon_2 I.$$

Multiplying this matrix inequality on the left and right by the positive definite matrix $F''(x)^{1/2}$ preserves the partial ordering $\preceq$:

$$-\epsilon_2 F''(x) \preceq E_2(x) \preceq \epsilon_2 F''(x).$$

Subtracting each quantity in the above matrix inequality from $F''(x)$ gives the required result. Since F is assumed nondegenerate, $F''(x)$ is positive definite, so F_2 is also. □

The outline for the remainder of this section is as follows. We first prove some preliminary perturbation results showing how various quantities are affected when the gradient and Hessian of F are replaced by the estimates $F_1(x)$ and $F_2(x)$. We then prove that for a primal-dual iterate $(x^k, w^k, s^k) \in \mathcal{N}(\theta)$, the Newton steps Δx^k and Δs^k in (4.2.1) are

bounded by a constant and a constant times the duality measure μ_k, respectively, in the appropriate norms (Lemma 4.3.5 and Corollary 4.3.6). Using such bounds we show that if the above-mentioned constants are less than one, then a full Newton step produces a strongly feasible primal-dual point (Lemma 4.3.7). Therefore no line search procedure is required. Hence our algorithm indeed belongs to the class of feasible-point algorithms: all iterates are (strongly) feasible if the starting point is. Next we derive formulas for the minimum and maximum rates of decrease of the duality measure at each iteration (Lemma 4.3.9). Since the sequence of positive duality measures generated by Algorithm short_step decreases linearly, they converge to zero, as is required for the algorithm to converge to an optimal solution of (4.1.1)–(4.1.2). The maximum rate of decrease of the duality measure is used to show that not only are all iterates strongly feasible, but they stay in an $\mathcal{N}(\theta)$ neighborhood of the central path. Thus our algorithm belongs to the class of "path-following" algorithms.

Since $F''(x)$ and $F_2(x)$ are positive definite on $\mathcal{X}(\theta)$, the square roots of these two matrices are well defined. Define

$$D(x) \;=\; F''(x)^{1/2}F_2(x)^{-1/2} = F''(x)^{1/2}(F''(x) - E_2(x))^{-1/2}. \tag{4.3.3}$$

for all $x \in \mathcal{X}(\theta)$. In the next lemma and corollary we give several technical results that will be used in our analysis of the above interior-point method.

Lemma 4.3.3. *Let $x \in \mathcal{X}(\theta)$, $s, z \in \mathcal{R}^n$ and $\tilde{\mu} \geq 0$. We have*

$$\begin{aligned}
\|D(x)\|_2^2 &\leq \frac{1}{1-\epsilon_2}, && (4.3.4)\\
\|D(x)^{-1}\|_2^2 &\leq 1+\epsilon_2, && (4.3.5)\\
(1+\epsilon_2)^{-1/2}\|z\|_{x,F}^* &\leq \|F_2(x)^{-1/2}z\|_2 \leq (1-\epsilon_2)^{-1/2}\|z\|_{x,F}^*, && (4.3.6)\\
(1-\epsilon_2)^{1/2}\|z\|_{x,F} &\leq \|F_2(x)^{1/2}z\|_2 \leq (1+\epsilon_2)^{1/2}\|z\|_{x,F}, && (4.3.7)\\
\|F_2(x)^{-1/2}(s+\tilde{\mu}F_1(x))\|_2 &\leq (1-\epsilon_2)^{-1/2}\big(\|s+\tilde{\mu}F'(x)\|_{x,F}^* + \tilde{\mu}\epsilon_1\big). && (4.3.8)
\end{aligned}$$

Proof. Throughout all parts of the proof, suppose that $x \in \mathcal{X}(\theta)$. We first prove (4.3.4).

$$\begin{aligned}
\|D(x)\|_2^2 = \|D(x)D(x)^T\|_2 &= \|F''(x)^{1/2}\big(F''(x) - E_2(x)\big)^{-1}F''(x)^{1/2}\|_2 \\
&= \|\big(I - F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\big)^{-1}\|_2 \\
&\leq \frac{1}{1 - \|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2} \\
&\leq \frac{1}{1-\epsilon_2},
\end{aligned}$$

where the inequalities follow from (4.3.2). We next prove (4.3.5).

$$\begin{aligned}
\|D(x)^{-1}\|_2^2 = \|D(x)^{-T}D(x)^{-1}\|_2 &= \|F''(x)^{-1/2}\big(F''(x) - E_2(x)\big)F''(x)^{-1/2}\|_2 \\
&= \|I - F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2 \\
&\leq 1 + \|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2 \\
&\leq 1 + \epsilon_2,
\end{aligned}$$

where the inequalities again follow from (4.3.2). Using (4.3.4) and (4.3.5), we have

$$\begin{aligned}
\|F_2(x)^{-1/2}z\|_2 &= \|D(x)^T F''(x)^{-1/2}z\|_2 \\
&\leq \|D(x)\|_2 \, \|F''(x)^{-1/2}z\|_2 \\
&\leq (1-\epsilon_2)^{-1/2}\|z\|_{x,F}^*,
\end{aligned}$$

and

$$\begin{aligned}
\|z\|_{x,F}^* &= \|F''(x)^{-1/2}z\|_2 \\
&= \|D(x)^{-T}F_2(x)^{-1/2}z\|_2 \\
&\leq \|D(x)^{-1}\|_2 \, \|F_2(x)^{-1/2}z\|_2 \\
&\leq (1+\epsilon_2)^{1/2}\|F_2(x)^{-1/2}z\|_2.
\end{aligned}$$

Combining these two inequalities gives (4.3.6). We now prove (4.3.7). Using (4.3.4) and

(4.3.5) again we have

$$\begin{aligned}\|F_2(x)^{1/2}z\|_2 &= \|D(x)^{-1}F''(x)^{1/2}z\|_2 \\ &\leq \|D(x)^{-1}\|_2 \, \|F''(x)^{1/2}z\|_2 \\ &\leq (1+\epsilon_2)^{1/2}\|z\|_{x,F},\end{aligned}$$

and

$$\begin{aligned}\|z\|_{x,F} &= \|F''(x)^{1/2}z\|_2 \\ &= \|D(x)F_2(x)^{1/2}z\|_2 \\ &\leq \|D(x)\|_2 \, \|F_2(x)^{1/2}z\|_2 \\ &\leq (1-\epsilon_2)^{-1/2}\|F_2(x)^{1/2}z\|_2.\end{aligned}$$

Combining these two inequalities gives (4.3.7). Finally we prove (4.3.8). In the following, (4.3.9a) follows from (4.3.6), (4.3.9b) follows from the definition of $E_1(x)$, and (4.3.9c) follows from (4.3.1).

$$\begin{aligned}\|F_2(x)^{-1/2}(s+\tilde{\mu}F_1(x))\|_2 &\leq (1-\epsilon_2)^{-1/2}\|s+\tilde{\mu}F_1(x)\|^*_{x,F} && \text{(4.3.9a)} \\ &\leq (1-\epsilon_2)^{-1/2}\|s+\tilde{\mu}F'(x)-\tilde{\mu}E_1(x)\|^*_{x,F} && \text{(4.3.9b)} \\ &\leq (1-\epsilon_2)^{-1/2}\big(\|s+\tilde{\mu}F'(x)\|^*_{x,F}+\tilde{\mu}\|E_1(x)\|^*_{x,F}\big) \\ &\leq (1-\epsilon_2)^{-1/2}\big(\|s+\tilde{\mu}F'(x)\|^*_{x,F}+\tilde{\mu}\epsilon_1\big). && \text{(4.3.9c)}\end{aligned}$$

□

Corollary 4.3.4. *Let $x \in \mathcal{X}(\theta)$ and $z \in \mathcal{R}^n$. We have*

$$\begin{aligned} F_1(x)^T F_2(x)^{-1} F_1(x) &\leq \frac{(\nu^{1/2}+\epsilon_1)^2}{1-\epsilon_2}, && (4.3.10)\\ |F_1(x)^T z| &\leq \frac{\nu^{1/2}+\epsilon_1}{(1-\epsilon_2)^{1/2}} \|F_2(x)^{1/2} z\|_2, && (4.3.11)\\ |E_1(x)^T z| &\leq \epsilon_1 \|z\|_{x,F}, && (4.3.12)\\ |E_1(x)^T x| &\leq \epsilon_1 \nu^{1/2}, && (4.3.13)\\ |x^T E_2(x) z| &\leq \epsilon_2 \nu^{1/2} \|z\|_{x,F}. && (4.3.14) \end{aligned}$$

Proof. To obtain the bound in (4.3.10), square the relation in (4.3.8), substitute $s = 0$ and $\tilde{\mu} = 1$, and use Lemma 3.3.2(e). We now prove (4.3.11).

$$\begin{aligned} |F_1(x)^T z| &= |(F_2(x)^{-1/2} F_1(x))^T F_2(x)^{1/2} z| \\ &\leq \|F_2(x)^{-1/2} F_1(x)\|_2 \; \|F_2(x)^{1/2} z\|_2 \\ &\leq \frac{\nu^{1/2}+\epsilon_1}{(1-\epsilon_2)^{1/2}} \|F_2(x)^{1/2} z\|_2, \end{aligned}$$

where the last inequality follows from (4.3.10). We now prove (4.3.12). Using (4.3.1), we have

$$\begin{aligned} |E_1(x)^T z| &= |(F''(x)^{-1/2} E_1(x))^T (F''(x)^{1/2} z)| \\ &\leq \|F''(x)^{-1/2} E_1(x)\|_2 \; \|F''(x)^{1/2} z\|_2 \\ &\leq \epsilon_1 \|z\|_{x,F}, \end{aligned}$$

where the last inequality follows from (4.3.1). The inequality in (4.3.13) follows from (4.3.12) with $z = x$, where Lemma 3.3.2(d) has also been used. Finally we prove (4.3.14). Using

Lemma 3.3.2(d) and (4.3.2), we have

$$\begin{aligned}|x^T E_2(x) z| &= |\big(F''(x)^{1/2}x\big)^T \big(F''(x)^{-1/2} E_2(x) F''(x)^{-1/2}\big)\big(F''(x)^{1/2}z\big)| \\ &\leq \|F''(x)^{1/2}x\|_2 \; \|F''(x)^{-1/2} E_2(x) F''(x)^{-1/2}\|_2 \; \|F''(x)^{1/2}z\|_2 \\ &\leq \nu^{1/2}\epsilon_2 \|z\|_{x,F}. \end{aligned}$$

□

It will be convenient to define the following three constants depending on $\theta, \tau, \epsilon_1, \epsilon_2$, and the complexity parameter $\nu \geq 1$.

$$\begin{aligned} \beta_0 &:= \frac{\theta + \epsilon_1 + (1-\tau)(\nu^{1/2} + \epsilon_1)}{(1-\epsilon_2)^{1/2}}, \\ \beta_1 &:= \left(\frac{1}{1-\epsilon_2}\right)^{1/2} \beta_0, \\ \beta_2 &:= \beta_0 \max\left\{\frac{(1+\epsilon_2)^{1/2}}{1-\theta}, \left(\frac{1}{1-\epsilon_2}\right)^{1/2}\right\}. \end{aligned} \tag{4.3.15}$$

Lemma 4.3.5. *Let $(x^k, w^k, s^k) \in \mathcal{N}(\theta)$ be the k-th iterate generated by Algorithm short_step. We have*

$$\mu_k^2 \|F_2(x^k)^{1/2}\Delta x^k\|_2^2 + \|F_2(x^k)^{-1/2}\Delta s^k\|_2^2 \leq \mu_k^2 \beta_0^2.$$

Proof. For convenience, we will omit all iteration subscripts and superscripts. Premultiply the third block equation in (4.2.1) by $(\mu F_2(x))^{-1/2}$:

$$(\mu F_2(x))^{1/2}\Delta x + (\mu F_2(x))^{-1/2}\Delta s = (\mu F_2(x))^{-1/2}(-\tau\mu F_1(x) - s). \tag{4.3.16}$$

It is seen from (4.2.1) that Δx lies in the nullspace of A, while Δs lies in the range space of A^T. Therefore Δx is orthogonal to Δs. Taking the square of the 2-norm of each side of (4.3.16) and using this orthogonality, we obtain

$$\|(\mu F_2(x))^{1/2}\Delta x\|_2^2 + \|(\mu F_2(x))^{-1/2}\Delta s\|_2^2 = \|(\mu F_2(x))^{-1/2}(-\tau\mu F_1(x) - s)\|_2^2.$$

Multiplying this equation by μ, we have

$$\mu^2\|F_2(x)^{1/2}\Delta x\|_2^2 + \|F_2(x)^{-1/2}\Delta s\|_2^2 = \|F_2(x)^{-1/2}(-\tau\mu F_1(x) - s)\|_2^2. \tag{4.3.17}$$

Let us now bound the right-hand side of (4.3.17). In the following, (4.3.18a) follows from (4.3.8), and (4.3.18b) follows from (4.3.10) and the fact that $x \in \mathcal{X}(\theta)$.

$$\begin{aligned}
&\|F_2(x)^{-1/2}(-\tau\mu F_1(x) - s)\|_2 \\
&\quad= \| - F_2(x)^{-1/2}(s + \mu F_1(x)) + F_2(x)^{-1/2}(1-\tau)\mu F_1(x)\|_2 \\
&\quad\le \| - F_2(x)^{-1/2}(s + \mu F_1(x))\|_2 + \|F_2(x)^{-1/2}(1-\tau)\mu F_1(x)\|_2 \\
&\quad\le (1-\epsilon_2)^{-1/2}\big(\|s + \mu F'(x)\|_{x,F}^* + \mu\epsilon_1\big) + \mu(1-\tau)\|F_2(x)^{-1/2}F_1(x)\|_2 \quad (4.3.18a) \\
&\quad\le (1-\epsilon_2)^{-1/2}\big(\theta\mu + \mu\epsilon_1\big) + \mu(1-\tau)\frac{\nu^{1/2}+\epsilon_1}{(1-\epsilon_2)^{1/2}} \quad (4.3.18b) \\
&\quad\le \mu\left[\frac{\theta + \epsilon_1 + (1-\tau)(\nu^{1/2}+\epsilon_1)}{(1-\epsilon_2)^{1/2}}\right] \\
&\quad= \mu\beta_0.
\end{aligned}$$

Combining this with (4.3.17) yields the required result. □

Corollary 4.3.6. *Let $(x^k, w^k, s^k) \in \mathcal{N}(\theta)$ be the k-th iterate generated by Algorithm short_step. We have*

$$\begin{aligned}
\|F_2(x^k)^{1/2}\Delta x^k\|_2 &\le \beta_0, \\
\|F_2(x^k)^{-1/2}\Delta s^k\|_2 &\le \mu_k\beta_0, \\
\|\Delta x^k\|_{x^k,F} &\le \beta_1, \\
\|\Delta s^k\|_{x^k,F}^* &\le (1+\epsilon_2)^{1/2}\mu_k\beta_0,
\end{aligned}$$

where β_0 and β_1 are defined in (4.3.15).

Proof. The first two bounds follow immediately from Lemma 4.3.5. The third inequality follows from the first inequality and (4.3.7), and the fourth inequality follows from the

second inequality and (4.3.6). □

We now study the convergence of Algorithm short_step. First we show that under a condition on the parameters θ, τ, ϵ_1, and ϵ_2, a full primal-dual Newton step is not only strongly feasible, justifying step (2) in the Iteration short_step, but the new iterate remains in the $\mathcal{N}(\theta)$ neighborhood of the central path. We also indicate the rate of decrease of sequence of duality measures $\{\mu_k\}$ to zero.

Lemma 4.3.7. *Let θ, τ, ϵ_1, and ϵ_2 be such that $\beta_2 < 1$ where β_2 is defined in* (4.3.15)*, and let $(x^k, w^k, s^k) \in \mathcal{N}(\theta)$. Then the point $(x^{k+1}, w^{k+1}, s^{k+1})$ generated by Algorithm short_step is a strongly feasible primal-dual point.*

Proof. As we have already noted, the equality constraints in (4.1.1) and (4.1.2) are satisfied by $(x^{k+1}, w^{k+1}, s^{k+1})$, since they are satisfied by (x^0, w^0, s^0), and any step from (x^k, w^k, s^k) in the direction $(\Delta x^k, \Delta w^k, \Delta s^k)$ will satisfy these constraints due to the first two block equations in (4.2.1). We now show that $x^{k+1} \in \text{int}(K)$ and $s^{k+1} \in \text{int}(K^*)$. For convenience, we now omit all iteration subscripts and superscripts. Since F is a nondegenerate self-concordant barrier (Assumption 4.1.4), Lemma 3.2.9(b) is applicable: if $\|\Delta x\|_{x,F} < 1$, then $x + \Delta x \in \text{int}(K)$. By Corollary 4.3.6, a sufficient condition for $\|\Delta x\|_{x,F} < 1$ is $\beta_1 < 1$, and this holds since $\beta_2 < 1$.

Similarly, in light of Lemma 3.3.6(b), if $\|\Delta s\|_{s,F_*} < 1$, then $s + \Delta s \in \text{int}(K^*)$. It remains to show that $\|\Delta s\|_{s,F_*} < 1$ under our assumption $\beta_2 < 1$. Since $(x, w, s) \in \mathcal{N}(\theta)$, we may apply Lemma 4.1.9 with $h = \Delta s$ to obtain

$$\|\Delta s\|_{s,F_*} \leq \frac{\frac{1}{\mu}\|\Delta s\|^*_{x,F}}{1-\theta} \leq \frac{(1+\epsilon_2)^{1/2}\beta_0}{1-\theta},$$

where the last inequality follows from Corollary 4.3.6. By the definition of β_2 in (4.3.15), we have $\|\Delta s\|_{s,F_*} \leq \beta_2 < 1$. □

It follows from Lemma 4.3.7 that a full step in the Newton direction for (4.2.1) is strongly feasible. Thus at each iteration the exact gradient and Hessian are well defined. Note that

it was not necessary to evaluate, or even estimate, values of the conjugate barrier function F_* or its derivatives in order to show that a full dual step $(\Delta w^k, \Delta s^k)$ was strongly feasible. Instead we estimated the gradient and Hessian of F_* using the gradient and Hessian of F and the results in Lemma 3.3.7.

Remark 4.3.8. *In contrast to our interior-point method, other primal-dual interior-point algorithms for conic optimization require the evaluation—or at least approximate evaluation—of the conjugate barrier function (or another normal barrier for K^*) or its gradient and Hessian. We note the recent work of Nesterov [32] which was made known to us as this research was concluding. Nesterov gave a primal-dual predictor-corrector algorithm for conic optimization that does not require exact evaluation of the conjugate barrier function or its derivatives, but uses an estimate of the conjugate barrier function. The reason is that his algorithm is based upon the use of a primal-dual "global proximity" measure*

$$\Psi(x, w, s) = F(x) + F_*(s) + \nu \log\left(\frac{x^T s}{\nu}\right) + \nu,$$

which by [33, Prop. 2.4.1] is nonnegative for any strongly feasible primal-dual pair (x, w, s), and zero if and only if (x, w, s) lies on the primal-dual central path. (This proximity function was first used in an interior-point method in [30].) By ensuring that $\Psi(x, w, s)$ is sufficiently small at all iterates produced by an interior-point algorithm, one stays sufficiently close to the central path to guarantee progress towards the optimal solution set.

We now study the behavior of the sequence of duality measures $\{\mu_k\}$.

Lemma 4.3.9. *Let $(x^k, w^k, s^k) \in \mathcal{N}(\theta)$ be the k-th iterate generated by Algorithm short_step. The duality measure μ_{k+1} of the next iterate $(x^{k+1}, w^{k+1}, s^{k+1})$ satisfies*

$$\underline{\delta}\mu_k \leq \mu_{k+1} \leq \bar{\delta}\mu_k,$$

where

$$\underline{\delta} = \tau - \frac{\tau\epsilon_1}{\nu^{1/2}} - \beta_0\frac{(1-\tau)(\nu^{1/2}+\epsilon_1)+\epsilon_1+\epsilon_2\nu^{1/2}}{\nu(1-\epsilon_2)^{1/2}} - \frac{1}{\nu}\beta_0^2, \quad (4.3.19)$$

$$\bar{\delta} = \tau + \frac{\tau\epsilon_1}{\nu^{1/2}} + \phi\frac{(1-\tau)(\nu^{1/2}+\epsilon_1)+\epsilon_1+\epsilon_2\nu^{1/2}}{\nu(1-\epsilon_2)^{1/2}} - \frac{1}{\nu}\phi^2, \quad (4.3.20)$$

$$\textit{with} \qquad \phi = \min\left\{\beta_0, \frac{(1-\tau)(\nu^{1/2}+\epsilon_1)+(\epsilon_1+\epsilon_2\nu^{1/2})}{2(1-\epsilon_2)^{1/2}}\right\}. \quad (4.3.21)$$

Proof. For ease of notation, we will write x, s, μ for x^k, s^k, μ_k, and x_+, s_+, μ_+ for $x^{k+1}, s^{k+1}, \mu_{k+1}$. Recalling that Δx is orthogonal to Δs, we have

$$\begin{aligned}\nu\mu_+ &= x_+^T s_+ \\ &= (x+\Delta x)^T(s+\Delta s) \\ &= x^T(s+\Delta s) + (\Delta x)^T s. \end{aligned} \quad (4.3.22)$$

From the third block equation in (4.2.1), we have $s + \Delta s = -\tau\mu F_1(x) - \mu F_2(x)\Delta x$, so

$$\begin{aligned} x^T(s+\Delta s) &= -x^T(\tau\mu F_1(x) + \mu F_2(x)\Delta x) \\ &= -\tau\mu x^T(F'(x) - E_1(x)) - \mu x^T(F''(x) - E_2(x))\Delta x \\ &= \tau\mu(\nu + E_1(x)^T x) + \mu(F'(x)^T\Delta x + x^T E_2(x)\Delta x) \\ &= \tau\mu\nu + \tau\mu E_1(x)^T x + \mu(F_1(x) + E_1(x))^T\Delta x + \mu x^T E_2(x)\Delta x, \end{aligned} \quad (4.3.23)$$

where we have used Lemma 3.3.2(b),(c). Since $(\Delta x)^T\Delta s = 0$, it also follows from the third block equation in (4.2.1) that

$$\begin{aligned}(\Delta x)^T s &= (\Delta x)^T(-\mu F_2(x)\Delta x - \tau\mu F_1(x)) \\ &= -\mu\|F_2(x)^{1/2}\Delta x\|_2^2 - \tau\mu F_1(x)^T\Delta x. \end{aligned} \quad (4.3.24)$$

Combining (4.3.22), (4.3.23), and (4.3.24) we have

$$\begin{aligned}\nu\mu_+ &= \tau\mu\nu + \tau\mu E_1(x)^T x + (1-\tau)\mu F_1(x)^T\Delta x + \mu E_1(x)^T\Delta x \\ &\quad +\mu x^T E_2(x)\Delta x - \mu\|F_2(x)^{1/2}\Delta x\|_2^2,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{\mu_+}{\mu} &= \tau + \frac{\tau}{\nu}E_1(x)^T x + \frac{1-\tau}{\nu}F_1(x)^T\Delta x + \frac{1}{\nu}E_1(x)^T\Delta x \\ &\quad +\frac{1}{\nu}x^T E_2(x)\Delta x - \frac{1}{\nu}\|F_2(x)^{1/2}\Delta x\|_2^2. \qquad (4.3.25)\end{aligned}$$

To reduce clutter, let $\hat{t} = \|F_2(x)^{1/2}\Delta x\|_2$ and $t = \|\Delta x\|_{x,F}$. Then by appealing to the results of Corollary 4.3.4, we obtain the following upper bound on μ_+/μ:

$$\begin{aligned}\frac{\mu_+}{\mu} &\leq \tau + \frac{\tau}{\nu}|E_1(x)^T x| + \frac{1-\tau}{\nu}|F_1(x)^T\Delta x| + \frac{1}{\nu}|E_1(x)^T\Delta x| \\ &\quad +\frac{1}{\nu}|x^T E_2(x)\Delta x| - \frac{1}{\nu}\|F_2(x)^{1/2}\Delta x\|_2^2 \\ &\leq \tau + \frac{\tau}{\nu}\epsilon_1\nu^{1/2} + \frac{1-\tau}{\nu}\frac{\nu^{1/2}+\epsilon_1}{(1-\epsilon_2)^{1/2}}\hat{t} + \frac{1}{\nu}\epsilon_1 t + \frac{1}{\nu}\epsilon_2\nu^{1/2}t - \frac{1}{\nu}\hat{t}^2 \\ &=: u(\hat{t},t).\end{aligned}$$

It follows from (4.3.7) and Corollary 4.3.6 that $\hat{t}$ and t satisfy the following conditions:

$$(1-\epsilon_2)^{1/2}t \leq \hat{t} \leq (1+\epsilon_2)^{1/2}t, \qquad 0 \leq \hat{t} \leq \beta_0. \qquad (4.3.26)$$

We now find the best upper bound on $u(\hat{t},t)$ by solving the problem

$$\max\{u(\hat{t},t) \mid (\hat{t},t) \text{ satisfies (4.3.26)}\}.$$

The function $u(\hat{t},t)$ is continuous over the bounded set of $(\hat{t},t)$ pairs satisfying (4.3.26). So there exists an optimal solution to our maximization problem. Suppose that $(\hat{t}^*,t^*)$ is one such optimal solution. In view of the positive coefficient of t in the formula for $u(\hat{t},t)$, it is

necessary that given $\hat{t}^*$, t^* be as large as possible, i.e., $(1-\epsilon_2)^{1/2}t^* = \hat{t}^*$. So it suffices to consider

$$u(\hat{t}, (1-\epsilon_2)^{-1/2}\hat{t}) = \tau + \frac{\tau}{\nu}\epsilon_1\nu^{1/2} + \frac{1-\tau}{\nu}\frac{\nu^{1/2}+\epsilon_1}{(1-\epsilon_2)^{1/2}}\hat{t} + \frac{\epsilon_1+\epsilon_2\nu^{1/2}}{\nu(1-\epsilon_2)^{1/2}}\hat{t} - \frac{1}{\nu}\hat{t}^2,$$

and the unconstrained maximizer is $\hat{t} = \hat{t}^*$, where

$$\hat{t}^* = \frac{(1-\tau)(\nu^{1/2}+\epsilon_1) + (\epsilon_1+\epsilon_2\nu^{1/2})}{2(1-\epsilon_2)^{1/2}}.$$

If this nonnegative solution satisfies the constraint $\hat{t}^* \leq \beta_0$, the maximum value of u is given by $u(\hat{t}^*, (1-\epsilon_2)^{-1/2}\hat{t}^*)$. Otherwise the maximum is $u(\beta_0, (1-\epsilon_2)^{-1/2}\beta_0)$. In either case, $\mu_+/\mu \leq \bar{\delta}$, where $\bar{\delta}$ is given by (4.3.20) and (4.3.21).

From (4.3.25) we can also obtain a lower bound on μ_+/μ:

$$\frac{\mu_+}{\mu} \geq \tau - \frac{\tau}{\nu}\epsilon_1\nu^{1/2} - \frac{1-\tau}{\nu}\frac{\nu^{1/2}+\epsilon_1}{(1-\epsilon_2)^{1/2}}\hat{t} - \frac{1}{\nu}\epsilon_1 t - \frac{1}{\nu}\epsilon_2\nu^{1/2}t - \frac{1}{\nu}\hat{t}^2 =: \ell(\hat{t}, t).$$

We seek the best lower bound for $\ell(\hat{t}, t)$ over $\hat{t}$ and t subject to the constraints in (4.3.26). It is clear that ℓ is minimized when $\hat{t}$ and t achieve their maximum values, viz., $\hat{t} = \beta_0$, $t = (1-\epsilon_2)^{-1/2}\beta_0$. It is readily verified that the resulting value of $\ell(\hat{t}, t)$ is $\underline{\delta}$ in (4.3.19). □

In the remainder of this section we shall use the following values for the parameters in Algorithm short_step:

$$\theta = 0.1, \quad \tau = 1 - \frac{1}{47\nu^{1/2}}, \quad 0 \leq \epsilon_1 \leq 0.01, \quad 0 \leq \epsilon_2 \leq 0.071. \tag{4.3.27}$$

The bounds on ϵ_1 and ϵ_2 mean that roughly speaking, the relative errors in the gradient and Hessian estimates are no more than 1% and 7.1% respectively. Using the values and bounds in (4.3.27), we proceed to bound $\beta_1, \beta_2, \underline{\delta}$, and $\bar{\delta}$ over $\nu \in [1, \infty)$. (Recall from Chapter 3 that any ν-normal barrier for a full cone has $\nu \geq 1$.)

