ABSTRACT

Title of dissertation: Simulating Risk Neutral Stochastic Processes with a View to Pricing Exotic Options

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Absence of arbitrage requires all claims to be priced as the expected value of cash flows under a risk neutral measure on the path space and every claim must be priced under the same measure. This motivates why we want to use the same measure to price vanilla options and path dependent products, and hence why we want to match marginal distributions.

There are many ways of matching marginal distributions. We present simulation methods for three stochastic processes that match prespecified marginal distributions at any continuous time: the Azéma and Yor solution to the Skorohod embedding problem, inhomogeneous Markov martingale processes with independent increments using subordinated Brownian motion, and a continuous martingale using Dupire's local volatility method. Then the question is which way is a good way of matching marginal distributions.

To make a judgement, we look at the properties of the processes. Since all vanilla options are already matched, we want to use exotic options to investigate properties of the processes. One of the properties that we investigate is whether for-

ward return distributions are close to spot return distributions as market structural features.

We price swaps associated with the first passage time to barrier levels on these processes and see which model gives the highest value of swaps, in other words, the shortest passage time to levels. Moreover, we price monthly reset arithmetic cliquets with local floors and global caps and with local caps and global floors. Then we check the model risks of these models and find how model risks change when caps or floors change. Finally, we price options on the realized quadratic variations to see how option prices move as maturity increases.

Simulating Risk Neutral Stochastic Processes with a view to Pricing Exotic Options

by

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DEDICATION

To my parents and my family

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I dedicate my thesis and my Ph.D. degree to my parents.

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Introduction

Originally, the concept of martingale was featured in betting strategies in eighteenth century France. If a coin came up heads, the bettor won his stake; if it came up tails, he lost. The gambler would double his bet after every loss, hoping that a single win would net a sum equal to the stake and all subsequent losses. Paul Pierre Lévy introduced the concept of martingale in probability theory and much of the development of these theories was carried out in the twentieth century by the American mathematician, Joseph Leo Doob. The martingale has become one of the most important concepts in stochastic processes and probability theory.

Martingale is a critical concept in mathematical finance, since discounted asset prices are martingales under a risk neutral measure or, alternatively, martingale measure. By the fundamental theorem of asset pricing, the existence of a risk neutral measure is equivalent to a no arbitrage condition in markets. A main assumption in the study of finance is that the market is free of arbitrage, since we interpret an arbitrage possibility as a case of mispricing in the market. Then the question arises naturally: How to construct martingales?

Three generic constructions of martingales that match with prespecified marginal densities and have the Markov property have been developed by Madan and Yor [24]. The first construction uses the Azéma and Yor [27] solution to the Skorohod embedding problem. The second generates inhomogeneous Markov martingale processes with independent increments using subordinated Brownian motion. The third constructs a continuous martingale using Dupire's method [14] for Markov martingales. After having constructed martingales with the general marginal distributions, they consider marginal distributions that satisfy a scaling property and then they construct martingales with unit time density.

The main contribution of this dissertation is the construction of martingales by simulating these three processes. We use an infinitesimal generator to investigate the properties of the process obtained from the Azéma and Yor solution to the Skorohod embedding problem. From the infinitesimal generator, we can conclude that the process is a pure jump process and obtain the analytical forms for jump times and jump sizes. To simulate this process, we use an inverse cumulative distribution method and an acceptance-rejection method to generate random jump sizes. For the time change Brownian motion, we show that this is a nonstationary process. Then, we simulate this process using inverse Laplace transformation and a Lévy measure. For the third construct, the continuous martingale, we use Milstein's higher order scheme. After we generate the three stochastic processes, we use the Kolmogorov-Smirnov test to check if the marginal distributions of the processes match with prespecified marginal distributions which, in this dissertation, are double negative exponential distributions.

The pricing of exotic options is always of interest. Since these processes all have the same marginal distributions across continuous time, the prices of European options are consistent at any maturity. But the prices of exotic options that depend on the paths may be different. We price swaps with barriers, locally floored and capped and globally floored and capped cliquets, and the options on the realized quadratic variations.

Another interesting study is about forward motion, the agreement of delivery being made in the future but the price determined on the initial trading day. For Lévy processes, the forward return distributions are in the same class of distributions as the spot return distribution. Many attempts at building path spaces adopt such a procedure. However, it may not be a reasonable request to make of processes in general. This dissertation investigates whether that assumption is valid for these processes with a specified spot return distribution (in this case, double negative exponential distribution).

The outline of this dissertation is as follows: In Chapter 2, we briefly introduce constructions of martingales for all three cases. Then, in Chapter 3, we consider the densities which have scaling properties. Chapter 4 is the main part, which shows how to simulate those processes. Examples of pricing exotic options, such as swaps, cliquets, options on the realized quadratic variation, and studies about forward motions appear in Chapter 5. Chapter 6 summarizes the dissertation and provides suggestions for future works.

Chapter 2

Construction of martingales

The martingale is a central concept in finance research. One of the reasons for this is that the price of an option can be expressed as an expectation under a martingale measure, in other words, a risk neutral measure. Moreover, stock prices are usually assumed to have Markov properties. The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past [32], conditional on the present state of the process.

2.1 General background

We start with a few definitions here. General references on martingales can be found in [4, 19]. Markov processes are discussed in [36].

Definition 2.1.1. A filtration $(\mathcal{F}_t)_{t\in T}$ is an increasing family ($\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t \in T$) of sub-σ-fields of $\mathcal{F}$ on a stochastic base $(\Omega, \mathcal{F}, P, (\mathcal{F}_t), T)$ with complete probability space $(\Omega, \mathcal{F}, P)$ and the filtration $(\mathcal{F}_t)_{t\in T}$ satisfies the following conditions:

1. $\mathcal{F}_0$ contains all P-null sets.

2. $(\mathcal{F}_t)$ is right-continuous: $\mathcal{F}_t = \mathcal{F}_{t^+} := \bigcap_{t<s} \mathcal{F}_s$ for all $t \in T$.

A process $(X_t)_{t\in T}$ is adapted to $(\mathcal{F}_t)$ if for each $t \in T$, X_t is $\mathcal{F}_t$-measurable.

Definition 2.1.2. A stochastic process $(X_t)_{t\in T}$ is called an $(\mathcal{F}_t)$-martingale (resp. supermartingale, submartingale) if the following conditions are satisfied.

1. $(X_t)_{t\in T}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

2. For all t, $X_t \in L^1$; i.e., $E[|X_t|] < \infty$.

3. For all s and t with $s \leq t$, the following relation holds

$$E[X_t|\mathcal{F}_s] = X_s \quad \text{a.s.}$$

$$(\text{resp}E[X_t|\mathcal{F}_s] \leq X_s, E[X_t|\mathcal{F}_s] \geq X_s).$$

Definition 2.1.3. A sequence of integral-valued random variables $\{X_n : n = 0, 1, 2, ...\}$ is called a discrete Markov chain if it has the Markov property:

$$P(X_{n+1} = j|X_n, X_{n-1}, ..., X_0) = P(X_{n+1} = j|X_n),$$

$$n = 0, 1, 2, ..., \quad j = 0, 1, 2, ...,$$

where, for each value of X_n, the probabilities $P(X_{n+1} = j|X_n)$ are called transition probabilities.

It is natural to generalize from discrete time to continuous time, especially for modelling stochastic processes.

Definition 2.1.4. Let $\{X(t) : t \geq 0\}$ be a continuous-time stochastic process with finite or countable state space $\mathcal{T}$; usually $\mathcal{T}$ is $\{0, 1, 2, ...\}$, or a subset thereof. We say $X\{(t)\}$ is a continuous-time Markov chain if the transition probabilities have the following property: For every $t, s \geq 0$ and $j \in \mathcal{T}$,

$$P\{X(s+t) = j|X(u); u \leq s\} = P\{X(s+t) = j|X(s)\}.$$

The stochastic processes that have the Markov property are called Markov processes and the martingales that have the Markov property are called Markov martingales.

Throughout this dissertation, we have Brownian motion, one of the most well-known stochastic process, in many places. In finance, Brownian motion was introduced in the Black-Scholes model in 1973 [5]. We present a definition of Brownian motion.

Definition 2.1.5. A stochastic process $\{B(t)\}$ is called a Brownian motion if the following conditions are satisfied.

1. $B(0) = 0$.
2. The process B has independent increments, i.e., if $r < s \leq t < u$, then $B(u) - B(t)$ and $B(s) - B(r)$ are independent stochastic variables.
3. For $s < t$, the stochastic variable $B(t) - B(s)$ has the Gaussian distribution $N[0, t - s]$.
4. B has continuous trajectories.

The following are well-known properties of Brownian motion.

Lemma 2.1.1. *Consider two points in time, s and t, with $s < t$ and*

$$\triangle t = t - s,$$

$$\triangle B(t) = B(t) - B(s).$$

Then

1. $E[\triangle B] = 0$.

2. $E[(\triangle B)^2] = \triangle t$.

3. $Var[\triangle B] = \triangle t$.

4. $Var[(\triangle B)^2] = 2(\triangle t)^2$.

Before we start constructing martingales, we present more theorems that are needed for the later part. Consider the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t. \tag{2.1}$$

where $dB_t = dB(t)$.

Definition 2.1.6. Given (2.1), the partial differential operator A, referred to as the infinitesimal operator (or, infinitesimal generator) of X, is defined for any function $f(x)$ with $f \in C^2(\mathbb{R})$, by

$$A_t f(x) = \mu(t, x)\frac{\partial f}{\partial x}(x) + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 f}{\partial x^2}(x).$$

This operator is also known as the Dynkin operator, the Itô operator, or the Kolmogorov backward operator.

Theorem 2.1.2 (Kolmogorov backward equation). *Let X be a solution to equation (2.1). Then the transition probabilities $P(s, y; t, B) = P(X_t \in B | X(s) = y)$ are given as the solution to the equation*

$$\begin{aligned} \left(\frac{\partial P}{\partial s} + AP\right)(s, y; t, B) &= 0, \quad (s, y) \in (0, t) \times \mathbb{R}^n, \\ P(t, y; t, B) &= I_B(y), \end{aligned}$$

where

$$I_B(y) \quad = \quad \begin{cases} 1, & \text{if} \quad y \in B \\ \\ 0, & \text{if} \quad y \notin B \end{cases}.$$

The following is the transition density version of the Kolmogorov backward equation.

Theorem 2.1.3 (Kolmogorov backward equation). *Let X be a solution to equation (2.1). Assume that the measure $P(s,y;t,dx)$ has a density $p(s,y;t,x)dx$. Then we have*

$$\begin{aligned} \Big(\frac{\partial p}{\partial s} + Ap\Big)(s,y;t,B) \quad &= \quad 0, \quad (s,y) \in (0,t) \times \mathbb{R}^n, \\ p(s,y;t,x) \quad &\to \quad \delta_x, \quad as \quad s \to t. \end{aligned}$$

where δ_x is the Dirac Delta function.

Define the adjoint operator $A^\star$ by

$$(A^\star f)(t,x) = -\frac{\partial}{\partial x}[\mu(t,x)f(t,x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(t,x)f(t,x)].$$

The next theorem is the Kolmogorov forward equation.

Theorem 2.1.4 (Kolmogorov forward equation). *Assume that the solution X of equation (2.1) has a transition density $p(s,y;t,x)$. Then p satisfies the Kolmogorov forward equation*

$$\begin{aligned} \frac{\partial}{\partial t}p(s,y;t,x) \quad &= \quad A^\star p(s,y;t,x), \quad (t,x) \in (0,T) \times \mathbb{R}, \\ p(s,y;t,x) \quad &\to \quad \delta_y, \quad as \quad t \downarrow s. \end{aligned}$$

Now, we start to construct martingales.

Suppose the density at time t is $g(y,t)$, for $y \in \mathbb{R}$ and

$$\int |y| g(y,t) dy < \infty,$$

$$\int y g(y,t) dy = 0,$$

and $B(t)$ is a Brownian motion. Hence these densities $g(y,t)$ are candidates for a martingale beginning at zero.

From the martingale property, we have the following Lemma.

Lemma 2.1.5. *Let $\phi(y)$ be a convex function and let $X(t)$ be a Markov martingale. Then, for $s < t$,*

$$E[\phi(X(t))] \geq E[\phi(X(s))]$$

Proof. Since $\phi(y)$ is a convex function,

$$E_s[\phi(X(t))] \geq \phi(E_s[(X(t))])$$

for $s < t$ where $E_s[\phi(X(t))]$ is the conditional expectation of $\phi(X(t))$ given $(\mathcal{F}_s)$. Since $X(t)$ is a Markov martingale,

$$E_s[(X(t))] = X(s).$$

Hence

$$E[\phi(X(t))] \geq E[\phi(X(s))].$$

□

Definition 2.1.7. Let X and Y be two random variables such that

$$E[\phi(X)] \leq E[\phi(Y)],$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided the expectations exist. Then X is said to be smaller than Y in the convex order.

The following theorem shows that if a Markov martingale $X(t)$ matching the marginal densities $g(y,t)$ exists, then the marginal densities across time have convexity order, and vice versa.

Theorem 2.1.6. *[24] Let $p(y,t)$ be a family of marginal densities, with finite first moment. The density at time t dominates the density at time s in the convex order for $s < t$ if and only if there exists a Markov process $X(t)$ with these marginal densities under which $X(t)$ is a submartingale. Furthermore, the means are independent of t if and only if $X(t)$is a martingale.*

2.2 Azéma and Yor's solution to the Skorohod embedding problem

The Skorohod embedding problem was formulated and solved by Skorohod in 1961 [35] and many people tried to find different ways to solve the Skorohod embedding problem. Azéma and Yor's solution was presented in 1979 [1]. In 2004, Jan Oblój presented all known solutions in his paper [27].

We start with the definition of stopping time and the definition of the Skorohod embedding problem.

Definition 2.2.1. A random variable $\tau : \Omega \to [0, \infty]$ is a stopping time for the filtration $(\mathcal{F}_t)$ if $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \in T$.

The Skorohod embedding problem is formulated using stopping times.

Definition 2.2.2. The Skorohod embedding problem or Skorohod stopping problem is as follows : For a given centered probability measure μ with finite second moment and a Brownian motion B, one looks for an integrable stopping time T such that the distribution of B_T is μ.

Now, we construct martingales with marginal distributions specified by densities $g(x,t)$ using Azéma and Yor's solution to the Skorohod embedding problem.

Define the family of barycentre functions by

$$\psi(x,t) = \frac{\int_x^\infty yg(y,t)dy}{\int_x^\infty g(y,t)dy},$$

and the stopping time τ by

$$M(t) = \sup_{0 \leq s \leq t} B(s),$$

$$\tau = \inf\{s|M(s) \geq \psi(B(s))\}.$$

Shaked and Shantikumar [33] showed that the increasing mean residual value (IMRV) property is stronger than convexity order. The mean residual value is commonly used in survival analysis and in the analysis of life tables.

Definition 2.2.3. $R(x)$ is called the mean residual life function at age x if

$$R(x) = E[X - x|X > x],$$

where X is a lifetime.

This Skorohod embedding solution constructs a martingale with the specified marginal densities $g(x,t)$ under the assumption that $\psi(x,t)$ is increasing in t for each x. Moreover, if $\psi(x,t)$ is increasing in t, then a family of zero expectation densities has the property of increasing mean residual value (IMRV)[24].

Let

$$T_t = \inf\{s | M(s) \geq \psi(B(s), t)\}.$$

Theorem 2.2.1. *[24] Under the IMRV property for a family of zero mean densities $g(y,t)$ on the real line, let $(B(u), u \geq 0)$ be a standard Brownian motion. Then there exists an increasing family of Brownian stopping times $(T_t, t \geq 0)$ such that:*

1. *$B(T_t)$ is a martingale.*

2. *$(B(T_t), t \geq 0)$ is an inhomogeneous Markov process.*

3. *For each t, the density of $B(T_t)$ is given by $g(y,t)$.*

The instructive way to investigate this process is to develop the infinitesimal generator of the inhomogeneous Markov process $B(T_t)$.

For the Azéma and Yor solution to the Skorohod embedding problem, the derivation of the infinitesimal generator is presented in [24]. The infinitesimal generator is

$$A_t(f)(b) = a_t(b)\Big\{\frac{1}{\psi_t(b) - b}\frac{\int\limits_b^\infty g(y,t)(f(y) - f(b))dy}{\int\limits_b^\infty g(y,t)dy} - f'(b)\Big\},$$

where $\psi_t(b) = \psi(b,t)$ and

$$a_t(b) = \frac{\frac{\partial}{\partial t}\psi(b,t)}{\frac{\partial}{\partial x}\psi(b,t)}.$$

So, we notice that this process is a one-sided jump process with jump intensities given by

$$\frac{a_t(b)}{\psi_t(b) - b} \frac{g(b+x,t)}{\int\limits_0^\infty g(b+x,t)dx}, \quad \text{for} \;\; x > 0 \tag{2.2}$$

where $b = B(T_t)$ and a drift factor of $-a_t(b)$, and again $g(x,t)$ is a prespecified density.

2.3 Inhomogeneous independent increments

The second way to construct martingales with prespecified marginals is to build subordinating Brownian motion by an independent increasing Markov process with independent increments. For this study, we need to start with Lévy processes. The general reference is [29].

Definition 2.3.1. Let ϕ be the characteristic function of a random variable X. Then X is self-decomposable if

$$\phi(u) = \phi(cu)\phi_c(u)$$

for all $u \in \mathbb{R}$ and all $c \in (0,1)$ and for some family of characteristic functions $\{\phi_c : c \in (0,1)\}$.

Definition 2.3.2. A process $X_t = \sum_{i=1}^n Z_i$, where the Z_i represent increments of X_t over intervals of length $\frac{t}{n}$ and i.i.d., is said to be infinitely divisible if for all n,

$$\Phi_{X_t}(u) = \Big(\Phi_{Z_i}(u)\Big)^n,$$

where $\Phi(u)$ is the characteristic function of a distribution.

We can simply rewrite the definition of infinitely divisible process as

$$\Phi_{X_t}(u) = \Big(\Phi_{X_1}(u)\Big)^t.$$

Definition 2.3.3. An adapted process $X = (X_t)_{0\leq t<\infty}$ is a Lévy process if

1. $X_0 = 0$ almost surely.
2. X has increments independent of the past: that is, $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$.
3. X has stationary increments: that is, $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t < \infty$.
4. X_t is continuous in probability: that is, $\lim_{t\to s} X_t = X_s$, where the limit is taken in probability.

So, the Lévy process is a class of stochastic processes which includes the Poisson process and Brownian motion as special cases. A random variable with self-decomposability law in Lévy's class is infinitely divisible.

Next, we define Lévy measure.

Let

$$N_t^\Lambda = \sum_{0<s\leq t} 1_\Lambda(\triangle X_s),$$

where Λ is a Borel set in $\mathbb{R}$ bounded away from 0, and $\triangle X_s = X_s - X_{s-}$, the jump at time s, and then we observe that N^Λ is a counting process without an explosion.

Definition 2.3.4. The measure ν defined by

$$\nu(\Lambda) = E[N_1^\Lambda] = E[\sum_{0<s\leq 1} 1 \wedge (\triangle X_s)]$$

is called the Lévy measure of the Lévy process X.

The following theorem gives us a formula for the Fourier transform of a Lévy process. This well known representation is called the Lévy-Khintchine formula.

Theorem 2.3.1. *Let X be a Lévy process with Lévy measure ν. Then*

$$E\left[e^{iuX_t}\right] = e^{-t\psi(u)}$$

where

$$\psi(u) = \frac{1}{2}\sigma^2 u^2 - i\gamma u + \int_{-\infty}^{\infty} (1 - \exp(iux) + iux 1_{\{|x|<1\}})\nu(dx), \tag{2.3}$$

and $\nu(dx)$ is a measure on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{-\infty}^{\infty} (1 \wedge x^2)\nu(dx) < \infty.$$

Moreover, given ν, σ^2, γ, the corresponding Lévy process is unique in distribution.

We note that Lévy-Khintchine formula has three components: a deterministic component with a drift coefficient γ, a diffusion coefficient σ and a pure jump component. If $\nu(dx) = k(x)dx$, then we call $k(x)$ a Lévy density.

Now, we construct martingale by seeking an increasing Markov process with inhomogeneous independent increments.

Let $L(t)$ be an increasing Markov process with inhomogeneous independent increments such that the process

$$X(t) = B(L(t))$$

has the prespecified marginals, where $B(u)$ is a Brownian motion independent of $(L(t), t \geq 0)$.

We identify Laplace transform of $L(t)$ by noting that

$$\begin{aligned} E[e^{iuX(t)}] &= E\Big[\exp\Big(\frac{-u^2}{2}L(t)\Big)\Big] \\ &= \int_{-\infty}^{\infty} e^{iuy} g(y,t)dy. \end{aligned}$$

The infinitesimal generator for this process can be derived by Laplace transformation for $L(t)$ written in infinitely divisible form as

$$E[\exp(-\lambda L(t))] = \exp\Big(\int_0^t \int_0^\infty (e^{-\lambda x} - 1)k_L(x,u)dxdu\Big). \tag{2.4}$$

where k_L is the Lévy density of L. Using Sato's [31] Theorems 30.1 and 31.5, we have

$$A_t(f)(x) = \int_{-\infty}^{\infty} (f(x+y) - f(x) - 1_{|y|\le 1} y f'(x))k_X(y,t)dy,$$

where

$$k_X(x,t) = \int_0^\infty \frac{1}{\sqrt{2\pi s}} \exp\Big(-\frac{x^2}{2s}\Big)k_L(s,t)ds.$$

2.4 Continuous martingales

The Black-Scholes model has been the most famous option pricing model since 1973 [5]. But its implied volatilities are dependent on the maturity and the strike of the European option. So, Merton suggested volatility be a time-dependent function in 1973 [26]. However, the strike-dependence of the implied volatility for a given maturity still remained as a big problem. In 1994, Dupire [14] introduced local volatility to solve this problem. Before we introduce local volatility, we need to know one of the most famous formulas, called Itô's formula [4].

Theorem 2.4.1 (Itô's formula). *Assume that the process X has a stochastic differential given by*

$$dX(t) = \mu(t)dt + \sigma(t)dB(t),$$

where μ and σ are adapted processes, and let f be a $C^{1,2}$-function. Define the processes Z by $Z(t) = f(t, X(t))$. Then Z has a stochastic differential given by

$$df(t, X(t)) = \left\{\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right\}dt + \sigma\frac{\partial f}{\partial x}dB(t). \tag{2.5}$$

The following is the brief sketch of the derivation of Dupire's local volatility.

Let $C(K, T)$ be arbitrage-free European call price of all strikes K and maturities T. Suppose the spot S follows risk-neutral process

$$dS = rSdt + \sigma(S, t)SdB.$$

Let X_t be a Markov process with stochastic differential equation

$$dX = \mu(X, t)dt + \sigma(X, t)dB,$$

and let $q(x, t, u, T)$ be the transition kernel, where x is a value of X at time t and u is a value of the process X at time T. The Kolmogorov forward equation is used for the derivation. Let us take a test function $f(u)$ and consider the martingale

$$V(x, t, T) = E[f(X(T))|X(t) = x] = \int_{-\infty}^{\infty} f(u)q(x, t, u, T)du.$$

Then

$$V_T(x, t, T) = \int_{-\infty}^{\infty} f(u)q_T(x, t, u, T)du. \tag{2.6}$$

By applying Itô's formula (Theorem 2.4.1) to $f(u)$ and using integration by parts,

$$V_T(x,t,T) = \int_{-\infty}^{\infty} f(u)\big(-\frac{\partial}{\partial u}\mu(u,T)q(x,t,u,T) + \frac{1}{2}\frac{\partial^2}{\partial u^2}\sigma^2(u,T)q(x,t,u,T)\big)du. \tag{2.7}$$

By (2.6) and (2.7), we have

$$q_T(x,t,u,T) = -\frac{\partial}{\partial u}\mu(u,T)q(x,t,u,T) + \frac{1}{2}\frac{\partial^2}{\partial u^2}\sigma^2(u,T)q(x,t,u,T). \tag{2.8}$$

This equation (2.8) is the Kolmogorov forward equation in u, T, and is expressed by

$$q_T(u,T) = -\frac{\partial}{\partial u}\mu(u,T)q(u,T) + \frac{1}{2}\frac{\partial^2}{\partial u^2}\sigma^2(u,T)q(u,T). \tag{2.9}$$

Using the option pricing formula

$$C(K,T) = e^{-rT}\int_{K}^{\infty}(u-K)q(u,T)du$$

and the derivatives of $C(K,T)$ with respect to K and T,

$$C_T = -re^{-rT}\int_{K}^{\infty}(u-K)q(u,T)du + e^{-rT}\int_{K}^{\infty}(u-K)q_T(u,T)du,$$

$$C_K = -e^{-rT}\int_{K}^{\infty}q(u,T)du,$$
$$C_{KK} = e^{-rT}q(K,T),$$

we find σ easily:

$$\sigma^2(K,T) = \frac{2\big(C_T + \eta C + (r-\eta)KC_K\big)}{K^2 C_{KK}}, \tag{2.10}$$

where $\mu = r - \eta$.

Hence, if σ is a local volatility, then the price of European option is consistent with that in market for all strikes and maturities.

Now, we are ready to construct a martingale with prespecified marginals $g(y, t)$. Define Markov martingale by

$$X(t) = \int_0^t \sigma(X(s), s) dB(s). \tag{2.11}$$

And $g(y, t)$ satisfies (2.9) with $\mu = 0$. Using the function

$$C(k, t) = \int_k^\infty (y - k) g(y, t) dy,$$

we get

$$\sigma^2(k, t) = \frac{2C_t}{C_k k}. \tag{2.12}$$

Hence, we have a continuous martingale representation with the prespecified marginals if the diffusion coefficients (2.12) are Lipschitz.

Chapter 3

Construction of Martingales on the Scaling Marginals

Many researchers have evaluated the relevance of scaling in describing asset returns. Initially, they used scaling properties for the sake of simplicity, but scaling properties have been very effective for many types of data and for the research. In this chapter, we construct martingales on the marginals which have a scaling property.

We suppose that marginals have a scaling property, specifically,

$$X(t) \cong \sqrt{t}X(1).$$

Let $h(x)$ be the unit time density. Then

$$\begin{aligned} P(X(t) < y) &= P(\sqrt{t}X(1) < y) \\ &= P(X(1) < \frac{y}{\sqrt{t}}). \end{aligned}$$

Let $G(y,t)$ and $H(y)$ be the cumulative distribution functions of $g(y,t)$ and $h(y)$. Then

$$G(y,t) = H(\frac{y}{\sqrt{t}}).$$

We take derivative in terms of y. Then the prespecified densities are expressed in terms of the unit time density as

$$g(y,t) = \frac{1}{\sqrt{t}}h\Big(\frac{y}{\sqrt{t}}\Big). \tag{3.1}$$

3.1 Skorohod embedding under scaling

Using (3.1), we get the family of barycentre functions

$$\psi_t(x) = \frac{\int_x^\infty y h\left(\frac{y}{\sqrt{t}}\right) dy}{\int_x^\infty g(y,t) dy} = \frac{\sqrt{t} \int_{x/\sqrt{t}}^\infty u h(u) du}{\int_{x/\sqrt{t}}^\infty h(u) du}.$$

where $\psi_t(x) = \psi(x,t)$.

By Theorem 2.1.6, we need that $\psi_t(x)$ be increasing in t. The following Lemma and Theorem give useful information to decide if we can construct martingales with scaling densities. The proofs of the following Lemma and Theorem are in [24].

Lemma 3.1.1. *The functions ψ_t are increasing with respect to t if and only if the function $\dfrac{a}{\psi_1(a)}$ is increasing in $a \in \mathbb{R}^+$.*

Theorem 3.1.2. *If $h(y) = \exp(-V(y))$ and $yV'(y)$ is increasing in $y > 0$, then $h(y)$ admits IMRV under scaling.*

3.2 Inhomogeneous independent increment process under scaling

We now construct martingales using subordinating Brownian motion by independent inhomogeneous increasing process $L(t)$ as

$$X(t) = B(L(t)).$$

Assume the increments have the scaling property

$$L(ct) \cong cL(t), \quad t > 0. \tag{3.2}$$

where $\cong$ means they are same in distribution.

Then we have the following theorem.

Theorem 3.2.1. [24] *Let $L(1) \geq 0$. Then the following three properties are equivalent:*

1. *There exists an increasing process with independent increments $(L(t), t \geq 0)$ which satisfies (3.2).*

2. *$L(1)$ is self-decomposable.*

3. *The Laplace transform of $L(1)$ is given by*

$$E[\exp(-\lambda L(1))] = \exp\Big(-\int_0^\infty (1 - e^{-\lambda l}\nu(dl))\Big)$$

where $\nu(dl) = (k(l)/l)dl$ and k is increasing.

3.3 Continuous martingales under scaling

By the assumption of the scaling property in the beginning of this chapter, we know that for any fixed $c > 0$,

$$(X_{ct}, t \geq 0) \cong (\sqrt{c}X_t, t \geq 0). \tag{3.3}$$

Then the following theorem gives the simple form of volatility.

Theorem 3.3.1. [24] *Assume that a process satisfies the scaling property (3.3) and simultaneously has the representation (2.11). Then we must have that*

$$\sigma^2(s, x) = a\big(\frac{x}{\sqrt{s}}\big), \tag{3.4}$$

$$a(y) = \frac{1}{h(y)} \int_{y}^{\infty} zh(z)dz. \tag{3.5}$$

Furthermore, if we have a density h and an associated function $a(y)$ that is Lipschitz, then there exists a continuous martingale satisfying the scaling property (3.3) and the Markov property, and for which

$$\langle X \rangle_t = \int_{0}^{t} a(\frac{X_s}{\sqrt{s}})ds.$$

Chapter 4

Simulation

Now, we simulate these three stochastic processes. For the prespecified density, we use double negative exponential densities,

$$g(x,t) = \frac{1}{\sigma\sqrt{2t}} \exp(-\frac{\sqrt{2}|x|}{\sigma\sqrt{t}}), \quad -\infty < x < \infty, \tag{4.1}$$

which have mean 0 and variance $\sigma^2 t$. Moreover, these densities have the scaling property.

