ORBIT PROPAGATION WITH LIE TRANSFER MAPS IN THE PERTURBED KEPLER PROBLEM

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Abstract. The *Lie transfer map* method may be applied to orbit propagation problems in celestial mechanics. This method, described in another paper, is a perturbation method applicable to Hamiltonian systems. In this paper, it is used to calculate orbits for zonal perturbations to the Kepler (two-body) problem, in both expansion in the eccentricity and closed form. In contrast with a normal form method like that of Deprit, the Lie transformations here are used to effect a propagation of phase space in time, and not to transform one Hamiltonian into another.

Key words: Lie transformation, orbit propagation, computer algebra

1. Introduction

The *Lie transfer map* (LTM) method was described in Healy (2001) as a way of computing the perturbation map given a known perturbed Hamiltonian and a solution to the unperturbed problem, with a perturbed harmonic oscillator as an illustration. In this paper, the technique is applied to zonal perturbations of the Kepler (two-body) problem.

The general zonal Hamiltonian is written

$$\mathcal{H} = -\frac{\mu^2}{2L^2} + \frac{\mu}{r}\sum_{\ell=2}^{\infty} J_\ell \left(\frac{\alpha}{r}\right)^{\ell} P_\ell(\sin\phi) \tag{1}$$

with ϕ the latitude of the satellite, and P_ℓ the Legendre polynomials. Transforming the argument of the Legendre polynomial to the more familiar Delaunay variables and writing the first few terms explicitly,

$$\begin{aligned}\mathcal{H} \; = \; & -\frac{\mu^2}{2L^2} + \delta\Bigg(J_2\alpha^2\mu\frac{1}{r^3}\left(\frac{3}{4}s^2 - \frac{1}{2} - \frac{3}{4}s^2\cos(2f+2g)\right) + \\ & + \; J_3\alpha^3\mu\frac{1}{r^4}\left(\left(\frac{15}{8}s^3 - \frac{3}{2}s\right)\sin(f+g) - \frac{5}{8}s^3\sin(3f+3g)\right) +\end{aligned}$$

$$+ J_4\alpha^4\mu\frac{1}{r^5}\left(\frac{105}{64}s^4 - \frac{15}{8}s^2 + \frac{3}{8} + \left(-\frac{35}{16}s^4 + \frac{15}{8}s^2\right)\times\right.$$
$$\left.\times\ \cos(2f+2g) + \frac{35}{64}s^4\cos(4f+4g)\right)\Bigg) \tag{2}$$

(Coffey et al., 1994) where the canonical variables are the Delaunay variables: the mean anomaly l, the argument of perigee g, the right ascension of the ascending node h, and the conjugate momenta, the Delaunay momentum L, the angular momentum G, and the third (z) component of angular momentum H. Other variables are the sine of the inclination $s = \sqrt{1-(H/G)^2}$, the gravitational constant μ, the radius of the earth α, and the perturbation parameters J_n for $n \geqslant 2$.

In the Lie transformation method, time evolution of the system under the Hamiltonian may be represented by the unperturbed solution $\mathcal{U}$ and a sequence of polynomials in the perturbation parameter(s) f_1, f_2, etc.,

$$\mathcal{M} = \cdots e^{:f_2:}e^{:f_1:}\mathcal{U}, \tag{3}$$

where the colon notation represents a Lie operator: $f{:}g = [f, g]$ and the exponential is represented by its Taylor series.

In the Hamiltonian under consideration here, the unperturbed part is expressed in Delaunay variables (specifically, L), but the perturbation part is more naturally expressed in Cartesian variables, specifically the true anomaly f and the radius r. Historically, this has presented a formidable obstacle in propagating orbits for this problem. The two sets are related through Kepler's equation,

$$l = E - e\sin E, \tag{4}$$

where E is the eccentric anomaly,

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \tag{5}$$

and e is the eccentricity. From the true anomaly, the radius may be defined as $r = p/(1+e\cos f)$. Because finding the true anomaly $f(l)$ requires solving a transcendental equation, it is difficult to solve problems directly in *closed form*; one way around that is to do an *eccentricity expansion*. The Lie transfer map method can indeed work without an explicit solution in the canonical variables for the unperturbed problem, because we need only the partial derivatives of the various quantities which are not difficult to come by, and the definite time integrals, which sometimes are. In order to do this, it is necessary to be more precise about the meaning of 'mean anomaly' and 'true anomaly' with regard to the time of evaluation; these terms are elucidated in Section 2.

A particular algebraic expression can usually be put in a number of different forms. In computing an integral, our goals in deciding on a form are threefold:

1. ease of numerical evaluation,
2. ability to confirm the result by differentiation,
3. closed form under computation of higher orders.

The last quality is the hardest to achieve. It means that the result of integration should in the same algebraic class as the integrand. For example, a polynomial $\sum c_n x^n$ for $n \geqslant 0$ and a Fourier series $\sum_n a\cos(nx) + b\sin(mx)$ for a, b constants meet this requirement, but x^n for any n does not, because the integral for $n = -1$ is $\log(x)$, not in the set of polynomials.

Celestial mechanics and astrodynamics have a long history of use of canonical transformations to solve orbit propagation problems. In the last 30 years, Lie algebraic techniques, such as those introduced by Hori (1966) and Deprit (1969) have become popular. In these methods, a normal form (Meyer, 1974), or, loosely, an averaged Hamiltonian, is computed from the actual Hamiltonian. There can be computed, at the same time, transformations that take coordinates in the actual space to those of the idealized Hamiltonian, and back. These coordinate transformations are near-identity, the deviation from identity being dependent on the perturbation, and the new Hamiltonian is simpler, usually by being cyclic in several or all coordinates, and dependent on the momentum only. In contrast, the Lie transfer map method applied here produces a map from phase space at the initial time to phase space at the final time, and not to transform one Hamiltonian into another; that is, it is a direct propagation.

It is possible to summarize the LTM algorithm as follows. First, one must decide on a phase space and time integration method. In the case of the Kepler problem, the choices to be made are not obvious; they are discussed later. Starting with the Hamiltonian (2) and the solution of the unperturbed problem $\mathcal{U}$ acting on a phase space ζ, compute an *interaction Hamiltonian*, $\mathcal{H}_R^{\mathrm{int}}(\zeta) = \mathcal{H}_R(\mathcal{U}\zeta)$ is the perturbed interaction Hamiltonian. From this, compute a series of *rest maps*, initially just the negative of the interaction Hamiltonian, $C_0 = -\mathcal{H}_R^{\mathrm{int}}$. Then compute the first polynomial $f_1 = \int P_1(C_0)\,\mathrm{d}t$, where P_n selects terms of rank n in the perturbation. Then the next rest map is $C_1 = C_0 - \mathrm{iex}(:f_1:)f_1$, with iex the integrated exponential explained in Section 4.3, and the next polynomial is computed from this, $f_2 = \int P_2(C_1)\,\mathrm{d}t$. The procedure goes on until the desired rank of perturbation is obtained. For more details, consult (Healy, 2001); in the language of that paper, the bracket grade Δ here is zero.

While this procedure is straightforward for some problems, like the perturbed harmonic oscillator, the Kepler problem is difficult. We can ease into this by starting with an eccentricity expansion; in a later section, we will treat the closed form problem.

2. Mean Anomaly and True Anomaly at Particular Times

Before computing the maps, we shall investigate closer the true anomaly and mean anomaly. Since we will be concerned with integration over a time interval, we will use the symbol f to indicate the true anomaly and l to indicate the mean anomaly at the beginning of the interval, and the Greek equivalents ϕ and λ to indicate the change in the corresponding quantities over the time interval (or to the 'current' time if time is changing, e.g. in an integral) of interest, so that the true and mean anomalies at the end of the time interval are $f + \phi$ and $l + \lambda$ respectively.[1] The true and mean anomalies at the ends of the intervals can be summarized as follows:

Angle	Beginning	End
Mean anomaly	l	$l + \lambda$
True anomaly	f	$f + \phi$

This means that with respect to time integrals and derivatives, f and l are constants, but ϕ and λ are dependent on time. On the other hand, with respect to the partial derivatives of the Delaunay variables, all except l are non-zero (Appendix A).

The time dependence of the mean anomaly is given by the solution to the unperturbed problem

$$l(t) = l + nt, \tag{6}$$

where $n = \mu^2/L^3$ is the mean motion, and the function $l(t)$ is to be distinguished from the quantity l. The time derivative is simple, $\mathrm{d}l(t)/\mathrm{d}t = n$. Thus $\lambda = nt$.

The time dependence of the true anomaly at an arbitrary time is

$$F(l(t), L, G), \tag{7}$$

where F is an unspecified function. In fact, because of Kepler's equation, F is transcendental and cannot be specified in closed form. That is not a problem for us, however; all we really care about are the partial derivatives of this function, and they are known. The partial derivatives of ϕ (as opposed to f) must take into account the dependence on the first argument of the true anomaly function (7). See Appendix A for more details.

3. Eccentricity Expansion

One approach to the problem of mixed variables in the Kepler problem and its perturbation is to do a formal expansion in the eccentricity in the perturbation.

[1] The usage of ϕ here differs from the more common meaning in the literature of the 'equation of center' $f - l$, and differs from the meaning as satellite latitude used earlier.

This eliminates the true anomaly and puts everything in the Hamiltonian in terms of canonical variables. It has some validity for orbits of small eccentricity; however, see the caution at the end of this section on treating the e series like a perturbation series. The true anomaly and radial terms $\sin f$, $\cos f$, f, and a/r are handled with Fourier–Bessel expansions (Battin, 1987, Section 5.3),

$$\cos f = -e + \frac{2(1-e^2)}{e} \sum_{k=1}^{\infty} J_k(ke) \cos kl, \tag{8a}$$

$$\sin f = 2\sqrt{1-e^2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{\mathrm{d}J_k(ke)}{\mathrm{d}e} \sin kl, \tag{8b}$$

$$\frac{a}{r} = 1 + 2\sum_{k=1}^{\infty} J_k(ke) \cos kl, \tag{8c}$$

$$f = l + 2\sum_{k=1}^{\infty} \frac{1}{k} \left[\sum_{n=-\infty}^{\infty} J_n(-ke) \beta^{|k+n|} \right] \sin kl, \tag{8d}$$

where $\beta = (1-\eta)/e$, and J_n here are Bessel functions of the first kind. Note that the expression for the secular f is particularly inconvenient to compute because of the double summation. The expansion to $\mathcal{O}(e^2)$ for these quantities is readily computed,

$$\cos f = \cos l + e(-1 + \cos 2l) + e^2\big(-\tfrac{9}{8}\cos l + \tfrac{9}{8}\cos 3l\big), \tag{9a}$$

$$\sin f = \sin l + e \sin 2l + e^2\big(-\tfrac{7}{8}\sin l + \tfrac{9}{8}\sin 3l\big), \tag{9b}$$

$$\frac{a}{r} = 1 + e\cos l + e^2 \cos 2l, \tag{9c}$$

$$f = l + 2e\sin l + \tfrac{5}{4}e^2 \sin 2l. \tag{9d}$$

In order to compute Poisson brackets, partial derivatives will be needed,

$$\frac{\partial p}{\partial G} = \frac{2p}{G}, \qquad \frac{\partial \lambda}{\partial L} = \frac{-3\lambda}{L}, \tag{10a}$$

$$\frac{\partial e}{\partial L} = \frac{G^2}{eL^3}, \qquad \frac{\partial e}{\partial G} = -\frac{G}{eL^2}, \tag{10b}$$

$$\frac{\partial \eta}{\partial L} = -\frac{\eta^2}{G}, \qquad \frac{\partial \eta}{\partial G} = \frac{\eta}{G}, \tag{10c}$$

$$\frac{\partial s}{\partial G} = \frac{1}{G}\left(\frac{1}{s} - s\right), \qquad \frac{\partial s}{\partial H} = -\frac{1}{H}\left(\frac{1}{s} - s\right), \tag{10d}$$

which may be found in the literature (e.g. Brouwer, 1959) or derived.