Lemma 4.3.10. *Let θ, τ, ϵ_1, and ϵ_2 satisfy* (4.3.27). *For all $\nu \geq 1$,*

$$\beta_1 < 0.1416, \qquad \beta_2 < 0.1569, \qquad \underline{\delta} > 1 - \frac{0.0642}{\nu^{1/2}} > 0, \qquad \bar{\delta} < 1 - \frac{0.00124}{\nu^{1/2}} < 1.$$

Proof. See Appendix A. □

Lemma 4.3.11. *Let θ, τ, ϵ_1, and ϵ_2 satisfy* (4.3.27), *and let $(x^k, w^k, s^k) \in \mathcal{N}(\theta)$. The primal-dual point $(x^{k+1}, w^{k+1}, s^{k+1})$ generated by Algorithm short_step also belongs to $\mathcal{N}(\theta)$.*

Proof. For ease of notation, we will write x, s, μ for x^k, s^k, μ_k, and x_+, s_+, μ_+ for $x^{k+1}, s^{k+1}, \mu_{k+1}$. We verified in Lemma 4.3.10 that for the values and bounds given in (4.3.27), $\beta_2 < 1$ for all $\nu \geq 1$, so Lemma 4.3.7 is applicable. Therefore it is sufficient to show that for all $\nu \geq 1$,

$$\|s + \mu F'(x)\|^*_{x,F} \leq \theta\mu \implies \|s_+ + \mu_+ F'(x_+)\|^*_{x_+,F} \leq \theta\mu_+.$$

From the third block equation in the linear system (4.2.1), we have

$$\begin{aligned} s_+ + \mu_+ F'(x_+) &= -\tau\mu F_1(x) - \mu F_2(x)\Delta x + \mu_+ F'(x_+) \\ &= -\tau\mu F'(x) - \mu F''(x)\Delta x + \mu_+ F'(x_+) + D_1 \\ &= \tau\mu(-F'(x) - F''(x)\Delta x + F'(x_+)) + D_2 + D_1 \\ &= D_3 + D_2 + D_1, \end{aligned} \tag{4.3.28}$$

where

$$\begin{aligned} D_1 &= \tau\mu E_1(x) + \mu E_2(x)\Delta x, \\ D_2 &= (\mu_+ - \tau\mu)F'(x_+) + \mu(\tau - 1)F''(x)\Delta x, \\ D_3 &= \tau\mu(-F'(x) - F''(x)\Delta x + F'(x_+)). \end{aligned}$$

We now bound $\|D_i\|^*_{x_+,F}$ for each i. Since $\|x_+ - x\|_{x,F} = \|\Delta x\|_{x,F} \leq \beta_1$ (Corollary 4.3.6) and $\beta_1 < 1$ (Lemma 4.3.10), we can use (3.2.6) to bound $\|D_1\|^*_{x_+,F}$ in terms of $\|D_1\|^*_{x,F}$; the

result is (4.3.29a). The inequality (4.3.29b) follows from the definition of D_1, and (4.3.29c) follows from (4.3.1). The inequality (4.3.29d) follows from (4.3.2) and Corollary 4.3.6.

$$\begin{aligned}
\|D_1\|^*_{x_+,F} &\leq \frac{1}{1-\beta_1}\|D_1\|^*_{x,F} && \text{(4.3.29a)}\\
&\leq \frac{1}{1-\beta_1}\big(\tau\mu\|E_1(x)\|^*_{x,F} + \mu\|E_2(x)\Delta x\|^*_{x,F}\big) && \text{(4.3.29b)}\\
&\leq \frac{1}{1-\beta_1}\big(\tau\mu\epsilon_1 + \mu\|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}F''(x)^{1/2}\Delta x\|_2\big) && \text{(4.3.29c)}\\
&\leq \frac{1}{1-\beta_1}\big(\tau\mu\epsilon_1 + \mu\|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2\ \|F''(x)^{1/2}\Delta x\|_2\big)\\
&\leq \frac{1}{1-\beta_1}(\tau\mu\epsilon_1 + \mu\epsilon_2\beta_1). && \text{(4.3.29d)}
\end{aligned}$$

Now in Lemma 4.3.9 we established that $\underline{\delta}\mu \leq \mu_+$, where $\underline{\delta}$ is given in (4.3.19). From Lemma 4.3.10 we see that $\underline{\delta} > 0$ for all $\nu \geq 1$, so

$$\begin{aligned}
\|D_1\|^*_{x_+,F} &\leq \frac{\tau\epsilon_1 + \epsilon_2\beta_1}{\underline{\delta}(1-\beta_1)}\mu_+\\
&=: d_1\mu_+. && \text{(4.3.30)}
\end{aligned}$$

We now bound $\|D_2\|^*_{x_+,F}$. First note that for $\underline{\delta}$ and $\bar{\delta}$ given in (4.3.19) and (4.3.20),

$$\underline{\delta} + \bar{\delta} < 2\tau + (\phi - \beta_0)\frac{(1-\tau)(\nu^{1/2}+\epsilon_1) + \epsilon_1 + \epsilon_2\nu^{1/2}}{\nu(1-\epsilon_2)^{1/2}} \leq 2\tau,$$

where the last inequality follows from $\phi \leq \beta_0$, which is a consequence of (4.3.21). It follows that $\bar{\delta} - \tau < \tau - \underline{\delta}$. Using this and the fact that $\underline{\delta} > 0$, we have

$$\frac{\tau}{\underline{\delta}} - 1 = \frac{\tau - \underline{\delta}}{\underline{\delta}} \geq \frac{\tau - \underline{\delta}}{\bar{\delta}} > \frac{\bar{\delta} - \tau}{\bar{\delta}} = 1 - \frac{\tau}{\bar{\delta}}.$$

That is,

$$\max\left\{\frac{\tau}{\underline{\delta}} - 1, 1 - \frac{\tau}{\bar{\delta}}\right\} = \frac{\tau}{\underline{\delta}} - 1. \tag{4.3.31}$$

In the following, (4.3.32a) follows from the definition of D_2, and (4.3.32b) follows from

Lemma 3.3.2(e) and (3.2.6). Furthermore (4.3.32c) follows from the fact that $\underline{\delta}\mu \leq \mu_+ \leq \bar{\delta}\mu$, and (4.3.32d) follows from (4.3.31) and Corollary 4.3.6.

$$\begin{aligned}
\|D_2\|^*_{x_+,F} &\leq |\mu_+ - \tau\mu| \|F'(x_+)\|^*_{x_+,F} + \mu(1-\tau)\|F''(x)\Delta x\|^*_{x_+,F} && \text{(4.3.32a)}\\
&\leq \mu_+ \left|1 - \tau\frac{\mu}{\mu_+}\right| \nu^{1/2} + \mu(1-\tau)\frac{1}{1-\beta_1}\|F''(x)\Delta x\|^*_{x,F} && \text{(4.3.32b)}\\
&\leq \mu_+ \max\left\{\frac{\tau}{\underline{\delta}} - 1, 1 - \frac{\tau}{\bar{\delta}}\right\}\nu^{1/2} + \mu(1-\tau)\frac{1}{1-\beta_1}\|F''(x)^{1/2}\Delta x\|^2_2 && \text{(4.3.32c)}\\
&\leq \mu_+\left(\frac{\tau}{\underline{\delta}} - 1\right)\nu^{1/2} + \frac{\mu_+}{\underline{\delta}}(1-\tau)\frac{1}{1-\beta_1}\beta_1 && \text{(4.3.32d)}\\
&=: d_2\mu_+. && \text{(4.3.32e)}
\end{aligned}$$

We now bound $\|D_3\|^*_{x_+,F}$. In what follows, we will be working with integrals of vectors and matrices. All such integrals are to be taken componentwise. From the Fundamental Theorem of Calculus for the gradient of F,

$$F'(x_+) - F'(x) = \int_0^1 F''(x + t\Delta x)\Delta x \, \mathrm{d}t.$$

Recalling the definition of $F'''(\cdot)[\cdot,\cdot]$ in (3.2.1), it follows from the Fundamental Theorem of Calculus for the Hessian of F that for any vectors h_1 and h_2,

$$\big(F''(x+h_1) - F''(x)\big)h_2 = \int_0^1 F'''(x+uh_1)[h_1,h_2] \, \mathrm{d}u.$$

Hence

$$D_3 = \tau\mu\int_0^1 \big(F''(x+t\Delta x) - F''(x)\big)\Delta x \, \mathrm{d}t = \tau\mu\int_0^1\int_0^t F'''(x+u\Delta x)[\Delta x, \Delta x] \, \mathrm{d}u \, \mathrm{d}t,$$

giving

$$\|D_3\|^*_{x_+,F} \leq \tau\mu\int_0^1\int_0^t \|F'''(x+u\Delta x)[\Delta x,\Delta x]\|^*_{x_+,F} \, \mathrm{d}u \, \mathrm{d}t.$$

As previously noted, $\beta_1 < 1$, so Lemma 3.2.12 is applicable. Applying this lemma, we get

$$\|F'''(x + u\Delta x)[\Delta x, \Delta x]\|^*_{x_+,F} \leq \frac{2\beta_1^2}{(1-\beta_1)(1-u\beta_1)},$$

giving

$$\begin{aligned}
\|D_3\|^*_{x_+,F} &\leq \tau\mu \int_0^1 \int_0^t \frac{2\beta_1^2}{(1-\beta_1)(1-u\beta_1)} \,\mathrm{d}u \,\mathrm{d}t \\
&= \tau\mu\left(2\log(1-\beta_1) + \frac{2\beta_1}{1-\beta_1}\right) \\
&\leq \frac{\tau}{\underline{\delta}}\left(2\log(1-\beta_1) + \frac{2\beta_1}{1-\beta_1}\right)\mu_+ \\
&=: d_3\mu_+. \qquad (4.3.33)
\end{aligned}$$

(Our bound on $\frac{1}{\tau\mu}\|D_3\|^*_{x_+,F}$ of $2\log(1-\beta_1) + \frac{2\beta_1}{1-\beta_1} = \beta_1^2 + \frac{4}{3}\beta_1^3 + \cdots$ can be improved upon in the case that K is a self-scaled cone: in [35, Theorem 4.3] a bound of no greater than β_1^2 is derived using special properties of self-scaled barriers and self-scaled cones.) Combining the bounds in (4.3.30), (4.3.32e), and (4.3.33) with (4.3.28), we obtain

$$\|s_+ + \mu_+ F'(x_+)\|^*_{x_+,F} \leq \|D_3\|^*_{x_+,F} + \|D_2\|^*_{x_+,F} + \|D_1\|^*_{x_+,F} \leq \mu_+(d_3 + d_2 + d_1),$$

where

$$\begin{aligned}
d_1 &:= \frac{\tau\epsilon_1 + \epsilon_2\beta_1}{\underline{\delta}(1-\beta_1)}, \\
d_2 &:= \left(\frac{\tau}{\underline{\delta}} - 1\right)\nu^{1/2} + \frac{(1-\tau)\beta_1}{\underline{\delta}(1-\beta_1)}, \\
d_3 &:= \frac{\tau}{\underline{\delta}}\left(2\log(1-\beta_1) + \frac{2\beta_1}{1-\beta_1}\right).
\end{aligned}$$

It suffices to show that for all $\nu \geq 1$,

$$d_1 + d_2 + d_3 \leq \theta. \qquad (4.3.34)$$

For the sake of generality, let us write, as we did in the proof of Lemma 4.3.10,

$$\tau = 1 - \frac{\kappa}{\nu^{1/2}},$$

where $\kappa \in (0,1)$. (We will eventually substitute $\kappa = 1/47$.) We now bound the d_i above in terms of $\theta, \kappa, \epsilon_1$, and ϵ_2. This will allow us to derive a uniform (with respect to ν, ϵ_1, and ϵ_2) bound on $d_1 + d_2 + d_3$. In the remainder of this proof we use the fact that for all $\nu \geq 1$, the parameters $\theta, \kappa, \epsilon_1$ and ϵ_2 satisfy $\beta_1 < 1$, $\tau > 0$, and $\underline{\delta} > 0$. It follows from Lemma 4.3.10 and $\kappa < 1$ that these facts hold for the specific parameters and bounds in (4.3.27). Using the relationship between τ and κ, we have

$$\begin{aligned} d_1 &= \frac{\tau\epsilon_1 + \epsilon_2\beta_1}{\underline{\delta}(1-\beta_1)} = \frac{\left(1 - \frac{\kappa}{\nu^{1/2}}\right)\epsilon_1 + \epsilon_2\beta_1}{\underline{\delta}(1-\beta_1)}, \\ d_3 &= \frac{\tau}{\underline{\delta}}\left(2\log(1-\beta_1) + \frac{2\beta_1}{1-\beta_1}\right) = \frac{1 - \frac{\kappa}{\nu^{1/2}}}{\underline{\delta}}\left(2\log(1-\beta_1) + \frac{2\beta_1}{1-\beta_1}\right). \end{aligned}$$

Let us now bound d_2. In (A.0-3) (see Appendix A) we derive a lower bound $\underline{\delta} \geq 1 - \frac{f}{\nu^{1/2}}$ on the ratio of successive duality measures, where f is defined in (A.0-4). Thus

$$\tau - \underline{\delta} \leq \left(1 - \frac{\kappa}{\nu^{1/2}}\right) - \left(1 - \frac{f}{\nu^{1/2}}\right) = \frac{f - \kappa}{\nu^{1/2}},$$

giving

$$\begin{aligned} d_2 &= \left(\frac{\tau - \underline{\delta}}{\underline{\delta}}\right)\nu^{1/2} + \frac{(1-\tau)\beta_1}{\underline{\delta}(1-\beta_1)} \\ &\leq \frac{f - \kappa}{\underline{\delta}} + \frac{\frac{\kappa}{\nu^{1/2}}\beta_1}{\underline{\delta}(1-\beta_1)} \\ &=: \hat{d}_2. \end{aligned}$$

The inequality used the assumption that $\underline{\delta} > 0$. Now let

$$\begin{aligned} a_1 &:= \frac{\epsilon_1 + (\epsilon_2 + 2)\beta_1}{1 - \beta_1} + f - \kappa + 2\log(1 - \beta_1), \\ a_2 &:= \frac{-\kappa(\beta_1 + \epsilon_1)}{1 - \beta_1} - 2\kappa \log(1 - \beta_1). \end{aligned}$$

From (A.0-4) in Appendix A, we have $f \geq \kappa$. Moreover, $2\log(1 - \beta_1) + \frac{2\beta_1}{1-\beta_1}$, being an upper bound on $\frac{1}{\tau\mu}\|D_3\|^*_{x+,F}$, is positive on $\beta_1 \in (0, 1)$. Hence

$$a_1 \geq \frac{2\beta_1}{1 - \beta_1} + f - \kappa + 2\log(1 - \beta_1) > 0.$$

It can be verified that

$$d_1 + \hat{d}_2 + d_3 = \frac{1}{\underline{\delta}}\left(a_1 + \frac{a_2}{\nu^{1/2}}\right).$$

Using $f \geq \kappa$ and $\beta_1 \in (0, 1)$ again, we obtain

$$a_2 + a_1 f \geq a_2 + a_1\kappa = \frac{\kappa\beta_1(\epsilon_2 + 1)}{1 - \beta_1} + \kappa(f - \kappa) > 0.$$

Hence $-a_2/a_1 < f$. It follows from this fact and the lower bound on $\underline{\delta}$ in (A.0-3) that

$$\begin{aligned} d_1 + \hat{d}_2 + d_3 &= a_1 \frac{1 - \frac{-a_2/a_1}{\nu^{1/2}}}{\underline{\delta}} \\ &\leq a_1 \frac{1 - \frac{-a_2/a_1}{\nu^{1/2}}}{1 - \frac{f}{\nu^{1/2}}}, \end{aligned}$$

and that the latter quantity is a decreasing function of ν on the interval $[1, \infty)$. Thus

$$\begin{aligned} d_1 + \hat{d}_2 + d_3 &\leq a_1 \frac{1 - (-a_2/a_1)}{1 - f} \\ &= \frac{a_1 + a_2}{1 - f} \\ &= \frac{1}{1 - f}\left[\frac{\epsilon_1(1 - \kappa) + \beta_1(\epsilon_2 + \kappa)}{1 - \beta_1} + f - \kappa + (1 - \kappa)\left(2\log(1 - \beta_1) + \frac{2\beta_1}{1 - \beta_1}\right)\right]. \end{aligned}$$

The latter quantity is an increasing function of ϵ_1, ϵ_2, and also β_1. (We have used the fact that $2\log(1-\beta_1)+\frac{2\beta_1}{1-\beta_1}$ is an increasing function of β_1 on $(0,1)$.) So

$$d_1+\hat{d}_2+d_3 \le \frac{1}{1-f}\left[\frac{\bar{\epsilon}_1(1-\kappa)+\beta_1^*(\bar{\epsilon}_2+\kappa)}{1-\beta_1^*}+f-\kappa+(1-\kappa)\left(2\log(1-\beta_1^*)+\frac{2\beta_1^*}{1-\beta_1^*}\right)\right],$$

where β_1^* is the maximum of β_1 over all $\nu \ge 1$, $\epsilon_1 \in [0,\bar{\epsilon}_1]$ and $\epsilon_2 \in [0,\bar{\epsilon}_2]$. From Lemma 4.3.10 we see that $\beta_1^* < 0.1416$, so $d_1+d_2+d_3 \le d_1+\hat{d}_2+d_3 < 0.09994 < \theta$. □

Theorem 4.3.12. *Let θ, τ, ϵ_1, and ϵ_2 be the values specified by* (4.3.27).

(a) *The sequence of duality measures generated by Algorithm short_step converges linearly to zero.*

(b) *Algorithm short_step is globally convergent: all limit points of the sequence of primal-dual iterates (x^k, w^k, s^k) generated by Algorithm short_step are primal-dual solutions of* (4.1.1)–(4.1.2).

(c) *An ε-optimal solution to* (4.1.1)–(4.1.2) *can be obtained in a polynomial number (in ν and $\log(\mu_0/\varepsilon)$) of iterations. Specifically, given $\varepsilon \in (0,1)$, there exists a number $k^* = \mathcal{O}(\nu^{1/2}\log(\mu_0/\varepsilon))$ such that $k \ge k^*$ implies $\mu_k \le \varepsilon$.*

Proof. By definition (see Lemma 4.3.9), for each k the positive ratio μ_{k+1}/μ_k is bounded above by a constant $\bar{\delta}$, and it was verified in Lemma 4.3.10 that $\bar{\delta} < 1$. Hence the duality measure decreases at least linearly. This proves (a). It follows from Lemma 4.3.11 that all iterates of Algorithm short_step remain in $\mathcal{N}(\theta)$. In fact $(x^k, w^k, s^k) \in \mathcal{N}(\theta, 0, \mu_k)$ for each k; see (4.1.16). Since $\mu_k \downarrow 0$ (from (a)), all limit points of the sequence $\{(x^k, w^k, s^k)\}$ lie in $\mathcal{N}(\theta, 0, 0)$, which is the primal-dual optimal solution set. This proves (b). The polynomial iteration bound in (c) follows from the bound on $\bar{\delta}$ in Lemma 4.3.10, and Lemma 2.4.2. □

Remark 4.3.13. *Worst-case complexity estimates of an optimization algorithm are typically phrased in terms of the problem size and some measure of accuracy of the generated solution. For the pair of problems* (4.1.1)–(4.1.2), *the parameters m and n measure the problem size. The connection to the parameter ν is seen via Lemma 4.1.2: for any full cone*

there exists a ν-normal barrier for K with $\nu = \mathcal{O}(n)$. The accuracy of the generated solution is related to $\log(\mu_0/\varepsilon)$, which is a rough indication of the number of digits of accuracy in the final iterate of the algorithm (relative to that of the initial iterate). At the time of writing, no convex optimization algorithm possessing a better worst-case iteration complexity bound than $\mathcal{O}(\nu^{1/2}\log(\mu_0/\varepsilon))$ has been constructed. This is the case even for algorithms designed only to solve linear optimization problems using exact evaluations of the barrier gradient and Hessian.

Corollary 4.3.14. *Provided the complexity parameter ν is polynomial in n, if the gradient and Hessian estimates $F_1(x^k)$ and $F_2(x^k)$ can each be computed in a polynomial number (in n) of arithmetic operations, then Algorithm short_step generates an ε-optimal solution in a polynomial number of arithmetic operations.*

Remark 4.3.15. *Here we sum up the various conditions given in this section on the parameters in order to guarantee that Algorithm short_step produces an ε-optimal solution in $\mathcal{O}(\nu^{1/2}\log(\mu_0/\varepsilon))$ iterations. It is sufficient for the parameters θ, τ, ϵ_1, and ϵ_2 to satisfy the following condition: There exists a constant $\alpha > 0$ independent of ν such that for every $\nu \geq 1$,*

$$\beta_2 < 1, \tag{4.3.35a}$$

$$\underline{\delta} := \tau - \frac{\tau\epsilon_1}{\nu^{1/2}} - \beta_0\frac{(1-\tau)(\nu^{1/2}+\epsilon_1)+\epsilon_1+\epsilon_2\nu^{1/2}}{\nu(1-\epsilon_2)^{1/2}} - \frac{1}{\nu}\beta_0^2 > 0, \tag{4.3.35b}$$

$$\tau + \frac{\tau\epsilon_1}{\nu^{1/2}} + \phi\frac{(1-\tau)(\nu^{1/2}+\epsilon_1)+\epsilon_1+\epsilon_2\nu^{1/2}}{\nu(1-\epsilon_2)^{1/2}} - \frac{1}{\nu}\phi^2 \leq 1 - \frac{\alpha}{\nu^{1/2}}, \tag{4.3.35c}$$

$$\textit{where} \qquad \phi = \min\left\{\beta_0, \frac{(1-\tau)(\nu^{1/2}+\epsilon_1)+(\epsilon_1+\epsilon_2\nu^{1/2})}{2(1-\epsilon_2)^{1/2}}\right\}, \tag{4.3.35d}$$

$$\frac{\tau\epsilon_1+\epsilon_2\beta_1}{\underline{\delta}(1-\beta_1)} + \left(\frac{\tau}{\underline{\delta}} - 1\right)\nu^{1/2} + \frac{(1-\tau)\beta_1}{\underline{\delta}(1-\beta_1)} + \frac{\tau}{\underline{\delta}}\left(2\log(1-\beta_1) + \frac{2\beta_1}{1-\beta_1}\right) \leq \theta. \tag{4.3.35e}$$

The condition in (4.3.35a) *is from Lemma 4.3.7, the conditions in* (4.3.35b), (4.3.35c), *and* (4.3.35d) *are from Lemma 4.3.9, and the condition in* (4.3.35e) *is from* (4.3.34) *in the proof of Lemma 4.3.11. The condition $\alpha > 0$ implies that for every $\nu \in [1, \infty)$ the ratio*

of subsequent duality measures μ_{k+1}/μ_k is bounded away from 1. We point out that some of the above conditions can be made tighter, leading to a slight enlargement in the set of permissible values of θ, τ, ϵ_1, and ϵ_2; see [47].

Remark 4.3.16. *The term $\nu^{1/2}$ in the worst-case complexity statement (Theorem 4.3.12(c)) arises from the condition $\beta_2 < 1$ in* (4.3.35a). *We assumed no a priori upper bound on the complexity parameter $\nu \geq 1$, so in order that $\beta_2 < 1$ it is clear from* (4.3.15) *that $(1-\tau)\nu^{1/2}$ needs to be bounded by a constant independent of ν. If τ is of the form $1 - \kappa/\nu^d$, then it is necessary that $d \geq 1/2$. So by choosing $d = 1/2$ the best worst-case complexity is obtained for our interior-point method, even when the gradient and Hessian are evaluated "exactly".*

Remark 4.3.17. *Suppose the quadruple of parameters $(\theta, \tau, \epsilon_1, \epsilon_2)$ satisfies the conditions in Remark 4.3.15. Does the quadruple $(\theta', \tau, \epsilon_1, \epsilon_2)$ also satisfy these conditions for any θ' satisfying $0 \leq \theta' < \theta$? The inequality* (4.3.35e) *tells us the answer is "no", since the left-hand side of this inequality does not tend to zero as $\theta \to 0^+$. Intuitively this makes sense since in the limiting case $\theta = 0$, a full Newton step from a point on the central path would need to end up on the central path. This is not possible however, since the central path is curved.*

How large can the errors in F_1 and F_2 be?

The parameters given in (4.3.27) are more restrictive than might be allowable in practice, because they were assumed to hold regardless of $\nu\,(\geq 1)$. For some classes of cones, one is able to obtain an upper bound or better lower bound on ν. For example, consider the class of conic optimization problems for which the barrier function whose derivatives we are estimating has complexity parameter satisfying, say, $\nu \geq 50$. The following parameters give rise to global convergence and an iteration complexity of $\mathcal{O}(\nu^{1/2}\log(\mu_0/\varepsilon))$ of Algorithm short_step:

$$\theta = 0.1, \quad \tau = 1 - \frac{0.031}{\nu^{1/2}}, \quad 0 \leq \epsilon_1 \leq 0.015, \quad 0 \leq \epsilon_2 \leq 0.084. \tag{4.3.36}$$

The value of θ is the same as that in (4.3.27). By making τ smaller, we were able to increase the range of allowable ϵ_1 and ϵ_2 values. The above parameter values and bounds are valid for a semidefinite optimization problem having a primal matrix variable whose order is at least 50, since the complexity parameter of the optimal barrier for the positive semidefinite cone equals the order of this matrix. It is known that in semidefinite optimization there are many ways of formulating the equations that define the central path, and each will have a different linearization. Depending on the linearization, it may be necessary to compute inverses or Cholesky decompositions of dense matrices in order to form the linear system encountered at each interior-point iteration. Given that such linear algebra operations are expensive, using an inexact interior-point method (that incorporates approximate inverses or Cholesky factorizations) is potentially advantageous.

To conclude this section, we indicate for various values of θ and κ, a set of (ϵ_1, ϵ_2) pairs such that the sufficient condition in Remark 4.3.15 holds. For $\theta = 0.05, 0.1, 0.15$ and 0.2, we show a set of permissible (ϵ_1, ϵ_2) pairs for $\tau = 1-\kappa/\nu^{1/2}$ with $\kappa = 0.005, 0.010, 0.015, 0.02, 0.03,$ and 0.04. The results are shown in Figures 4.2, 4.3, 4.4, and 4.5. Observe that in the case $\theta = 0.1$ and $\tau = 1 - 0.02/\nu^{1/2}$ (see Figure 4.3), the point $(0.01, 0.071)$ is close to the upper boundary of the shaded region, showing that for the parameter values in (4.3.27), our upper bounds on ϵ_1 and ϵ_2 of 0.01 and 0.071 are close to the "Pareto optimal" set. (Note that the parameter $\tau = 1 - 1/(47\nu^{1/2})$ in (4.3.27) is slightly different from $1 - 0.02/\nu^{1/2}$.) We should point out that the actual sets of permissible (ϵ_1, ϵ_2) pairs may be larger than those indicated in Figures 4.2, 4.3, 4.4, and 4.5, since the conditions from which the plots are produced can be tightened. Furthermore, no a priori information about the complexity parameter ν was assumed, and the errors $E_1(x^k)$ and $E_2(x^k)$ in the gradient and Hessian estimates are assumed to act in the worst possible directions; cf. the results in Corollary 4.3.4. In practice it is very unlikely that all the bounds derived in this corollary are tight.

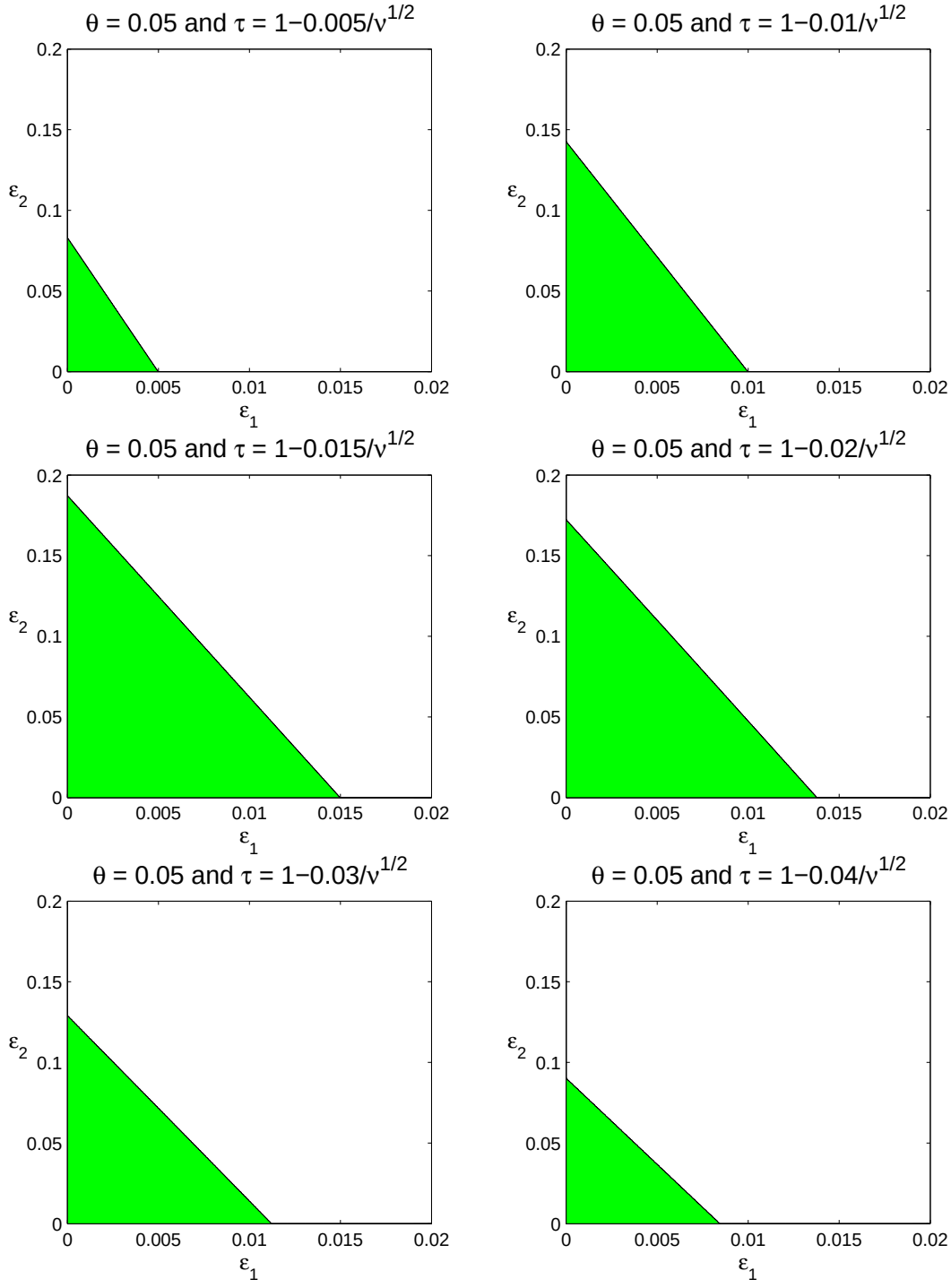

Figure 4.2: For the parameters $\theta = 0.05$ and $\tau = 1 - \kappa/\nu^{1/2}$ with six values of κ, each shaded region shows a set of permissible (ϵ_1, ϵ_2) pairs for which Algorithm short_step is globally convergent and has polynomial worst-case complexity for any $\nu \geq 1$.

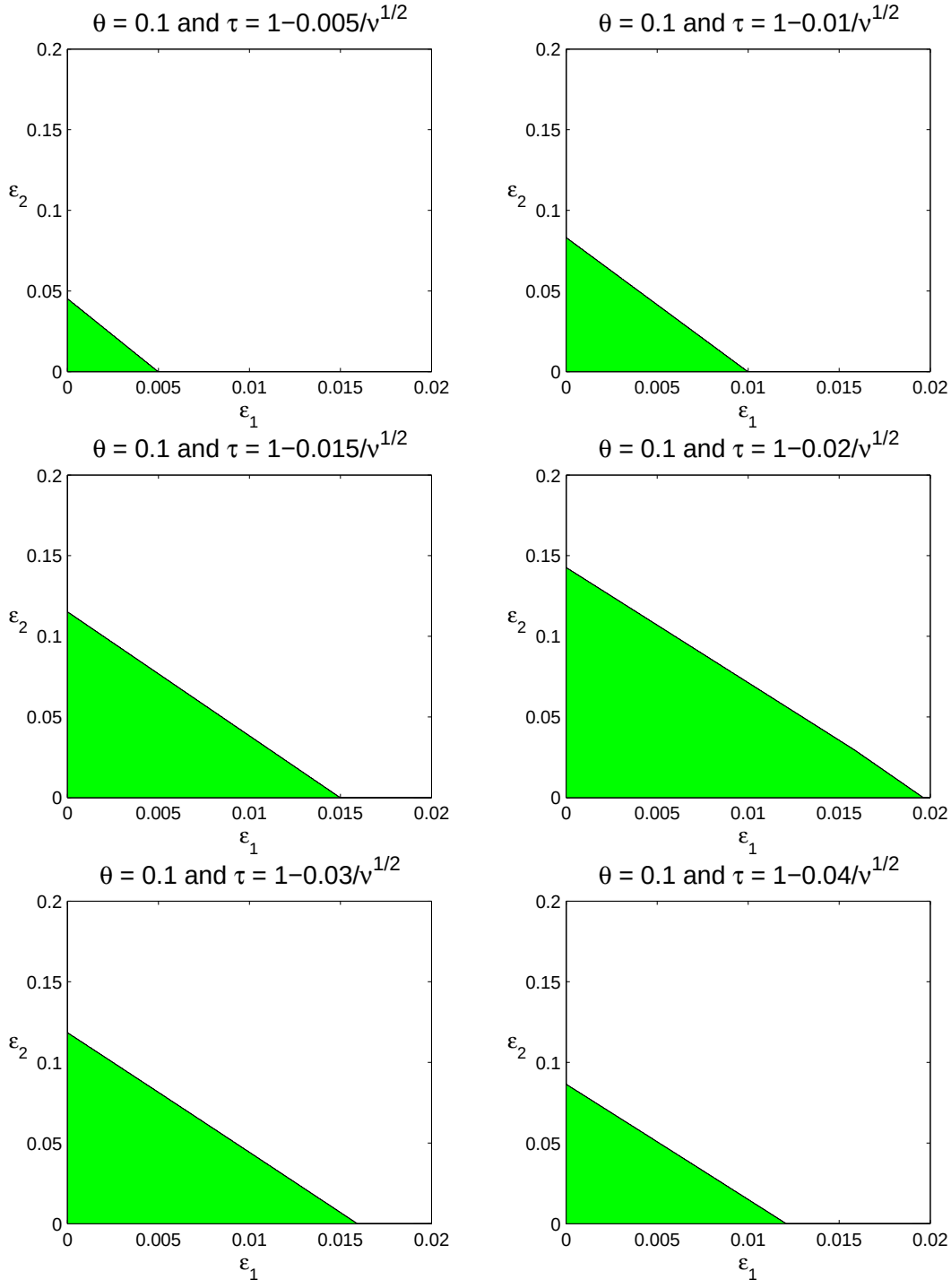

Figure 4.3: For the parameters $\theta = 0.1$ and $\tau = 1 - \kappa/\nu^{1/2}$ with six values of κ, each shaded region shows a set of permissible (ϵ_1, ϵ_2) pairs for which Algorithm short_step is globally convergent and has polynomial worst-case complexity for any $\nu \geq 1$.

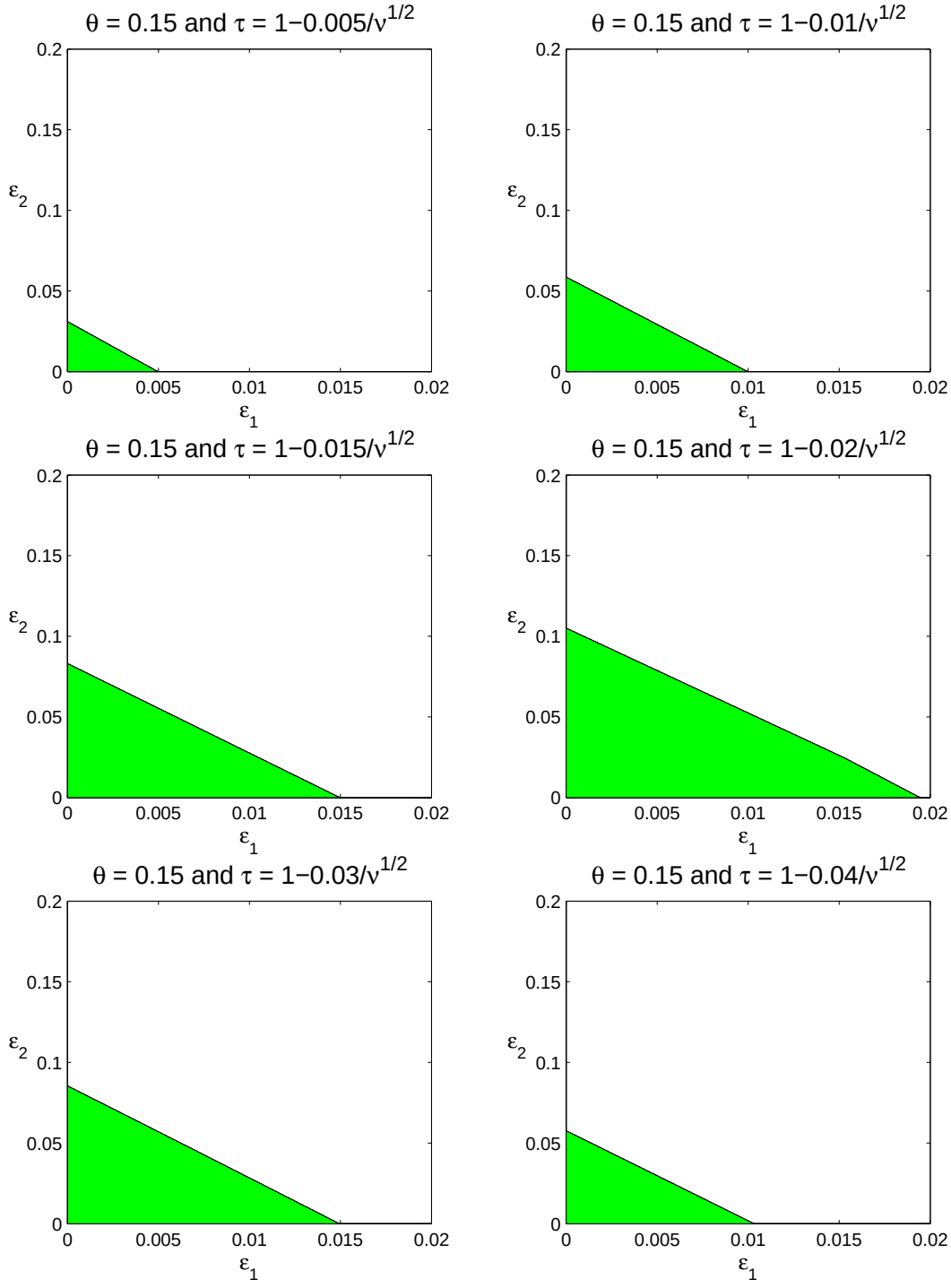

Figure 4.4: For the parameters $\theta = 0.15$ and $\tau = 1 - \kappa/\nu^{1/2}$ with six values of κ, each shaded region shows a set of permissible (ϵ_1, ϵ_2) pairs for which Algorithm short_step is globally convergent and has polynomial worst-case complexity for any $\nu \geq 1$.