4.1 Skorohod embedding

Assume that the process is at level b at time t, i.e., $B(T_t) = b$. By (2.2),

$$\frac{a_t(b)dt}{\psi_t(b) - b}$$

is arrival rate of a Poisson jump in the interval $(t, t+dt)$. And if there is a jump, then it has a jump size x drawn from distribution

$$\frac{g(y,t)}{\int\limits_b^\infty g(y,t)dx}, \quad y > b.$$

If we use double negative exponential functions as marginals, the family of barycentre functions is

$$\psi(x,t) = \frac{-\frac{1}{2}x\exp\left(\frac{\sqrt{2}x}{\sigma\sqrt{t}}\right) + \frac{\sigma\sqrt{t}}{2\sqrt{2}}\exp\left(\frac{\sqrt{2}x}{\sigma\sqrt{t}}\right)}{1-\frac{1}{2}\exp\left(\frac{\sqrt{2}x}{\sigma\sqrt{t}}\right)}, \text{ if } x < 0$$

$$\psi(x,t) = \frac{\frac{1}{2}x\exp\left(-\frac{\sqrt{2}x}{\sigma\sqrt{t}}\right) + \frac{\sigma\sqrt{t}}{2\sqrt{2}}\exp\left(-\frac{\sqrt{2}x}{\sigma\sqrt{t}}\right)}{\frac{1}{2}\exp\left(-\frac{\sqrt{2}x}{\sigma\sqrt{t}}\right)}, \text{ if } x \geq 0.$$

And the jump size distributions are

$$\frac{g(x+b,t)}{\int\limits_0^\infty g(x+b,t)dx} = \begin{cases} \frac{\sqrt{2}}{\sigma\sqrt{t}}\exp\left(-\frac{\sqrt{2}x}{\sigma\sqrt{t}}\right) \text{ if } \quad b \geq 0 \\ \frac{\frac{1}{\sigma\sqrt{2t}}\exp\left(-\frac{\sqrt{2}|x+b|}{\sigma\sqrt{t}}\right)}{1-\frac{1}{2}\exp\left(\frac{\sqrt{2}b}{\sigma\sqrt{t}}\right)} \text{ if } \quad b < 0 \end{cases}. \tag{4.2}$$

Let

$$C = 1 - \frac{1}{2}\exp\left(\frac{\sqrt{2}b}{\sigma\sqrt{t}}\right).$$

To get the random jump size, we can use either inverse CDF method or acceptance-rejection method.

4.1.1 Simulating using Inverse CDF method

The most well known method of generating random variable from given distribution is inverse cumulative distribution function method. First, we use this method to generate random numbers which represent jump sizes of Azéma and Yor's solution to the Skorohod embedding problem.

Let $f(x)$ be a probability distribution function and $F(x)$ be a cumulative distribution function. We want to generate random numbers x from $F(x)$.

The algorithm of the inverse cumulative distribution function method is as follows.

1. Generate an uniform random number $u \sim U(0,1)$.

2. Let $x = F^{-1}(u)$.

3. Repeat step one and two.

Then x has a probability distribution f.

After the rigorous derivation work, the random jump size using inverse cdf method is

$$x = \begin{cases} -\dfrac{\sigma\sqrt{t}}{\sqrt{2}}\ln(1-p), \quad \text{if} \quad b \geq 0 \\ \dfrac{\sigma\sqrt{t}}{\sqrt{2}}\ln\Big[2pC\exp\big(-\dfrac{\sqrt{2}b}{\sigma\sqrt{t}}\big)+1\Big], \quad \text{if } b<0 \text{ and } p < \dfrac{1}{2C}\big(1-\exp\big(\dfrac{\sqrt{2}b}{\sigma\sqrt{t}}\big)\big) \\ -\dfrac{\sigma\sqrt{t}}{\sqrt{2}}\ln\left[\dfrac{2Cp-2+\exp\big(\frac{\sqrt{2}b}{\sigma\sqrt{t}}\big)}{-\exp\big(-\frac{\sqrt{2}b}{\sigma\sqrt{t}}\big)}\right], \quad \text{if } b<0 \text{ and } p \geq \dfrac{1}{2C}\big(1-\exp\big(\dfrac{\sqrt{2}b}{\sigma\sqrt{t}}\big)\big) \end{cases}.$$

where p is a uniformly distributed random number in $[0,1]$.

4.1.2 Simulating using Rejection method

Even though, the inverse cdf method is quite simple to understand, it may be very complicated or sometimes impossible to derive the inverse of a cumulative distribution function. So, the rejection method, which is based on a simple geometry, was introduced. The rejection method does not require one to know the cumulative distribution function.

Let $f(x)$ be the probability density function we wish to generate random numbers from. We choose another probability density function $g(x)$ such that

$$f(x) \leq cg(x) \quad \text{for all} \quad x,$$

for some constant c. We call $g(x)$ the comparison function.

The algorithm appears below [17].

1. Generate a random number x from $g(x)$.

2. Let
$$r = \frac{cg(x)}{f(x)}.$$

3. Generate uniform random number $u \sim U(0,1)$.

4. If $ur < 1$, return x.

5. Otherwise, repeat steps from 1.

The expected number of iterations is c.

Note that the jump size distribution with double negative exponential distribution is (4.2). Then the possible comparison function $g(x)$ and the constant c are as follows.

1. For $b \geq 0$,
$$g(x) = \frac{\sqrt{2}}{\sigma\sqrt{t}} \exp\left(-\frac{\sqrt{2}x}{\sigma\sqrt{t}}\right),$$

$$c = \frac{1}{0.5\exp\left(-\frac{\sqrt{2}b}{\sigma\sqrt{t}}\right)}.$$

2. For $b < 0$,

$$g(x) = \frac{1}{d}\exp\left(-\frac{x}{d}\right),$$

$$c = \exp\left(-\frac{b}{d}\right),$$

where

$$d = \sigma\sqrt{2t}\Big(1 - 0.5\exp\left(\frac{\sqrt{2}b}{\sigma\sqrt{t}}\right)\Big).$$

4.1.3 Measure change

Since the arrival rate of jumps in $(t, t + dt)$ is

$$\frac{a_t(b)dt}{\psi_t(b) - b},$$

we may have a very small number of jumps if dt is very small. In that case, we can use change of measure to make jumps occur more often. Before we derive measure change of Skorohod embedding process, we introduce general definitions and theorems about measure change. The reference about measure change is in [29].

Definition 4.1.1. Let X, Y be semimartingales. The quadratic variation process of X, denoted $[X, X] = ([X, X]_t)_{t\geq 0}$, is defined by:

$$[X, X] = X^2 - 2\int X_- dX.$$

The quadratic covariation of X, Y, also called the bracket process of X, Y, is defined by:

$$[X, Y] = XY - \int X_{-} dY - \int Y_{-} dX.$$

It is clear that the operation $(X, Y) \to [X, Y]$ is bilinear and symmetric.

Theorem 4.1.1. [29] *Let X be a semimartingale with $X_0 = 0$. Then there exists a (unique) semimartingale Z that satisfies the equation:*

$$Z_t = 1 + \int_0^t Z_{s-} dX_s.$$

And Z is given by

$$Z_t = \exp\Big(X_t - \frac{1}{2}[X, X]_t\Big) \prod_{0<s\leq t} (1 + \triangle X_s) \exp\Big(- \triangle X_s + \frac{1}{2}(\triangle X_s)^2\Big) \tag{4.3}$$

where the infinite product converges and $\triangle X_s = X_s - X_{s-}$, jump of the process X at time s.

Definition 4.1.2. For a semimartingale X with $X_0 = 0$, the stochastic exponential of X, written $\mathcal{E}(X)$, is the (unique) semimartingale Z that is a solution of

$$Z_t = 1 + \int_0^t Z_{s-} dX_s.$$

The stochastic exponential is also known as the Doléans-Dade exponential.

Simply, if X is a continuous semimartingale with $X_0 = 0$, then

$$\mathcal{E}(X)_t = \exp\big(X_t - \frac{1}{2}[X, X]_t\big).$$

Now, we are ready to derive measure change (in other words, Radon-Nikodym derivative) in the Skorohod embedding case.

Let the true compensator ν_P be

$$\nu_P(dx,dt) = \frac{a_t(b)}{\psi_t(b) - b} \frac{g(b+x,t)}{\int\limits_0^\infty g(b+x,t)dx},$$

Define the reference compensator, ν_R by

$$\nu_P(dx,dt) = y(t)\nu_R(dx,dt).$$

Assume $\nu_R(dx,dt)$ has fixed arrival rate λ with jump size distribution (4.2). Then

$$\nu_R(dx,dt) = \lambda \frac{g(b+x,t)}{\int\limits_0^\infty g(b+x,t)dx}.$$

Let

$$\theta_t = \frac{a_t(b)}{\psi_t(b) - b}.$$

Then

$$y(t) = \frac{\theta_t}{\lambda}.$$

By the equivalence of measure,

$$E_P[\frac{dR}{dP}F(X_s, s \geq t)] = E_R\big[F(X_s, s \geq t)\big].$$

with stochastic exponential

$$\frac{dR}{dP} = \mathcal{E}\big((y(t) - 1) * (\rho - \nu_P)\big).$$

where ρ is the integer valued random measure associated with the jumps of a process with compensator ν_P. So, in this case, the stochastic exponential is

$$
\begin{aligned}
\frac{dR}{dP} &= \mathcal{E}\big((y(s)-1)*(\rho-\nu_P)\big) \\
&= \mathcal{E}\Big(\int_0^t\int_{-\infty}^{\infty}(\frac{\theta_s}{\lambda}-1\big)(\rho(dx,dt)-\nu_P(dx,dt))dxds\Big) \\
&= \mathcal{E}\Big(\int_0^t\int_{-\infty}^{\infty}(\frac{\theta_s}{\lambda}-1\big)\rho(dx,dt)dxds-\int_0^t\int_{-\infty}^{\infty}(\frac{\theta_s}{\lambda}-1\big)y(s)\nu_R(dx,dt)dxds\Big) \\
&= \mathcal{E}\Big(\sum_{s\le t}(\frac{\theta_s}{\lambda}-1\big)\Delta N_s-\int_0^t(\frac{\theta_s}{\lambda}-1)y(s)\int_{-\infty}^{\infty}\nu_R(dx,dt)dxds\Big) \\
&= \mathcal{E}\Big(\sum_{s\le t}(\frac{\theta_s}{\lambda}-1\big)\Delta N_s-\int_0^t(\frac{\theta_s}{\lambda}-1)y(s)\lambda ds\Big) \\
&= \mathcal{E}\Big(\sum_{s\le t}(\frac{\theta_s}{\lambda}-1\big)\Delta N_s-\int_0^t(\frac{\theta_s}{\lambda}-1)\theta_s ds\Big) \\
&= \prod_{s\le t}\Big((\frac{\theta_s}{\lambda}-1\big)\Delta N_s+1\Big)\exp\big(-\int_0^t(\frac{\theta_s^2}{\lambda}-\theta_s)ds\big),
\end{aligned}
$$

where

$$
\Delta N_s = \begin{cases} 1 & \text{if jump occurs at time } \ s \\ 0 & \text{if no jump occurs at time } \ s \end{cases} .
$$

4.2 Inhomogeneous independent increment processes

To simulate this case, first we get the arrival rate of jumps and distribution of jump size. Let $L(a)$ be the local time at zero of a Brownian motion up to the first passage time of this Brownian motion to the level a. Then, it is well known that $L(a)$ is an exponential random variable with mean $2a$ [30]. So, the Laplace transform of $L(a)$ is

$$
\begin{aligned}
\mathcal{L}(L(a)) &= E\left[e^{-\lambda L(a)}\right] \\
&= \frac{1}{1+2a\lambda}.
\end{aligned} \tag{4.4}
$$

The characteristic function for an independent Brownian motion at $L(a)$ is

$$
E\left[\exp(iuB(L(a)))\right] = \frac{1}{1+au^2}.
$$

With comparing characteristic function of the double negative exponential density, we see

$$
X(t) = B(L(\sigma^2 t)). \tag{4.5}
$$

To simulate (4.5), we have to know what kind of process $L(\sigma^2 t)$ is.

Without loss of generality, we consider the process $L(t)$ instead of $L(\sigma^2 t)$.

4.2.1 Simulation using Inverse Laplace Transformation

First, we simulate the inhomogeneous independent increment process using inverse Laplace transformations.

Let's consider discrete times between zero and one, $0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = 1$, where $n > 1$ is an integer. Let $t_{i-1} = a$ and $t_i = b$, $i = 1, 2, ..., n$.

Since $L(t)$ is increasing, let

$$
L(b) = L(a) + x,
$$

where x is an independent increment.

We take Laplace transform. Then

$$\mathcal{L}(L(b)) = \mathcal{L}(L(a))\mathcal{L}(x).$$

So, by (4.4)

$$\begin{aligned}
\mathcal{L}(x) &= \frac{\mathcal{L}(L(b))}{\mathcal{L}(L(a))} \\
&= \frac{1+2a\lambda}{1+2b\lambda}.
\end{aligned}$$

So, the inverse Laplace transform of $\mathcal{L}(x)$ is

$$\mathcal{L}^{-1}(s) = \frac{a}{b}\delta(s) + \frac{b-a}{2b^2}\exp\bigl(-\frac{1}{2b}s\bigr). \tag{4.6}$$

where $\delta(s)$ is Dirac delta.

It can be easily checked that (4.6) is a probability density function. So the cumulative distribution function is

$$y = 1 - \frac{b-a}{b}\exp\bigl(-\frac{1}{2b}x\bigr).$$

We can use inverse CDF method to simulate this case.

4.2.2 Simulation using Lévy measure

Now we simulate inhomogeneous independent increment process using Lévy measure. So, time is continuous for this case.

Using the Lévy measure $k_u(x)$ and (2.4), we can rewrite Laplace transformation of $L(t)$ as

$$\frac{1}{1+2t\lambda} = \exp\Big(\int_0^t \int_0^\infty \big(e^{-\lambda x} - 1\big) k_u(x) dx du\Big).$$

Taking the logarithm and differentiating with respect to t and λ, we see that

$$\frac{2}{(1+2t\lambda)^2} = \int_0^\infty x \exp(-\lambda x) k_t(x) dx.$$

To get $k_t(x)$, we use gamma distribution. Note that

$$\int_0^\infty \frac{c^\gamma x^{\gamma-1} e^{-cx}}{\Gamma(\gamma)} dx = 1.$$

Let $\gamma = 2$, and $c = 1 + 2t\gamma$.

Then we get the Lévy measure after we change t to $t\sigma^2$,

$$k_t(x) = \frac{2}{(2t\sigma^2)^2} \frac{e^{-\frac{1}{2t\sigma^2}x}}{\Gamma(2)}. \tag{4.7}$$

So, the rate of jumps occurred at time t is

$$\int_0^\infty k_t(x) dx = \frac{1}{\sigma^2 t \Gamma(2)}, \tag{4.8}$$

and the distribution of jump size is

$$\frac{k_t(x)}{\int_0^\infty k_t(x) dx} = \frac{1}{2t\sigma^2} e^{-\frac{1}{2t\sigma^2}x}. \tag{4.9}$$

4.2.3 Simulating Nonstationary Poisson Processes

As we find from the Lévy measure and rate of occurance of jumps, this process $L(t)$ is a nonstationary Poisson process. To simulate this, we follow the method in Law and Kelton, Chapter 8 [21].

Let $\lambda(t)$ be a rate function at time t. Then the expectation function

$$\Lambda(t) = \int_0^t \lambda(y)dy$$

is always a continuous function of t and $\Lambda(t)$ is the expected number of arrivals between time 0 and time t. Using (4.8),

$$\Lambda(t) = \frac{1}{\Gamma(2)\sigma^2}\big(\ln(t) - \ln(t_1)\big),$$

where t_1 is very close to 0. We can use t_1 close enough to 0 since the jump size density is negligible when t is very close to 0 as we see in (4.9).

Now, we use the following recursive algorithm:

1. Generate a Poisson arrival time $t_0^{'}$ at rate 1.
2. Set $t_0 = \Lambda^{-1}(t_0^{'})$.
3. For $i \geq 1$, generate $U \sim U(0,1)$.
4. Set $t_i^{'} = t_{i-1}^{'} - \ln U$.
5. Return $t_i = \Lambda^{-1}(t_i^{'})$.

4.3 Continuous martingales

Again, Dupire's approach for a continuous Markov martingale is

$$X(t) = \int_0^t \sigma(X(s), s)dB(s). \tag{4.10}$$

Then (4.10) is equivalent to

$$dX(t) = \sigma(X(t), t)dB(t). \tag{4.11}$$

We use Milstein's higher order method [18] to simulate this case. Let us consider the stochastic differential equation

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t).$$

Then Milstein's method defines $\{Y_n\}_{n=0}^N$ by

$$Y_0 = X(0),$$

$$Y_{n+1} = Y_n + \mu(Y_n, t_n)h_n + \sigma(Y_n, t_n)\triangle B_n + \frac{1}{2}\sigma(Y_n, t_n)\sigma'(Y_n, t_n)((\triangle B_n)^2 - h_n),$$

where

$$h_n = t_{n+1} - t_n, \quad \triangle B_n = B(t_{n+1}) - B(t_n).$$

Furthermore, it converges with strong order h:

$$E[|Y_n - X(t_n)|] \leq Ch,$$

where $C \in \mathbb{R}$.

For continuous martingale case, $\mu = 0$. By (3.4), we determine the function $a(y)$ corresponding to the double negative exponential distribution (4.1). Then

$$a(y) = \frac{\sqrt{2}}{2}\sigma\big(|y| + \frac{\sigma}{\sqrt{2}}\big).$$

Then by (3.5),

$$\begin{aligned} \sigma(x, s) &= \sqrt{\frac{\sqrt{2}}{2}\sigma\Big(\frac{|x|}{\sqrt{s}} + \frac{\sigma}{\sqrt{2}}\Big)} \\ &= \sqrt{\frac{\sigma^2}{2} + \sigma\frac{|x|}{\sqrt{2s}}}. \end{aligned}$$

4.4 Hypothesis test for matching marginals

In this section, we show that the marginal distributions obtained by simulations match the prespecified marginal distributions, in this case, double negative exponential distributions, at any points. One of the most widely used for tests of goodness of fit is the χ^2 test. The advantage of using χ^2 is that we can apply this test to discrete distributions, such as binomial or Poisson. But for the continuous data, the value of the chi-square test statistic depends on how the data are binned. Another disadvantage of the chi-square test is that it requires a large sample size in order for the chi-square approximation to be valid (see [38] chapter 1.3.5).

An alternative way to test goodness of fit is the Kolmogorov-Smirnov test (K-S test). An attractive feature of this test is that the distribution of the K-S test statistic itself does not depend on the underlying cumulative distribution function being tested as long as it is continuous. We use the Kolmogorov-Smirnov test in this dissertation. The one-sample test is following.

Let $x_1, x_2, ...x_n$ be observations on continuous i.i.d. random variables, $X_1, X_2, ..., X_n$ with a c.d.f. F. We want to test the hypothesis

$$H_0 : F(x) = F_0(x), \quad \text{for all } x,$$

where F_0 is a known c.d.f.

The Kolmogorov-Smirnov test statistic D_n is defined by

$$D_n = \sup_{x \in \mathbb{R}} |\hat{F}(x) - F_0(x)|,$$

where $\hat{F}$ is the empirical cumulative distribution function defined as

$$\hat{F}(x) = \frac{\#(i : x_i \leq x)}{n}.$$

We extend the one-sample test to a two-sample test.

Let

$$x_1, x_2, ...x_m \text{ be observations on i.i.d. rvs } X_1, X_2, ..., X_m \text{ with CDF } \quad F_1,$$

$$y_1, y_2, ...y_m \text{ be observations on i.i.d. rvs } Y_1, Y_2, ..., Y_m \text{ with CDF } \quad F_2.$$

The null and alternative hypotheses for this test are

$$H_0 : F_1(x) = F_2(x), \quad \text{for all } x,$$

$$H_1 : F_1(x) \neq F_2(x).$$

Then the Kolmogorov-Smirnov test statistic is

$$D_{m,n} = \sup_t |\hat{F}_1(t) - \hat{F}_2(t)|,$$

where $\hat{F}_1$ and $\hat{F}_2$ are empirical CDF of $X's$ and $Y's$, respectively.

In the interest of brevity, we will refer to the Azéma and Yor solution to the Skorohod embedding problem, the inhomogeneous Markov martingale processes with independent increments using subordinated Brownian motion, and the continuous martingale using Dupire's method as the Azéma and Yor process, the Sato process, the Dupire process, repectively.

Figures 4.1 to 4.3 show the paths of the simulated processes, sample histogram of marginal density at time 1 by simulation, sample histogram of double negative exponential density at time 1 and cumulative distributions of two histograms, respectively. Tables 4.1 to 4.3 present the Kolmogorov-Smirnov test between that processes which were simulated and double negative exponential distribution at any time points. Seven time points were chosen to check hypothesis test, p-value, and Kolmogorov-Smirnov distance.

As we can see in the table, the null hypothesis is accepted for every case.

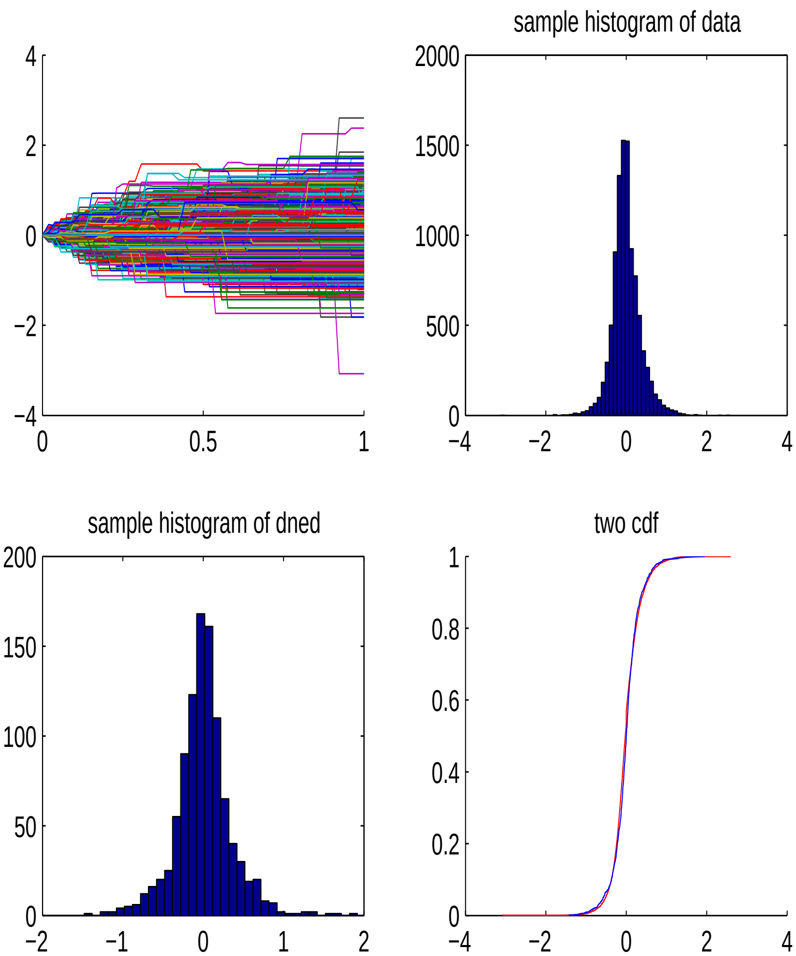


Figure 4.1: CDF match for Azéma and Yor process.

Table 4.1: Kolmogorov-Smirnov test for the Azéma and Yor process.

	1day	1week	1month	3months	6months	9months	12months
H	0	0	0	0	0	0	0
p_value	0.2197	0.2399	0.7953	0.5802	0.6972	0.3693	0.2582
KS distance	0.0347	0.0340	0.0214	0.0257	0.0234	0.0303	0.0334

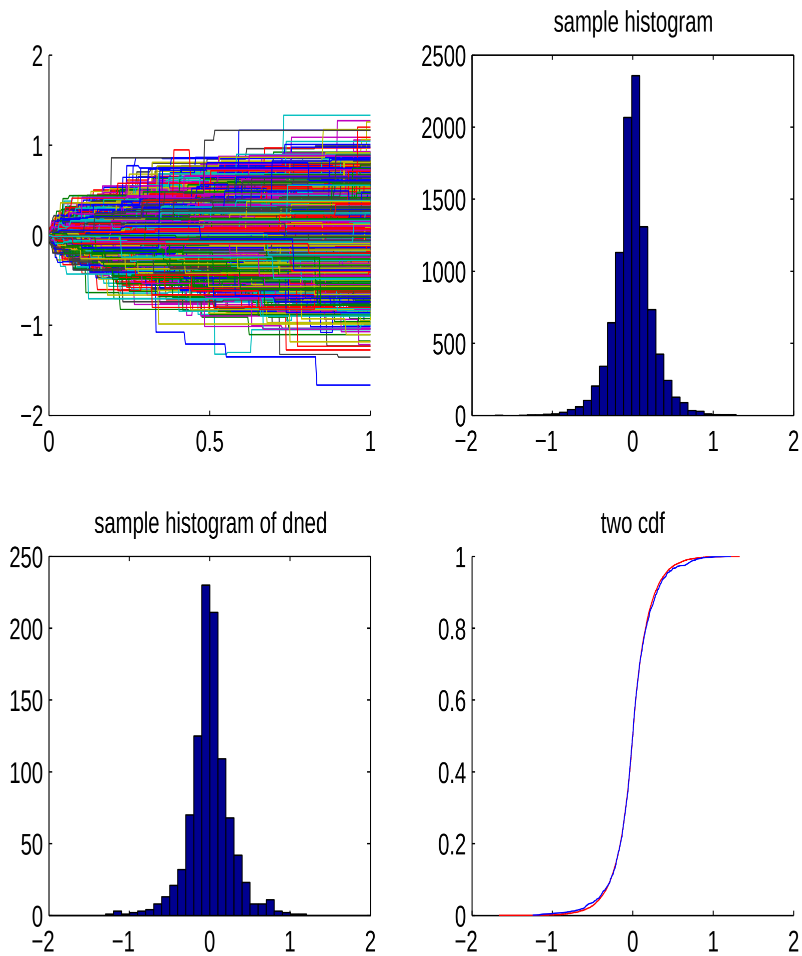


Figure 4.2: CDF match for Sato process.

Table 4.2: Kolmogorov-Smirnov test for the Sato process.

	1day	1week	1month	3months	6months	9months	12months
H	0	0	0	0	0	0	0
p_value	0.5602	0.5953	0.7953	0.7022	0.4828	0.7953	0.9390
KS distance	0.0261	0.0254	0.0214	0.0233	0.0277	0.0214	0.0176

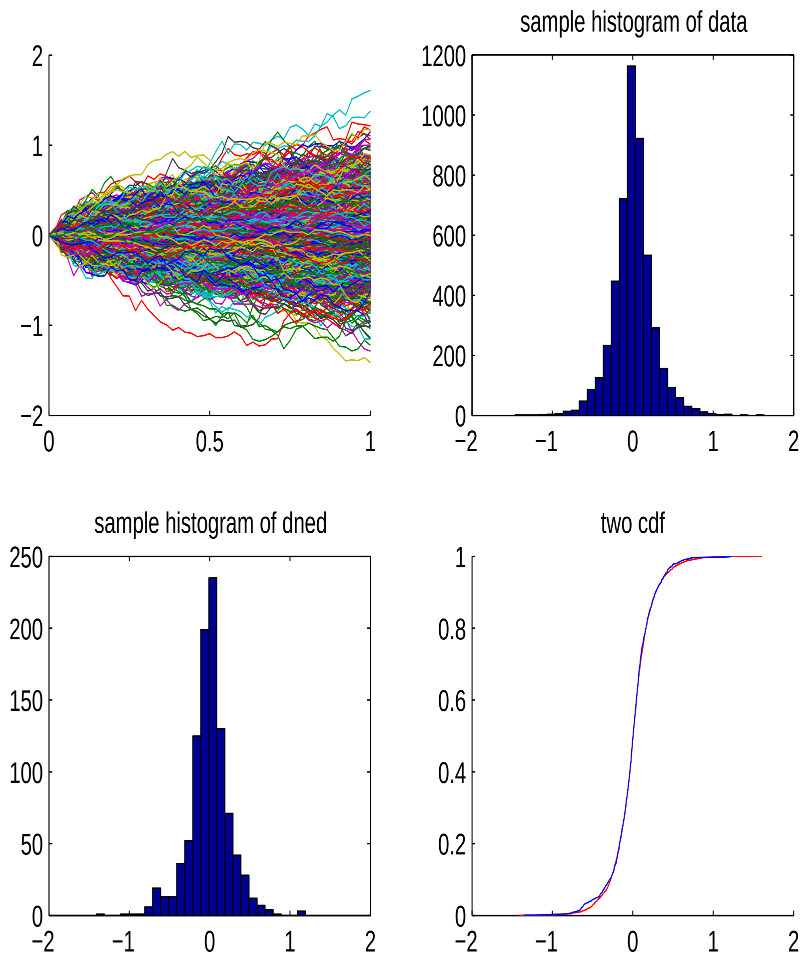


Figure 4.3: CDF match for Dupire process.

Table 4.3: Kolmogorov-Smirnov test for the Dupire process.

	1day	1week	1month	3months	6months	9months	12months
H	0	0	0	0	0	0	0
p_value	0.0675	0.5113	0.7123	0.7373	0.6666	0.6156	0.9090
KS distance	0.0430	0.0271	0.0231	0.0226	0.0240	0.0250	0.0186

Chapter 5

Pricing exotic options

This chapter is a comparative investigation of the resulting stochastic processes as exemplified through the pricing of exotic options. For this study, we assume that all spot return distributions are double negative exponential distributions at any continuous time and consider the unit time volatility σ to be fixed at 0.25.

First, we study the forward return distribution to determine if it is the double negative exponential distribution used as the spot return distribution in this study. We also study the implied volatility curves for forward return distribution for all three stochastic processes.

Second, we obtain swap rates to study the first passage times of certain barriers. Here, we consider swap rate for the credit default swap with up and in barriers and down and in barriers.

Third, we consider the prices of locally floored and capped and globally floored and capped cliquets on monthly resets.

Lastly, we consider options on the realized quadratic variation. For this study, we use daily increments of spot return distributions.

The basic concepts will be presented in each section.

5.1 Forward return

Now, we investigate forward return distributions on three stochastic processes. A forward contract, made at t, is an agreement between two parties, a buyer and a seller, to buy an asset or currency at a later delivery date at a fixed price [12]. The holder of contract pays a deterministic amount and receives a stochastic amount at delivery date. We assume the contract is made at time $t = 0.5$. Nothing is paid or received at time 0.5, although the price of forward contract is determined at this time.

Let X_1 denote the Azéma-Yor process, let X_2 denote the Sato process, and let X_3 denote the Dupire process. Let X_j^x be a random variable generated by the conditional law of $X_j(1) - X_j(0.5)$ given that $X_j(0.5) = x$ for $j = 1, 2, 3$. We note that by construction, the forward return distribution has a zero conditional mean. We measure its distance from the double negative exponential. Clearly, the Sato process is independent of x since it is an independent increment process but the Azéma-Yor process and the Dupire process depend on x.