With the above expansions carried to e^2, the Hamiltonian (2) may be written

$$\begin{aligned}
\mathcal{H} \;=\; & -\frac{\mu^2}{2L^2} + \delta\bigg(J_2\alpha^2\mu^4\frac{1}{L^6}\bigg(e^2\bigg(\frac{9}{8}s^2 - \frac{3}{4}\bigg) + \frac{3}{4}s^2 - \frac{1}{2} + \\
& + e\bigg(\frac{9}{4}s^2 - \frac{3}{2}\bigg)\cos l + \frac{3}{8}s^2 e\cos(l+2g) + e^2\bigg(\frac{27}{8}s^2 - \frac{9}{4}\bigg) \times \\
& \times\ \cos 2l + \bigg(\frac{15}{8}s^2e^2 - \frac{3}{4}s^2\bigg)\cos(2l+2g) - \\
& - \frac{21}{8}s^2 e\cos(3l+2g) - \frac{51}{8}s^2e^2\cos(4l+2g)\bigg) + \\
& + J_3\alpha^3\mu^5\frac{1}{L^8}\bigg(e\bigg(\frac{15}{8}s^3 - \frac{3}{2}s\bigg)\sin g + e^2\bigg(-\frac{165}{64}s^3 + \frac{33}{16}s\bigg) \times \\
& \times\ \sin(l-g) + \bigg(e^2\bigg(\frac{15}{4}s^3 - 3s\bigg) + \frac{15}{8}s^3 - \frac{3}{2}s\bigg)\sin(l+g) - \\
& - \frac{5}{64}s^3e^2\sin(l+3g) + e\bigg(\frac{45}{8}s^3 - \frac{9}{2}s\bigg)\sin(2l+g) + \\
& + \frac{5}{8}s^3 e\sin(2l+3g) + e^2\bigg(\frac{795}{64}s^3 - \frac{159}{16}s\bigg)\sin(3l+g) + \\
& + \bigg(\frac{15}{4}s^3e^2 - \frac{5}{8}s^3\bigg)\sin(3l+3g) - \frac{25}{8}s^3 e\sin(4l+3g) - \\
& - \frac{635}{64}s^3e^2\sin(5l+3g)\bigg) +\ J_4\alpha^4\mu^6\frac{1}{L^{10}}\bigg(e^2\bigg(\frac{525}{64}s^4 - \frac{75}{8}s^2 + \\
& + \frac{15}{8}\bigg) + \frac{105}{64}s^4 - \frac{15}{8}s^2 + \frac{3}{8} + e^2\bigg(-\frac{105}{64}s^4 + \frac{45}{32}s^2\bigg)\cos 2g + \\
& + e\bigg(\frac{525}{64}s^4 - \frac{75}{8}s^2 + \frac{15}{8}\bigg)\cos l + e\bigg(-\frac{35}{32}s^4 + \frac{15}{16}s^2\bigg) \times \\
& \times\ \cos(l+2g) + e^2\bigg(\frac{525}{32}s^4 - \frac{75}{4}s^2 + \frac{15}{4}\bigg)\cos 2l + \\
& + \bigg(e^2\bigg(-\frac{35}{16}s^4 + \frac{15}{8}s^2\bigg) - \frac{35}{16}s^4 + \frac{15}{8}s^2\bigg)\cos(2l+2g) + \\
& + \frac{35}{128}s^4e^2\cos(2l+4g) + e\bigg(-\frac{315}{32}s^4 + \frac{135}{16}s^2\bigg)\cos(3l+2g) - \\
& - \frac{105}{128}s^4 e\cos(3l+4g) + e^2\bigg(-\frac{1855}{64}s^4 + \frac{795}{32}s^2\bigg)\cos(4l+2g) + \\
& + \bigg(-\frac{385}{64}s^4e^2 + \frac{35}{64}s^4\bigg)\cos(4l+4g) + \frac{455}{128}s^4 e\cos(5l+4g) + \\
& + \frac{1785}{128}s^4e^2\cos(6l+4g)\bigg)\bigg) \qquad (11)
\end{aligned}$$

with J_2, J_3, J_4 now the zonal coefficients. The unperturbed solution maps all Delaunay variables unchanged except the mean anomaly; if the initial mean anomaly is l, at the end of the time interval t (Section 2) it is $l + \lambda$. The interaction Hamiltonian is therefore computed by transforming the Hamiltonian by the unperturbed solution, that is, substituting $l + \lambda$ for l. The perturbation transfer map at first rank is then computed (Healy, 2001) as the time integral of the interaction Hamiltonian,

$$\begin{aligned}
\mathcal{P} \;=\; & \exp{:}\mathcal{O}(\delta^2){:}\exp{:}\delta\Bigg(J_2\alpha^2\mu^2\frac{1}{L^3}\Bigg(\lambda\Bigg(e^2\Bigg(-\frac{9}{8}s^2+\frac{3}{4}\Bigg)-\\
& -\frac{3}{4}s^2+\frac{1}{2}\Bigg)+e\Bigg(\frac{9}{4}s^2-\frac{3}{2}\Bigg)\sin l+\frac{3}{8}s^2e\sin(l+2g)+\\
& +e\Bigg(-\frac{9}{4}s^2+\frac{3}{2}\Bigg)\sin(l+\lambda)-\frac{3}{8}s^2e\sin(l+\lambda+2g)+\\
& +e^2\Bigg(\frac{27}{16}s^2-\frac{9}{8}\Bigg)\sin 2l+\Bigg(\frac{15}{16}s^2e^2-\frac{3}{8}s^2\Bigg)\sin(2l+2g)+\\
& +e^2\Bigg(-\frac{27}{16}s^2+\frac{9}{8}\Bigg)\sin(2l+2\lambda)+\Bigg(-\frac{15}{16}s^2e^2+\frac{3}{8}s^2\Bigg)\times\\
& \times\ \sin(2l+2\lambda+2g)-\frac{7}{8}s^2e\sin(3l+2g)+\frac{7}{8}s^2e\times\\
& \times\ \sin(3l+3\lambda+2g)-\frac{51}{32}s^2e^2\sin(4l+2g)+\frac{51}{32}s^2e^2\times\\
& \times\ \sin(4l+4\lambda+2g)\Bigg)+J_3\alpha^3\mu^3\frac{1}{L^5}\Bigg(\lambda e\Bigg(-\frac{15}{8}s^3+\frac{3}{2}s\Bigg)\sin g+\\
& +e^2\Bigg(\frac{165}{64}s^3-\frac{33}{16}s\Bigg)\cos(l-g)+\Bigg(e^2\Bigg(-\frac{15}{4}s^3+3s\Bigg)-\\
& -\frac{15}{8}s^3+\frac{3}{2}s\Bigg)\cos(l+g)+\frac{5}{64}s^3e^2\cos(l+3g)+\\
& +e^2\Bigg(-\frac{165}{64}s^3+\frac{33}{16}s\Bigg)\cos(l+\lambda-g)+\Bigg(e^2\Bigg(\frac{15}{4}s^3-3s\Bigg)+\\
& +\frac{15}{8}s^3-\frac{3}{2}s\Bigg)\cos(l+\lambda+g)-\frac{5}{64}s^3e^2\cos(l+\lambda+3g)+\\
& +e\Bigg(-\frac{45}{16}s^3+\frac{9}{4}s\Bigg)\cos(2l+g)-\frac{5}{16}s^3e\cos(2l+3g)+\\
& +e\Bigg(\frac{45}{16}s^3-\frac{9}{4}s\Bigg)\cos(2l+2\lambda+g)+\frac{5}{16}s^3e\cos(2l+2\lambda+3g)+\\
& +e^2\Bigg(-\frac{265}{64}s^3+\frac{53}{16}s\Bigg)\cos(3l+g)+\\
& +\Bigg(-\frac{5}{4}s^3e^2+\frac{5}{24}s^3\Bigg)\cos(3l+3g)+e^2\Bigg(\frac{265}{64}s^3-\frac{53}{16}s\Bigg)\times
\end{aligned}$$

$$
\begin{aligned}
&\times\ \cos(3l+3\lambda+g)+\left(\frac{5}{4}s^3e^2-\frac{5}{24}s^3\right)\cos(3l+3\lambda+3g)+\\
&+\frac{25}{32}s^3e\cos(4l+3g)-\frac{25}{32}s^3e\cos(4l+4\lambda+3g)+\\
&+\frac{127}{64}s^3e^2\cos(5l+3g)-\frac{127}{64}s^3e^2\cos(5l+5\lambda+3g)\Bigg)+\\
&+J_4\alpha^4\mu^4\frac{1}{L^7}\Bigg(\lambda\Bigg(e^2\left(-\frac{525}{64}s^4+\frac{75}{8}s^2-\frac{15}{8}\right)-\\
&-\frac{105}{64}s^4+\frac{15}{8}s^2-\frac{3}{8}\Bigg)+\lambda e^2\left(\frac{105}{64}s^4-\frac{45}{32}s^2\right)\cos 2g+\\
&+e\left(\frac{525}{64}s^4-\frac{75}{8}s^2+\frac{15}{8}\right)\sin l+e\left(-\frac{35}{32}s^4+\frac{15}{16}s^2\right)\sin(l+2g)+\\
&+e\left(-\frac{525}{64}s^4+\frac{75}{8}s^2-\frac{15}{8}\right)\sin(l+\lambda)+e\left(\frac{35}{32}s^4-\frac{15}{16}s^2\right)\times\\
&\times\ \sin(l+\lambda+2g)+e^2\left(\frac{525}{64}s^4-\frac{75}{8}s^2+\frac{15}{8}\right)\sin 2l+\\
&+\left(e^2\left(-\frac{35}{32}s^4+\frac{15}{16}s^2\right)-\frac{35}{32}s^4+\frac{15}{16}s^2\right)\sin(2l+2g)+\\
&+\frac{35}{256}s^4e^2\sin(2l+4g)+e^2\left(-\frac{525}{64}s^4+\frac{75}{8}s^2-\frac{15}{8}\right)\times\\
&\times\ \sin(2l+2\lambda)+\left(e^2\left(\frac{35}{32}s^4-\frac{15}{16}s^2\right)+\frac{35}{32}s^4-\frac{15}{16}s^2\right)\times\\
&\times\ \sin(2l+2\lambda+2g)-\frac{35}{256}s^4e^2\sin(2l+2\lambda+4g)+\\
&+e\left(-\frac{105}{32}s^4+\frac{45}{16}s^2\right)\sin(3l+2g)-\frac{35}{128}s^4e\sin(3l+4g)+\\
&+e\left(\frac{105}{32}s^4-\frac{45}{16}s^2\right)\sin(3l+3\lambda+2g)+\frac{35}{128}s^4e\times\\
&\times\ \sin(3l+3\lambda+4g)+e^2\left(-\frac{1855}{256}s^4+\frac{795}{128}s^2\right)\sin(4l+2g)+\\
&+\left(-\frac{385}{256}s^4e^2+\frac{35}{256}s^4\right)\sin(4l+4g)+e^2\left(\frac{1855}{256}s^4-\frac{795}{128}s^2\right)\times\\
&\times\ \sin(4l+4\lambda+2g)+\left(\frac{385}{256}s^4e^2-\frac{35}{256}s^4\right)\sin(4l+4\lambda+4g)+\\
&+\frac{91}{128}s^4e\sin(5l+4g)-\frac{91}{128}s^4e\sin(5l+5\lambda+4g)+\\
&+\frac{595}{256}s^4e^2\sin(6l+4g)-\frac{595}{256}s^4e^2\sin(6l+6\lambda+4g)\Bigg)\Bigg): \qquad (12)
\end{aligned}
$$

The author has computed the map to second order and checked that the result solves Hamilton's equations of motion for the map (see Appendix B for the procedure). The Lie polynomial has 897 terms at this order. The third order polynomial has 18541 terms.

A note of caution is in order. While a formal expansion in a small parameter is sensible, one cannot assume that *derivatives* of that parameter are small. Indeed, derivatives such as $\partial e/\partial L$ have a factor e^{-1}, so that the smaller e gets, the larger its derivative gets. Therefore thinking of the series as being like the Taylor series in the perturbation is potentially hazardous: the series may not be cut off arbitrarily in the course of the calculation. Starting with the Hamiltonian expanded through a given order in e, one must maintain all resulting terms in e until the end, because Poisson brackets may 'pull down' the e order and thus make significant a term that would have been cut off. This does not happen in the case of perturbation expansions because satisfying the *perturbation transformation* axioms insure map expansions can be made consistently (Healy, 2001).

4. Closed Form

The specific zonal Hamiltonian used for the closed form factorization is given above (2). As with the e expansion form of the map, the map computation is essentially an application of the procedure described in Healy (2001). The difficulty here is that the partial derivatives and time integral, because of the implicit form of the expressions, are not straightforward. We thus begin by turning our attention to these operators.

4.1. TIME INTEGRATION

Because everything is explicit in the e-expanded form, time integration is not difficult: the mean anomaly is a straightforward proxy for time. In the closed form development, where dependence on canonical variables is implicit through the true anomaly, it is more difficult. Prior to developing the closed form results, this section will elaborate the time integration considerations.