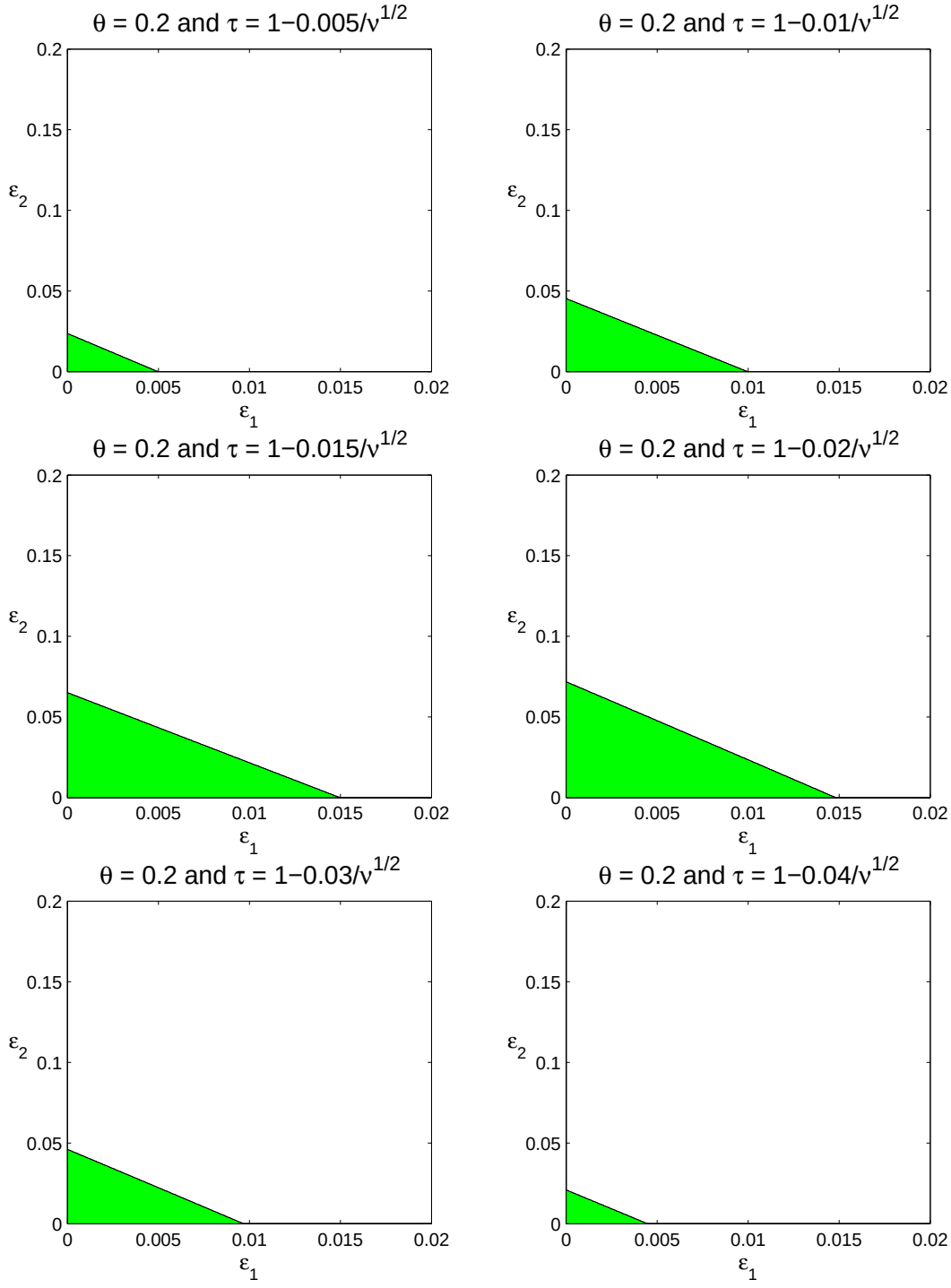

Figure 4.5: For the parameters $\theta = 0.2$ and $\tau = 1 - \kappa/\nu^{1/2}$ with six values of κ, each shaded region shows a set of permissible (ϵ_1, ϵ_2) pairs for which Algorithm short_step is globally convergent and has polynomial worst-case complexity for any $\nu \geq 1$.

4.4 Using the exact gradient and Hessian of F

In this section we will suppose that the gradient and Hessian of the ν-normal barrier function F are computed "exactly", i.e., $\epsilon_1 = \epsilon_2 = 0$. Consequently the results in this section are special cases of results in Section 4.3. For the iterates generated by Algorithm short_step the following corollary of Theorem 4.3.12 holds.

Corollary 4.4.1. *Let θ and τ be the values specified by* (4.3.27)*, and let $\epsilon_1 = \epsilon_2 = 0$.*

(a) *The sequence of duality measures generated by Algorithm short_step converges linearly to zero.*

(b) *Algorithm short_step is globally convergent: all limit points of the sequence of primal-dual iterates (x^k, w^k, s^k) generated by Algorithm short_step are primal-dual solutions of* (4.1.1)–(4.1.2)*.*

(c) *An ε-optimal solution to* (4.1.1)–(4.1.2) *can be obtained in a polynomial number (in ν and $\log(\mu_0/\varepsilon)$) of iterations. Specifically, given $\varepsilon \in (0,1)$, there exists a number $k^* = \mathcal{O}(\nu^{1/2}\log(\mu_0/\varepsilon))$ such that $k \geq k^*$ implies $\mu_k \leq \varepsilon$.*

Intuitively a stronger result than Corollary 4.4.1 should be possible for the case that $\epsilon_1 = \epsilon_2 = 0$, since Theorem 4.3.12 was proven under the assumption that $0 \leq \epsilon_1 \leq 0.01$ and $0 \leq \epsilon_2 \leq 0.071$. First we explain what "stronger" means in our context. We observed in Section 4.3 that in order for our inexact interior-point method to possess global convergence and polynomial iteration complexity, it is necessary that the parameters θ, τ, ϵ_1, and ϵ_2 be chosen appropriately. More specifically, θ, ϵ_1, and ϵ_2 must be small enough (notwithstanding Remark 4.3.17), and τ must be sufficiently close to (but less than) 1. We expect that if $\epsilon_1 = \epsilon_2 = 0$, then more flexibility can be exercised in choosing appropriate values of θ and τ.

From a theoretical point of view, we are interested in an algorithm having polynomial iteration complexity. The worst-case iteration complexity derived in Theorem 4.3.12 is of the form $\mathcal{O}(\nu^{1/2}\log(\mu_0/\varepsilon))$, where the order constant depends on θ, τ, and the maximum allowable values of ϵ_1 and ϵ_2 (but is independent of ν and ε.) By setting the maximum

allowable values of ϵ_1 and ϵ_2 to be zero, this constant is lowered, but the complexity is still $\mathcal{O}(\nu^{1/2}\log(\mu_0/\varepsilon))$. So setting $\epsilon_1 = \epsilon_2 = 0$ cannot improve the worst-case complexity. Hence the fundamental results of Corollary 4.4.1—linear convergence of the sequence $\{\mu_k\}$, global convergence of the sequence of triples $\{(x^k, w^k, s^k)\}$, and $\mathcal{O}(\nu^{1/2}\log(\mu_0/\varepsilon))$ worst-case iteration complexity—cannot be strengthened by setting $\epsilon_1 = \epsilon_2 = 0$.

However from a practical point of view, computational experience for convex optimization has shown that path-following interior-point methods require fewer iterations when the iterates are permitted to lie in a wider neighborhood of the central path. This is intuitively reasonable since the larger neighborhoods associated with larger values of θ give more freedom for points to move toward an optimal solution. So if setting $\epsilon_1 = \epsilon_2 = 0$ permits larger values of θ than are otherwise possible to be used in our algorithm, then in a practical sense the resulting algorithm can be said to represent an improvement. Furthermore we can ensure a faster reduction in the sequence of duality measures by choosing τ to be further from 1. Letting τ take the form $1 - \kappa/\nu^{1/2}$, as in (4.3.27), it can be verified that for $\epsilon_1 = \epsilon_2 = 0$, the bounds on the ratio of successive duality measures in Lemma 4.3.9 are given by

$$\underline{\delta} = 1 - \frac{\kappa}{\nu^{1/2}} - \frac{(\theta+\kappa)(\theta+2\kappa)}{\nu} \leq \frac{\mu_{k+1}}{\mu_k} \leq 1 - \frac{\kappa}{\nu^{1/2}} + \frac{\kappa^2}{4\nu} = \bar{\delta}.$$

The condition $\beta_2 < 1$ (see (4.3.35a)) implies that $\kappa < 1 - 2\theta$. As κ increases from 0 to $1 - 2\theta$, the above lower and upper bounds on μ_{k+1}/μ_k decrease monotonically. Note also that for fixed κ, as θ increases, the lower bound on μ_{k+1}/μ_k decreases and the upper bound remains constant. We conclude that if the barrier gradient and Hessian can be computed "exactly", then larger values of θ and κ are to be preferred in Algorithm short_step. In the rest of this section we improve the choices of θ and κ over those in (4.3.27). The proofs of these results follow from those in Section 4.3, so they are omitted.

In the case that $\epsilon_1 = \epsilon_2 = 0$ and $\tau = 1 - \frac{\kappa}{\nu^{1/2}}$, the constants β_1 and β_2 in (4.3.15) become

$$\begin{aligned} \beta_1 &:= \theta + \kappa, \\ \beta_2 &:= \frac{\theta + \kappa}{1 - \theta}. \end{aligned} \tag{4.4.1}$$

In the remainder of this section we shall use the following values for the parameters in Algorithm short_step:

$$\theta = 0.25, \qquad \tau = 1 - \frac{1}{32\nu^{1/2}}. \tag{4.4.2}$$

Using these values, let us bound the values of $\beta_1, \beta_2, \underline{\delta}$, and $\bar{\delta}$.

Lemma 4.4.2. *Let θ and τ satisfy* (4.4.2). *For all $\nu \geq 1$,*

$$\beta_1{=}0.2813, \qquad \beta_2 = 0.3750, \qquad \underline{\delta} > 1 - \frac{0.1192}{\nu^{1/2}} > 0, \qquad \bar{\delta} < 1 - \frac{0.0310}{\nu^{1/2}} < 1.$$

Proof. Similar to that of Lemma 4.3.10. □

We now present a convergence and complexity result for Algorithm short_step with exact gradient and Hessian information. It is similar to Theorem 4.3.12.

Theorem 4.4.3. *Let θ and τ be the values specified by* (4.4.2).

(a) *The sequence of duality measures generated by Algorithm short_step converges linearly to zero.*

(b) *Algorithm short_step is globally convergent: all limit points of the sequence of primal-dual iterates (x^k, w^k, s^k) generated by Algorithm short_step are primal-dual solutions of* (4.1.1)–(4.1.2).

(c) *An ε-optimal solution to* (4.1.1)–(4.1.2) *can be obtained in a polynomial number (in ν and $\log(\mu_0/\varepsilon)$) of iterations. Specifically, given $\varepsilon \in (0, 1)$, there exists a number $k^* = \mathcal{O}(\nu^{1/2} \log(\mu_0/\varepsilon))$ such that $k \geq k^*$ implies $\mu_k \leq \varepsilon$.*

Proof. Similar to that of Theorem 4.3.12. □

The disparity between the parameters in (4.4.2) and those in (4.3.27) reflects the fact that in order to guarantee that the interior-point iterates remain inside $\mathcal{N}(\theta)$, less flexibility is allowed when unstructured perturbations are permitted. If in (4.3.27) θ was increased to 0.25 but the bounds on ϵ_1 and ϵ_2 were unchanged, then it would not be possible to guarantee that all iterates remain in $\mathcal{N}(\theta)$; i.e., Lemma 4.3.11 may no longer hold.

The parameters given in (4.4.2) are more restrictive than might be allowable in practice, due to the assumption that $\nu \geq 1$. For some classes of cones, one is able to obtain a better lower bound on ν. Consider the class of conic optimization problems for which the barrier function whose derivatives we are estimating has complexity parameter $\nu \geq 50$. The following parameters give rise to global convergence and a polynomial iteration complexity of $\mathcal{O}(\nu^{1/2} \log(\mu_0/\varepsilon))$ for Algorithm short_step:

$$\theta = 0.35, \quad \tau = 1 - \frac{1}{20\nu^{1/2}}. \tag{4.4.3}$$

Note that the values of θ and τ are superior (in the sense described earlier in this section) to those in (4.4.2). The above parameter values are valid for a semidefinite optimization problem having a matrix variable of order at least 50, since the complexity parameter of the optimal barrier for the positive semidefinite cone equals the order of this matrix.

To conclude this section, we indicate in Figure 4.6 the possible values of θ and κ such that for $\epsilon_1 = \epsilon_2 = 0$ and for any $\nu \geq 1$, Algorithm short_step is globally convergent and has polynomial worst-case complexity.

4.5 Structured perturbations in the gradient and Hessian of F

In this section we will consider a class of *structured* perturbations in the gradient and Hessian. In other words, the errors in the gradient and Hessian estimates are assumed to be related. Specifically, throughout the rest of this section it will be assumed that the ν-normal

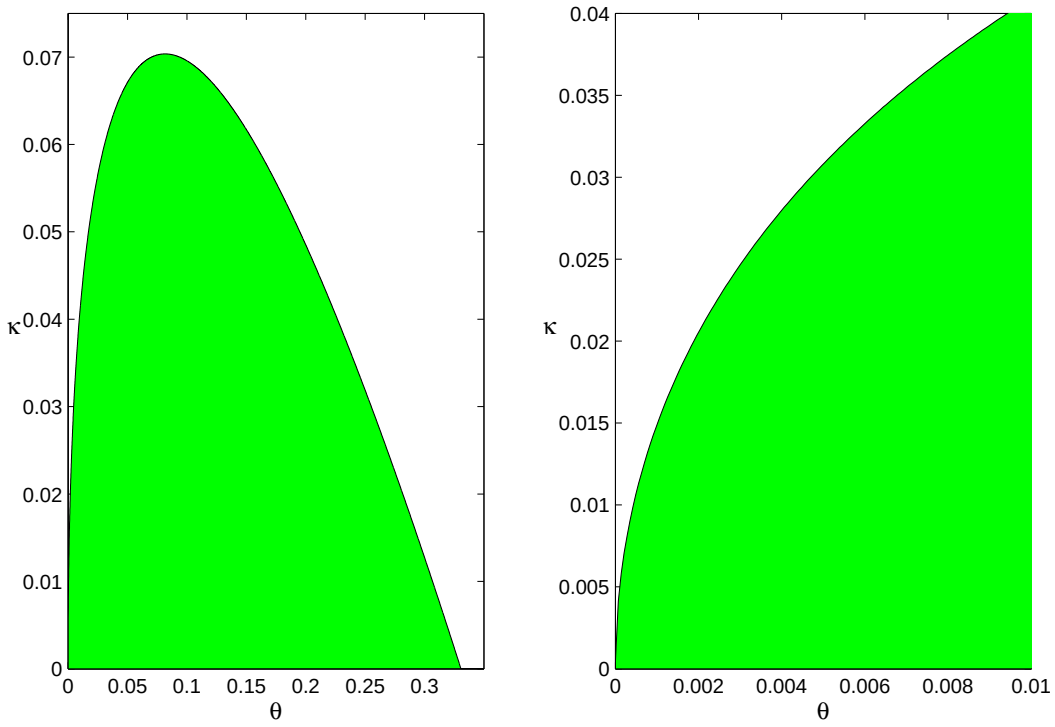

Figure 4.6: For the case that exact gradient and Hessian information is computed, i.e., $\epsilon_1 = \epsilon_2 = 0$, the shaded region in the plot on the left shows a set of permissible (θ, κ) pairs (where $\tau = 1 - \kappa/\nu^{1/2}$) for which Algorithm short_step is globally convergent and has polynomial worst-case iteration complexity for any $\nu \geq 1$. The region of permissible pairs excludes the axes. The plot on the right is close-up of the plot on the left when θ is small. It shows that for fixed κ, if θ is too small, we cannot guarantee that the algorithm will converge to a solution and have polynomial worst-case iteration complexity. Cf. Remark 4.3.17.

barrier F is estimated by a function $\hat{F} : \text{int}(K) \to \mathcal{R}$ possessing the following properties:

(a) $\hat{F}$ is twice continuously differentiable on $\text{int}(K)$;

(b) $\hat{F}$ is a nondegenerate convex function on $\text{int}(K)$;

(c) $\hat{F}$ satisfies the logarithmic-homogeneity property, i.e., for every $t > 0$ and $x \in \text{int}(K)$,

$$\hat{F}(tx) = \hat{F}(x) - \nu \log(t). \tag{4.5.1}$$

These properties are enough for the results in Lemma 3.3.2 to apply, even though $\hat{F}$ is not assumed to be logarithmically homogeneous. That is, we do not assume $\hat{F}$ satisfies the barrier property. Consequently it is not possible to suppose that $\hat{F}$ can approximate F well arbitrarily close to the boundary of K: there may exist a point x on the boundary of K such that for some sequence of points $x^i \in \text{int}(K)$ converging to x, $\|F''(x^i)\|_2 \to \infty$, yet $\|\hat{F}''(x)\|_2$ is finite. So we will assume that $\hat{F}$ approximates F well only inside a restricted neighborhood $\mathcal{X}(\theta, \varepsilon, \mu_0)$ of $\text{int}(K)$, where

$$\mathcal{X}(\theta, \underline{\mu}, \bar{\mu}) := \{x \mid (x, w, s) \in \mathcal{N}(\theta, \underline{\mu}, \bar{\mu}) \text{ for some } w, s\}, \tag{4.5.2}$$

and $\mathcal{N}(\theta, \underline{\mu}, \bar{\mu})$ is defined in (4.1.16). Here $\varepsilon > 0$ is fixed and μ_0 is the duality measure of the strongly feasible starting point in Algorithm short_step. All results in this section hold in such a neighborhood, but may fail to hold outside it. Therefore we can only run Algorithm short_step while its iterates are in $\mathcal{N}(\theta, \varepsilon, \mu_0)$. In other words, the termination criterion of the algorithm is determined by ε. We have seen in Section 4.3 that for an appropriate choice of the parameters θ and τ, and certain ranges of the errors ϵ_1 and ϵ_2, we can ensure all iterates remain in the neighborhood $\mathcal{N}(\theta)$. Moreover the sequence of duality measures $\{\mu_k\}$ strictly decreases to zero. So if our barrier estimate $\hat{F}$ is such that the errors ϵ_1 and ϵ_2 satisfy the conditions derived in Section 4.3, then Algorithm short_step will run until the duality measure falls below ε, thereby yielding an ε-optimal solution.

We will estimate the gradient and Hessian of F by the gradient and Hessian of $\hat{F}$, so that when we refer to Algorithm short_step in this section, it will be assumed that $\hat{F}'$ and

$\hat{F}''$ will take the place of F_1 and F_2 in that algorithm.

Denote the errors in our estimates of the gradient and Hessian of $F(x)$ by

$$E_1(x) = F'(x) - \hat{F}'(x), \qquad E_2(x) = F''(x) - \hat{F}''(x).$$

By definition $E_2(x)$ is a symmetric matrix. We first note two relationships between $E_1(x)$ and $E_2(x)$.

Lemma 4.5.1. *The errors in $\hat{F}'(x)$ and $\hat{F}''(x)$ are related by $E_2(x)x = -E_1(x)$.*

Proof. It follows from the logarithmic homogeneity of F that $F''(x)x = -F'(x)$ (see Lemma 3.3.2(c)), and since $\hat{F}$ satisfies the logarithmic-homogeneity relation (4.5.1), $\hat{F}''(x)x = -\hat{F}'(x)$. Subtracting these equations gives the required result. □

Lemma 4.5.2. *Suppose that $x \in \text{int}(K)$. Then*

$$\|E_1(x)\|^*_{x,F} \;\leq\; \|F''(x)^{-1/2} E_2(x) F''(x)^{-1/2}\|_2 \; \nu^{1/2}.$$

Proof. It follows from Lemma 4.5.1 that

$$\big(F''(x)^{-1/2} E_2(x) F''(x)^{-1/2}\big)\big(F''(x)^{1/2} x\big) = -F''(x)^{-1/2} E_1(x).$$

Taking the norm of each side and using Lemma 3.3.2(d), we obtain

$$\|F''(x)^{-1/2} E_2(x) F''(x)^{-1/2}\|_2 \; \nu^{1/2} \geq \|F''(x)^{-1/2} E_1(x)\|_2 \equiv \|E_1(x)\|^*_{x,F} \; . \qquad \square$$

The relation in Lemma 4.5.1 indicates that if the norm of $E_2(x)$ is small, the norm of $E_1(x)$ is also small. Let $\theta \in (0,1)$ and $\epsilon_1, \epsilon_2, \varepsilon > 0$ be given. As in Section 4.3, we will assume that the "relative errors" in $E_1(x)$ and $E_2(x)$ are sufficiently small. Specifically,

$$x \in \mathcal{X}(\theta, \varepsilon, \mu_0) \quad \Longrightarrow \quad \|E_1(x)\|^*_{x,F} \equiv \|F''(x)^{-1/2} E_1(x)\|_2 \leq \epsilon_1 < 1, \tag{4.5.3}$$

$$x \in \mathcal{X}(\theta, \varepsilon, \mu_0) \quad \Longrightarrow \quad \|F''(x)^{-1/2} E_2(x) F''(x)^{-1/2}\|_2 \leq \epsilon_2 < 1. \tag{4.5.4}$$

As a consequence of Lemma 4.5.2, it is not essential to assume separate unrelated bounds ϵ_1 and ϵ_2 on the errors in the gradient and Hessian estimates, because if $E_2(x)$ satisfies (4.5.4), then (4.5.3) is guaranteed to hold, provided $\epsilon_2 \nu^{1/2} < 1$. However this is unsatisfactory, since $\epsilon_2 \nu^{1/2} < 1 \, \forall \, \nu \geq 1$ implies that $\epsilon_2 = \mathcal{O}(\nu^{-1/2})$ would be required. However, in the case of unstructured perturbations in Section 4.3—of which the structured perturbations considered here are a special case—the maximum value of ϵ_2 given in (4.3.27) is $\mathcal{O}(1)$.

As another consequence of our structured perturbations being a special case of the unstructured perturbations considered in Section 4.3, the convergence and worst-case complexity results in Theorem 4.3.12 hold here. The rest of this section is devoted to showing that the relationship between the estimates $\hat{F}'$ and $\hat{F}''$ allows one to strengthen Theorem 4.3.12 in the sense that if θ and τ are fixed, the maximum allowable "relative errors" ϵ_1 and ϵ_2, are increased. The analysis is similar to that in Section 4.3, so we include it in Appendix B, giving only the main results here.

It will be convenient to define the following three constants depending on $\theta, \tau, \epsilon_1, \epsilon_2$ and $\nu \geq 1$ (cf. (4.3.15)).

$$\begin{aligned} \beta_0 &:= \frac{\theta + \epsilon_1}{(1-\epsilon_2)^{1/2}} + (1-\tau)\nu^{1/2}, \\ \beta_1 &:= \left(\frac{1}{1-\epsilon_2} \right)^{1/2} \beta_0, \\ \beta_2 &:= \beta_0 \max \left\{ \frac{(1+\epsilon_2)^{1/2}}{1-\theta}, \left(\frac{1}{1-\epsilon_2} \right)^{1/2} \right\}. \end{aligned} \tag{4.5.5}$$

In the remainder of this section we shall use the following values for the parameters in Algorithm short_step:

$$\theta = 0.1, \quad \tau = 1 - \frac{1}{47\nu^{1/2}}, \quad 0 \leq \epsilon_1 \leq 0.015, \quad 0 \leq \epsilon_2 \leq 0.12. \tag{4.5.6}$$

Using these values, let us bound the values of $\beta_1, \beta_2, \underline{\delta}$, and $\bar{\delta}$.

Lemma 4.5.3. *Let θ, τ, ϵ_1, and ϵ_2 satisfy* (4.5.6). *For all $\nu \geq 1$,*

$$\beta_1 < 0.1534, \qquad \beta_2 < 0.1692, \qquad \underline{\delta} > 1 - \frac{0.0451}{\nu^{1/2}} > 0, \qquad \bar{\delta} < 1 - \frac{0.0211}{\nu^{1/2}} < 1.$$

Proof. See Appendix B and in particular Lemma B.0.5. □

We now present the convergence and complexity result for Algorithm short_step with structured perturbations in the gradient and Hessian of a normal barrier.

Theorem 4.5.4. *Let θ, τ, ϵ_1, and ϵ_2 be the values specified by* (4.5.6).

(a) *The sequence of duality measures generated by Algorithm short_step converges linearly to zero.*

(b) *All limit points of the sequence of primal-dual iterates (x^k, w^k, s^k) generated by Algorithm short_step are ε-optimal solutions of* (4.1.1)–(4.1.2).

(c) *An ε-optimal solution to* (4.1.1)–(4.1.2) *can be obtained in a polynomial number (in ν and $\log(\mu_0/\varepsilon)$) of iterations. Specifically, given $\varepsilon \in (0, 1)$, there exists a number $k^* = \mathcal{O}(\nu^{1/2} \log(\mu_0/\varepsilon))$ such that $k \geq k^*$ implies $\mu_k \leq \varepsilon$.*

Proof. Similar to that of Theorem 4.3.12. □

The values of θ and τ in (4.5.6) are the same as those used in the analogous result (Theorem 4.3.12) for the case of unstructured perturbations. Observe however that the allowable range of ϵ_1 and ϵ_2 values has increased by at least 50%, showing the benefits of structure.

Remark 4.5.5. *The parameters given in* (4.5.6) *are more restrictive than might be allowable in practice, due to the assumption that $\nu \geq 1$. For some classes of cones, one is able to obtain a better lower bound on ν. Consider the class of conic optimization problems for which the barrier function whose derivatives we are estimating has complexity parameter $\nu \geq 50$. The following parameters give an iteration complexity for Algorithm short_step of $\mathcal{O}(\nu^{1/2} \log(\mu_0/\varepsilon))$:*

$$\theta = 0.1, \quad \tau = 1 - \frac{0.031}{\nu^{1/2}}, \quad 0 \leq \epsilon_1 \leq 0.02, \quad 0 \leq \epsilon_2 \leq 0.147. \tag{4.5.7}$$

The values of θ and τ are the same as those in (4.3.36)*, where $\nu \geq 50$ and unstructured perturbations were assumed, but the range of allowable ϵ_1 and ϵ_2 values is now significantly increased. The above parameter values and bounds are valid for a semidefinite optimization problem having a primal matrix variable whose order is at least* 50*, since the complexity parameter of the optimal barrier for the positive semidefinite cone equals the order of this matrix.*

Remark 4.5.6. *Here we sum up the various conditions given in this section on the parameters in order to guarantee that Algorithm short_step produces an ε-optimal solution in $\mathcal{O}(\nu^{1/2}\log(\mu_0/\varepsilon))$ iterations. It is sufficient for the parameters θ, τ, ϵ_1, and ϵ_2 to satisfy the following condition: There exists a constant $\alpha > 0$ independent of ν such that for every $\nu \geq 1$,*

$$\begin{aligned}
\beta_2 &< 1, && \text{(4.5.8a)}\\
\underline{\delta} := \tau - \frac{1}{\nu^{1/2}}(1-\tau)\beta_0 - \frac{1}{\nu}\beta_0^2 &> 0, && \text{(4.5.8b)}\\
\frac{(1+\tau)^2}{4} &\leq 1 - \frac{\alpha}{\nu^{1/2}}, && \text{(4.5.8c)}\\
\frac{\tau\epsilon_1 + \epsilon_2\beta_1}{\underline{\delta}(1-\beta_1)} + \left(\frac{\tau}{\underline{\delta}} - 1\right)\nu^{1/2} + \frac{(1-\tau)\beta_1}{\underline{\delta}(1-\beta_1)} + \frac{\tau}{\underline{\delta}}\left(2\log(1-\beta_1) + \frac{2\beta_1}{1-\beta_1}\right) &\leq \theta. && \text{(4.5.8d)}
\end{aligned}$$

We point out that some of the above conditions can be made tighter, leading to a slight enlargement in the set of permissible values of θ, τ, ϵ_1, and ϵ_2; see [47].

Chapter 5

An application: using the universal barrier function in an inexact interior-point method for conic optimization problems

Contents

5.1 Introduction

Recall from Lemma 4.1.2 that for any full cone $K \subset \mathcal{R}^n$, there exists a ν-normal barrier F for K, with $\nu = \mathcal{O}(n)$. The proof of this result is constructive in that a cone-dependent ν-normal barrier was given. This function was named the *universal barrier function* because it can be constructed for *any* full cone. It follows that for any full cone K, at least two normal barrier functions are known (up to an additive and a multiplicative constant), and for some cones these are the only known normal barriers. One is the universal barrier function for K. The other, in light of Lemma 3.3.5, is the conjugate of the universal barrier function for K^*. Since evaluation of the latter function may require the solution of the optimization problem in (3.3.2), evaluation of the former function is usually to be preferred. Therefore in order to solve some conic optimization problems using an interior-point method, one may need to evaluate, or at least estimate, the universal barrier function or its gradient and Hessian.

As it was given in [33], the universal barrier function F for a cone K was written in terms of a multidimensional integral whose integrand depends on K. Due to the complicated nature of this integral, for some classes of cones K, F and its derivatives do not appear to be available in an easily computable form, which is problematic if we wish to use these quantities in an interior-point method. In addition, the complexity parameter ν of the universal barrier function, which appears in the upper bound on the number of iterations of an interior-point method required to obtain a near-optimal solution, can be much larger than the optimal value. The optimal value is the smallest ψ satisfying the properties defining a self-concordant barrier function (see Definition 3.2.3). This disparity was illustrated in [16, Section 7]. To our knowledge, a systematic procedure for computing the optimal complexity parameter—or even a good upper bound on it—has not been found. This is another reason why the universal barrier function has not been used in a general conic

optimization algorithm. We do, however, point out the work in [9], where the universal barrier function for the cone K generated by a *Chebyshev system* was studied. This cone is such that the evaluation of the gradient and Hessian of the associated universal barrier function reduces to the computation of several one-dimensional integrals. In some cases, it is not possible to evaluate exactly such integrals, so a sequence of polyhedral outer-approximations of the cone generated by a Chebyshev system is designed. This sequence of cones converges to K in an appropriate sense, and the universal barrier function for each polyhedral approximation is easily computable; see [8]. As the polyhedral outer-approximations of K are defined on "finer meshes", the universal barrier function may be approximated to arbitrary precision.

As an application of our work on inexact interior-point methods in Chapters 4, we consider the effect of estimating, by a *Monte Carlo approximation*, the gradient and Hessian of the universal barrier function F for any convex cone. It is shown that the resulting estimate of F satisfies several important properties that make F itself a a suitable barrier function in the context of interior-point methods. The Monte Carlo sampling can be done in such way that the errors in the estimated gradient and Hessian of F are related, and this additional structure permits larger errors than would otherwise be the case. In other words, we have an application of the structured perturbations that were considered in Section 4.5. Since for many classes of convex cones, F cannot be computed in a finite number of arithmetic operations, the problem of using estimates of the gradient and Hessian of F in an interior-point method is an application of our theory on the inexact evaluation of barrier functions. A key aspect of the application is to obtain a bound on the Monte Carlo sample size such that our interior-point method is globally convergent and has polynomial worst-case iteration complexity. Ideally we would also like *each iteration* to require only a polynomial amount of computational effort, but this may be unrealistic in general.