As we mentioned in the introductory chapter, there are many attempts at building path spaces with condition that the forward return distributions are in the same class of distributions as the spot return distributions. It is true for Lèvy processes since they are stationary independent increment processes. We are investigating whether our paths follow this condition.

For that purpose, we estimate best fitting volatility of double negative exponential distribution to the forward return distribution using the maximum likelihood method. We also investigate the distance between the forward return distribution and the initial spot return distribution that, in this case, is the double negative exponential with volatility $\sigma\sqrt{0.5} = 0.1768$.

Table 5.1 to 5.3 show the Kolmogorov-Smirnov distances between the forward return distribution and the double negative exponential distribution with best fitting volatility and the Kolmogorov-Smirnov distance between the forward return distribution and the spot return distribution with theoretical volatility $\sigma\sqrt{0.5}$. We choose 5 values of the level x at time 0.5 to condition on: $0, \pm\sigma\sqrt{0.5}, \pm 2\sigma\sqrt{0.5}$. The volatilities at time 0.5 are given in the table. Again, we are testing

$$H_0 : \text{distribution of } \ X(1) - x = \text{distribution of } \ X(0.5),$$

$$H_1 : \text{distribution of } \ X(1) - x \neq \text{distribution of } \ X(0.5).$$

As we can see in the table, the null hypothesis is rejected in all cases except Dupire process at $X(0.5) = 0$. Histograms of the forward return density, best fitting double negative exponential density, double negative exponential density with volatility $0.25 \times \sqrt{0.5} = 0.1768$ and cumulative distribution functions of these three histograms are presented in Appendix A.

The conclusion we get from this section is that the attempts at building path spaces with the condition that the forward return distributions are in the same class of distributions as the spot return distribution are not reasonable, in general. The forward return distribution is skewed to the left if x is negative and skewed to the right if x is positive except for the Sato process case.

Additionally, we investigate the departures form the initial spot return in

Table 5.1: Kolmogorov-Smirnov test for the forward motion and double negative exponential on the Azéma and Yor process.

forward motion and best fitting

x	-0.3536	-0.1768	0	0.1768	0.3536
volatility	0.3998	0.2349	0.1273	0.1261	0.1258
H	1	1	1	1	1
p_value	1.2320e-05	0.0012	0.0052	3.1603e-04	3.1235e-05
KS distance	0.0809	0.0636	0.0570	0.0691	0.0777

forward motion and theoretical case

x	-0.3536	-0.1768	0	0.1768	0.3536
H	1	1	1	1	1
p_value	5.4930e-37	2.7234e-08	7.8386e-05	1.9144e-05	1.8846e-06
KS distance	0.2143	0.0994	0.0744	0.0794	0.0870

terms of double negative exponential implied volatility curves. If the forward return distribution is in the same double negative exponential family as the initial spot return density, then its implied volatility curve should be flat.

We consider $K = x + u$ for a range of strikes with $u \in [-2\sigma\sqrt{0.5}, 2\sigma\sqrt{0.5}]$ and assume the interest rate r is equal to 0. Then the value of the European option price with strike K with maturity 0.5 is

Table 5.2: Kolmogorov-Smirnov test for the forward motion and double negative exponential on the Sato process.

forward motion and best fitting

x	-0.3536	-0.1768	0	0.1768	0.3536
volatility	0.1131	0.1131	0.1131	0.1131	0.1131
H	1	1	1	1	1
p_value	5.9277e-56	5.9277e-56	5.9277e-56	5.9277e-56	5.9277e-56
KS distance	0.2641	0.2641	0.2641	0.2641	0.2641

forward motion and theoretical case

x	-0.3536	-0.1768	0	0.1768	0.3536
H	1	1	1	1	1
p_value	4.8466e-62	4.8466e-62	4.8466e-62	4.8466e-62	4.8466e-62
KS distance	0.2782	0.2782	0.2782	0.2782	0.2782

$$
\begin{aligned}
E[(Y-K)^+] &= \int_K^\infty (y-K)g(y)dy \\
&= \begin{cases} \dfrac{\sigma}{4}\exp\left(-\frac{2}{\sigma}K\right), & \text{if} \quad K \geq 0 \\[2ex] \dfrac{\sigma}{4}\exp\left(\frac{2}{\sigma}K\right) - K, & \text{if } K < 0 \end{cases}
\end{aligned}
$$

where the distribution of the random variable Y is double negative exponential. This comments on the departures that can occur in forward return implied volatility curves from the initial spot implied volatility curve. The graphs in Appendix B

Table 5.3: Kolmogorov-Smirnov test for the forward motion and double negative exponential on the Dupire process.

forward motion and best fitting

x	-0.3536	-0.1768	0	0.1768	0.3536
volatility	0.2519	0.2033	0.1580	0.2031	0.2518
H	1	1	0	1	1
p_value	4.3226e-07	1.6905e-08	0.3939	3.5117e-08	1.0213e-06
KS distance	0.0915	0.1007	0.0297	0.0987	0.0889

forward motion and theoretical case

x	-0.3536	-0.1768	0	0.1768	0.3536
H	1	1	0	1	1
p_value	1.7707e-16	5.2727e-12	0.2945	2.6401e-11	6.1144e-19
KS distance	0.1420	0.1206	0.0323	0.1169	0.1525

present the call price curves and implied volatility curves for forward return at 5 different x values for the three stochastic processes. As we see in the graphs, we also get the same result that the forward return distribution is not in the same class of the spot return distribution.

5.2 Swaprate

Along with basic instruments such as options, forwards, and futures, in derivative markets, one of the most popular contracts is the swap. A swap is a derivative contract in which two parties agree to exchange cash flows calculated according to different formulas [12]. One of the most popular swaps is the interest rate swap: fixed interest rate payments are exchanged for a stream of floating interest payments. Some other examples of swaps are commodity swap, currency swap, credit default swap, variance swap, etc. The swap rate is the fixed rate that a swap dealer will pay or receive on a swap. The notional is the fixed amount that a swap dealer will pay or receive on a swap.

In this section, we price credit default swaps. The credit default swap means that one party makes regular payments to the counterparty but receives nothing except in the event of a default on some other contract. The event of this swap is to hit the barrier. We consider the swap which pays coupons monthly and receives notional if the process hits the barrier.

Of interest to the prices of barrier options and the prices of swaps associated with first passage to levels is the distribution of the running supremum or infimum of the process. Let τ denote the first passage time of the process X to the barrier B. Then

$$P(U_t \le B) = P(\tau > t),$$

where $U_t = \sup_{0\le s\le t} X_s$.

Let r be the annual interest rate continuously compounded, c the fixed coupon payment, N the notional, and T the settlement date. Let $t_1, t_2, ..., t_{12}$ be the monthly coupon payment dates, i.e., $t_i = \frac{i}{12}$ where $i = 1, 2, ..., 12$. Assume the first passage time τ is in the interval $[t_{i-1}, t_i]$ where $t_i \le T$. Then the present values of the two cash flows are

$$
\begin{aligned}
V(1) &= \sum_{j=1}^{i-1} \frac{CN}{12} e^{-rt_j} + CN(\tau - t_{i-1}) e^{-r\tau} && (5.1) \\
V(2) &= N e^{-r\tau} && (5.2)
\end{aligned}
$$

The last term in (5.1) reflects the accrued interest, i.e., accumulated since the last interest payment up to but not including the settlement date. If the pathes don't hit the barrier, then the present values of two cash flows are

$$
\begin{aligned}
V(1) &= \sum_{j=1}^{T} \frac{CN}{12} e^{-rT} && (5.3) \\
V(2) &= 0. && (5.4)
\end{aligned}
$$

Then the probability of $V(1)$ is equal to the probability of $V(2)$.

After we investigate the swaprate on three processes, we conclude this section by saying that the swap prices are higher on the continuous process than on the jump processes. The graphs are presented in Appendix C.

The following graphs (5.1) show the average of the first passage time of each barrier. We can see that the paths of the Dupire process hits the barrier faster than the other two processes. The reason is that the two jump processes have the features: if time is close to zero, there are a lot of jumps but jump sizes are very small, and if time is increasing, there are very small number of jumps occurred.

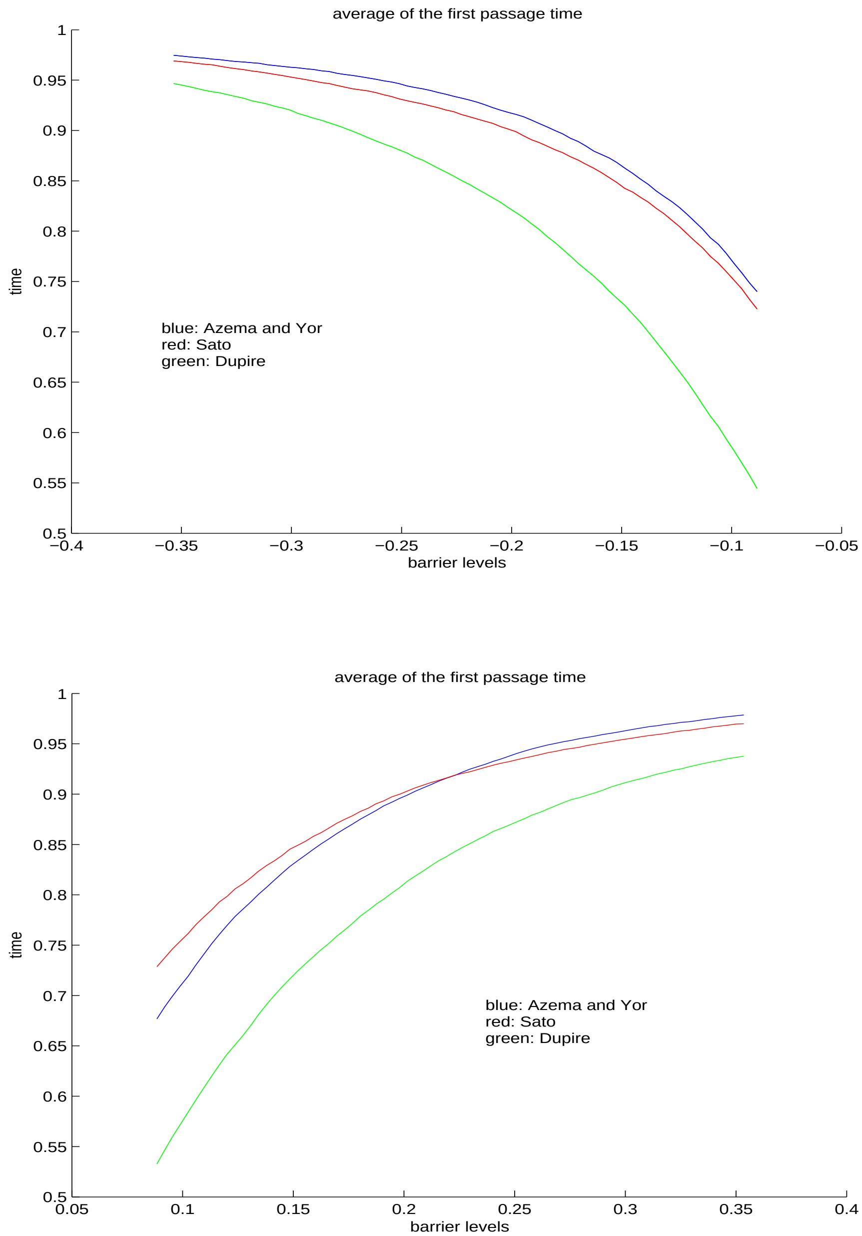


Figure 5.1: Average of the first passage time of the barrier.

5.3 Cliquets

Cliquet options are financial derivative contracts that provide a minimum return in exchange for capping the maximum return. The cap is an option transaction in which a party borrowing at a floating rate pays a premium to another party, which reimburses the borrower in the event that the borrower's interest costs exceed a certain level, thus making the effective interest paid on a floating rate loan have a cap or maximum amount [12]. The floor is same as cap except lender's interest rate are below a certain level to make it effective for lender. These options are attractive since they can protect against loss using the floor, even though they have to cap their return. Cliquet options periodically reset the strikes at the spot. So, it is a series of at-the-money options, but the price is determined in advance.

We consider the prices of locally floored and capped and globally floored and capped cliquets on monthly resets. Define

$$Z_i = X_{\frac{i}{12}} - X_{\frac{i-1}{12}}$$

for $i = 1, 2, ..., 12$. Let a, b be local floors and caps and let A, B be the global floors and caps. Then the cash flow of the locally floored and capped and globally floored and capped arithmetic cliquet is

$$c(a, b, A, B) = notional \times \Big[\Big(\sum_{i=1}^{12}\big((Z_i \vee a) \wedge b\big) \vee A\Big) \wedge B\Big].$$

In this dissertation, we consider two cases, local caps with global floors and local floors with global caps. Figures 5.2, 5.4, and 5.6 are local caps with global floors and the figures 5.3, 5.5 and 5.7 are local floors with global caps.

To see the model risk, we use spread defined as maximum minus minimum. Figure 5.8 is the spread for locally capped globally floored cliquets and Figure 5.9 is the spread for locally floored globally capped cliquets.

As we see in the figures, the model risk on the locally capped globally floored cliquets is decreasing as the local cap is increasing. And the model risk on the locally floored globally capped cliquets is increasing as the local floor is increasing.

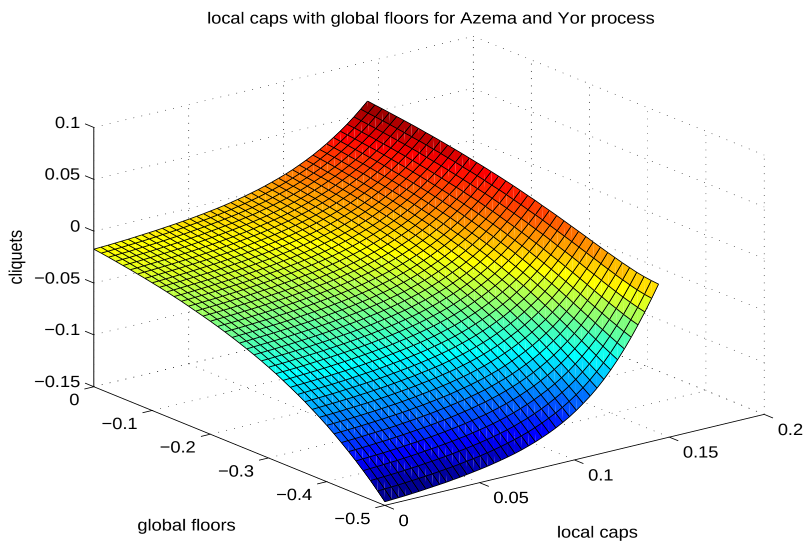


Figure 5.2: Local capped and global floored Cliquets for Azéma and Yor process

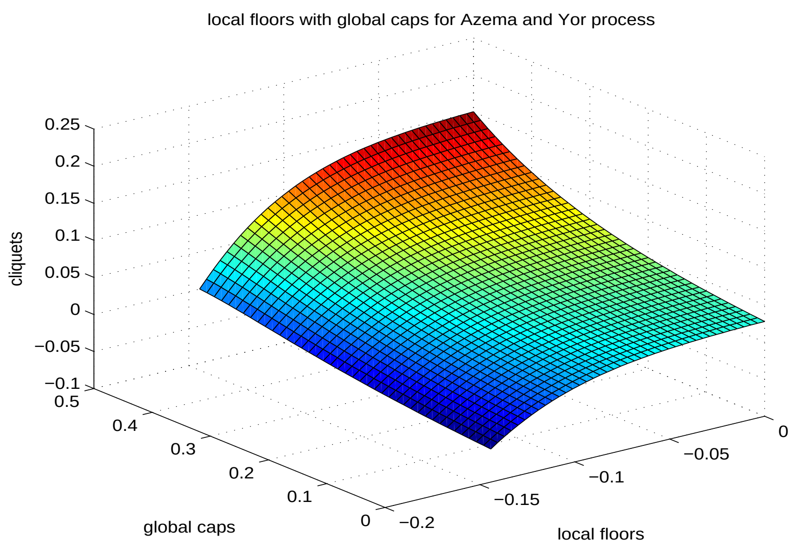


Figure 5.3: Local floored and global capped Cliquets for Azéma and Yor process

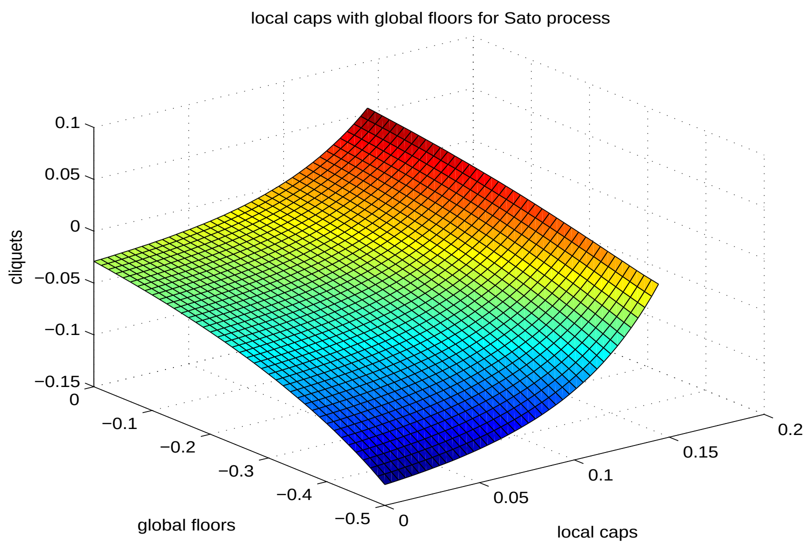


Figure 5.4: Local capped and global floored Cliquets for Sato process

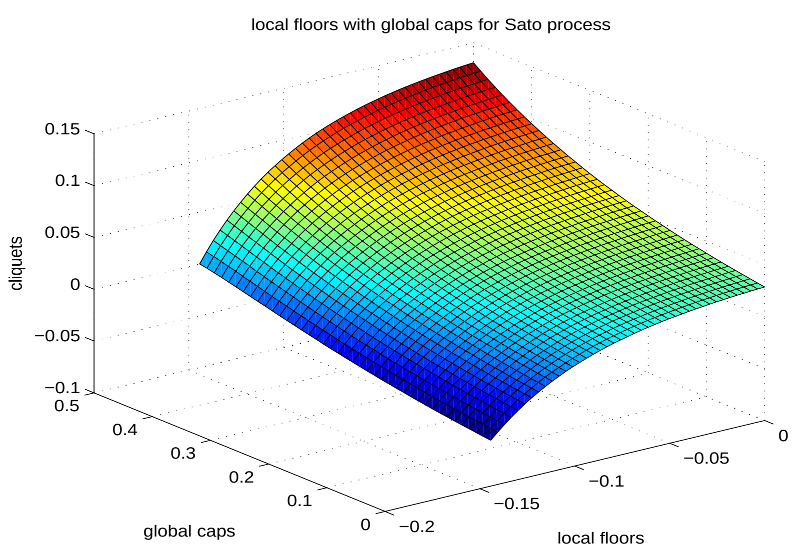


Figure 5.5: Local floored and global capped Cliquets for Sato process

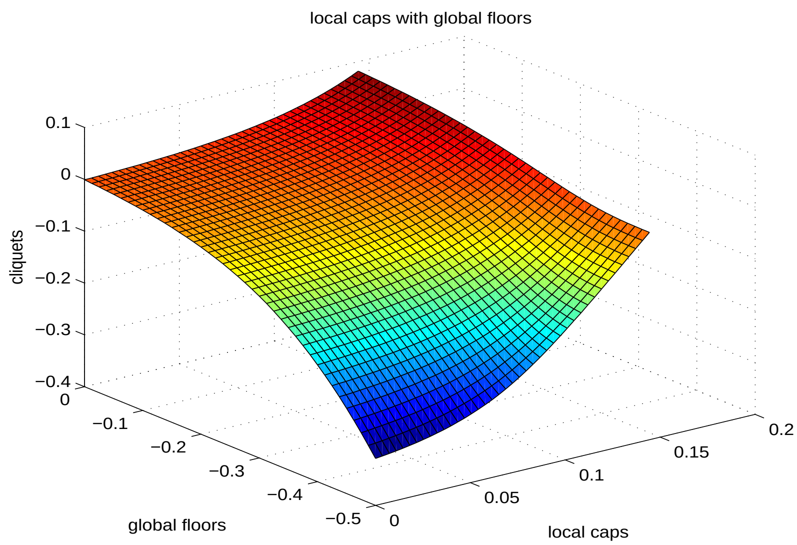


Figure 5.6: Local capped and global floored Cliquets for Dupire process

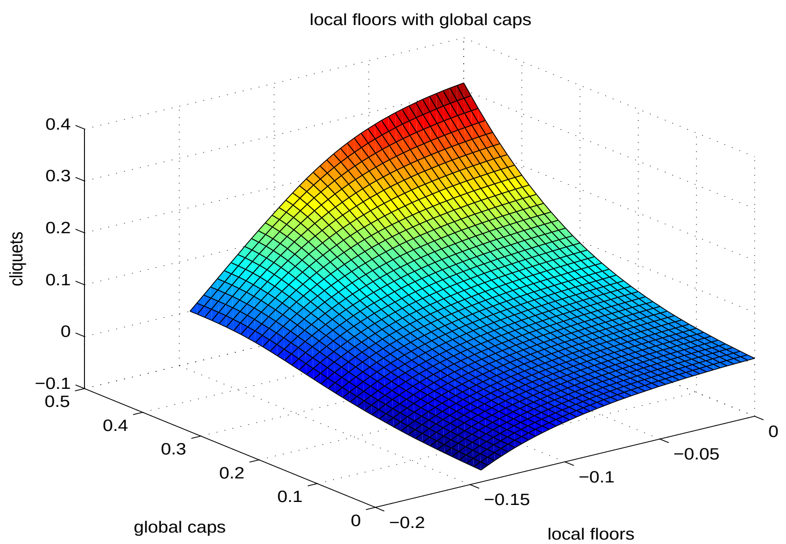


Figure 5.7: Local floored and global capped Cliquets for Dupire process

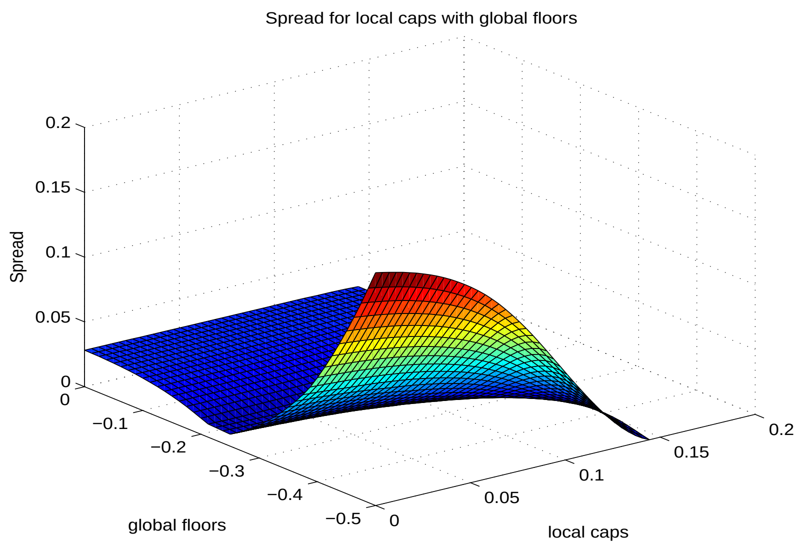


Figure 5.8: Spread for local capped and global floored Cliquets

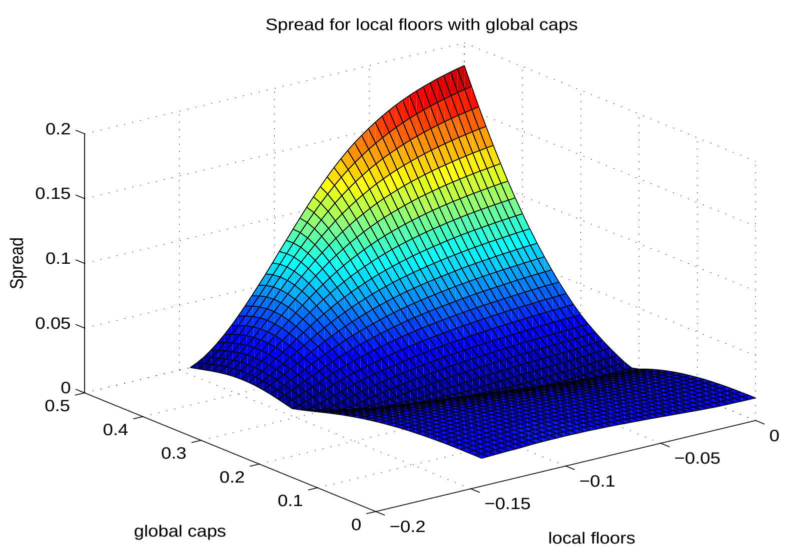


Figure 5.9: Spread for local floored and global capped Cliquets

5.4 Options on the realized quadratic variations

The quadratic variation process of a jump process is defined by

$$Q(t) = \sum_{s \le t} (\Delta X_s)^2.$$

where ΔX_s is the jump of X at time s.

We consider an option on the quadratic variation with strike K and maturity t given by

$$(Q(t) - K)^+.$$

So, the value of this option at time zero is

$$E\big[e^{-rt}(Q(t) - K)^+\big].$$

But continuous processes don't have jumps. So we consider daily increments as

$$R_i = X_{t_i} - X_{t_{i-1}},$$

where $t_i = \frac{i}{252}$, $0 \le i \le 252$.

For this study, we consider the option that pays

$$\begin{aligned} w(K,t) &= E\big[(\sqrt{\frac{252}{N} Q(t)} - K)^+\big] \\ &= E\Big[\Big(\sqrt{\frac{252}{N} \sum_{t_i \le t} (R_i)^2} - K\Big)^+\Big], \end{aligned} \tag{5.5}$$

where N is the number of days in time t and the daily increments are used to get values of option.

Figures 5.10 to 5.12 represent the option values on the realized quadratic variations. It shows that the values of options on the realized quadratic variation are slightly decreasing as maturities are increasing. The data1 is the value of options with maturity 3 months, data2 6 months, data3 9 months and data4 12 months. Figure 5.13 represent the value of options on the realized quadratic variation with the notional $10,000$.

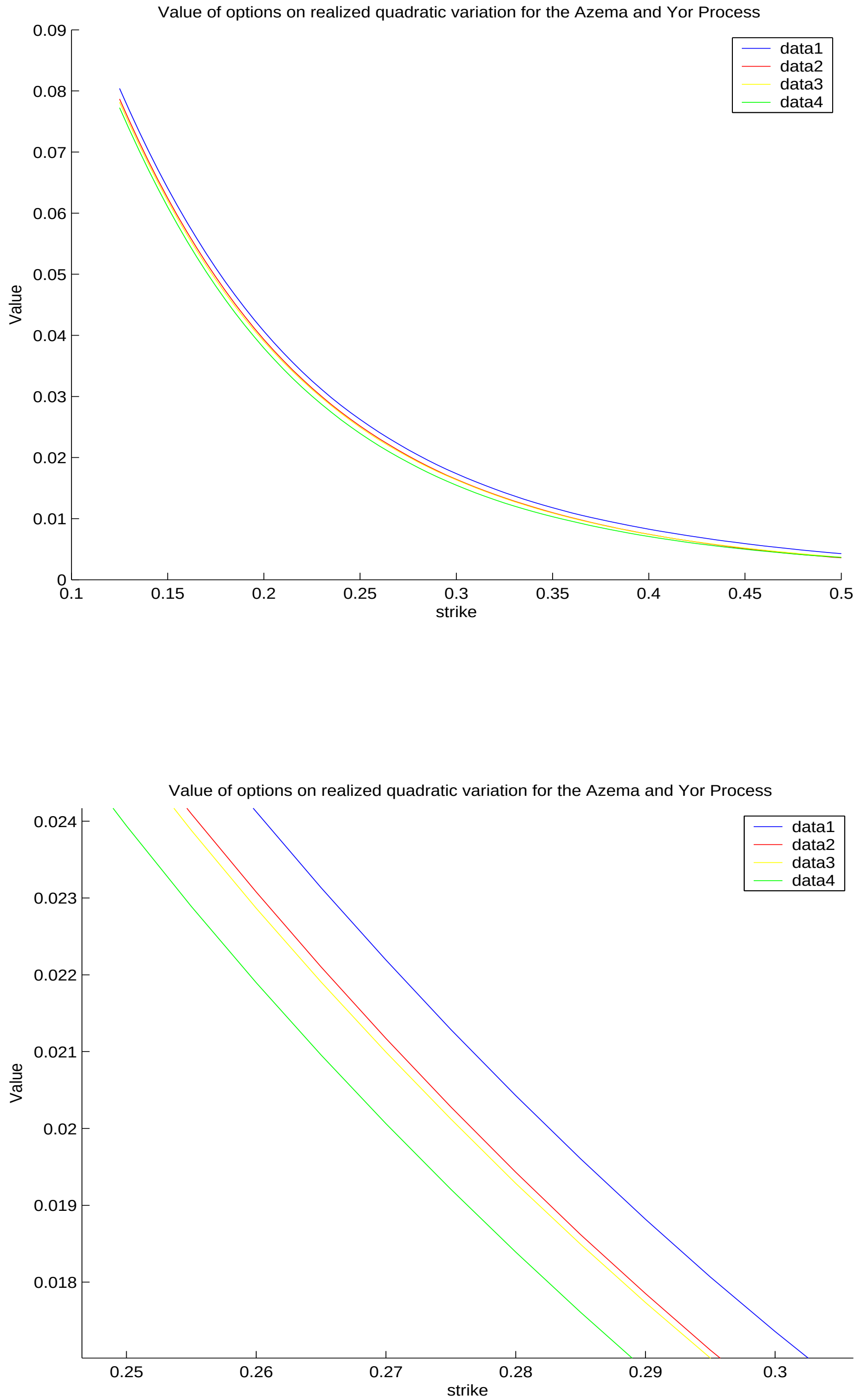


Figure 5.10: Value of options on realized quadratic variation for Azéma and Yor process.

