ABSTRACT

Title of dissertation: ACHIEVABLE RATES, OPTIMAL SIGNALLING SCHEMES AND RESOURCE ALLOCATION FOR FADING WIRELESS CHANNELS

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The proliferation of services involving the transmission of high rate data traffic over wireless channels makes it essential to overcome the detrimental effects of the wireless medium, such as fading and multiuser interference. This thesis is devoted to obtaining optimal resource allocation policies which exploit the transmitters' and receiver's knowledge about the fading to the network's advantage, to attain information theoretic capacity limits of fading wireless channels.

The major focus of the thesis is on capacity results for fading code division multiple access (CDMA) channels, which have proved to be a robust way of combatting the multiuser interference in practical wireless networks. For these channels, we obtain the capacity region achievable with power control, as well as the power control policies that achieve the desired rate points on the capacity region. We provide practical one-user-at-a-time iterative algorithms to compute the optimal power distributions as functions of the fading. For the special case of sum capacity, some properties of the optimal policy, such as the number of simultaneously transmitting users, are obtained.

We also investigate the effects of limited feedback on the capacity, and demonstrate that very coarse channel state information (CSI) is sufficient to benefit from power control as a means of increasing the capacity.

The selection of the signature sequences also plays an important role in determining the capacity of CDMA systems. This thesis addresses the problem of jointly optimizing the signature sequences and power levels to maximize the sum capacity. The resulting policies are shown to be simple, consisting of orthogonal transmissions in time or signal space, and requiring only local CSI. We also provide an iterative way of updating the joint resource allocation policy, and extend our results to asynchronous, and multi-antenna CDMA systems.

Rather than treating the received signal at the transmitters as interference, it is possible to treat it as free side information and use it for cooperation. The final part of the thesis provides power allocation policies for a fading Gaussian multiple access channel with user cooperation, which maximize the rates achievable by block Markov superposition coding, and also simplify the coding strategy.

Achievable Rates, Optimal Signalling Schemes and Resource Allocation for Fading Wireless Channels

by

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DEDICATION

To my parents Rezzan and Fikret Kaya, and to Başak.

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Chapter 1

Introduction

Increasing demand for higher rates in wireless communication systems continues to trigger major research efforts aimed to characterize and approach the capacity limits of such systems. The wireless medium brings along its unique challenges such as fading and multiuser interference, which make the analysis of the communication systems more complicated. On the other hand, the same challenging properties of such systems are what give rise to the concepts such as diversity which play a vital role in the design of the wireless systems.

The presence of multiple paths and reflectors in the wireless channel causes fluctuations in the received signal, that may cause loss in signal quality, or even the loss of entire communication. The channel fluctuation introduced by these phenomena is called *fading*. Fading may be an important limiting factor in wireless communication networks unless appropriate resource allocation is applied to exploit the variations in the channel gains to the advantage of the network capacity. The capacity limits for communication systems subject to fading have recently drawn significant attention, and in the last decade, several results regarding the information theoretic capacity for many channel models have been reported. The particularly interesting types of channel models are those where the transmitter(s) and receiver(s) are able to track the variations in the channel, and therefore are capable of allocating the system resources and adapting their coding and decoding strategies to the variations in the channel, in order to improve the capacity.

The focus of this thesis is on resource allocation for fading channels, where the fading is modelled as a stationary and ergodic random process, whose statistics, as well as the instantaneous realization are known to the communicating parties. For the most part, the development will be focused on Code Division Multiple Access (CDMA) networks, which provide a practical powerful way to combat the other very important limiting factor in the design of wireless systems: multiuser interference. In the final portion of the thesis, we will take a step back and look at resource allocation policies for wireless channels, while regarding the interference as side information, which allows for user cooperation.

How the resource allocation needs to be performed for a specific wireless network strongly depends on the underlying application. The initial wireless networks that have been deployed in practice were invariably intended to carry voice traffic, and therefore had to make sure that a desired quality of service is attained, so that the delay sensitive voice application is not disrupted. Such networks need to find ways to combat the possible deep attenuations caused by the fading at any given point in time. However, recently, the growing volume of data traffic (email, files, etc.) and the emergence of wireless networks as a medium for communicating higher rate, but less delay sensitive traffic has made it essential to investigate resource allocation policies that are more efficient and opportunistic.

Consider a wireless system, in which the transmitted signals are modulated by a randomly varying channel gain, a realization of which is illustrated in Figure 1.1. Here, the fading is assumed to have Rayleigh statistics, which translates to exponential power attenuation. If the signal transmission is required to be completed within a short period of time, as seen in Figure 1.2, these statistics cannot be observed through the course of the transmission, and the prior knowledge about the distribution of the fading cannot be used accurately to efficiently allocate the resources. However, if the transmission window is long enough, and more delay can be tolerated, it becomes possible to observe the channel and cleverly schedule the transmissions in time. This is illustrated in Figures 1.3 and 1.4, where, by allowing for a sufficiently long time window, all the possible realizations of the fading are observed with their prescribed probabilities, and the long term ergodic properties of the fading process are observed. In this case, if the transmitter faces one of the deep fade levels illustrated in Figure 1.3, knowing that the better channel states are to be realized in the future, it can save its available resources, for example transmit power, to the upcoming favorable states. This type of approach leads to worse instantaneous performance at some channel states, but is expected to improve the average performance metrics, such as the ergodic Shannon capacity. In the remainder of the thesis, this intuitive argument will be made precise.

Power control is perhaps the most common type of resource allocation in wireless communication systems. Following from the ideas above, in the literature, power control for wireless systems have been treated mainly in two different contexts: one based on meeting certain quality of service requirements, and the other based on

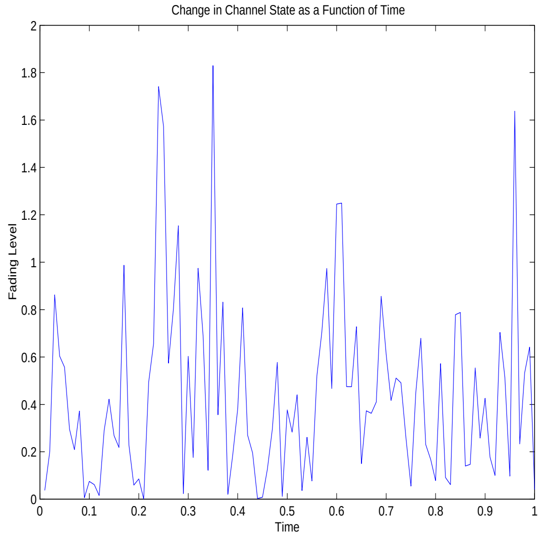


Figure 1.1: Channel fluctuations for a short transmission period.

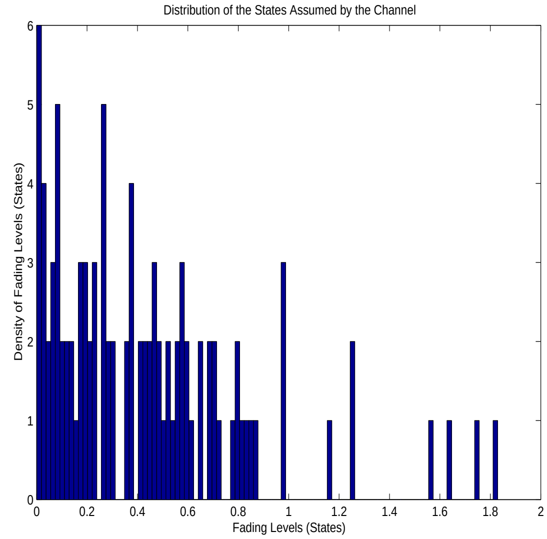


Figure 1.2: Relative frequency of the realized fading levels.

maximizing the information theoretic capacity.

The quality-of-service based power control approaches assign transmit powers to the users so that all users satisfy their signal-to-interference-ratio (SIR) requirements while transmitting with the least amount of power. The SIR-based power control assigns powers to the users with the aim of *compensating* for the variations in the channel; it assigns more power to the users with bad channel states, and less power to the users with good channel states [1–4].

The problem of power allocation in the presence of fading in order to maximize the information theoretic capacity was first studied for a single user channel in [5], where it was shown that, subject to an average power constraint and under the ergodicity assumption on the fading process, the ergodic capacity of the channel is maximized by allocating the total power of the user according to a waterfilling strategy, where the user "waterfills" its power in time, over the inverse of the channel states.

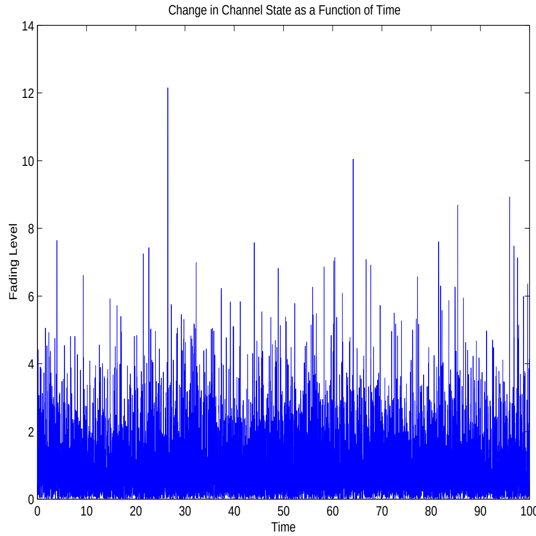


Figure 1.3: Channel fluctuations for a long transmission period, a broader range of fading values observed.

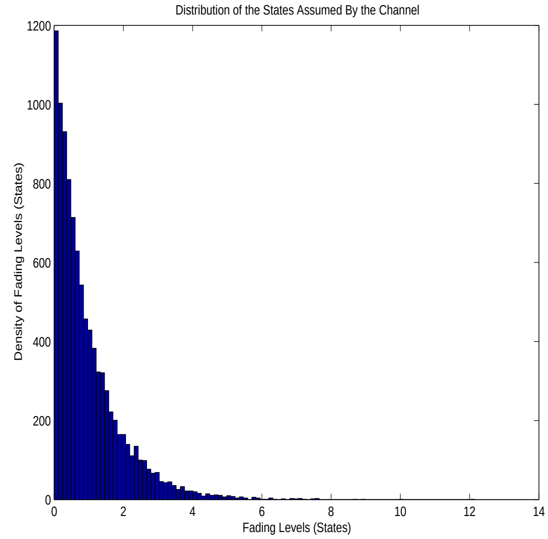


Figure 1.4: Relative frequency of the realized fading levels, as it converges to the underlying statistical distribution.

For multiple access channels (MAC), the capacity region is defined as the set of achievable rate tuples. For a scalar MAC, [6] solved the power allocation problem with the goal of achieving a special rate tuple on the capacity region, the one that achieves ergodic sum capacity. There, it was shown that in order to achieve the sum capacity, only the strongest user may transmit at any given time, and the power control policy is again waterfilling, over disjoint sets of channel states.

The entire capacity region, and the corresponding power control policies for the scalar MAC were characterized in [7]. The capacity region is shown to be a union of the capacity regions (polymatroids) achievable by all valid power allocation policies (i.e., the policies that satisfy the average power constraints). The optimal power allocation policy for each rate tuple on the capacity region is obtained by a greedy algorithm, which compares certain marginal utility functions, and makes use of the generalized symmetry properties of the rank function of the polymatroid corresponding to the rate tuple in question.

There has also been some recent work on power control for vector MACs and their associated capacities. The capacity region for a non-fading vector MAC, where the total average power of the components of the transmitted vectors are constrained, is given by [8]. There, also an iterative waterfilling algorithm which allocates the powers over the components of the transmitted vector in order to maximize the sum capacity was proposed. The capacity region of a non-fading CDMA channel was established in [9]. The power allocation problem for a fading vector MAC was considered in [10], again with the aim of maximizing the sum capacity. It was shown that, the optimal power allocation in the fading case satisfies the Karush-Kuhn-Tucker (KKT) conditions, which can also be interpreted as simultaneous waterfilling, where the water levels are matrices. Also, a relationship between the maximum number of active transmit and receive antennas was given. The problem of maximizing the sum capacity as a function of the transmit powers in a vector multiple access channel, such as a CDMA or multiple transmit antenna system, in fading channels, is studied for the case of large systems and random transmit vectors (signature sequences) in [11] where a simple single-user waterfilling strategy is proposed and shown to be asymptotically optimal.

Despite the recent results on the sum capacity of power controlled vector MACs, the characterization of the capacity region of such channels as well as the resource (e.g., power) allocation schemes that achieve arbitrary rate tuples on the boundary of the capacity region remain as important open problems. A significant portion of the proposed research is devoted to solving these aforementioned problems, for a specific type of vector MAC: a CDMA channel.

CDMA is a multiple accessing scheme that is widely accepted and employed in practice, due to the many desirable features it provides for a wireless multiuser system [12]. Therefore, the problem of characterizing the achievable rates, and the corresponding resource allocation policies for CDMA systems is of great importance, since the solution to this problem will provide both an upper limit on the capacity, which will be a benchmark for the performance of practical CDMA systems, and a method of allocating the resources, i.e., the powers, to attain that limit. Moreover, the study of CDMA systems may give new insight to the problem of obtaining the capacity region for the general fading vector MAC.

The methodology we follow in this thesis for characterizing the capacity region and solving for the optimum power allocation policies is parallel to the above presented evolution of the capacity results for the fading channels in the literature. We first solve the problem of maximizing the sum capacity for fading CDMA, and find the corresponding optimal power allocation policy [13, 14]. We investigate the properties of this policy as it directly relates to the optimum medium access strategies of the users. We then attack the problem of characterizing the entire capacity region [14, 15], and we also solve for the power allocation policies that achieve arbitrary desired rate tuples on the boundary of the capacity region [16, 17]. The assumption of perfect CSI at the transmitters, although useful in analytically characterizing the performance limits of the CDMA network, is hard to realize in practice, due to the limitations on the feedback channel. Therefore, we also investigate the effects of limited feedback on the capacity region.

The capacity region of a CDMA network also depends on the choice of the signature sequences employed by the users (see, for instance, the capacity expression in [9]). Therefore it is possible to consider a problem where the users are allowed to select their signature sequences, and the capacity region is obtained accordingly, by taking into account all possible signature sequences and finding the corresponding rate regions. In the literature, this setting is treated only for the non-fading case, and only for the sum capacity point(s) on the capacity region boundary, where the sum capacity of a non-fading CDMA network is optimized as a function of the signature sequences. When each user has an average power constraint, and there is no fading in the system, [18] shows that when the number of users is less than or equal to the processing gain, the optimal strategy is to allocate orthogonal signature sequences to all users, and when the number of users is greater than the processing gain, with all users having the same average power constraints, the optimal strategy is to allocate Welch Bound Equality (WBE) [19] sequences. Reference [20] generalizes [18] to arbitrary (unequal) average power constraints, and gives the optimal signature sequence allocation as a function of the power constraints of the users. Specifically, for the case in which the number of users is greater than the processing gain, when a user has a relatively larger power constraint then the others, it is called "oversized", and such users are allocated orthogonal signature sequences; whereas the "non-oversized" users are allocated the so-called Generalized Welch-Bound-Equality (GWBE) sequences. In [21], the authors extend their results of [20] to colored noise.

The possibility of improving the capacity of the CDMA systems by choosing the signature sequences motivates us to seek a jointly optimum signature sequence and power allocation scheme in the fading case, in order to further expand the capacity region that is obtained by power allocation only. To this end, we solve the jointly optimum power and signature sequence allocation policy that maximizes the sum capacity of fading CDMA [22, 23]. We observe that, although the joint optimization problem looks more complicated than the power optimization only problem, its solution turns out to have a much simpler and nicer structure; namely, the sum capacity achieving signature sequences turn out to be orthogonal, and the powers follow a single user waterfilling solution over disjoint sets of channel states. We also develop an iterative algorithm, which iterates between the optimal sequence update and the optimal single user power update, for jointly optimizing the signature sequences and the powers. We prove the convergence of this algorithm to the optimum resource allocation.

We extend the solution for jointly optimal power and signature sequence optimization problem for synchronous CDMA to an asynchronous CDMA system. We note that the optimal power allocation policy does not change, and the conditions on the signature sequences are modified as in [24] so as to preserve orthogonality in the asynchronous case.

Another interesting and very active line of research regarding the capacity of wireless systems is on multiple input multiple output (MIMO) wireless channels. For single user MIMO channels, it was shown in [25,26] that the spatial diversity provided by multiple antennas both at the transmitters and the receivers provide a significant improvement in the channel capacity. Inspired by this work, there have been many efforts to quantify the capacity limits of several MIMO systems, including multiuser systems. A comprehensive summary of the results on the capacity of MIMO channels, as well as a list of references and some open problems on this topic can be found in [27].

Motivated by these promising results for MIMO channels, we also address the sum capacity maximization problem for the uplink of fading CDMA which employs multiple antennas at the receiver. Such a system can be considered as an artificial MIMO channel, since each user sends its information symbol by spreading its power over different dimensions, but each of these dimensions experience the same fading level, as opposed to a multi-antenna system. Yet, as for the single antenna system, the joint signature sequence and power allocation provides possibility to choose the transmit power in each signalling direction, thereby making it possible to achieve spatial diversity, which increases the capacity. We provide an iterative algorithm, similar in spirit to our algorithms in the single antenna case, to improve the sum capacity of the multi-antenna CDMA system. This algorithm iterates over the users, and also for each user, it iterates between the best single user power update for given sequences and the eigen-update for that user's sequence for given powers. We demonstrate by simulations that this algorithm attains significant capacity gains for the multi-antenna system.