We will integrate in terms of the true anomaly using the relation

$$\mathrm{d}F = \frac{1}{\eta^3}\frac{p^2}{r^2}\,\mathrm{d}l = \frac{\mu^2}{L^3\eta^3}\frac{p^2}{r^2}\,\mathrm{d}t = \frac{G}{r^2}\,\mathrm{d}t, \tag{13}$$

or $\mathrm{d}t = (r^2/G)\,\mathrm{d}F$ (Brouwer, 1959). Notice that there is a factor of r^2 which if expressed in terms of the true anomaly would produce $(1 + e\cos(f + \phi))^2$ in the denominator; thus a time integration does not close in the algebra of Fourier series in the true anomaly. If we insist that the integrand have a factor of r^m for $m \leqslant -2$, integration will then close in that algebra. In fact, the requirement is more stringent than this, as we shall see.

Because the partial derivative $\partial\phi/\partial L$ produces explicit time terms, that is, secular terms in λ (35), and thus explicit mean anomaly terms (Section 2), we must be prepared to handle mixed mean and true anomaly expressions. Specifically, we must be able to integrate a term like $l^n \operatorname{trig} m\phi$ where $m, n \geqslant 0$ are integers and 'trig' is shorthand for sine or cosine.

The integration algorithm treats terms differently depending on the coefficients and exponents of the true and mean anomalies, and the exponent of r. If the true anomaly does not occur and the exponent of r is zero, the only time-dependent terms are in the mean anomaly, and integration is straightforward. If the mean anomaly does not occur and the exponent of r is no greater than -2, the integration can proceed via the change of variables (13); the bound on the exponent of r insures that an integrable term remains. The remaining cases are more difficult. As we shall see (14), we need to treat terms with a linear secular component in the mean anomaly, harmonic in the true anomaly, and $r^{-\nu}$. In computing the map at second order in the perturbation, $\nu = m + 1$ for the zonal harmonic J_m.

An alternative to this approach is to integrate the individual terms, with a goal of reducing the expression to a minimal set of non-integrable functions (Jefferys, 1971). The integral has mixed linear secular mean anomaly and a trigonometric function of the true anomaly,

$$I = \int \frac{\lambda}{r^\nu} \operatorname{trig} m\phi \, \mathrm{d}t = n \int \frac{t}{r^\nu} \operatorname{trig} m\phi \, \mathrm{d}t. \tag{14}$$

Integrating by parts, we evaluate a new quantity J,

$$\begin{aligned} J &= \int \frac{\operatorname{trig} m\phi}{r^\nu} \, \mathrm{d}t = \frac{1}{G} \int \frac{\operatorname{trig} m\phi}{r^{\nu-2}} \, \mathrm{d}\phi \\ &= \frac{1}{Gp^{\nu-2}} \int \operatorname{trig} m\phi (1 + e\cos(f + \phi))^{\nu-2} \, \mathrm{d}\phi, \end{aligned} \tag{15}$$

and this integral over ϕ is easy to do, provided $\nu \geqslant 2$. For the second term, we must perform a time integral of J. The terms involving the true anomaly must be synchronized before being integrated; this issue is addressed below. The terms $m \neq 0$ in the integrand of (15) can be categorized as $r^0 \cos m\phi$ and $r^0 \sin m\phi$ which can be recast in a canonical form (Healy, 2000): $r^0 \cos mF$ can be written as the sum of terms of the form $r^0 \cos 0F$, $r^0 \cos F$, and $(p^2/r^2) \cos mF$, for integers m, and $r^0 \cos mF$ can be written as the sum of terms of the form $r^0 \sin F$, $r^0 \sin 2F$, and $(p^2/r^2) \sin mF$.

The integrals of quantities that result from this canonical form may all be computed. Of course, the $(p^2/r^2) \sin kF$ and $(p^2/r^2) \cos kF$ terms are easy to integrate with respect to time by making the change of variables (13). The integral of constant terms $\int r^0 \cos 0F \, \mathrm{d}l = l$ gives a secular mean anomaly (time) term. The three remaining integrals are (Kelly, 1989):

$$\int \cos F \, \mathrm{d}l = -el + \frac{r}{p}\eta^3 \sin F, \tag{16a}$$

$$\int \sin F \, \mathrm{d}l = -\frac{r}{p}\eta(e + \cos F), \tag{16b}$$

$$\int \sin 2F \, \mathrm{d}l = \frac{2\eta}{e^2}\left[-\eta^2 \log\left(\frac{p}{r}\right) + \frac{r}{p}e(e + \cos F)\right]. \tag{16c}$$

In order to compute the map, it will be necessary to take Poisson brackets, and thus partial derivatives, of the Lie polynomials, and thus of these quantities for higher ranks.

Calculating higher ranks of the map presents a problem however: the expression r/p which occurs in all three is $1/(1 + e\cos F)$, and we thus no longer have a Fourier series in the true anomaly f. In addition, the $\sin 2F$ integration has $\log(1 + e\cos F)$ which is even more difficult to integrate.

The case $m = 0$ (with trig $=$ cos) is treated separately. In this case, J will have a secular term in ϕ. Thus the time integral of J will require the evaluation of the integral $\Gamma = \int \phi \, \mathrm{d}t$. There is no representation for Γ in terms of elementary functions.

Any expression where all angles correspond to a particular time will be called *synchronous*. Specifically, an expression is synchronous either at the beginning of the time interval, with the multipliers of ϕ and λ zero, or at the end of the interval with the multipliers of f and ϕ equal and the multipliers of l and λ equal. For example, $(l + \lambda)\cos(2f + 2\phi)$ is synchronous at the end of the interval. Using elementary rules of trigonometry, it is possible to factor trigonometric functions into the sum of products of synchronous terms. For example, $\cos(f - 5\phi) = \cos(5f + 5\phi)\cos 6f + \sin(5f + 5\phi)\sin 6f$. We may integrate a non-synchronous expression by factoring into a synchronous-beginning and synchronous-end and integrating the latter; because the synchronous-beginning terms are not time-dependent, they may be treated as constants.

One may proceed with integration in two steps. First, the expression is factorized synchronously as described in the previous paragraph. Before recombining, the terms synchronous at the end of the interval are integrated. This integration is performed by putting the expression in the canonical form (Healy, 2000, Section 7); cosines of the true anomaly can be expressed as a linear combination of a constant term, $\cos f$, and p^2/r^2 times cosines of the true anomaly, and sines of the true anomaly can be expressed as a linear combination of $\sin f$, $\sin 2f$ and p^2/r^2 times cosines of the true anomaly. Once in this form, the constant term gives a secular mean anomaly, the straight sines and cosines are treated as above (16), and any terms with p^2/r^2 are easily integrated with the relation (13).

4.2. PARTIAL DERIVATIVES

Unlike the eccentricity expansion case, the partial derivatives have a subtlety because of the implicit time dependence (Section 2). The derivatives of the true anomaly depend on what true anomaly is the subject of discussion: the true anomaly at the start of the interval f, or the change in true anomaly over the interval ϕ. The latter has an explicit dependence on time via the mean anomaly (7); see Appendix A. With true and mean anomaly, the time of evaluation has been adequately addressed so that confusion should be reduced. Problems arise, however, in quantities like the radial distance r whose time of evaluation is ambiguous. The distance r_0 at the beginning of an interval, $p/r_0 = 1 + e\cos f$, is clearly distinguished from the distance r_1 at the end, $p/r_1 = 1 + e\cos(f+\phi)$, but also, the time derivatives of the latter will need to take account of the explicit time dependence of ϕ.

The set of partial derivatives used depends on the context of the differentiation. For a time differentiation, we will need derivatives of Γ with respect to time and its partial derivatives with respect to L, G, and H. Also of course $\partial\lambda/\partial t = n$, and $\partial\phi/\partial t = G/r^2$ where r is evaluated at the end of the interval.

An important aspect of computing in the simplest terms and checking the validity of a result analytically is to put expressions into a canonical form, that is, have a single representation for any of a set of equivalent expressions. The Poisson brackets in this computation have the canonical form as follows. First, the expression is reduced to having a common factor of L, G, and H by insertion of factors of $\eta = G/L = \sqrt{1-e^2}$ and $1 - s^2 = H/G$. Although this is not a canonical form, it helps to simplify the form and allow terms to cancel. Second, the factors of e and η are reduced to a canonical form where the exponents of η are non-negative and e has an exponent no higher than 1 (Healy, 2000). Finally, we reduce to the minimum exponent of L by extracting an appropriate factor of η.

4.3. THE MAP

With the definitions above, we have the tools at hand to compute the map in closed form, that is, with no eccentricity expansion, to first order in J_2. The map is computed by the procedure outlined in Section 1; the first step is to transform the Hamiltonian (2) using the unperturbed solution which maps the mean anomaly to $l + \lambda$ and the true anomaly to $f + \phi$ to get the interaction Hamiltonian,

$$
\begin{aligned}
\mathcal{H}^{\text{int}} = {} & -\frac{\mu^2}{2L^2} + \delta\Bigg(J_2\alpha^2\mu\frac{1}{r^3}\left(\frac{3}{4}s^2 - \frac{1}{2} - \frac{3}{4}s^2\cos(2f + 2\phi + 2g)\right) + \\
& + J_3\alpha^3\mu\frac{1}{r^4}\left(\left(\frac{15}{8}s^3 - \frac{3}{2}s\right)\sin(f + \phi + g) - \frac{5}{8}s^3 \times \right. \\
& \left. \times \sin(3f + 3\phi + 3g)\right) + J_4\alpha^4\mu\frac{1}{r^5}\left(\frac{105}{64}s^4 - \frac{15}{8}s^2 + \frac{3}{8} + \right.
\end{aligned}
$$

$$+ \left(-\frac{35}{16}s^4 + \frac{15}{8}s^2 \right) \cos(2f + 2\phi + 2g) +$$
$$+ \frac{35}{64}s^4 \cos(4f + 4\phi + 4g) \Bigg) \Bigg) + \mathcal{O}(\delta^3). \tag{17}$$

Because of the transcendental relation between the mean and true anomalies, it is useful to keep both in an extended phase space, that is, f, l, g, h, L, G, H. Of course, the real phase space is only the latter six variables, but with the correct derivatives, maps for f as well can be computed.

From the interaction Hamiltonian, computation of the time integral (Section 4.1) gives the map as a factored product expansion to first order

$$\begin{aligned}
f_1 = {} & \delta\Bigg(J_2 \alpha^2 \mu^2 \frac{1}{G^3} \Bigg(\phi\left(-\frac{3}{4}s^2 + \frac{1}{2} \right) + e\left(\frac{3}{4}s^2 - \frac{1}{2} \right) \sin f - \\
& -\frac{3}{8}s^2 e \sin(f + 2g) + e\left(-\frac{3}{4}s^2 + \frac{1}{2} \right) \sin(f + \phi) + \\
& + \frac{3}{8}s^2 e \sin(f + \phi + 2g) - \frac{3}{8}s^2 \sin(2f + 2g) + \\
& + \frac{3}{8}s^2 \sin(2f + 2\phi + 2g) - \frac{1}{8}s^2 e \sin(3f + 2g) + \\
& + \frac{1}{8}s^2 e \sin(3f + 3\phi + 2g) \Bigg) + \\
& + J_3 \alpha^3 \mu^3 \frac{1}{G^5} \Bigg(\phi e\left(-\frac{15}{8}s^3 + \frac{3}{2}s \right) \sin g + \\
& + e^2\left(\frac{15}{32}s^3 - \frac{3}{8}s \right) \cos(f - g) + \left(e^2\left(-\frac{15}{16}s^3 + \frac{3}{4}s \right) - \right. \\
& \left. - \frac{15}{8}s^3 + \frac{3}{2}s \right) \cos(f + g) + \frac{5}{32}s^3 e^2 \cos(f + 3g) + \\
& + e^2\left(-\frac{15}{32}s^3 + \frac{3}{8}s \right) \cos(f + \phi - g) + \left(e^2\left(\frac{15}{16}s^3 - \frac{3}{4}s \right) + \right. \\
& \left. + \frac{15}{8}s^3 - \frac{3}{2}s \right) \cos(f + \phi + g) - \frac{5}{32}s^3 e^2 \cos(f + \phi + 3g) + \\
& + e\left(-\frac{15}{16}s^3 + \frac{3}{4}s \right) \cos(2f + g) + \frac{5}{16}s^3 e \cos(2f + 3g) + \\
& + e\left(\frac{15}{16}s^3 - \frac{3}{4}s \right) \cos(2f + 2\phi + g) - \\
& - \frac{5}{16}s^3 e \cos(2f + 2\phi + 3g) + e^2\left(-\frac{5}{32}s^3 \frac{1}{8}s \right) \times
\end{aligned}$$