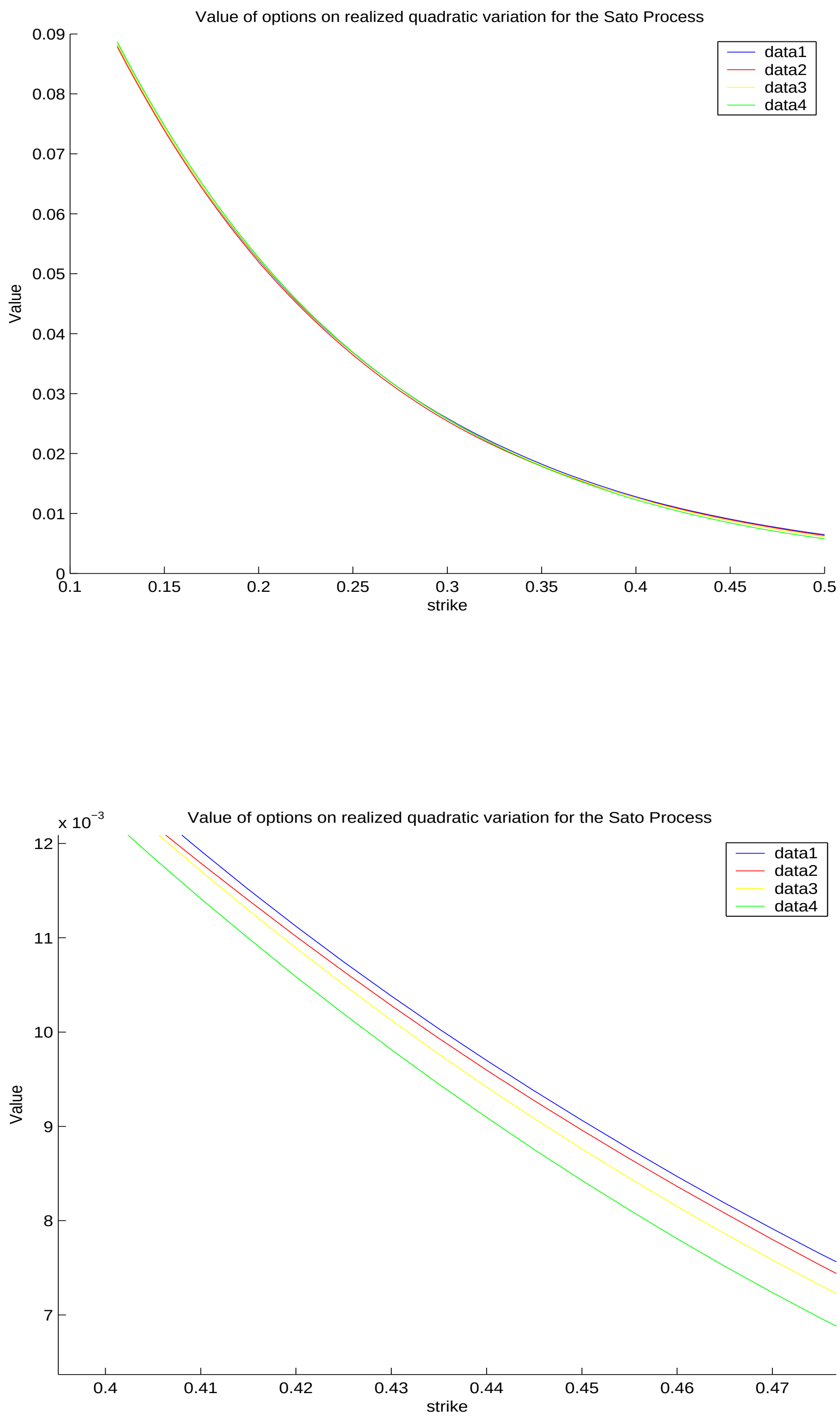
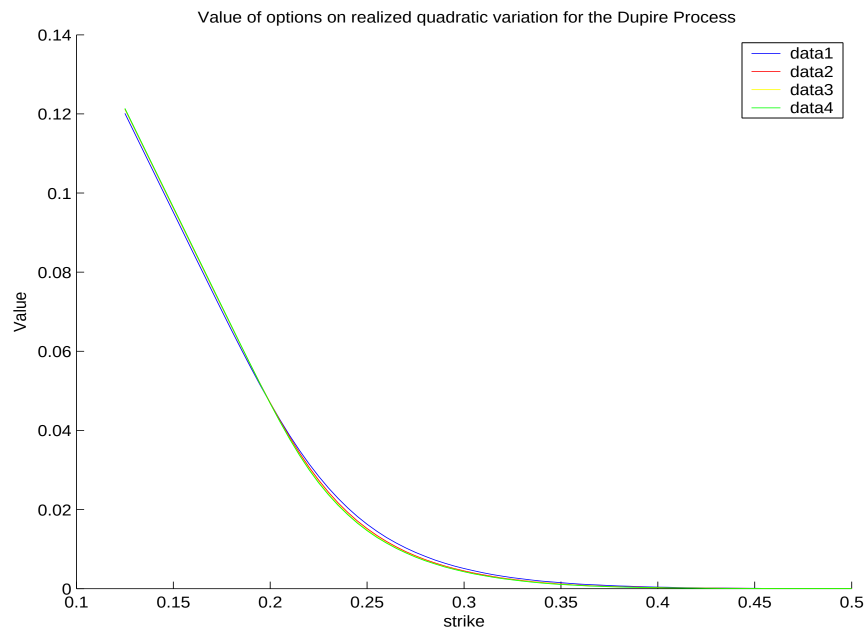


Figure 5.11: Value of options on realized quadratic variation for Sato process.



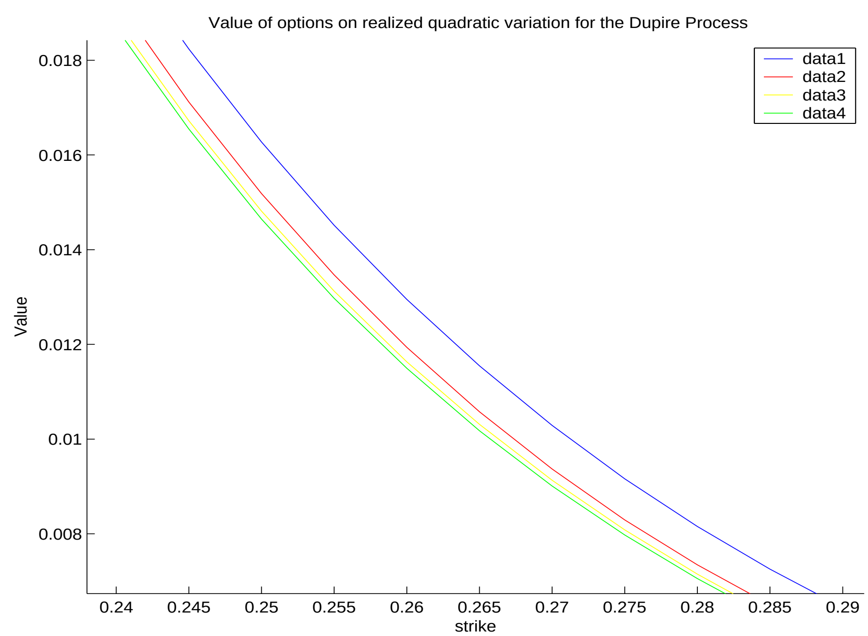


Figure 5.12: Value of options on realized quadratic variation for Dupire process.

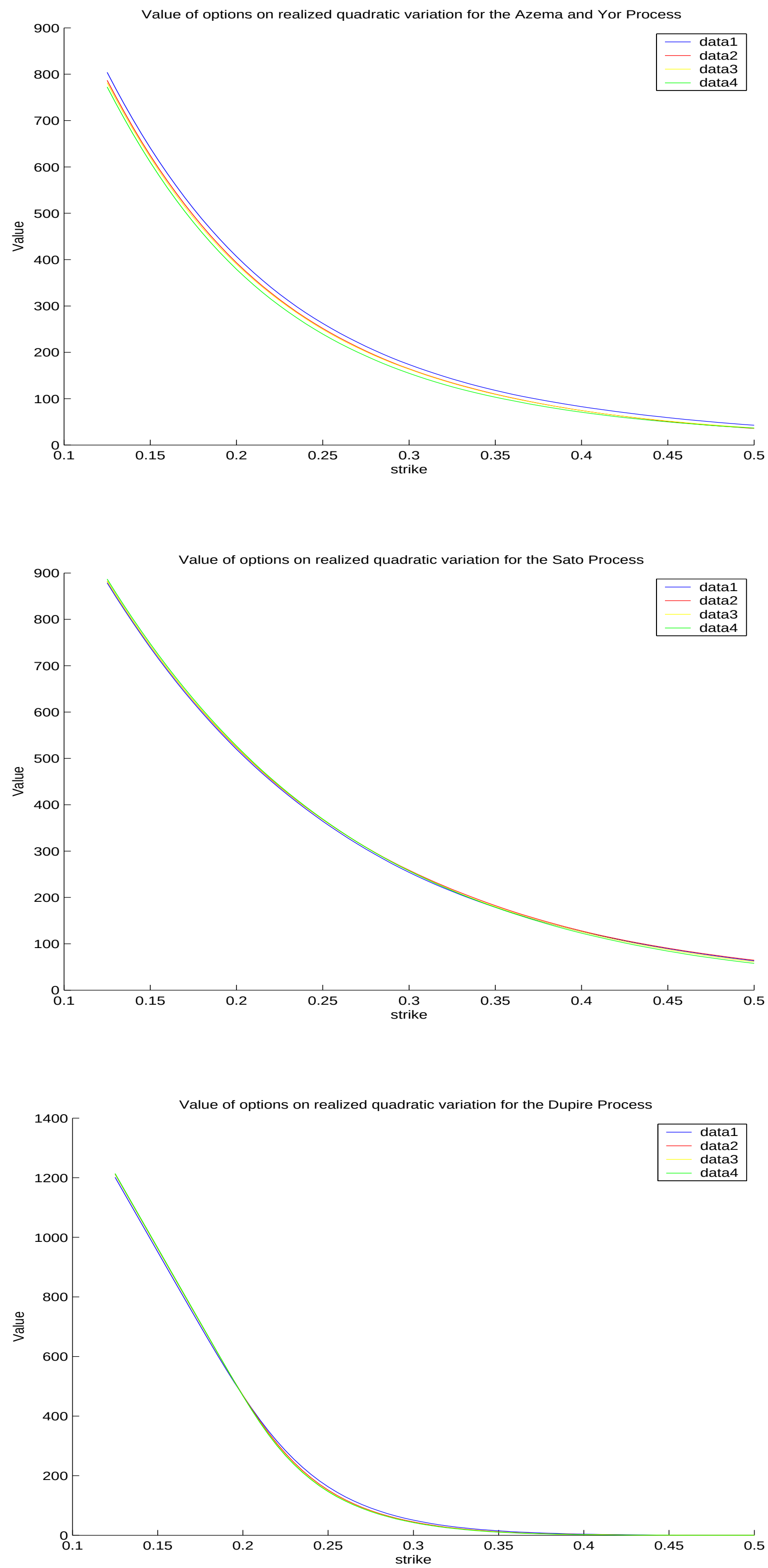


Figure 5.13: Value of options on realized quadratic variation with notional 10,000.

Chapter 6

Conclusion

In this chapter, we summarize this dissertation and provide suggestions for future work.

6.1 Conclusion

We began by introducing three ways to construct Markov martingales that meet prespecified marginal distributions: the Azéma and Yor solution to the Skorohod embedding problem, the inhomogeneous Markov martingale processes with independent increments using subordinated Brownian motion, and the continuous martingale using Dupire's method. We then considered the marginal distributions that have scaling properties.

Next, we investigated the simulation methods for each process, for this study, we used double negative exponential distributions as the prespecified marginals. For the Aźema and Yor solution to the Skorohod embedding problem, we used the inverse cumulative distribution method and the rejection method. By the infinitesimal generator,

$$\frac{a_t(b)dt}{\psi_t(b) - b}$$

is arrival rate of a Poisson jump in the interval $(t, t + dt)$ and the jump size x is drawn from the distribution

$$\frac{g(y,t)}{\int_b^\infty g(y,t)dx},\ y > b.$$

We found that if dt is very small, then jumps occur rarely. In order to increase the number of jumps, we could use measure change from P to R

$$\begin{aligned}\frac{dR}{dP} &= \mathcal{E}\big((y(s)-1)*(\mu-\nu_P)\big)\\ &= \prod_{s\le t}\Big(\big(\frac{\theta_s}{\lambda}-1\big)\Delta N_s+1\Big)\exp\big(-\int_0^t\big(\frac{\theta_s^2}{\lambda}-\theta_s\big)ds\big).\end{aligned}$$

For the inhomogeneous Markov martingale processes with independent increments using subordinated Brownian motion,

$$X(t) = B(L(t)),$$

we used inverse Laplace transformation and a Lévy measure. Let x be the independent increment of $L(t)$. The inverse Laplace transform of $\mathcal{L}(x)$ is

$$\mathcal{L}^{-1}(s) = \frac{a}{b}\delta(s) + \frac{b-a}{2b^2}\exp\big(-\frac{1}{2b}s\big).$$

And, using the Lévy measure, we showed that the rate of jumps occurrence at time t is

$$\int_0^\infty k_t(x)dx = \frac{1}{\sigma^2 t\Gamma(2)},$$

and the distribution of jump size is

$$\frac{k_t(x)}{\int_0^\infty k_t(x)dx} = \frac{1}{2t\sigma^2}e^{-\frac{1}{2t\sigma^2}}.$$

For the continuous martingale using the Dupire method,

$$X(t) = \int_0^t \sigma(X(s), s) dB(s),$$

we used Milstein's higher order method with

$$\sigma(x) = \sqrt{\frac{\sigma^2}{2} + \sigma \frac{|x|}{\sqrt{2s}}}.$$

For the application of these simulations, we priced exotic options on these processes. First, we confirmed that the assumption that forward return distributions are in the same class of distributions as the spot return distributions is not valid except for the Dupire process with values of the process equal to zero at time 0.5 when double negative exponential distributions are used as the spot return distributions. The forward motion is skewed to the left if x is negative and skewed to the right if x is positive, except for the Sato process case.

Second, we priced credit default swaps with barriers at maturities 3, 6, 9 and 12 months. We found that the swap prices are higher on the continuous process than those on the jump processes. Also, the average time of the first passage time to the barrier is shorter on the continuous process, the Dupire process, than on the jump processes, the Azéma and Yor process and the Sato process.

Third, we priced cliquets with local caps and global floors and with local floors and global caps. Then we checked model risk using spread. The model risk on the locally capped globally floored cliquets decreased as the local cap increased. On the locally floored globally capped cliquets, the model risk increases as the local floor increased.

Last, we priced options on the realized quadratic variations. We noticed that the values are slightly decreasing as maturities are increasing.

6.2 Future work

In this dissertation, we used double negative exponential distributions as the prespecified marginal distributions. However, one could use other distribution functions for the marginal distributions, such as Gaussian. Likewise, one could price other kinds of exotic options, such as forward starting options and barrier options. Furthermore, it could be possible to construct a martingale using inhomogeneous Markov martingale processes with independent increments using subordinated Lévy processes. Moreover, one might also construct a martingale using other solutions to the Skorohod embedding problem. Any of these variations would provide areas for future research.

Appendix A

Histograms of forward motion

For the value of $x = X(0.5)$, we choose 5 different points:

$$-2\sigma\sqrt{0.5}, \;\; -\sigma\sqrt{0.5}, \;\; 0, \;\; \sigma\sqrt{0.5}, \;\; 2\sigma\sqrt{0.5},$$

and σ was chosen to be 0.25.

Figures A.1 to A.5 display histograms of forward return, best fitting double negative exponential, double negative exponential with unit time volatility 0.25, and the three corresponding cumulative distributions for the Skorohod embedding process at each value of x.

Since the Sato process is an independent increment process, the forward motion is independent of $X(0.5)$. So, the results for all 5 different values of x agree and appear in Figure A.6.

Figures A.7 to A.11 present same graphs for the Dupire process.