Today's multiuser communication systems are designed to avoid the multiuser interference inherently caused by the medium. There are many multiple accessing techniques, of which CDMA is a popular one, that simply try to minimize the effect of the additive nature of the signals from multiple users propagating in the air. These policies however, fail to exploit a very important property of the wireless medium: free overheard information. In the last part of the thesis, we turn our attention to systems in which the transmitters take on a more active role in the communication: they cooperate, by decoding messages from other users and employing coding and decoding strategies that yield better rates for all participating users. The channel models in which the transmitters have some level of access to the channel's output(s) were investigated in the early 1980s, and many information theoretic results on achievable rates or capacity regions of variations of such channels were obtained [28–32].

More recently, [33] applied some of these results, mainly the ones regarding achievable rates for MACs with generalized feedback [32], to the Gaussian MAC with cooperating encoders. It was shown that, the rates achievable by what is called *block Markov superposition coding* improve significantly on the traditional MAC capacity region. These results however, although obtained in a fading setting, do not address the possibility of allocation the resources jointly optimally with user cooperation. In the final portion of the thesis, we consider the problem of finding the power control polices that are optimal in conjunction with the block Markov superposition coding. We show that, power control significantly simplifies the original block Markov superposition coding, by eliminating the need to transmit some of the components of the codewords, depending on the realization of the channel states. This also provides nice structural properties for the objective function such as concavity, which would be otherwise absent. We use sub-gradient methods to find the optimal solutions to the simplified power optimization problem, and obtain the resulting improved rate regions.

The remainder of this thesis is organized as follows. In Chapter 2 [13, 14], we introduce the CDMA system model and we find the optimum power allocation policies that maximize the sum capacity of a fading CDMA channel, as well as the properties of these policies and their implications in terms of medium accessing. We also obtain an iterative waterfilling algorithm to solve for the optimal power levels. In Chapter 3 [14–17], we attack the more general problem of finding the entire capacity region for fading CDMA, and power control policies that achieve arbitrary rate tuples on the capacity region boundary. The power levels are obtained through a generalized iterative waterfilling algorithm. We provide some structural properties of the capacity region, such as its non-strict convexity. We then relax the somewhat impractical assumption of perfect CSI and investigate the effects of limited feedback. Chapter 4 [22,23] broadens the sum capacity optimization problem by also including signature sequences as design variables, and we obtain the jointly optimum signature sequences and power levels as a function of the channel states. We propose an algorithm that iterates between the sequences and the powers, and converges to the optimal solution. We extend our results and methods to asynchronous CDMA and systems equipped with multiple receive antennas. In Chapter 5 [34], we solve the optimum power allocation problem for fading Gaussian MAC with user cooperation, and show that the coding scheme is simplified by resource allocation, and the achievable rates are significantly improved. Some concluding remarks are provided in Chapter 6.

Chapter 2

Sum Capacity of Fading CDMA and Optimum Power Allocation

2.1 Introduction

In this chapter, we focus on the sum capacity of a fading CDMA channel where the number of users and the processing gain are finite and arbitrary, and the users are assigned arbitrary deterministic signature sequences. Sum capacity quantifies the maximum reliable rate of information flow in a network, and therefore it is a commonly used performance metric for MACs [6, 10, 11]. Our goal is to find the sum capacity maximizing power control policy for the fading CDMA system.

We consider a symbol synchronous CDMA system with processing gain N where all K users transmit to a single receiver site. In the presence of fading and additive white Gaussian noise (AWGN), the received signal within each symbol interval of length T is given by

$$r(t) = \sum_{i=1}^{K} \sqrt{p_i h_i} b_i s_i(t) + n(t), \qquad 0 < t < T \tag{2.1}$$

where, for user i, b_i denotes the information symbol with $\mathrm{E}[b_i^2] = 1$, $s_i(t)$ denotes the unit energy signature waveform, $\sqrt{h_i}$ denotes the random channel gain, and p_i denotes the transmit power; $n(t)$ denotes the AWGN with zero-mean and power spectral density σ^2. In our model (2.1) we assume quasi-static fading where the channel gain is constant over the symbol duration T, i.e., $h_i(t) = h_i$, $0 \leq t < T$, but changes at random from one symbol interval to the next. The signature waveforms can be represented by N orthonormal basis waveforms $\{\psi_j\}_{j=1}^N$, such that $s_i(t) = \sum_{j=1}^N s_{ij}\psi_j(t)$, where $s_{ij} = \langle s_i(t), \psi_j(t)\rangle$. Projecting the received signal onto the basis waveforms, i.e., $r_j = \langle r(t), \psi_j(t)\rangle$, we obtain the sufficient statistics $\{r_j\}_{j=1}^N$. Therefore, the continuous channel in (2.1) can be represented in an equivalent vector form as [12],

$$\mathbf{r} = \sum_{i=1}^{K} \sqrt{p_i h_i} b_i \mathbf{s}_i + \mathbf{n} \tag{2.2}$$

where $\mathbf{s}_i = [s_{i1}, \cdots, s_{iN}]^\top$ is the signature sequence of user i, and $\mathbf{n}$ is a zero-mean Gaussian random vector with covariance $\sigma^2\mathbf{I}_N$. We assume that the receiver and all of the transmitters have perfect knowledge of the channel states of all users represented as a vector $\mathbf{h} = [h_1, \cdots, h_K]^\top$, as well as their statistics, which are assumed to be independent across the users. We further assume that although the fading is slow enough to ensure constant channel gain in a symbol interval, it is fast enough so that within the transmission time of a block of symbols the long term ergodic properties of the fading process can be observed [35].

Our problem reduces to K independent Goldsmith-Varaiya problems [5] when the signature sequences are chosen to be orthogonal, and to a Knopp-Humblet problem [6] when the signature sequences are chosen to be identical. We show that the optimum power allocation policy is a *simultaneous waterfilling* policy that requires the solution of a set of highly nonlinear equations. We develop an iterative power allocation policy, where, at each step, only one user allocates its power optimally over all joint channel states when the power allocations of all other users are fixed. The power allocation of each user in this iterative process is a waterfilling where the *base level of the water tank* is determined by the inverse of the SIR the user would obtain at the output of a minimum mean squared error (MMSE) receiver if it transmitted with unit power. We prove the convergence of our algorithm to an optimum solution, and provide conditions for the uniqueness of that solution.

The information theoretic approach to power allocation has proved to have the intriguing property that the capacity optimal power control policies for multi-access channels have specific forms which automatically dictate optimal multiple access strategies (i.e., channel adaptive time division of [6]), by assigning zero powers to some of the users based on their channel states, thereby combining the medium access control layer and the physical layer design of communication systems. Therefore, one of the questions of interest, for a CDMA network with an arbitrary set of signature sequences, is whether there exists a set of channel states having a non-zero probability where either all or some of the users transmit simultaneously. In the case of orthogonal signature sequences, for instance, all users transmit simultaneously in an orthant of the space of all channel states where the channel states of all users exceed their corresponding thresholds; and, clearly, this region has a non-zero probability. In the case of identical signature sequences however, users transmit simultaneously only on a half-line in the space of all channel states; and, this region has a zero probability [6]. In the most general case, the existence of a region of channel states having non-zero probability where all (or more than one) users transmit simultaneously depends on the number of users, the dimensionality of the signal space (processing gain), and the set of signature sequences being used. We identify the conditions under which such a non-zero probability region of channel states exists. These conditions turn out to be very mild; for instance, if the number of users is less than the processing gain and the sequences are linearly independent, a simultaneous transmit region for all users is guaranteed to exist. This region also exists even when the number of users is larger than the processing gain so long as the signature sequences satisfy certain properties. Also, even if these conditions are not satisfied for all users, there may be a subset of users that are guaranteed to transmit simultaneously. This is a result of the fact that CDMA scheme with non-identical signature sequences provides users with multiple degrees of freedom; therefore, the users do not have to *avoid* each other completely in the space of all channel states (as in the case of scalar channels), that is, multiple users can *share* some of the channel states that are favorable to all of them.

The existence of simultaneous transmit regions is of interest to us since it provides a sense of fairness, in that while maximizing the overall average rate achieved by the system, it allows users to access the medium more frequently. This is in contrast to the scalar channels where each user has to wait until its channel is the best in order to transmit [6].

2.2 Optimal Power Control via Iterative Waterfilling

For a given set of signature sequences and a fixed set of channel gains, $\mathbf{h}$, the sum capacity $C_{\text{sum}}(\mathbf{h})$ is [9]

$$C_{\text{sum}}(\mathbf{h}) = \frac{1}{2} \log \left| \mathbf{I}_N + \sigma^{-2} \sum_{i=1}^{K} h_i \bar{p}_i \mathbf{s}_i \mathbf{s}_i^\top \right| \tag{2.3}$$

where $\bar{p}_i$ is the average power of user i, and $|\cdot|$ denotes the determinant of its argument. When the channel state is modelled as a random vector, the quantity $C_{\text{sum}}(\mathbf{h})$ is random as well. If a constant (non-adaptive) power policy is applied, the ergodic sum capacity is found as the expected value of $C_{\text{sum}}(\mathbf{h})$ over all channel states [35],

$$C_{\text{sum}} = \frac{1}{2} E_{\mathbf{h}} \left[\log \left| \mathbf{I}_N + \sigma^{-2} \sum_{i=1}^{K} h_i \bar{p}_i \mathbf{s}_i \mathbf{s}_i^\top \right| \right] \tag{2.4}$$

where the expectation is with respect to the joint probability density function $f(\mathbf{h})$ of the components of the channel state vector $\mathbf{h}$. In (2.4), the transmit power of user i is fixed to $\bar{p}_i$, its average power constraint. Our goal is to choose the transmit powers of the users as a function of the channel state $p_i(\mathbf{h})$, $i = 1, \cdots, K$, with the aim of maximizing the ergodic sum capacity of the system subject to average transmit power constraints for all users. We formulate the problem as,

$$\begin{aligned} &\max_{\{p_i(\mathbf{h})\}} E_{\mathbf{h}} \left[\log \left| \mathbf{I}_N + \sigma^{-2} \sum_{i=1}^{K} h_i p_i(\mathbf{h}) \mathbf{s}_i \mathbf{s}_i^\top \right| \right] \\ &\text{s.t.} E_{\mathbf{h}} \left[p_i(\mathbf{h}) \right] = \bar{p}_i, \qquad p_i(\mathbf{h}) \geq 0, \qquad i = 1, \cdots, K \end{aligned} \tag{2.5}$$

For arbitrary signature sequences, no closed form solution for this problem is known. It is interesting to note that, (2.5) reduces to the Knopp-Humblet problem [6] if the signature sequences are identical, i.e., $\mathbf{s}_i = \mathbf{s}$ for all i, and it reduces to K separable Goldsmith-Varaiya [5] problems, if the signature sequences are orthogonal, i.e., $\mathbf{s}_i^\top \mathbf{s}_j = 0$ for $i \neq j$, in which case each problem can be solved independently of the others. Our aim is to find the optimal power allocation for the most general case where the signature sequences are arbitrarily correlated, i.e., $\mathbf{s}_i^\top \mathbf{s}_j$ is not restricted to be zero or one.

Using the matrix inversion lemma [12] together with the fact that $\det(\mathbf{I} + \mathbf{AB}) = \det(\mathbf{I} + \mathbf{BA})$ we can isolate the contribution of user k to the sum capacity as follows

$$
\begin{aligned}
C_{\text{sum}} &= \frac{1}{2} E_{\mathbf{h}} \left[\log \left| \mathbf{I}_N + \sigma^{-2} \sum_{i=1}^{K} h_i p_i(\mathbf{h}) \mathbf{s}_i \mathbf{s}_i^\top \right| \right] \\
&= \frac{1}{2} E_{\mathbf{h}} \left[\log \left| \sigma^{-2} p_k(\mathbf{h}) h_k \mathbf{s}_k \mathbf{s}_k^\top + \mathbf{I}_N + \sigma^{-2} \sum_{i \neq k} h_i p_i(\mathbf{h}) \mathbf{s}_i \mathbf{s}_i^\top \right| \right] \\
&= \frac{1}{2} E_{\mathbf{h}} \left[\log \left| \sigma^{-2} \mathbf{A}_k \left(\mathbf{I}_N + p_k(\mathbf{h}) h_k \mathbf{A}_k^{-1} \mathbf{s}_k \mathbf{s}_k^\top \right) \right| \right] \\
&= \frac{1}{2} E_{\mathbf{h}} \left[\log \left| \sigma^{-2} \mathbf{A}_k \right| + \log \left(1 + p_k(\mathbf{h}) h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k \right) \right] \qquad (2.6)
\end{aligned}
$$

Hence, we can express the ergodic sum capacity, the objective function of (2.5), as

$$
C_{\text{sum}} = C_k + \overline{C}_k \qquad (2.7)
$$

where

$$
C_k = \frac{1}{2} E_{\mathbf{h}} \left[\log \left(1 + h_k p_k(\mathbf{h}) \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k \right) \right] \qquad (2.8)
$$

represents the contribution of the kth user to the sum capacity when the transmit powers of all other users at all channel states are fixed, and

$$\overline{C}_k = \frac{1}{2} E_{\mathbf{h}} \left[\log \left| \mathbf{I}_N + \sigma^{-2} \sum_{i \neq k} h_i p_i(\mathbf{h}) \mathbf{s}_i \mathbf{s}_i^\top \right| \right] \tag{2.9}$$

represents the sum capacity of the remaining users when the kth user is removed from the system. In (2.8), $\mathbf{A}_k$ is defined as

$$\mathbf{A}_k = \sigma^2 \mathbf{I}_N + \sum_{i \neq k} h_i p_i(\mathbf{h}) \mathbf{s}_i \mathbf{s}_i^\top \tag{2.10}$$

It is worth noting that C_{sum}, the objective function in (2.5), is a concave function of the powers, and moreover, provided that the matrices $\{\mathbf{s}_i \mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent, it is a strictly concave function of the powers [11]. Also, the constraint set in (2.5) is convex. Therefore, the optimization problem in (2.5) has a unique global optimum when $\{\mathbf{s}_i \mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent; and all local optimums yield the same objective function value, otherwise. Let us associate the Lagrange multipliers λ_k with the equality constraints and μ_k with the inequality constraints. The optimum power allocation policy satisfies the extended Karush-Kuhn-Tucker (KKT) conditions with mixed constraints [36, Chap. 13], which, after taking the derivatives and employing the complementary slackness conditions $p_k \mu_k = 0$, simplify to

$$\frac{h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k}{1 + p_k(\mathbf{h}) h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k} \leq \lambda_k, \qquad k = 1, \cdots, K, \qquad \forall\ \mathbf{h} \in R^K \tag{2.11}$$

which is satisfied with equality if and only if $p_k > 0$. Using the fact that $p_k \geq 0$ for all i, (2.11) implies that the capacity maximizing power allocation policy satisfies

$$p_k(\mathbf{h}) = \left(\frac{1}{\lambda_k} - \frac{1}{h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k}\right)^+, \qquad k = 1, \cdots, K \tag{2.12}$$

for any realization of the channel $\mathbf{h}$.

Here the Lagrange multipliers λ_k are determined by inserting (2.12) into the average power constraints in (2.5). The values of λ_ks depend on the statistical characterization of the channel and the choice of signature sequences. This solution is similar in structure to the solution in [11], however it is more general in that it is valid for any continuous joint fading distribution, any power constraints and any finite number of deterministic signature sequences, as opposed to the symmetric and asymptotical situation in [11]. Note that, even though the continuity and independence assumptions on fading will be needed in order to prove the simultaneous transmission conditions for the optimal power allocation policy in Section 2.3, the characterization of optimal power allocation policy in (2.12) does not require these assumptions.

For arbitrary signature sequences, the set of equations (2.12) is highly nonlinear. Although it is possible to solve for the optimum powers and transmit regions in a simple system with few users, it seems intractable for systems with large numbers of users. It is worth noting that $h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k$ is the SIR of user k at the output of an MMSE receiver if it transmitted with $p_k = 1$. Therefore, all users should *simultaneously* waterfill on the "base levels" of the inverse of the SIRs they would obtain if they transmitted with unit powers. Since solving for the simultaneous waterfilling solution for all users seems intractable, we devise an iterative algorithm. Consider optimizing for the power of *only* user k over all channel states, given the powers of all other users at all channel states,

$$\begin{aligned} p_k^{n+1} &= \arg\max_{p_k} \; C_{\text{sum}}\left(p_1^{n+1}, \cdots, p_{k-1}^{n+1}, p_k, p_{k+1}^n \cdots, p_K^n\right) \\ &= \arg\max_{p_k} \; C_k\left(p_k\right) \end{aligned} \tag{2.13}$$

where $C_k(p_k)$ denotes the contribution of user k to C_{sum}, as defined in (2.8). $C_k(p_k)$ depends on the power distributions and signature sequences of other users through $\mathbf{A}_k$'s which change as a function of the channel state. We have already noted that the objective function C_{sum} is a concave function of the powers, and also that $C_k(p_k)$ given by (2.8) is a strictly concave function of p_k. The constraint set for powers over which the maximization is to be performed is convex, and has a Cartesian product structure among the users. The solution of (2.13) can be found as a single-user waterfilling over all channel states of the system,

$$p_k(\mathbf{h}) = \left(\frac{1}{\tilde{\lambda}_k} - \frac{1}{h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k}\right)^+ \tag{2.14}$$

If we let only one user allocate its power over all channel states using (2.14), and iterate over all users sequentially, this iterative *one-user-at-a-time algorithm* is guaranteed to converge to the global optimum solution of (2.5) [37, Prop. 3.9].

A snapshot from one of the iterations of the one-user-at-a-time algorithm is illustrated in Figure 2.1, where user 1 allocates its power by filling the tank, the base of which is determined by the interference caused by user 2. The level of the water is determined by the average power constraint, and is equal to $\tilde{\lambda}$ at that iteration.