$$
\begin{aligned}
&\times\ \cos(3f+g) + \left(\frac{5}{48}s^3e^2 + \frac{5}{24}s^3\right)\times\\
&\times\ \cos(3f+3g) + e^2\left(\frac{5}{32}s^3 - \frac{1}{8}s\right)\cos(3f+3\phi+g) +\\
&+ \left(-\frac{5}{48}s^3e^2 - \frac{5}{24}s^3\right)\cos(3f+3\phi+3g) +\\
&+ \frac{5}{32}s^3e\cos(4f+3g) - \frac{5}{32}s^3e\cos(4f+4\phi+3g) +\\
&+ \frac{1}{32}s^3e^2\cos(5f+3g) - \frac{1}{32}s^3e^2\cos(5f+5\phi+3g)\Big) +\\
&+ J_4\alpha^4\mu^4\frac{1}{G^7}\left(\phi\left(e^2\left(-\frac{315}{128}s^4 + \frac{45}{16}s^2 - \frac{9}{16}\right) -\right.\right.\\
&- \frac{105}{64}s^4 + \frac{15}{8}s^2 - \frac{3}{8}\Big) + \phi e^2\left(\frac{105}{64}s^4 - \frac{45}{32}s^2\right)\cos 2g +\\
&+ e^3\left(-\frac{35}{128}s^4 + \frac{15}{64}s^2\right)\sin(f-2g) +\\
&+ \left(e^3\left(\frac{315}{256}s^4 - \frac{45}{32}s^2 + \frac{9}{32}\right) + e\left(\frac{315}{64}s^4 - \frac{45}{8}s^2 + \frac{9}{8}\right)\right)\times\\
&\times\ \sin f + \left(e^3\left(-\frac{105}{128}s^4 + \frac{45}{64}s^2\right) + e\left(-\frac{105}{32}s^4 + \frac{45}{16}s^2\right)\right)\times\\
&\times \sin(f+2g) + \frac{35}{512}s^4e^3\sin(f+4g) + e^3\left(\frac{35}{128}s^4 - \frac{15}{64}s^2\right)\times\\
&\times\ \sin(f+\phi-2g) + \left(e^3\left(-\frac{315}{256}s^4 + \frac{45}{32}s^2 - \frac{9}{32}\right) +\right.\\
&+ e\left(-\frac{315}{64}s^4 + \frac{45}{8}s^2 - \frac{9}{8}\right)\Big)\sin(f+\phi) +\\
&+ \left(e^3\left(\frac{105}{128}s^4 - \frac{45}{64}s^2\right) + e\left(\frac{105}{32}s^4 - \frac{45}{16}s^2\right)\right)\sin(f+\phi+2g) -\\
&-\frac{35}{512}s^4e^3\sin(f+\phi+4g) + e^2\left(\frac{315}{256}s^4 - \frac{45}{32}s^2 + \frac{9}{32}\right)\sin 2f +\\
&+ \left(e^2\left(-\frac{105}{64}s^4 + \frac{45}{32}s^2\right) - \frac{35}{32}s^4 + \frac{15}{16}s^2\right)\times\\
&\times\ \sin(2f+2g) + \frac{105}{512}s^4e^2\sin(2f+4g) +\\
&+ e^2\left(-\frac{315}{256}s^4 + \frac{45}{32}s^2 - \frac{9}{32}\right)\sin(2f+2\phi) +
\end{aligned}
$$

$$
\begin{aligned}
&+\left(e^2\left(\frac{105}{64}s^4-\frac{45}{32}s^2\right)+\frac{35}{32}s^4-\frac{15}{16}s^2\right)\times\\
&\times\ \sin(2f+2\phi+2g)-\frac{105}{512}s^4e^2\times\\
&\times\ \sin(2f+2\phi+4g)+e^3\left(\frac{35}{256}s^4-\frac{5}{32}s^2+\frac{1}{32}\right)\sin 3f+\\
&+\left(e^3\left(-\frac{35}{128}s^4+\frac{15}{64}s^2\right)+e\left(-\frac{35}{32}s^4+\frac{15}{16}s^2\right)\right)\times\\
&\times\ \sin(3f+2g)+\left(\frac{35}{512}s^4e^3+\frac{35}{128}s^4e\right)\sin(3f+4g)+\\
&+e^3\left(-\frac{35}{256}s^4+\frac{5}{32}s^2-\frac{1}{32}\right)\sin(3f+3\phi)+\\
&+\left(e^3\left(\frac{35}{128}s^4-\frac{15}{64}s^2\right)+e\left(\frac{35}{32}s^4-\frac{15}{16}s^2\right)\right)\sin(3f+3\phi+2g)+\\
&+\left(-\frac{35}{512}s^4e^3-\frac{35}{128}s^4e\right)\sin(3f+3\phi+4g)+\\
&+e^2\left(-\frac{105}{256}s^4+\frac{45}{128}s^2\right)\sin(4f+2g)+\left(\frac{105}{512}s^4e^2+\frac{35}{256}s^4\right)\times\\
&\times\ \sin(4f+4g)+e^2\left(\frac{105}{256}s^4-\frac{45}{128}s^2\right)\sin(4f+4\phi+2g)+\\
&+\left(-\frac{105}{512}s^4e^2-\frac{35}{256}s^4\right)\sin(4f+4\phi+4g)+\\
&+e^3\left(-\frac{7}{128}s^4+\frac{3}{64}s^2\right)\sin(5f+2g)+\left(\frac{21}{512}s^4e^3+\frac{21}{128}s^4e\right)\times\\
&\times\ \sin(5f+4g)+e^3\left(\frac{7}{128}s^4-\frac{3}{64}s^2\right)\sin(5f+5\phi+2g)+\\
&+\left(-\frac{21}{512}s^4e^3-\frac{21}{128}s^4e\right)\sin(5f+5\phi+4g)+\\
&+\frac{35}{512}s^4e^2\sin(6f+4g)-\frac{35}{512}s^4e^2\sin(6f+6\phi+4g)+\\
&+\frac{5}{512}s^4e^3\sin(7f+4g)-\frac{5}{512}s^4e^3\sin(7f+7\phi+4g)\Big)\Big). \qquad (18)
\end{aligned}
$$

In the Delaunay variables as shown, the presence of the eccentricity to an odd power indicates that Poisson brackets, such as would be computed in the transformation of the variables, will involve e to a negative power. This presents the $e = 0$ singularity characteristic of Delaunay variables. In Whittaker variables, the computation, like the normal form computation, should be free of the $e = 0$

singularity. However, the change in true anomaly over the interval, ϕ, presents a problem in Whittaker variables.

To extend the map to second rank including the J_2 term only (second order in J_2), one applies the integrated exponential operator iex $(:f_1:)$ where

$$\mathrm{iex}(w) = \int_0^1 e^{tw}\,\mathrm{d}t = \sum_{m=0}^{\infty} \frac{w^m}{(m+1)!}. \tag{19}$$

This leaves a 'rest map' C_1 (Healy, 2001). The second rank part of this expression is then integrated with respect to time to produce the Lie polynomial f_2. Because of the Poisson brackets that arise in computing C_1, the mixed secular λ, ϕ terms I (14) arise. The time integral of these terms produces the Γ, r/p, and $\log(p/r)$ terms discussed above.

The expression for f_2 is too long to reproduce here, but it can be summarized as follows. All terms have $\delta^2 J_2^2 \alpha^4$

Common factor	Number of terms
$\mu^6 L^2 G^{-12} \Gamma$	102
$\mu^4 r p^{-1} L^2 G^{-9}$	354
$\mu^4 \log(p/r) L^2 G^{-9}$	87
$\mu^4 L^2 G^{-9}$	2368
$\mu^2 r^{-2} L^2 G^{-9}$	138

a total of 3049 terms. It is possible to reexpress this with the negative powers of r eliminated; the expression will then have a total of 3134 terms.

4.4. TIME DERIVATIVES

The time derivatives of r and $\log(p/r)$ are useful for checking the computed maps. Write r as

$$r = \frac{p}{1 + e\cos f}. \tag{20}$$

The time derivative is easily computed because the only thing time dependent on the right-hand side is f,

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= p\frac{\mathrm{d}(1+e\cos f)^{-1}}{\mathrm{d}t} = -p(1+e\cos f)^{-2}(-e\sin f)\frac{\mathrm{d}f}{\mathrm{d}t} \\ &= \frac{Ge}{p}\sin f, \end{aligned} \tag{21}$$

using $\mathrm{d}f/\mathrm{d}t = G/r^2$. The derivative of $\log(a/r)$ is computed similarly

$$\frac{\mathrm{d}\log(a/r)}{\mathrm{d}t} = \frac{r}{a}\frac{\mathrm{d}(a/r)}{\mathrm{d}t} = -\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{Ge\sin f}{rp}. \tag{22}$$

These results refer to the true anomaly generically; to stay consistent with our current–time notation, we would map $f \to f + \phi$.

The method of strained coordinates, or Poincaré–Lighthill method (Zwillinger, 1989, Chapter 133 and references therein) is a way of eliminating false secular terms in perturbation calculations. These secular terms arise because the perturbation affects the frequency of the system; by making a perturbation correction on that frequency, the secular terms can be eliminated and the long-term accuracy improved. In the present case, one would hope that a strained coordinate calculation would eliminate the secular term at first order, thus making the Lie polynomial at second order easier to calculate. Alas, the problem secular terms at second order arise not from the secular terms at first order, but from the Poisson bracket $[\phi, L]$ which has the secular λ term. Still, it is possible that a strained coordinate method could improve our prediction accuracy over a long time span, if not the task of map computation.

5. Comparison of Closed Form and Eccentricity Expansion

One way to check the validity of the computed map is to see if it satisfies Hamilton's equations (Healy, 2001); the details of this calculation in the LTM formalism are discussed in Appendix B. However, in the present instance, it is possible to compare the results of computing the map from the eccentricity-expanded Hamiltonian and eccentricity expansion of the map of the closed form Hamiltonian. If one has been validated by showing that it solves Hamilton's equations, then the other must be correct also.

As noted in Section 3, eccentricity expansion is a tricky business. One is tempted to treat it like a perturbation series: assume that a given expression expanded in the eccentricity has been truncated consistently at a certain order. One may then be use the map when eccentricity is low, having less confidence in the results for satellites whose eccentricity is nearer one than zero. This misses the essential fact that a small eccentricity does not imply a small derivative of eccentricity; the opposite is true. Because the Hamiltonian (and the Lie polynomials) themselves mean anything only insofar as their partial derivatives mean something, it is these derivatives to which we must attend. Specifically, for each potential partial derivative, one must allow an extra order of e. Because at the lowest rank, the Lie transfer map computation involves no partial derivatives, and each additional rank adds another partial derivative, one may conclude that when comparing the two forms at first rank, the eccentricity order should be the same, at second rank, it should be one higher for the eccentricity-expanded Hamiltonian than for ultimate e-order of interest for the closed form map, and so on.

To make this expansion, it is necessary to substitute all e-dependent terms in the closed form result. Specifically, the radius r, if it appears, and the true anomaly f will be dependent on the eccentricity and must be substituted with an explicit expansion. Substitution of the e-dependent terms in the closed-form map (18) should give the eccentricity expanded map (12).

The validity of the closed form map (12) may be confirmed for all zonal harmonics by this comparison, because the eccentricity-expanded map is checked by satisfaction of Hamilton's equations.