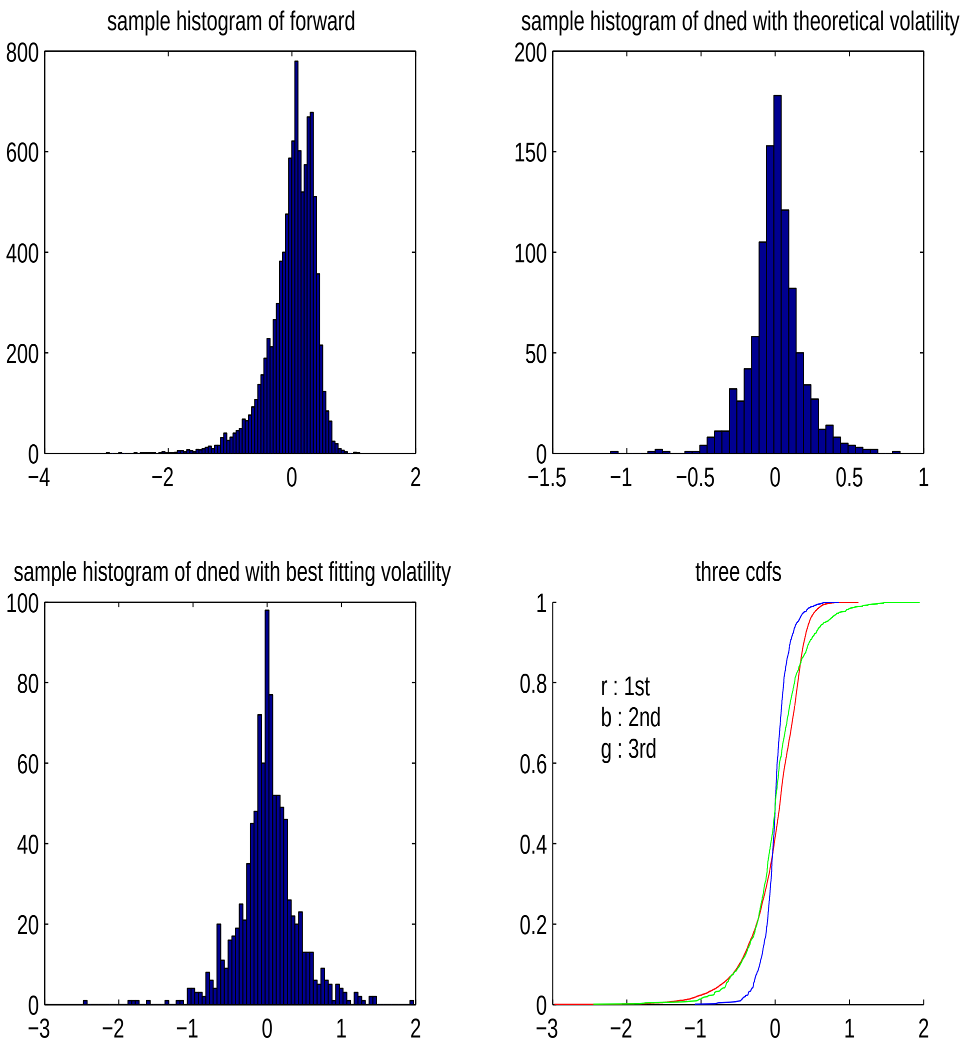


Figure A.1: Histograms and cdfs for Azéma and Yor process at x=-0.3536

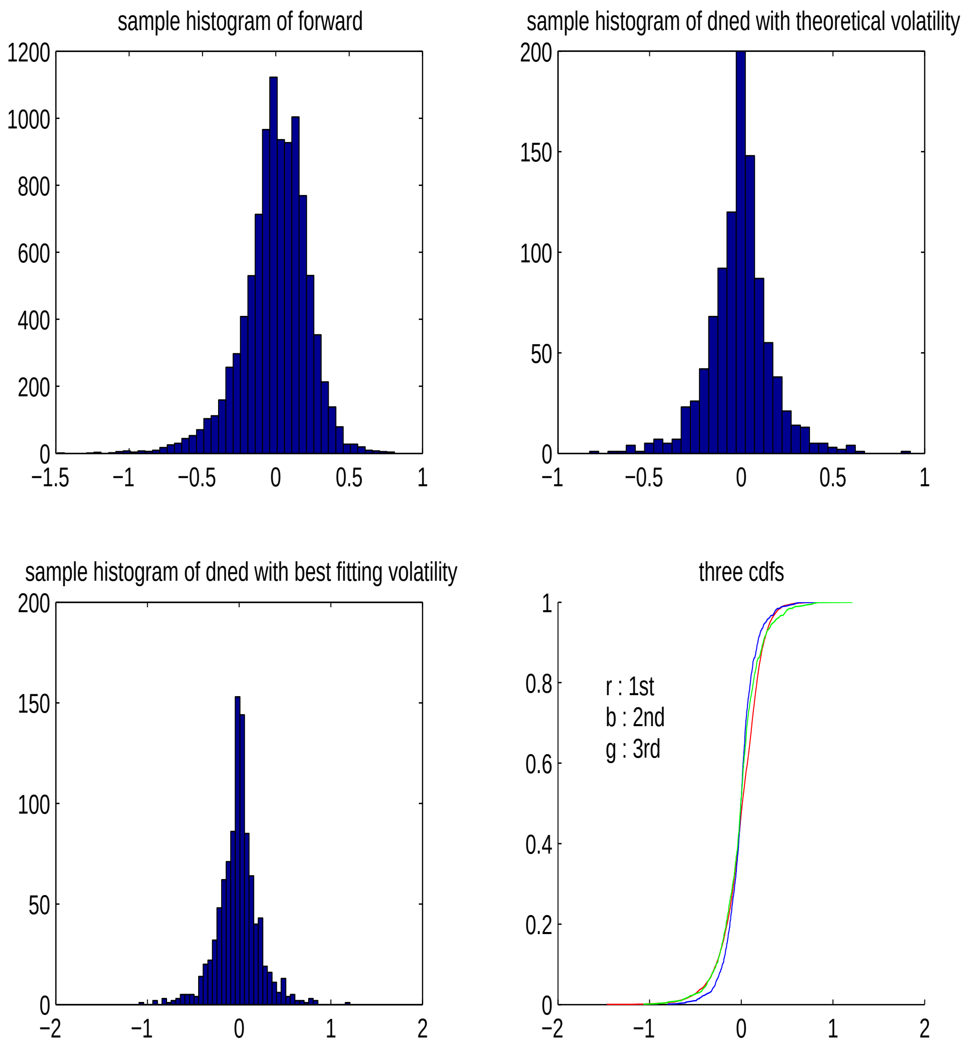


Figure A.2: Histograms and cdfs for Azéma and Yor process at x=-0.1768

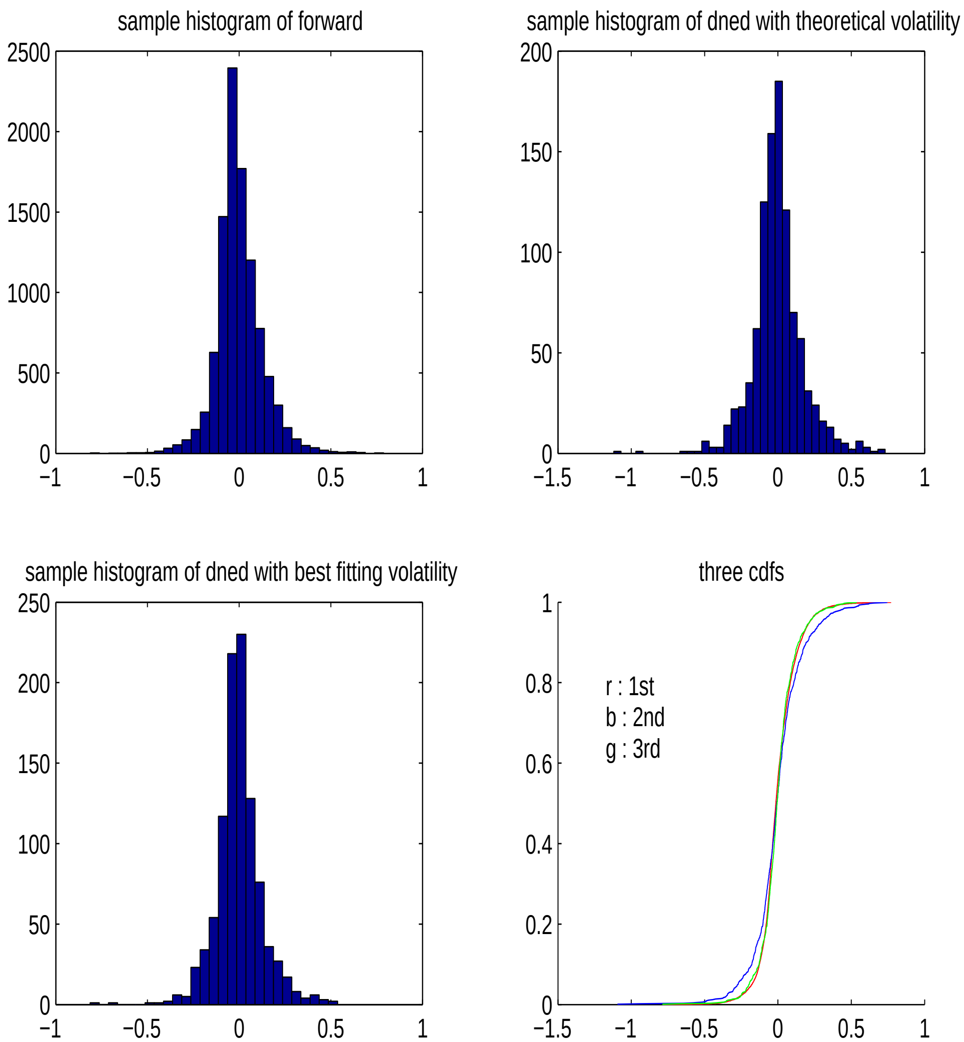


Figure A.3: Histograms and cdfs for Azéma and Yor process at x=0

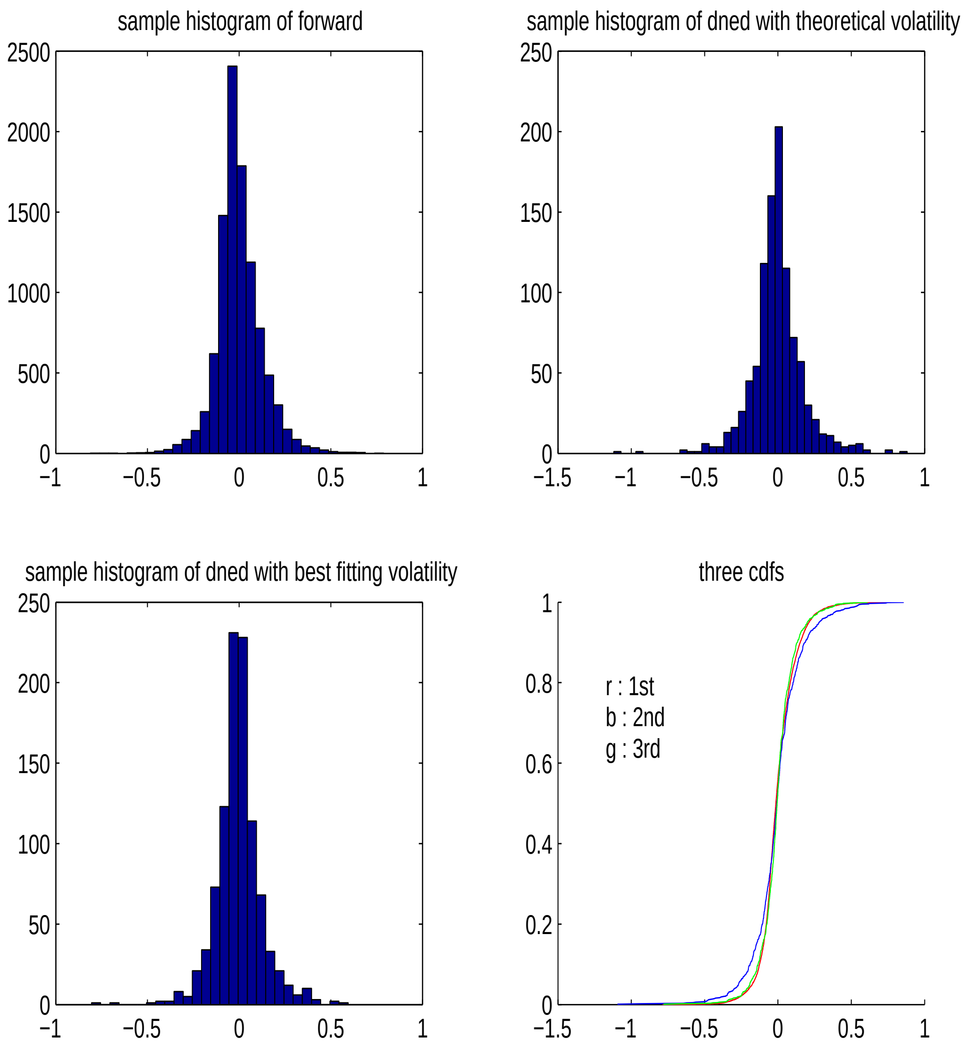


Figure A.4: Histograms and cdfs for Azéma and Yor process at x=0.1768

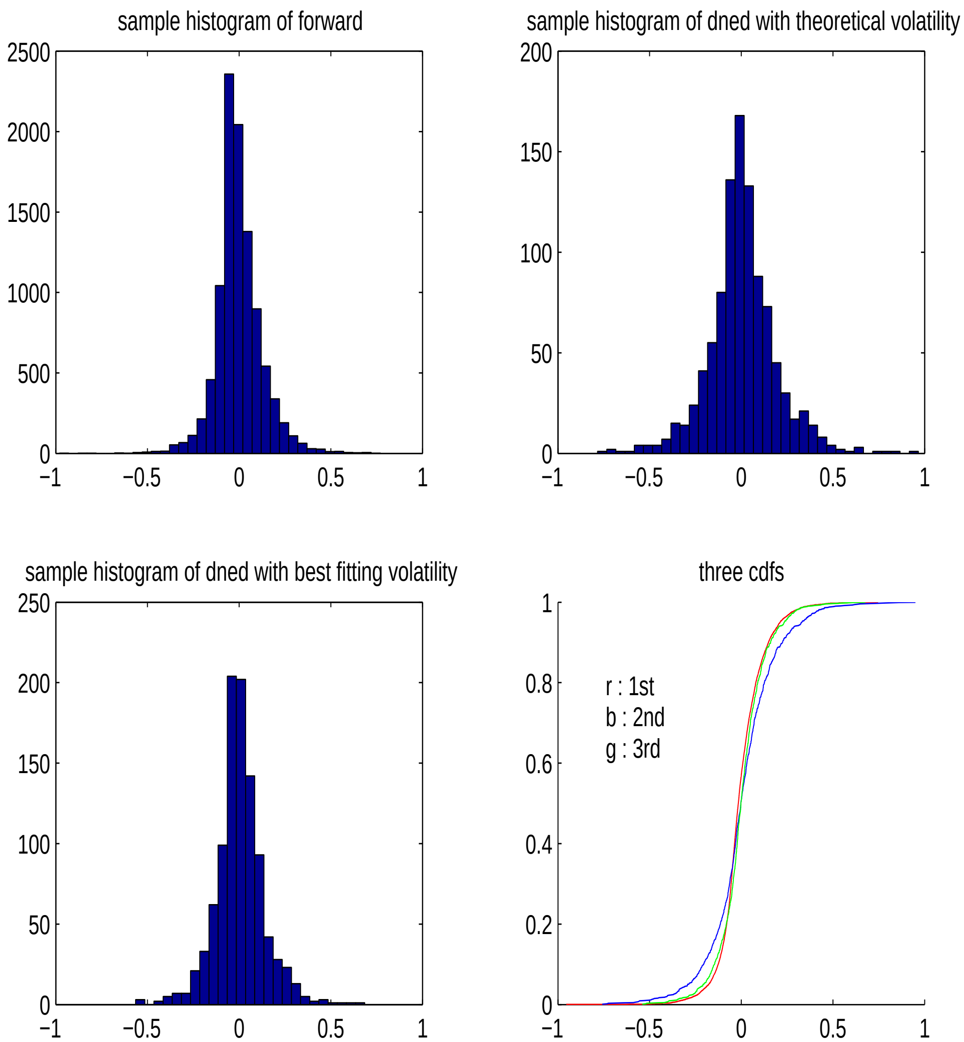


Figure A.5: Histograms and cdfs for Azéma and Yor process at x=0.3536

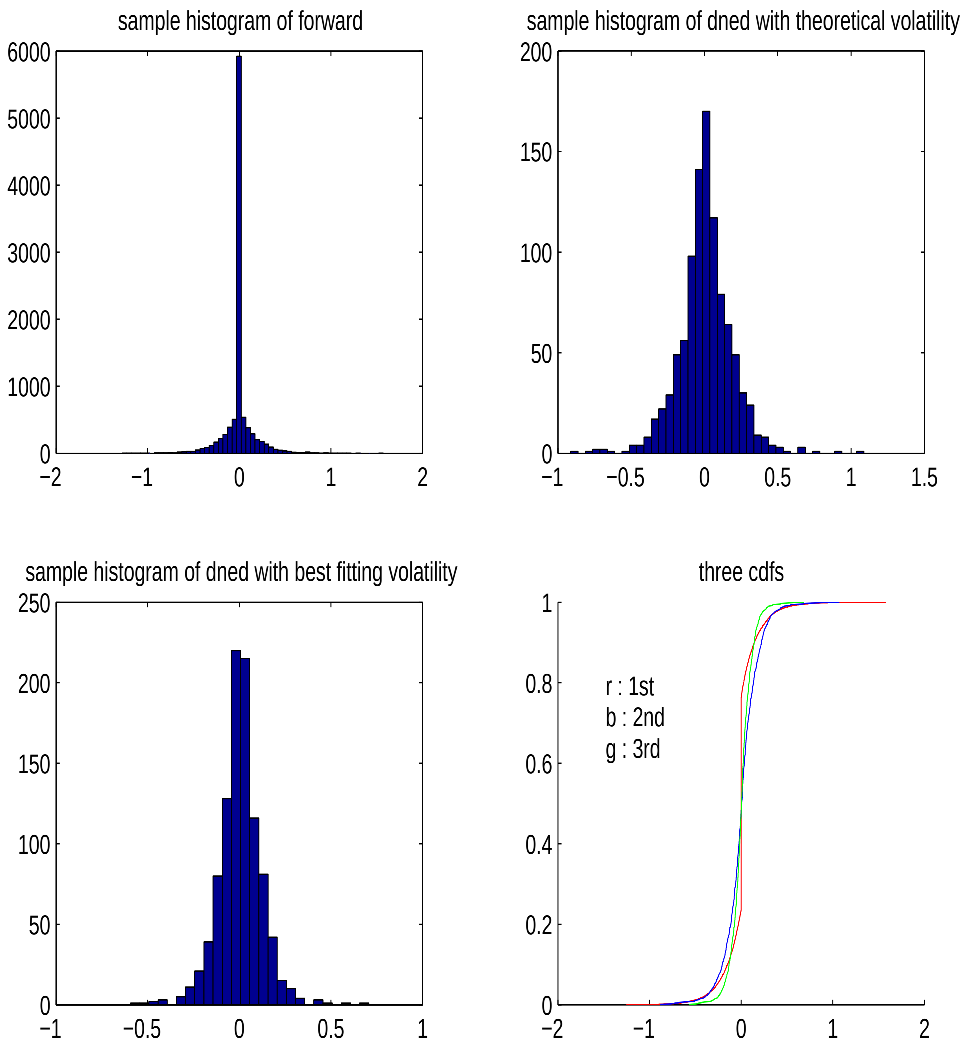


Figure A.6: Histograms and cdfs for Sato process

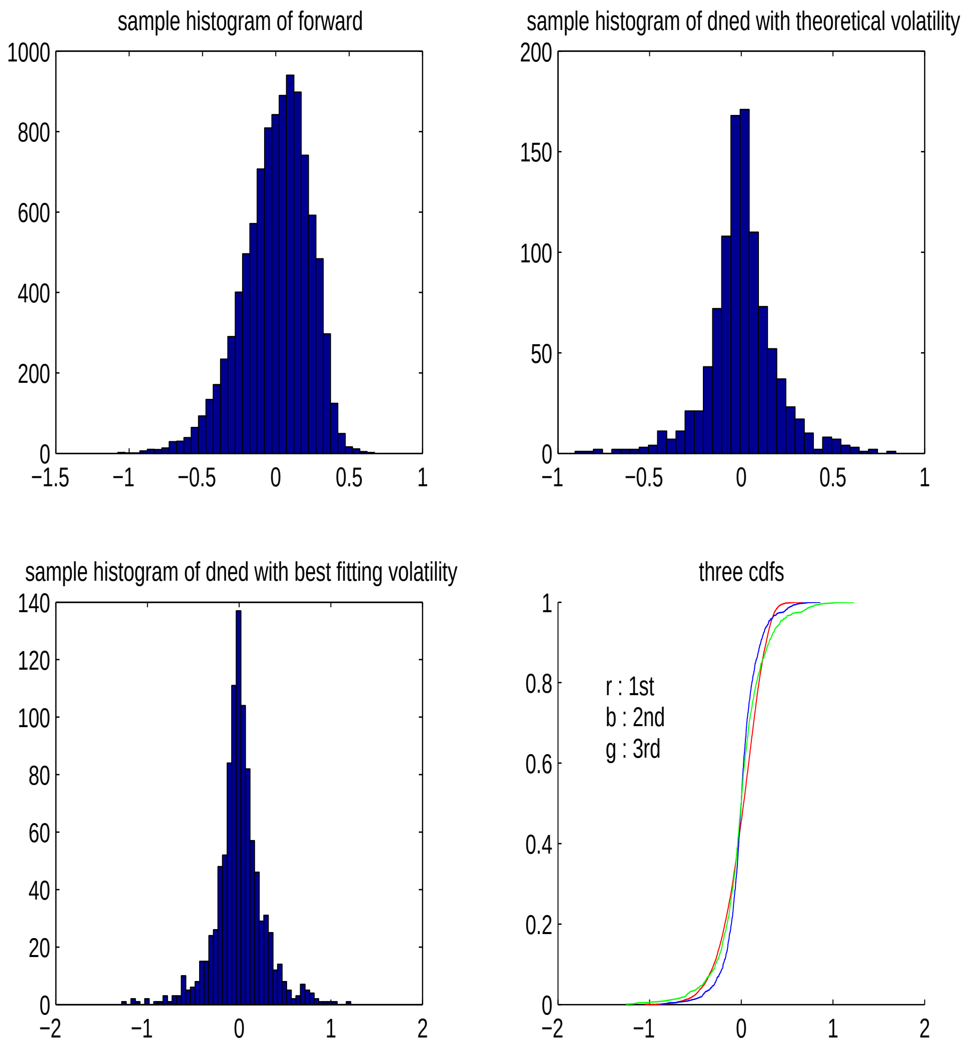


Figure A.7: Histograms and cdfs for Dupire process at x=-0.3536

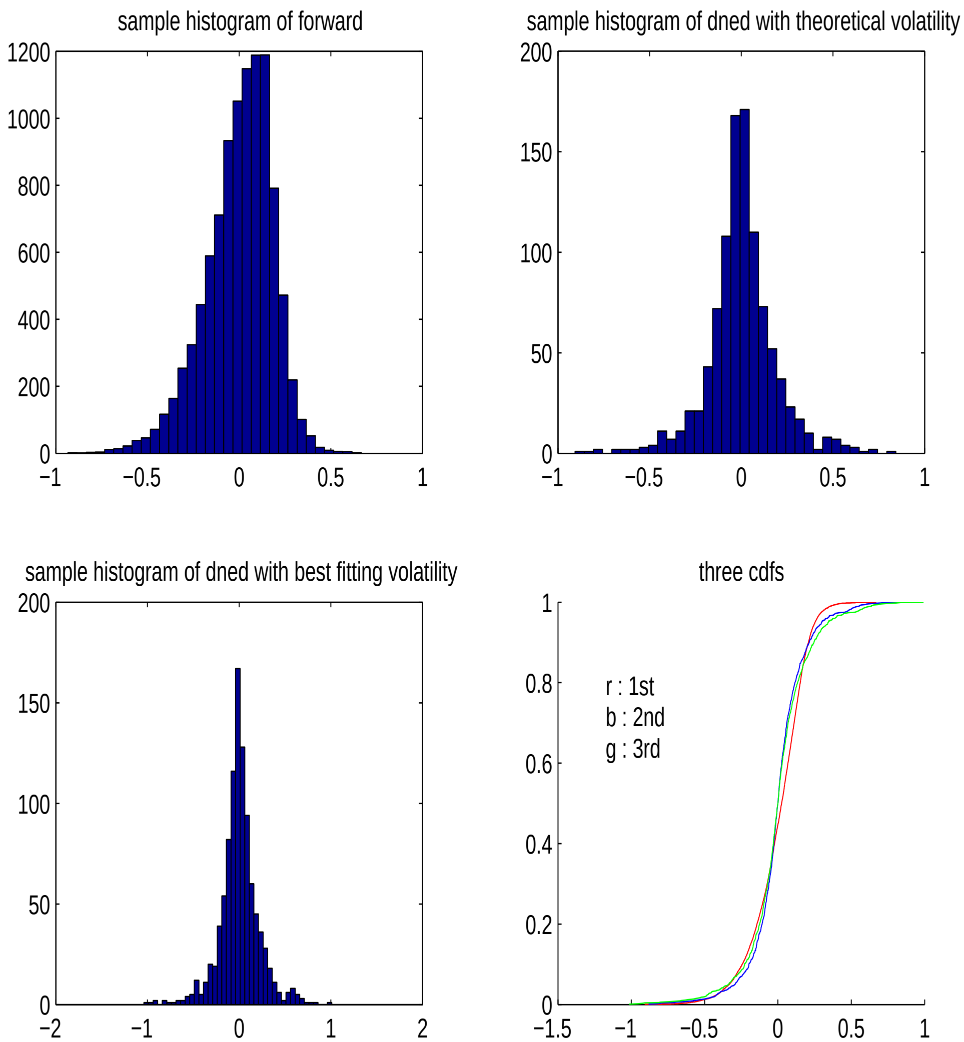


Figure A.8: Histograms and cdfs for Dupire process at x=-0.1768

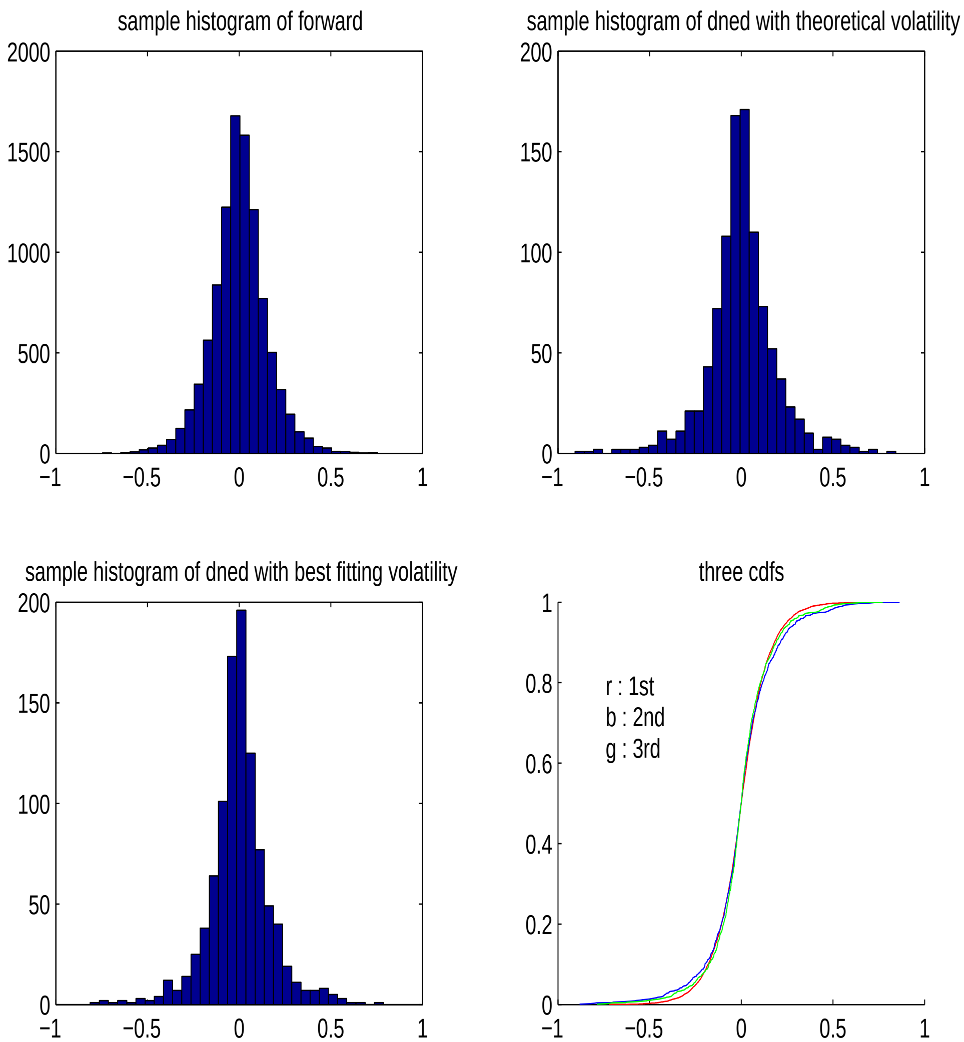


Figure A.9: Histograms and cdfs for Dupire process at x=0

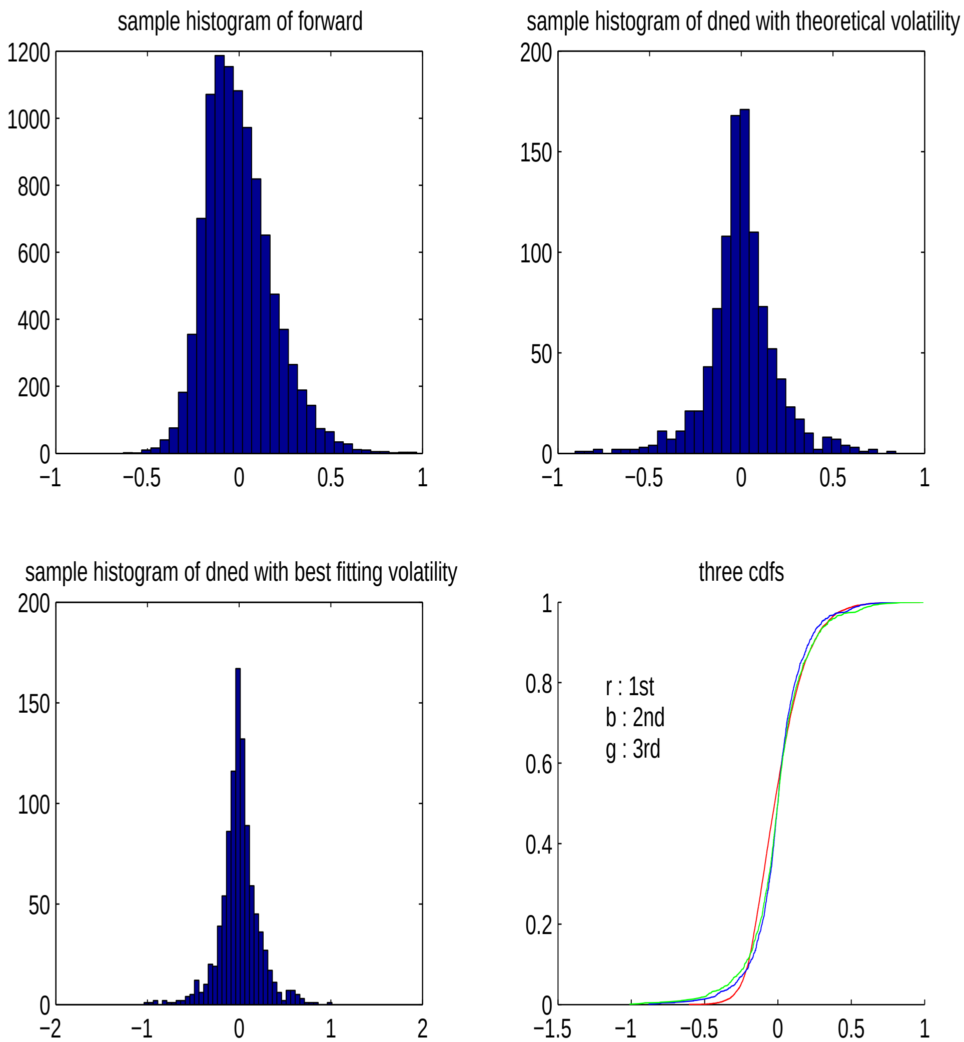


Figure A.10: Histograms and cdfs for Dupire process at x=0.1768

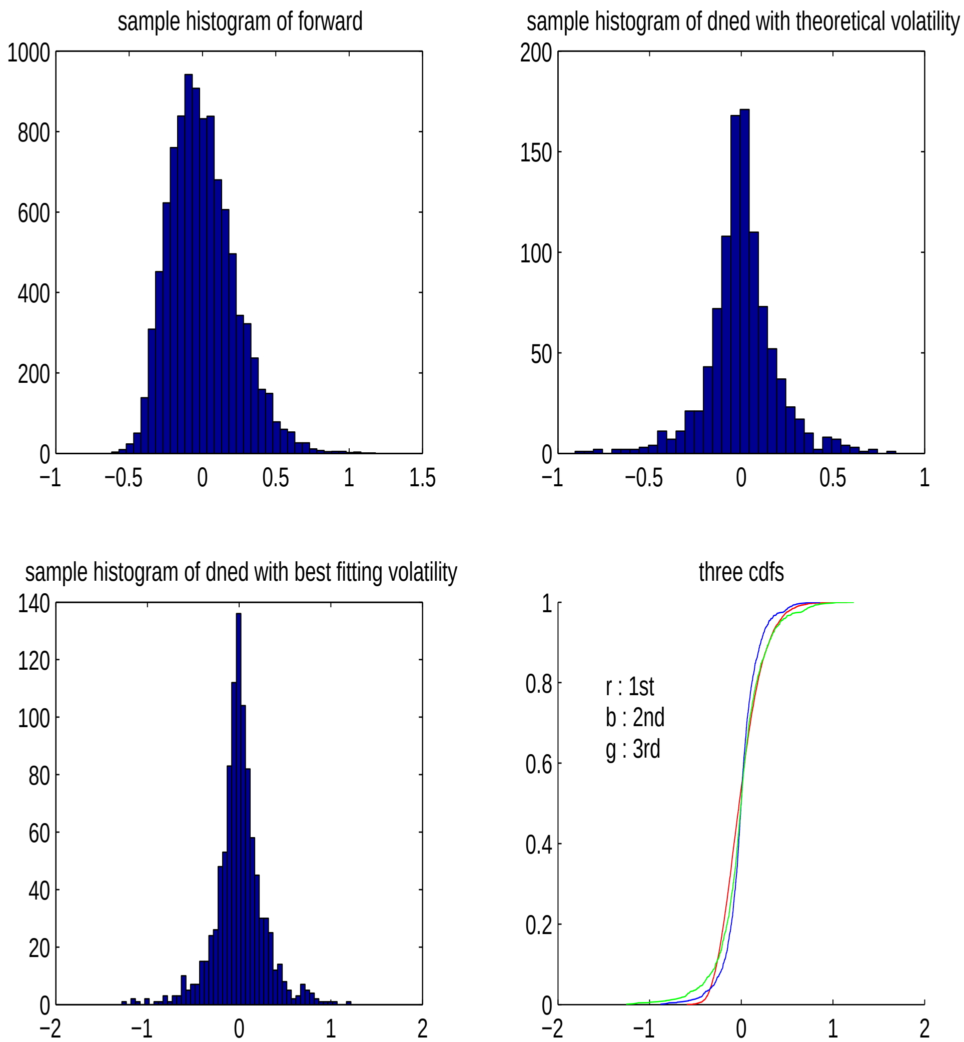


Figure A.11: Histograms and cdfs for Dupire process at x=0.3536

Appendix B

Implied volatilities of forward return distributions

Figures B.1, B.2 and B.3 present the call price curves and implied volatility curves for the Azéma and Yor process, the Sato process and the Dupire process, respectively, when the value at time 0.5 is -0.3536. Similarly, Figures B.4 to B.6 present call price curves and implied volatility curves for the Azéma and Yor process, the Sato process and the Dupire process, respectively when the value at time 0.5 is -0.1768, and so on.

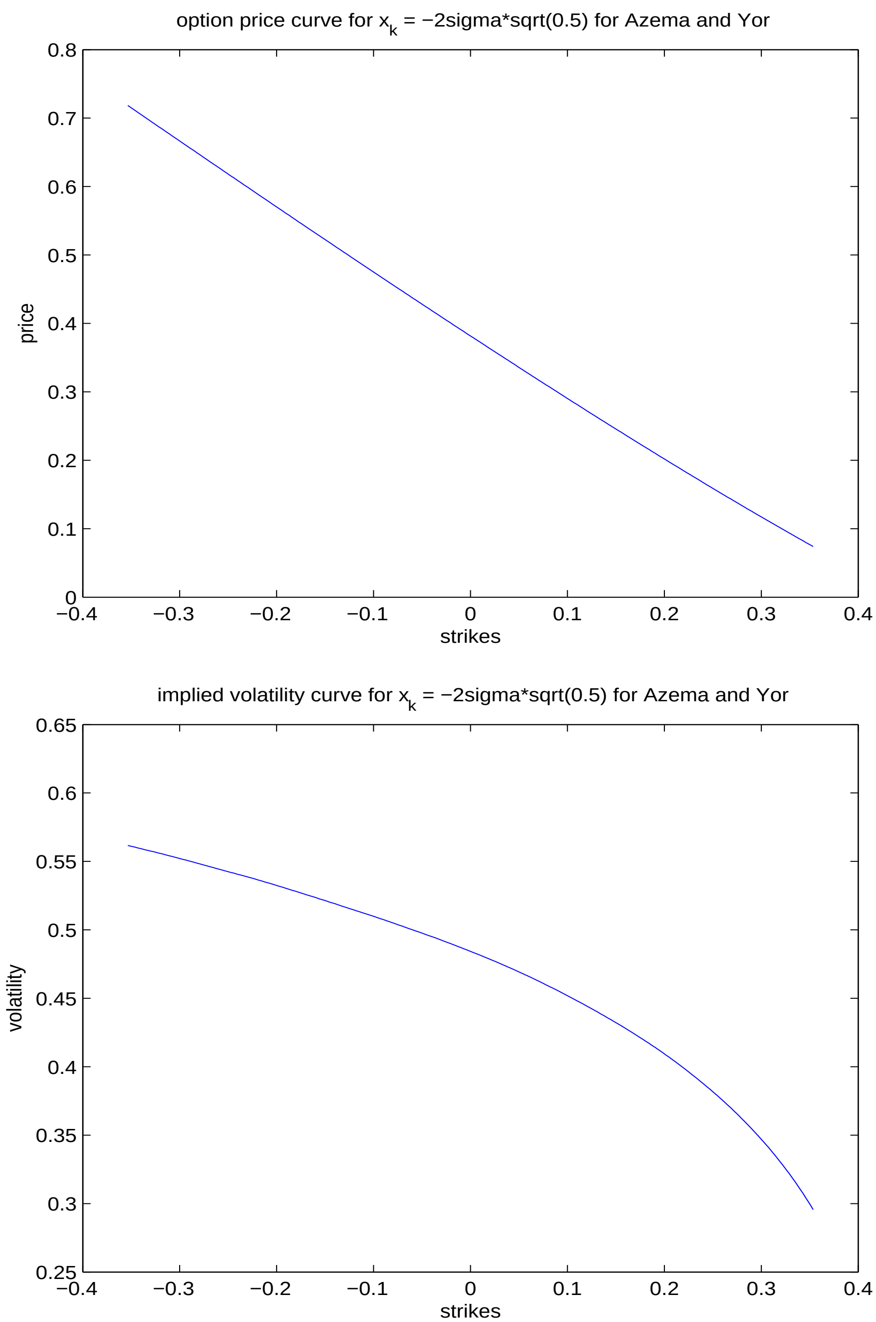
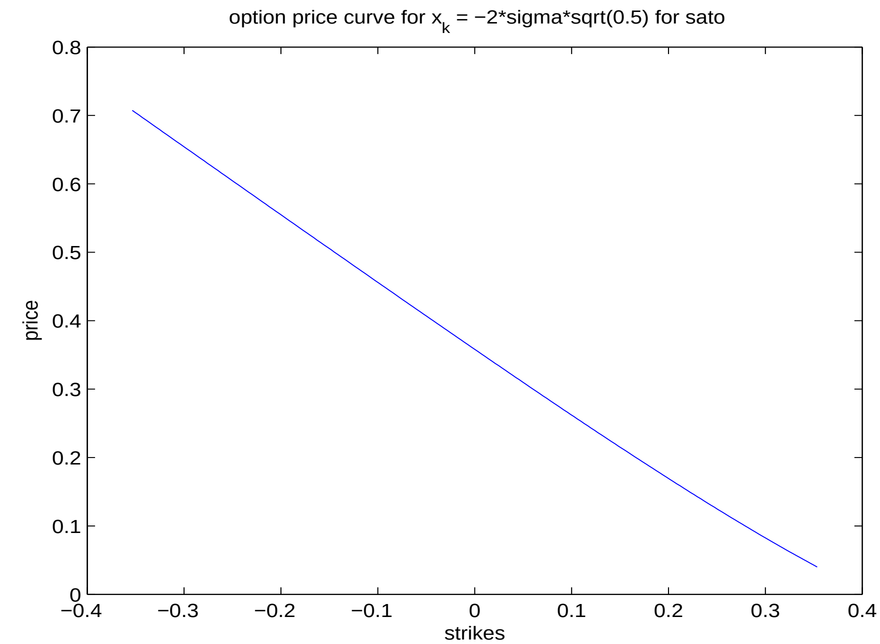


Figure B.1: Call price and implied volatility curve for Azéma and Yor process at x=-0.3536



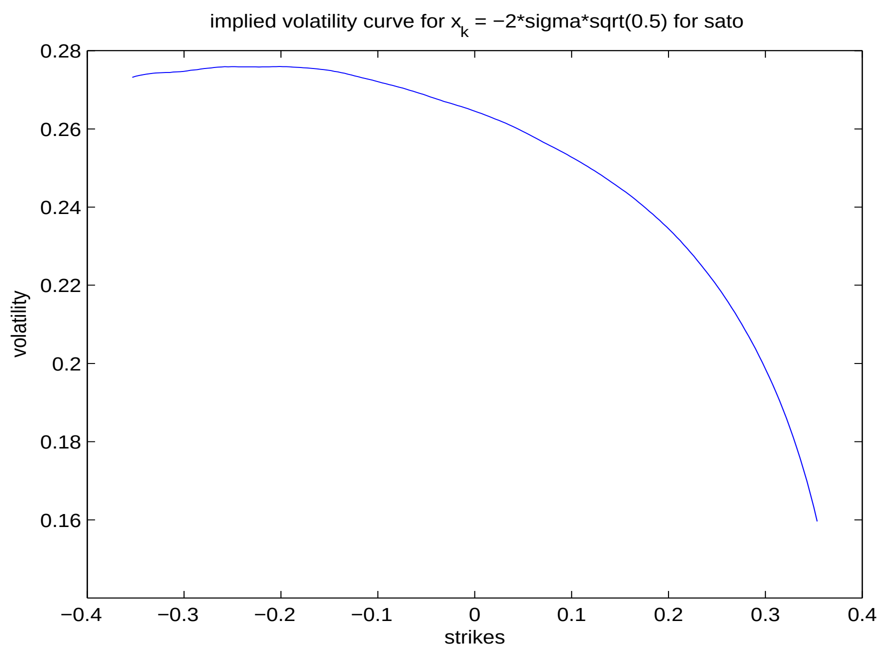


Figure B.2: Call price and implied volatility curve for Sato process at x=-0.3536