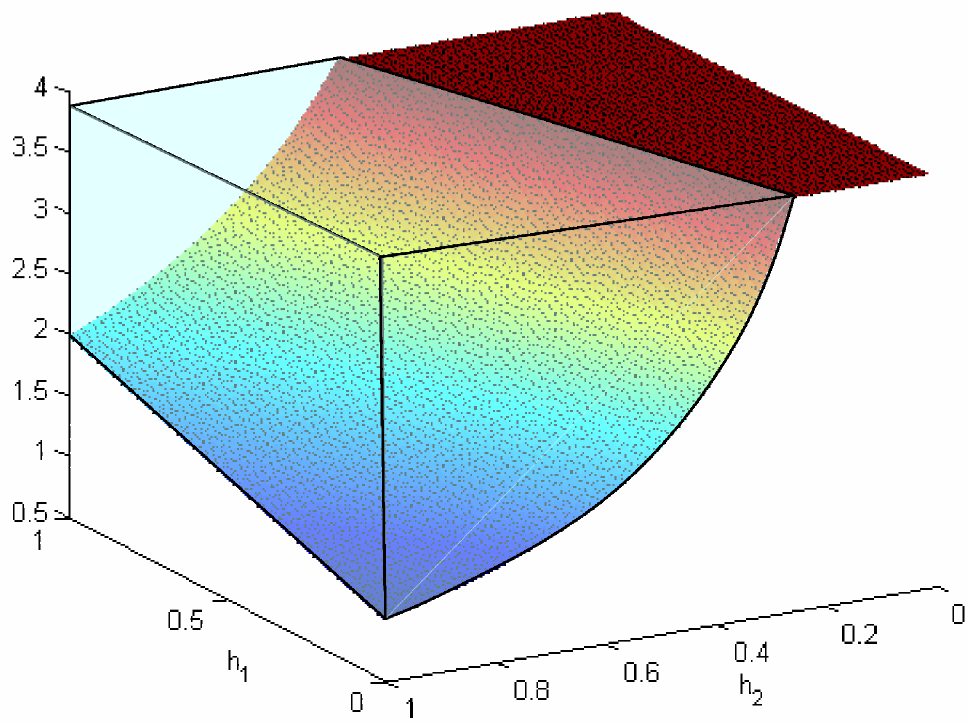


Figure 2.1: Illustration of waterfilling for user 1 over the channel state space. The base level for the tank is determined by the power level of user 2 which is kept fixed for this iteration.

As noted earlier, $h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k$ denotes the SIR user k would obtain at the output of an MMSE filter if it transmitted with unit power. Therefore, at any given iteration, a user waterfills over the inverse of the SIRs it would obtain at all channel states if it transmitted with unit power, given the current power allocations of all other users: the user puts more power into states where its expected SIR with unit transmit power is larger. When the signature sequences of the users are all orthogonal, iteration in

(2.14) reduces to

$$p_k(\mathbf{h}) = \left(\frac{1}{\tilde{\lambda}_k} - \frac{\sigma^2}{h_k} \right)^+ \tag{2.15}$$

and converges to the optimum solution found in [5] in one step. When the signature sequences of the users are identical, iteration in (2.14) becomes

$$p_k(\mathbf{h}) = \left(\frac{1}{\tilde{\lambda}_k} - \frac{\sigma^2 + \sum_{i \neq k} h_i p_i(\mathbf{h})}{h_k} \right)^+ \tag{2.16}$$

and converges to the solution found in [6]. Note here that, the SIR in (2.16) takes the familiar form of the SIR at the output of a MF, since in this case MMSE filters reduce to scaled MFs. Finally, we note that, the iterative implementation of the "simultaneous waterfilling in time" presented in this chapter is analogous to the iterative implementation of the "simultaneous waterfilling over parallel channels" in [8].

2.3 Properties of the Optimal Power Allocation

Let us now consider the inverse problem of finding the channel state of the system for a given transmit power vector with non-zero components. Since all components of the power vector are non-zero, this means that all users transmit simultaneously at this particular channel state, and (2.11) should be satisfied with equality for all k. Therefore, given any arbitrary power vector $\mathbf{p}$ with $0 < p_i < 1/\lambda_i$, the channel state where this power vector is used can be found by solving

$$\mathbf{h} = \mathbf{f}(\mathbf{h}) \tag{2.17}$$

where the vector function $\mathbf{f}(\mathbf{h})$ is defined as

$$f_k(\mathbf{h}) = \frac{\lambda_k}{(1-\lambda_k p_k)} \frac{1}{\mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k}, \qquad k = 1, \cdots, K \tag{2.18}$$

Our first goal is to show that there exists a unique vector $\mathbf{h}$ of channel states corresponding to any given non-zero solution $\mathbf{p}$ to the power control problem. To this end, we first need to prove some properties of the function $\mathbf{f}(\mathbf{h})$.

Definition 2.1 ([3]) $\mathbf{f}(\mathbf{h})$ *is standard, if for all* $\mathbf{h} \geq \mathbf{0}$, *the following properties are satisfied.*

- *Positivity:* $\mathbf{f}(\mathbf{h}) > \mathbf{0}$
- *Monotonicity: If* $\mathbf{h} \geq \mathbf{h}'$ *then* $\mathbf{f}(\mathbf{h}) \geq \mathbf{f}(\mathbf{h}')$
- *Scalability: For all* $\alpha > 1$, $\alpha \mathbf{f}(\mathbf{h}) > \mathbf{f}(\alpha \mathbf{h})$

Lemma 2.1 *Let* $0 < p_k < 1/\lambda_k$, *for all* k. *Then,* $\mathbf{f}(\mathbf{h})$ *is standard.*

Proof: For notational convenience, let us define

$$\begin{aligned} g_k(\mathbf{h}, \mathbf{c}_k) &= \frac{\lambda_k}{(1-\lambda_k p_k)} \frac{\sum_{i \neq k} p_i h_i \left(\mathbf{c}_k^\top \mathbf{s}_i\right)^2 + \sigma^2 \left(\mathbf{c}_k^\top \mathbf{c}_k\right)}{\left(\mathbf{c}_k^\top \mathbf{s}_k\right)^2} && (2.19) \\ &= \frac{\lambda_k}{(1-\lambda_k p_k)} \frac{\mathbf{c}_k^\top \mathbf{A}_k \mathbf{c}_k}{\left(\mathbf{c}_k^\top \mathbf{s}_k\right)^2} && (2.20) \end{aligned}$$

Then, interpreting $\mathbf{c}_k$ as a linear receiver filter, we can relate $f_k(\mathbf{h})$ to $g_k(\mathbf{h}, \mathbf{c}_k)$ by

$$f_k(\mathbf{h}) = \min_{\mathbf{c}_k} g_k(\mathbf{h}, \mathbf{c}_k) \tag{2.21}$$

where the filter that minimizes $g_k(\mathbf{h}, \mathbf{c}_k)$ is $\mathbf{c}_k^* = \mathbf{A}_k^{-1}\mathbf{s}_k$, i.e., a scaled version of the well-known MMSE filter.

For $0 < p_k < 1/\lambda_k$, $g_k(\mathbf{h}, \mathbf{c}_k) > 0$ for any $\mathbf{c}_k$, due to non-zero noise variance. Then, $f_k(\mathbf{h}) = \min_{\mathbf{c}_k} g_k(\mathbf{h}, \mathbf{c}_k) > 0$, proving the positivity.

For monotonicity, let $\mathbf{h} \geq \mathbf{h}'$,

$$\begin{aligned} f_k(\mathbf{h}) &= \min_{\mathbf{c}_k} g_k(\mathbf{h}, \mathbf{c}_k) && (2.22)\\ &= g_k(\mathbf{h}, \mathbf{c}_k^*) && (2.23)\\ &\geq g_k(\mathbf{h}', \mathbf{c}_k^*) && (2.24)\\ &\geq \min_{\mathbf{c}_k} g_k(\mathbf{h}', \mathbf{c}_k) = f_k(\mathbf{h}') && (2.25) \end{aligned}$$

Inequality (2.24) follows from (2.19) noting that $\mathbf{h} \geq \mathbf{h}'$ and $\mathbf{c}_k$ is fixed.

For scalability, we pick $\alpha > 1$,

$$\begin{aligned} \alpha f_k(\mathbf{h}) &= \alpha \min_{\mathbf{c}_k} g_k(\mathbf{h}, \mathbf{c}_k) && (2.26)\\ &= \alpha g_k(\mathbf{h}, \mathbf{c}_k^*) && (2.27)\\ &> g_k(\alpha\mathbf{h}, \mathbf{c}_k^*) && (2.28)\\ &\geq \min_{\mathbf{c}_k} g_k(\alpha\mathbf{h}, \mathbf{c}_k) = f_k(\alpha\mathbf{h}) && (2.29) \end{aligned}$$

Inequality (2.28) follows from (2.19) noting that $\alpha > 1$ and $\mathbf{c}_k$ is fixed. $\square$

Note that, since $\mathbf{f}(\mathbf{h})$ is standard, if there is a solution for (2.17), it is unique [3]. In fact, one can devise an iterative algorithm to find this solution,

$$\mathbf{h}(n+1) = \mathbf{f}(\mathbf{h}(n)) \quad (2.30)$$

The problem in (2.17) with the definition of $\mathbf{f}(\mathbf{h})$ in (2.18) is very similar to the joint power control and linear receiver filter design problem in [38]. In [38], the problem is to solve for the componentwise smallest power vector $\mathbf{p}$ and the receiver filters $\{\mathbf{c}_i\}_{i=1}^K$ such that all users satisfy their SIR based quality of service requirements. For a single receiver site (e.g., single-cell) system, the problem becomes that of finding componentwise smallest power vector and receiver filters that satisfy

$$\mathrm{SIR}_k = \frac{p_k h_k \left(\mathbf{c}_k^\top \mathbf{s}_k\right)^2}{\sum_{i \neq k} p_i h_i \left(\mathbf{c}_k^\top \mathbf{s}_i\right)^2 + \sigma^2 \left(\mathbf{c}_k^\top \mathbf{c}_k\right)} \geq \beta_k \tag{2.31}$$

where β_k, $k = 1, \cdots, K$ are the SIR targets.

When there are no maximum power constraints, solving for optimum transmit powers $\mathbf{p}$, and received powers $\mathbf{q}$ where $q_k = p_k h_k$, are equivalent. The optimum transmit powers can be found using the optimum received powers via $p_k^* = q_k^*/h_k$. Then, from (2.31) and (2.10),

$$\mathrm{SIR}_k = \frac{q_k \left(\mathbf{c}_k^\top \mathbf{s}_k\right)^2}{\mathbf{c}_k^\top \mathbf{A}_k \mathbf{c}_k} \geq \beta_k \tag{2.32}$$

For any given set of powers, $\mathbf{c}_k$ should be chosen to be the MMSE filter as it maximizes the SIR [38]. Using the MMSE filters $\mathbf{c}_k = \alpha_k A_k^{-1} \mathbf{s}_k$, the problem becomes equivalent to solving for $\mathbf{q}$ in

$$q_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k = \beta_k \tag{2.33}$$

While [38] developed a distributed iterative algorithm that converges to the optimum powers (and receivers) assuming that the problem is feasible, [39] found the conditions on the SIR targets $\{\beta_i\}_{i=1}^K$ and the signature sequences $\{\mathbf{s}_i\}_{i=1}^K$ that guarantee that the problem is feasible, i.e., positive q_ks that satisfy (2.33) exist. The SIR targets $\beta_1, \cdots, \beta_k$ are feasible if and only if [39, Thm. 10],

$$\sum_{k \in U} \frac{\beta_k}{1+\beta_k} < \text{rank}(\mathbf{S}(U)), \qquad \forall\, U \subset \{1, \cdots, K\} \tag{2.34}$$

where $\mathbf{S}(U)$ is the matrix containing the sequences of the users in the subset U.

In our problem, the channel gains are found for any given power vector by solving (2.17),

$$h_k p_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k = \frac{\lambda_k p_k}{(1-\lambda_k p_k)} \tag{2.35}$$

Since there are no maximum constraints on the channel gains, solving for h_k and $q_k = h_k p_k$ are equivalent, as we can obtain the solution for h_k using the solution for q_k via $h_k^* = q_k^*/p_k$. Thus, our problem is equivalent to (2.33) where β_k are given by

$$\beta_k = \frac{\lambda_k p_k}{(1-\lambda_k p_k)}, \qquad k = 1, \cdots, K \tag{2.36}$$

and are determined by the given power vector. The set of feasible powers can then be found by inserting (2.36) into (2.34)

$$\sum_{k \in U} \lambda_k p_k < \text{rank}(\mathbf{S}(U)), \qquad \forall\, U \subset \{1, \cdots, K\} \tag{2.37}$$

Therefore, once we fix a power vector satisfying (2.37), (2.17) has a unique solution, since $\mathbf{f}(\mathbf{h})$ is standard. That is, the power vector we chose is a possible candidate for the optimum power allocation at the channel state obtained by solving (2.17). This means that, corresponding to a set of feasible power vectors, there always exists a set of channel states where all of the users in the system transmit simultaneously. This set however can have zero probability as in [6], which is the result of the fact that, although we can find a unique channel state for a feasible power vector, multiple feasible power vectors may correspond to the same channel state, i.e., there may be multiple optimum power vectors with the same C_{sum}. That is, the mapping between the powers and the channel states is not one-to-one, in general.

The significance of (2.37) for our purposes is that the set of feasible power vectors constitutes a volume in K dimensional space. For the set of feasible power vectors satisfying (2.37), and having strictly positive components, if the set of corresponding channel states found by solving (2.17) have a non-zero measure, then we can conclude that all users transmit simultaneously with a positive probability.

Theorem 2.1 *There exists a non-zero probability region of fading states* $\mathbf{h}$ *where all* K *users transmit simultaneously, if and only if* $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ *are linearly independent.*

Proof: It is clear that the set of feasible powers as given by (2.37) constitutes a volume V in R^K. Let us then pick any point $\mathbf{p}_0 > \mathbf{0}$ in this set, and compute the channel state which corresponds to this particular solution of powers. By feasibility of $\mathbf{p}_0$, the resulting channel state $\mathbf{h}_0$ is unique, and the original vector $\mathbf{p}_0$ satisfies the KKT conditions at $\mathbf{h}_0$. Given that $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent, we know that there exists a unique global maximum for C_{sum} since it is strictly concave. Therefore, the water-filling solution we get at the fading state $\mathbf{h}_0$ should be equal to $\mathbf{p}_0$, as it is a possible solution to the problem, and the problem has a unique global optimum. Hence, we obtain a unique fading state for a power level, and a unique power for a fading state, for a set of powers satisfying (2.37). This implies that there exists a one-to-one mapping from the space of feasible strictly positive powers to the space of fading states. This one-to-one mapping maps the volume $V \subset R^K$ of feasible powers to a volume of fading states $\tilde{V} \subset R^K$ implying that the resulting set of fading states where K users transmit simultaneously has non-zero probability. This completes the proof of the if part.

For the only if part, consider the case where $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly dependent. For all K users to transmit simultaneously with non-zero powers, (2.11) must be satisfied with equality for all k. By applying matrix inversion lemma, and defining $\mathbf{A} = \sigma^2\mathbf{I}_N + \mathbf{SPS}^\top$, which contains all users' powers and signatures, (2.11) can be written in the alternative form

$$h_k\mathbf{s}_k^\top\mathbf{A}^{-1}\mathbf{s}_k = \lambda_k, \qquad k = 1, \cdots, K \tag{2.38}$$

Each of these equations can also be rewritten as,

$$h_k\mathrm{tr}\left(\mathbf{A}^{-1}\mathbf{s}_k\mathbf{s}_k^\top\right) = \lambda_k, \qquad k = 1, \cdots, K \tag{2.39}$$

If $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly dependent, then any one of the elements of this set, say $\mathbf{s}_k\mathbf{s}_k^\top$, can be written as a linear combination of the others, say, with coefficients α_i, not all equal to zero. Thus,

$$h_k\mathrm{tr}(\mathbf{A}^{-1}\sum_{i\neq k}\alpha_i\mathbf{s}_i\mathbf{s}_i^\top) = h_k\sum_{i\neq k}\alpha_i\mathbf{s}_i^\top\mathbf{A}^{-1}\mathbf{s}_i = \lambda_k \tag{2.40}$$

and using (2.38) in (2.40), we get

$$\sum_{i \neq k} \alpha_i \frac{\lambda_i}{h_i} = \frac{\lambda_k}{h_k} \tag{2.41}$$

This means that, if $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly dependent, then regardless of the power levels, for all users to transmit simultaneously, the channel states should satisfy (2.41). Since the channel states are continuous random variables, this event has zero probability. Therefore, given that $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly dependent, all K users transmit simultaneously only with zero probability. □

Therefore, the necessary and sufficient condition for all K users to transmit simultaneously with non-zero probability is that the signature sequences are such that the matrices $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent. Our first corollary below states that if the signature sequences $\{\mathbf{s}_i\}_{i=1}^K$ are linearly independent, then $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent and all users transmit simultaneously with non-zero probability.

Corollary 2.1 *When $K \leq N$, for a set of K linearly independent signature sequences, there always exists a non-zero probability region of channel states where all K users transmit simultaneously.*

Proof: Assume that $\{\mathbf{s}_i\}_{i=1}^K$ are linearly independent. For $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ to be linearly dependent, we should be able to write

$$\mathbf{s}_k\mathbf{s}_k^\top = \sum_{i \neq k} \alpha_i \mathbf{s}_i\mathbf{s}_i^\top \tag{2.42}$$

with at least two non-zero α_is; if only one α_i is non-zero, this implies that two signature sequences are the same violating the fact that $\{\mathbf{s}_i\}_{i=1}^K$ are linearly independent. The ranks of both sides of (2.42) have to be equal. As $\{\mathbf{s}_i\}_{i=1}^K$ are linearly independent, the rank of the right hand side is equal to at least two, whereas that of the left hand side is always one. Therefore, the set $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent for linearly independent signature sequences, and the result follows from Theorem 2.1.