5.2 Definition and characterization of the universal barrier function

5.2.1 Introduction

In this section, given a full cone K, we will study the *universal barrier function* $F : \mathrm{int}(K) \to \mathcal{R}$, defined in [33, p. 50] as

$$F(x) = C \log \mathrm{vol}_n(K^o(x)), \tag{5.2.1}$$

where $C > 0$ is an appropriately chosen constant,

$$K^o(x) := \{y \in \mathcal{R}^n \mid y^T(z - x) \leq 1 \, \forall \, z \in K\}$$

is the polar set of K at x, and $\mathrm{vol}_n(\cdot)$ denotes n-dimensional Lebesgue measure. For each $x \in \mathrm{int}(K)$ the polar set $K^o(x)$ is compact, convex, and has nonempty interior.

As noted in Lemma 4.1.2, it was shown in [33] that there exists a Cn-normal barrier for every cone K, for some constant C independent of n and K. The normal barrier constructed in [33] is in fact the universal barrier function in (5.2.1), and the constant C in (5.2.1) is that in Lemma 4.1.2. The logarithmic homogeneity of the universal barrier function was noted in [33, Remark 2.5.1]. In [33] the self-concordancy of this function was used to establish that *in principle* (4.1.1) may be solved by certain classes of interior-point methods in a polynomial number of iterations. More specifically, such methods will produce a feasible x such that $c^T x - v_P \leq \varepsilon$ in a number of iterations that is polynomial in n and $\log(1/\varepsilon)$.[1] In the proof of [33, Theorem 2.5.1], an alternative equivalent formula for the universal barrier

[1] In Chapter 4 we defined an ε-optimal solution in a slightly different way—in terms of the duality gap $c^T x - b^T w$ associated with a primal-dual point (x, w, s). In [33] the main concern was solving the primal problem (4.1.1), so the "gap to optimality" is defined differently, viz., independently of the dual variables.

function in (5.2.1) is suggested:

$$F(x) = C \log \frac{1}{n}\varphi(x), \tag{5.2.2}$$

where

$$\varphi(x) = \int_{K^* \cap S^{n-1}} \frac{1}{(x^T y)^n} \, \mathrm{d}y, \tag{5.2.3}$$

and S^{n-1} denotes the unit sphere in $\mathcal{R}^n$. (It is easily verified from (5.2.2) and (5.2.3) that F does in fact satisfy the logarithmic-homogeneity relation (3.3.1) with parameter $\nu = Cn$.) Since $F(x)$ is a normal barrier, $\log \varphi(x)$ is a nondegenerate convex function (on $\mathrm{int}(K)$), hence a strictly convex function. It follows that $\varphi(x)$ is logarithmically strictly convex, hence strictly convex. We now discuss the value of C. It follows from the definition of a self-concordant barrier function that the function $C \log \frac{1}{n}\varphi(x)$ is a normal barrier, provided C is chosen large enough. However the cost of increasing C is an increase in the worst-case iteration complexity of interior-point methods that use $C \log \frac{1}{n}\varphi(x)$ as a barrier function. Unfortunately it is difficult to find the optimal (minimal) universal constant C. It seems difficult to even find a good upper bound on the optimal C. Moreover, even if one can find the optimal C, there is no guarantee that Cn is the smallest complexity parameter of any normal barrier for K, since other normal barriers for K may exist. We point out that in practice it is not essential to know a universal constant C that works for any cone K having any dimension; it is enough to know a C such that $C \log \frac{1}{n}\varphi(x)$ is a normal barrier for the particular cone in (4.1.1). One may ask if there is a relationship between the various barriers for a given full cone K, so that given one barrier, some or all of the others might be generated. As far as we know this is an open question. However it has been shown (see [17]) that the self-scaled barrier functions (see Definition 3.3.8) defined on self-scaled cones are precisely the universal barrier function up to a homothetic transformation. That is, given a (irreducible) self-scaled cone K and a self-scaled barrier function $\tilde{F}$ for K, there exist constants $c_1 > 0$ and c_2 such that $\tilde{F}(x) = c_1 F(x) + c_2$, where F is defined in (5.2.2).

An equivalent expression for the universal barrier function can be derived. For any full cone K and $x \in \mathrm{int}(K)$ we have

$$\begin{aligned}
\int_{K^*} e^{-x^T y} \, \mathrm{d}y &= \int_{R=0}^{\infty} \int_{K^* \cap \{y : \|y\|_2 = R\}} e^{-x^T y} \, \mathrm{d}y \, \mathrm{d}R \\
&= \int_{R=0}^{\infty} \int_{K^* \cap S^{n-1}} e^{-x^T (Ry)} \, R^{n-1} \, \mathrm{d}y \, \mathrm{d}R \\
&= \int_{K^* \cap S^{n-1}} \int_{R=0}^{\infty} e^{-R x^T y} \, R^{n-1} \, \mathrm{d}R \, \mathrm{d}y \\
&= \int_{K^* \cap S^{n-1}} \frac{(n-1)!}{(x^T y)^n} \, \mathrm{d}y.
\end{aligned}$$

The second equality follows from a change of coordinates from Cartesian to radial—e.g., hyperspherical—coordinates, a linear scaling of the radial variable by R, and then another change of variables from radial back to Cartesian coordinates.[2] The fourth equality follows from integration by parts on the inner (one-dimensional) integral in R. It uses the fact that $x^T y > 0$ for all $y \in K^* \cap S^{n-1}$, which is true for any $x \in \mathrm{int}(K)$. Note that the new region of integration $K^* \cap S^{n-1}$ is bounded. So we have the following equivalent formula for the universal barrier function:

$$F(x) = C \log \frac{1}{n!} \int_{K^*} e^{-x^T y} \, \mathrm{d}y. \tag{5.2.4}$$

In fact the relation (5.2.4) was proven in [16, Theorem 4.1], but the details of the proof are different from those we have given. In [22] (see also [23, Chapter 1]) the *characteristic function* of a certain class of convex "cones" U was defined to be $\int_{U^*} e^{-\langle x,y \rangle} \, \mathrm{d}y$, with $x \in U$.[3] It will be convenient for us to instead define $\varphi(x)$ as the characteristic function of K. It

[2] To explain this in more detail, suppose that we use the hyperspherical coordinate representation explained in Section 5.4. Given a Cartesian point $y \in K^*$ with $\|y\|_2 = R$, one generates a hyperspherical point $(R, \alpha) \in \mathcal{R}_+ \times \mathcal{R}^{n-1}$. The factor of R^{n-1} in the transformed integrand comes from the Jacobian of the coordinate transformation; see (5.4.8). Geometrically, it is a consequence of the fact that an $(n-1)$-dimensional sphere of radius R has $(n-1)$-dimensional volume R^{n-1} times that of the unit $(n-1)$-dimensional sphere. Now perform the inverse transformation by converting the hyperspherical point $(1, \alpha)$ into a Cartesian point y with $\|y\|_2 = 1$.

[3] More precisely, U was assumed to be the interior of a self-dual cone, and therefore excludes the origin. Hence by our definition of a cone, U is not a cone. Nevertheless U is a cone according to the definition used in [22]; see footnote 1 on page 16.

can be seen that our characteristic function is $\int_{U^*} e^{-x^T y}\,\mathrm{d}y$ up to a constant factor, where $U = K$.

Yet another equivalent expression for the characteristic function shows that it can be written as a one-dimensional integral. Given a full cone K, a vector $x \in \mathrm{int}(K)$, and a scalar $t \geq 0$, define

$$\begin{aligned} \mathcal{H}(t,x) &= \left\{ y \in K^* \cap S^{n-1} \;\middle|\; \frac{x^T y}{\|x\|_2} \leq t \right\}, \\ h(t,x) &= \mathrm{vol}_{n-1}(\mathcal{H}(t,x)), \\ \gamma(x) &= \min_{y \in K^* \cap S^{n-1}} \frac{x^T y}{\|x\|_2} > 0. \end{aligned} \tag{5.2.5}$$

Noting that $x^T y/\|x\|_2$ is the cosine of the angle between x and $y \in S^{n-1}$, $\gamma(x)$ measures the maximum angle between x and vectors in K^*. We see that $h(t,x) = 0$ for $0 \leq t \leq \gamma(x)$ and $h(t,x) = \mathrm{vol}_{n-1}(K^* \cap S^{n-1})$ for $t \geq 1$. The function $h(t,x)$ is a monotone increasing function of t, but fails to be differentiable everywhere. We can write the characteristic function as a one-dimensional integral in terms of $h(t,x)$:

$$\begin{aligned} \varphi(x) &= \frac{1}{\|x\|_2^n} \int_{K^* \cap S^{n-1}} \left(\frac{\|x\|_2}{x^T y} \right)^n \mathrm{d}y \\ &= \frac{1}{\|x\|_2^n} \int_{K^* \cap S^{n-1}} \int_0^{\left(\|x\|_2 / x^T y\right)^n} \mathrm{d}t\,\mathrm{d}y \\ &= \frac{1}{\|x\|_2^n} \int_0^\infty \int_{\mathcal{H}(t^{-1/n},x)} \mathrm{d}y\,\mathrm{d}t \\ &= \frac{1}{\|x\|_2^n} \int_0^\infty h(t^{-1/n}, x)\ \mathrm{d}t \\ &= \frac{1}{\|x\|_2^n} \int_0^{\gamma(x)^{-1/\alpha}} \alpha n u^{\alpha n - 1} h(u^{-\alpha}, x)\ \mathrm{d}u, \qquad \alpha > 0. \end{aligned} \tag{5.2.6}$$

The third equality follows from the definition of $\mathcal{H}$, the fourth follows from the definition of h, and the last follows from the change of variables $u = t^{\alpha n}$.

5.2.2 Derivatives of the universal barrier function

In Chapter 4 we presented an interior-point method that uses estimates of the first and

second derivatives of a suitable barrier function. We now compute these derivatives for the universal barrier function. We first give formulas for the first and second derivatives of φ. Since this requires the differentiation of the integral in (5.2.3), we will first need to verify that it is valid to interchange the differential and integral operators.

Consider the function $f : \mathrm{int}(K) \times (K^* \cap S^{n-1}) \to \mathcal{R}$ given by $f(x,y) = \frac{1}{(x^T y)^n}$. For a fixed $x \in \mathrm{int}(K)$, $y \in K^* \cap S^{n-1}$ is bounded away from orthogonality to x; cf. (5.2.5). Hence for all $y \in K^* \cap S^{n-1}$,

$$
\begin{aligned}
& & f &\le \frac{1}{(\gamma(x)\|x\|_2)^n}, \\
\frac{\partial f}{\partial x_i} &= \frac{-n y_i}{(x^T y)^{n+1}}, & \left|\frac{\partial f}{\partial x_i}\right| &\le \frac{n|y_i|}{(\gamma(x)\|x\|_2)^{n+1}} \le \frac{n}{(\gamma(x)\|x\|_2)^{n+1}}, \\
\frac{\partial^2 f}{\partial x_i \partial x_j} &= \frac{n(n+1) y_i y_j}{(x^T y)^{n+2}}, & \left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right| &\le \frac{n(n+1)|y_i y_j|}{(\gamma(x)\|x\|_2)^{n+2}} \le \frac{0.5n(n+1)}{(\gamma(x)\|x\|_2)^{n+2}}.
\end{aligned}
$$

It follows that for each $x \in \mathrm{int}(K)$, $f(x,\cdot)$ and its first and second partial derivatives with respect to the x_i exist and are bounded functions on $K^* \cap S^{n-1}$. Since $K^* \cap S^{n-1}$ has finite $(n-1)$-dimensional volume, it follows that $f(x,\cdot)$ and its first and second partial derivatives are L^1 integrable on $K^* \cap S^{n-1}$. So for each i, we can interchange the differential operators $\frac{\partial}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ with the integral operator in (5.2.3) to obtain the partial derivatives of φ:

$$
\begin{aligned}
\nabla\varphi(x) &= \int_{K^* \cap S^{n-1}} \nabla_x \frac{1}{(x^T y)^n}\, \mathrm{d}y \\
&= \int_{K^* \cap S^{n-1}} \frac{-n\, y}{(x^T y)^{n+1}}\, \mathrm{d}y \\
&= \frac{-n}{\|x\|^{n+1}} \int_{K^* \cap S^{n-1}} \int_0^{\left(\|x\|/x^T y\right)^{n+1}} y\, \mathrm{d}t\, \mathrm{d}y,
\end{aligned}
\tag{5.2.7}
$$

and

$$\begin{aligned}\nabla^2\varphi(x) &= \int_{K^*\cap S^{n-1}} \nabla_x^2 \frac{1}{(x^Ty)^n}\,\mathrm{d}y \\ &= \int_{K^*\cap S^{n-1}} \frac{n(n+1)\,yy^T}{(x^Ty)^{n+2}}\,\mathrm{d}y \\ &= \frac{n(n+1)}{\|x\|^{n+2}}\int_{K^*\cap S^{n-1}}\int_0^{\left(\|x\|/x^Ty\right)^{n+2}} yy^T\,\mathrm{d}t\,\mathrm{d}y.\end{aligned} \tag{5.2.8}$$

The Cn-normal barrier F in (5.2.2) may be written terms of its complexity parameter $\nu = Cn$ rather than C, i.e.,

$$F(x) = \frac{\nu}{n}\log\frac{1}{n}\varphi(x). \tag{5.2.9}$$

The derivatives of the universal barrier function are given by

$$F'(x) = \frac{\nu}{n}\frac{\nabla\varphi(x)}{\varphi(x)}, \qquad F''(x) = \frac{\nu}{n}\frac{\varphi(x)\nabla^2\varphi(x) - \nabla\varphi(x)\nabla\varphi(x)^T}{\varphi(x)^2}. \tag{5.2.10}$$

Therefore to determine the gradient and Hessian of F, it is necessary to compute $(1 + n + \frac{n(n+1)}{2})$ n-dimensional integrals: one for $\varphi(x)$, one for each component of $\nabla\varphi(x)$, and one for each independent component of $\nabla^2\varphi(x)$. For some cones these integrals can be computed cheaply, but in general this does not appear to be the case. In Section 5.2.3 we will give some examples of cones for which these integrals can be computed cheaply, but before doing so, we give a lemma relating the Hessian of the barrier function F to that of the characteristic function $\varphi(x)$.

Lemma 5.2.1. *Let $x \in \mathrm{int}(K)$, $\varphi(x)$ be the characteristic function defined in* (5.2.3)*, and F the universal barrier function in* (5.2.9)*. We have*

$$\frac{\nu}{n(n+1)}\frac{\nabla^2\varphi(x)}{\varphi(x)} \preceq F''(x) \preceq \frac{\nu}{n}\frac{\nabla^2\varphi(x)}{\varphi(x)}, \tag{5.2.11}$$

$$\|F''(x)^{-1/2}\frac{\nabla^2\varphi(x)}{\varphi(x)}F''(x)^{-1/2}\|_2 \leq \frac{n(n+1)}{\nu}. \tag{5.2.12}$$

Proof. Since F is a ν-normal barrier function, it follows from Definition 3.2.3(c) that for all $x \in \text{int}(K)$ and $h \in \mathcal{R}^n$, $(F'(x)^T h)^2 \leq \nu h^T F''(x) h$, i.e., $F'(x)F'(x)^T \preceq \nu F''(x)$ for all $x \in \text{int}(K)$. The latter inequality is equivalent to, in light of the relations in (5.2.10),

$$\frac{\nabla\varphi(x)\nabla\varphi(x)^T}{\varphi(x)^2} \preceq \frac{n}{n+1}\frac{\nabla^2\varphi(x)}{\varphi(x)}.$$

It follows that

$$\begin{aligned}
\frac{\nu}{n(n+1)}\frac{\nabla^2\varphi(x)}{\varphi(x)} &= \frac{\nu}{n}\left(\frac{\nabla^2\varphi(x)}{\varphi(x)} - \frac{n}{n+1}\frac{\nabla^2\varphi(x)}{\varphi(x)}\right) \\
&\preceq \frac{\nu}{n}\left(\frac{\nabla^2\varphi(x)}{\varphi(x)} - \frac{\nabla\varphi(x)\nabla\varphi(x)^T}{\varphi(x)^2}\right) \\
&= F''(x).
\end{aligned}$$

This proves the left-hand inequality in (5.2.11). The right-hand inequality follows immediately from (5.2.10), since $\nabla\varphi(x)\nabla\varphi(x)^T \succeq 0$. To prove (5.2.12), multiply the left-hand inequality in (5.2.11) on the left and right by $F''(x)^{-1/2}$. The result is

$$\frac{\nu}{n(n+1)}F''(x)^{-1/2}\frac{\nabla^2\varphi(x)}{\varphi(x)}F''(x)^{-1/2} \preceq I.$$

Noting that the matrix on the left-hand side is positive definite, taking the matrix 2-norm of each side gives the required result. □

5.2.3 The universal barrier function and its conjugate for various cones

We list in this section the characteristic function φ, the universal barrier function F, and the gradient and Hessian of F for various full cones K. We will suppose that the constant in (5.2.2) satisfies $C = 1$, and will determine if $F(x) = \log \frac{1}{n}\varphi(x)$ is a normal barrier for K. If it is, we indicate, where possible, how C can be chosen so that the complexity parameter of $CF(x)$ is optimal. Where possible, we also give the conjugate of the universal barrier function. The formulas for φ and F are similar to those derived in [16, Section 7]. See also [6, Chapter 1].

1. **The nonnegative orthant** $K = \mathcal{R}^n_+$. The nonnegative orthant is self-dual (under the Euclidean inner product), i.e., $K = K^*$. Let $e \in \mathcal{R}^n$ denote the vector of ones, and let X denote the diagonal matrix whose diagonal is the vector x. For $x \in \mathrm{int}(K)$ and $s \in \mathrm{int}(K^*)$ we have

$$\begin{aligned}
\varphi(x) &= \frac{1}{\prod_{i=1}^n x_i}, \\
F(x) &= -\sum_{i=1}^n \log(x_i) + \text{constant}, \\
F'(x) &= -X^{-1}e, \\
F''(x) &= X^{-2}, \\
F_*(s) &= -\sum_{i=1}^n \log(s_i) + \text{constant}.
\end{aligned}$$

F is an n-normal barrier function for K. The complexity parameter n is optimal according to [33, Proposition 2.3.6].

2. **The second-order cone** $K = \{x \mid x_n \geq \|x_{1:n-1}\|_2\}$. The second-order cone is self-dual. For $x \in \mathrm{int}(K)$ and $s \in \mathrm{int}(K^*)$, there exists a constant $c_1(n)$ depending on n such that

$$\begin{aligned}
\varphi(x) &= \frac{c_1(n)}{\left(x_n^2 - \|x_{1:n-1}\|_2^2\right)^{n/2}}, \\
F(x) &= -\frac{n}{2}\log\left(x_n^2 - \|x_{1:n-1}\|_2^2\right) + \text{constant}, \\
F'(x) &= \frac{n}{x_n^2 - \|x_{1:n-1}\|_2^2}\begin{bmatrix} x_{1:n-1} \\ -x_n \end{bmatrix}, \\
F''(x) &= \frac{n}{x_n^2 - \|x_{1:n-1}\|_2^2}\begin{bmatrix} I_n & 0 \\ 0 & -1 \end{bmatrix} + \frac{2n}{\left(x_n^2 - \|x_{1:n-1}\|_2^2\right)^2}\begin{bmatrix} x_{1:n-1} \\ -x_n \end{bmatrix}\begin{bmatrix} x_{1:n-1} \\ -x_n \end{bmatrix}^T, \\
F_*(s) &= -\frac{n}{2}\log(s_n^2 - \|s_{1:n-1}\|_2^2) + \text{constant}.
\end{aligned}$$

It can be verified that F is an n-normal barrier function for K. The complexity parameter n is not optimal since the scaled function $\frac{2}{n}F$ is a 2-normal barrier function for K [33, Proposition 5.4.3]. In the formula for the conjugate function, we used Lemma 3.3.5(c) together with the fact that the conjugate barrier for the 2-normal barrier function $-\log(x_n^2 - \|x_{1:n-1}\|_2^2)$ is $-\log(s_n^2 - \|s_{1:n-1}\|_2^2) - 2 + 2\log(2)$. The latter formula can be found in, e.g., [35, Section 2].

3. **The positive semidefinite cone** $K = \{X \in \mathcal{S}^n \mid X \succeq 0\}$, where $\mathcal{S}^n$ is the vector space of symmetric matrices of order n equipped with the trace inner product. The positive semidefinite cone is self-dual under this inner product. Since the dimension of K is $\frac{n(n+1)}{2}$, using K in (4.1.1) gives a problem with $\frac{n(n+1)}{2}$ rather than n variables. For $X \in \operatorname{int}(K)$ and $S \in \operatorname{int}(K^*)$, there exists a constant $c_2(n)$ depending on n such that

$$\begin{aligned}
\varphi(X) &= \frac{c_2(n)}{(\det(X))^{(n+1)/2}}, \\
F(X) &= -\frac{n+1}{2}\log(\det(X)) + \text{constant}, \\
F'(X) &= -\frac{n+1}{2}X^{-1}, \\
F''(X)Y &= \frac{n+1}{2}X^{-1}YX^{-1}, \qquad Y \in \mathcal{S}^n, \\
F_*(S) &= -\frac{n+1}{2}\log(\det(S)) + \text{constant},
\end{aligned}$$

where $\det(X)$ denotes the determinant of X. It can be verified that F is a $\frac{n(n+1)}{2}$-normal barrier function for K. The complexity parameter $\frac{n(n+1)}{2}$ is not optimal since the scaled function $\frac{2}{n+1}F$ is an n-normal barrier function for K [33, Proposition 5.4.5]. A complexity parameter of $\frac{n(n+1)}{2}$ for F appears to contradict Lemma 4.1.2, which shows that the complexity parameter is $\mathcal{O}(n)$. The explanation for the apparent contradiction is that the dimension of the vector space $\mathcal{S}^n$ is $\frac{n(n+1)}{2}$ rather than n. In the formula for the conjugate function, we used Lemma 3.3.5(c) together with the fact that the conjugate barrier for the n-normal barrier $-\log(\det(X))$ is $-\log(\det(S)) - n$. The latter formula can be found in, e.g., [35, Section 2].

Note that for each of the above cones, the conjugate barrier function F_* has the same

form as F up to an additive constant. Each of these cones is self-scaled.

Recall from (2.3.3) the definition of the cone $K(S) \subset \mathcal{R}^{p+1}$ fitted to a convex set $S \subset \mathcal{R}^p$. We now compute the universal barrier function for the cone fitted to two convex (non-conic) sets. We first discuss the relationship between self-concordant barriers for S and those for $K(S)$. In the seminal work of Nesterov and Nemirovski, it was shown that if F is a ψ-self-concordant barrier for S, then one can construct a self-concordant barrier for $K(S)$. Unfortunately the complexity parameter of the latter function was 800ψ, which for many cones is much larger than that of the optimal barrier. A superior bound was proven in [14], and this was improved recently in [31, Theorem 1], where it was shown that

$$\tilde{F}(z,t) = \left(1.54 + \frac{1.785}{\psi^{1/2}}\right)^2 \big(F(z/t) - 4\psi \log(t)\big) : \mathrm{int}(K(S)) \to \mathcal{R}.$$

is a $(3.08\psi^{1/2} + 3.57)^2$-self-concordant barrier for $K(S)$.

In the following two examples involving cones fitted to convex sets, we omit formulas for the conjugate function $F_*(s)$.

4. **The cone fitted to the ℓ_1 unit ball** $S = \{x \in \mathcal{R}^n \,|\, \|x\|_1 \leq 1\}$. The cone fitted to S is $K(S) = \{(x,t) \in \mathcal{R}^{n+1} \,|\, \|x\|_1 \leq t\}$. Its dual is $K(S)^* = \{(s,u) \in \mathcal{R}^{n+1} \,|\, \|s\|_\infty \leq u\}$, which is the cone fitted to the ℓ_∞ unit ball. Both $K(S)$ and its dual are full polyhedral cones. For $(x,t) \in \mathrm{int}(K(S))$ we have

$$\begin{aligned}
\varphi(x,t) &= \frac{1}{\prod_{i=1}^n |x_i|} \sum_{e_i = \pm 1} \frac{\prod_{i=1}^n e_i}{t - \sum_{i=1}^n e_i |x_i|}, \\
F(x,t) &= -\sum_{i=1}^n \log(|x_i|) + \log\left(\sum_{e_i = \pm 1} \frac{\prod_{i=1}^n e_i}{t - \sum_{i=1}^n e_i |x_i|} \right) - \log(n).
\end{aligned}$$

We omit the cumbersome formulas for the gradient and Hessian of F. The work required to evaluate F and its derivatives at a particular point (x,t) is exponential in n. An "appropriately large" multiple of F is a normal barrier for $K(S)$, but we are unable to determine the "appropriately large" multiple, and hence the complexity parameter of such a normal

barrier. We do know from Lemma 4.1.2 that the complexity parameter is $\mathcal{O}(n+1)$. Note that the constraint $(x,t) \in K(S)$ can be written as a system of linear inequality constraints. This can be done by using the fact that for any $x_i \in \mathcal{R}$, there exist scalars $y_i, z_i \geq 0$ such that $x_i = y_i - z_i$ and $|x_i| = y_i + z_i$. So $K(S)$ can be embedded in the higher dimensional space $\mathcal{R}^{2n+1}$ as

$$\left\{(y,z,t) \;\middle|\; \sum_{i=1}^{n}(y_i+z_i) \leq t,\ y \geq 0, z \geq 0\right\}.$$

It follows from Example 3.2.5 that the logarithmic barrier

$$F(y,z,t) = -\log\left(t - \sum_{i=1}^{n}(y_i+z_i)\right) - \sum_{i=1}^{n}\log(y_i) - \sum_{i=1}^{n}\log(z_i)$$

for this polyhedral description of $K(S)$ has complexity parameter $2n+1$.

5. **The cone fitted to the ℓ_∞ unit ball** $S = \{x \in \mathcal{R}^n \,|\, \|x\|_\infty \leq 1\}$. The cone fitted to S is $K(S) = \{(x,t) \in \mathcal{R}^{n+1} \,|\, \|x\|_\infty \leq t\}$. Its dual is $K(S)^* = \{(s,u) \in \mathcal{R}^{n+1} \,|\, \|s\|_1 \leq u\}$, which is the cone fitted to the ℓ_1 unit ball discussed above. For $(x,t) \in \mathrm{int}(K(S))$ we have

$$\begin{aligned}
\varphi(x,t) &= \frac{2^n t^{n-1}}{\prod_{i=1}^{n}(t^2-x_i^2)}, \\
F(x,t) &= (n-1)\log(t) - \sum_{i=1}^{n}\log(t^2-x_i^2) + \text{ constant}, \\
F'(x,t) &= \begin{bmatrix} v \\ \frac{n-1}{t} - \sum_{i=1}^{n}\frac{2t}{t^2-x_i^2} \end{bmatrix}, \qquad v_i = \frac{2x_i}{t^2-x_i^2}, \\
F''(x,t) &= \begin{bmatrix} \mathrm{diag}(z) & -2tv^T \\ -2tv & -\frac{n-1}{t^2} + \sum_{i=1}^{n} z_i \end{bmatrix}, \qquad z_i = \frac{2(t^2+x_i^2)}{(t^2-x_i^2)^2}.
\end{aligned}$$

It can be verified that F is an $(n+1)$-normal barrier for the $(n+1)$-dimensional full polyhedral cone $K(S)$. For the purpose of comparison with other cones, we need the cone $K(S)$ to be n-dimensional rather than $(n+1)$-dimensional; this is achieved by fitting a cone

to the ℓ_∞ unit ball in $\mathcal{R}^{n-1}$ rather than $\mathcal{R}^n$. The resulting universal barrier function is an n-normal barrier. The complexity parameter of n is optimal in light of [33, Proposition 2.3.6].

5.2.4 Computing the universal barrier function

We see that easily computable formulas for the gradient and Hessian of the universal barrier function are known in the cases that K is a polyhedral cone, the second-order cone, the positive semidefinite cone, the cone fitted to the ℓ_∞ unit ball, or the direct products of such cones (see Lemma 3.2.4). More precisely, the gradient and Hessian are computable in a polynomial number of arithmetic operations. For cones such that this is not the case, it may be necessary to approximate the integral in (5.2.3), since this might represent the easiest way of using gradient and Hessian information in an interior-point method. Even if the gradient and Hessian *can* be evaluated in polynomial time, it may still be advantageous to use estimates of these quantities if such estimates are cheaper to compute. In the next two sections we study one approach to this approximation problem.

5.3 Monte Carlo methods

In this section, we give a brief review of Monte Carlo methods for the approximation of integrals, following [37, Chapter 1]. It is well known that traditional deterministic numerical quadrature methods for estimating integrals suffer from the "curse of dimensionality", meaning that the number of function evaluations required to guarantee a fixed level of accuracy is exponential in the problem size. Therefore such methods are computationally tractable only for problems in very low dimensions. If we instead generate a uniformly distributed random sample—rather than a deterministic sample—of points in the domain of integration, then a probabilistic bound—rather than a deterministic worst-case bound—on the error in the integral estimate is obtained. However the probabilistic bound can still depend on the problem dimension, even though Monte Carlo methods are typically advertised as having a rate of convergence independent of the dimension.

Suppose that the set $Y \subset \mathcal{R}^n$ has finite positive p-dimensional Lebesgue measure, i.e., $0 < \mathrm{vol}_p(Y) < \infty$. (Of course $p \leq n$.) Let $f : Y \to \mathcal{R}$ be square integrable, i.e., $\int_Y f(y)^2 \mathrm{d}y < \infty$. Introduce a random vector y that is uniformly distributed on Y. The function $f(y)$ is a random variable. We define the *expected value* $\mathbf{E}(f)$ of this random variable by

$$\mathbf{E}(f) := \frac{1}{\mathrm{vol}_p(Y)} \int_Y f(y) \, \mathrm{d}y, \tag{5.3.1}$$

and the *variance* $\sigma(f)^2$ of f by

$$\begin{aligned} \sigma(f)^2 &:= \mathbf{E}([f - \mathbf{E}(f)]^2) \\ &= \mathbf{E}(f^2) - \mathbf{E}(2f\mathbf{E}(f)) + \mathbf{E}(\mathbf{E}(f)^2) \\ &= \mathbf{E}(f^2) - 2\mathbf{E}(f)^2 + \mathbf{E}(f)^2 \\ &= \mathbf{E}(f^2) - \mathbf{E}(f)^2. \end{aligned} \tag{5.3.2}$$

Since f is square-integrable, the variance of f is finite. The quantity $\sigma(f)$ is called the standard deviation of f. Now take a uniformly distributed random sample of N points $y^1, \cdots, y^N$ from the set Y. The Monte Carlo estimate of $\mathbf{E}(f)$ is given by

$$\mathbf{E}(f) \approx \frac{1}{N} \sum_{k=1}^{N} f(y^k). \tag{5.3.3}$$

The strong law of large numbers (see e.g., [10, Section VII.8]) implies that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(y^k) = \mathbf{E}(f)$$

"almost surely". That is, the above relation holds with probability (Lebesgue measure) 1, where the Lebesgue measure here is the product measure of countably infinite copies of one-dimensional Lebesgue measure.