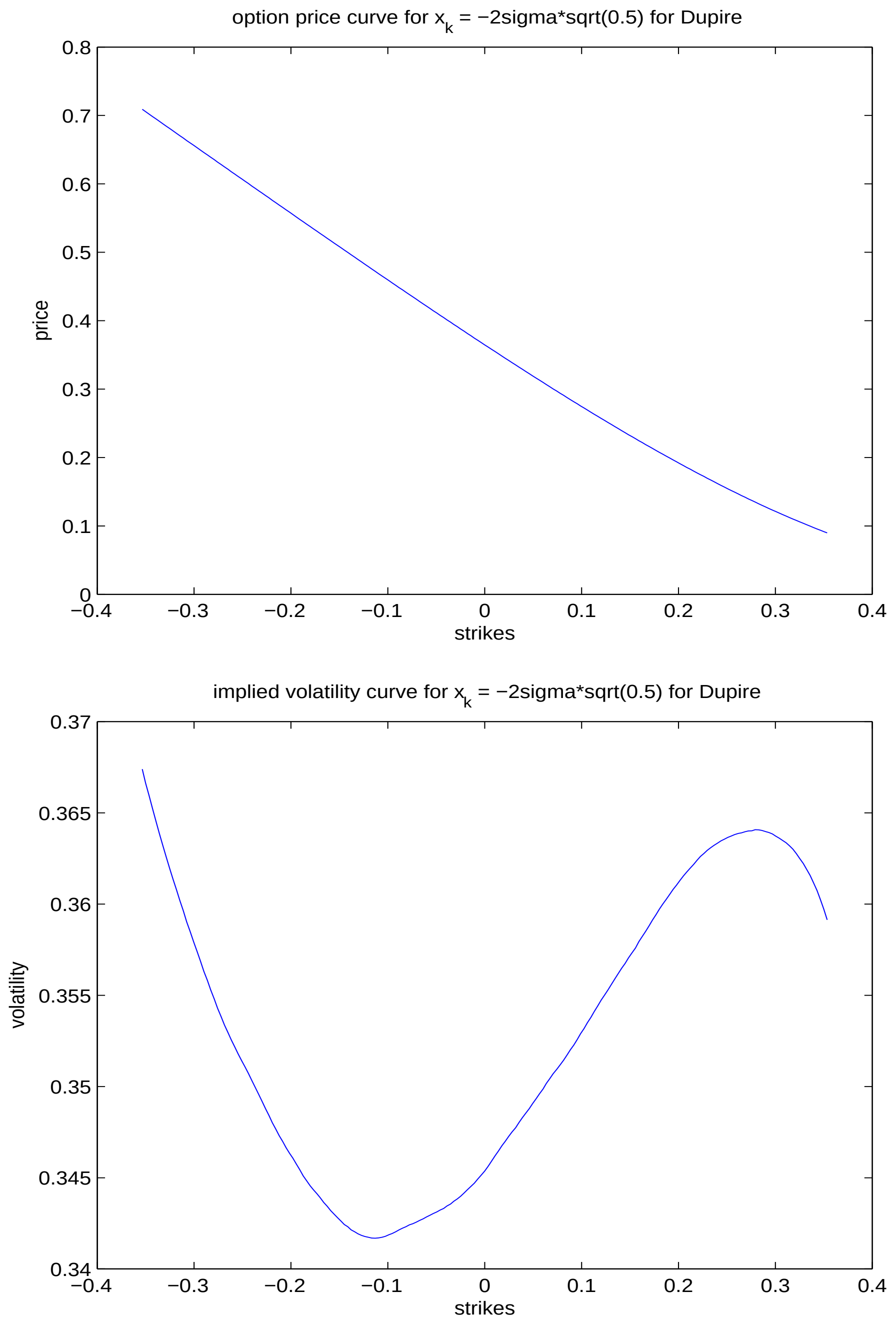


Figure B.3: Call price and implied volatility curve for Dupire process at x=-0.3536

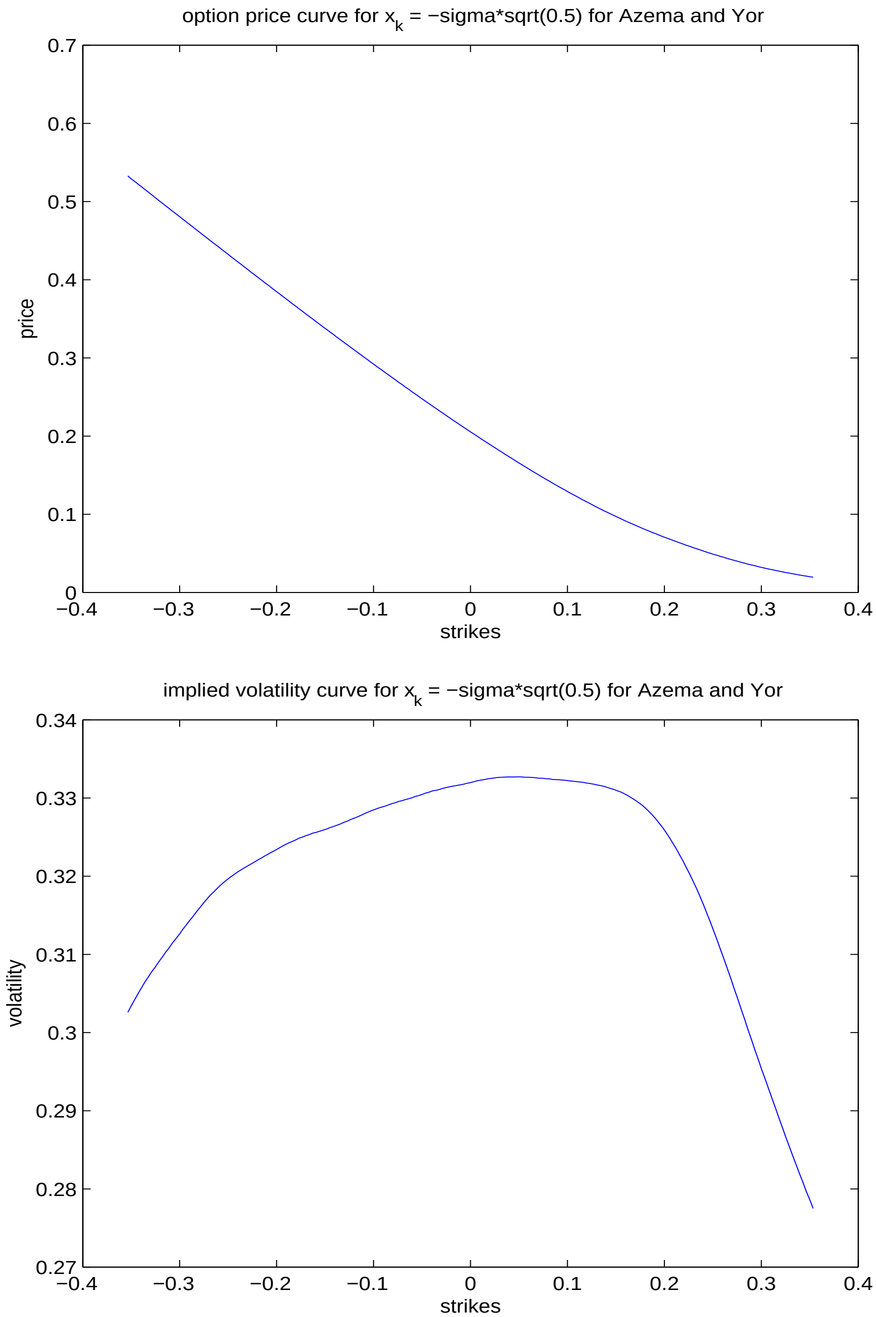


Figure B.4: Call price and implied volatility curve for Azéma and Yor process at x=-0.1768

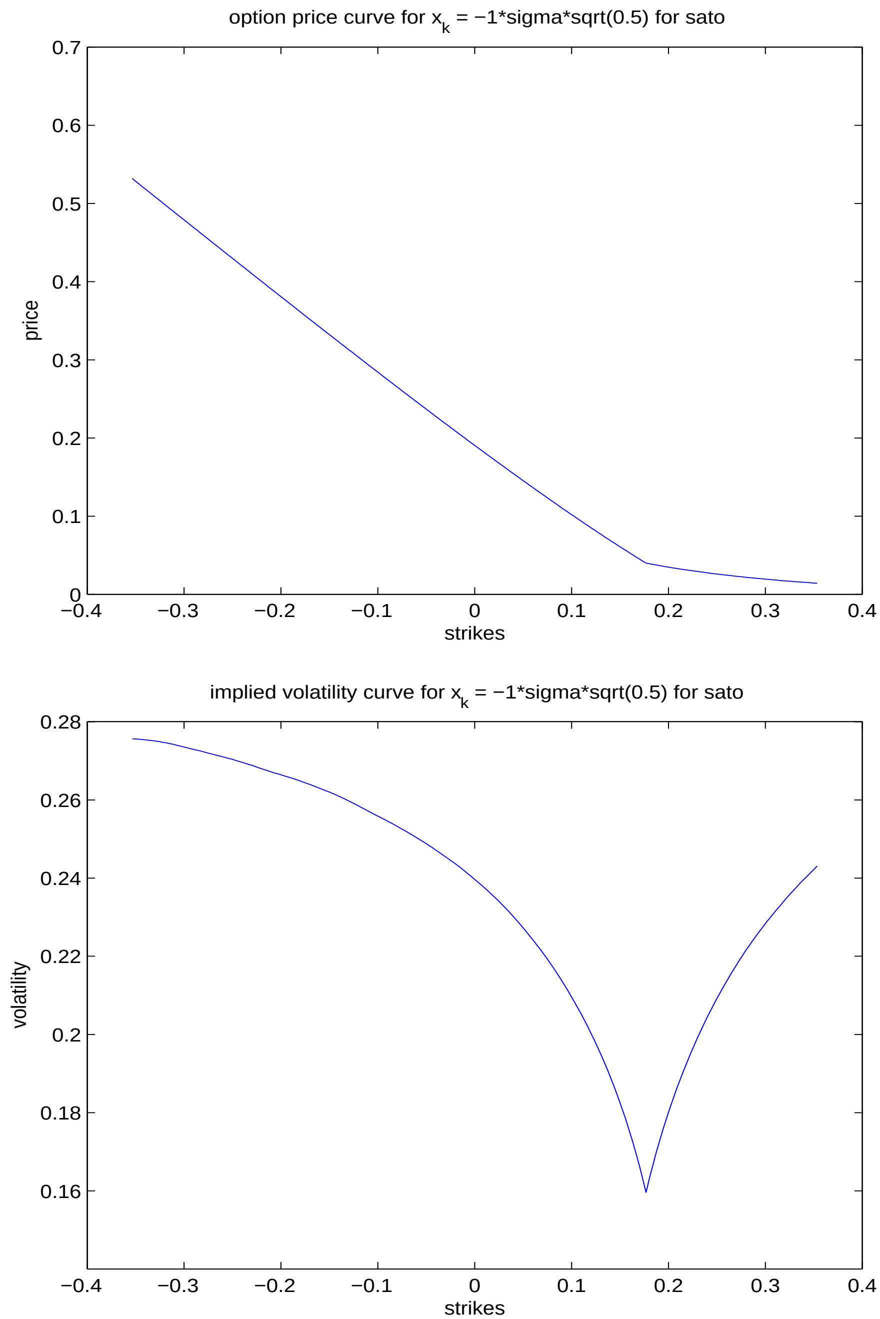


Figure B.5: Call price and implied volatility curve for Sato process at x=-0.1768

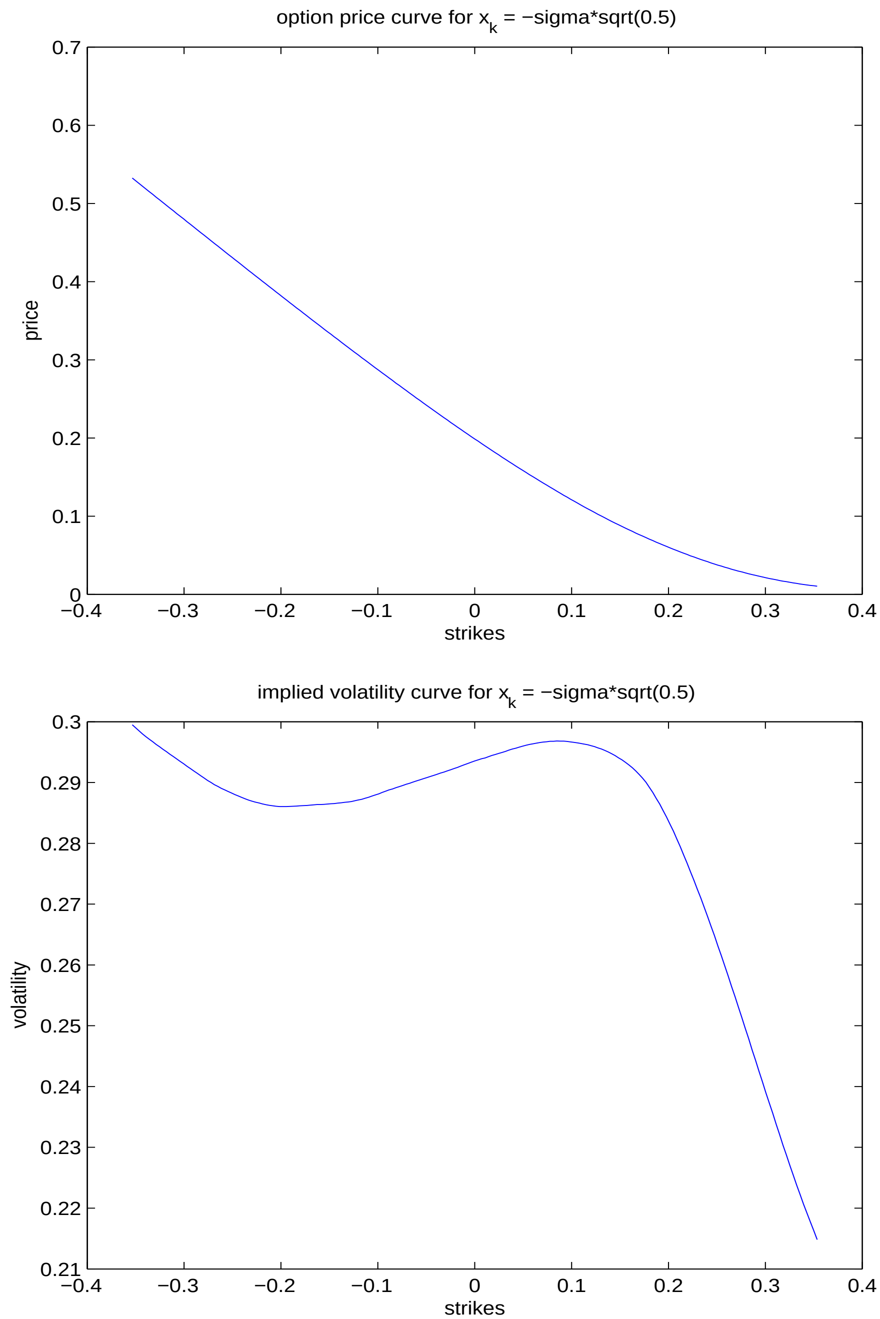


Figure B.6: Call price and implied volatility curve for Dupire process at x=-0.1768

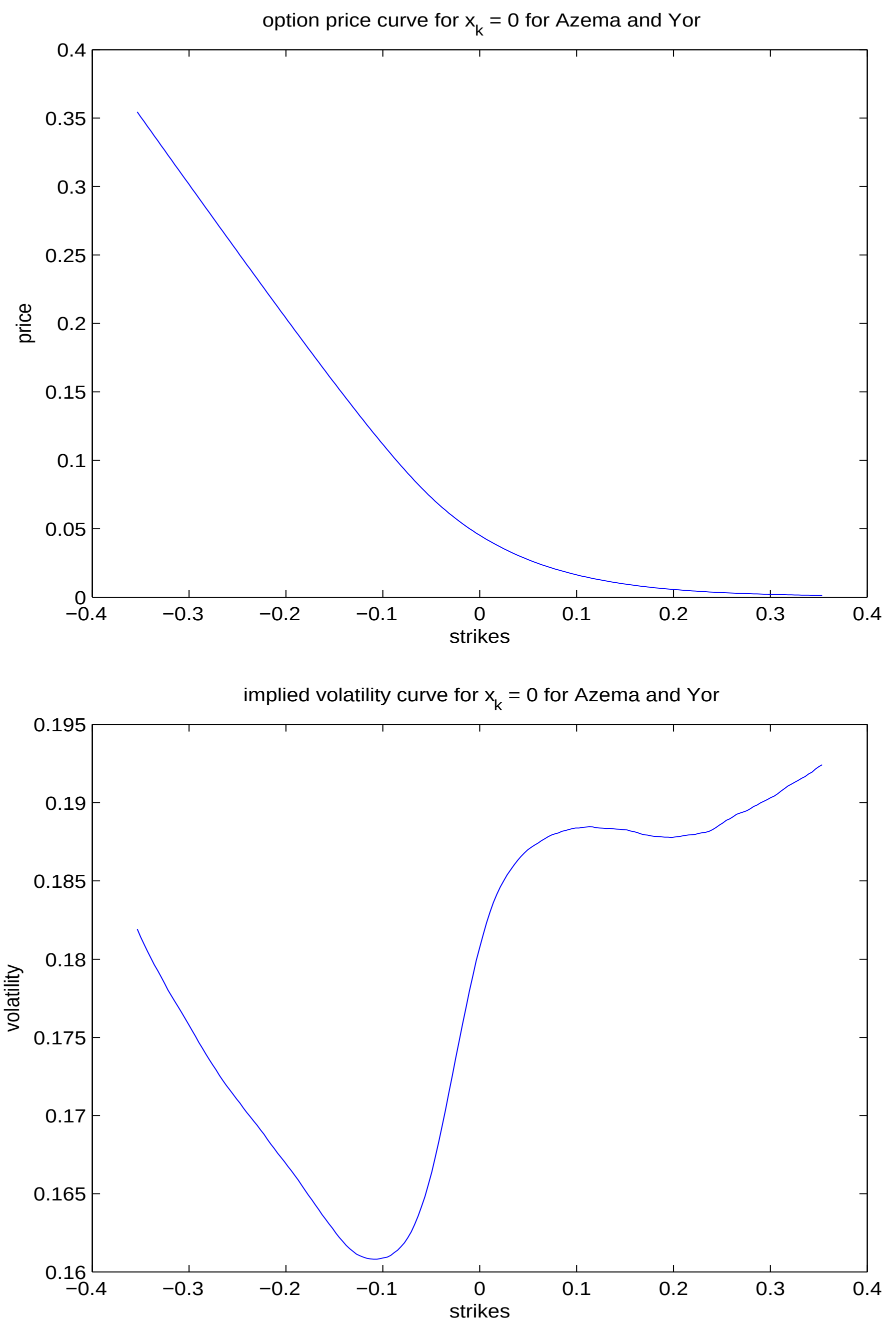


Figure B.7: Call price and implied volatility curve for Azéma and Yor process at x=0

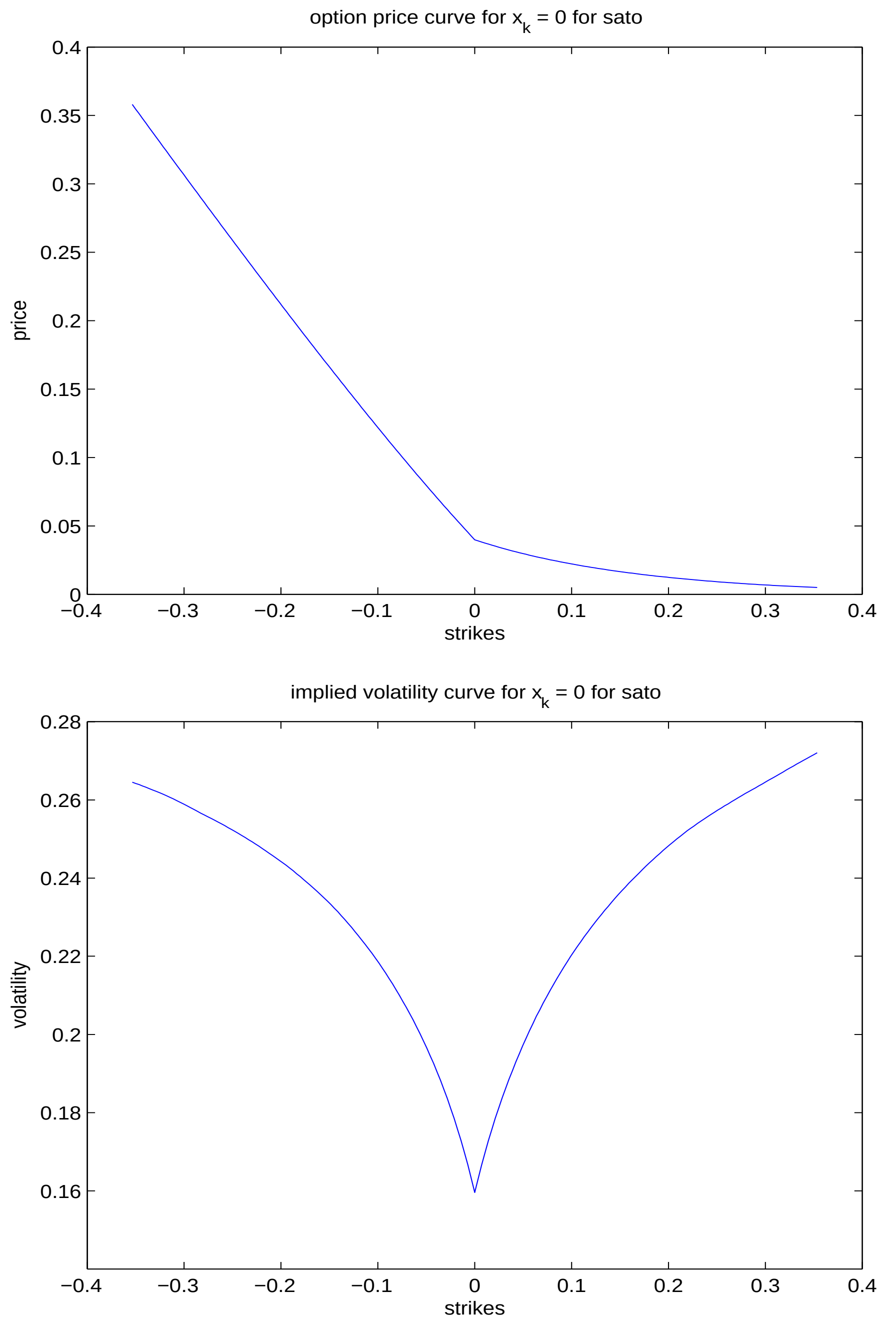


Figure B.8: Call price and implied volatility curve for Sato process at x=0

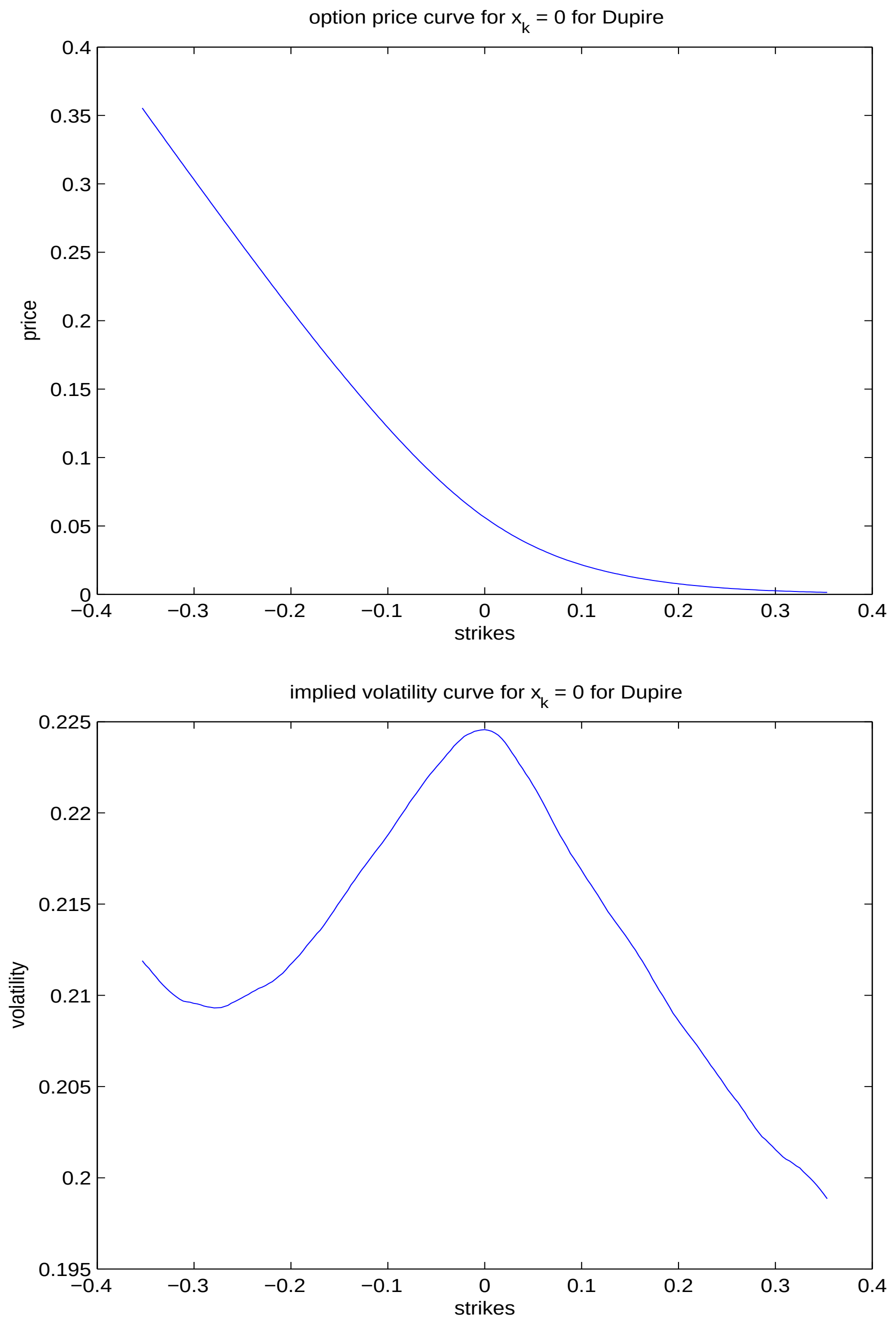


Figure B.9: Call price and implied volatility curve for Dupire process at x=0

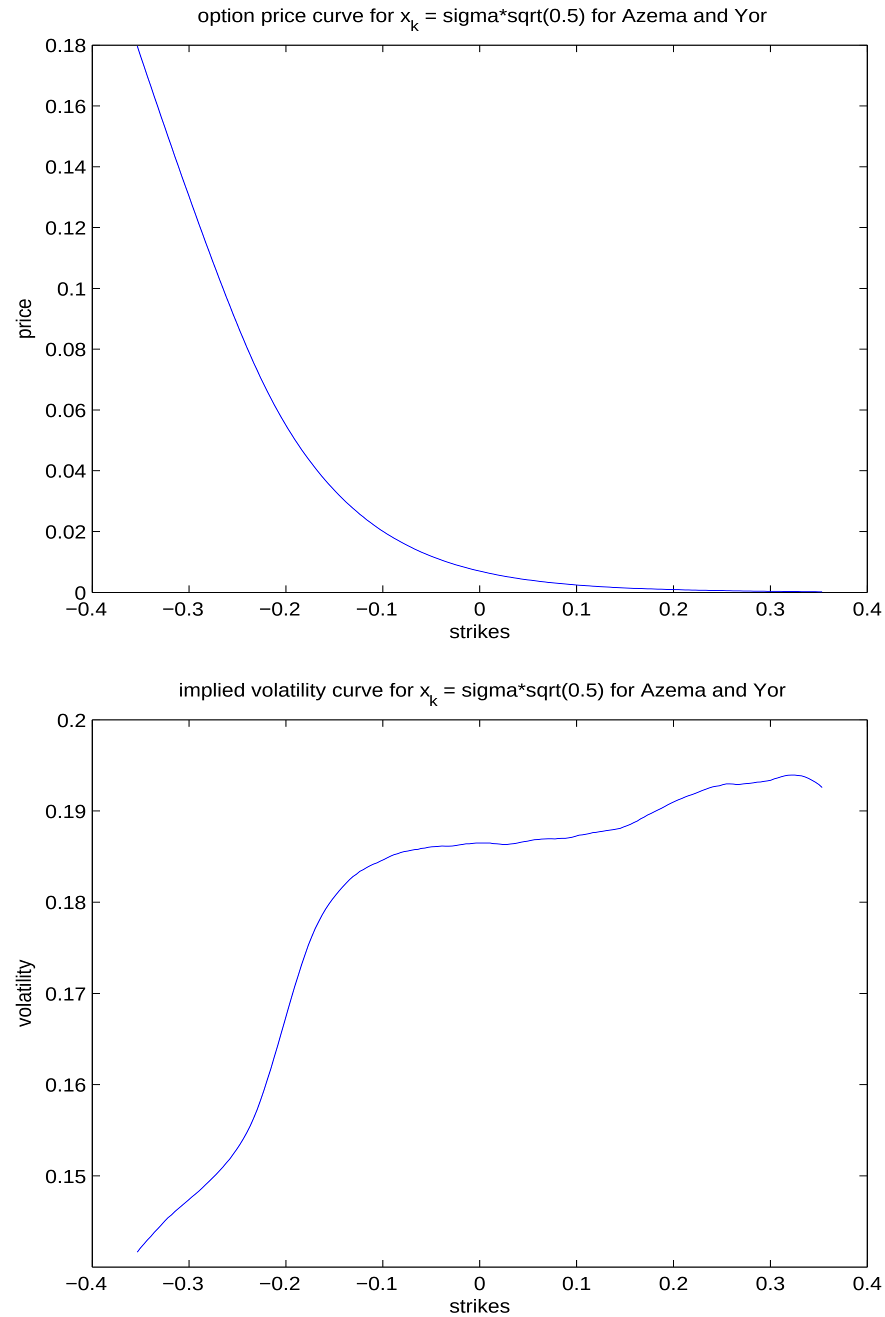


Figure B.10: Call price and implied volatility curve for Azéma and Yor process at x=0.1768