□

It is hard to find closed form expressions for the region of the channel gains where all users transmit simultaneously. For a simple two-user system, it can be shown that both users transmit with non-zero powers when $\mathbf{h}$ belongs to a region expressed by

$$h_1 > \frac{\lambda_1\sigma^2 h_2}{h_2\,(1-\rho^2)+\rho^2\sigma^2\lambda_2}, \qquad h_2 > \frac{\lambda_2\sigma^2 h_1}{h_1\,(1-\rho^2)+\rho^2\sigma^2\lambda_1} \tag{2.43}$$

where $\rho = \mathbf{s}_1^\top\mathbf{s}_2$ denotes the cross correlation between the signature sequences of the users. This region is depicted in Figure 2.2.

It is interesting to note that when h_2 goes to infinity, the lower bound on h_1 approaches the limit $\lambda_1\sigma^2/(1-\rho^2)$, and as h_1 goes to infinity, the lower bound on h_2 goes to $\lambda_2\sigma^2/(1-\rho^2)$. These are the two (horizontal and vertical) asymptotes shown in Figure 2.2. For more than two users, even though the exact expressions for the boundaries of the simultaneous transmission region are nonlinear and complex, we can describe an "orthant" of the space of all channel states where all users transmit simultaneously. This orthant is a subset of the actual simultaneous transmission region.

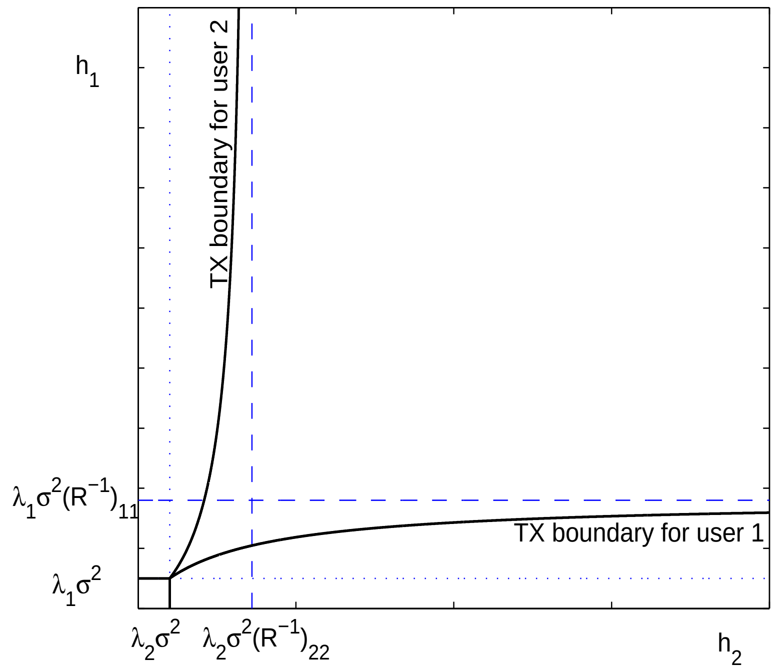


Figure 2.2: Transmit region boundaries for two users with correlated signature sequences.

Theorem 2.2 *For a set of K linearly independent signature sequences, the region of channel states where all users transmit simultaneously includes an "orthant" in R^K described by,*

$$h_k > \lambda_k \sigma^2 (\mathbf{R}^{-1})_{kk}, \qquad k = 1, \cdots, K \tag{2.44}$$

where $\mathbf{R} = \mathbf{S}^\top \mathbf{S}$ is the correlation matrix of the signature sequences.

Proof: From (2.11), user k transmits when its channel state h_k satisfies

$$h_k = \frac{\lambda_k}{(1 - \lambda_k p_k)} \frac{1}{\mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k} \tag{2.45}$$

The transmit power of the user satisfies $0 < p_k < 1/\lambda_k$. Therefore, user k transmits with non-zero power if and only if

$$h_k > \frac{\lambda_k}{\mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k} \tag{2.46}$$

Comparing the right hand side of (2.46) with (2.18), it is easy to see that it is a standard function and is increasing in $p_i h_i$, $i \neq k$. Thus, from the monotonicity of $\lambda_k / \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k$ we have,

$$\lambda_k \sigma^2 \leq \frac{\lambda_k}{\mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k} \leq \lambda_k \sigma^2 (\mathbf{R}^{-1})_{kk} \tag{2.47}$$

where the first inequality is satisfied with equality when the received powers $p_i h_i$ of all other users are zero, and the second inequality follows from the fact that the SIR of the linear MMSE detector is always larger than or equal to the SIR of the decorrelating detector. In fact, the upper bound becomes tight as $p_i h_i$, $i \neq k$ go to infinity for a fixed noise variance σ^2, as the MMSE detector converges to the decorrelator [12]. Now, if the channel gains are such that

$$h_k > \lambda_k \sigma^2 (\mathbf{R}^{-1})_{kk}, \qquad k = 1, \cdots, K \tag{2.48}$$

using (2.47) we get

$$h_k > \lambda_k \sigma^2 (\mathbf{R}^{-1})_{kk} \geq \frac{\lambda_k}{\mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k}, \qquad k = 1, \cdots, K \tag{2.49}$$

and conclude that all users transmit in the region of channel states where $h_k > \lambda_k \sigma^2 (\mathbf{R}^{-1})_{kk}$, $k = 1, \cdots, K$. $\square$

It is worth mentioning that Theorem 2.2 could also have been used to prove Corollary 2.1, by noting

$$P\left[\text{all users transmit}\right] \geq P\left[\mathbf{h} : h_k > \lambda_k \sigma^2 (\mathbf{R}^{-1})_{kk}\right] > 0 \tag{2.50}$$

Figure 2.2 illustrates the statement of Theorem 2.2, for two users with correlated signature sequences. The orthant described in the theorem in this case is the infinite rectangle $(\lambda_1\sigma^2(\mathbf{R}^{-1})_{11}, \infty) \times (\lambda_2\sigma^2(\mathbf{R}^{-1})_{22}, \infty)$.

Since, as stated by Theorem 2.1, for all K users to transmit simultaneously $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ should be linearly independent, the number of users transmitting simultaneously with non-zero powers cannot be arbitrarily large. The following corollary to Theorem 2.1 gives a bound on the maximum number of users that can transmit simultaneously.

Corollary 2.2 *For a set of K signature sequences and processing gain N, let the rank of the signature sequence matrix $\mathbf{S}$ be $M \leq min\{K, N\}$. Then the number of users that can transmit simultaneously cannot be larger than $min\{K, M(M+1)/2\}$.*

Proof: If $K \leq M(M+1)/2$, the bound is trivial. Let's focus on the case $K > M(M+1)/2$. If rank$(\mathbf{S}) = M$, the signature sequences can be written as,

$$\mathbf{s}_k = \sum_{i=1}^{M} a_{ki}\mathbf{v}_i \tag{2.51}$$

where the $N \times 1$ vectors $\{\mathbf{v}_i\}_{i=1}^M$ constitute an orthonormal basis spanning the signature sequences. Then,

$$\begin{aligned}
\sum_{k=1}^{K} \alpha_k \mathbf{s}_k\mathbf{s}_k^\top &= \sum_{k=1}^{K}\sum_{i=1}^{M}\sum_{j=1}^{M} \alpha_k a_{ki} a_{kj} \mathbf{v}_i\mathbf{v}_j^\top && (2.52)\\
&= \sum_{i=1}^{M}\sum_{j=1}^{M} \mathbf{v}_i\mathbf{v}_j^\top \sum_{k=1}^{K} \alpha_k a_{ki} a_{kj} && (2.53)\\
&= \sum_{i=1}^{M}\sum_{j=1}^{M} \beta_{ij}\mathbf{v}_i\mathbf{v}_j^\top = \mathbf{V}\mathbf{B}\mathbf{V}^\top && (2.54)
\end{aligned}$$

where $\mathbf{V}$ is a matrix with columns $\mathbf{v}_i$ and $\mathbf{B}_{ij} = \beta_{ij}$, with β_{ij} defined by (2.54). Therefore, $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent if and only if the equality $\mathbf{VBV}^\top = \mathbf{0}_{N\times N}$ implies $\alpha_k = 0, \; k = 1, \cdots, K$. Note that $\mathbf{V}$ is an orthonormal matrix by construction, and if $\mathbf{VBV}^\top = \mathbf{0}_{N\times N}$ then multiplying this by $\mathbf{V}^\top$ and $\mathbf{V}$ from left and right we obtain $\mathbf{B} = \mathbf{0}_{M\times M}$. This dictates,

$$\begin{aligned} \sum_{k=1}^{K} \alpha_k a_{ki} a_{kj} &= 0 \qquad i, j \in \{1, \cdots, M\} && (2.55) \\ \sum_{k=1}^{K} \alpha_k \mathbf{a}_k \mathbf{a}_k^\top &= \mathbf{0}_{M\times M} && (2.56) \end{aligned}$$

where $\mathbf{a}_k = [a_{k1}, \cdots, a_{kM}]^\top$. The dimensionality of the space of $M \times M$ symmetric matrices is $M(M+1)/2$, therefore if $K > M(M+1)/2$, we can find α_k not all zero, such that (2.56) is satisfied, and $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are guaranteed to be linearly dependent, and the result follows from Theorem 2.1. □

So far, we have established results that relate to the simultaneous transmission of *all users* in the system. The following corollary to Theorem 2.1 is an extension of the simultaneous transmission result given for all users by Theorem 2.1 to an arbitrary subset of $\{1, \cdots K\}$.

Corollary 2.3 *The sum capacity maximizing power control policy dictates that there exists a non-zero probability region of fading states* $\mathbf{h}$ *where a subset* $E \subset \{1, \cdots, K\}$ *of users transmit simultaneously, if and only if* $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i\in E}$ *are linearly independent.*

Proof: The only if part follows immediately from the proof of Theorem 2.1, by letting $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i\in E}$ be linearly dependent, and writing any $\mathbf{s}_k\mathbf{s}_k^\top$, $k \in E$ as a linear combination of the remaining matrices $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i\in E,\ i\neq k}$. This, together with the KKT conditions for optimality, gives the following relation between the channel gains,

$$\sum_{i\in E,\ i\neq k} \alpha_i \frac{\lambda_i}{h_i} = \frac{\lambda_k}{h_k} \tag{2.57}$$

which is a zero probability event by virtue of the channel states being continuous random variables. This proves the only if part.

We show the if part by proving that the probability $\mathrm{P}\{p_i(\mathbf{h}) > 0,\ i \in E\}$ is bounded away from zero for linearly independent $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i\in E}$.

$$\begin{aligned}
&\mathrm{P}\left[p_i(\mathbf{h}) > 0,\ i \in E\right] \\
&\geq \mathrm{P}\left[p_i(\mathbf{h}) > 0,\ i \in E,\ p_j(\mathbf{h}) = 0,\ j \notin E\right] && (2.58)\\
&\geq \mathrm{P}\left[p_i(\mathbf{h}) > 0,\ i \in E,\ h_j \leq \sigma^2\lambda_j,\ j \notin E\right] && (2.59)\\
&= \mathrm{P}\left[h_i > \frac{\lambda_i}{\mathbf{s}_i^\top \mathbf{A}_i^{-1}\mathbf{s}_i},\ i \in E,\ h_j \leq \sigma^2\lambda_j,\ j \notin E\right] && (2.60)\\
&= \mathrm{P}\left[h_i > \frac{\lambda_i}{\mathbf{s}_i^\top\left\{\mathbf{I}_N + \sum_{k\in E, k\neq i}\frac{h_k p_k(\mathbf{h})}{\sigma^2}\mathbf{s}_k\mathbf{s}_k^\top\right\}^{-1}\mathbf{s}_i},\ i \in E,\ h_j \leq \sigma^2\lambda_j,\ j \notin E\right] && (2.61)\\
&= \mathrm{P}\left[h_i > \frac{\lambda_i}{\mathbf{s}_i^\top\left\{\mathbf{I}_N + \sum_{k\in E, k\neq i}\frac{h_k p_k(\mathbf{h}_E)}{\sigma^2}\mathbf{s}_k\mathbf{s}_k^\top\right\}^{-1}\mathbf{s}_i},\ i \in E\right]\mathrm{P}\left[\ h_j \leq \sigma^2\lambda_j,\ j \notin E\right] \\
& && (2.62)\\
&> 0 && (2.63)
\end{aligned}$$

Here, (2.58) follows from the fact that the set on the right hand side is a subset of that on the left hand side, (2.59) follows because user j does not transmit regardless of the powers of other users if $h_j \leq \sigma^2\lambda_j$, (2.60) follows from (2.46), and (2.61) follows because users $j \notin E$ have zero powers, within the set in (2.60). Note that in (2.61), the powers $p_k(\mathbf{h})$, which are given by (2.12), actually depend only on the channel states of users in E. Thus, we define

$$p_k(\mathbf{h}_E) = p_k(\mathbf{h})|_{p_j(\mathbf{h})=0}, \quad k \in E, \quad j \notin E \tag{2.64}$$

Then, (2.62) follows from independence of the channel states for different users, where the vector $\mathbf{h}_E$ is defined as the vector of channel states for users in E. Clearly, the second term on the right hand side of (2.62) is positive. In order to prove (2.63), we will interpret the first term in (2.62) as the probability that all users transmit simultaneously in an equivalent $|E|$ user problem. To accomplish this, consider a fictitious problem, where we have only the users $k \in E$ in another CDMA system, and users $k \in E$ still employ the signature sequences $\mathbf{s}_k$. The noise variance σ^2 is also the same as in our original problem (2.2). Say we would like to maximize the sum capacity for the new system with $|E|$ users, and let each user $i \in E$ have a power constraint given by

$$\bar{p}_i' = E_{\mathbf{h}_E}\left[p_i(\mathbf{h}_E)\right] \tag{2.65}$$

Then, the power allocation $\{p_i(\mathbf{h}_E)\}_{i\in E}$ is optimal in the sense of maximizing the sum capacity for the fictitious sub-problem. Consequently, the first term in (2.62) is simply the probability that all users transmit with non-zero powers for this new problem, and by Theorem 2.1, this probability is greater than zero as long as $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i\in E}$ are linearly independent, which establishes the if part. $\square$

2.4 Simulation Results

In this section, we give some simple numerical examples to support our analysis. Figures 2.3 and 2.4 give an example for the two user case where the signature sequences are correlated with $\mathbf{s}_1^\top \mathbf{s}_2 = 0.966$. In this example, the processing gain is $N = 2$, the channel is an i.i.d. Rayleigh channel with parameter 1, that is h_k, $k = 1, \cdots, K$ are exponential random variables (squares of Rayleigh random variables) with mean 1. Figure 2.3 shows the power of user 1 for each fading level. In this figure, the transmit power of user 1 is represented by gray levels, lighter colors corresponding to more power. Note that, user 1 performs a single user waterfilling wherever user 2 does not transmit. In this region, the transmit power of user 1 for a fixed h_1 is constant (independent of h_2). However, once user 2 starts transmitting, the "base level of the water tank" is increased, decreasing the power level of user 1 with increasing h_2. Figure 2.4 shows the transmit regions in the space of channel states of the two users. The small dark region near the origin corresponds to the channel states where both users have zero power. Gray regions marked by "user 1" and "user 2" show the channel states where only one of the users transmits, whereas the white region shows the simultaneous transmit region. The simulated system corresponds to the setting in Corollary 2.1, and Theorem 2.2.

We have noted earlier that the optimal power allocation depends on the fading distribution only through the thresholds λ_k. Therefore, the choice of channel fading distribution should not affect the structure of the transmit regions, except for possible shifts and scalings. To show this, we repeat our simulations for a channel where h_k

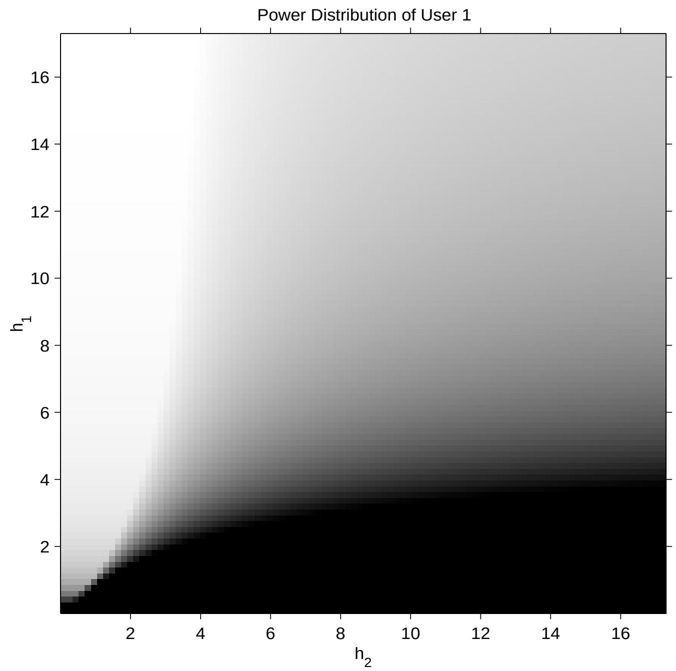


Figure 2.3: Power distribution of user 1 in Rayleigh fading.

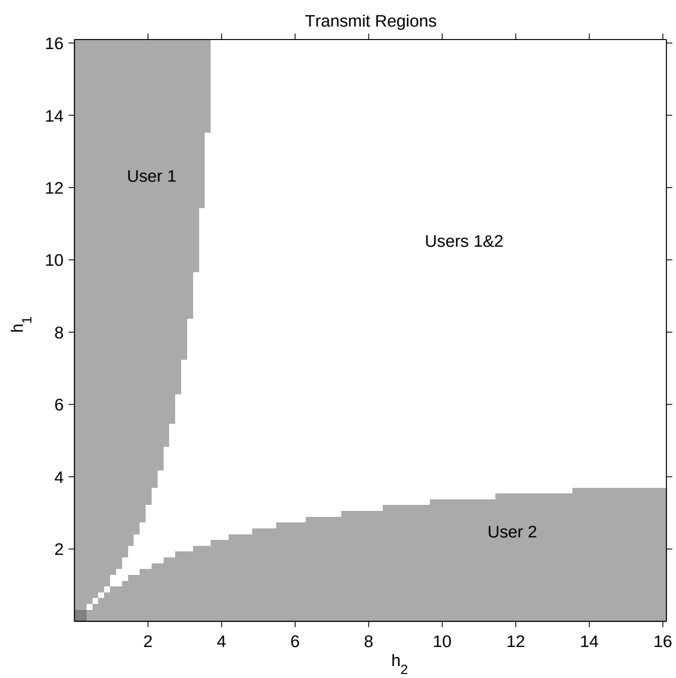


Figure 2.4: Transmit regions for Rayleigh fading.