6. Computation of the Explicit Map

With the Lie transfer map (18) in hand, derivation of the explicit map is straightforward by transformation of each phase space variable by the procedure described in Appendix B. Essentially, one expands the exponential of the Lie operator in a Taylor series, computes and sums the Poisson brackets appropriately. This series may be truncated at any order; a natural order at which to stop would be the same order through which the map has been computed. In this case, we compute the map through first order in J_2.

$$\begin{aligned}
f \;\leftarrow\; & f + \phi + \delta J_2 \alpha^2 \mu^2 \frac{L^5}{G^9}\Bigg(\lambda\bigg(\eta^4\bigg(-\frac{27}{16}s^2 + \frac{9}{8}\bigg) + \eta^2\bigg(\frac{63}{8}s^2 - \frac{21}{4}\bigg) - \\
& - \frac{135}{16}s^2 + \frac{45}{8}\bigg) + \lambda\bigg(\frac{27}{32}s^2\eta^4 - \frac{27}{8}s^2\eta^2 + \frac{81}{32}s^2\bigg)\cos 2g + \\
& + \bigg(\frac{27}{32}s^2\eta^5 - \frac{27}{32}s^2\eta^3\bigg)\sin 2g + \lambda\bigg(\frac{27}{32}s^2\eta^4 - \frac{81}{16}s^2\eta^2 + \frac{135}{32}s^2\bigg)\times \\
& \times \cos(\phi - 2g) + \bigg(\frac{9}{16}s^2\eta^5 - \frac{9}{16}s^2\eta^3\bigg)\sin(\phi - 2g) + \\
& + \lambda\bigg(\eta^4\bigg(-\frac{27}{16}s^2 + \frac{9}{8}\bigg) + \eta^2\bigg(\frac{81}{8}s^2 - \frac{27}{4}\bigg) - \frac{135}{16}s^2 + \frac{45}{8}\bigg)\times \\
& \times \cos\phi + \eta^3\bigg(\frac{9}{2}s^2 - 3\bigg)\sin\phi + \lambda\bigg(\frac{9}{32}s^2\eta^4 - \frac{9}{16}s^2\eta^2 + \frac{9}{32}s^2\bigg)\times \\
& \times \cos(\phi + 2g) + \bigg(\frac{3}{16}s^2\eta^5 - \frac{3}{16}s^2\eta^3\bigg)\sin(\phi + 2g) + \lambda\bigg(\frac{27}{32}s^2\eta^4 - \\
& - \frac{9}{4}s^2\eta^2 + \frac{45}{32}s^2\bigg)\cos(2\phi - 2g) + \bigg(\frac{15}{32}s^2\eta^5 - \frac{15}{32}s^2\eta^3\bigg)\times \\
& \times \sin(2\phi - 2g) + \lambda\bigg(\eta^4\bigg(-\frac{27}{32}s^2 + \frac{9}{16}\bigg) + \eta^2\bigg(\frac{27}{16}s^2 - \frac{9}{8}\bigg) - \frac{27}{32}s^2 + \\
& + \frac{9}{16}\bigg)\cos 2\phi + \bigg(\eta^5\bigg(-\frac{9}{16}s^2 + \frac{3}{8}\bigg) + \eta^3\bigg(\frac{9}{16}s^2 - \frac{3}{8}\bigg)\bigg)\sin 2\phi +
\end{aligned}$$

$$
\begin{aligned}
&+\lambda\left(e\eta^4\left(-\frac{9}{64}s^2+\frac{3}{32}\right)+e\eta^2\left(\frac{9}{32}s^2-\frac{3}{16}\right)+e\left(-\frac{9}{64}s^2+\frac{3}{32}\right)\right)\times\\
&\times\cos(f-2\phi)+\left(e\eta^5\left(\frac{3}{32}s^2-\frac{1}{16}\right)+e\eta^3\left(-\frac{3}{32}s^2+\frac{1}{16}\right)\right)\times\\
&\times\sin(f-2\phi)+\lambda\left(\frac{27}{128}s^2e\eta^4-\frac{81}{64}s^2e\eta^2+\frac{135}{128}s^2e\right)\times\\
&\times\cos(f-2\phi+2g)+\left(-\frac{1}{8}s^2e\eta^5+\frac{9}{16}s^2e\eta^3\right)\sin(f-2\phi+2g)+\\
&+\lambda\left(e\eta^2\left(\frac{27}{8}s^2-\frac{9}{4}\right)+e\left(-\frac{27}{8}s^2+\frac{9}{4}\right)\right)\cos(f-\phi)+\\
&+e\eta^3\left(-\frac{9}{4}s^2+\frac{3}{2}\right)\sin(f-\phi)+\lambda\left(-\frac{27}{8}s^2e\eta^2+\frac{45}{8}s^2e\right)\times\\
&\times\cos(f-\phi+2g)+\frac{15}{8}s^2e\eta^3\sin(f-\phi+2g)+\lambda\left(\frac{9}{64}s^2e\eta^4-\right.\\
&\left.-\frac{9}{16}s^2e\eta^2+\frac{27}{64}s^2e\right)\cos(f-2g)+\left(-\frac{3}{64}s^2e\eta^5+\frac{9}{64}s^2e\eta^3\right)\times\\
&\times\ \sin(f-2g)+\lambda\left(e\eta^4\left(-\frac{27}{32}s^2+\frac{9}{16}\right)+e\eta^2\left(\frac{27}{4}s^2-\frac{9}{2}\right)+\right.\\
&\left.+e\left(-\frac{405}{32}s^2+\frac{135}{16}\right)\right)\cos f+\left(e\eta^5\left(-\frac{9}{32}s^2+\frac{3}{16}\right)+\right.\\
&\left.+e\eta^3\left(-\frac{9}{32}s^2+\frac{3}{16}\right)+e\eta\left(\frac{9}{4}s^2-\frac{3}{2}\right)+\frac{\eta}{e}\left(-\frac{9}{4}s^2+\frac{3}{2}\right)\right)\sin f+\\
&+\lambda\left(\frac{27}{64}s^2e\eta^4-\frac{27}{8}s^2e\eta^2+\frac{405}{64}s^2e\right)\cos(f+2g)+\left(\frac{15}{64}s^2e\eta^5-\right.\\
&\left.-\frac{57}{64}s^2e\eta^3+\frac{3}{8}s^2e\eta-\frac{3}{8}s^2\frac{\eta}{e}\right)\sin(f+2g)+\lambda\left(-\frac{27}{16}s^2e\eta^2+\frac{27}{16}s^2e\right)\times\\
&\times\cos(f+\phi-2g)-\frac{3}{32}s^2e\eta^5\sin(f+\phi-2g)+\lambda\left(e\eta^2\left(\frac{27}{4}s^2-\frac{9}{2}\right)+\right.\\
&\left.+e\left(-\frac{45}{4}s^2+\frac{15}{2}\right)\right)\cos(f+\phi)+\left(e\eta^5\left(\frac{3}{16}s^2-\frac{1}{8}\right)+e\eta^3\left(\frac{3}{2}s^2-1\right)+\right.\\
&\left.+e\eta\left(-\frac{9}{4}s^2+\frac{3}{2}\right)+\frac{\eta}{e}\left(\frac{9}{4}s^2-\frac{3}{2}\right)\right)\sin(f+\phi)+\lambda\left(-\frac{27}{16}s^2e\eta^2+\right.\\
&\left.+\frac{27}{16}s^2e\right)\cos(f+\phi+2g)+\left(-\frac{5}{32}s^2e\eta^5-\frac{3}{2}s^2e\eta^3-\frac{3}{8}s^2e\eta+\right.\\
&\left.+\frac{3}{8}s^2\frac{\eta}{e}\right)\sin(f+\phi+2g)+\lambda\left(\frac{27}{128}s^2e\eta^4-\frac{81}{64}s^2e\eta^2+\frac{135}{128}s^2e\right)\times
\end{aligned}
$$

$$\times\cos(f+2\phi-2g)+\left(\frac{9}{64}s^2e\eta^5-\frac{9}{64}s^2e\eta^3\right)\sin(f+2\phi-2g)+$$
$$+\lambda\left(e\eta^4\left(-\frac{27}{64}s^2+\frac{9}{32}\right)+e\eta^2\left(\frac{81}{32}s^2-\frac{27}{16}\right)+e\left(-\frac{135}{64}s^2+\frac{45}{32}\right)\right)\times$$
$$\times\cos(f+2\phi)+e\eta^3\left(\frac{9}{8}s^2-\frac{3}{4}\right)\sin(f+2\phi)+\lambda\left(\frac{9}{128}s^2e\eta^4-\right.$$
$$\left.-\frac{9}{64}s^2e\eta^2+\frac{9}{128}s^2e\right)\cos(f+2\phi+2g)+\left(\frac{3}{64}s^2e\eta^5-\frac{3}{64}s^2e\eta^3\right)\times$$
$$\times\sin(f+2\phi+2g)+\lambda\left(\frac{27}{64}s^2\eta^4-\frac{27}{32}s^2\eta^2+\frac{27}{64}s^2\right)\times$$
$$\times\cos(2f-2\phi+2g)+\left(-\frac{9}{32}s^2\eta^5+\frac{9}{32}s^2\eta^3\right)\sin(2f-2\phi+2g)+$$
$$+\lambda\left(\eta^4\left(-\frac{9}{16}s^2+\frac{3}{8}\right)+\eta^2\left(\frac{9}{8}s^2-\frac{3}{4}\right)-\frac{9}{16}s^2+\frac{3}{8}\right)\cos(2f-\phi)+$$
$$+\left(\eta^5\left(\frac{3}{8}s^2-\frac{1}{4}\right)+\eta^3\left(-\frac{3}{8}s^2+\frac{1}{4}\right)\right)\sin(2f-\phi)+\lambda\left(\frac{27}{32}s^2\eta^4-\right.$$
$$\left.-\frac{81}{16}s^2\eta^2+\frac{135}{32}s^2\right)\cos(2f-\phi+2g)+\left(-\frac{1}{2}s^2\eta^5+\frac{9}{4}s^2\eta^3\right)\times$$
$$\times\sin(2f-\phi+2g)+\lambda\left(\eta^4\left(-\frac{27}{16}s^2+\frac{9}{8}\right)+\eta^2\left(\frac{27}{4}s^2-\frac{9}{2}\right)-\right.$$
$$\left.-\frac{81}{16}s^2+\frac{27}{8}\right)\cos 2f+\left(\eta^5\left(\frac{9}{16}s^2-\frac{3}{8}\right)+\eta^3\left(-\frac{27}{16}s^2+\frac{9}{8}\right)\right)\times$$
$$\times\sin 2f+\lambda\left(\frac{27}{16}s^2\eta^4-\frac{63}{8}s^2\eta^2+\frac{135}{16}s^2\right)\cos(2f+2g)+$$
$$+\lambda\left(\frac{9}{32}s^2\eta^4-\frac{9}{16}s^2\eta^2+\frac{9}{32}s^2\right)\cos(2f+\phi-2g)+\lambda\left(\eta^4\left(-\frac{27}{16}s^2+\right.\right.$$
$$\left.\left.+\frac{9}{8}\right)+\eta^2\left(\frac{81}{8}s^2-\frac{27}{4}\right)-\frac{135}{16}s^2+\frac{45}{8}\right)\cos(2f+\phi)+$$
$$+\left(\eta^5\left(-\frac{9}{8}s^2+\frac{3}{4}\right)+\eta^3\left(\frac{9}{8}s^2-\frac{3}{4}\right)\right)\sin(2f+\phi)+$$
$$+\lambda\left(\frac{27}{32}s^2\eta^4-\frac{81}{16}s^2\eta^2+\frac{135}{32}s^2\right)\cos(2f+\phi+2g)+$$
$$+\left(\frac{3}{2}s^2\eta^5-\frac{9}{4}s^2\eta^3\right)\sin(2f+\phi+2g)+\lambda\left(\frac{27}{64}s^2\eta^4-\frac{27}{32}s^2\eta^2+\right.$$
$$\left.+\frac{27}{64}s^2\right)\cos(2f+2\phi-2g)+\lambda\left(\eta^4\left(-\frac{27}{16}s^2+\frac{9}{8}\right)+\right.$$
$$\left.+\eta^2\left(\frac{9}{2}s^2-3\right)-\frac{45}{16}s^2+\frac{15}{8}\right)\cos(2f+2\phi)+\left(\eta^5\left(\frac{3}{16}s^2-\frac{1}{8}\right)+\right.$$