We now study the error in the approximation (5.3.3). The Central Limit theorem used

in probability theory says that the mean of a random sample, itself a random variable, approaches a Gaussian distribution as the sample size increases. In our notation,

$$\lim_{N\to\infty} \text{Prob}\left(\ell_1 \frac{\sigma(f)}{\sqrt{N}} \leq \frac{1}{N}\sum_{k=1}^{N} f(y^k) - \mathbf{E}(f) \leq \ell_2 \frac{\sigma(f)}{\sqrt{N}}\right) = \frac{1}{\sqrt{2\pi}} \int_{\ell_1}^{\ell_2} e^{-t^2/2}\, \mathrm{d}t,$$

where $\ell_1 \leq \ell_2$ and $\text{Prob}(\cdot)$ denotes probability. For our purposes it is desirable to know a bound on the absolute error $\left|\frac{1}{N}\sum_{k=1}^{N} f(y^k) - \mathbf{E}(f)\right|$ in terms of the sample size N, viz., we wish to quantify the rate at which the distribution of the sample mean converges to a Gaussian probability density function. An answer is provided by the Berry-Esséen theorem, which was discovered independently by Berry and Esséen in the 1940s. We will use the version of this theorem from [10, Section XVI.5]. It applies to independently distributed random variables having mean zero. The function $f - \mathbf{E}(f)$ is such a random variable: its mean is $\mathbf{E}(f - \mathbf{E}(f)) = \mathbf{E}(f) - \mathbf{E}(f) = 0$. Its second moment is $\mathbf{E}([f - \mathbf{E}(f)]^2) = \sigma(f)^2$. If the third absolute moment of $f - \mathbf{E}(f)$,

$$\rho(f) := \mathbf{E}(|f - \mathbf{E}(f)|^3), \tag{5.3.4}$$

is finite, the following probabilistic bound holds:

$$\left|\text{Prob}\left(\frac{1}{N}\sum_{k=1}^{N} f(y^k) - \mathbf{E}(f) \leq \ell \frac{\sigma(f)}{\sqrt{N}}\right) - \int_{-\infty}^{\ell} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, \mathrm{d}t\right| \leq \frac{0.7655\rho(f)}{\sigma(f)^3\sqrt{N}}, \tag{5.3.5}$$

where $\ell > 0$. Observe that the above-mentioned convergence is described in terms of the maximum deviation between two distribution functions. To our knowledge, the constant 0.7655 from [50], which uses the fact that the y^k are identically distributed, is the best known.[4] Denote the standardized sample mean by Z and the integral of the Gaussian

[4] If the samples are not drawn from identical distributions, (5.3.5) is known to hold with constant 0.7915 instead of 0.7655.

density function between two points ℓ_1 and ℓ_2 with $\ell_1 \leq \ell_2$ by $\Phi(\ell_1, \ell_2)$:

$$Z = \frac{\frac{1}{N}\sum_{k=1}^{N} f(y^k) - \mathbf{E}(f)}{\sigma(f)/\sqrt{N}}, \qquad \Phi(\ell_1, \ell_2) = \int_{\ell_1}^{\ell_2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{d}t.$$

Then for $\ell > 0$,

$$\begin{aligned}
\mathrm{Prob}(|Z| \leq \ell) &= \mathrm{Prob}(Z \leq \ell) - \mathrm{Prob}(Z \leq -\ell) \\
&= \Phi(-\ell, \ell) + \big(\mathrm{Prob}(Z \leq \ell) - \Phi(-\infty, \ell)\big) - \big(\mathrm{Prob}(Z \leq -\ell) - \Phi(-\infty, -\ell)\big) \\
&\geq \Phi(-\ell, \ell) - |\mathrm{Prob}(Z \leq \ell) - \Phi(-\infty, \ell)| - |\mathrm{Prob}(Z \leq -\ell) - \Phi(-\infty, -\ell)| \\
&\geq \Phi(-\ell, \ell) - \frac{0.7655\rho(f)}{\sigma(f)^3\sqrt{N}} - \frac{0.7655\rho(f)}{\sigma(f)^3\sqrt{N}} \\
&= \Phi(-\ell, \ell) - \frac{1.531\rho(f)}{\sigma(f)^3\sqrt{N}}.
\end{aligned}$$

So an upper bound on the probability that the Monte Carlo estimate

$$\int_Y f(y) \, \mathrm{d}y \approx \frac{\mathrm{vol}_p(Y)}{N} \sum_{k=1}^{N} f(y^k) \tag{5.3.6}$$

lies within ℓ standard deviations of its expected value is given by

$$\mathrm{Prob}\left(\frac{\left| \frac{\mathrm{vol}_p(Y)}{N} \sum_{k=1}^{N} f(y^k) - \int_Y f(y) \, \mathrm{d}y \right|}{\mathrm{vol}_p(Y)\sigma(f)/\sqrt{N}} \leq \ell \right) \geq \int_{-\ell}^{\ell} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{d}t - \frac{1.531\rho(f)}{\sigma(f)^3\sqrt{N}}. \tag{5.3.7}$$

We now prove a lemma about the tail of a Gaussian probability density function. The lemma involves a constant Ω, which we will later choose to be proportional to an upper bound on the number of iterations of our interior-point method, Algorithm short_step, that uses $\hat{F}'$ and $\hat{F}''$ as estimates of the gradient and Hessian of a suitable barrier function. The lemma will allow us to compute a lower bound on the probability that our interior-point method has polynomial iteration complexity.

Lemma 5.3.1. *Let $n \geq 3$ be an integer, $\Omega \geq 50$, $\ell(n,\Omega) = 0.08\Omega(\log(n))^{1/2}$, and*

$$\mathcal{G}(n,\Omega) = \int_{\ell(n,\Omega)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{d}t.$$

We have

$$\Omega n^{1/2}(n+1)(n+2)\mathcal{G}(n,\Omega) \quad < \quad \frac{11}{n^{5.5}}.$$

Proof. Let $f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$. Then $f'(t) = -tf(t)$ for all t. So for all $t \neq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{-f(t)}{t}\right) = \frac{-tf'(t) + f(t)}{t^2} = \left(1 + \frac{1}{t^2}\right) f(t).$$

Since $f(t)$ is nonnegative everywhere, it follows that

$$\mathcal{G}(n,\Omega) = \int_{\ell(n,\Omega)}^{\infty} f(t) \, \mathrm{d}t \leq \int_{\ell(n,\Omega)}^{\infty} \left(1 + \frac{1}{t^2}\right) f(t) \, \mathrm{d}t = \left.\frac{-f(t)}{t}\right|_{t=\ell(n,\Omega)}^{\infty} = \frac{f(\ell(n,\Omega))}{\ell(n,\Omega)}. \quad (5.3.8)$$

We now find a bound on $\Omega n^{1/2}(n+1)(n+2)\mathcal{G}(n,\Omega)$. Equation (5.3.9a) follows from (5.3.8), (5.3.9b) uses the assumption that $\Omega \geq 50$, and (5.3.9c) uses the fact that the function being

maximized is strictly decreasing for $n \geq 3$.

$$\begin{aligned}
\Omega n^{1/2}(n+1)(n+2)\mathcal{G}(n,\Omega) &\leq \Omega n^{1/2}(n+1)(n+2)\frac{f(\ell(n,\Omega))}{\ell(n,\Omega)} && \text{(5.3.9a)}\\
&= \Omega n^{1/2}(n+1)(n+2)\frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0.08\Omega(\log(n))^{1/2})^2}}{0.08\Omega(\log(n))^{1/2}} \\
&= \frac{n^{1/2}(n+1)(n+2)}{0.08(\log(n))^{1/2}}\ \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0.08\Omega)^2\log(n)} \\
&\leq \frac{n^{1/2}(n+1)(n+2)}{0.08(\log(n))^{1/2}}\ \frac{1}{\sqrt{2\pi}}e^{-8\log(n)} && \text{(5.3.9b)}\\
&= \frac{1}{n^{5.5}}\cdot\frac{(n+1)(n+2)}{0.08\sqrt{2\pi}(\log(n))^{1/2}n^2} \\
&\leq \frac{1}{n^{5.5}}\cdot\max_{n\geq 3}\ \frac{(n+1)(n+2)}{0.08\sqrt{2\pi}(\log(n))^{1/2}n^2} \\
&= \frac{1}{n^{5.5}}\cdot\frac{(n+1)(n+2)}{0.08\sqrt{2\pi}(\log(n))^{1/2}n^2}\Bigg|_{n=3} && \text{(5.3.9c)}\\
&< \frac{11}{n^{5.5}}. && \square
\end{aligned}$$

5.4 Using a Monte Carlo method to estimate F' and F''

In (5.2.9) a formula for the universal barrier function is given. One method of estimating the gradient and Hessian of F at $x \in \text{int}(K)$ would be to first estimate the characteristic function $\varphi(x)$ and its first two derivatives. The latter can be done by generating a uniform random sample of points in the bounded set $K^* \cap S^{n-1}$, and then applying a Monte Carlo method to estimate the integrals in (5.2.3), (5.2.7), and (5.2.8). One way of generating a uniform sample of points on the sphere S^{n-1} is explained in [29]: generate n numbers z_i sampled from a Gaussian distribution with mean 0 and variance 1, and let $z = (z_1, \cdots, z_n)$. Then the normalized point $z/||z||_2$ lies in S^{n-1}. Repeat this process $\tilde{N}$ times to generate a sample of size $\tilde{N}$. Suppose that N of these points $\{y^k\}_{k=1}^N$ also lie in K^*. We shall assume that $N \geq n$, which is always true in practice. Then the points $\{y^k\}_{k=1}^N$ almost surely span $\mathcal{R}^n$.

Denote the Monte Carlo estimates of $\varphi(x)$, $\nabla\varphi(x)$, and $\nabla^2\varphi(x)$ by $\hat{\varphi}(x)$, $\hat{g}(x)$, and

$\hat{H}(x)$ respectively. A $\hat{}$ will denote an estimate throughout this chapter, and the quantities $\hat{g}$ and $\hat{H}$ should remind the reader that we are estimating the gradient and Hessian of $\varphi(x)$. Letting $V = \mathrm{vol}_{n-1}(K^* \cap S^{n-1})$, we have from (5.2.3), (5.2.7), and (5.2.8) (cf. 5.3.3)

$$\hat{\varphi}(x) = V \frac{1}{N} \sum_{k=1}^{N} \frac{1}{(x^T y^k)^n}, \tag{5.4.1}$$

$$\hat{g}(x) = V \frac{-n}{N} \sum_{k=1}^{N} \frac{y^k}{(x^T y^k)^{n+1}}, \tag{5.4.2}$$

$$\hat{H}(x) = V \frac{n(n+1)}{N} \sum_{k=1}^{N} \frac{y^k (y^k)^T}{(x^T y^k)^{n+2}}. \tag{5.4.3}$$

Observe that $\hat{g}(x)$ and $\hat{H}(x)$—estimates of the gradient and Hessian of $\varphi(x)$—are themselves the gradient and Hessian of the estimate $\hat{\varphi}(x)$ of $\varphi(x)$. We note that the greatest contribution to $\hat{\varphi}(x)$, $\hat{g}(x)$, and $\hat{H}(x)$ comes from the points y^k that are closest to being orthogonal to x.

An efficient Monte Carlo code in MATLAB will compute $\hat{\varphi}(x)$, $\hat{g}(x)$, and $\hat{H}(x)$ in the following way. Let `u` be the column vector of $x^T y^k$ values and `Y` be the matrix whose k-th column is y^k. The values of `phi`, `g`, and `H` in the following algorithm are, after multiplication by V/N, the values of $\hat{\varphi}(x)$, $\hat{g}(x)$, and $\hat{H}(x)$ respectively.[5]

```
Inputs: n,N,u,Y.
  beta = 1 ./ (u.^n);
  phi = sum(beta);
  g = -n * Y * (beta ./ u);
  % spdiags(beta./u.^2,0,N,N) is a diagonal matrix with components equal
  % to the components of the vector beta./(u.^2) .
  H = n * (n+1) * Y * spdiags((beta./u.^2),0,N,N) * Y';
```

With estimates $\hat{\varphi}(x)$, $\hat{g}(x)$, and $\hat{H}(x)$ in hand, we now estimate the universal barrier function

[5] We have used the MATLAB notation `a./b` to denote componentwise division of two vectors `a` and `b`, and `a.^m` to denote componentwise exponentiation of the vector `a`.

in (5.2.9) as

$$\hat{F}(x) := \frac{\nu}{n} \log \frac{1}{n} \hat{\varphi}(x) = \frac{\nu}{n} \log \left(\sum_{k=1}^{N} \frac{1}{(x^T y^k)^n} \right) + \frac{\nu}{n} \log \left(\frac{V}{nN} \right). \tag{5.4.4}$$

A simple computation shows that the gradient and Hessian of this estimate are given by

$$\hat{F}'(x) = \frac{\nu}{n} \frac{\hat{g}(x)}{\hat{\varphi}(x)}, \qquad \hat{F}''(x) = \frac{\nu}{n} \frac{\hat{\varphi}(x)\hat{H}(x) - \hat{g}(x)\hat{g}(x)^T}{\hat{\varphi}(x)^2}. \tag{5.4.5}$$

(Cf. (5.2.10).) In light of this comparison and our earlier observation that the Monte Carlo estimates of the gradient and Hessian of $\varphi(x)$ are the gradient and Hessian, respectively, of the Monte Carlo estimate of $\varphi(x)$, we conclude that the gradient and Hessian of the Monte Carlo approximation to F are the Monte Carlo approximations of the gradient and Hessian, respectively, of F. That is, our barrier gradient and Hessian estimates have two interpretations. They can be thought of derivatives of our estimate of the barrier function $F(x)$, or as direct estimates, via a Monte Carlo method, of $F'(x)$ and $F''(x)$.

It is seen that the gradient and Hessian of $\hat{F}$ as given in (5.4.5) are independent of V, so it is not necessary to estimate this quantity. However we need to estimate V in our error analysis.[6] This can be done by using the estimate (5.3.3), where N becomes $\tilde{N}$, $Y = S^{n-1}$, and $f : Y \to \mathcal{R}$ is now defined to be the indicator function of $K^* \cap S^{n-1}$, i.e.,

$$f(y) = \begin{cases} 1 & : \ y \in K^* \cap S^{n-1}, \\ 0 & : \ \text{otherwise.} \end{cases}$$

We obtain

$$\frac{V}{\text{vol}_{n-1}(S^{n-1})} \approx \frac{N}{\tilde{N}}. \tag{5.4.6}$$

It is known (see e.g., [46, page xii]) that $\text{vol}_{n-1}(S^{n-1}) = n\pi^{n/2}/\Gamma(1 + n/2)$, where $\Gamma(\cdot)$ is

[6] The error analysis depends on the quantities in (5.3.7), including $\text{vol}_p(Y)$, which in the present context is V.

the gamma function, given by $\Gamma(u) = \int_0^\infty x^{u-1}e^{-x}\,\mathrm{d}x$, $u > 0$. This function satisfies the recursion $\Gamma(u) = (u-1)\Gamma(u-1)$ and $\Gamma(0.5) = \sqrt{\pi}$.

Note that in estimating the gradient and Hessian of F, the only information about the cones K and K^* that was used is (a) K, hence K^*, is a full cone (Assumption 4.1.1), and (b) we have a *membership oracle* (also known as an inclusion oracle) for K^*. Such an oracle takes as its input a vector x, and gives as output "yes" if $y \in K^*$ and "no" if $y \notin K^*$. The membership oracle is used to generate a uniformly distributed sample in the set $K^* \cap S^{n-1}$. We do not need to know how far a given point lies from the boundary of K or K^*, nor do we use any information about the boundary structure of these cones. However, since we wish to use the estimates of the gradient and Hessian of F in our feasible-point algorithm, Algorithm short_step, and this algorithm requires a strongly feasible initial point, it is also necessary to have membership oracles for $\mathrm{int}(K)$ and $\mathrm{int}(K^*)$.

We intend to estimate the gradient and Hessian of the universal barrier function (5.2.2) at each iteration of an interior-point method applied to the problem (4.1.1). Pseudorandomly generated points obtained *before* running the interior-point method will be used at each iteration of the method.

We mention that other ways to estimate the gradient and Hessian of the universal barrier function are possible. For example, we may write the characteristic function (5.2.3) in terms of *hyperspherical coordinates*. Following [40], a point $y \in \mathcal{R}^n$ is represented in hyperspherical coordinates as $(R, \alpha_1, \cdots, \alpha_{n-1})$, where

$$
\begin{aligned}
R &= \|y\|_2, && \text{(5.4.7a)}\\
\alpha_k &= \arccos\left(\frac{y_k}{\left(R^2 - \sum_{i=1}^{k-1} y_i^2\right)^{1/2}}\right), \qquad k = 1, \cdots, n-2, && \text{(5.4.7b)}\\
\alpha_{n-1} &= \mathrm{arctan2}(y_n, y_{n-1}), && \text{(5.4.7c)}
\end{aligned}
$$

where $0 \le \alpha_k \le \pi$, $k = 1, \cdots, n-2$, $0 \le \alpha_{n-1} < 2\pi$, and $\sum_1^0 = 0$. We define $\mathrm{arctan2}(y_n, y_{n-1})$ to be the angle $\xi \in [0, 2\pi)$ measured counter-clockwise from the positive real axis (in two dimensional space) to the point (y_n, y_{n-1}). That is, $y_n = \cos(\xi)$ and

$y_{n-1} = \sin(\xi)$. Using the convention that $\prod_1^0 = 1$, the inverse transformation is

$$y_k = R\cos(\alpha_k)\prod_{i=1}^{k-1}\sin(\alpha_i),\ k = 1, \cdots, n-1, \qquad y_n = R\prod_{i=1}^{n-1}\sin(\alpha_i).$$

We see that the conversion of a n-dimensional Cartesian point to a point in hyperspherical coordinates, or vice versa, requires the evaluation of $\mathcal{O}(n)$ sines, cosines, inverse cosines, and square roots. It can be shown that for the inverse transformation, the magnitude of the Jacobian determinant is given by

$$J(R, \alpha) := R^{n-1}\prod_{i=1}^{n-2}(\sin(\alpha_i))^{(n-i-1)}. \tag{5.4.8}$$

Let us now represent (5.2.3) as an integral in hyperspherical coordinates. Let

$$\Theta = \{(\alpha_1, \cdots, \alpha_{n-1}) \mid y(\alpha) \in K^*\} \subseteq [0, \pi]^{n-2} \times [0, 2\pi),$$

where $y(\alpha)$ is the Cartesian point corresponding to the hyperspherical point $(1, \alpha_1, \cdots, \alpha_{n-1}) =: (1, \alpha)$. The characteristic function, its gradient, and its Hessian, expressed as integrals in hyperspherical coordinates, are then

$$\begin{aligned}
\varphi(x) &= \int_\Theta \frac{J(1,\alpha)}{(x^T y(\alpha))^n}\ \mathrm{d}\alpha, \\
\nabla\varphi(x) &= -n\int_\Theta \frac{J(1,\alpha)y(\alpha)}{(x^T y(\alpha))^{n+1}}\ \mathrm{d}\alpha, \\
\nabla^2\varphi(x) &= n(n+1)\int_\Theta \frac{J(1,\alpha)y(\alpha)y(\alpha)^T}{(x^T y(\alpha))^{n+2}}\ \mathrm{d}\alpha.
\end{aligned}$$

We then use estimates of these quantities to obtain estimates of $F'(x)$ and $F''(x)$ via (5.4.5). Our Monte Carlo sample now consists of points in the set Θ. Such points are found by first generating a random sample of points uniformly distributed in the box $[0, \pi]^{n-2} \times [0, 2\pi)$, and then rejecting those points whose Cartesian equivalents lie outside K^*. The Monte Carlo estimates of $\varphi(x)$ and its derivatives are the same using either Cartesian or hyperspherical

coordinate systems, if the same Cartesian points $\{y^k\}$ are used in each case. (That is, the hyperspherical points in Θ are generated by converting the $\{y^k\}$ using (5.4.7a)–(5.4.7c).) However the variance and other moments of the respective integrands, considered as random variables, will be different.

One may also ask if it would be better to apply a Monte Carlo method to (5.2.6), in which $\varphi(x)$ is written as a one-dimensional rather than a multi-dimensional integral. Given $x \in \text{int}(K)$ and a sample $\{y^k\}$, we can estimate $h(t, x)$ in (5.2.6) as a function of t, and use this estimate to estimate the integral in (5.2.6). The result is the same as that in (5.4.1), but again, the variance and other moments of the integrand, considered as a random variable, will be different. Due to the complicated nature of the integrals involved, it appears inconclusive as to whether one representation is always superior to another, i.e., is superior for every full cone and at every $x \in \text{int}(K)$. Finally, we point out that the hyperspherical representation of the domain of integration allows for the use of *quasi-Monte Carlo* methods to estimate $\varphi(x)$ and its derivatives: sampling now takes place from a box rather than from on a sphere.

5.4.1 Properties of the Monte Carlo estimate of the universal barrier function

As we have already discussed, the universal barrier function F has several desirable properties that make it suitable for use in an interior-point method. In this section, we will show that the estimate $\hat{F}$ inherits some of these properties. In particular, we show that it satisfies the assumptions made on our barrier function estimate at the beginning of Section 4.5: $\hat{F}$ is twice continuously differentiable on $\text{int}(K)$, is nondegenerate and convex on $\text{int}(K)$, and satisfies the logarithmic-homogeneity relation. For all $x \in \text{int}(K)$ and positive integers k, $x^T y^k > 0$, so $\hat{\varphi}(x) > 0$, and $\hat{F}$ is well defined on $\text{int}(K)$. It is clear from the definition (5.4.4) that $\hat{F}$ is smooth on $\text{int}(K)$. We now show that (4.5.1) holds.

Lemma 5.4.1. *The barrier estimate $\hat{F}$ given in* (5.4.4) *satisfies the logarithmic-homogeneity relation* (4.5.1) *with parameter ν.*

Proof. From the definition of $\hat{\varphi}(x)$, we see that for all $x \in \text{int}(K)$ and $t > 0$, $\hat{\varphi}(tx) = \hat{\varphi}(x)/t^n$. Hence $\hat{F}(tx) - \hat{F}(x) = \frac{\nu}{n} \log 1/t^n = -\nu \log t$. □

We now prove $\hat{F}''$ is almost surely positive definite on $\text{int}(K)$.

Lemma 5.4.2. *For all* $x \in \text{int}(K)$,

$$\hat{F}''(x) \succeq \frac{\nu}{n(n+1)} \frac{\hat{H}(x)}{\hat{\varphi}(x)}. \tag{5.4.9}$$

Furthermore, $\hat{F}''(x)$ *is almost surely positive definite on* $\text{int}(K)$.

Proof. Applying the Cauchy-Schwarz inequality (Lemma 2.4.1(b)) with $a_k = 1/(x^T y^k)^{n/2}$ and $b_k = h^T y^k/(x^T y^k)^{n/2+1}$, we have for all $h \in \mathcal{R}^n$ and all $x \in \text{int}(K)$,

$$\left(\sum_{k=1}^{N} \frac{h^T y^k}{(x^T y^k)^{n+1}} \right)^2 \leq \left(\sum_{k=1}^{N} \frac{|h^T y^k|}{(x^T y^k)^{n+1}} \right)^2 \leq \sum_{k=1}^{N} \frac{1}{(x^T y^k)^n} \sum_{k=1}^{N} \frac{(h^T y^k)^2}{(x^T y^k)^{n+2}}.$$

Hence

$$\left(V \frac{-n}{N} \sum_{k=1}^{N} \frac{h^T y^k}{(x^T y^k)^{n+1}} \right)^2 \leq \frac{n}{n+1} \left[V \frac{1}{N} \sum_{k=1}^{N} \frac{1}{(x^T y^k)^n} \right] \left[V \frac{n(n+1)}{N} \sum_{k=1}^{N} \frac{(h^T y^k)^2}{(x^T y^k)^{n+2}} \right].$$

In light of (5.4.1), (5.4.2), and (5.4.3), this inequality may be written as

$$(h^T \hat{g}(x))^2 \leq \frac{n}{n+1} \hat{\varphi}(x) \, h^T \hat{H}(x) h.$$

Since $(h^T \hat{g}(x))^2 = h^T \hat{g}(x) \hat{g}(x)^T h$, the inequality amounts to

$$\hat{g}(x) \hat{g}(x)^T \preceq \frac{n}{n+1} \hat{\varphi}(x) \, \hat{H}(x).$$

It follows from (5.4.5) that

$$\begin{aligned}
\hat{F}''(x) &= \frac{\nu}{n}\frac{\hat{\varphi}(x)\hat{H}(x) - \hat{g}(x)\hat{g}(x)^T}{\hat{\varphi}(x)^2} \\
&\succeq \frac{\nu}{n}\frac{\hat{\varphi}(x)\hat{H}(x) - \frac{n}{n+1}\hat{\varphi}(x)\hat{H}(x)}{\hat{\varphi}(x)^2} \\
&= \frac{\nu\,\hat{H}(x)}{n(n+1)\hat{\varphi}(x)}.
\end{aligned}$$

This proves (5.4.9). Now from (5.4.3), $\hat{H}(x)$ can be written as the matrix product YDY^T, where $Y \in \mathcal{R}^{n\times N}$ is the matrix whose columns are $y^1, \cdots, y^N$, and D is a positive definite diagonal matrix. Since the points $\{y^k\}$ almost surely span $\mathcal{R}^n$, then Y almost surely has full row rank, and hence $\hat{H}(x)$ is almost surely positive definite on $\text{int}(K)$. It follows from (5.4.9) that the same is true of $\hat{F}''(x)$. □

Note that the inequality in (5.4.9) is similar to that proven for $F''(x)$ in (5.2.11). We now show that each $y^k \in K^* \cap S^{n-1}$ almost surely lies in $\text{int}(K^*)$ rather than in $\text{bnd}(K^*)$. Consider the following result of Ewald, Larman, and Rogers regarding the measure of the boundary of a *convex body*. A convex body is a nonempty compact convex set.

Lemma 5.4.3 ([5, Theorem 1]). *Let $K' \subseteq \mathcal{R}^n$ be a convex body. The set of directions on the unit sphere S^{n-1} that are parallel to a line segment in the boundary of K' has σ-finite $(n-2)$-dimensional Hausdorff measure.*

We first remark that a set having σ-finite measure can be written as a countable union of sets having finite measure. Hence the set of directions in Lemma 5.4.3 has zero $(n-1)$-dimensional Hausdorff measure. Now in Euclidean space endowed with the usual Euclidean distance function, $(n-1)$-dimensional Hausdorff measure is proportional to $(n-1)$-dimensional Lebesgue (outer) measure, where the constant of proportionality depends only on n; see e.g., [45, Theorem 30]. Let K' be the intersection of the full cone K^* with the unit ball B^n in $\mathcal{R}^n$. Clearly, K' is a convex body, so Lemma 5.4.3 implies that set of directions on the unit sphere S^{n-1} that are parallel to a line segment in the boundary of $K^* \cap B^n$ has zero $(n-1)$-dimensional Lebesgue measure. Since all points in the set $\text{bnd}(K^*) \cap S^{n-1}$

lie in S^{n-1} and are parallel to a line segment in the boundary of $K^* \cap B^n$, then the set $\text{bnd}(K^*) \cap S^{n-1}$ also has zero $(n-1)$-dimensional Lebesgue measure. On the other hand, the set $K^* \cap S^{n-1}$ has positive $(n-1)$-dimensional Lebesgue measure. We conclude that a point randomly generated in $K^* \cap S^{n-1}$ will almost surely lie in $\text{int}(K^*)$.

We now show that $\hat{F}'(x)$ almost surely maps $\text{int}(K)$ into $-\text{int}(K^*)$.

Lemma 5.4.4. *For any $x \in \text{int}(K)$, $-\hat{F}'(x) \in \text{int}(K^*)$ almost surely.*

Proof. Observe from (5.4.5) together with (5.4.1) and (5.4.2), that for every $x \in \text{int}(K)$, $\hat{F}'(x)$ is a negative linear combination of the $\{y^k\}$. Hence $\hat{F}'$ maps $\text{int}(K)$ to the cone generated by the points $\{-y^k\}$, which is a subset of the cone $-K^*$. From the discussion preceding this lemma, each y^k almost surely lies in $\text{int}(K^*)$, so almost surely any finite negative linear combination of the $\{y^k\}$ also lies in $\text{int}(K^*)$. □

Finally, it is worth noting that $\hat{F}$ almost surely fails to satisfy the barrier property. In order for the barrier property to hold, it is necessary that for every feasible boundary point x' of K, one of the $\{y^k\}$ is orthogonal to x'. Necessarily such a y^k must lie in $\text{bnd}(K^*)$, but from the considerations above, this will almost surely fail to be the case. Since $\hat{F}$ almost surely fails to satisfy the barrier property, $\hat{F}$ almost surely fails to approximate F well close to $\text{bnd}(K)$.

5.4.2 Relationships among the errors in $\hat{\varphi}(x)$, $\hat{g}(x)$, and $\hat{H}(x)$

We now use the probabilistic error bounds on the Monte Carlo estimates in Section 5.3 to bound the errors in the estimates $\hat{\varphi}(x)$, $\hat{g}(x)$, and $\hat{H}(x)$. We then proceed to bound the errors in the estimates $\hat{F}'(x)$ and $\hat{F}''(x)$. For consistency, we will now denote the exact gradient $\nabla\varphi(x)$ and Hessian $\nabla^2\varphi(x)$ of the characteristic function by $g(x)$ and $H(x)$ respectively, since we have denoted estimates of these quantities by $\hat{g}(x)$ and $\hat{H}(x)$ respectively. Given

$x \in \text{int}(K)$, define the absolute errors

$$\begin{aligned} \delta\varphi(x) &= \varphi(x) - \hat{\varphi}(x), \\ \delta g(x) &= g(x) - \hat{g}(x), \\ \delta H(x) &= H(x) - \hat{H}(x). \end{aligned}$$

Denote the components of the vector $\delta g(x)$ by $\delta g_i(x)$ and the components of the matrix $\delta H(x)$ by $\delta H_{ij}(x)$. Likewise $g_i(x), \hat{g}_i(x), H_{ij}(x)$, and $\hat{H}_{ij}(x)$ refer to the i-th or (i,j)-components of the relevant vector or matrix.

Since F and $\hat{F}$ satisfy the logarithmic-homogeneity property (3.3.1), it is immediate from the formulas for the characteristic function and its gradient and Hessian ((5.2.3), (5.2.7), and (5.2.8)) and for their estimates ((5.4.1), (5.4.2), and (5.4.3)), that

$$\begin{aligned} g(x)^T x &= -n\varphi(x), \qquad & H(x)x &= -(n+1)g(x), \\ \hat{g}(x)^T x &= -n\hat{\varphi}(x), \qquad & \hat{H}(x)x &= -(n+1)\hat{g}(x). \end{aligned}$$

It follows that the absolute errors in the estimates of the characteristic function, its gradient, and its Hessian, are related by

$$\delta g(x)^T x = -n\ \delta\varphi(x), \tag{5.4.10}$$

$$\delta H(x)x = -(n+1)\ \delta g(x),$$

$$x^T \delta H(x) x = n(n+1)\ \delta\varphi(x). \tag{5.4.11}$$

Now it follows from (5.4.10) that

$$\big(F''(x)^{-1/2}\delta g(x)\big)^T \big(F''(x)^{1/2}x\big) = -n\delta\varphi(x),$$

and taking the absolute value of each side while using Lemma 3.3.2(d) gives

$$\|\delta g(x)\|_{x,F}^{*}\ \nu^{1/2} \quad \geq \quad n|\delta\varphi(x)|. \tag{5.4.12}$$

From (5.4.11) we have

$$\big(F''(x)^{1/2}x\big)^{T}\big(F''(x)^{-1/2}\delta H(x)F''(x)^{-1/2}\big)\big(F''(x)^{1/2}x\big) \quad = \quad n(n+1)\ \delta\varphi(x).$$

Taking the absolute value of each side and again using Lemma 3.3.2(d) gives

$$\|F''(x)^{-1/2}\delta H(x)F''(x)^{-1/2}\|_2\ \nu \quad \geq \quad n(n+1)|\delta\varphi(x)|. \tag{5.4.13}$$

We now present deterministic bounds on the "relative errors" in the Monte Carlo estimates of $F'(x)$ and $F''(x)$ under the condition that the relative error in the estimate $\hat{\varphi}(x)$ is not too large. This will pave the way for probabilistic bounds, because the relative error in $\hat{\varphi}(x)$ is a random variable.