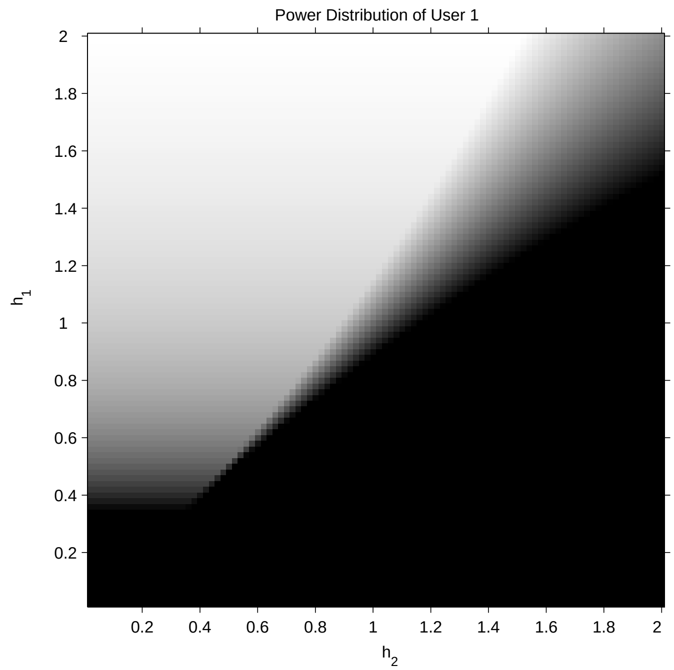


Figure 2.5: Power distribution of user 1 in uniform fading.

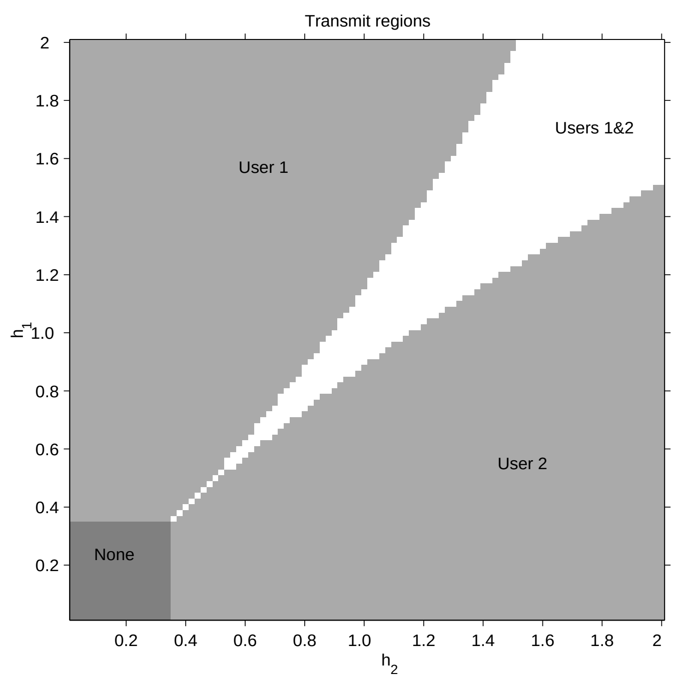


Figure 2.6: Transmit regions for uniform fading.

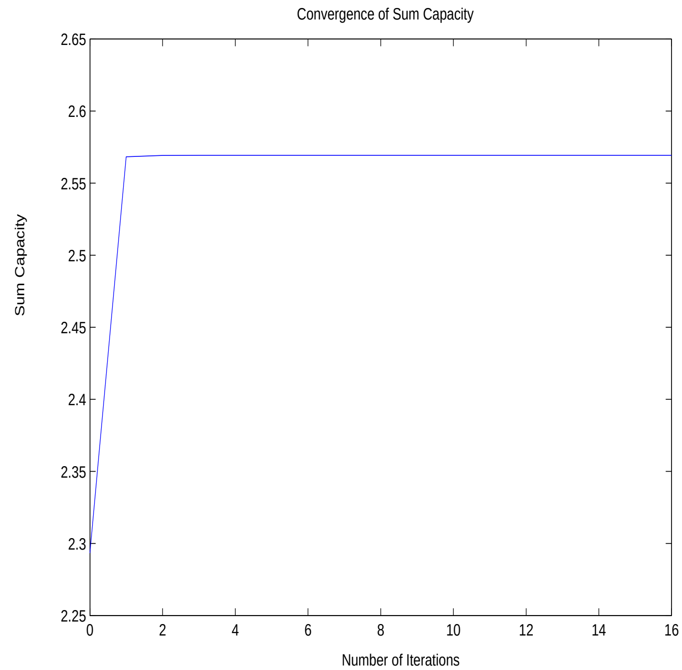


Figure 2.7: Sum capacity versus number of iterations.

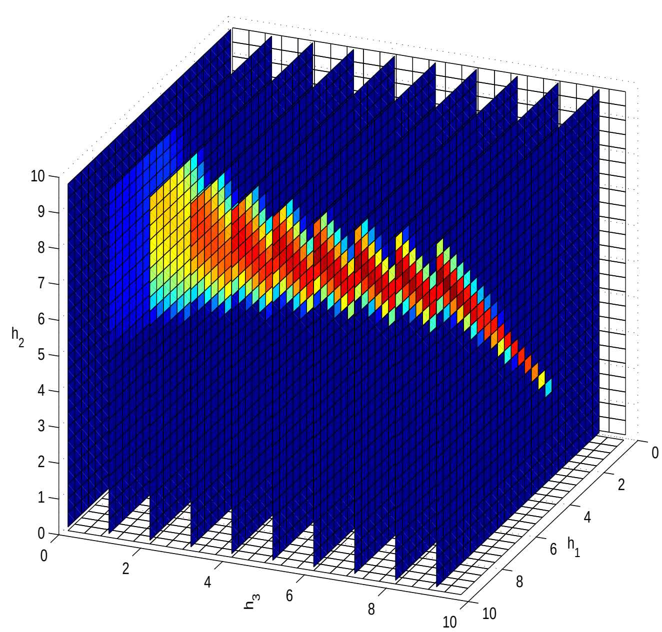


Figure 2.8: Transmit region for all three users when $K = 3$ and $N = 2$.

are uniform i.i.d. random variables in (0,2], all other parameters being the same. Figures 2.5 and 2.6 show the corresponding power levels and transmit regions, for this narrower span of possible channel states to emphasize better all four of the transmit regions. We see that the λ_k value is slightly changed by the change in channel distribution, but the transmit regions and power distribution are very similar to the previous case.

Figure 2.7 illustrates the convergence of the iterative waterfilling algorithm to the maximum sum capacity of the system under uniform fading U(0,2], with average transmit powers equal to $\bar{p}_k = 1$ and noise variance equal to $\sigma^2 = 0.1$; the convergence is quite fast as suggested by the plot.

A consequence of Theorem 2.1 is that we can have multiple users transmit simultaneously with non-zero probability, even when the signature sequences are linearly dependent, as long as we can have the linear independence of $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$. Figure 2.8 shows the region where all users transmit for $K = 3$ and $N = 2$, the colored portions correspond to the states where all three users transmit simultaneously, in the 3-D channel state space.

In general, the probability that all users transmit simultaneously, i.e., the probability of the colored region, depends on the cross-correlations between the signature sequences, fading statistics and power constraints. As an example, for a system with $K = 3$, $N = 2$, $\bar{p} = 1$, $\sigma^2 = 0.1$, uniform U(0,1) fading, and the correlations between the sequences $\rho_{12} = 0.898$, $\rho_{13} = 0.645$, $\rho_{23} = 0.916$; the probability that all users transmit simultaneously is 0.245.

2.5 Summary and Conclusions

We characterized the optimum power allocation policy that achieves the sum capacity of a fading CDMA system that employs fixed deterministic signature sequences. The optimum allocation is shown to be a simultaneous waterfilling of the powers of all users, which in general does not have a tractable analytical solution. Therefore, we devised an algorithm that computes the optimum transmit powers of the users at all channel states. The algorithm is an iterative waterfilling of powers of all users over all fading states treating at each step all other users' signals as additional colored noise. We showed that this iterative strategy converges to a globally optimum solution, and that the global optimum is unique if the signature sequence set is such that the matrices $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent.

We also showed that, the optimum power allocation scheme in the vector MAC of interest dictates more than one user to transmit simultaneously at some channel states, and the set of such channel states has a non-zero probability under certain mild conditions on the signature sequences. In fact, all K users in the system are shown to transmit simultaneously with non-zero probability, if and only if $\{\mathbf{s}_i\mathbf{s}_i^\top\}_{i=1}^K$ are linearly independent. An immediate implication of this is that, if the signature sequences $\{\mathbf{s}_i\}_{i=1}^K$ are linearly independent, then all users transmit simultaneously in a non-zero probability region of the channel states. We extended this simultaneous transmit condition for all users to one for an arbitrary subset of users. We further showed that if the signature sequence matrix $\mathbf{S}$ of the users in the system has rank M, the number of users transmitting simultaneously with nonzero probability cannot be larger than $\min\{K, M(M+1)/2\}$.

The results of this chapter have been published in [13, 14].

Chapter 3

Capacity Region with Fixed Sequences and Adaptive Powers

3.1 Introduction

For a fading CDMA network, the sum capacity maximization problem was addressed in Chapter 2, where the optimal power allocation policy and some of its properties were characterized. Sum capacity is perhaps the most desirable metric from the network operator's point of view, as it is the maximum reliable rate of information flow in the network. However, in practical communication systems, the varying demand of various users may dictate a design which allows for multiple rate classes, where different users are assigned different priorities. Therefore, in a multi-access system the goal is not always to maximize the sum of achievable rates, it may as well be to achieve an arbitrary rate tuple on the capacity region boundary, i.e., to maximize a weighted sum of rates.

The capacity region for a MAC is defined as the set of all rate tuples at which all transmitters can communicate their messages to the receiver reliably, i.e., with probability of error arbitrarily close to zero. For traditional scalar MACs in the absence of fading, the capacity region has been studied extensively, and was established in the classical works by Ahlswede, Liao, Cover and Wyner [40–43]. For *fading* scalar MAC, the capacity region, and the corresponding power control policies for the scalar MAC were characterized more recently in [7]. The capacity region is shown to be a union of the capacity regions (polymatroids) achievable by all valid power allocation policies, i.e., the policies that satisfy the average power constraints. The optimal power allocation policy for each rate tuple on the capacity region is obtained by a greedy algorithm, which compares certain marginal utility functions, and makes use of the generalized symmetry properties of the rank function of the polymatroid corresponding to the rate tuple in question.

The capacity region for a non-fading *vector* MAC, where the total average powers of the components of the transmitted vectors are constrained, is given by [8]. There, also an iterative waterfilling algorithm which allocates the powers over the components of the transmitted vector in order to maximize the *sum* capacity was proposed. The power allocation problem for a fading vector MAC was considered in [10], again with the aim of maximizing the *sum* capacity. It was shown that, the optimal power allocation in the fading case as well satisfies the Karush-Kuhn-Tucker (KKT) conditions, which can also be interpreted as simultaneous waterfilling, where the water levels are matrices.

In Chapter 2, where we have obtained the sum capacity of a power controlled fading CDMA channel with perfect CSI at the transmitters and the receiver, we have somewhat bypassed the underlying encoding and decoding strategy that indeed gives the sum capacity expression (2.5). Instead, we treated the problem mainly from an optimization point of view, by regarding the power levels as variables, and generalizing the objective function trivially from its scalar counterpart [6]. In this chapter, we precisely characterize the capacity region of a fading CDMA system with fixed sequences, and perfect CSI at the transmitters and the receiver. We show that, treating the powers as the only design parameters as we did in Chapter 2 is in fact the most one can do, establishing precisely that the objective function (2.5) was indeed on the capacity region boundary, and also that all other points on the capacity region can be achieved by appropriately choosing the allocated powers.

Also, inspired by the findings in [7], we investigate strict convexity of the capacity region. Strict convexity of the capacity region plays an important role in decoding. For a strictly convex capacity region (i.e., the one in [7]) any point on the boundary of the capacity region may be achieved without timesharing, whereas if there is a flat portion on the capacity region, the rate tuples falling in this region need to be achieved by methods like timesharing [44] or rate splitting [45] among different successive decoding points. We show that unless the signature sequences of all users are identical or orthogonal, the capacity region is not strictly convex, and there are infinitely many rate tuples that achieve the sum capacity.

Next, we consider the problem of solving for the power allocation policy that achieves an *arbitrary rate tuple* on the capacity region of fading CDMA. As in [7] and [8], this problem is equivalent to a maximization of a weighted sum of rates, subject to average power constraints. However, the algorithm proposed in [7] to find the power allocation policies that achieve the boundary points of the scalar MAC does not generalize to the CDMA case. This is due to the fact that the generalized symmetry properties of the rank functions that describe the capacity region of a scalar

MAC does not carry over to an arbitrary CDMA system, in which non-identical signature sequences are employed. Instead, we make use of the concavity of the objective function and the convexity of the constraints, and write the KKT conditions at each fading state, for a given set of weights. We then develop a "generalized" waterfilling approach, where we gradually pour some power at some or all channel states until all the KKT conditions are satisfied. Using this approach, we propose a one-user-at-a-time algorithm which is similar in spirit to those in [8,13,14], and show that it converges to the optimum power allocation for any given point on the boundary of the capacity region. This algorithm, while providing a systematic solution to the capacity achieving power allocation problem in fading CDMA, also provides as a special case, an intuitive approach to the power allocation for scalar MAC in [7].

We also relax the somewhat impractical assumption of perfect CSI at the transmitters, and investigate the effects of limited feedback rates from the receiver to the transmitters. We show that even with very low feedback rates, it is possible to achieve rates very close to the capacity region boundary.

3.2 Capacity Region of Fading CDMA

We first provide the capacity region of a fading CDMA channel where users have perfect CSI, and are able to choose their transmit powers as a function of these channel states, subject to average power constraints. The capacity region is obtained by a simple extension of [7], which deals with an equivalent problem in the case of scalar MAC. As in the scalar fading MAC [7], the capacity region of the fading CDMA is a union of capacity regions obtained for each valid power allocation policy.

For the CDMA system given by (2.2), let the transmitters be able to choose their powers as a function of the channel state, subject to the average power constraints $E_{\mathbf{h}}[p_i(\mathbf{h})] \le \bar{p}_i$. We first characterize the set of long term achievable rates, i.e., the capacity region, for fading CDMA. Hanly and Tse [7, Thm. 2.1] have characterized the capacity region for a power controlled scalar multi-access channel. Both forward and converse parts of the proof of this theorem can be directly generalized to the CDMA channel, also by incorporating the methods and results from [9, Prop. 1] and [46, Thm. 1]. Therefore, we state the capacity region of the fading CDMA channel in the following theorem, without providing a proof.

Theorem 3.1 *Let $\mathcal{P} = \{\mathbf{p}(\mathbf{h}) : E_{\mathbf{h}}[p_i(\mathbf{h})] \le \bar{p}_i, \forall i\}$ denote the family of valid power allocation policies. The capacity region $\mathcal{C}$ of a fading CDMA channel under additive white Gaussian noise, where users have perfect CSI and allocate their powers as a function of the CSI is given by,*

$$\bigcup_{\mathbf{p}(\mathbf{h}) \in \mathcal{P}} \left\{ \mathbf{R} : \sum_{i \in \Gamma} R_i \le E_{\mathbf{h}} \left[\frac{1}{2} \log \left| \mathbf{I}_N + \sigma^{-2} \sum_{i \in \Gamma} h_i p_i(\mathbf{h}) \mathbf{s}_i \mathbf{s}_i^\top \right| \right], \quad \forall \Gamma \subset \{1, \cdots, K\} \right\} \tag{3.1}$$

Figure 3.1 illustrates a typical capacity region for some fixed signature sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ in a two user setting. Each of the pentagons corresponds to a valid power allocation policy. Note the flat portion on the capacity region, which in fact is the dominant face of one of the pentagons. Unlike scalar multi-access channel capacity region [7], the capacity region for fading CDMA may contain such a flat region, and

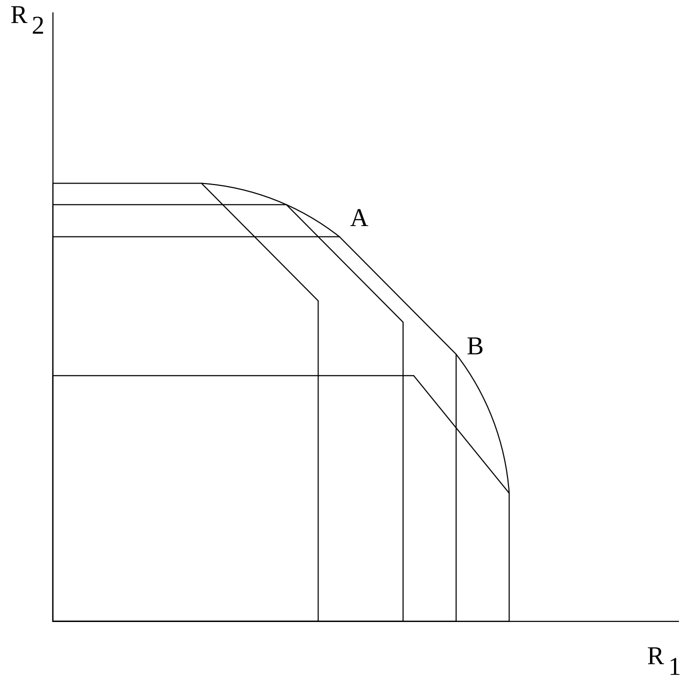


Figure 3.1: Sample two user capacity region.

in general is not strictly convex. That is, the rate pairs on the line segment $|AB|$ in the figure are in general achieved by time-sharing between the points A and B. This can be shown by noting that the pentagon containing $|AB|$ corresponds to the power control policy that maximizes the sum capacity, and then proving that for correlated signature sequences, there are infinitely many rate tuples that give the same sum rate. This is stated more precisely in the following theorem.