$$+\eta^3\left(\frac{15}{16}s^2-\frac{5}{8}\right)\Bigg)\sin(2f+2\phi)+\lambda\left(\frac{27}{64}s^2\eta^4-\frac{27}{32}s^2\eta^2+\frac{27}{64}s^2\right)\times$$
$$\times\cos(2f+2\phi+2g)+\left(-\frac{23}{32}s^2\eta^5-\frac{9}{32}s^2\eta^3\right)\sin(2f+2\phi+2g)+$$
$$+\lambda\left(\frac{9}{128}s^2e\eta^4-\frac{9}{64}s^2e\eta^2+\frac{9}{128}s^2e\right)\cos(3f-2\phi+2g)+$$
$$+\left(-\frac{3}{64}s^2e\eta^5+\frac{3}{64}s^2e\eta^3\right)\sin(3f-2\phi+2g)+\lambda\Bigg(-\frac{27}{16}s^2e\eta^2+$$
$$+\frac{27}{16}s^2e\Bigg)\cos(3f-\phi+2g)+\frac{9}{8}s^2e\eta^3\sin(3f-\phi+2g)+$$
$$+\lambda\left(e\eta^4\left(-\frac{9}{32}s^2+\frac{3}{16}\right)+e\eta^2\left(\frac{9}{8}s^2-\frac{3}{4}\right)+e\left(-\frac{27}{32}s^2+\frac{9}{16}\right)\right)\times$$
$$\times\cos 3f+\left(e\eta^5\left(\frac{3}{32}s^2-\frac{1}{16}\right)+e\eta^3\left(-\frac{9}{32}s^2+\frac{3}{16}\right)\right)\sin 3f+$$
$$+\lambda\left(\frac{27}{64}s^2e\eta^4-\frac{27}{8}s^2e\eta^2+\frac{405}{64}s^2e\right)\cos(3f+2g)+\Bigg(\frac{1}{64}s^2e\eta^5+$$
$$+\frac{25}{64}s^2e\eta^3-\frac{7}{8}s^2e\eta+\frac{7}{8}s^2\frac{\eta}{e}\Bigg)\sin(3f+2g)+\lambda\Bigg(e\eta^2\left(\frac{27}{8}s^2-\frac{9}{4}\right)+$$
$$+e\left(-\frac{27}{8}s^2+\frac{9}{4}\right)\Bigg)\cos(3f+\phi)+\lambda\left(-\frac{27}{8}s^2e\eta^2+\frac{45}{8}s^2e\right)\times$$
$$\times\cos(3f+\phi+2g)-\frac{15}{8}s^2e\eta^3\sin(3f+\phi+2g)+\lambda\Bigg(\frac{9}{128}s^2e\eta^4-$$
$$-\frac{9}{64}s^2e\eta^2+\frac{9}{128}s^2e\Bigg)\cos(3f+2\phi-2g)+\lambda\Bigg(e\eta^4\left(-\frac{27}{64}s^2+\frac{9}{32}\right)+$$
$$+e\eta^2\left(\frac{81}{32}s^2-\frac{27}{16}\right)+e\left(-\frac{135}{64}s^2+\frac{45}{32}\right)\Bigg)\cos(3f+2\phi)+$$
$$+\left(e\eta^5\left(-\frac{9}{32}s^2+\frac{3}{16}\right)+e\eta^3\left(\frac{9}{32}s^2-\frac{3}{16}\right)\right)\sin(3f+2\phi)+$$
$$+\lambda\left(\frac{27}{128}s^2e\eta^4-\frac{81}{64}s^2e\eta^2+\frac{135}{128}s^2e\right)\cos(3f+2\phi+2g)+$$
$$+\left(\frac{3}{8}s^2e\eta^5-\frac{9}{16}s^2e\eta^3\right)\sin(3f+2\phi+2g)+e\eta^5\left(\frac{3}{16}s^2-\frac{1}{8}\right)\times$$
$$\times\sin(3f+3\phi)+\left(-\frac{11}{32}s^2e\eta^5+\frac{7}{8}s^2e\eta^3+\frac{7}{8}s^2e\eta-\frac{7}{8}s^2\frac{\eta}{e}\right)\times$$
$$\times\sin(3f+3\phi+2g)+\lambda\left(\frac{9}{32}s^2\eta^4-\frac{9}{16}s^2\eta^2+\frac{9}{32}s^2\right)\cos(4f-\phi+2g)+$$
$$+\left(-\frac{3}{16}s^2\eta^5+\frac{3}{16}s^2\eta^3\right)\sin(4f-\phi+2g)+\lambda\Bigg(\frac{27}{32}s^2\eta^4-\frac{27}{8}s^2\eta^2+$$

$$+\frac{81}{32}s^2\Big)\cos(4f+2g)+\Big(-\frac{9}{32}s^2\eta^5+\frac{27}{32}s^2\eta^3\Big)\sin(4f+2g)+$$
$$+\lambda\Big(\eta^4\Big(-\frac{9}{16}s^2+\frac{3}{8}\Big)+\eta^2\Big(\frac{9}{8}s^2-\frac{3}{4}\Big)-\frac{9}{16}s^2+\frac{3}{8}\Big)\cos(4f+\phi)+$$
$$+\lambda\Big(\frac{27}{32}s^2\eta^4-\frac{81}{16}s^2\eta^2+\frac{135}{32}s^2\Big)\cos(4f+\phi+2g)+\Big(\frac{9}{16}s^2\eta^5-$$
$$-\frac{9}{16}s^2\eta^3\Big)\sin(4f+\phi+2g)+\lambda\Big(\eta^4\Big(-\frac{27}{32}s^2+\frac{9}{16}\Big)+$$
$$+\eta^2\Big(\frac{27}{16}s^2-\frac{9}{8}\Big)-\frac{27}{32}s^2+\frac{9}{16}\Big)\cos(4f+2\phi)+\lambda\Big(\frac{27}{32}s^2\eta^4-\frac{9}{4}s^2\eta^2+$$
$$+\frac{45}{32}s^2\Big)\cos(4f+2\phi+2g)+\Big(\frac{15}{32}s^2\eta^5-\frac{15}{32}s^2\eta^3\Big)\sin(4f+2\phi+2g)-$$
$$-\frac{9}{16}s^2\eta^5\sin(4f+4\phi+2g)+\lambda\Big(\frac{9}{64}s^2e\eta^4-\frac{9}{16}s^2e\eta^2+\frac{27}{64}s^2e\Big)\times$$
$$\times\cos(5f+2g)+\Big(-\frac{3}{64}s^2e\eta^5+\frac{9}{64}s^2e\eta^3\Big)\sin(5f+2g)+$$
$$+\lambda\Big(-\frac{27}{16}s^2e\eta^2+\frac{27}{16}s^2e\Big)\cos(5f+\phi+2g)+\lambda\Big(e\eta^4\Big(-\frac{9}{64}s^2+$$
$$+\frac{3}{32}\Big)+e\eta^2\Big(\frac{9}{32}s^2-\frac{3}{16}\Big)+e\Big(-\frac{9}{64}s^2+\frac{3}{32}\Big)\Big)\cos(5f+2\phi)+$$
$$+\lambda\Big(\frac{27}{128}s^2e\eta^4-\frac{81}{64}s^2e\eta^2+\frac{135}{128}s^2e\Big)\cos(5f+2\phi+2g)+$$
$$+\Big(\frac{9}{64}s^2e\eta^5-\frac{9}{64}s^2e\eta^3\Big)\sin(5f+2\phi+2g)-\frac{3}{32}s^2e\eta^5\times$$
$$\times\sin(5f+5\phi+2g)+\lambda\Big(\frac{9}{32}s^2\eta^4-\frac{9}{16}s^2\eta^2+\frac{9}{32}s^2\Big)\times$$
$$\times\cos(6f+\phi+2g)+\lambda\Big(\frac{27}{64}s^2\eta^4-\frac{27}{32}s^2\eta^2+\frac{27}{64}s^2\Big)\times$$
$$\times\cos(6f+2\phi+2g)+\lambda\Big(\frac{9}{128}s^2e\eta^4-\frac{9}{64}s^2e\eta^2+$$
$$+\frac{9}{128}s^2e\Big)\cos(7f+2\phi+2g)\Big)+\mathcal{O}(\delta^2). \tag{23}$$

$$l \leftarrow l+\lambda+$$
$$+\delta J_2\alpha^2\mu^2\frac{L^2}{G^6}\Big(\lambda\Big(\eta^2\Big(\frac{27}{8}s^2-\frac{9}{4}\Big)-\frac{45}{8}s^2+\frac{15}{4}\Big)+\lambda\Big(-\frac{27}{16}s^2\eta^2+$$
$$+\frac{27}{16}s^2\Big)\cos 2g+\lambda\Big(-\frac{9}{32}s^2e\eta^2+\frac{9}{32}s^2e\Big)\cos(f-2g)+\frac{3}{32}s^2e\eta^3\times$$