Lemma 5.4.5. *Let $x \in \mathrm{int}(K)$ and let F and $\hat{F}$ be defined as in* (5.2.9) *and* (5.4.4)*. Let $E_1(x) = F'(x) - \hat{F}'(x)$ and $E_2(x) = F''(x) - \hat{F}''(x)$ be the absolute errors in the gradient and Hessian estimates, and suppose that the relative error in $\hat{\varphi}(x)$ is less than one. Then the following bounds hold:*

(a)

$$\|E_1(x)\|_{x,F}^{*} \quad \leq \quad \frac{2\nu\|\delta g(x)\|_2}{n\sqrt{\lambda_{\min}(F''(x))}(\varphi(x) - |\delta\varphi(x)|)}\ , \tag{5.4.14}$$

(b)

$$\begin{aligned}\|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2 \quad \leq \quad & \frac{2\nu\|\delta H(x)\|_2}{n\lambda_{\min}(F''(x))(\varphi(x) - |\delta\varphi(x)|)} \\ & + \frac{n}{\nu}\big(\|E_1(x)\|_{x,F}^{*}\big)^2 + \frac{2n}{\nu^{1/2}}\|E_1(x)\|_{x,F}^{*}\ .\end{aligned} \tag{5.4.15}$$

Proof. To reduce clutter, we shall omit the argument x from the quantities $\varphi(x), g(x)$, and $H(x)$, their estimates $\hat{\varphi}(x), \hat{g}(x)$, and $\hat{H}(x)$, and the errors $\delta\varphi(x)$, $\delta g_i(x)$, and $\delta H_{ij}(x)$. The assumption on the relative error implies that $|\delta\varphi(x)| < \varphi(x)$, so $0 < \frac{1}{\varphi(x)-\delta\varphi(x)} \leq \frac{1}{\varphi(x)-|\delta\varphi(x)|}$. By (5.2.10) and (5.4.5),

$$\begin{aligned} E_1(x) &= \frac{\nu}{n}\left(\frac{g}{\varphi} - \frac{g-\delta g}{\varphi-\delta\varphi}\right) \\ &= \frac{\nu}{n}\left(\frac{\varphi(\delta g) - (\delta\varphi)g}{\varphi(\varphi-\delta\varphi)}\right) \\ &= \frac{1}{\varphi-\delta\varphi}\left(\frac{\nu}{n}(\delta g) - (\delta\varphi)F'(x)\right). \end{aligned}$$

Now take the $\|\cdot\|^*_{x,F}$ norm of each side. The inequality in (5.4.16a) follows from (5.4.12) and Lemma 3.3.2(e), and (5.4.16b) follows from (3.2.4). The equality in (5.4.16c) follows from the relation $\|M^{-1}\|_2 = \lambda_{\max}(M^{-1}) = 1/\lambda_{\min}(M)$ for a symmetric positive definite matrix M, and the equality in (5.4.16d) follows from the relation $\lambda_{\min}(M^{1/2}) = (\lambda_{\min}(M))^{1/2}$. (These eigenvalue relations were proven in Section 2.1.)

$$\begin{aligned} \|E_1(x)\|^*_{x,F} &\leq \frac{1}{\varphi-|\delta\varphi|}\left(\frac{\nu}{n}\|\delta g\|^*_{x,F} + |\delta\varphi|\|F'(x)\|^*_{x,F}\right) \\ &\leq \frac{1}{\varphi-|\delta\varphi|}\left(\frac{\nu}{n}\|\delta g\|^*_{x,F} + \frac{\|\delta g\|^*_{x,F}\ \nu^{1/2}}{n}\nu^{1/2}\right) && \text{(5.4.16a)} \\ &= \frac{2\nu\|\delta g\|^*_{x,F}}{n(\varphi-|\delta\varphi|)} \\ &\leq \frac{2\nu\|F''(x)^{-1/2}\|_2\ \|\delta g\|_2}{n(\varphi-|\delta\varphi|)} && \text{(5.4.16b)} \\ &= \frac{2\nu\|\delta g\|_2}{n\lambda_{\min}(F''(x)^{1/2})\ (\varphi-|\delta\varphi|)} && \text{(5.4.16c)} \\ &= \frac{2\nu\|\delta g\|_2}{n\sqrt{\lambda_{\min}(F''(x))}\ (\varphi-|\delta\varphi|)}\ . && \text{(5.4.16d)} \end{aligned}$$

This proves the bound on $\|E_1(x)\|^*_{x,F}$. We now prove the bound on $\|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2$ in (5.4.15). Observe that the Hessians of F and $\hat{F}$ (given in (5.2.10) and (5.4.5)) can be

written as

$$F''(x) = \frac{\nu}{n}\frac{H}{\varphi} - \frac{n}{\nu}F'(x)F'(x)^T, \qquad \hat{F}''(x) = \frac{\nu}{n}\frac{\hat{H}}{\hat{\varphi}} - \frac{n}{\nu}\hat{F}'(x)\hat{F}'(x)^T. \tag{5.4.17}$$

Hence

$$\begin{aligned} E_2(x) &= \frac{\nu}{n}\left(\frac{H}{\varphi} - \frac{H-\delta H}{\varphi-\delta\varphi}\right) + \frac{n}{\nu}\big(\hat{F}'(x)\hat{F}'(x)^T - F'(x)F'(x)^T\big) \\ &= \frac{\nu}{n}\left(\frac{\varphi(\delta H) - (\delta\varphi)H}{\varphi(\varphi-\delta\varphi)}\right) + \frac{n}{\nu}\big(\hat{F}'(x)\hat{F}'(x)^T - F'(x)F'(x)^T\big), \end{aligned}$$

giving

$$F''(x)^{-1/2}E_2(x)F''(x)^{-1/2} = \frac{\nu}{n}\Upsilon_1(x) + \frac{n}{\nu}\Upsilon_2(x), \tag{5.4.18}$$

where

$$\begin{aligned} \Upsilon_1(x) &:= F''(x)^{-1/2}\left(\frac{\varphi(\delta H) - (\delta\varphi)H}{\varphi(\varphi-\delta\varphi)}\right)F''(x)^{-1/2}, \\ \Upsilon_2(x) &:= F''(x)^{-1/2}\big(\hat{F}'(x)\hat{F}'(x)^T - F'(x)F'(x)^T\big)F''(x)^{-1/2}. \end{aligned}$$

In order to obtain a bound on $\|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2$, let us bound $\Upsilon_1(x)$ and $\Upsilon_2(x)$ in the 2-norm. The inequality in (5.4.19a) follows from $|\delta\varphi| < \varphi$. The relation in (5.4.19b) follows from (5.2.12) and the inequality in (5.4.19c) follows from (5.4.13). The relation in (5.4.19d) follows from Lemma 2.1.1(a)(ii) and (5.4.19e) follows from the relation $\|M^{-1}\|_2 =$

$\lambda_{\max}(M^{-1}) = 1/\lambda_{\min}(M)$ for a symmetric positive definite matrix M.

$$\begin{aligned}
\|\Upsilon_1(x)\|_2 &\le \frac{1}{\varphi - |\delta\varphi|}\left\|F''(x)^{-1/2}\left(\delta H - \frac{(\delta\varphi)H}{\varphi}\right)F''(x)^{-1/2}\right\|_2 && \text{(5.4.19a)}\\
&\le \frac{1}{\varphi - |\delta\varphi|}\left(\|F''(x)^{-1/2}\delta H F''(x)^{-1/2}\|_2 + \frac{|\delta\varphi|}{\varphi}\|F''(x)^{-1/2}HF''(x)^{-1/2}\|_2\right)\\
&\le \frac{1}{\varphi - |\delta\varphi|}\left(\|F''(x)^{-1/2}\delta H F''(x)^{-1/2}\|_2 + |\delta\varphi|\frac{n(n+1)}{\nu}\right) && \text{(5.4.19b)}\\
&\le \frac{1}{\varphi - |\delta\varphi|}\left(\|F''(x)^{-1/2}\delta H F''(x)^{-1/2}\|_2 + \|F''(x)^{-1/2}\delta HF''(x)^{-1/2}\|_2\right) && \text{(5.4.19c)}\\
&\le \frac{2\|F''(x)^{-1}\|_2\ \|\delta H\|_2}{\varphi - |\delta\varphi|} && \text{(5.4.19d)}\\
&= \frac{2\|\delta H\|_2}{\lambda_{\min}(F''(x))(\varphi - |\delta\varphi|)}. && \text{(5.4.19e)}
\end{aligned}$$

We now prove a bound on $\|\Upsilon_2(x)\|_2$. First observe that

$$\begin{aligned}
\hat{F}'(x)\hat{F}'(x)^T - F'(x)F'(x)^T &= (F'(x) - E_1(x))(F'(x) - E_1(x))^T - F'(x)F'(x)^T\\
&= E_1(x)E_1(x)^T - E_1(x)F'(x)^T - F'(x)E_1(x)^T.
\end{aligned}$$

Hence

$$\begin{aligned}
\Upsilon_2(x) &= F''(x)^{-1/2}\big(E_1(x)E_1(x)^T - E_1(x)F'(x)^T - F'(x)E_1(x)^T\big)F''(x)^{-1/2}\\
&= \big(F''(x)^{-1/2}E_1(x)\big)\big(F''(x)^{-1/2}E_1(x)\big)^T - \big(F''(x)^{-1/2}E_1(x)\big)\big(F''(x)^{-1/2}F'(x)\big)^T\\
&\quad -\big(F''(x)^{-1/2}F'(x)\big)\big(F''(x)^{-1/2}E_1(x)\big)^T.
\end{aligned}$$

Taking the norm of each side and using Lemma 3.3.2(e), we obtain

$$\begin{aligned}
\|\Upsilon_2(x)\|_2 &\le \|F''(x)^{-1/2}E_1(x)\|_2^2 + \|F''(x)^{-1/2}E_1(x)\|_2\ \|F''(x)^{-1/2}F'(x)\|_2\\
&\quad +\|F''(x)^{-1/2}F'(x)\|_2\ \|F''(x)^{-1/2}E_1(x)\|_2\\
&\le \big(\|E_1(x)\|_{x,F}^*\big)^2 + \|E_1(x)\|_{x,F}^*\nu^{1/2} + \nu^{1/2}\|E_1(x)\|_{x,F}^*.
\end{aligned}$$

Combining this with (5.4.18) and the bound on $\|\Upsilon_1(x)\|_2$ in (5.4.19e) establishes (5.4.15).

□

We next study the relationship between the size of our Monte Carlo sample, i.e., the number of points generated in the set $K^* \cap S^{n-1}$, and the errors in the Monte Carlo estimates of $\varphi(x), g(x)$, and $H(x)$. We use these relationships to relate the size of our Monte Carlo sample to the errors in our estimates of $F'(x)$ and $F''(x)$.

5.4.3 Error estimates in terms of the sample size

For a fixed $x \in \text{int}(K)$, let $f(y) = 1/(x^T y)^n$ where the vector y is uniformly distributed in $K^* \cap S^{n-1}$. Consider the random variable

$$f(y) - \mathbf{E}(f) = \frac{1}{(x^T y)^n} - \frac{1}{V} \int_{K^* \cap S^{n-1}} \frac{1}{(x^T y)^n} \, \mathrm{d}y = \frac{1}{(x^T y)^n} - \frac{\varphi(x)}{V}$$

defined on $K^* \cap S^{n-1}$. Let us denote the second moment and third absolute moment of $f(y) - \mathbf{E}(f)$ at x by $\sigma_0^2(x)$ and $\rho_0(x)$ respectively. To reduce clutter, denote $\frac{\partial f}{\partial x_i}(y)$ by $f_{1i}(y)$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}(y)$ by $f_{2ij}(y)$. Let us also denote the second moment and third absolute moment of $f_{1i}(y) - \mathbf{E}(f_{1i})$ by $\sigma_{1i}^2(x)$ and $\rho_{1i}(x)$ respectively, and the second moment and third absolute moment of $f_{2ij}(y) - \mathbf{E}(f_{2ij})$ by $\sigma_{2ij}^2(x)$ and $\rho_{2ij}(x)$ respectively.

From the definitions in (5.3.1), (5.3.2), and (5.3.4), and exploiting the positivity of f, hence $\mathbf{E}(f)$, we have:

$$\begin{aligned}
\mathbf{E}(f^p) &= \frac{1}{V} \int_{K^* \cap S^{n-1}} \frac{1}{(x^T y)^{pn}} \, \mathrm{d}y, \qquad p \geq 1, \\
\sigma_0^2(x) &= \mathbf{E}(f^2) - \mathbf{E}(f)^2, \qquad (5.4.20) \\
\rho_0(x) &= \mathbf{E}(|f - \mathbf{E}(f)|^3) \\
&= \frac{1}{V} \int_{\substack{K^* \cap S^{n-1} \\ f(y) \geq \mathbf{E}(f)}} (f(y) - \mathbf{E}(f))^3 \, \mathrm{d}y + \frac{1}{V} \int_{\substack{K^* \cap S^{n-1} \\ f(y) \leq \mathbf{E}(f)}} (\mathbf{E}(f) - f(y))^3 \, \mathrm{d}y \\
&\leq \frac{1}{V} \int_{\substack{K^* \cap S^{n-1} \\ f(y) \geq \mathbf{E}(f)}} f(y)^3 \, \mathrm{d}y + \frac{1}{V} \int_{\substack{K^* \cap S^{n-1} \\ f(y) \leq \mathbf{E}(f)}} \mathbf{E}(f)^3 \, \mathrm{d}y \\
&\leq \mathbf{E}(f^3) + \mathbf{E}(f)^3. \qquad (5.4.21)
\end{aligned}$$

We wish to apply the bound (5.3.7) resulting from the Berry Esséen theorem to the random variable $f - \mathbf{E}(f)$. Before doing so, we check that the hypotheses of this theorem are satisfied. First, the points y^k are generated independently, and as already noted, the expected value of the random variable $f - \mathbf{E}(f)$ is zero. We now show that the third absolute moment is finite. Recall from (5.2.5) the quantity

$$\gamma(x) = \min_{y \in K^* \cap S^{n-1}} \frac{x^T y}{\|x\|_2} > 0$$

For $p \geq 1$ we have the bound

$$\begin{aligned} \mathbf{E}(f^p) &= \frac{1}{V} \int_{K^* \cap S^{n-1}} \frac{1}{(x^T y)^{pn}} \, \mathrm{d}y \\ &\leq \frac{1}{V} \int_{K^* \cap S^{n-1}} \frac{1}{(\gamma(x)\|x\|_2)^{pn}} \, \mathrm{d}y \\ &= \frac{1}{(\gamma(x)\|x\|_2)^{pn}} \\ &< \infty. \end{aligned}$$

It follows from (5.4.21) that the absolute third moment $\rho_0(x)$ is finite. Now fix $\ell > 0$ to be the number of standard deviations separating the estimate $\hat{\varphi}(x)$ from its expected value $\varphi(x)$. From (5.3.7) and the definitions of $\delta\varphi(x)$, $\sigma_0(x)$, and $\rho_0(x)$, we have

$$\mathrm{Prob}\left(\frac{|\delta\varphi(x)|}{V\sigma_0(x)/\sqrt{N}} \leq \ell \right) \geq \int_{-\ell}^{\ell} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{d}t - \frac{1.531\rho_0(x)}{\sigma_0(x)^3\sqrt{N}}. \tag{5.4.22}$$

Let us now bound the errors in our estimates $\hat{g}_i(x)$ of the first partial derivatives $g_i(x)$ of $\varphi(x)$. In light of (5.2.7) we now need to study the random variables $f_{1i}(y) - \mathbf{E}(f_{1i})$ on

$K^* \cap S^{n-1}$. (By definition, $f_{1i}(y) = -ny_i/(x^T y)^{n+1}$.)

$$\begin{aligned}
\mathbf{E}(f_{1i}^p) &= \frac{1}{V}\int_{K^*\cap S^{n-1}} \left(\frac{-ny_i}{(x^T y)^{n+1}}\right)^p \,\mathrm{d}y, \qquad p \geq 1, \\
\sigma_{1i}^2(x) &= \mathbf{E}(f_{1i}^2) - \mathbf{E}(f_{1i})^2, \qquad (5.4.23) \\
\rho_{1i}(x) &= \mathbf{E}(|f_{1i} - \mathbf{E}(f_{1i})|^3) \\
&= \mathbf{E}(|f_{1i}^3 - 3\mathbf{E}(f_{1i})f_{1i}^2 + 3\mathbf{E}(f_{1i})^2 f_{1i} - \mathbf{E}(f_{1i})^3|) \\
&\leq \mathbf{E}(|f_{1i}|^3) + 3|\mathbf{E}(f_{1i})|\mathbf{E}(f_{1i}^2) + 3\mathbf{E}(f_{1i})^2|\mathbf{E}(f_{1i})| + |\mathbf{E}(f_{1i})|^3. \qquad (5.4.24)
\end{aligned}$$

Note that when obtaining a bound on the third absolute moment of each $f_{1i} - \mathbf{E}(f_{1i})$, we were unable to assume positivity of the $f_{1i}(y)$. We showed above that $\rho_0(x)$ is finite for each x. We can also show that $\mathbf{E}(|f_{1i}|^p)$ is finite for all $p \geq 1$ and all x, so that $\rho_{1i}(x)$ is also finite for each x. Therefore, from (5.3.7) and the definitions of $\delta g_i(x)$, $\sigma_{1i}(x)$, and $\rho_{1i}(x)$, we have

$$\mathrm{Prob}\left(\frac{|\delta g_i(x)|}{V\sigma_{1i}(x)/\sqrt{N}} \leq \ell\right) \geq \int_{-\ell}^{\ell} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,\mathrm{d}t - \frac{1.531\rho_{1i}(x)}{\sigma_{1i}(x)^3\sqrt{N}}. \qquad (5.4.25)$$

Finally we obtain error bounds for the estimates $\hat{H}_{ij}(x)$ of the second partial derivatives $H_{ij}(x)$. In light of (5.2.8) we now need to study the random variables $f_{2ij}(y) - \mathbf{E}(f_{2ij})$ on $K^* \cap S^{n-1}$. (By definition, $f_{2ij}(y) = n(n+1)y_i y_j/(x^T y)^{n+2}$.) We obtain

$$\begin{aligned}
\mathbf{E}(f_{2ij}^p) &= \frac{1}{V}\int_{K^*\cap S^{n-1}} \left(\frac{n(n+1)y_i y_j}{(x^T y)^{n+2}}\right)^p \,\mathrm{d}y, \qquad p \geq 1, \\
\sigma_{2ij}^2(x) &= \mathbf{E}(f_{2ij}^2) - \mathbf{E}(f_{2ij})^2, \qquad (5.4.26) \\
\rho_{2ij}(x) &= \mathbf{E}(|f_{2ij} - \mathbf{E}(f_{2ij})|^3) \\
&= \mathbf{E}(|f_{2ij}^3 - 3\mathbf{E}(f_{2ij})f_{2ij}^2 + 3\mathbf{E}(f_{2ij})^2 f_{2ij} - \mathbf{E}(f_{2ij})^3|) \\
&\leq \mathbf{E}(|f_{2ij}|^3) + 3|\mathbf{E}(f_{2ij})|\mathbf{E}(f_{2ij}^2) + 3\mathbf{E}(f_{2ij})^2|\mathbf{E}(f_{2ij})| + |\mathbf{E}(f_{2ij})|^3. \qquad (5.4.27)
\end{aligned}$$

(A better bound, analogous to that in (5.4.21), can be obtained for ρ_{2ij} in the case that $i = j$, since $f_{2ii}(y) \geq 0$ for each i.) We can show that for all i and j, $\rho_{2ij}(x)$ is finite for

each x, so from (5.3.7) and the definitions of $\delta H_{ij}(x)$, $\sigma_{2ij}(x)$, and $\rho_{2ij}(x)$, we have

$$\text{Prob}\left(\frac{|\delta H_{ij}(x)|}{V\sigma_{2ij}(x)/\sqrt{N}} \le \ell\right) \ge \int_{-\ell}^{\ell} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, \mathrm{d}t - \frac{1.531\rho_{2ij}(x)}{\sigma_{2ij}(x)^3\sqrt{N}}. \tag{5.4.28}$$

5.4.4 How large need the sample size be?

We now find a lower bound on the probability that the estimates of φ, g_i, and H_{ij} *all* lie within ℓ standard deviations of their expected value. Let $\{X_i\}$ denote a countable sequence of events, and denote the negation of the event X_i by $\bar{X}_i$. We will apply the *Bonferroni inequality*

$$\text{Prob}(\cap_i X_i) \ge 1 - \sum_i \text{Prob}(\bar{X}_i), \tag{5.4.29}$$

with each X_i being the event that the absolute error in one component of $\delta\varphi(x)$, $\delta g(x)$, or $\delta H(x)$, is no larger than ℓ standard deviations. Using also the symmetry of the matrix δH, we obtain the following lower bound on the probability that *all* components of the errors lie within ℓ standard deviations.

$$\begin{aligned}
q^* &:= \text{Prob}\Big(\frac{|\delta\varphi(x)|}{V\sigma_0(x)/\sqrt{N}} \le \ell,\ \frac{|\delta g_i(x)|}{V\sigma_{1i}(x)/\sqrt{N}} \le \ell\ \forall\, i,\ \frac{|\delta H_{ij}(x)|}{V\sigma_{2ij}(x)/\sqrt{N}} \le \ell\ \forall\, i,j\Big) \\
&\ge 1 - \text{Prob}\Big(\frac{|\delta\varphi(x)|}{V\sigma_0(x)/\sqrt{N}} \ge \ell\Big) - \sum_{i=1}^{n} \text{Prob}\Big(\frac{|\delta g_i(x)|}{V\sigma_{1i}(x)/\sqrt{N}} \ge \ell\Big) \\
&\quad - \sum_{i,j=1}^{n} \text{Prob}\Big(\frac{|\delta H_{ij}(x)|}{V\sigma_{2ij}(x)/\sqrt{N}} \ge \ell\Big) \\
&= 1 - \text{Prob}\Big(\frac{|\delta\varphi(x)|}{V\sigma_0(x)/\sqrt{N}} \ge \ell\Big) - \sum_{i=1}^{n} \text{Prob}\Big(\frac{|\delta g_i(x)|}{V\sigma_{1i}(x)/\sqrt{N}} \ge \ell\Big) \\
&\quad - \sum_{1\le i\le j\le n} \text{Prob}\Big(\frac{|\delta H_{ij}(x)|}{V\sigma_{2ij}(x)/\sqrt{N}} \ge \ell\Big) \\
&= 1 - \Big[1 - \text{Prob}\Big(\frac{|\delta\varphi(x)|}{V\sigma_0(x)/\sqrt{N}} \le \ell\Big)\Big] - \sum_{i=1}^{n} \Big[1 - \text{Prob}\Big(\frac{|\delta g_i(x)|}{V\sigma_{1i}(x)/\sqrt{N}} \le \ell\Big)\Big] \\
&\quad - \sum_{1\le i\le j\le n} \Big[1 - \text{Prob}\Big(\frac{|\delta H_{ij}(x)|}{V\sigma_{2ij}(x)/\sqrt{N}} \le \ell\Big)\Big].
\end{aligned}$$

Therefore, in light of the bounds in (5.4.22), (5.4.25), and (5.4.28), we may write

$$q^* \geq 1 - \frac{\xi(x)}{\sqrt{N}} - \zeta(\ell), \tag{5.4.30}$$

where

$$\zeta(\ell) = \left(1 + n + \frac{n(n+1)}{2}\right) \int_{|t| \geq \ell} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{d}t, \tag{5.4.31}$$

$$\xi(x) = \frac{1.531\rho_0(x)}{\sigma_0(x)^3} + \sum_{i=1}^{n} \frac{1.531\rho_{1i}(x)}{\sigma_{1i}(x)^3} + \sum_{1 \leq i \leq j \leq n} \frac{1.531\rho_{2ij}(x)}{\sigma_{2ij}(x)^3}. \tag{5.4.32}$$

We now give a definition that is important in light of the probabilistic statements we will make throughout this chapter.

Definition 5.4.6 (High probability). *Let $t_0 > 0$. An event U_t is said to occur with* high probability (with respect to the parameter t) *if there exist positive constants α_1 and α_2 such that*

$$t \geq t_0 \implies \mathrm{Prob}(U_t) \geq 1 - \frac{\alpha_1}{t^{\alpha_2}}.$$

Throughout the rest of this chapter we will also make the following assumption.

Assumption 5.4.7. (a) *The number of variables n in the primal problem* (4.1.1) *is at least three.*

(b) *For every $x \in \mathrm{int}(K)$, suppose that the Monte Carlo sample size N satisfies $N \geq \lceil N_\xi(x) \rceil$ where*

$$N_\xi(x) := \big(2n^4 \ell(n, \Omega) \xi(x)\big)^2. \tag{5.4.33}$$

If $n = 1$ or 2, then (4.1.1) and (4.1.2) are trivial linear optimization problems. The reasoning behind Assumption 5.4.7(b) is as follows. The lower bound on the Monte Carlo sample size ensures that with high probability the relative error in $\hat{\varphi}$ is less than one, so Lemma 5.4.5 can be applied. Furthermore, it ensures that the term $\xi(x)/\sqrt{N}$ in (5.4.30) decreases to zero as $n \to \infty$, so that with high probability, the estimates of $\varphi(x)$, $g(x)$, and $H(x)$ are

sufficiently accurate. Here and throughout this chapter, "high probability" is with respect to the problem size n of (4.1.1), where according to Assumption (5.4.7), $n \geq 3$.

We wish to find the probability that the estimates of φ, g_i, and H_{ij} lie within $\ell(n, \Omega)$ standard deviations of their expected values, where

$$\ell(n, \Omega) = 0.08\Omega(\log(n))^{1/2}, \tag{5.4.34}$$

and $\Omega \geq 50$ is a constant. As was explained in the discussion preceding Lemma 5.3.1, Ω is proportional to an upper bound on the number of iterations of our interior-point method that uses $\hat{F}'$ and $\hat{F}''$ as estimates of the gradient and Hessian of a suitable barrier function. Using the symmetry of the Gaussian probability density function, we have from (5.4.31)

$$\begin{aligned}
\zeta(\ell(n, \Omega)) &= \left(1 + n + \frac{n(n+1)}{2}\right) \int_{|t| \geq \ell(n,\Omega)} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{d}t \\
&= \left(1 + n + \frac{n(n+1)}{2}\right) 2 \int_{\ell(n,\Omega)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{d}t \\
&= (n+1)(n+2) \int_{\ell(n,\Omega)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, \mathrm{d}t \\
&< \frac{11}{\Omega \, n^6},
\end{aligned}$$

where the inequality follows from Lemma 5.3.1. Now under Assumption 5.4.7, $\ell(n, \Omega) \geq 0.08\Omega(\log(3))^{1/2}$ and

$$\frac{\xi(x)}{\sqrt{N}} \leq \frac{\xi(x)}{\sqrt{N_\xi(x)}} \leq \frac{1}{2n^4 \ell(n, \Omega)} \leq \frac{1}{0.16(\log(3))^{1/2}} \frac{1}{\Omega \, n^4} < \frac{6}{\Omega \, n^4}.$$

So from (5.4.30), the probability q^* that the estimates of φ, g_i, and H_{ij} *all* lie within $\ell(n, \Omega)$ standard deviations of their expected values is bounded by

$$q^* > 1 - \frac{6}{\Omega \, n^4} - \frac{11}{\Omega \, n^6}. \tag{5.4.35}$$

Hence with high probability the estimates of φ and each g_i and H_{ij} all lie within $\ell(n, \Omega)$

standard deviations of their expected values.

Using the bounds on $\|E_1(x)\|^*_{x,F}$ and $\|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2$ in Lemma 5.4.5, we will give an upper bound on the Monte Carlo sample size in order that with high probability these quantities are bounded above by ϵ_1 and ϵ_2 respectively, as required by our interior-point method; cf. (4.5.3) and (4.5.4). First we show that if N is large enough, i.e., satisfies Assumption 5.4.7, so that with high probability our Monte Carlo estimates of $\varphi(x)$, $g(x)$, and $H(x)$ lie within $\ell(n,\Omega)$ standard deviations of their expected values, then with high probability the relative error in the estimate of $\varphi(x)$ is less than one, as was assumed in Lemma 5.4.5.

Lemma 5.4.8. *Let $x \in \mathrm{int}(K)$. Under Assumption 5.4.7, with high probability the relative error in the estimate $\hat{\varphi}(x)$ is less than one.*

Proof. We have already shown in Section 5.4.3 that the third absolute moment of f is finite, where $f(y) = 1/(x^T y)^n$ for $y \in K^* \cap S^{n-1}$ (and x is fixed). It follows that the first absolute moment is also finite. Therefore we may apply the Cauchy-Schwarz inequality (Lemma 2.4.1(a)) with $a_1(y) = \frac{1}{V^{1/2}}|f(y) - \mathbf{E}(f)|^{1/2}$ and $a_2(y) = \frac{1}{V^{1/2}}|f(y) - \mathbf{E}(f)|^{3/2}$:

$$\begin{aligned}
\frac{\mathbf{E}(|f-\mathbf{E}(f)|^3)}{\mathbf{E}(|f-\mathbf{E}(f)|^2)^{1.5}} &= \frac{\frac{1}{V}\int_{K^*\cap S^{n-1}} |f(y)-\mathbf{E}(f)|^3 \, \mathrm{d}y}{\left(\frac{1}{V}\int_{K^*\cap S^{n-1}} |f(y)-\mathbf{E}(f)|^2 \, \mathrm{d}y\right)^{1.5}} \\
&\geq \frac{\left(\frac{1}{V}\int_{K^*\cap S^{n-1}} |f(y)-\mathbf{E}(f)|^2 \, \mathrm{d}y\right)^{0.5}}{\frac{1}{V}\int_{K^*\cap S^{n-1}} |f(y)-\mathbf{E}(f)| \, \mathrm{d}y}. \qquad (5.4.36)
\end{aligned}$$

A simple upper bound on the first absolute moment of $f - \mathbf{E}(f)$ is

$$\begin{aligned}
\frac{1}{V}\int_{K^*\cap S^{n-1}} |f(y)-\mathbf{E}(f)| \, \mathrm{d}y &\leq \frac{1}{V}\int_{K^*\cap S^{n-1}} f(y) \, \mathrm{d}y + \frac{1}{V}\int_{K^*\cap S^{n-1}} \mathbf{E}(f) \, \mathrm{d}y \\
&= \frac{\varphi(x)}{V} + \mathbf{E}(f) \\
&= \frac{2\varphi(x)}{V}. \qquad (5.4.37)
\end{aligned}$$

It now follows from (5.4.32), (5.4.36), and (5.4.37) that

$$\begin{aligned}
\xi(x) &\geq \frac{1.531\rho_0(x)}{\sigma_0(x)^3} \\
&= 1.531\frac{\mathbf{E}(|f-\mathbf{E}(f)|^3)}{\mathbf{E}(|f-\mathbf{E}(f)|^2)^{1.5}} \\
&\geq 1.531\frac{\left(\frac{1}{V}\int_{K^*\cap S^{n-1}}|f(y)-\mathbf{E}(f)|^2\,\mathrm{d}y\right)^{0.5}}{\frac{1}{V}\int_{K^*\cap S^{n-1}}|f(y)-\mathbf{E}(f)|\,\mathrm{d}y} \\
&\geq 1.531\frac{\sigma_0(x)}{2\varphi(x)/V}. \qquad (5.4.38)
\end{aligned}$$

In the discussion preceding Lemma 5.4.8 it was shown under Assumption 5.4.7 that with high probability, the estimates of φ and each g_i and H_{ij} all lie within $\ell(n,\Omega)$ standard deviations of their expected values. It follows that with high probability the estimate of φ alone lies within $\ell(n,\Omega)$ standard deviations of its expected value. That is, with high probability the absolute error $|\delta\varphi(x)|$ in $\hat{\varphi}(x)$ satisfies

$$|\delta\varphi(x)| \leq \frac{V\ell(n,\Omega)\sigma_0(x)}{\sqrt{N}}. \qquad (5.4.39)$$

In the following, (5.4.40a) follows from (5.4.39), and (5.4.40b) and (5.4.40c) follow from Assumption 5.4.7(b). Finally, (5.4.40d) follows from (5.4.38). With high probability,

$$\begin{aligned}
|\delta\varphi(x)| &\leq \frac{V\ell(n,\Omega)\sigma_0(x)}{\sqrt{N}} \qquad (5.4.40a) \\
&\leq \frac{V\ell(n,\Omega)\sigma_0(x)}{\sqrt{N_\xi(x)}} \qquad (5.4.40b) \\
&= \frac{V\ell(n,\Omega)\sigma_0(x)}{2n^4\ell(n,\Omega)\xi(x)} \qquad (5.4.40c) \\
&\leq \frac{1}{1.531n^4}\varphi(x). \qquad (5.4.40d)
\end{aligned}$$

(To be precise, the first inequality holds with high probability; the other inequalities and equality are true with probability one.) Since $n \geq 3$ (Assumption 5.4.7) we conclude that with high probability $|\delta\varphi(x)| < \varphi(x)$, i.e., the relative error $|\delta\varphi(x)|/\varphi(x)$ is less than one. □

Now for fixed $x \in \text{int}(K)$ and $\epsilon \in (0,1)$, define

$$\alpha_1 := \frac{2\nu\ell(n,\Omega)V\|\sigma_2(x)\|_2}{n\lambda_{\min}(F''(x))} + \frac{2n}{\nu^{1/2}}\left(\frac{2\nu\ell(n,\Omega)V\|\sigma_1(x)\|_2}{n\sqrt{\lambda_{\min}(F''(x))}}\right), \tag{5.4.41}$$

$$\alpha_2 := \frac{n}{\nu}\left(\frac{2\nu\ell(n,\Omega)V\|\sigma_1(x)\|_2}{n\sqrt{\lambda_{\min}(F''(x))}}\right)^2, \tag{5.4.42}$$

$$N_1(x,\epsilon) := \left(\frac{2\nu\ell(n,\Omega)V\|\sigma_1(x)\|_2}{n\epsilon\varphi(x)\sqrt{\lambda_{\min}(F''(x))}} + \frac{\ell(n,\Omega)V\sigma_0(x)}{\varphi(x)}\right)^2, \tag{5.4.43}$$

$$N_2(x,\epsilon) := \left(\frac{\alpha_1 + \left(\alpha_1^2 + 4\epsilon\alpha_2\right)^{1/2}}{2\epsilon\varphi(x)} + \frac{\ell(n,\Omega)V\sigma_0(x)}{\varphi(x)}\right)^2. \tag{5.4.44}$$

The quantities $N_1(x,\epsilon)$ and $N_2(x,\epsilon)$ represent "critical" values of the Monte Carlo sample size. If N exceeds $N_1(x,\epsilon_1)$, then with high probability the "relative error" in $\hat{F}'$ (cf. (4.5.3)) is less than ϵ_1. Similarly, if N exceeds $N_2(x,\epsilon_2)$, then with high probability the "relative error" in $\hat{F}''$ (cf. (4.5.4)) is less than ϵ_2. Denote by $\sigma_1(x)$ and $\sigma_2(x)$ the vector and matrix whose components are $\sigma_{1i}(x)$ and $\sigma_{2ij}(x)$ respectively.