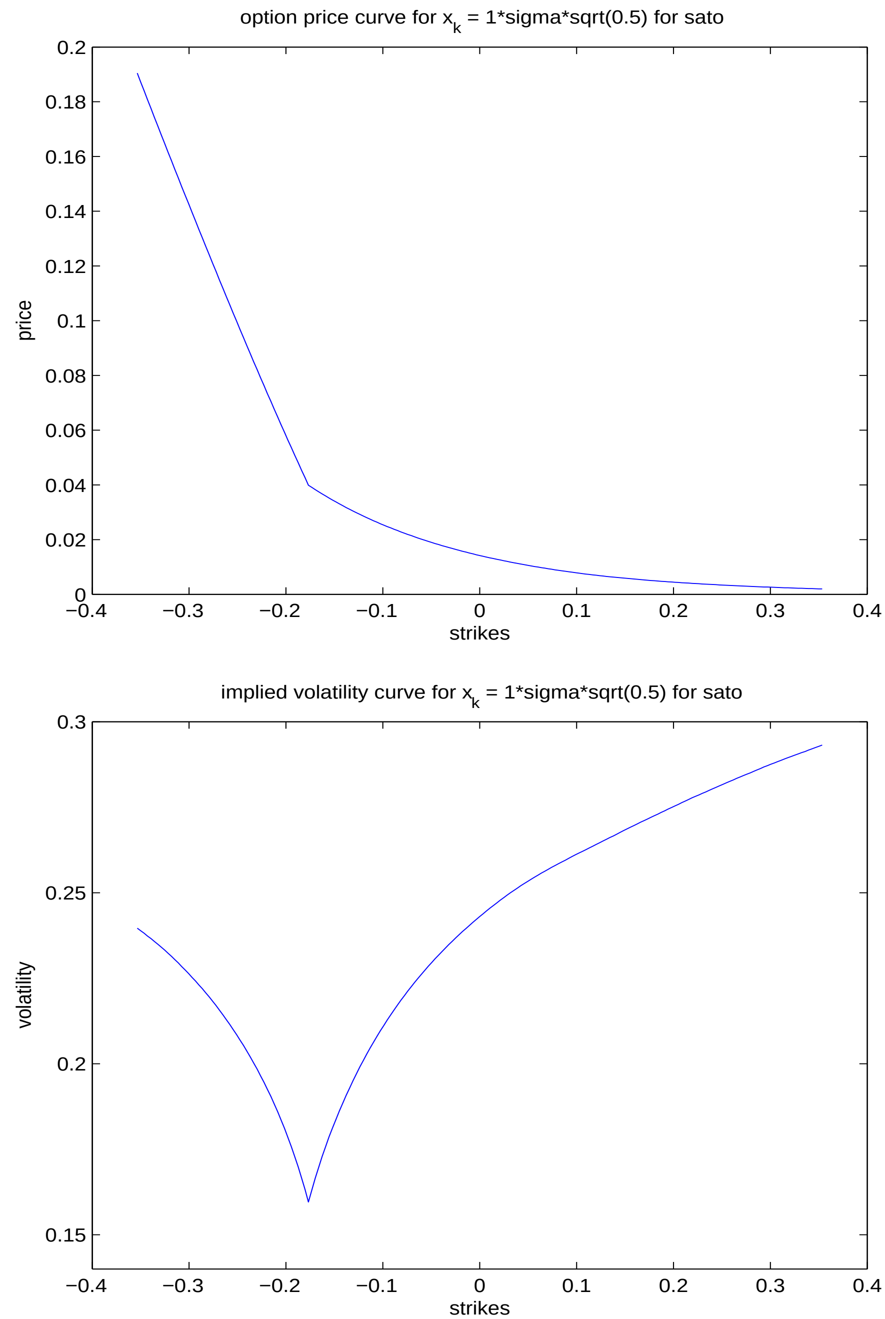


Figure B.11: Call price and implied volatility curve for Sato process at x=0.1768

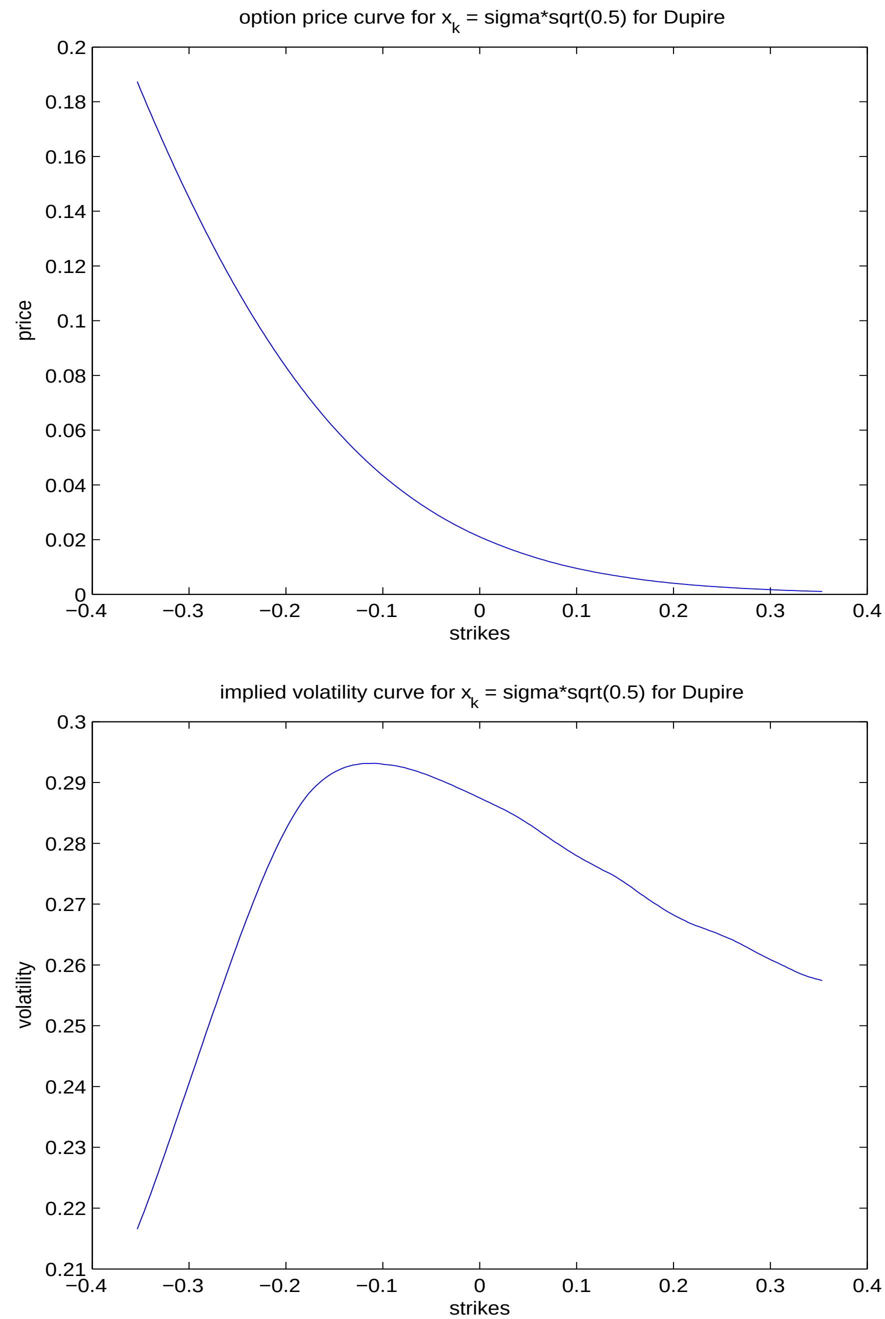
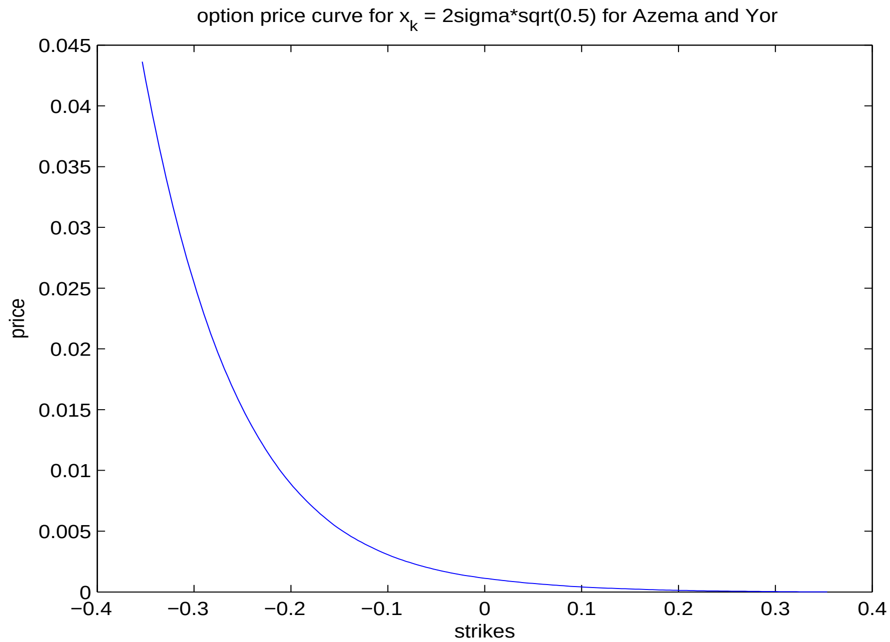


Figure B.12: Call price and implied volatility curve for Dupire process at x=0.1768



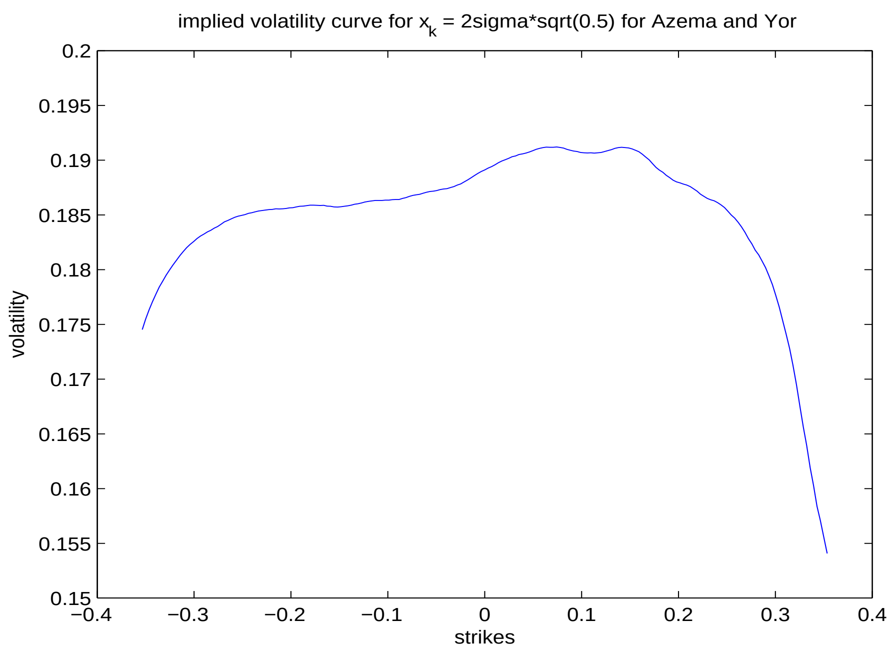


Figure B.13: Call price and implied volatility curve for Azéma and Yor process at x=0.3536

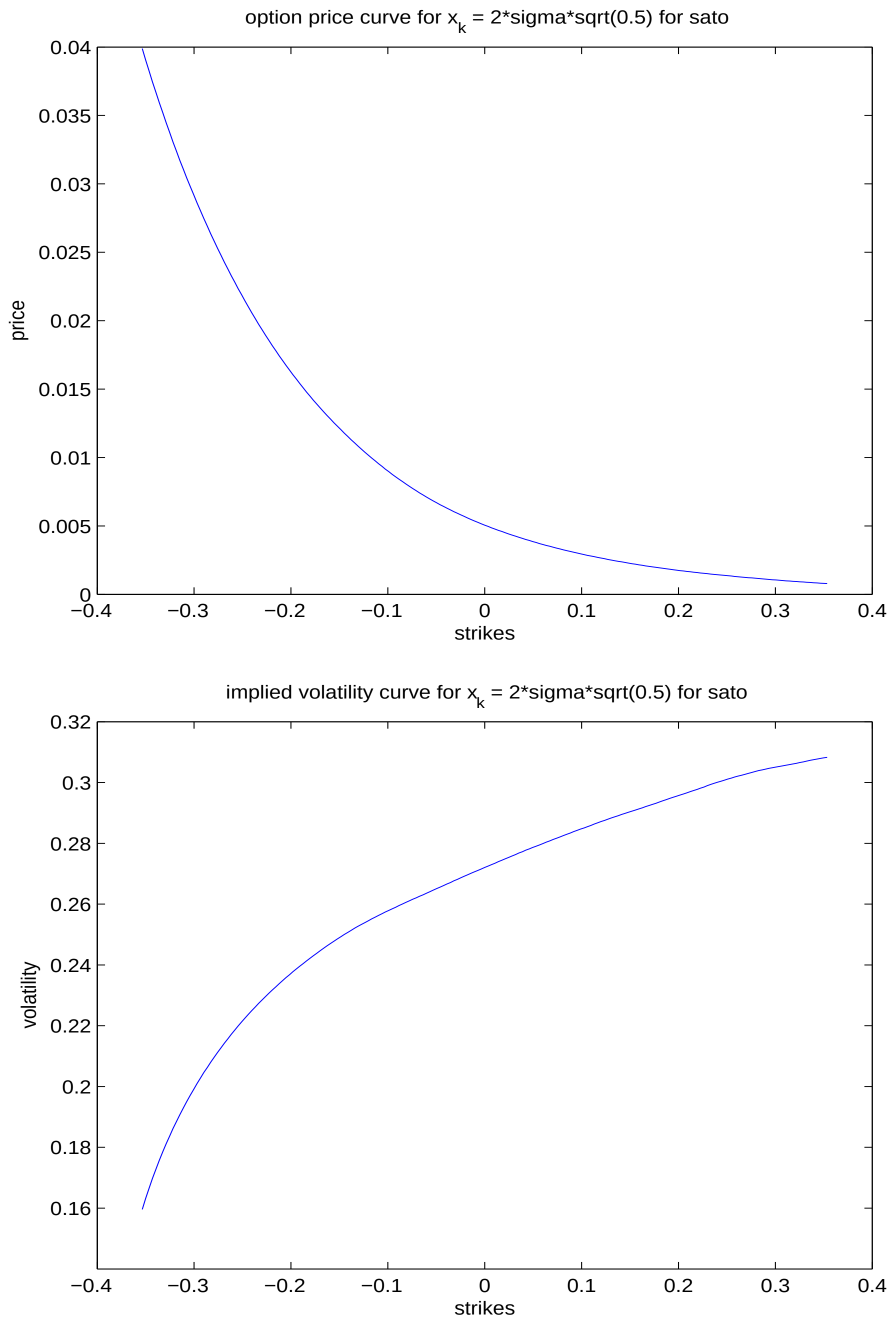


Figure B.14: Call price and implied volatility curve for Sato process at x=0.3536

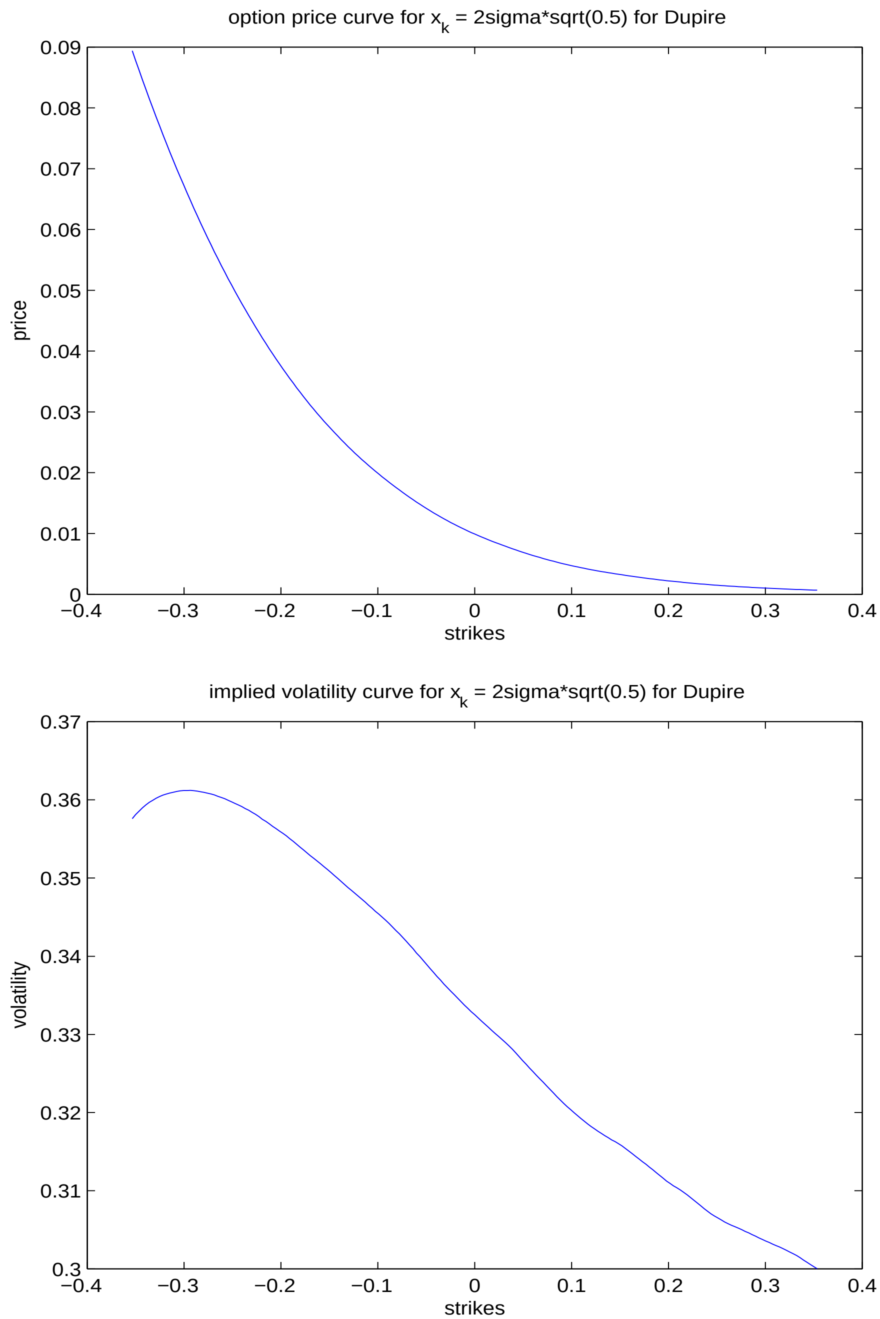


Figure B.15: Call price and implied volatility curve for Dupire process at x=0.3536

Appendix C

Graphs for the swap rates

We suppose the coupon payments are monthly and the interest rate is 3% annually continuously compounded. Using (5.1) and (5.3) and the values of two cash flows at initial time are same, we get the monthly coupon payment in percent which we call swaprate, and we get the following graphs. We choose four different settlement dates: 3 months, 6 months, 9 months, and 12 months. Figures C.1, C.2 and C.3 represents 3 month swaprate for the Azéma and Yor, Sato, and Dupire processes, respectively. Figures C.4, C.5 and C.6 represent 6 month swaprate for the Azéma and Yor, Sato and Dupire processes, respectively, and so on.

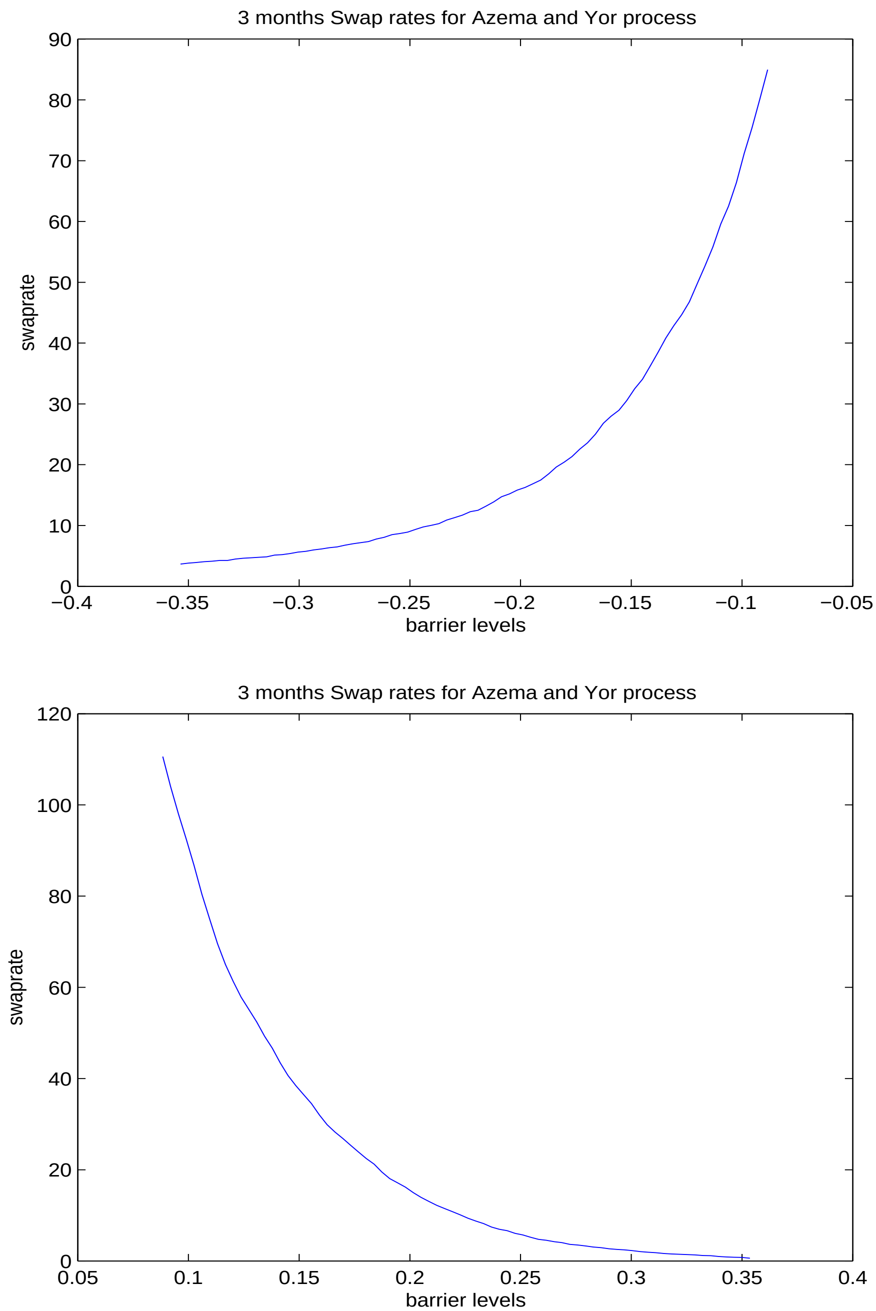


Figure C.1: 3 month swaprate for Azéma and Yor process

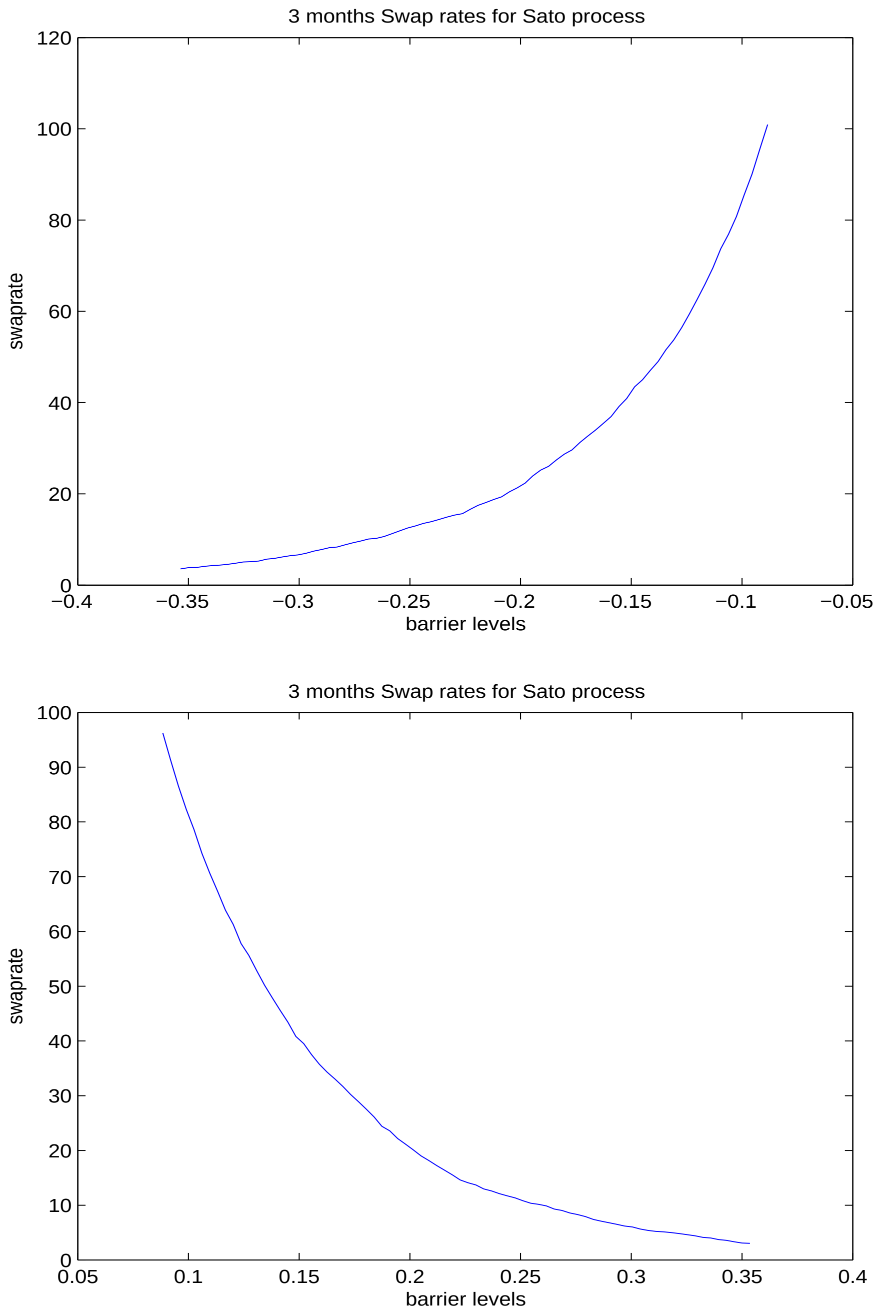


Figure C.2: 3 month swaprate for Sato process

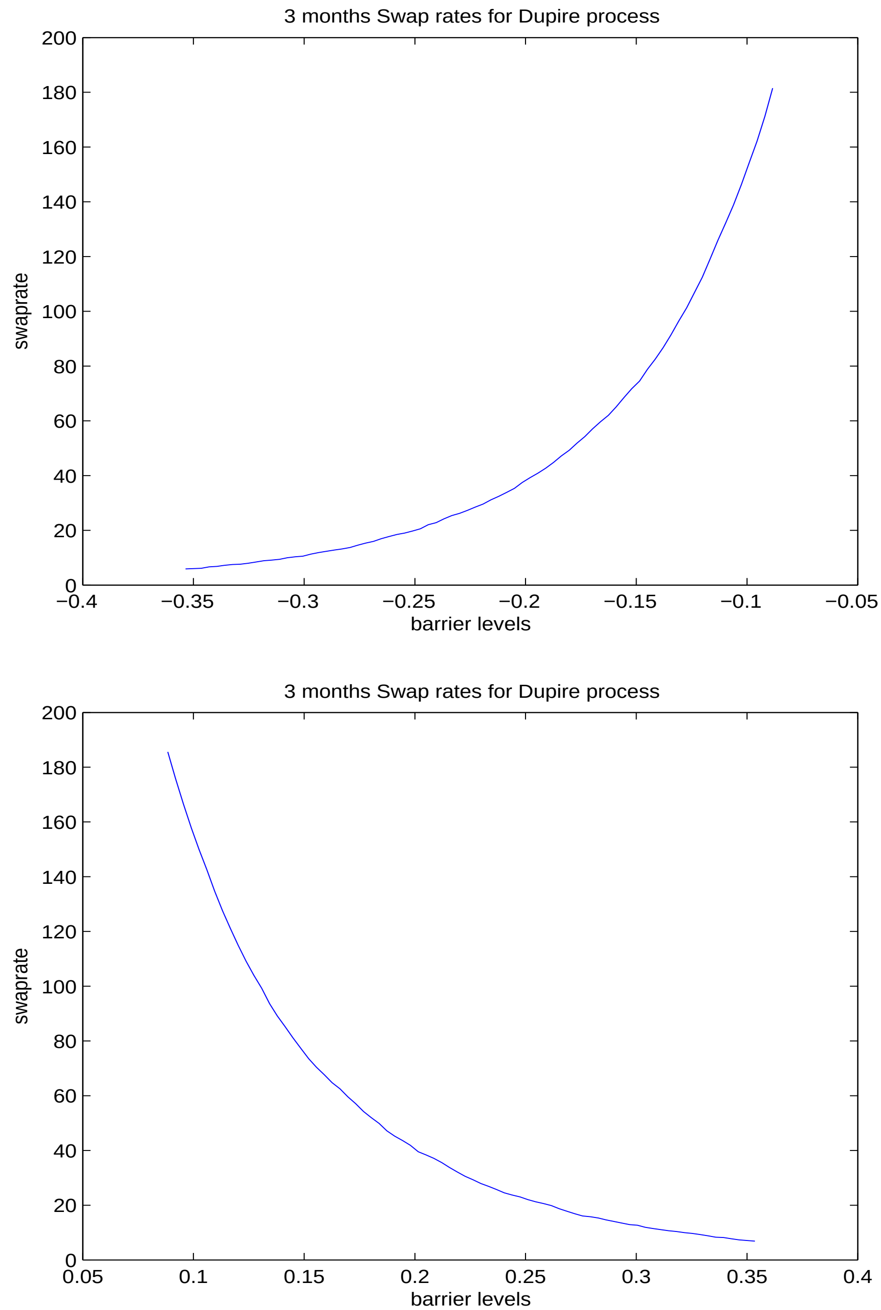
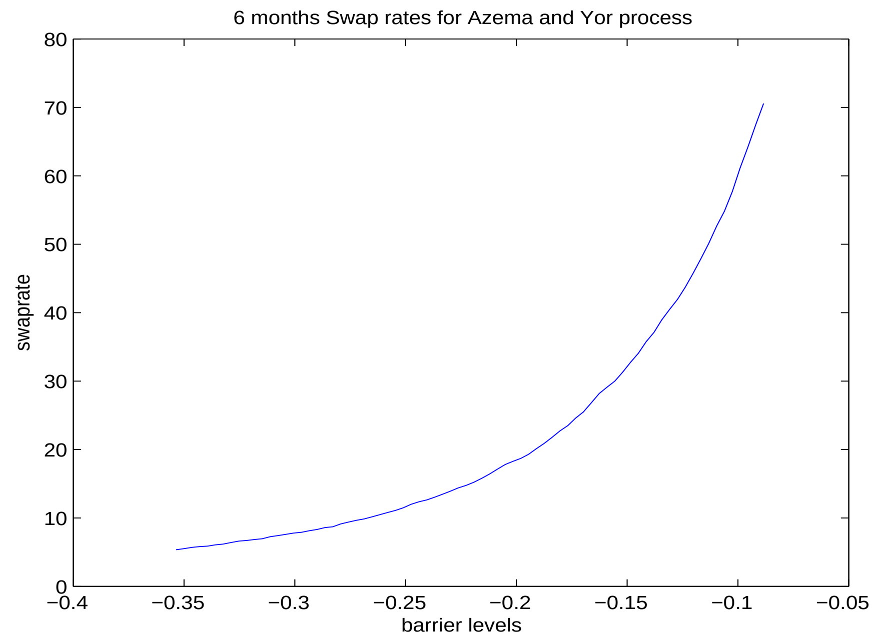