Theorem 3.2 *The capacity region of a power controlled fading CDMA channel is not strictly convex, provided* $\exists\, i, j \in \{1, \cdots, K\}$ *such that* $i \neq j$ *and* $0 < \mathbf{s}_i^\top \mathbf{s}_j < 1$.

Proof: Let $\mathcal{P}(\mathbf{h}) = \{p_1^*(\mathbf{h}), \cdots, p_K^*(\mathbf{h}), \quad \forall \mathbf{h}\}$ be the power control policy that maximizes the sum of rates of all users, i.e., the *sum capacity.* The capacity region corresponding to this particular power control policy is a polymatroid $\mathcal{G}_{\mathcal{P}(\mathbf{h})}$, with corners in the positive "quadrant" given by

$$R_{\mathbf{\Gamma}(k+1)} = E_{\mathbf{h}}\left[\frac{1}{2}\log\frac{\left|\mathbf{I}_N + \sigma^{-2}\mathbf{S}_{\Gamma_{k+1}}\mathbf{D}_{\Gamma_{k+1}}(\mathbf{h})\mathbf{S}_{\Gamma_{k+1}}^{\top}\right|}{\left|\mathbf{I}_N + \sigma^{-2}\mathbf{S}_{\Gamma_k}\mathbf{D}_{\Gamma_k}(\mathbf{h})\mathbf{S}_{\Gamma_k}^{\top}\right|}\right], \quad k = 0, \cdots, K-1 \qquad (3.2)$$

where $\mathbf{S} = [\mathbf{s}_1 \cdots \mathbf{s}_K]$, $\mathbf{D}(\mathbf{h}) = \text{diag}[p_1^*(\mathbf{h})h_1, \cdots, p_K^*(\mathbf{h})h_K]$, $\mathbf{\Gamma} \triangleq [\Gamma(1), \cdots, \Gamma(K)]$ is any permutation of $\{1, \cdots, K\}$, $\Gamma_k \triangleq [\Gamma(1), \cdots, \Gamma(k)]$ for $k = 1, \cdots, K$, and $\Gamma_0 \triangleq \varnothing$. $\mathbf{D}_{\Gamma_k}$ and $\mathbf{S}_{\Gamma_k}$ refer to sub-matrices containing only the received powers and signature sequences of the users in the subset Γ_k. Each one of the $K!$ possible permutations correspond to a corner point of the polymatroid $\mathcal{G}_{\mathcal{P}(\mathbf{h})}$, and the polygon formed by the convex hull of all these points is called the dominant face of $\mathcal{G}_{\mathcal{P}(\mathbf{h})}$. Note that, since any point on the dominant face of $\mathcal{G}_{\mathcal{P}(\mathbf{h})}$ achieves the maximum sum capacity, it should also lie on the surface of the overall capacity region $\mathcal{C}$. That is, the dominant face of $\mathcal{G}_{\mathcal{P}(\mathbf{h})}$ constitutes a portion of the surface of $\mathcal{C}$. Therefore, for the surface of $\mathcal{C}$ to be strictly convex, we need all the corners (3.2) of the dominant face to collapse to a single point. It is easy to see that this condition can be summarized by

$$E_{\mathbf{h}}\left[\log\left|\mathbf{I}_N + \sigma^{-2}\mathbf{S}_E\mathbf{D}_E(\mathbf{h})\mathbf{S}_E^{\top}\right|\right] = \sum_{i\in E} E_{\mathbf{h}}\left[\log\left(1 + \sigma^{-2}p_i^*(\mathbf{h})h_i\right)\right], \quad \forall E \subset \{1, \cdots, K\} \qquad (3.3)$$

Define $\mathbf{Q}_E(\mathbf{h}) \triangleq \mathbf{S}_E\mathbf{D}_E(\mathbf{h})^{1/2}$. Then for all $\mathbf{h}$, we have

$$\begin{aligned}
\log\left|\mathbf{I}_N + \sigma^{-2}\mathbf{S}_E\mathbf{D}_E(\mathbf{h})\mathbf{S}_E^{\top}\right| &= \log\left|\mathbf{I}_N + \sigma^{-2}\mathbf{Q}_E(\mathbf{h})\mathbf{Q}_E(\mathbf{h})^{\top}\right| && (3.4)\\
&= \log\left|\mathbf{I}_{|E|} + \sigma^{-2}\mathbf{Q}_E(\mathbf{h})^{\top}\mathbf{Q}_E(\mathbf{h})\right| && (3.5)\\
&\le \sum_{i\in E}\log\left(1 + \sigma^{-2}p_i^*(\mathbf{h})h_i\right) && (3.6)
\end{aligned}$$

where the last step follows from Hadamard's inequality [44], and the equality is achieved if and only if $\mathbf{W}_E(\mathbf{h}) \triangleq \mathbf{Q}_E(\mathbf{h})^\top \mathbf{Q}_E(\mathbf{h})$ is diagonal. Since (3.6) holds for all $\mathbf{h}$, (3.3) holds when and only when $\mathbf{W}_E(\mathbf{h})$ is diagonal for almost all $\mathbf{h}$ (i.e., with probability 1). For equality in (3.6), we need

$$\left[\mathbf{Q}_E(\mathbf{h})^\top \mathbf{Q}_E(\mathbf{h})\right]_{i,j} = \sqrt{p_i^*(\mathbf{h}) p_j^*(\mathbf{h}) h_i h_j} \mathbf{s}_i \mathbf{s}_j = 0, \quad \forall\, i \neq j \tag{3.7}$$

or equivalently,

$$p_i^*(\mathbf{h}) p_j^*(\mathbf{h}) = 0 \quad \vee \quad \mathbf{s}_i^\top \mathbf{s}_j = 0, \quad \forall\, i \neq j, \quad \forall\, \mathbf{h} \tag{3.8}$$

Note that, this condition is readily satisfied if $K \leq N$ and the signature sequences of all users are orthogonal, in which case the sum rate is achieved at a single point rather than on a polygon (i.e., the dominant face of the corresponding polymatroid). Therefore, let us focus on non-orthogonal sequences. Let $\mathbf{s}_i^\top \mathbf{s}_j \neq 0$ for $i \neq j$. Then, for strict convexity of $\mathcal{C}$, we need $p_i^*(\mathbf{h}) p_j^*(\mathbf{h}) = 0$ for almost all $\mathbf{h}$, i.e., except over a zero probability subset of channel states. In other words, the optimal power allocation policy which achieves the sum capacity should dictate no more than one user transmit simultaneously with non-zero probability. But by Theorem 2.1, this is true if only if the signature sequences of all users are identical, which establishes that $\mathcal{C}$ is not strictly convex unless all signature sequences are identical or orthogonal. $\square$

In proving Theorem 3.2, we made use of the properties of sum capacity achieving power allocation policy, which is yet another justification of the importance of the treatment of the sum capacity in Chapter 2. Sum capacity is often considered as a figure of merit for multiuser systems, because of the ease with which it can be handled as an objective function, as opposed to the more difficult to handle arbitrary rate tuples on the boundary of the capacity region. Note that, although in this section we have characterized the capacity region for the fading CDMA with perfect CSI at the transmitters and the receiver, we have not yet been able to solve for the optimal power allocation policies that will achieve an arbitrary point on the capacity region as we did in the symmetric sum capacity case. The problem of solving for such policies is addressed in the next section.

3.3 Power Optimization for Weighted Sum of Rates

We have shown in Section 3.2 that the capacity region for fading CDMA is in general not strictly convex, and there may be a flat portion on the boundary of the capacity region, which coincides with the dominant face of the rate region corresponding to the sum capacity maximizing power control policy. Now, note that, for any given pair of non-negative numbers μ_1 and μ_2, there exists a point (or there exist points) (R_1^*, R_2^*) on the boundary of the capacity region, such that the line $\mu_1 R_1 + \mu_2 R_2 = C$ is tangent to the capacity region for some $C = C^*(\mu_1, \mu_2)$, and in fact $C^*(\mu_1, \mu_2)$ is the maximum achievable value of $\mu_1 R_1 + \mu_2 R_2$ (Figure 3.2). Therefore, the problem of finding the power control policy that corresponds to the rate pair (R_1^*, R_2^*) is equivalent to maximizing $\mu_1 R_1 + \mu_2 R_2$ subject to the average power constraints. Here, μ_is can be interpreted as the priorities assigned to each

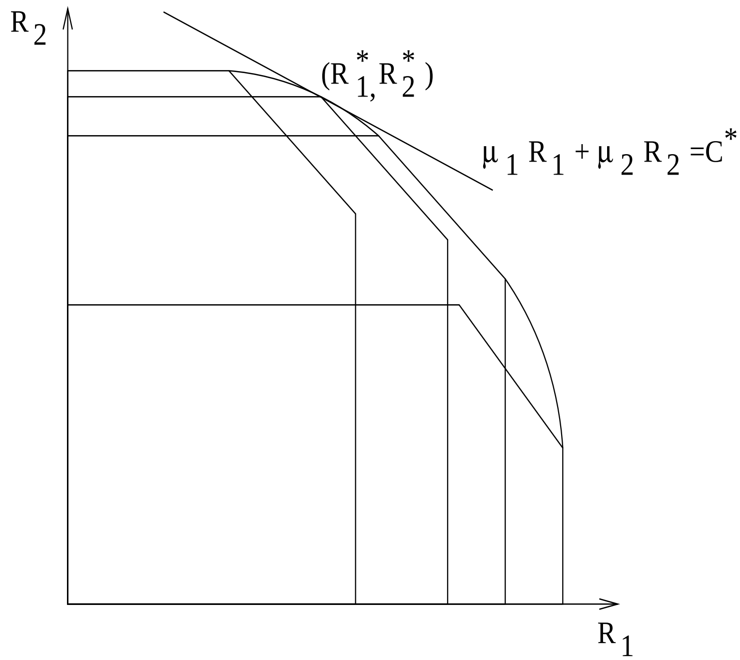


Figure 3.2: Maximization of the rate tuples on the capacity region boundary.

user. The boundary of the capacity region can be traced by varying these priorities μ_i. The desired rate pair (R_1^*, R_2^*) is either the corner of one of the pentagons specified by a power allocation policy as in (3.1), or it lies on one of the flat portions. If it is a corner, its coordinates can be written as a function of the power allocation policy using (3.1), and the maximization can be carried out. The case where (R_1^*, R_2^*) lies on one of the flat portions corresponds to either the rather easier case where we want to maximize the sum capacity, which is solved in [13, 14], or the trivial case where one of the μ_is is zero, and the problem reduces to a single user problem.

Having introduced the reasoning in the simple two user case, we now define our problem in the general K user case. Without loss of generality, assume $\mu_K > \cdots > \mu_1$. Then, the optimum power allocation policy for $\{\mu_i\}_{i=1}^K$ is the solution to the maximization problem,

$$\max_{\mathbf{p}(\mathbf{h})} \quad \frac{1}{2}E_{\mathbf{h}}\left[\mu_1 \log\left|\mathbf{I}_N + \sigma^{-2}\mathbf{S}\mathbf{D}(\mathbf{h})\mathbf{S}^\top\right| + \sum_{i=2}^{K}(\mu_i - \mu_{i-1})\log\left|\mathbf{I}_N + \sigma^{-2}\mathbf{S}_{E_i}\mathbf{D}_{E_i}(\mathbf{h})\mathbf{S}_{E_i}^\top\right|\right]$$

$$\text{s.t.} \quad E_{\mathbf{h}}[p_i(\mathbf{h})] \leq \bar{p}_i, \qquad i = 1, \cdots, K$$

$$p_i(\mathbf{h}) \geq 0, \quad \forall\ \mathbf{h}, \quad i = 1, \cdots, K \tag{3.9}$$

where $\mathbf{S} = [\mathbf{s}_1 \cdots \mathbf{s}_K]$, $\mathbf{D}(\mathbf{h}) = \text{diag}[p_1(\mathbf{h})h_1, \cdots, p_K(\mathbf{h})h_K]$, $E_i = \{i, \cdots, K\}$ and $\mathbf{p}(\mathbf{h}) = [p_1(\mathbf{h}), \cdots p_K(\mathbf{h})]$. Here, $\mathbf{D}_{E_i}$ and $\mathbf{S}_{E_i}$ refer to sub-matrices containing only the received powers and signature sequences of the users in the subset E_i. Note that, this is the fading CDMA version of equation (3) in [8], and is similar to equation (17) for the scalar case in [7].

3.4 Generalized Iterative Waterfilling

Let us denote the objective function in (3.9) by $C_{\boldsymbol{\mu}}(p_1(\mathbf{h}), \cdots, p_K(\mathbf{h}))$, where $\boldsymbol{\mu} = [\mu_1, \cdots, \mu_K]$. In order to solve (3.9), we first note that the objective function is concave in the power vector $\mathbf{p}(\mathbf{h})$, and further, it is strictly concave in the individual components $p_i(\mathbf{h})$ of $\mathbf{p}(\mathbf{h})$. The constraint set is convex (in fact, affine). Therefore, the unique global solution to the maximization problem in (3.9) should satisfy the extended KKT conditions, which can be shown to reduce to,

$$\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h}) + p_k(\mathbf{h})} \leq \lambda_k,\ \forall\ \mathbf{h},\ k = 1, \cdots, K \tag{3.10}$$

where, we have defined $\mu_0 \triangleq 0$ for notational convenience. Here, $a_{ki}(\mathbf{h})$ for $i \leq k \leq K$ is given by,

$$a_{ki}(\mathbf{h}) = \frac{1}{\sigma^{-2} h_k \mathbf{s}_k^\top \left(\mathbf{I}_N + \sigma^{-2} \sum_{j=i, j \neq k}^{K} p_j(\mathbf{h}) h_j \mathbf{s}_j \mathbf{s}_j^\top \right)^{-1} \mathbf{s}_k} \tag{3.11}$$

Note that, this quantity can be identified as the inverse of the SIR user k would obtain at the output of an MMSE filter if it transmitted with unit power, when users $i, i+1, \cdots, K$ are active. The condition in (3.10) is satisfied with equality at some $\mathbf{h}$, if $p_k(\mathbf{h}) > 0$. Since the optimum power allocation policy for a given $\boldsymbol{\mu}$ should simultaneously satisfy all the conditions given by (3.10), and the optimum power of each user k at each fading state $\mathbf{h}$ depends on the power allocations of all other users at that state through $a_{ki}(\mathbf{h})$, it is hard to analytically solve for the optimum policy from the KKT conditions. Therefore, to proceed, we devise an iterative algorithm. Consider optimizing the power of *only* user k over all channel states, given the powers of all other users at all channel states,

$$\begin{aligned} p_k^{n+1} &= \arg\max_{p_k} \; C_{\boldsymbol{\mu}} \left(p_1^{n+1}, \cdots, p_{k-1}^{n+1}, p_k, p_{k+1}^{n} \cdots, p_K^{n} \right) \\ &= \arg\max_{p_k} \; C_{\boldsymbol{\mu}}^{k} \left(p_k \right) \end{aligned} \tag{3.12}$$

where $C_{\boldsymbol{\mu}}^{k}(p_k)$ denotes the first k terms in (3.9), i.e., $i = 1, \cdots, k$, that contain contributions from user k to $C_{\boldsymbol{\mu}}(\mathbf{p}(\mathbf{h}))$.

The convergence of such an algorithm has been proved for the case of sum capacity in Chapter 2 for fading channels, and in [8] for non-fading channels. The objective function here satisfies the same concavity and strict concavity properties as the sum capacity, i.e., it is concave in the power vector $\mathbf{p}(\mathbf{h})$ and strictly concave in its individual components $p_k(\mathbf{h})$, and the constraint set is the same as in Chapter 2. Therefore, the proof in Section 2.2 immediately applies to the case of unequal μ_is here, and the update (3.12) converges to the optimal power allocation by [37, Prop. 3.9]. Thus, it is sufficient to consider separately finding the solution $p_k(\mathbf{h})$ that satisfies the kth KKT condition in (3.10) for each user k, while keeping the powers of all other users $j \neq k$ as fixed and known quantities.

Let us concentrate on user k, and fix $p_j(\mathbf{h})$, $j \neq k$. It can be shown that, the solution to (3.12) subject to the average power constraint on $p_k(\mathbf{h})$ should satisfy the KKT condition for the single user problem,

$$\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h}) + p_k(\mathbf{h})} \leq \tilde{\lambda}_k, \quad \forall\, \mathbf{h} \tag{3.13}$$

We note here that $\tilde{\lambda}_k$ is in general different from the Lagrange multiplier λ_k in (3.10), since the powers we have fixed for the other users need not be the optimal powers. Eventually, since the iterative algorithm converges to the optimal powers, we know that $\tilde{\lambda}_k$ will converge to λ_k.