Lemma 5.4.9. *Let $\epsilon_1, \epsilon_2 \in (0,1)$ and $x \in \text{int}(K)$. Then with probability exceeding $1 - \frac{6}{\Omega n^4} - \frac{11}{\Omega n^6}$, the following statements hold:*

(a) *If $N \geq N_1(x,\epsilon_1)$ then $\|E_1(x)\|^*_{x,F} \leq \epsilon_1$.*

(b) *If $N \geq N_2(x,\epsilon_2)$ then $\|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2 \leq \epsilon_2$.*

Proof. Let $x \in \text{int}(K)$. It follows from Lemma 5.4.8 that with high probability the relative error in $\hat{\varphi}(x)$ is less than one. Hence the results in Lemma 5.4.5 hold. We now determine how large N might be in order that the bounds on the "relative errors" in (5.4.14) and (5.4.15) be no greater than ϵ_1 and ϵ_2 respectively. In the discussion preceding Lemma 5.4.8 (see (5.4.35)) it was established that with probability exceeding $1 - \frac{6}{\Omega n^4} - \frac{11}{\Omega n^6}$, the errors $\delta\varphi(x) \in \mathcal{R}$, $\delta g(x) \in \mathcal{R}^n$, and $\delta H(x) \in \mathcal{S}^n$ lie in the set

$$\mathcal{T} := \left\{(\delta\varphi, \delta g, \delta H) \;\middle|\; \frac{|\delta\varphi|}{V\sigma_0(x)/\sqrt{N}} \leq \ell(n,\Omega),\ \frac{|\delta g_i|}{V\sigma_{1i}(x)/\sqrt{N}} \leq \ell(n,\Omega)\ \forall\, i,\ \frac{|\delta H_{ij}|}{V\sigma_{2ij}(x)/\sqrt{N}} \leq \ell(n,\Omega)\ \forall\, i,j\right\}.$$

We will now determine the largest possible magnitude of each of these errors. Specifically, we seek the maximum values of $|\delta\varphi|$, $\|\delta g\|_2$, and $\|\delta H\|_2$ over all triples $(\delta\varphi, \delta g, \delta H)$ belonging to $\mathcal{T}$. The maximum value of $|\delta\varphi|$ is clearly $\frac{V\ell(n,\Omega)\sigma_0(x)}{\sqrt{N}}$ and the maximum value of $\|\delta g\|_2$ over $(\delta\varphi, \delta g, \delta H) \in \mathcal{T}$ is clearly $\frac{V\ell(n,\Omega)\|\sigma_1(x)\|_2}{\sqrt{N}}$. So for all $(\delta\varphi, \delta g, \delta H) \in \mathcal{T}$,

$$\begin{aligned}
\frac{2\nu\|\delta g(x)\|_2}{n\sqrt{\lambda_{\min}(F''(x))}(\varphi(x) - |\delta\varphi(x)|)} &\leq \frac{2\nu\frac{V\ell(n,\Omega)\|\sigma_1(x)\|_2}{\sqrt{N}}}{n\sqrt{\lambda_{\min}(F''(x))}(\varphi(x) - \frac{V\ell(n,\Omega)\sigma_0(x)}{\sqrt{N}})} \\
&\leq \frac{2\nu V\ell(n,\Omega)\|\sigma_1(x)\|_2}{n\sqrt{\lambda_{\min}(F''(x))}(\sqrt{N}\varphi(x) - V\ell(n,\Omega)\sigma_0(x))} \\
&=: r_1(x,N).
\end{aligned}$$

So in light of (5.4.14), a sufficient condition for $\|E_1(x)\|^*_{x,F} \leq \epsilon_1$ is that $r_1(x,N) \leq \epsilon_1$, and it can be verified from (5.4.43) that the latter inequality is equivalent to $N \geq N_1(x,\epsilon_1)$. This proves (a). Now by Lemma 2.1.2, the maximum value of $\|\delta H(x)\|_2$ over $(\delta\varphi, \delta g, \delta H) \in \mathcal{T}$ is $\frac{V\ell(n,\Omega)\|\sigma_2(x)\|_2}{\sqrt{N}}$. So for all $(\delta\varphi, \delta g, \delta H) \in \mathcal{T}$,

$$\begin{aligned}
&\frac{2\nu\|\delta H(x)\|_2}{n\lambda_{\min}(F''(x))(\varphi(x) - |\delta\varphi(x)|)} + \frac{n}{\nu}\big(\|E_1(x)\|^*_{x,F}\big)^2 + \frac{2n}{\nu^{1/2}}\|E_1(x)\|^*_{x,F} \\
&\qquad \leq \frac{2\nu\frac{V\ell(n,\Omega)\|\sigma_2(x)\|_2}{\sqrt{N}}}{n\lambda_{\min}(F''(x))(\varphi(x) - \frac{V\ell(n,\Omega)\sigma_0(x)}{\sqrt{N}})} + \frac{n}{\nu}r_1(x,N)^2 + \frac{2n}{\nu^{1/2}}r_1(x,N). \quad (5.4.45)
\end{aligned}$$

Now gathering powers of $1/(\sqrt{N}\varphi(x) - V\ell(n,\Omega)\sigma_0(x))$, and using the definition of α_1 and α_2 in (5.4.41) and (5.4.42), it can be verified that the quantity on the right-hand side of (5.4.45) is

$$r_2(x,N) := \frac{\alpha_1}{\sqrt{N}\varphi(x) - V\ell(n,\Omega)\sigma_0(x)} + \frac{\alpha_2}{\big(\sqrt{N}\varphi(x) - V\ell(n,\Omega)\sigma_0(x)\big)^2}.$$

So in light of (5.4.15), a sufficient condition for $\|F''(x)^{-1/2}E_2(x)F''(x)^{-1/2}\|_2 \leq \epsilon_2$ is that $r_2(x,N) \leq \epsilon_2$, and the latter inequality is equivalent to

$$\epsilon_2\big(\sqrt{N}\varphi(x) - V\ell(n,\Omega)\sigma_0(x)\big)^2 - \alpha_1\big(\sqrt{N}\varphi(x) - V\ell(n,\Omega)\sigma_0(x)\big) - \alpha_2 \geq 0.$$

It is easily verified this quadratic inequality holds when $N \geq N_2(x, \epsilon_2)$, proving (b). $\square$

In Lemma 5.4.9 we indicated how much work is sufficient to ensure that at each iteration the Monte Carlo estimates of the gradient and Hessian of the universal barrier function are accurate enough for use in our interior-point method. We now show how much work is required for the algorithm to generate an ε-optimal solution. We will assume that the parameters θ, τ, ϵ_1, and ϵ_2 satisfy the conditions in Remark 4.5.6, so that by Theorem 4.5.4, Algorithm short_step is globally convergent and has iteration complexity $\mathcal{O}(\nu^{1/2} \log(\mu_0/\varepsilon))$. We will explain in Section 5.4.5 how an upper bound on the order constant can be determined. Since the complexity parameter of the universal barrier function is given by $\nu = \mathcal{O}(n)$ (Lemma 4.1.2), the iteration complexity of Algorithm short_step can be written as $\mathcal{O}(\nu^{1/2} \log(\mu_0/\varepsilon)) = \Omega n^{1/2}$, where the constant Ω depends on $\theta, \tau, \epsilon_1, \epsilon_2, \varepsilon$, and μ_0, but not (directly) on n or ν. In other words, $\Omega n^{1/2}$ is the maximum number of iterations required by the algorithm in order to generate an ε-optimal solution.

Theorem 5.4.10. *Suppose that the parameters θ, τ, ϵ_1, and ϵ_2 satisfy the conditions in Remark 4.5.6. Let $\varepsilon \in (0, 1)$ and suppose that the constant $\Omega = \Omega(\theta, \tau, \epsilon_1, \epsilon_2, \varepsilon, \mu_0)$ in the discussion preceding this theorem satisfies $\Omega \geq 50$. Let μ_0 be the duality measure of the initial primal-dual point in Algorithm short_step. Define*

$$N_* := \sup_{x \in \mathcal{X}(\theta, \varepsilon, \mu_0)} \max\{N_1(x, \epsilon_1), N_2(x, \epsilon_2), N_\xi(x)\},$$

where $\mathcal{X}(\cdot)$ was defined in (4.5.2)*. Note that N_* depends on $\theta, \epsilon_1, \epsilon_2, \mu_0, \varepsilon, \nu$, and n. Suppose that the Monte Carlo estimates in* (5.4.5) *of the gradient and Hessian of the universal barrier function, with $N = N_*$ points, are used in Algorithm short_step. With high probability, this algorithm generates an ε-optimal solution in no more than $\mathcal{O}(n^{2.5} N_* \Omega)$ arithmetic operations, in addition to the work required to generate N_* points in the set $K^* \cap S^{n-1}$.*

Proof. It follows from Lemma 5.4.9 that if the Monte Carlo sample size is at least

$$\max\{N_1(x, \epsilon_1), N_2(x, \epsilon_2), N_\xi(x)\},$$

then with probability exceeding $1 - \frac{6}{\Omega n^4} - \frac{11}{\Omega n^6}$, the "relative errors" in $\hat{F}'$ and $\hat{F}''$ are no greater than ϵ_1 and ϵ_2 respectively. We now derive a lower bound on the probability that this will be the case at *every* iteration of Algorithm short_step. Suppose that the algorithm actually runs for $\Omega_0 n^{1/2}$ iterations, where $\Omega_0 \leq \lfloor \Omega \rfloor$. Using (5.4.29), where X_i is the event that at iteration i the "relative errors" in $\hat{F}'$ and $\hat{F}''$ are no greater than ϵ_1 and ϵ_2 respectively, we obtain

$$\begin{aligned}
\text{Prob}\left(\bigcap_{i=1}^{\Omega_0 n^{1/2}} X_i \right) &\geq 1 - \sum_{i=1}^{\Omega_0 n^{1/2}} \text{Prob}(\bar{X}_i) \\
&\geq 1 - \sum_{i=1}^{\Omega_0 n^{1/2}} \left[1 - \left(1 - \frac{6}{\Omega\, n^4} - \frac{11}{\Omega\, n^6} \right) \right] \\
&= 1 - \Omega_0 n^{1/2} \frac{1}{\Omega n^{1/2}} \left(\frac{6}{n^{3.5}} + \frac{11}{n^{5.5}} \right) \\
&\geq 1 - \frac{6}{n^{3.5}} - \frac{11}{n^{5.5}}. \qquad (5.4.46)
\end{aligned}$$

Therefore with high probability, the "relative errors" in $\hat{F}'$ and $\hat{F}''$ will be small enough at every iteration of Algorithm short_step.

Since θ, τ, ϵ_1, and ϵ_2 are assumed to satisfy the conditions in Remark 4.5.6, then with high probability Algorithm short_step will produce iterates lying in the neighborhood $\mathcal{N}(\theta)$, and the associated sequence of duality measures will be strictly decreasing. Since the algorithm terminates once $\mu_k \leq \varepsilon$, all iterates except the last lie in the neighborhood $\mathcal{X}(\theta, \varepsilon, \mu_0)$. So if a Monte Carlo sample size of N_* points is used at each iteration, then with high probability Algorithm short_step will generate an ε-optimal solution to (4.1.1)–(4.1.2).

It is necessary to generate N_* points lying in $K^* \cap S^{n-1}$, and this may be done before running the interior-point method, i.e., can be considered as a one-off cost. The generation of such points first requires the generation of a sufficiently large number, say $\tilde{N}$, of points in S^{n-1}. Then the membership oracle for K^* must be called for as many of the $\tilde{N}$ points as is needed to produce N_* points in $K^* \cap S^{n-1}$. The cost of generating the $\tilde{N}$ points in S^{n-1} includes that of generating $n\tilde{N}$ standard Gaussian variates in addition to the cost of

normalization, which is $\mathcal{O}(n\tilde{N})$ arithmetic operations; see the first paragraph of Section 5.4.

Assuming that the $\{y^k\}$ are generated before running the interior-point method, the cost of performing one iteration of Algorithm short_step is dominated by the cost of computing $\hat{\varphi}(x)$, $\hat{g}(x)$, and $\hat{H}(x)$.[7] From (5.4.1), (5.4.2), and (5.4.3) we see that these quantities require $\mathcal{O}(nN_*)$, $\mathcal{O}(nN_*)$, and $\mathcal{O}(n^2N_*)$ arithmetic operations respectively. So the total number of arithmetic operations required to compute $\hat{F}'$ and $\hat{F}''$ over all $\Omega_0 n^{1/2}$ iterations is $\mathcal{O}(n^{2.5}N_*\Omega_0) = \mathcal{O}(n^{2.5}N_*\Omega)$. □

5.4.5 Comments on Theorem 5.4.10

In this section we discuss several issues relating to Theorem 5.4.10, some of them practical in nature.

1. Why can we assume $\Omega \geq 50$, where the upper bound on the number of iterations of Theorem 5.4.10 is given by $\Omega n^{1/2}$? In the proof of Lemma B.0.5 (see Appendix B), we derived an upper bound

$$\bar{\delta} \leq 1 - \frac{\kappa - \kappa^2/4}{\nu^{1/2}}$$

on the ratio of successive duality measures, where κ is related to the centering parameter τ via $\tau = 1 - \kappa/\nu^{1/2}$. It then follows from Lemma 2.4.2 that the algorithm will generate an ε-optimal solution in at most

$$\frac{1}{\kappa - \kappa^2/4} \log\left(\frac{\mu_0}{\varepsilon}\right)\nu^{1/2}$$

iterations. How small can $\frac{1}{\kappa-\kappa^2/4}\log(\mu_0/\varepsilon)$ be? Examining the conditions in Remark 4.5.6, we see that for fixed θ, the largest value of κ occurs when $\epsilon_1 = \epsilon_2 = 0$. This case was studied in Section 4.4, since it corresponds to the case that the exact gradient and Hessian of F are used in Algorithm short_step. It can be shown that $\kappa < 0.1$, and with more effort, a tighter upper bound can be obtained. Figure 4.6 indicates that $\kappa \lessapprox 0.07$. The

[7] From (5.4.5) we can show that the cost of computing the estimates $\hat{F}'$ and $\hat{F}''$ from $\hat{\varphi}(x)$, $\hat{g}(x)$, and $\hat{H}(x)$ is only $\mathcal{O}(n^2)$ at each iteration.

quantity $\log(\mu_0/\varepsilon)$, which measures the number of digits of accuracy gained by running the algorithm, is generally between 5 and 10 in practice, depending on the accuracy requirement and the quality of the initial point in the algorithm. Hence a reasonable lower bound for $\frac{1}{\kappa-\kappa^2/4}\log(\mu_0/\varepsilon)\nu^{1/2}$ is $50\nu^{1/2}$. Recall from Lemma 4.1.2 that $\nu = \mathcal{O}(n)$ for the universal barrier function. We will assume that $\nu = Cn$, where $C \geq 1$. This assumption holds for the nonnegative orthant, the positive semidefinite cone, and the second-order cone, which are all self-scaled cones, and also the cone fitted to the ℓ_∞ ball, which is not a self-scaled cone. (These facts were noted in Section 5.2.3.) Thus we may bound the worst-case iteration complexity by $50n^{1/2}$.

2. In practice there is dependence between the errors in $\hat{F}'$ and $\hat{F}''$ at each iteration of Algorithm short_step, because the same Monte Carlo sample is used to compute these estimates. We already expressed in Lemma 4.5.1 this relationship: $E_2(x)x = -E_1(x)$, indicating that if $E_2(x)$ is small in norm, then $E_1(x)$ will be also. Numerical experiments indicate that this is generally the case.

3. The quantity N_* is finite since the set $\mathcal{X}(\theta, \varepsilon, \mu_0)$ is bounded away from the boundary of K.

4. The number of arithmetic operations given in Theorem 5.4.10 is extremely pessimistic, since it assumes that the Monte Carlo sample size N_* used at the final iteration is also used in earlier iterations. However, since x is further from the boundary of K in earlier iterations, the moments in Section 5.4.3 will be smaller, so a smaller sample size suffices. (Closeness to the boundary is measured by the quantity $\gamma(x)$ in (5.2.5).) A better bound on the complexity of Algorithm short_step can be obtained by studying how $\max\{N_1(x,\epsilon_1), N_2(x,\epsilon_2), N_\xi(x)\}$ varies as x approaches the boundary of K. Alternatively, if one uses N_* points at *each* iteration, the expected errors in $\hat{F}'$ and $\hat{F}''$ at the beginning iterations of the interior-point method are much smaller—something we have not taken into account.

5. We have assumed that the N_* points are generated only once, rather than having to generate a new sample at each iteration. This requires much less computational effort, but the errors from iteration to iteration will no longer independent. Our analysis did not

require independence of the errors between iterations.

6. The lower bound on the probability of Algorithm short_step generating an ε-optimal solution is pessimistic, because the worst-case complexity is usually significantly worse than what is achieved in practice, and the Bonferroni inequality (5.4.29) and Gaussian tail bound (Lemma 5.3.1) are weak bounds.

7. A procedure for estimating a positive lower bound on $\lambda_{\min}(F''(x))$ is required in order to estimate $N_1(x, \epsilon_1)$ and $N_2(x, \epsilon_2)$.

8. In the proof of Lemma 5.4.5, we used the bounds

$$\begin{aligned} \|F''(x)^{-1/2}\delta g(x)\|_2 &\leq \|F''(x)^{-1/2}\|_2 \, \|\delta g(x)\|_2, \\ \|F''(x)^{-1/2}\delta H(x)F''(x)^{-1/2}\|_2 &\leq \|F''(x)^{-1}\|_2 \, \|\delta H(x)\|_2. \end{aligned}$$

These bounds may be very loose. To strengthen them, we need to determine at each interior-point iterate $x = x^k$, (reasonable upper bounds on) the smallest values of $\chi_1(x)$ and $\chi_2(x)$ such that

$$\begin{aligned} \frac{|\delta g_i(x)|}{V\sigma_{1i}(x)/\sqrt{N}} \leq \ell(n,\Omega) \ \forall\, i \quad &\Longrightarrow \quad \|F''(x)^{-1/2}\delta g(x)\|_2^2 \leq \chi_1(x), \\ \frac{|\delta H_{ij}(x)|}{V\sigma_{2ij}(x)/\sqrt{N}} \leq \ell(n,\Omega) \ \forall\, i,j \quad &\Longrightarrow \quad \|F''(x)^{-1/2}\delta H(x)F''(x)^{-1/2}\|_2 \leq \chi_2(x). \end{aligned}$$

The problem of finding the smallest values $\chi_1^*(x)$ and $\chi_2^*(x)$ of $\chi_1(x)$ and $\chi_2(x)$ respectively, can be formulated as semidefinite optimization problems, the second of which has an infinite

number of constraints:

$$\chi_1^*(x) = \min\Bigg\{\chi_1(x) \;\Bigg|\; \chi_1(x)F''(x) \succeq \delta g(x)\delta g(x)^T \quad \forall\, \delta g(x) \in \mathcal{R}^n \text{ such that } |\delta g_i(x)| \leq \frac{\ell(n,\Omega)}{\sqrt{N}}\sigma_{1i}(x)\ \forall\, i\Bigg\}, \tag{5.4.47}$$

$$\chi_2^*(x) = \min\Bigg\{\chi_2(x) \;\Bigg|\; \chi_2(x)\begin{bmatrix} F''(x) & \\ & F''(x) \end{bmatrix} \succeq \begin{bmatrix} \delta H(x) & \\ & -\delta H(x) \end{bmatrix} \quad \forall\, \delta H(x) \in \mathcal{S}^n \text{ such that } |\delta H_{ij}(x)| \leq \frac{\ell(n,\Omega)}{\sqrt{N}}\sigma_{2ij}(x)\ \forall\, i,j\Bigg\}. \tag{5.4.48}$$

However these problems cannot be solved directly since (5.4.47) and (5.4.48) involve the exact Hessian $F''(x)$, which is assumed unknown.

Chapter 6

Conclusions and future directions

Convex optimization problems have numerous applications, so it is of interest to study algorithms that can solve such problems efficiently. We have stated and analyzed an interior-point method for convex optimization. One of the main ingredients of interior-point methods is a self-concordant barrier function. Assuming that the convex problem has been written in conic form, so that the objective function is now linear, Newton's method can be applied to the problem of minimizing this linear objective function plus a multiple of a self-concordant barrier function for the underlying cone. Newton's method requires the evaluation of the gradient and Hessian of this combined function, but in some cases these quantities cannot be computed exactly. Our interior-point algorithm makes use of *inexact* barrier gradient and Hessian information, hence our algorithm is said to be an inexact interior-point method. We gave conditions on the inexactness in our estimates of the exact gradient and Hessian, under which our algorithm is globally convergent and has polynomial worst-case iteration complexity.

Practically speaking, the most efficient interior-point methods are *primal-dual* methods. In these methods, a given (primal) optimization problem is solved together with its Lagrangian dual, which is also a convex optimization problem. Intuitively, such methods need barrier information for the primal cone as well as the dual cone. Even if an easily computable barrier function for the primal cone is known, it may be difficult or impossible to determine an easily computable barrier function for the dual cone. This limits the class

of conic problems for which a practical interior-point method can be applied. One contribution of our primal-dual algorithm is that evaluation of the dual barrier function and its derivatives is not required. In fact it is not even necessary to know estimates of such functions. Therefore our algorithm requires less barrier information than other interior-point methods for conic optimization in the literature.

Our feasible-point path-following algorithm might be generalized in various ways. For example, it may be of interest to study the effect of varying the allowable errors in the gradient and Hessian estimates as the algorithm progresses. Since the maximum allowable errors are constrained by the size of the neighborhood of the central path as well as the centering parameter, it may be helpful to consider whether these quantities should also be adaptively chosen. Another generalization is to *infeasible-point* algorithms, in which the iterates are not required to be feasible. Such methods may perform better in practice, since they can search more of the space of variables, and do not require the computation of a feasible starting point, which may take some effort.

In our inexact interior-point method, a means of computing estimates of the exact barrier gradient and Hessian is required. In general, such a means depends on the specific form of the exact barrier gradient and Hessian. For some cones, only two self-concordant barriers are known; both are written in terms of a multidimensional integral called the universal barrier function. On the plus side, the universal barrier function can be computed for *any* full cone, hence the name "universal". Unfortunately the multidimensional integral defining the universal barrier function is sometimes difficult to evaluate accurately. By proposing a Monte Carlo method to estimate this integral, we have given a systematic means of computing inexact barrier gradient and Hessian information for conic optimization problems. We gave an upper bound on the size of the Monte Carlo sample required in order to guarantee that our inexact algorithm generates a near-optimal primal-dual solution to a pair of dual conic optimization problems with high probability. It is not necessary to know an algebraic description of the cones K and K^*; membership oracles for these cones and their interiors is sufficient. This allows for the study of many problems that might otherwise

be intractable using other classes of convex optimization algorithms. For example, even for some *linear* optimization problems, our inexact algorithm may be useful, because the polyhedral feasible set and its dual may not have a simple algebraic description, making the task of finding a suitable barrier function difficult. (An example of a simple algebraic description would be an explicit list of halfspaces whose intersection is the feasible set.) Such linear optimization problems may result from relaxations of discrete optimization problems.

A future direction is to *implement* our algorithm with the above-mentioned Monte Carlo estimates of the barrier gradient and Hessian. At the time of writing we have conducted numerical experiments to observe the relationship between the size of the Monte Carlo sample and the relative errors in the gradient and Hessian estimates. (In this work we only gave an upper bound on how large the sample size must be.) The experiments were performed in the cases that the cone K is the nonnegative orthant and the second-order cone. In both cases the exact barrier function and its derivatives are cheap to compute, so the relative errors are also. The results show promise, and it is likely that significant improvements will be possible once sophisticated sampling strategies are incorporated.

As the primal iterate x approaches an optimal solution (on the boundary of K), the Hessian of the characteristic function becomes singular, so ill-conditioning is an issue. Such ill-conditioning needs to be better understood insofar as it affects our Monte Carlo estimates. According to the standard Monte Carlo error estimates, for a given x, one can obtain a sufficiently small expected error in the Monte Carlo estimate of the gradient and Hessian of $F(x)$ by using a sufficiently large sample, but this sample size increases significantly as x approaches a primal optimal solution.

Some Monte Carlo sampling strategies for the estimation of F' and F'' merit further investigation. Given a sample that is uniformly distributed over the domain of integration, we may consider the integrand of the characteristic function as a random variable. Due to the nature of the integrand, its variance can be large. This is undesirable, because a large sample size is then required in order for the expected error to be small. Variance reduction techniques are common in statistical sampling, and they will surely be of assistance to us

here. Specifically, we may use importance sampling by recognizing that a small part of the domain of integration contributes much to the Monte Carlo estimate of the characteristic function. As our interior-point progresses, this "region of importance" becomes smaller, and it becomes necessary to sample this shrinking region accurately. This motivates adaptive sampling techniques. Instead of computing the Monte Carlo sample *before* running the interior-point method, it may be better to compute additional points as the method progresses. The cost in computing additional points may be outweighed by the fact that these points can be generated in the most important region of the domain of integration. There are several ways in which additional points could be added. For example, we might add random perturbations to existing points in order to generate new points. One way of doing this is via random walks. Although our domain of integration is the intersection of the dual convex cone K^* and the unit sphere, hence is nonconvex, we may perform a random walk inside the intersection of K^* with the unit ball, which is a convex set, and then project each point on the random walk onto the unit sphere. Provided the random walk is performed in an appropriate way, this results in a uniform distribution of points in our domain of integration.

Finally, we mention an important question regarding the complexity parameter of the universal barrier function for a particular full cone K. It is known that this parameter is of the order of the number of variables in the primal problem. Moreover there exists a universal order constant that is valid for all K. However, as far we know, the only known bounds on this universal constant are very large. This is unfortunate, since for several cones for which the optimal complexity parameter is known, it is quite reasonable. One direct approach for finding a bound on this universal constant is to manipulate inequalities involving multidimensional integrals; we refer to the inequalities defining a self-concordant barrier function. By using Monte Carlo approximations of these integrals, it may be possible to instead manipulate inequalities involving summations, which is generally easier. This would result in probabilistic rather than deterministic bounds on the complexity parameter, but such bounds may be an acceptable price to pay if the resulting estimate is of good quality. One

may ask whether it is necessary to find a universal order constant that holds for all full cones K, when we might only be interested in solving conic problems for a specific class of full cones. It may be easier to find a bound on the order constant for a specific class of full cones, but it is of more interest to find a bound that holds for all cones, since such would obviously have universal applicability.

Appendix A

Proof of Lemma 4.3.10

In this appendix we give a proof of Lemma 4.3.10. We restate here the parameter values and bounds in (4.3.27), as well as Lemma 4.3.10 itself, for ease of reference.

$$\theta = 0.1, \quad \tau = 1 - \frac{1}{47\nu^{1/2}}, \quad 0 \le \epsilon_1 \le 0.01, \quad 0 \le \epsilon_2 \le 0.071. \tag{A.0-1}$$

Lemma A.0.1. *Let θ, τ, ϵ_1, and ϵ_2 satisfy* (4.3.27)*. Then for all $\nu \ge 1$,*

$$\beta_1 < 0.1416, \qquad \beta_2 < 0.1569, \qquad \underline{\delta} > 1 - \frac{0.0642}{\nu^{1/2}} > 0, \qquad \bar{\delta} < 1 - \frac{0.00124}{\nu^{1/2}} < 1.$$

Proof. Rather than giving the simplest proofs of the desired bounds, we shall give longer but more general proofs showing how $\beta_1, \beta_2, \underline{\delta}$, and $\bar{\delta}$ depend on the specific values of the parameters in (4.3.27). We will develop general bounds on these quantities in terms of θ, κ, and the largest allowable values of ϵ_1 and ϵ_2, which we will call $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ respectively. It will be assumed that $\bar{\epsilon}_1, \bar{\epsilon}_2 \in [0, 1)$. (For ϵ_1 and ϵ_2 given in (4.3.27), $\bar{\epsilon}_1 = 0.01$ and $\bar{\epsilon}_2 = 0.071$.) In addition, we will develop the proof in such a way that it can be easily adapted if additional information about ν, i.e., an upper bound or a superior lower bound

(to 1), is known. Suppose that

$$\tau = 1 - \frac{\kappa}{\nu^{1/2}} \tag{A.0-2}$$

for some constant $\kappa \in (0,1)$. For τ given in (4.3.27), $\kappa = 1/47$.[1] From (4.3.15) we have the following bound that is uniform with respect to ϵ_1, ϵ_2, and ν:

$$\begin{aligned}
\beta_1 &= \frac{\theta + \epsilon_1 + \frac{\kappa}{\nu^{1/2}}(\nu^{1/2} + \epsilon_1)}{1 - \epsilon_2} \\
&\leq \frac{\theta + \epsilon_1 + \kappa + \kappa\epsilon_1}{1 - \epsilon_2} \\
&\leq \frac{\theta + \bar{\epsilon}_1 + \kappa + \kappa\bar{\epsilon}_1}{1 - \bar{\epsilon}_2} \\
&< 0.1416 \ .
\end{aligned}$$

Also

$$\begin{aligned}
\beta_2 &= \beta_0 \max\left\{\frac{(1+\epsilon_2)^{1/2}}{1-\theta}, \left(\frac{1}{1-\epsilon_2}\right)^{1/2}\right\} \\
&= \frac{\theta + \epsilon_1 + \frac{\kappa}{\nu^{1/2}}(\nu^{1/2} + \epsilon_1)}{(1-\epsilon_2)^{1/2}} \max\left\{\frac{(1+\epsilon_2)^{1/2}}{1-\theta}, \left(\frac{1}{1-\epsilon_2}\right)^{1/2}\right\} \\
&\leq \frac{\theta + \bar{\epsilon}_1 + \kappa + \kappa\bar{\epsilon}_1}{(1-\bar{\epsilon}_2)^{1/2}} \max\left\{\frac{(1+\bar{\epsilon}_2)^{1/2}}{1-\theta}, \left(\frac{1}{1-\bar{\epsilon}_2}\right)^{1/2}\right\} \\
&< 0.1569,
\end{aligned}$$

proving the bound on β_2. We now prove the lower bound on $\underline{\delta}$. Using (4.3.19) together with the definition of β_0 in (4.3.15) and the relation (A.0-2) linking τ and κ, we have

$$\begin{aligned}
\underline{\delta} &= \tau - \frac{\tau\epsilon_1}{\nu^{1/2}} - \beta_0 \frac{(1-\tau)(\nu^{1/2} + \epsilon_1) + \epsilon_1 + \epsilon_2\nu^{1/2}}{\nu(1-\epsilon_2)^{1/2}} - \frac{1}{\nu}\beta_0^2 \\
&= 1 - \frac{f_1}{\nu^{1/2}} - \frac{f_2}{\nu} - \frac{f_3}{\nu^{3/2}} - \frac{f_4}{\nu^2},
\end{aligned}$$

[1] In order to guarantee convergence of Algorithm short_step, it is necessary to assume $\tau \in (0,1)$. Since we do not assume that any information about ν is known (other than $\nu \geq 1$), we restrict κ to the interval $(0,1)$. If more information about ν is known, larger values of κ may be possible.

where

$$\begin{aligned}
f_1 &= \kappa + \epsilon_1 + \frac{\epsilon_2(\theta+\epsilon_1+\kappa)}{1-\epsilon_2} \le \kappa + \bar{\epsilon}_1 + \frac{\bar{\epsilon}_2(\theta+\bar{\epsilon}_1+\kappa)}{1-\bar{\epsilon}_2} =: \hat{f}_1, \\
f_2 &= -\kappa\epsilon_1 + \frac{1}{1-\epsilon_2}\big((\theta+\epsilon_1+\kappa)(\theta+2\epsilon_1+2\kappa) + \kappa\epsilon_1\epsilon_2\big) \\
&\le -\kappa\bar{\epsilon}_1 + \frac{1}{1-\bar{\epsilon}_2}\big((\theta+\bar{\epsilon}_1+\kappa)(\theta+2\bar{\epsilon}_1+2\kappa) + \kappa\bar{\epsilon}_1\bar{\epsilon}_2\big) \\
&=: \hat{f}_2, \\
f_3 &= \frac{\kappa\epsilon_1}{1-\epsilon_2}(3\theta+4\epsilon_1+4\kappa) \le \frac{\kappa\bar{\epsilon}_1}{1-\bar{\epsilon}_2}(3\theta+4\bar{\epsilon}_1+4\kappa) =: \hat{f}_3, \\
f_4 &= \frac{2(\kappa\epsilon_1)^2}{1-\epsilon_2} \le \frac{2(\kappa\bar{\epsilon}_1)^2}{1-\bar{\epsilon}_2} =: \hat{f}_4.
\end{aligned}$$

(The inequality in the relation $f_2 \le \hat{f}_2$ follows from the fact that the coefficient of $\kappa\epsilon_1$ in the formula for f_2 is nonnegative.) Since f_2, f_3, and f_4 are nonnegative, we see that the ratio of successive duality measures is bounded below by

$$\underline{\delta} \ge 1 - \frac{f}{\nu^{1/2}}, \tag{A.0-3}$$

where

$$\begin{aligned}
f &= \hat{f}_1 + \hat{f}_2 + \hat{f}_3 + \hat{f}_4 \\
&= \kappa + \bar{\epsilon}_1 - \kappa\bar{\epsilon}_1 + \frac{\theta+\bar{\epsilon}_1+\kappa+\kappa\bar{\epsilon}_1}{1-\bar{\epsilon}_2}\big(2\kappa+2\kappa\bar{\epsilon}_1+2\bar{\epsilon}_1+\theta+\bar{\epsilon}_2\big).
\end{aligned} \tag{A.0-4}$$

Using the values and bounds in (4.3.27), this gives

$$\underline{\delta} > 1 - \frac{0.0642}{\nu^{1/2}}.$$

Finally we prove the bound on $\bar{\delta}$. Suppose that $\epsilon_2 > 0$, hence $\bar{\epsilon}_2 > 0$. (The case $\epsilon_2 = 0$ will be handled later by taking the limit as $\epsilon_2 \to 0$.) Noting the complicated behavior of ϕ as a function of ν in (4.3.21), we consider separately the cases $\phi = \beta_0$ and $\phi = \frac{(1-\tau)(\nu^{1/2}+\epsilon_1)+(\epsilon_1+\epsilon_2\nu^{1/2})}{2(1-\epsilon_2)^{1/2}}$. For each case we will obtain a bound of the form $\bar{\delta} \le 1 - \frac{d}{\nu^{1/2}}$

for some positive constant d over a certain range of ν values. We then combine these two bounds into one bound that holds for all $\nu \geq 1$. The final bound will be independent of ϵ_1 and ϵ_2, i.e., it holds uniformly over all $\epsilon_1 \in [0, \bar{\epsilon}_1]$ and $\epsilon_2 \in [0, \bar{\epsilon}_2]$, as does the lower bound in (A.0-3).