Figure C.3: 3 month swaprate for Dupire process



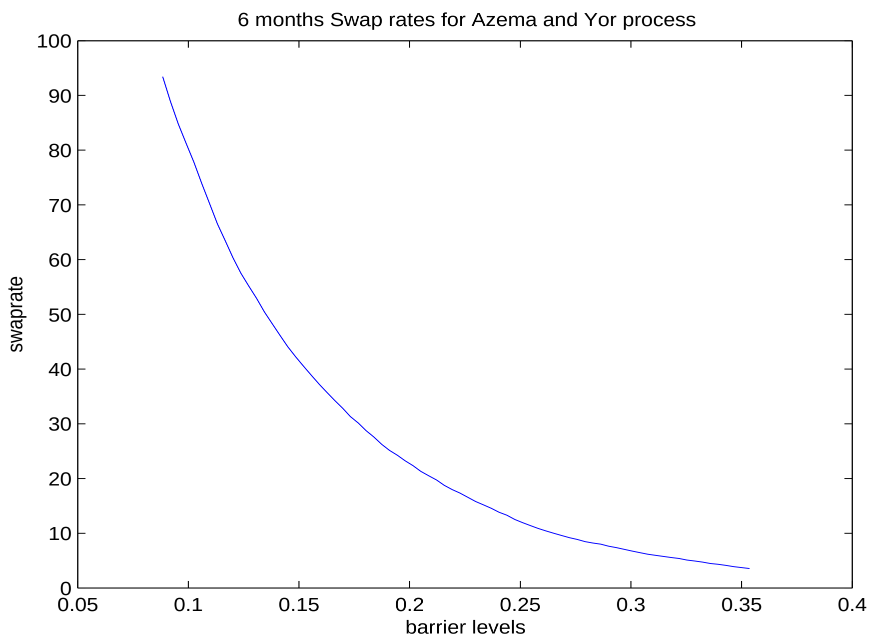
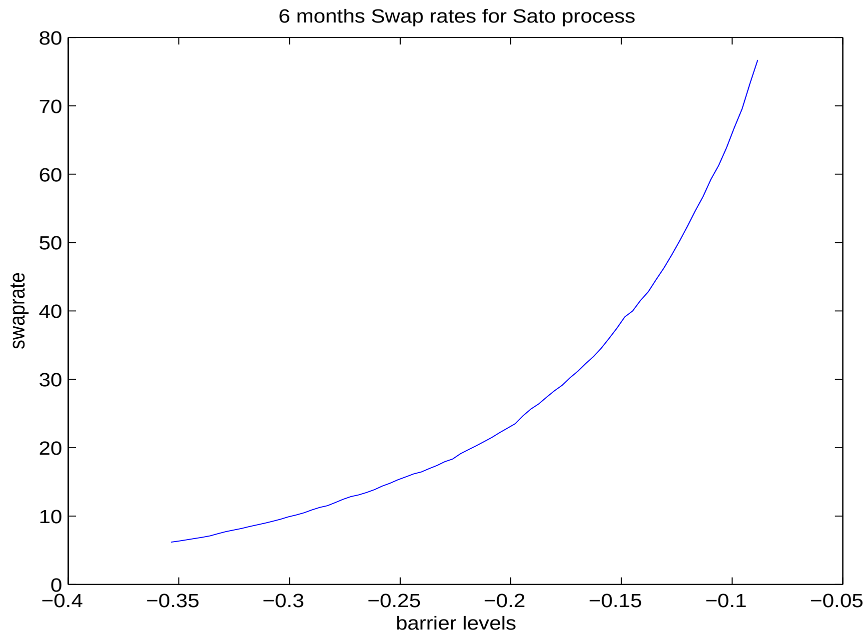


Figure C.4: 6 month swaprate for Azéma and Yor process



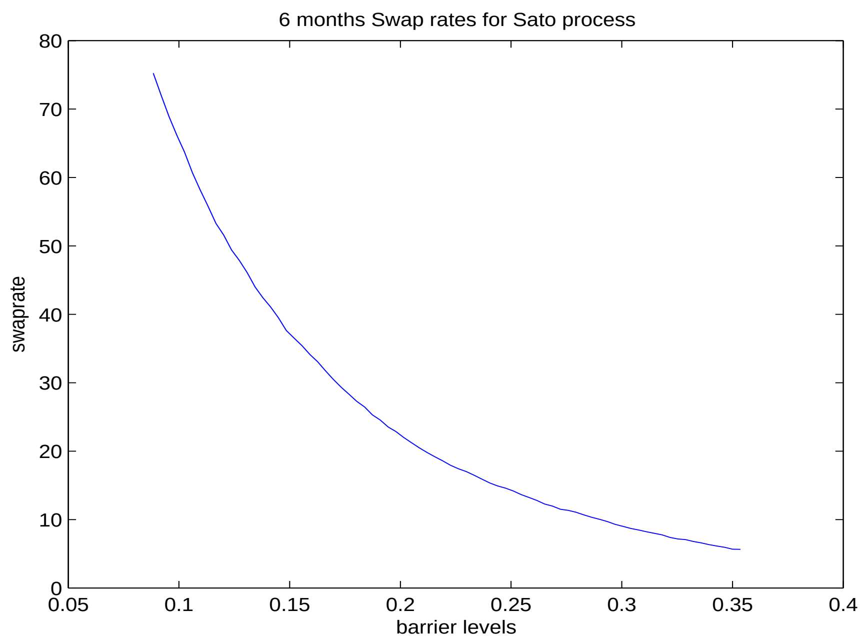
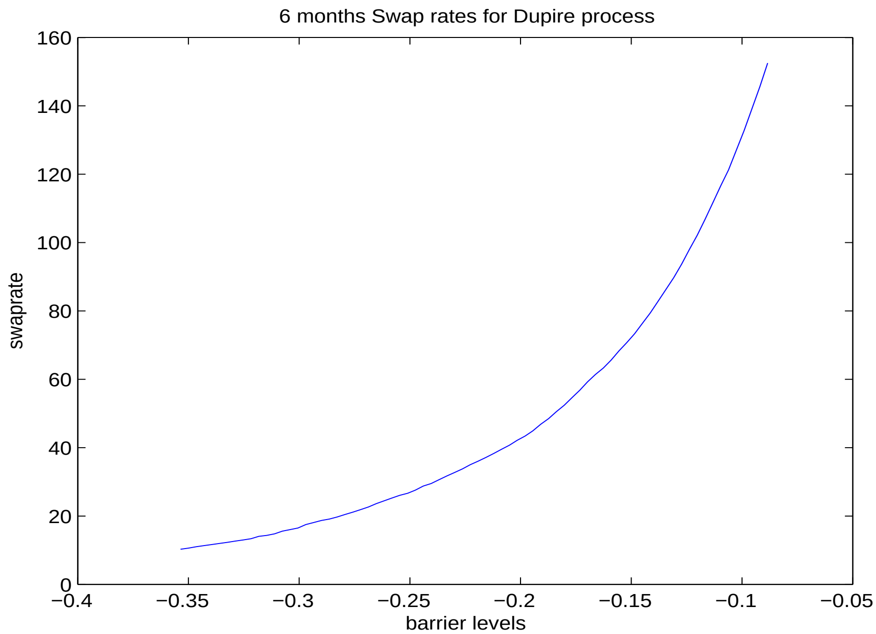


Figure C.5: 6 month swaprate for Sato process



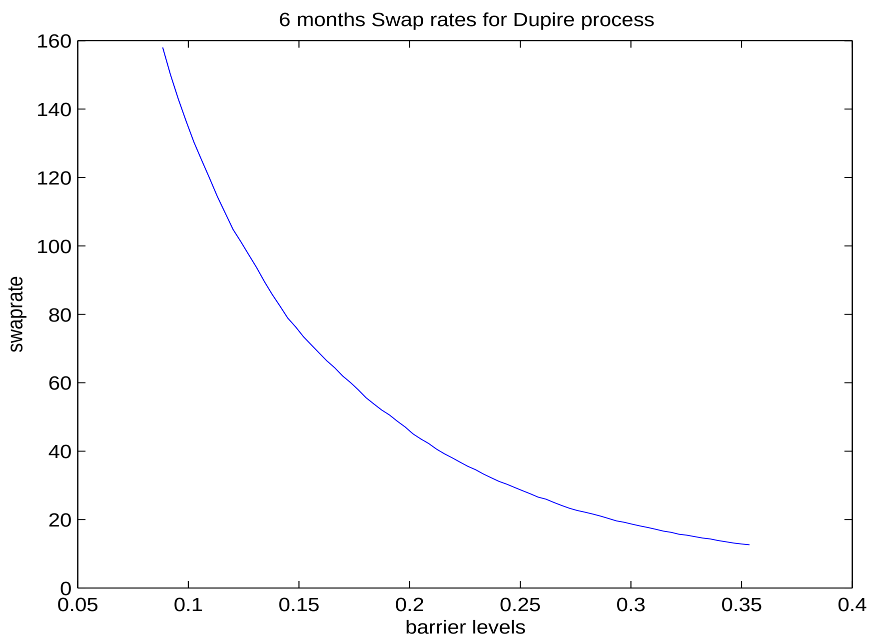


Figure C.6: 6 month swaprate for Dupire process

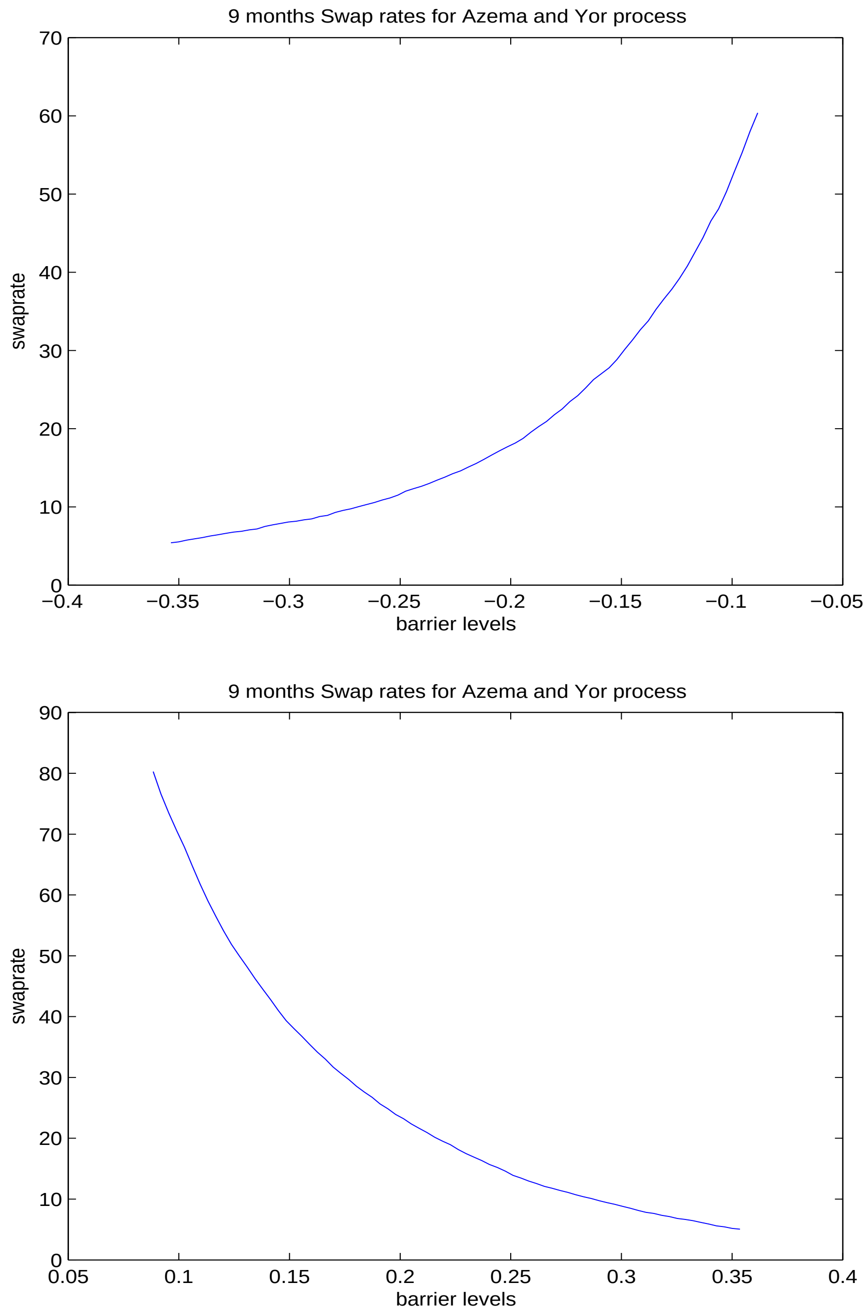


Figure C.7: 9 month swaprate for Azéma and Yor process

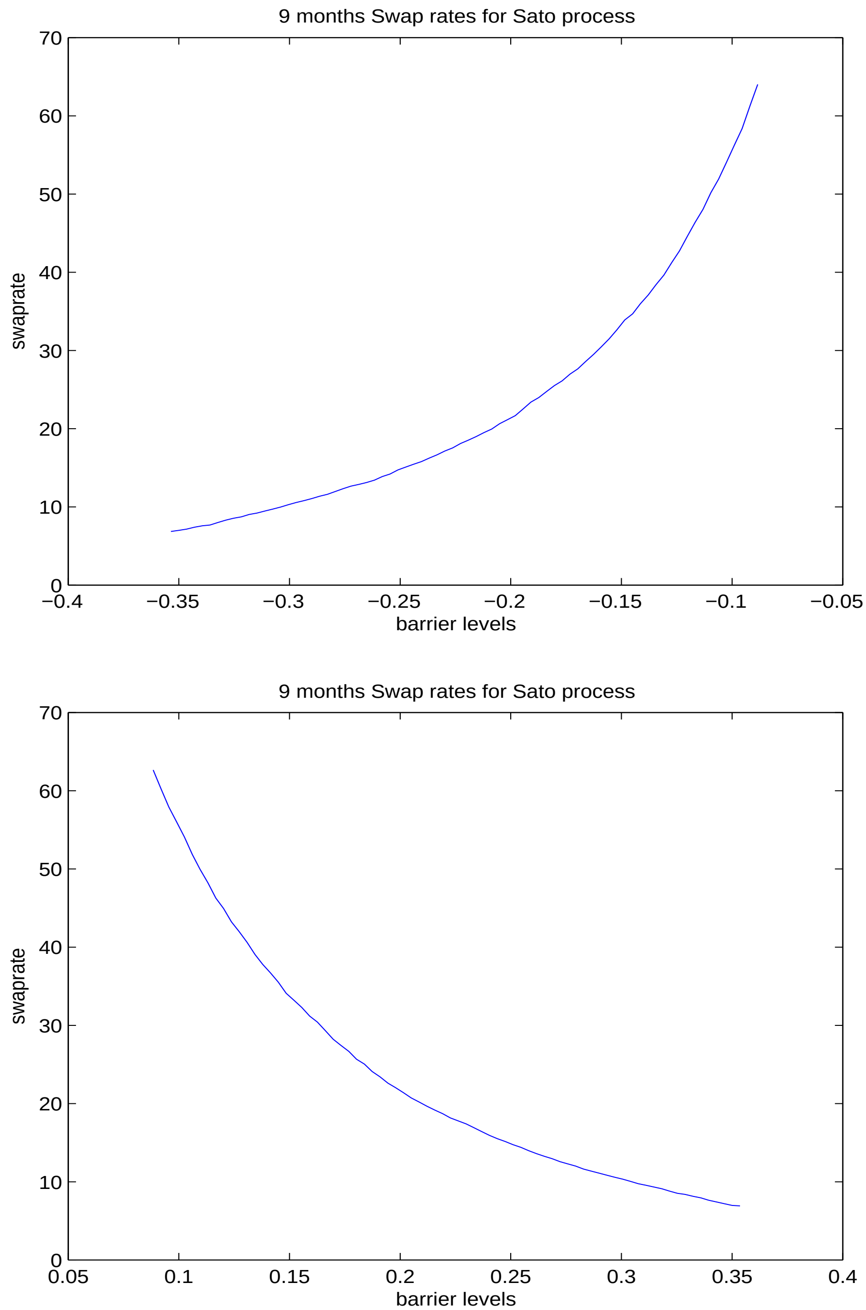
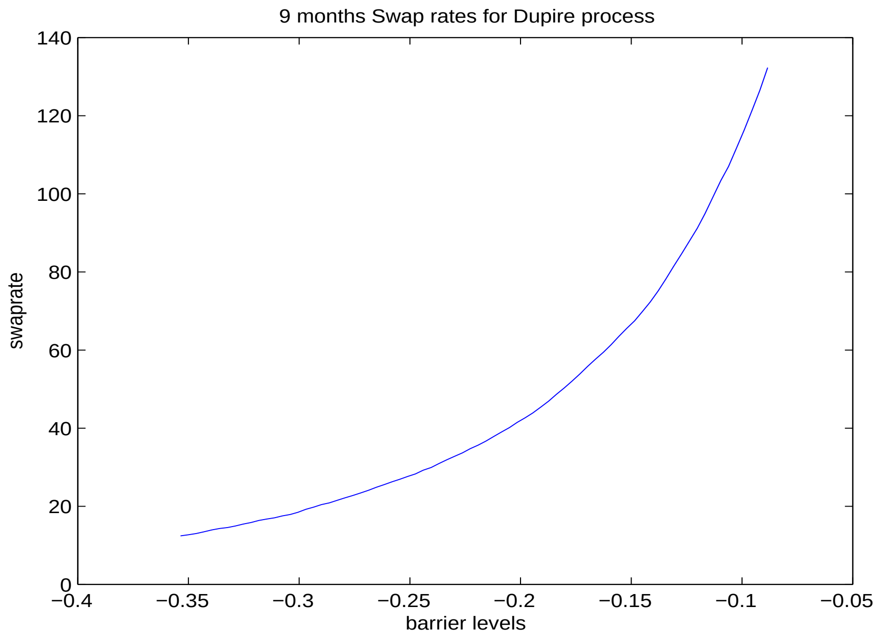


Figure C.8: 9 month swaprate for Sato process



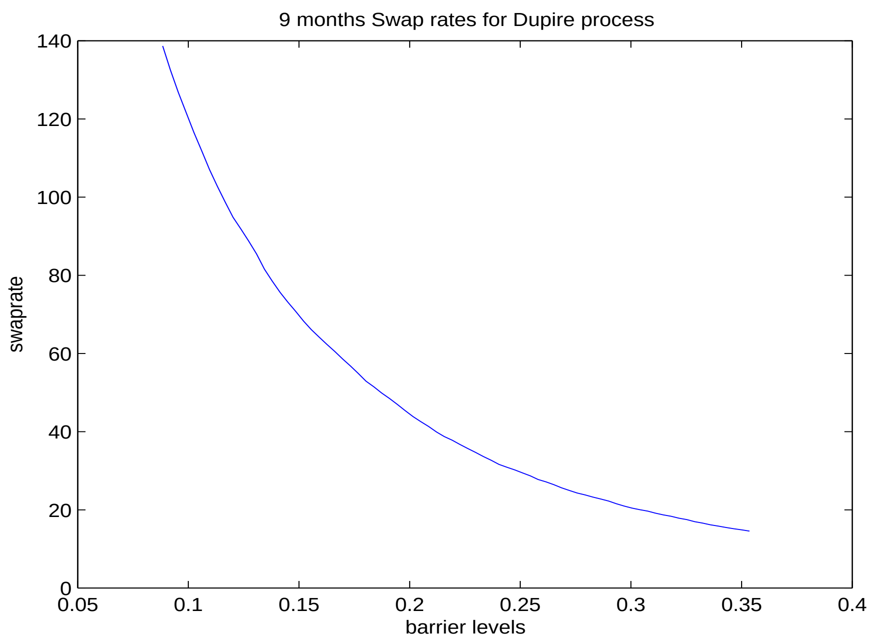


Figure C.9: 9 month swaprate for Dupire process

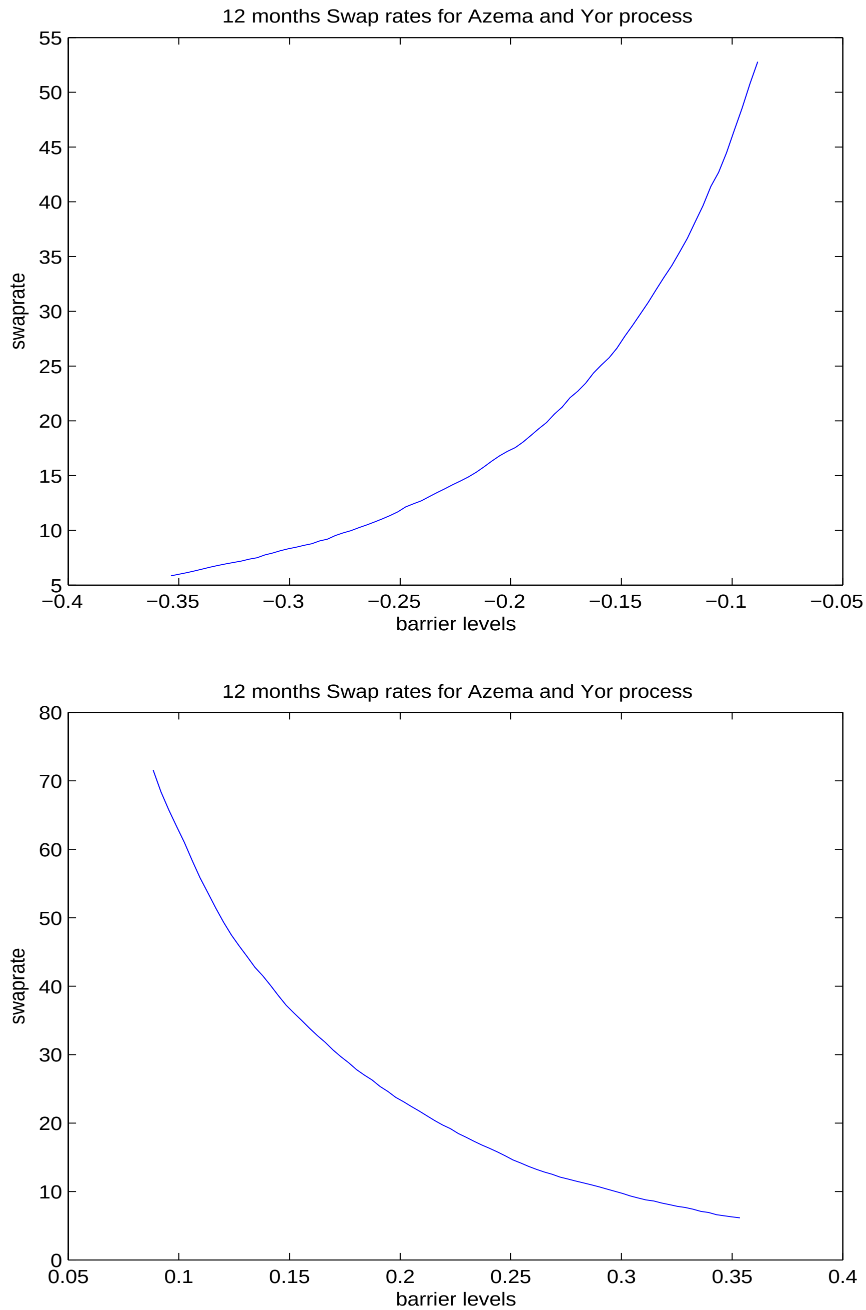


Figure C.10: 12 month swaprate for Azéma and Yor process

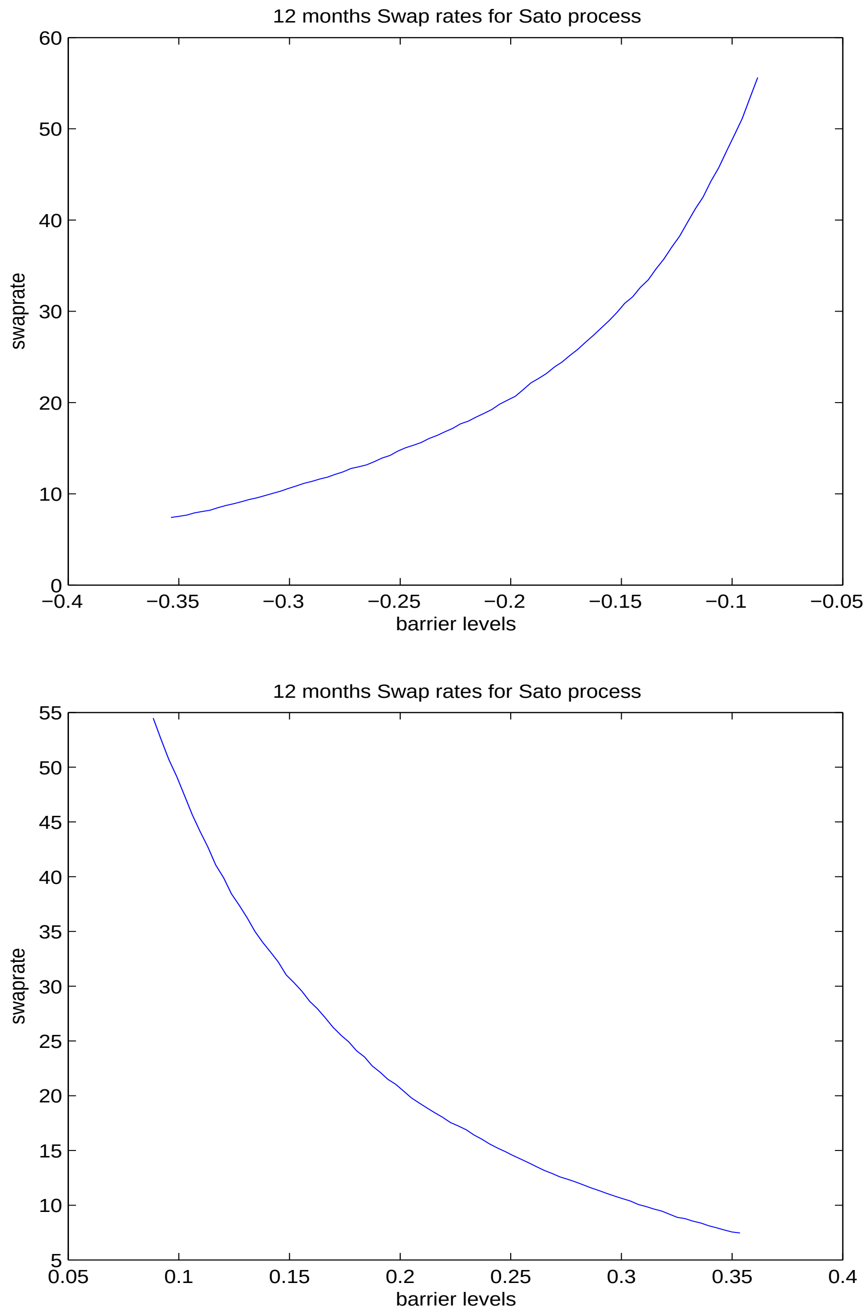


Figure C.11: 12 month swaprate for Sato process

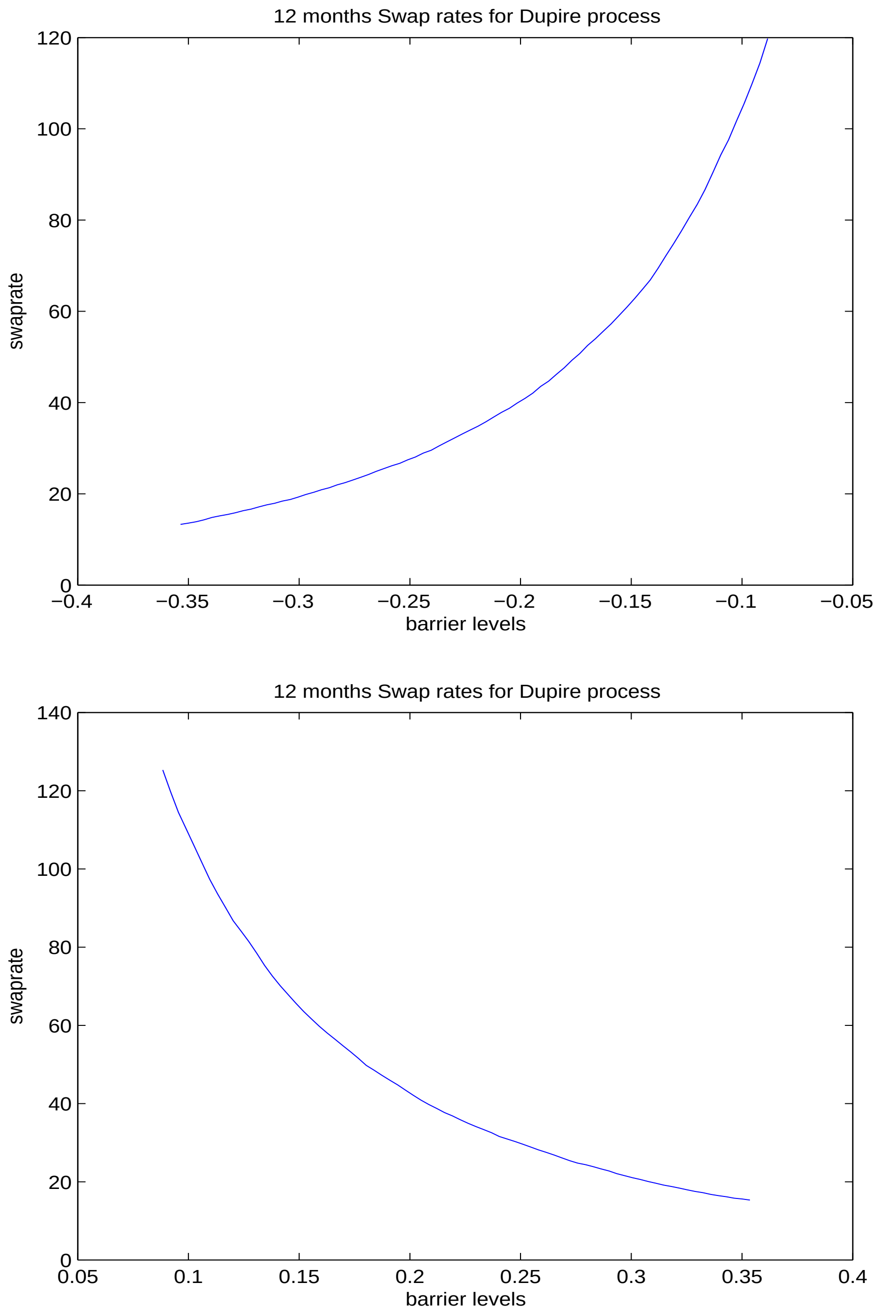


Figure C.12: 12 month swaprate for Dupire process

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