We will next argue how this condition can be interpreted as a "generalized" waterfilling. First assume no power has yet been allocated to any channel state. Define the inverse of the left hand side of (3.13) evaluated at $p_k(\mathbf{h}) = 0$ for all $\mathbf{h}$ by,

$$b_k(\mathbf{h}) = \left(\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h})} \right)^{-1} \tag{3.14}$$

Then, sort $b_k(\mathbf{h})$ over all channel states $\mathbf{h}$ in increasing order. Since user k has to satisfy its average power constraint, it has to put some power to a non-zero probability subset, say Ω, of all possible channel states. At the channel states where user k transmits with positive power, (3.13) needs to be satisfied with equality. Let user k start pouring some of its available power to the state which gives the lowest $b_k(\mathbf{h})$, say $\mathbf{h}'$. Next, pick another state $\mathbf{h}''$, such that $b_k(\mathbf{h}') < b_k(\mathbf{h}'')$. User k starts transmitting at $\mathbf{h}''$ only if (i) it has already poured some powers $q_k(\mathbf{h})$ to all states $\mathbf{h}$ such that $b_k(\mathbf{h}) < b_k(\mathbf{h}'')$, (ii) it still has some power left to allocate, and (iii) the already allocated powers satisfy

$$\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h}) + q_k(\mathbf{h})} = b_k^{-1}(\mathbf{h}''), \quad \forall\ \mathbf{h} : b_k(\mathbf{h}) \leq b_k(\mathbf{h}'') \tag{3.15}$$

Before going any further, using the current construction, let us revisit the sum capacity case in Chapter 2, where μ_i, $i = 1, \cdots, K$, are all equal to 1. In this case, from (3.14), $b_k(\mathbf{h}) = a_{k1}(\mathbf{h})$, and it can be easily seen that the described procedure produces the ordinary waterfilling solution; user k will pour its power over $a_{k1}(\mathbf{h}) = b_k(\mathbf{h})$, until all the available power is used. The optimal power value at $\mathbf{h}$ is the difference between the water level $1/\tilde{\lambda}_k$ and the base level $b_k(\mathbf{h})$, whenever the difference is positive; it is zero otherwise, i.e.,

$$p_k(\mathbf{h}) = \left(\frac{1}{\tilde{\lambda}_k} - b_k(\mathbf{h}) \right)^+ \tag{3.16}$$

The main subtlety in solving for the optimal powers in the arbitrary μ_is case is that, there are more than one terms that involve $p_k(\mathbf{h})$ on the left hand side of (3.13), and thus the optimal $p_k(\mathbf{h})$ is no longer given by a nice expression such as (3.16), but is rather the solution to a polynomial equation, obtained by equating the denominators in (3.13). Therefore, the optimal power levels lose their traditional waterfilling interpretation. However, we can still see the procedure described here as a type of waterfilling, as it gradually equalizes the base levels $b_k(\mathbf{h})$, and solves for the power levels required for such equalization, hence the name "generalized" waterfilling.

Generalized waterfilling yields the optimum power allocation because of the fact that by construction, the KKT conditions are satisfied when all average power is used. To see this, let us denote the left hand side of (3.15) by $L(\mathbf{h}, q_k(\mathbf{h}))$. We keep increasing $q_k(\mathbf{h}'')$ on the left hand side of (3.15) gradually. Letting $p_k(\mathbf{h}) = q_k(\mathbf{h})$ when the solution $q_k(\mathbf{h})$ obtained from (3.15) satisfies the average power constraint, and taking $\tilde{\lambda}_k \triangleq L(\mathbf{h}, p_k(\mathbf{h}))$, we see that the solution $p_k(\mathbf{h})$ satisfies the KKT conditions and it is optimal.

In order to better visualize how the generalized waterfilling is performed, we consider a simple example with $K = 2$ and with discrete joint channel states $\mathbf{h}^i$, $i = 1, \cdots, M$. Without loss of generality, let us assume $b_k(\mathbf{h}^1) < \cdots < b_k(\mathbf{h}^M)$. Figure 3.3 shows the generalized waterfilling procedure. The ordered values $b_k(\mathbf{h}^i)$ are illustrated in Figure 3.3(a). First, using (3.15), we solve for the amount of power $q_k(\mathbf{h}^1)$ that will level $L(\mathbf{h}^1, q_k(\mathbf{h}^1))$ and $b_k(\mathbf{h}^2)$, so that the water level is $b_k(\mathbf{h}^2)$, as shown in Figure 3.3(b). It can be easily shown that $q_k(\mathbf{h}^1)$ is the only non-negative solution to a k^{th} order polynomial equation, obtained from (3.15). In this particular example, the available average power is not yet completely used in this first step, so we repeat the same procedure at both $\mathbf{h}^1$ and $\mathbf{h}^2$, i.e., we solve for $q_k(\mathbf{h}^1)$ and $q_k(\mathbf{h}^2)$ that will level $L(\mathbf{h}^1, q_k(\mathbf{h}^1))$, $L(\mathbf{h}^2, q_k(\mathbf{h}^2))$ and $b_k(\mathbf{h}^3)$ (see Figure 3.3(c)). We continue this procedure until we see that although the water levels can be made equal at $b_k(\mathbf{h}^{t-1})$ while satisfying the average power constraint, it is not possible to equalize the water levels at $b_k(\mathbf{h}^t)$, since the available average power falls short of the required average power that is needed for such equalization. At this point, we know that the final water level, i.e., the true value of $1/\tilde{\lambda}_k$ that will satisfy the KKT conditions together with $q_k(\mathbf{h}^i)$ obtained from (3.15) should lie between $b_k(\mathbf{h}^{t-1})$ and $b_k(\mathbf{h}^t)$, and we can find it by searching between these two values until the $q_k(\mathbf{h}^i)$, $i = 1, \cdots, t-1$, satisfy the average power constraint with equality. Figure 3.3(d) illustrates this last step, and the final value of $\tilde{\lambda}_k$ that satisfies the KKT conditions.

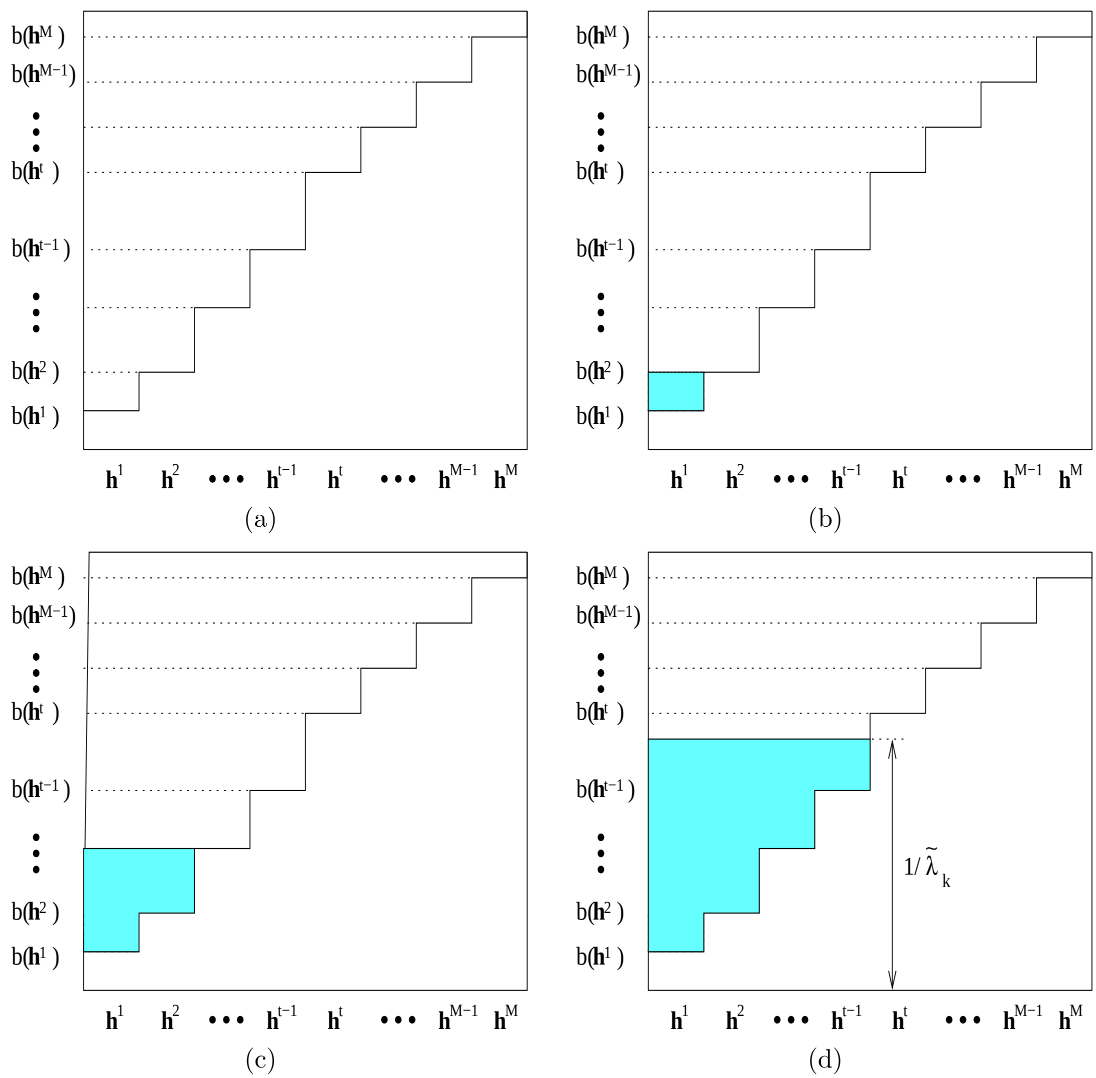


Figure 3.3: Illustration of the generalized waterfilling.

Note that, by letting $\mu_1 = \cdots = \mu_K$, we recover the traditional waterfilling solution in [13, 14], since only the first term survives in the KKT conditions. On the other hand, if we let $\mathbf{s}_i = 1$ for $i = 1, \cdots, K$, the generalized waterfilling algorithm solves the resource allocation problem in [7] for scalar MAC.

3.5 Power Control with Limited Feedback

In Section 3.2, we have characterized the capacity region of the fading CDMA channel with the assumption of perfect channel state information at the transmitters and the receiver. This scenario is well suited for the theoretical treatment of the CDMA system, and in fact gives the utmost limit one can achieve in terms of reliable communication rates. On the other hand, perfect knowledge of the channel state at the transmitter is not a practical assumption, since it would require an infinite rate feedback link. The question that arises naturally is, should the side information be perfectly accurate in order for power control to be an effective means of increasing the capacity? Here, we will consider the power control problem from a more practical point of view, and demonstrate the effects of limited feedback on the set of achievable rates.

In particular, we will now consider the fact that the feedback link from the receiver to the transmitters is limited in rate, and therefore only part of the information that is available to the receiver can be communicated back to the transmitters to aid the resource allocation. On the other hand, we still assume that the feedback is instantaneous and error free.

Let us assume that the feedback link has the limitation that it allows reliable transmission of at most L bits per user. Then the receiver can inform the transmitter which one of up to 2^L transmit power levels to use, depending on the observed channel state. This requires a mapping from the channel state space to a discrete set of power levels, i.e., the problem of maximizing the weighted sum of rates as a function of a finite number of transmit power levels, can be formulated as a vector quantization problem

$$\begin{aligned} &\max \sum_{j=1}^{2^L} \int_{\gamma_j(\mathbf{h})} \sum_{i=1}^{K} (\mu_i - \mu_{i-1}) \log \left| \mathbf{I}_N + \sigma^{-2} \sum_{k=i}^{K} p_k(j) h_k \mathbf{s}_k \mathbf{s}_k^\top \right| f(\mathbf{h}) d\mathbf{h} \\ &\text{s.t.} \sum_{j=1}^{2^L} \int_{\gamma_j(\mathbf{h})} p_k(j) f(\mathbf{h}) d\mathbf{h} = \bar{p}_k \end{aligned} \tag{3.17}$$

where the channel state space, say $\mathbb{R}_+^K$, is partitioned into 2^L subsets $\gamma_j(\mathbf{h})$, and each of these partitions are mapped onto the element $p_k(j)$ from the codebook $\{p_k(1), \cdots, p_k(2^L)\}$. It is often a very tedious, if not impossible, task to find an optimal vector quantizer analytically for a given probability distribution of the quantizer input, even in the case of the traditional quantization where the goal is to represent a random vector as closely as possible. In fact, even much easier scalar quantization problems do not lend themselves to such solutions. On the other hand, conditions for optimality of a quantizer lead to algorithmic solutions that yield "good" quantizers, which achieve local optima to the minimization problem [47]. Probably the most popular such algorithm is the Lloyd method which iterates between the partitioning and codebook selection [47].

It is possible to use the generalized Lloyd algorithm for vector quantization [47] to solve (3.17), by suitably defining an unconventional "distortion" function as the negative of the Lagrangian of the optimization problem in (3.17). However, this approach is still not guaranteed to obtain an absolutely optimal quantizer. In what follows, we will simplify the problem by limiting ourselves to scalar quantizers, where the goal is to represent the channel state, or optimal power level of each user as closely as possible. Namely, we will consider two settings: (i) a quantized version of the CSI is fed back to the transmitters and the optimal power allocation is determined at the transmitters, and (ii) the optimal power levels are computed at the receiver, and are then quantized and fed back to the transmitters.

For the first case, let the quantizer $Q_i(h_i)$ be defined by the codebook $\hat{H}_i = \{\hat{h}_i^1 < \cdots < \hat{h}_i^{2^L}\}$, and the partition $W_i = \{0 = w_i^0 < w_i^1 < \cdots < w_i^{2^L-1} < \cdots < w_i^{2^L} = \infty\}$, such that

$$Q_i(h_i) = \hat{h}_i^j, \quad w_i^{j-1} \le h_i < w_i^j, \quad j = 1, \cdots, 2^L, \quad i = 1, \cdots, K. \tag{3.18}$$

The quantization of a random variable is often performed subject to a fidelity criterion. In this particular case we consider the widely used mean square distortion as the fidelity criterion,

$$D(Q(h_i)) = E\left[(h_i - \hat{h}_i)^2\right], \quad i = 1, \cdots, K. \tag{3.19}$$

and we use Lloyd's algorithm [47, 48] to perform the quantization of the variables of interest. Note that, although more advanced quantization techniques, including vector quantization, could be used to more accurately represent the original random variables, our purpose is to demonstrate how the power control performs for systems with quantized feedback when compared to ones with perfect (infinite precision) feedback, rather than finding a powerful quantizer. For our purposes, we will see that Lloyd's algorithm with the mean square distortion gives satisfactory enough results in terms of getting close to the perfect feedback capacity.

To avoid extra notation, we will assume that $Q_i(h_i)$ defined in (3.18) is a good quantizer obtained by running Lloyd's algorithm. When the channel state $\mathbf{h}$ is measured at the receiver, its components are quantized using $Q_i(h_i)$, $i = 1, \cdots, K$, and fed back to the transmitters. Then, for given priorities μ_i, the transmitters solve the optimal power allocation problem for the set of discrete channel states $\hat{h}_i^j$ using the generalized waterfilling algorithm, to get $p_i^*(\hat{\mathbf{h}})$, which yields a corresponding rate tuple $\hat{\mathbf{R}}$ given by the expectations in (3.1). One should note that the expectation in (3.1) is still with respect to the actual (unquantized) channel state $\mathbf{h}$.

For the second case, the quantizer for the power levels is similarly defined with the codebook $\hat{P}_i = \{\hat{p}_i^1 < \cdots < \hat{p}_i^{2^L}\}$, and the partition $v_i = \{0 = v_i^0 < v_i^1 < \cdots < v_i^{2^L-1} < \cdots < v_i^{2^L} = \infty\}$, such that

$$Q_i(p_i(\mathbf{h})) = \hat{p}_i^j, \quad v_i^{j-1} \leq p_i(\mathbf{h}) < v_i^j, \quad j = 1, \cdots, 2^L, \quad i = 1, \cdots, K. \tag{3.20}$$

The mean square distortion function is also defined accordingly. Now, the receiver first uses the generalized waterfilling algorithm to obtain the optimal power levels $p_i^*(\mathbf{h})$ for given priorities, then quantizes them using $Q_i(p_i^*(\mathbf{h}))$, which is the quantizer obtained by the Lloyd's algorithm, and sends the quantized power level $\hat{p}_i^*$, to transmitter i, for $i = 1, \cdots, K$. These power levels can be used by the transmitters without the knowledge of the channel state, to obtain the rate tuple $\hat{\hat{\mathbf{R}}}$ which can be computed again by taking the expectation in (3.1).

The two feedback approaches differ in that while the total amount of receiver feedback is the same, the amount of feedback bits required by the transmitters is clearly different, since the transmitters need all KL feedback bits in case (i), whereas they need only their own power level, i.e., L bits, in case (ii). Also, the rate tuples $\hat{\mathbf{R}}$ and $\hat{\hat{\mathbf{R}}}$ are in general different. The corresponding achievable rate regions are given in the following section.

3.6 Simulation Results

In this section, we present some simulation results for the generalized iterative water-filling algorithm. In our simulations, we pick the number of users $K = 2$, so that our results such as the capacity regions and the optimum power allocations can be easily visualized. The processing gain is chosen to be $N = 2$, the noise variance is $\sigma^2 = 1$, and both users have an average power constraint equal to 1.