Let us first find the critical value(s) of ν where ϕ switches between the two above-mentioned cases. That is, we seek ν such that

$$(\beta_0 \equiv) \frac{\theta + \epsilon_1 + (1-\tau)(\nu^{1/2} + \epsilon_1)}{(1-\epsilon_2)^{1/2}} = \frac{(1-\tau)(\nu^{1/2} + \epsilon_1) + (\epsilon_1 + \epsilon_2 \nu^{1/2})}{2(1-\epsilon_2)^{1/2}}, \tag{A.0-5}$$

where we have expressed β_0 in terms of the parameters θ, τ, ϵ_1 and ϵ_2 using (4.3.15). Writing τ in terms of κ via (A.0-2), it can be verified that (A.0-5) is equivalent to the following quadratic equation in $\nu^{1/2}$:

$$\epsilon_2 \nu - (2\theta + \epsilon_1 + \kappa)\nu^{1/2} - \kappa\epsilon_1 = 0.$$

This equation has a unique (positive) solution

$$\nu_+ = \left(\frac{(2\theta + \epsilon_1 + \kappa) + \left((2\theta + \epsilon_1 + \kappa)^2 + 4\kappa\epsilon_1\epsilon_2\right)^{1/2}}{2\epsilon_2} \right)^2, \tag{A.0-6}$$

giving

$$\epsilon_2 \nu_+^{1/2} = \frac{(2\theta + \epsilon_1 + \kappa) + \left((2\theta + \epsilon_1 + \kappa)^2 + 4\kappa\epsilon_1\epsilon_2\right)^{1/2}}{2}.$$

Since the complexity parameter ν is at least 1, we are only interested in solutions satisfying $\nu_+ \geq 1$. As $\epsilon_2 \to 0^+$, we see from (A.0-6) that $\nu_+ \to \infty$. So bearing in mind our desire to obtain uniform bounds over all $\epsilon_1 \in [0, \bar{\epsilon}_1]$ and $\epsilon_2 \in [0, \bar{\epsilon}_2]$, we see that regardless of $\bar{\epsilon}_2 > 0$, for some pairs (ϵ_1, ϵ_2) it will be the case that $\nu_+ > 1$.

Taking the infimum and supremum of $\epsilon_2 \nu_+^{1/2}$ over $\epsilon_1 \in [0, \bar{\epsilon}_1]$ and $\epsilon_2 \in (0, \bar{\epsilon}_2]$, we obtain

$$\alpha_1 := 2\theta + \kappa \leq \epsilon_2 \nu_+^{1/2} \leq \frac{(2\theta + \bar{\epsilon}_1 + \kappa) + \left((2\theta + \bar{\epsilon}_1 + \kappa)^2 + 4\kappa\bar{\epsilon}_1\bar{\epsilon}_2\right)^{1/2}}{2} =: \alpha_2. \tag{A.0-7}$$

It can be verified using (A.0-5) and (A.0-6) that ϕ in (4.3.21) is given by

$$\phi = \begin{cases} \frac{(1-\tau)(\nu^{1/2}+\epsilon_1)+(\epsilon_1+\epsilon_2\nu^{1/2})}{2(1-\epsilon_2)^{1/2}} & : \ \nu \in [1, \nu_+], \\ \beta_0 & : \ \nu \in (\nu_+, \infty). \end{cases} \tag{A.0-8}$$

Now suppose $\nu > \nu_+$, so that $\phi = \beta_0$. Using (4.3.20), together with the relation (A.0-2) linking τ and κ, and the definition of β_0 in (4.3.15), we have

$$\begin{aligned} \bar{\delta} &= \tau + \frac{\tau\epsilon_1}{\nu^{1/2}} + \beta_0 \frac{(1-\tau)(\nu^{1/2}+\epsilon_1)+\epsilon_1+\epsilon_2\nu^{1/2}}{\nu(1-\epsilon_2)^{1/2}} - \frac{\beta_0^2}{\nu} \\ &= 1 - \frac{g_1}{\nu^{1/2}} - \frac{g_2}{\nu} - \frac{g_3}{\nu^{3/2}}, \end{aligned}$$

where

$$\begin{aligned} g_1 &= \kappa - \epsilon_1 - \frac{\epsilon_2(\theta+\epsilon_1+\kappa)}{1-\epsilon_2}, \\ g_2 &= \kappa\epsilon_1 + \frac{\theta(\theta+\epsilon_1+\kappa) - \kappa\epsilon_1\epsilon_2}{1-\epsilon_2}, \\ g_3 &= \frac{\kappa\epsilon_1\theta}{1-\epsilon_2}. \end{aligned}$$

Since g_2 and g_3 are nonnegative, we have

$$\bar{\delta} \leq 1 - \frac{g_1}{\nu^{1/2}} \leq 1 - \frac{b_1}{\nu^{1/2}}, \tag{A.0-9}$$

where

$$b_1 := \kappa - \bar{\epsilon}_1 - \frac{\bar{\epsilon}_2(\theta+\bar{\epsilon}_1+\kappa)}{1-\bar{\epsilon}_2}. \tag{A.0-10}$$

Since $\nu \geq 1$, the only other possibility is that $\nu \in [1, \nu_+]$. Therefore let us assume that $\nu_+ > 1$. (For the parameters in (4.3.27), we have $\nu_+ > 9.7$.) From (A.0-8) we have $\phi = \frac{(1-\tau)(\nu^{1/2}+\epsilon_1)+(\epsilon_1+\epsilon_2\nu^{1/2})}{2(1-\epsilon_2)^{1/2}}$, so the upper bound on the ratio of successive duality measures

in (4.3.20) is given by the following, where we have again used (A.0-2).

$$\begin{aligned}
\bar{\delta} &= \tau + \frac{\tau\epsilon_1}{\nu^{1/2}} + \frac{\left((1-\tau)(\nu^{1/2}+\epsilon_1)+\epsilon_1+\epsilon_2\nu^{1/2}\right)^2}{4\nu(1-\epsilon_2)} \\
&= \left(1-\frac{\kappa}{\nu^{1/2}}\right) + \left(1-\frac{\kappa}{\nu^{1/2}}\right)\frac{\epsilon_1}{\nu^{1/2}} + \frac{\left(\frac{\kappa}{\nu^{1/2}}(\nu^{1/2}+\epsilon_1)+\epsilon_1+\epsilon_2\nu^{1/2}\right)^2}{4\nu(1-\epsilon_2)} \\
&\leq \left(1-\frac{\kappa}{\nu^{1/2}}\right) + \left(1-\frac{\kappa}{\nu^{1/2}}\right)\frac{\epsilon_1}{\nu^{1/2}} + \frac{\left(\kappa+\kappa\epsilon_1+\epsilon_1+\epsilon_2\nu^{1/2}\right)^2}{4\nu(1-\epsilon_2)} \\
&= 1 - \frac{b(\nu)}{\nu^{1/2}},
\end{aligned}$$

where

$$b(\nu) := \kappa - \epsilon_1 + \frac{\kappa\epsilon_1}{\nu^{1/2}} - \frac{\left(\kappa+\kappa\epsilon_1+\epsilon_1+\epsilon_2\nu^{1/2}\right)^2}{4\nu^{1/2}(1-\epsilon_2)}.$$

Let us now find the minimum value of $b(\nu)$ in the interval $[1,\nu_+]$. The derivative of b is given by

$$b'(\nu) = \frac{-\epsilon_2^2\nu + (\kappa+\kappa\epsilon_1+\epsilon_1)^2 - 4\kappa\epsilon_1(1-\epsilon_2)}{8\nu^{3/2}(1-\epsilon_2)}.$$

Hence b has a unique stationary point at $\nu = \nu_{\rm opt}$ where

$$\nu_{\rm opt} = \frac{(\kappa+\kappa\epsilon_1+\epsilon_1)^2 - 4\kappa\epsilon_1(1-\epsilon_2)}{\epsilon_2^2},$$

assuming $\nu_{\rm opt} \in [1,\nu_+]$. However $\nu_{\rm opt}$ is a local maximizer since

$$b''(\nu_{\rm opt}) = \frac{-\epsilon_2^2}{8\nu_{\rm opt}^{3/2}(1-\epsilon_2)} < 0.$$

Hence the minimum of $b(\nu)$ on $[1,\nu_+]$ is $\min(b(1), b(\nu_+))$. Let us now obtain bounds on $b(1)$

and $b(\nu_+)$ in terms of θ, κ, $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$. Firstly,

$$\begin{aligned} b(1) &= \kappa - \epsilon_1(1-\kappa) - \frac{(\kappa + \kappa\epsilon_1 + \epsilon_1 + \epsilon_2)^2}{4(1-\epsilon_2)} \\ &\geq \kappa - \bar{\epsilon}_1(1-\kappa) - \frac{(\kappa + \kappa\bar{\epsilon}_1 + \bar{\epsilon}_1 + \bar{\epsilon}_2)^2}{4(1-\bar{\epsilon}_2)} \\ &=: b_2. \end{aligned} \tag{A.0-11}$$

(The inequality holds since $b(1)$ is a decreasing function of ϵ_1 and ϵ_2.) We now obtain a bound on $b(\nu_+)$ by writing b as a function of $z = \epsilon_2 \nu_+^{1/2}$:

$$\begin{aligned} b(\nu_+) &= \kappa - \epsilon_1 + \frac{\kappa\epsilon_1}{\nu_+^{1/2}} - \frac{(\kappa + \kappa\epsilon_1 + \epsilon_1 + \epsilon_2\nu_+^{1/2})^2}{4\nu_+^{1/2}(1-\epsilon_2)} \\ &= \kappa - \epsilon_1 + \frac{\epsilon_2\kappa\epsilon_1}{\epsilon_2\nu_+^{1/2}} - \frac{\epsilon_2(\kappa + \kappa\epsilon_1 + \epsilon_1 + \epsilon_2\nu_+^{1/2})^2}{4\epsilon_2\nu_+^{1/2}(1-\epsilon_2)} \\ &= \kappa - \epsilon_1 + \frac{\epsilon_2\kappa\epsilon_1}{z} - \frac{\epsilon_2(\kappa + \kappa\epsilon_1 + \epsilon_1 + z)^2}{4z(1-\epsilon_2)} \\ &= \kappa - \epsilon_1 - \frac{\epsilon_2[-4\kappa\epsilon_1(1-\epsilon_2) + (\kappa + \kappa\epsilon_1 + \epsilon_1 + z)^2]}{4z(1-\epsilon_2)} \\ &=: \bar{b}(z). \end{aligned} \tag{A.0-12}$$

Let us now find the worst (minimum) possible value of $\bar{b}(z)$ over the interval of possible z values. In view of (A.0-7), this interval is $[\alpha_1, \alpha_2]$. Since the functions b and $\bar{b}$ have a similar form, the procedure for minimizing $\bar{b}$ is similar to that above for minimizing b. We find that the minimizer again lies at an endpoint, in this case, α_1 or α_2. The resulting lower bound on $\bar{b}$ is

$$\bar{b}(z) \geq \min(\bar{b}(\alpha_1), \bar{b}(\alpha_2)) \geq \min(b_3, b_4),$$

where

$$\begin{aligned} b_3 &:= \inf\{\bar{b}(\alpha_1) \mid \epsilon_1 \in [0, \bar{\epsilon}_1],\ \epsilon_2 \in (0, \bar{\epsilon}_2]\}, \\ b_4 &:= \inf\{\bar{b}(\alpha_2) \mid \epsilon_1 \in [0, \bar{\epsilon}_1],\ \epsilon_2 \in (0, \bar{\epsilon}_2]\}. \end{aligned}$$

Now the term in the square brackets in (A.0-12) can be written as

$$-4\kappa\epsilon_1(1-\epsilon_2)+(\kappa+\kappa\epsilon_1+\epsilon_1+z)^2=(\kappa+\kappa\epsilon_1+z)^2+\epsilon_1^2+2\epsilon_1(-\kappa+\kappa\epsilon_1+z)+4\kappa\epsilon_1\epsilon_2.$$

Regardless of whether $z=\alpha_1$ or $z=\alpha_2$, $-\kappa+\kappa\epsilon_1+z$ is positive, so the term in square brackets is an increasing function of ϵ_1 and ϵ_2. It follows that $\bar{b}(z)$ is a decreasing function of ϵ_1 and ϵ_2, so

$$b_3=\kappa-\bar{\epsilon}_1-\frac{\bar{\epsilon}_2[-4\kappa\bar{\epsilon}_1(1-\bar{\epsilon}_2)+(\kappa+\kappa\bar{\epsilon}_1+\bar{\epsilon}_1+2\theta+\kappa)^2]}{4(2\theta+\kappa)(1-\bar{\epsilon}_2)}, \tag{A.0-13}$$

$$b_4=\kappa-\bar{\epsilon}_1-\frac{\bar{\epsilon}_2\big[-4\kappa\bar{\epsilon}_1(1-\bar{\epsilon}_2)+\big(\kappa+\kappa\bar{\epsilon}_1+\bar{\epsilon}_1+\frac{(2\theta+\bar{\epsilon}_1+\kappa)+[(2\theta+\bar{\epsilon}_1+\kappa)^2+4\kappa\bar{\epsilon}_1\bar{\epsilon}_2]^{1/2}}{2}\big)^2\big]}{4\big(\frac{(2\theta+\bar{\epsilon}_1+\kappa)+[(2\theta+\bar{\epsilon}_1+\kappa)^2+4\kappa\bar{\epsilon}_1\bar{\epsilon}_2]^{1/2}}{2}\big)(1-\bar{\epsilon}_2)}. \tag{A.0-14}$$

Therefore, when $\epsilon_2>0$ and $\nu\in[1,\nu_+]$, a bound on $\bar{\delta}$ is given by

$$\bar{\delta} \;\leq\; 1-\frac{\min(b_2,b_3,b_4)}{\nu^{1/2}}. \tag{A.0-15}$$

If the parameters were such that $\nu_+\leq 1$, then the bound in (A.0-15) would be vacuous. However, as already noted, $\nu_+\to\infty$ as $\epsilon_2\to 0^+$, so for any $\bar{\epsilon}_2>0$, $\nu_+>1$ for some (sufficiently small positive) values of ϵ_2. Combining (A.0-9) and (A.0-15), we have the following upper bound on $\bar{\delta}$ over $\nu\in[1,\infty)$:

$$\bar{\delta}\leq\begin{cases}1-\frac{b_1}{\nu^{1/2}} & :\ \epsilon_2>0,\ \nu\in(\nu_+,\infty)\\ 1-\frac{\min(b_2,b_3,b_4)}{\nu^{1/2}} & :\ \epsilon_2>0,\ \nu\in[1,\nu_+].\end{cases} \tag{A.0-16}$$

Since $\bar{\delta}$ from (4.3.20) is a continuous function of ϵ_2 for ϵ_2 small enough, and the bounds on $\bar{\delta}$ in (A.0-16) are independent of ϵ_2, these bounds are also valid when $\epsilon_2=0$, and hence when $\bar{\epsilon}_2=0$. We conclude that

$$\bar{\delta}\leq 1-\frac{\min(b_1,b_2,b_3,b_4)}{\nu^{1/2}}\qquad\forall\ \nu\in[1,\infty),\ \epsilon_1\in[0,\bar{\epsilon}_1],\ \epsilon_2\in[0,\bar{\epsilon}_2]. \tag{A.0-17}$$

Of course the bound in (A.0-17) is only meaningful when the b_i are positive, and this restricts the values of $\theta, \kappa, \bar{\epsilon}_1$, and $\bar{\epsilon}_2$. For the parameters and bounds in (4.3.27), $\min(b_1, b_2, b_3, b_4) = b_1 > 0.00124$, hence $\bar{\delta} < 1 - \frac{0.00124}{\nu^{1/2}}$. This upper bound on $\bar{\delta}$ is clearly less than one for all $\nu \geq 1$. □

Appendix B

Proof of some intermediate results for structured perturbations

In this appendix we state and prove some results used in Section 4.5, where it is assumed that an estimate $\hat{F}(x)$ of a ν-normal barrier function F is used. Recall that $\hat{F}(x)$ is assumed to satisfy the three properties stated at the beginning of Section 4.5. Many of the results in this appendix are similar to—or special cases of—analogous results for the more general case of unstructured perturbations. We first bound the quantities $\|\Delta x^k\|_{x^k,\hat{F}}$ and $\|\Delta s^k\|^*_{x^k,\hat{F}}$. The proof is similar to that of Lemma 4.3.5.

Lemma B.0.1. *For each k,*

$$\mu_k^2\|\Delta x^k\|^2_{x^k,\hat{F}} + \left(\|\Delta s^k\|^*_{x^k,\hat{F}}\right)^2 \leq \mu_k^2\beta_0^2.$$

Proof. The proof is mostly the same as that of Lemma 4.3.5, except that F_1 and F_2 are replaced by $\hat{F}'$ and $\hat{F}''$ respectively. In fact the proof is the same up to (4.3.18a):

$$\|\hat{F}''(x)^{-1/2}(-\tau\mu\hat{F}'(x) - s)\|_2 \leq (1-\epsilon_2)^{-1/2}\big(\|s + \mu F'(x)\|^*_{x,F} + \mu\epsilon_1\big) + \mu(1-\tau)\|\hat{F}''(x)^{-1/2}\hat{F}'(x)\|_2.$$

Since $\hat{F}$ satisfies the logarithmic-homogeneity property, we can appeal to Lemma 3.3.2(e) to obtain $\|\hat{F}''(x)^{-1/2}\hat{F}'(x)\|_2 = \nu^{1/2}$. This is seen to be an improvement over $\frac{\nu^{1/2}+\epsilon_1}{(1-\epsilon_2)^{1/2}}$, which is the bound we obtained for the case of unstructured perturbations; cf. (4.3.10). We

conclude that

$$\|\hat{F}''(x)^{-1/2}(-\tau\mu\hat{F}'(x) - s)\|_2 \leq \mu\beta_0,$$

where β_0 is defined in (4.5.5). The rest of the proof imitates that of Lemma 4.3.5. □

Corollary B.0.2. *For each k,*

$$\begin{aligned}
\|\Delta x^k\|_{x^k,\hat{F}} &\leq \beta_0, \\
\|\Delta s^k\|^*_{x^k,\hat{F}} &\leq \mu_k\beta_0, \\
\|\Delta x^k\|_{x^k,F} &\leq \beta_1, \\
\|\Delta s^k\|^*_{x^k,F} &\leq (1+\epsilon_2)^{1/2}\mu_k\beta_0,
\end{aligned}$$

where β_0 and β_1 are defined in (4.5.5).

Proof. Similar to that of Corollary 4.3.6. □

Let us study the convergence of Algorithm short_step. First we show that under a condition on the parameters θ, τ, ϵ_1, and ϵ_2, a full primal-dual Newton step is not only strongly feasible, thereby verifying the validity of step (2) in the algorithm, but the new iterate remains in the $\mathcal{N}(\theta, \varepsilon, \mu_0)$ neighborhood of the central path, except at the final iteration, when the duality measures first falls below ε. Then we show that the sequence of duality measures $\{\mu_k\}$ decreases linearly to ε.

Lemma B.0.3. *Let θ, τ, ϵ_1, and ϵ_2 be such that $\beta_2 < 1$ where β_2 is defined in* (4.5.5)*, and let $(x^k, w^k, s^k) \in \mathcal{N}(\theta, \varepsilon, \mu_0)$. Then the point $(x^{k+1}, w^{k+1}, s^{k+1})$ generated by Algorithm short_step is a strongly feasible primal-dual point.*

Proof. Similar to that of Lemma 4.3.7. □

We now study the behavior of the sequence of duality measures $\{\mu_k\}$.

Lemma B.0.4. *Let $(x^k, w^k, s^k) \in \mathcal{N}(\theta, \varepsilon, \mu_0)$ be the k-th iterate generated by Algorithm short_step. The duality measure μ_{k+1} of the next iterate $(x^{k+1}, w^{k+1}, s^{k+1})$ satisfies*

$$\underline{\delta}\mu_k \leq \mu_{k+1} \leq \bar{\delta}\mu_k,$$

where

$$\underline{\delta} = \tau - \frac{1}{\nu^{1/2}}(1-\tau)\beta_0 - \frac{1}{\nu}\beta_0^2, \tag{B.0-1}$$

$$\bar{\delta} = \frac{(1+\tau)^2}{4}. \tag{B.0-2}$$

Proof. For ease of notation, we will write x, s, μ for x^k, s^k, μ_k, and x_+, s_+, μ_+ for $x^{k+1}, s^{k+1}, \mu_{k+1}$. Recalling that Δx is orthogonal to Δs, we have

$$\begin{aligned} \nu\mu_+ &= x_+^T s_+ \\ &= (x+\Delta x)^T(s+\Delta s) \\ &= x^T(s+\Delta s) + (\Delta x)^T s. \end{aligned} \tag{B.0-3}$$

From the third block equation in (4.2.1), we have $s + \Delta s = -\tau\mu\hat{F}'(x) - \mu\hat{F}''(x)\Delta x$, so

$$\begin{aligned} x^T(s+\Delta s) &= -x^T(\tau\mu\hat{F}'(x) + \mu\hat{F}''(x)\Delta x) \\ &= \nu\tau\mu + \mu\hat{F}'(x)^T\Delta x, \end{aligned} \tag{B.0-4}$$

where we have used Lemma 3.3.2(b),(c). Since $(\Delta x)^T\Delta s = 0$, it also follows from the third block equation in (4.2.1) that

$$\begin{aligned} (\Delta x)^T s &= (\Delta x)^T(-\mu\hat{F}''(x)\Delta x - \tau\mu\hat{F}'(x)) \\ &= -\mu\|\Delta x\|^2_{x,\hat{F}} - \tau\mu\hat{F}'(x)^T\Delta x. \end{aligned} \tag{B.0-5}$$

Combining (B.0-3), (B.0-4), and (B.0-5), we have

$$\nu\mu_+ = \nu\tau\mu + (1-\tau)\mu\hat{F}'(x)^T\Delta x - \mu\|\Delta x\|^2_{x,\hat{F}}.$$

Now

$$\begin{aligned} |\hat{F}'(x)^T\Delta x| &= |\big(\hat{F}''(x)^{-1/2}\hat{F}'(x)\big)^T(\hat{F}''(x)^{1/2}\Delta x)| \\ &\leq \|\hat{F}''(x)^{-1/2}\hat{F}'(x)\|_2 \; \|\hat{F}''(x)^{1/2}\Delta x\|_2 \\ &= \nu^{1/2}\|\Delta x\|_{x,\hat{F}}, \end{aligned}$$

where the last equality follows from Lemma 3.3.2(e). So we have the following bounds on the ratio of successive duality measures:

$$\frac{\mu_+}{\mu} \leq \tau + \frac{1}{\nu^{1/2}}(1-\tau)\|\Delta x\|_{x,\hat{F}} - \frac{1}{\nu}\|\Delta x\|^2_{x,\hat{F}}, \tag{B.0-6}$$

$$\frac{\mu_+}{\mu} \geq \tau - \frac{1}{\nu^{1/2}}(1-\tau)\|\Delta x\|_{x,\hat{F}} - \frac{1}{\nu}\|\Delta x\|^2_{x,\hat{F}}. \tag{B.0-7}$$

By maximizing the upper bound in (B.0-6) over $0 \leq \|\Delta x\|_{x,\hat{F}} \leq \beta_0$ (see Corollary B.0.2), we obtain the best upper bound

$$\frac{\mu_+}{\mu} \leq \frac{(1+\tau)^2}{4},$$

which is (B.0-2). To obtain the best lower bound on μ_+/μ, we minimize the lower bound in (B.0-7) over $0 \leq \|\Delta x\|_{x,\hat{F}} \leq \beta_0$. The result is that

$$\frac{\mu_+}{\mu} \geq \tau - \frac{1}{\nu^{1/2}}(1-\tau)\beta_0 - \frac{1}{\nu}\beta_0^2,$$

which is (B.0-1). □

We can now give a proof of Lemma 4.5.3. Due to the logarithmic homogeneity of $\hat{F}$, the analysis is simpler than that in the proof of Lemma 4.3.10. For ease of reference, we restate here the values of the parameters and bounds in Section 4.5, in addition to Lemma 4.5.3

itself.

$$\theta = 0.1, \quad \tau = 1 - \frac{1}{47\nu^{1/2}}, \quad 0 \le \epsilon_1 \le 0.015, \quad 0 \le \epsilon_2 \le 0.12.$$

Lemma B.0.5. *Let θ, τ, ϵ_1, and ϵ_2 satisfy* (4.5.6)*. For all $\nu \ge 1$,*

$$\beta_1 < 0.1534, \qquad \beta_2 < 0.1692, \qquad \underline{\delta} > 1 - \frac{0.0451}{\nu^{1/2}} > 0, \qquad \bar{\delta} < 1 - \frac{0.0211}{\nu^{1/2}} < 1.$$

Proof. For the sake of generality, let

$$\tau = 1 - \frac{\kappa}{\nu^{1/2}} \tag{B.0-8}$$

for some constant $\kappa \in (0, 1)$, and let $\bar{\epsilon}_1 (= 0.015)$ and $\bar{\epsilon}_2 (= 0.12)$ denote the largest allowable values of ϵ_1 and ϵ_2 respectively. The following uniform bound on β_0 over $[1, \infty)$ follows from (4.5.5):

$$\begin{aligned} \beta_0 &= \frac{\theta + \epsilon_1}{(1 - \epsilon_2)^{1/2}} + \kappa \\ &\le \frac{\theta + \bar{\epsilon}_1}{(1 - \bar{\epsilon}_2)^{1/2}} + \kappa. \end{aligned} \tag{B.0-9}$$

We now prove the bound on β_1:

$$\begin{aligned} \beta_1 &= \left(\frac{1}{1 - \epsilon_2}\right)^{1/2} \beta_0 \\ &\le \frac{\theta + \bar{\epsilon}_1}{1 - \bar{\epsilon}_2} + \frac{\kappa}{(1 - \bar{\epsilon}_2)^{1/2}} \\ &< 0.1534. \end{aligned}$$

Also

$$\begin{aligned}
\beta_2 &= \beta_0 \max\left\{\frac{(1+\epsilon_2)^{1/2}}{1-\theta}, \left(\frac{1}{1-\epsilon_2}\right)^{1/2}\right\} \\
&\leq \left(\frac{\theta+\bar{\epsilon}_1}{(1-\bar{\epsilon}_2)^{1/2}}+\kappa\right) \max\left\{\frac{(1+\bar{\epsilon}_2)^{1/2}}{1-\theta}, \left(\frac{1}{1-\bar{\epsilon}_2}\right)^{1/2}\right\} \\
&< 0.1692,
\end{aligned}$$

proving the bound on β_2. We now prove the lower bound on $\underline{\delta}$. Using (B.0-1) and (B.0-8), we have

$$\begin{aligned}
\underline{\delta} &= \tau - \frac{1}{\nu^{1/2}}(1-\tau)\beta_0 - \frac{1}{\nu}\beta_0^2 \\
&= \tau - \frac{1}{\nu}\beta_0(\nu^{1/2}(1-\tau)+\beta_0) \\
&= \left(1-\frac{\kappa}{\nu^{1/2}}\right) - \frac{1}{\nu}\left[\frac{\theta+\epsilon_1}{(1-\epsilon_2)^{1/2}}+\kappa\right]\left[\kappa+\frac{\theta+\epsilon_1}{(1-\epsilon_2)^{1/2}}+\kappa\right] \\
&\geq 1-\frac{\kappa}{\nu^{1/2}} - \frac{1}{\nu^{1/2}}\left[\frac{\theta+\epsilon_1}{(1-\epsilon_2)^{1/2}}+\kappa\right]\left[\frac{\theta+\epsilon_1}{(1-\epsilon_2)^{1/2}}+2\kappa\right] \\
&= 1-\frac{1}{\nu^{1/2}}\left(\kappa+\left[\frac{\theta+\epsilon_1}{(1-\epsilon_2)^{1/2}}+\kappa\right]\left[\frac{\theta+\epsilon_1}{(1-\epsilon_2)^{1/2}}+2\kappa\right]\right) \\
&\geq 1-\frac{1}{\nu^{1/2}}\left(\kappa+\left[\frac{\theta+\bar{\epsilon}_1}{(1-\bar{\epsilon}_2)^{1/2}}+\kappa\right]\left[\frac{\theta+\bar{\epsilon}_1}{(1-\bar{\epsilon}_2)^{1/2}}+2\kappa\right]\right).
\end{aligned}$$

Substituting $\theta = 0.1$, $\kappa = 1/47$, $\bar{\epsilon}_1 = 0.015$, and $\bar{\epsilon}_2 = 0.12$ gives the required lower bound on $\underline{\delta}$. Finally, from (B.0-2) and (B.0-8), we have

$$\begin{aligned}
\bar{\delta} &= \frac{(1+\tau)^2}{4} \\
&= 1-\frac{\kappa}{\nu^{1/2}}+\frac{\kappa^2}{4\nu} \\
&\leq 1-\frac{\kappa}{\nu^{1/2}}+\frac{\kappa^2}{4\nu^{1/2}} \\
&= 1-\frac{1}{\nu^{1/2}}\left(\kappa-\frac{\kappa^2}{4}\right) \\
&< 1-\frac{0.0211}{\nu^{1/2}}.
\end{aligned}$$

□

Lemma B.0.6. *Let θ, τ, ϵ_1, and ϵ_2 satisfy* (4.5.6)*, and let $(x^k, w^k, s^k) \in \mathcal{N}(\theta, \varepsilon, \mu_0)$. Ex-*

cept at the final iteration, the primal-dual point $(x^{k+1}, w^{k+1}, s^{k+1})$ *generated by Algorithm short_step also belongs to* $\mathcal{N}(\theta, \varepsilon, \mu_0)$.

Proof. The proof is similar to that of Lemma 4.3.11. In that proof we assumed that the parameter values were such that $\beta_2 < 1$, $\tau > 0$, and $\underline{\delta} > 0$ for all $\nu \geq 1$. We see from Lemma B.0.5 that these assumptions do indeed hold. Given $(x^k, w^k, s^k) \in \mathcal{N}(\theta)$, we showed in the proof of Lemma 4.3.11 that a sufficient condition for $(x^{k+1}, w^{k+1}, s^{k+1}) \in \mathcal{N}(\theta)$ was

$$\frac{1}{1-f}\left[\frac{\bar{\epsilon}_1(1-\kappa)+\beta_1^*(\bar{\epsilon}_2+\kappa)}{1-\beta_1^*}+f-\kappa+(1-\kappa)\left(2\log(1-\beta_1^*)+\frac{2\beta_1^*}{1-\beta_1^*}\right)\right] \leq \theta, \quad \text{(B.0-10)}$$

where β_1^* is the maximum of β_1 over all $\nu \geq 1$, $\epsilon_1 \in [0, \bar{\epsilon}_1]$ and $\epsilon_2 \in [0, \bar{\epsilon}_2]$. In obtaining this condition it was assumed that $f \geq \kappa$, where f is such that $\underline{\delta} \geq 1 - \frac{f}{\nu^{1/2}}$ is a lower bound on the ratio of successive duality measures. From Lemma B.0.5 we see that a valid f is 0.0451, which exceeds $\kappa = 1/47$. Also from Lemma B.0.5 we have $\beta_1^* < 0.1534$, so the left-hand side of (B.0-10) is less than 0.0999, which is less than θ. Hence $(x^k, w^k, s^k) \in \mathcal{N}(\theta, \varepsilon, \mu_0)$ implies that $(x^{k+1}, w^{k+1}, s^{k+1}) \in \mathcal{N}(\theta)$. Since $\{\mu_k\}$ is a strictly decreasing sequence, and at all iterations except the last, $\mu_{k+1} > \varepsilon$, then $(x^{k+1}, w^{k+1}, s^{k+1})$ lies inside the restricted neighborhood $\mathcal{N}(\theta, \varepsilon, \mu_0)$. □

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