$$
\begin{aligned}
&\times \sin(f-2g) + \lambda\biggl(e\eta^2\biggl(\frac{27}{16}s^2 - \frac{9}{8}\biggr) + e\biggl(-\frac{135}{16}s^2 + \frac{45}{8}\biggr)\biggr)\cos f + \\
&+\biggl(e\eta^3\biggl(\frac{9}{16}s^2 - \frac{3}{8}\biggr) + e\eta\biggl(\frac{9}{4}s^2 - \frac{3}{2}\biggr) + \frac{\eta}{e}\biggl(-\frac{9}{4}s^2 + \frac{3}{2}\biggr)\biggr)\sin f + \\
&+\lambda\biggl(-\frac{27}{32}s^2e\eta^2 + \frac{135}{32}s^2e\biggr)\cos(f+2g) + \biggl(-\frac{15}{32}s^2e\eta^3 + \frac{3}{8}s^2e\eta - \\
&-\frac{3}{8}s^2\frac{\eta}{e}\biggr)\sin(f+2g) - \frac{3}{32}s^2e\eta^3\sin(f+\phi-2g) + \biggl(e\eta^3\biggl(-\frac{9}{16}s^2 + \\
&+\frac{3}{8}\biggr) + e\eta\biggl(-\frac{9}{4}s^2 + \frac{3}{2}\biggr) + \frac{\eta}{e}\biggl(\frac{9}{4}s^2 - \frac{3}{2}\biggr)\biggr)\sin(f+\phi) + \biggl(\frac{15}{32}s^2e\eta^3 - \\
&-\frac{3}{8}s^2e\eta + \frac{3}{8}s^2\frac{\eta}{e}\biggr)\sin(f+\phi+2g) + \lambda\biggl(\eta^2\biggl(\frac{27}{8}s^2 - \frac{9}{4}\biggr) - \frac{27}{8}s^2 + \\
&+\frac{9}{4}\biggr)\cos 2f + \eta^3\biggl(-\frac{9}{8}s^2 + \frac{3}{4}\biggr)\sin 2f + \lambda\biggl(-\frac{27}{8}s^2\eta^2 + \frac{45}{8}s^2\biggr)\times \\
&\times\cos(2f+2g) + \eta^3\biggl(\frac{9}{8}s^2 - \frac{3}{4}\biggr)\sin(2f+2\phi) + \lambda\biggl(e\eta^2\biggl(\frac{9}{16}s^2 - \\
&-\frac{3}{8}\biggr) + e\biggl(-\frac{9}{16}s^2 + \frac{3}{8}\biggr)\biggr)\cos 3f + \\
&+e\eta^3\biggl(-\frac{3}{16}s^2 + \frac{1}{8}\biggr)\sin 3f + \lambda\biggl(-\frac{27}{32}s^2e\eta^2 + \frac{135}{32}s^2e\biggr)\times \\
&\times\cos(3f+2g) + \biggl(-\frac{1}{32}s^2e\eta^3 - \frac{7}{8}s^2e\eta + \frac{7}{8}s^2\frac{\eta}{e}\biggr)\times \\
&\times\sin(3f+2g) + e\eta^3\biggl(\frac{3}{16}s^2 - \frac{1}{8}\biggr)\sin(3f+3\phi) + \\
&+\biggl(\frac{1}{32}s^2e\eta^3 + \frac{7}{8}s^2e\eta - \frac{7}{8}s^2\frac{\eta}{e}\biggr)\sin(3f+3\phi+2g) + \\
&+\lambda\biggl(-\frac{27}{16}s^2\eta^2 + \frac{27}{16}s^2\biggr)\cos(4f+2g) + \frac{9}{16}s^2\eta^3\times \\
&\times\sin(4f+2g) - \frac{9}{16}s^2\eta^3\sin(4f+4\phi+2g) + \\
&+\lambda\biggl(-\frac{9}{32}s^2e\eta^2 + \frac{9}{32}s^2e\biggr)\cos(5f+2g) + \frac{3}{32}s^2e\eta^3\times \\
&\times\sin(5f+2g) - \frac{3}{32}s^2e\eta^3\sin(5f+5\phi+2g)\biggr) + \mathcal{O}(\delta^2).
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
g \;\leftarrow\; & g + \delta J_2 \alpha^2 \mu^2 \frac{1}{G^4}\biggl(\phi\biggl(-\frac{15}{4}s^2 + 3\biggr) - \frac{3}{32}s^2 e \sin(f - 2g) + \\
& + \biggl(e\biggl(\frac{51}{16}s^2 - \frac{21}{8}\biggr) + \frac{1}{e}\biggl(\frac{9}{4}s^2 - \frac{3}{2}\biggr)\biggr)\sin f + \biggl(e\biggl(-\frac{45}{32}s^2 + \frac{3}{4}\biggr) + \\
& + \frac{3}{8}s^2\frac{1}{e}\biggr)\sin(f + 2g) + \frac{3}{32}s^2 e \sin(f + \phi - 2g) + \\
& + \biggl(e\biggl(-\frac{51}{16}s^2 + \frac{21}{8}\biggr) + \frac{1}{e}\biggl(-\frac{9}{4}s^2 + \frac{3}{2}\biggr)\biggr)\sin(f + \phi) + \\
& + \biggl(e\biggl(\frac{45}{32}s^2 - \frac{3}{4}\biggr) - \frac{3}{8}s^2\frac{1}{e}\biggr)\sin(f + \phi + 2g) + \\
& + \biggl(\frac{9}{8}s^2 - \frac{3}{4}\biggr)\sin 2f + \biggl(-\frac{15}{8}s^2 + \frac{3}{4}\biggr)\sin(2f + 2g) + \\
& + \biggl(-\frac{9}{8}s^2 + \frac{3}{4}\biggr)\sin(2f + 2\phi) + \biggl(\frac{15}{8}s^2 - \frac{3}{4}\biggr)\sin(2f + 2\phi + 2g) + \\
& + e\biggl(\frac{3}{16}s^2 - \frac{1}{8}\biggr)\sin 3f + \biggl(e\biggl(-\frac{19}{32}s^2 + \frac{1}{4}\biggr) - \frac{7}{8}s^2\frac{1}{e}\biggr) \times \\
& \times \sin(3f + 2g) + e\biggl(-\frac{3}{16}s^2 + \frac{1}{8}\biggr)\sin(3f + 3\phi) + \\
& + \biggl(e\biggl(\frac{19}{32}s^2 - \frac{1}{4}\biggr) + \frac{7}{8}s^2\frac{1}{e}\biggr)\sin(3f + 3\phi + 2g) - \\
& - \frac{9}{16}s^2 \sin(4f + 2g) + \frac{9}{16}s^2 \sin(4f + 4\phi + 2g) - \\
& - \frac{3}{32}s^2 e \sin(5f + 2g) + \frac{3}{32}s^2 e \sin(5f + 5\phi + 2g)\biggr) + \mathcal{O}(\delta^2).
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
h \;\leftarrow\; & h + \delta J_2 \alpha^2 \mu^2 \frac{1}{G^3 H}\biggl(\phi\biggl(\frac{3}{2}s^2 - \frac{3}{2}\biggr) + e\biggl(-\frac{3}{2}s^2 + \frac{3}{2}\biggr)\sin f + \\
& + e\biggl(\frac{3}{4}s^2 - \frac{3}{4}\biggr)\sin(f + 2g) + e\biggl(\frac{3}{2}s^2 - \frac{3}{2}\biggr)\sin(f + \phi) + \\
& + e\biggl(-\frac{3}{4}s^2 + \frac{3}{4}\biggr)\sin(f + \phi + 2g) + \biggl(\frac{3}{4}s^2 - \frac{3}{4}\biggr)\sin(2f + 2g) + \\
& + \biggl(-\frac{3}{4}s^2 + \frac{3}{4}\biggr)\sin(2f + 2\phi + 2g) + e\biggl(\frac{1}{4}s^2 - \frac{1}{4}\biggr)\sin(3f + 2g) + \\
& + e\biggl(-\frac{1}{4}s^2 + \frac{1}{4}\biggr)\sin(3f + 3\phi + 2g)\biggr) + \mathcal{O}(\delta^2).
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
L \;\leftarrow\; & L + \delta J_2 \alpha^2 \mu^2 \frac{L^3}{G^6}\Bigg(\bigg(\frac{3}{32}s^2 e\eta^2 - \frac{3}{32}s^2 e\bigg)\cos(f-2g) + \\
& + \bigg(e\eta^2\Big(-\frac{9}{16}s^2 + \frac{3}{8}\Big) + e\Big(\frac{45}{16}s^2 - \frac{15}{8}\Big)\bigg)\cos f + \\
& + \bigg(\frac{9}{32}s^2 e\eta^2 - \frac{45}{32}s^2 e\bigg)\cos(f+2g) + \bigg(-\frac{3}{32}s^2 e\eta^2 + \frac{3}{32}s^2 e\bigg) \times \\
& \times \cos(f+\phi-2g) + \bigg(e\eta^2\Big(\frac{9}{16}s^2 - \frac{3}{8}\Big) + e\Big(-\frac{45}{16}s^2 + \frac{15}{8}\Big)\bigg) \times \\
& \times \cos(f+\phi) + \bigg(-\frac{9}{32}s^2 e\eta^2 + \frac{45}{32}s^2 e\bigg)\cos(f+\phi+2g) + \\
& + \bigg(\eta^2\Big(-\frac{9}{8}s^2 + \frac{3}{4}\Big) + \frac{9}{8}s^2 - \frac{3}{4}\bigg)\cos 2f + \bigg(\frac{9}{8}s^2\eta^2 - \frac{15}{8}s^2\bigg) \times \\
& \times \cos(2f+2g) + \bigg(\eta^2\Big(\frac{9}{8}s^2 - \frac{3}{4}\Big) - \frac{9}{8}s^2 + \frac{3}{4}\bigg)\cos(2f+2\phi) + \\
& + \bigg(-\frac{9}{8}s^2\eta^2 + \frac{15}{8}s^2\bigg)\cos(2f+2\phi+2g) + \bigg(e\eta^2\Big(-\frac{3}{16}s^2 + \frac{1}{8}\Big) + \\
& + e\Big(\frac{3}{16}s^2 - \frac{1}{8}\Big)\bigg)\cos 3f + \bigg(\frac{9}{32}s^2 e\eta^2 - \frac{45}{32}s^2 e\bigg)\cos(3f+2g) + \\
& + \bigg(e\eta^2\Big(\frac{3}{16}s^2 - \frac{1}{8}\Big) + e\Big(-\frac{3}{16}s^2 + \frac{1}{8}\Big)\bigg)\cos(3f+3\phi) + \\
& + \bigg(-\frac{9}{32}s^2 e\eta^2 + \frac{45}{32}s^2 e\bigg)\cos(3f+3\phi+2g) + \bigg(\frac{9}{16}s^2\eta^2 - \frac{9}{16}s^2\bigg) \times \\
& \times \cos(4f+2g) + \bigg(-\frac{9}{16}s^2\eta^2 + \frac{9}{16}s^2\bigg)\cos(4f+4\phi+2g) + \\
& + \bigg(\frac{3}{32}s^2 e\eta^2 - \frac{3}{32}s^2 e\bigg)\cos(5f+2g) + \bigg(-\frac{3}{32}s^2 e\eta^2 + \frac{3}{32}s^2 e\bigg) \times \\
& \times \cos(5f+5\phi+2g)\Bigg) + \mathcal{O}(\delta^2). \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
G \;\leftarrow\; & G + \delta J_2 \alpha^2 \mu^2 \frac{1}{G^3}\bigg(-\frac{3}{4}s^2 e\cos(f+2g) + \frac{3}{4}s^2 e\cos(f+\phi+2g) - \\
& - \frac{3}{4}s^2\cos(2f+2g) + \frac{3}{4}s^2\cos(2f+2\phi+2g) - \frac{1}{4}s^2 e \times \\
& \times \cos(3f+2g) + \frac{1}{4}s^2 e\cos(3f+3\phi+2g)\bigg) + \mathcal{O}(\delta^2). \qquad (28) \\
H \;\leftarrow\; & H + \mathcal{O}(\delta^2).
\end{aligned}
$$

Because of the singular nature of Delaunay variables for circular and equatorial orbits, this transfer map is not always useful; it may compute $G > L$ for some orbits and propagation times, which is not a physical result (corresponding to imaginary eccentricity). A traditional circumvention of this problem is to use non-singular variables such as Poincaré variables instead of the Delaunay variables.

Nevertheless, it is possible to use this result to validate the computation of the map numerically. By picking orbital elements and propagation times that insure that singularities are not an issue, the map may be compared numerically with a known correct map, for example, one obtained from numerical integration. A numerical evaluation of the closed form map at second rank will require the calculation of partial derivatives of Γ. These may be calculated by taking the derivatives inside the integral,

$$\begin{aligned}\frac{\partial \Gamma}{\partial l} &= \frac{\partial}{\partial l}\int \phi \,\mathrm{d}t \int \frac{\partial \phi}{\partial l}\,\mathrm{d}t = \int \frac{L^3}{G^3}[(1+e\cos(f+\phi))^2 \\ &\quad - (1+e\cos f)^2]\,\mathrm{d}t \end{aligned} \tag{29}$$

which can then be reduced to integrals of the type (16).

7. Conclusion

Using the formalism developed by the author for computing Lie transfer maps for perturbed Hamiltonian systems, it is possible to construct a map using Lie transformations that gives a transfer map of the perturbed two-body Kepler motion in celestial mechanics. No normal form is computed. This map may be computed easily to first order in the zonal harmonics. At higher order, functions not integrable in simple terms appear. Expanding the exponential of the Lie transformation, explicit functions giving the transformation of the Delaunay variables to first order in the zonal coefficients have been shown. These functions may be used for numerical propagation of satellite orbits.

Appendix A. True Anomaly Over Time Interval

As discussed above, the change in true anomaly from the beginning of the interval ϕ needs to be handled differently than the true anomaly at the beginning of the interval f.

The derivative of true anomaly

$$\frac{\partial f}{\partial L} = \left(1+\frac{p}{r}\right)\frac{\sin f}{Le} = \frac{(2+e\cos f)\sin f}{Le} \tag{30}$$

needs elaboration. Let $F(l, L, G)$ be the function that gives the true anomaly in terms of mean anomaly l at a particular time, Delaunay momentum L, and angular momentum G. The true anomaly for some value of mean anomaly $l + \lambda$ at an arbitrary time after the start of the interval would be written $f+\phi = F(l+\lambda, L, G)$. Therefore, based on the true anomaly at the start of the interval when the mean anomaly is l, $f = F(l, L, G)$, the change in true anomaly over the interval is given by $\phi = F(l+\lambda, L, G) - F(l, L, G)$. Because of the transcendental nature of Kepler's equation, it is not possible to write this function explicitly. Nevertheless, we may compute derivatives.

Partial derivatives of the true anomaly are really derivatives of the function $f = F(l, L, G)$,

$$\frac{\partial f}{\partial \xi} = D_1 F(l, L, G)\frac{\partial l}{\partial \xi} + D_2 F(l, L, G)\frac{\partial L}{\partial \xi} + D_3 F(l, L, G)\frac{\partial G}{\partial \xi}, \tag{31}$$

where ξ represents one of l, L, or G, and D_n represents the partial derivative with respect to the nth argument. Therefore, the derivatives of ϕ may be derived from them,

$$\begin{aligned}\frac{\partial f}{\partial \xi} &= D_1 F(l+\phi, L, G)\frac{\partial (l+\lambda)}{\partial \xi} + D_2 F(l+\phi, L, G)\frac{\partial L}{\partial \xi} + \\ &\quad + D_3 F(l+\phi, L, G)\frac{\partial G}{\partial \xi} - D_1 F(l, L, G)\frac{\partial l}{\partial \xi} - \\ &\quad - D_2 F(l, L, G)\frac{\partial L}{\partial \xi} - D_3 F(l, L, G)\frac{\partial G}{\partial \xi}.\end{aligned} \tag{32}$$

Because λ is dependent only on L, the derivatives of ϕ are the derivatives of f, except for $\partial\phi/\partial L$. In this case, an additional term appears,

$$\frac{\partial \phi}{\partial L} = D_1 F(l+\lambda, L, G)\frac{\partial \lambda}{\partial L} + D_2 F(l+\lambda, L, g) - D_2 F(l, L, G). \tag{33}$$

Since $\lambda = nt$, it derivative with respect to L is secular in time,

$$\frac{\partial \lambda}{\partial L} = -\frac{3\lambda}{L}. \tag{34}$$

Putting this together with the result of the previous section and substituting for p/r, we have the derivative of ϕ with respect to L,

$$\begin{aligned}\frac{\partial \phi}{\partial L} &= -\frac{3\lambda}{L}\frac{1}{\eta^3}(1 + e\cos(f+\phi))^2 + \\ &\quad + \frac{(2 + e\cos(f+\phi))\sin(f+\phi)}{Le} - \frac{(2 + e\cos f)\sin f}{Le}.\end{aligned} \tag{35}$$

The other derivatives of ϕ are straightforward; they are simply the difference of derivatives given in the previous section evaluated at the ends of the time interval,

$$\frac{\partial \phi}{\partial l} = \frac{L^3}{G^3}\left[(1 + e\cos(f+\phi))^2 - (1 + e\cos f)^2\right], \tag{36}$$

$$\frac{\partial \phi}{\partial G} = -\frac{(2 + e\cos(f+\phi))\sin(f+\phi)}{Ge} + \frac{(2 + e\cos f)\sin f}{Ge}. \tag{37}$$

Appendix B. Lie Transfer Map Application

With the perturbation map in the form of a factored Lie product

$$\mathcal{M}\boldsymbol{\zeta} = \ldots e^{:f_{\Delta+1}:} e^{:f_\Delta:}\boldsymbol{\zeta} \tag{38}$$

(Healy, 2001), we have a compact representation of the Hamiltonian evolution. It is not, however, immediately practical for computation: the Lie polynomials themselves do not give the transfer map explicitly. In order to obtain the explicit transformation, we will need to apply the transformations. Then, we will be able to propagate a particular initial condition, and, incidentally, check the validity of the computed map analytically or numerically.

B.1. TRANSFER MAP COMPUTATION

There are two ways to go about propagation of a point in phase space under a factored Lie transformation. One is to apply the map to the symbolic set of phase space variables $\zeta_1, \ldots, \zeta_{2d}$ to compute an expansion to the desired order of the transformed variables $\bar{\zeta}_1, \ldots, \bar{\zeta}_{2d}$. The alternative is to apply each Lie transformation (and the unperturbed map) to the numerical set of values representing the phase space point. This has two potential advantages: first, we save ourselves the necessity of computing the effect of transforming the variables analytically. Although PGLT will do this computation analytically, and it is then easy to generate numerical code for evaluation, we may end up with very large expressions. Second, in some circumstances, it is desirable to propagate a point such that the result is *symplectic* to machine precision, even if the result is not *accurate* to machine precision. There are methods to symplectify the map numerically based on the individual Lie transformations.

It also has a disadvantage. Surprisingly, when computing numerically, the homogeneous Lie transformations are applied left to right, and therefore, the map as computed from the Lie transfer map algorithm needs to be transformed from the descending form (38) to the ascending form (Healy, 2001),

$$\mathcal{M}\boldsymbol{\zeta} = e^{:g_\Delta:} e^{:g_{\Delta+1}:} \ldots \boldsymbol{\zeta}. \tag{39}$$

To see that this is so, we need to be less elliptic in our notation; the polynomials f that constitute the Lie transformations $e^{:f:}$ are really functions of phase space, so that they should be written $e^{:f(\zeta):}$. We need the theorem that governs the exchange of transformations, which may be expressed as

$$e^{:f:}e^{:g:} = e^{:e^{:f:}g:}e^{:f:}. \tag{40}$$

Then, consider two successive Lie transformations that are to be numerically evaluated at the phase space point v,

$$\bar{v} = e^{:a(\zeta):}e^{:b(\zeta):}\zeta\big|_{\zeta=v}, \tag{41a}$$

$$\bar{v} = e^{:b(e^{:a(\zeta):}\zeta):}e^{:a(\zeta):}\zeta\big|_{\zeta=v}, \tag{41b}$$

$$\bar{v} = e^{:b(\xi):}\xi\big|_{\xi=e^{:a(\zeta):}\zeta}\big|_{\zeta=v}. \tag{41c}$$

Algorithmically, this expression means we transform with a first,

$$v' = e^{:a(\zeta):}\zeta\big|_{\zeta=v}, \tag{42}$$

and then with b,

$$\bar{v} = e^{:b(\zeta):}\zeta\big|_{\zeta=v'}. \tag{43}$$

Thus, the left-hand transformations are applied first in a numerical computation.

This means that a factored Lie product in the descending form (38) that we get from the Hamiltonian factorization algorithm is not useful, because the high rank transformations would need to be applied first. In contrast, in the ascending form (39), the lower rank terms are applied first, giving us the freedom to decide when to stop on the spot.

To reformulate from descending to ascending (or vice versa), we need to use the rule (40), or concatenate and pull out Lie transformations in the right order. This is described in more detail in a forthcoming publication by the author.

B.2. ANALYTICAL COMPUTATION OF TRANSFER MAP

The analytical computation of propagation with the right-hand transformation first, and proceeds right to left. The application of a Lie transformation to a phase space variable produces an analytical expression for the transformation of that variable:

$$\bar{x} = e^{:f:}x = x + [f, x] + \tfrac{1}{2}[f, [f, x]] + \dots, \tag{44}$$

this analytical expression constitutes a function that gives the propagated position and momentum of a phase space point. If the rank of f is at or equal the bracket grade $r(f) \geqslant \Delta$ as required by the perturbation theory, then each term of this transformation has the same or higher rank as the previous, but never lower. This allows us to control the series, terminating it when terms can be neglected.

B.3. EXAMPLE

Using the computed map of the quartic harmonic oscillator discussed in Healy (2001),

$$\begin{aligned} f_1 &= \exp\colon \delta\mu J^2\Big(-\frac{3}{8}\frac{t}{\omega_0^2} + \frac{1}{\omega_0^3}\Big(-\frac{1}{4}\sin 2\theta + \frac{1}{4}\sin(2\theta + 2\phi_0) + \\ &\quad + \frac{1}{32}\sin 4\theta - \frac{1}{32}\sin(4\theta + 4\phi_0)\Big)\Big)\colon \end{aligned} \tag{45}$$

and

$$\begin{aligned} f_2 &= \exp\colon \delta^2\mu^2 J^3\Big(\frac{t}{\omega_0^5}\Big(\frac{17}{64} + \frac{3}{16}\cos 2\theta + \frac{3}{16}\cos(2\theta + 2\phi_0) - \\ &\quad - \frac{3}{64}\cos 4\theta - \frac{3}{64}\cos(4\theta + 4\phi_0)\Big) + \frac{1}{\omega_0^6}\Big(-\frac{1}{8}\sin 2\phi_0 - \\ &\quad - \frac{1}{256}\sin 4\phi_0 - \frac{3}{128}\sin(2\theta - 2\phi_0) + \frac{33}{128}\sin 2\theta - \\ &\quad - \frac{33}{128}\sin(2\theta + 2\phi_0) + \frac{3}{128}\sin(2\theta + 4\phi_0) - \frac{3}{128}\sin 4\theta + \\ &\quad + \frac{3}{128}\sin(4\theta + 4\phi_0) - \frac{1}{384}\sin 6\theta + \frac{1}{128}\sin(6\theta + 2\phi_0) - \\ &\quad - \frac{1}{128}\sin(6\theta + 4\phi_0) + \frac{1}{384}\sin(6\theta + 6\phi_0)\Big)\Big)\colon \end{aligned} \tag{46}$$

we can compute a transfer map to any rank. We apply the computed Lie transformations to each phase space variable,

$$\bar{\theta} = \dots e^{:f_2:}e^{:f_1:}\theta, \tag{47}$$

$$\bar{J} = \dots e^{:f_2:}e^{:f_1:}J. \tag{48}$$

To compute through a particular rank, we must expand these exponentials, and then truncate at the appropriate point. Suppose we wish to have the transfer map through rank 2; then we need to expand the f_1 exponential through second order, and the f_2 exponential through first order:

$$\dots e^{:f_2:}e^{:f_1:} = I + : f_1 : + : f_1 :^2 + : f_2 : + \mathcal{O}(\delta^3), \tag{49}$$

where I is the identity map and the 'order' term indicates a operators of rank-raise indicated. Keep in mind this formula holds for zero bracket grade $\Delta = 0$ only, with $f_0 = 0$. The map for the quartic harmonic oscillator may be computed using the polynomials above (45) and (46)

$$
\begin{aligned}
\theta \;\leftarrow\; & \theta + \phi_0 + \delta\mu J\Bigg(\frac{3}{4}\frac{t}{\omega_0^2} + \frac{1}{\omega_0^3}\Bigg(\frac{1}{2}\sin 2\theta - \frac{1}{2}\sin(2\theta + 2\phi_0) - \\
& - \frac{1}{16}\sin 4\theta + \frac{1}{16}\sin(4\theta + 4\phi_0)\Bigg)\Bigg) + \delta^2\mu^2 J^2\Bigg(\frac{t}{\omega_0^5}\Bigg(- \frac{51}{64} - \\
& - \frac{3}{8}\cos 2\theta - \frac{3}{4}\cos(2\theta + 2\phi_0) + \frac{3}{32}\cos 4\theta + \frac{3}{16}\cos(4\theta + 4\phi_0)\Bigg) + \\
& + \frac{1}{\omega_0^6}\Bigg(\frac{3}{8}\sin 2\phi_0 + \frac{3}{256}\sin 4\phi_0 + \frac{1}{16}\sin(2\theta - 2\phi_0) - \frac{49}{64}\sin 2\theta + \\
& + \frac{25}{32}\sin(2\theta + 2\phi_0) - \frac{5}{64}\sin(2\theta + 4\phi_0) + \frac{17}{128}\sin 4\theta - \\
& - \frac{1}{8}\sin(4\theta + 2\phi_0) - \frac{1}{128}\sin(4\theta + 4\phi_0) + \frac{1}{64}\sin 6\theta + \\
& + \frac{3}{64}\sin(6\theta + 4\phi_0) - \frac{1}{32}\sin(6\theta + 6\phi_0) + \frac{1}{512}\sin 8\theta - \\
& - \frac{1}{256}\sin(8\theta + 4\phi_0) + \frac{1}{512}\sin(8\theta + 8\phi_0)\Bigg)\Bigg) + \mathcal{O}(\delta^3). \qquad (50) \\
J \;\leftarrow\; & J + \delta\mu J^2\frac{1}{\omega_0^3}\Bigg(- \frac{1}{2}\cos 2\theta + \frac{1}{2}\cos(2\theta + 2\phi_0) + \frac{1}{8}\cos 4\theta - \\
& - \frac{1}{8}\cos(4\theta + 4\phi_0)\Bigg) + \delta^2\mu^2 J^3\Bigg(\frac{t}{\omega_0^5}\Bigg(- \frac{3}{4}\sin(2\theta + 2\phi_0) + \\
& + \frac{3}{8}\sin(4\theta + 4\phi_0)\Bigg) + \frac{1}{\omega_0^6}\Bigg(\frac{17}{32} - \frac{1}{2}\cos 2\phi_0 - \frac{1}{32}\cos 4\phi_0 + \\
& + \frac{3}{32}\cos(2\theta - 2\phi_0) + \frac{3}{8}\cos 2\theta - \frac{21}{32}\cos(2\theta + 2\phi_0) + \\
& + \frac{3}{16}\cos(2\theta + 4\phi_0) - \frac{3}{32}\cos 4\theta + \frac{3}{32}\cos(4\theta + 4\phi_0) + \\
& + \frac{1}{32}\cos(6\theta + 2\phi_0) - \frac{1}{16}\cos(6\theta + 4\phi_0) + \\
& + \frac{1}{32}\cos(6\theta + 6\phi_0)\Bigg)\Bigg) + \mathcal{O}(\delta^3).
\end{aligned}
$$

B.4. CHECKING THE COMPUTED MAP

The most obvious way of checking any map obtained (with this method or any other) is to see that it satisfies Hamilton's equations of motion for the map

$$\dot{\mathcal{M}} = \mathcal{M} : -\mathcal{H}\left(\boldsymbol{\zeta}^{\text{in}}, t\right) :, \qquad (51)$$

and the initial condition, which is that it must reduce to the identity at $t = 0$. Because it is a differential equation on an operator, it is not easily tested in the form given. Instead, we must turn this into a differential equation of a phase space function.

Pick an arbitrary phase space function $g(\boldsymbol{\zeta}, t)$. The map $\mathcal{M}$ that we have obtained propagates this from time t_0 to time t, that is, $g(\boldsymbol{\zeta}, t) = \mathcal{M}(t, t_0)g(\boldsymbol{\zeta}, t_0)$. The result should satisfy Hamilton's equations of motion for a phase space function,

$$\dot{g}(\boldsymbol{\zeta}, t) = [g(\boldsymbol{\zeta}, t), \mathcal{H}(\boldsymbol{\zeta}, t)]. \tag{52}$$

For a given function g, this is easy to check. Proof that the map is correct depends on showing this for all functions; to satisfy this requirement, we need check only each of the phase space variables themselves, that is, $g = \zeta_1, \zeta_2, \ldots, \zeta_{2d}$ in turn. These form the explicit representation of the map.

Consider the factored Lie transformations computed in Healy (2001). To first order, both sides of the differential equation (52) with $g = \theta$ are

$$\omega_0 + \delta\mu J \frac{1}{\omega_0^2}\left(\frac{3}{4} + \cos(2\theta + 2\phi_0) + \frac{1}{4}\cos(4\theta + 4\phi_0)\right) + \mathcal{O}(\delta^2) \tag{53}$$

and for $g = J$ are

$$\delta\mu J^2 \frac{1}{\omega_0^2}\left(\sin(2\theta + 2\phi_0) + \frac{1}{2}\sin(4\theta + 4\phi_0)\right) + \mathcal{O}(\delta^2). \tag{54}$$

Therefore, we know the computed map is correct to first order. A similar calculation on the second order map will confirm that the second rank polynomial (46) and the resultant explicit transformation (50) are correct.

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