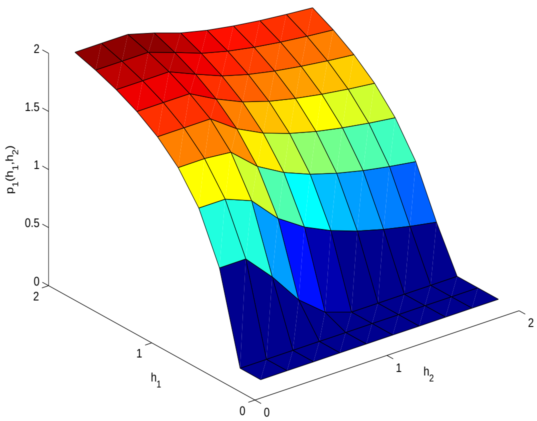
First, in order to observe the effect of the priorities μ_i on the optimum power allocation, we plot the optimum power allocation policies for both users for two different sets of (μ_1, μ_2) values. We fix the signature sequences of the users to be $\mathbf{s}_1 = [1/\sqrt{2} \;\; 1/\sqrt{2}]^\top$, and $\mathbf{s}_2 = [1 \;\; 0]^\top$. The channel states h_1 and h_2 are chosen to be independent uniform random variables, each taking values from the discrete set {0.2:0.2:2}. Figures 3.4(a) and 3.4(b) correspond to the sum capacity maximizing power control policies, i.e., to $(\mu_1, \mu_2) = (1, 1)$. In each figure, the height of the surface corresponds to the power allocated to each channel state. We see that the two users perform simultaneous waterfilling, which was also observed in [13, 14]. Here, each of the users tend to transmit with less power over the channel states where the other user is stronger, and due to the symmetry of the problem, the power allocation policies are symmetric. When we choose unequal priorities $(\mu_1, \mu_2) = (1, 2)$, we observe in Figures 3.4(c) and 3.4(d) that the power allocation for user 1 does not change significantly, but user 2 pours more power to channel states where it transmitted with considerably less power in the symmetric priorities case. If we increase μ_2 even further, and solve for the case when $(\mu_1, \mu_2) = (1, 10)$, we see in Figure 3.4(f) that the power allocation policy of user 2 converges nearly to single user waterfilling. Since the priority of user 1 is much less than that of user 2, user 2, while trying to maximize the weighted sum of rates, acts as if it is alone in the system in allocating its power. The power allocation of user 1, given in Figure 3.4(e), is not significantly different from the previous two cases.

In Figure 3.5, we give the capacity region of the fading CDMA channel, for different values of correlations between the signature sequences. The regions are formed by finding the optimal power allocation policies for a large set of (μ_1, μ_2) values, and then by using these allocation policies to compute the corresponding (R_1, R_2) pairs. The case when the correlation is $\rho = 1$ corresponds to the identical signature sequences case, in which case the boundary of the capacity region is strictly convex, and each point on the surface can be achieved by a power control policy, without timesharing. Note that, this setting also covers the scalar MAC case in [7], and the properties derived in [7] for the capacity region are observed here. When we decrease the correlation between the sequences, we begin to observe a flat portion on the capacity region, which agrees with the findings of [14]. As we further decrease the correlation, eventually the sequences become orthogonal and the capacity region becomes the rectangular region whose boundaries are single user limits, as expected.

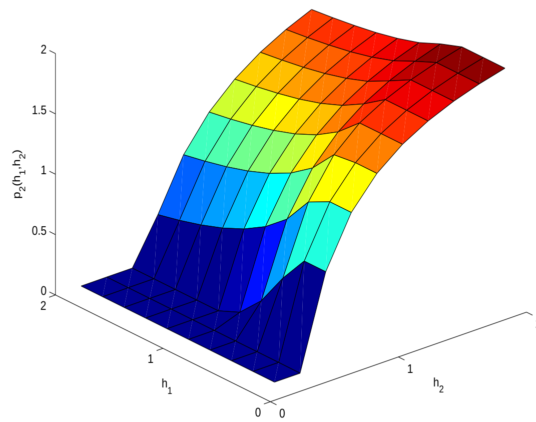
In Figure 3.6, we show an example of the convergence of the generalized iterative waterfilling algorithm for the simple system considered here; the powers converge after only three iterations, and the optimum weighted capacity value is almost attained after one round of iterations.

We now turn our attention to the systems in Section 3.5, and investigate the achievable rates for systems with limited feedback. Here, we consider i.i.d. Rayleigh channel fading (exponential channel states h_i) with mean 0.63. The signature sequences are fixed with correlation equal to $\rho = 0.95$.

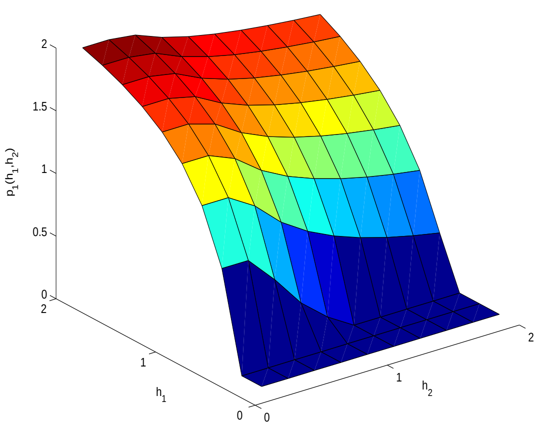
The achievable rates for systems with 1, 2 and 3-bit quantization of each h_i are illustrated in Figure 3.7. We observe that, even with the very low feedback rate of 1-bit, the achievable rate region is remarkably improved when compared to a system with no feedback, thanks to the possibility of employing power control. We further see that, the amount of feedback, though it enlarges the region of achievable rates, is not very significant in terms of improving the set of achievable rates. We conclude that the feedback cost of power control may be kept to a minimum 1-bit per user, i.e., a total of K bits, while still achieving much higher information rates than no



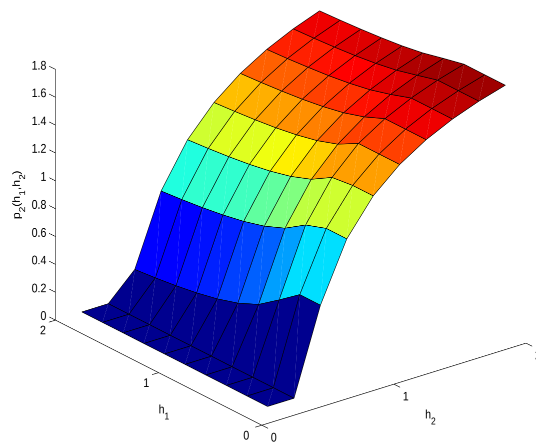
(a) Power allocation for user 1, $\mu_1 = \mu 2 = 1$.



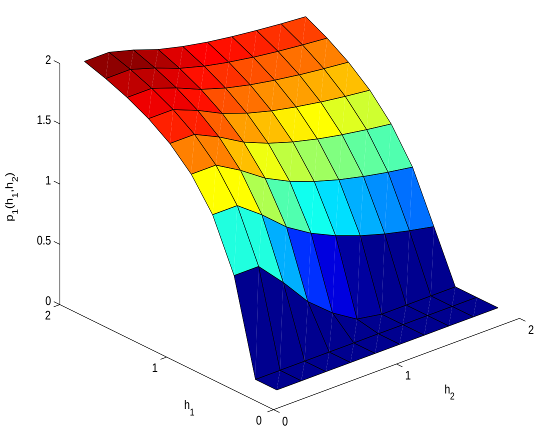
(b) Power allocation for user 2, $\mu_1 = \mu 2 = 1$.



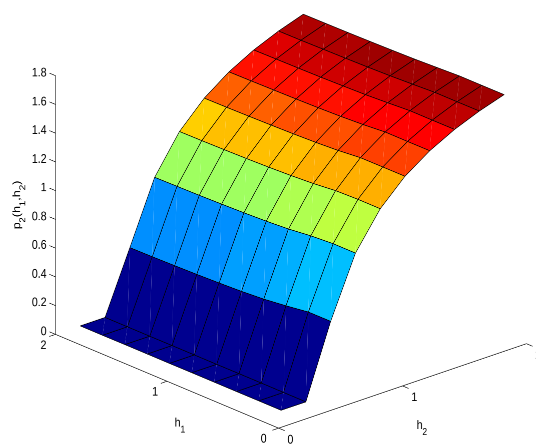
(c) Power allocation for user 1, $\mu_1 = 1$, $\mu 2 = 2$.



(d) Power allocation for user 2, $\mu_1 = 1$, $\mu 2 = 2$.



(e) Power allocation for user 1, $\mu_1 = 1$, $\mu 2 = 10$.



(f) Power allocation for user 2, $\mu_1 = 1$, $\mu 2 = 10$.

Figure 3.4: Power distributions for different values of priorities.

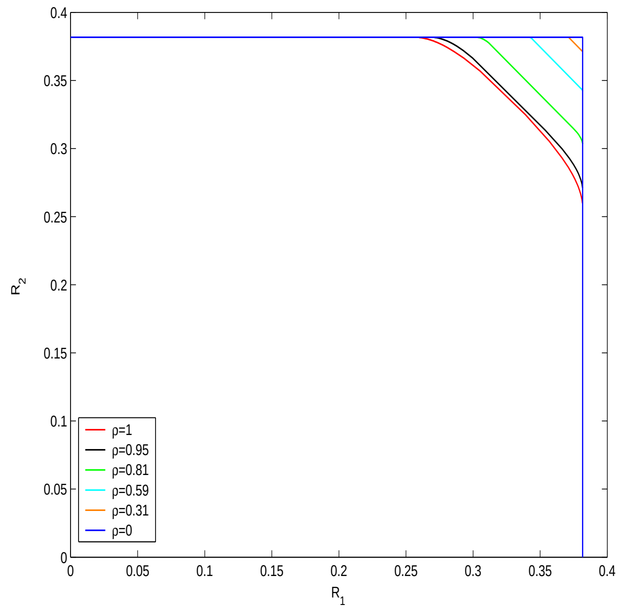


Figure 3.5: Capacity region of a two user fading CDMA channel for several correlation values among the signature sequences.

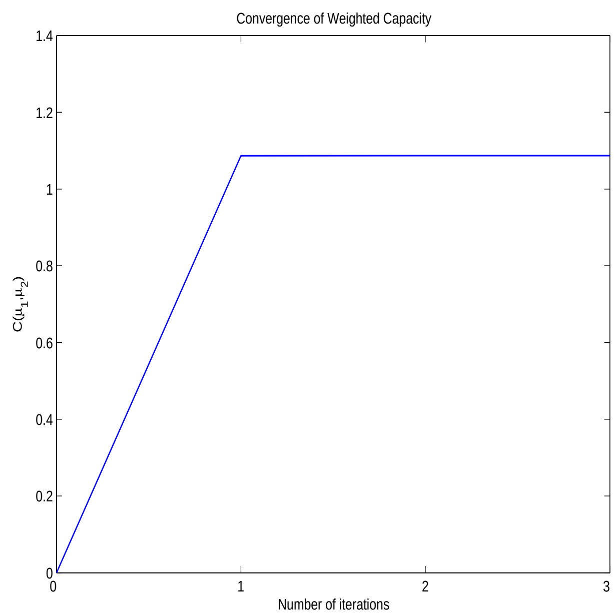


Figure 3.6: Convergence of the generalized iterative waterfilling algorithm.

power control, in fact, rates very close to the perfect feedback capacity region. As discussed in Section 3.5, the set of achievable rates may further be improved by using more advanced quantization techniques.

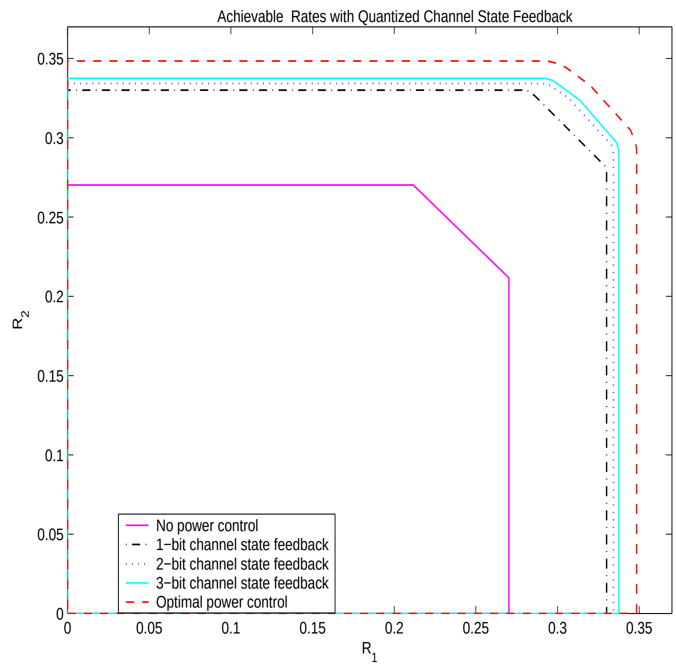


Figure 3.7: Achievable rates with quantized channel state feedback.

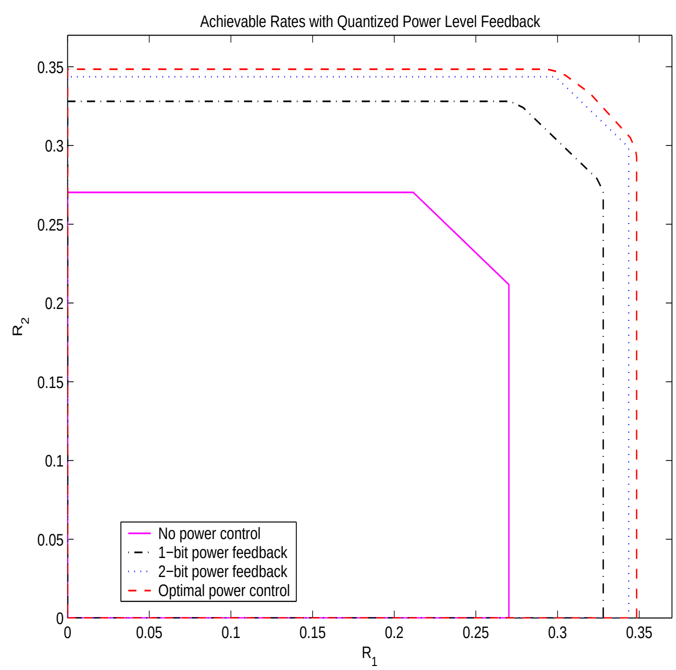


Figure 3.8: Achievable rates with quantized power level feedback.

For the case of directly feeding back a quantized power level, the achievable rates are shown in Figure 3.8. We observe that while 1-bit feedback gives similar achievable rates to those for channel state feedback, by 2-bit feedback we obtain a much more significant improvement when compared to the channel state feedback. In fact, by employing only four power levels, i.e., 2-bit feedback, it is possible to get very close to the capacity region with perfect feedback. Both feedback schemes show that the significant performance gains due to power control do not require very accurate feedback information, which is very promising in terms of possible implementation of the developed algorithm in practical systems. We would like to note that these results agree with earlier findings regarding the capacity of single user channels with limited CSI feedback [49], and the throughput of time varying MACs with limited CSI feedback [50], which also demonstrated that the capacity does not depend strongly on the accuracy of the CSI feedback.

3.7 Summary and Conclusions

We provided the capacity region for a power controlled fading CDMA system, and proved that unless all users have orthogonal or identical sequences, it has a flat portion on which the sum capacity is maximized; i.e., it is not strictly convex. This yields the important result that, sum capacity may be achieved by infinitely many rate tuples, implying that one has flexibility in choosing the individual rates of the users while keeping the sum capacity constant at its maximum.

We have characterized the power allocation policies that achieve arbitrary rate tuples on the boundary of the capacity region of a fading CDMA channel. The optimal power allocation policy for a given set of priorities μ_i is the joint solution to the extended KKT conditions for all users. Since the KKT inequalities appear difficult to solve analytically, we have provided a one-user-at-a-time iterative power allocation algorithm that converges to the optimum solution. We showed that, each iteration of this algorithm corresponds to solving for the power levels of the user of interest at all fading states, so that the power allocation satisfies the single-user KKT conditions. We have also provided a "generalized" waterfilling interpretation for the power allocation procedure as it operates by gradually equating the levels of "water" poured on top of certain base levels, which are functions of the channel states, power levels of other users, and the priorities μ_i. We then investigated the effect of limited feedback on the capacity region of the CDMA channel. We demonstrated that, even with very low rate feedback, rates very close to the boundary of the capacity region are achievable.

The results of this Chapter were published in [14–16], and have been submitted for publication in [17].

Chapter 4

Jointly Optimal Power and Signature Sequence Allocation for Fading CDMA

4.1 Introduction

So far, we have considered CDMA systems where the signature sequences assigned to the users are fixed throughout the transmission, and we used the CSI to the advantage of the network capacity only through allocating the powers. The sum capacity of a CDMA network can also be optimized as a function of the signature sequences. When each user has an average power constraint, and there is no fading in the system, [18] shows that when the number of users is less than or equal to the processing gain, the optimal strategy is to allocate orthogonal signature sequences to all users, and when the number of users is greater than the processing gain, with all users having the same average power constraints, the optimal strategy is to allocate Welch Bound Equality (WBE) [19] sequences. Reference [20] generalizes [18] to arbitrary (unequal) average power constraints, and gives the optimal signature sequence allocation as a function of the power constraints of the users, by making use of some results from the theory of majorization in matrix analysis [51, 52]. Specifically, for the case in which the number of users is greater than the processing gain, when a user has a "relatively larger" power constraint then the others, it is called "oversized", and such users are allocated orthogonal signature sequences; whereas the "non-oversized" users are allocated the so-called Generalized Welch-Bound-Equality (GWBE) sequences.

In this chapter, we present the solution to the problem of joint power and signature sequence optimization in order to maximize the ergodic sum capacity of a fading CDMA system. Specifically, we adapt the set of signature sequences and transmit powers of all users as a function of the CSI, in order to maximize the ergodic sum capacity. At each fading state, for any given arbitrary power allocation, results of [20] can be used to allocate the optimal sequences. Among those power allocations, with signature sequences chosen optimally, we find the best power allocation strategy.

We show that the optimal strategy is still a waterfilling strategy for each user, and very strikingly, at each fading state, that strategy dictates that we allocate (at most) N orthogonal signature sequences to the users with best (at most) N channel states (scaled by a factor as in [6]). Moreover, the other users with worse channel states than the users with orthogonal sequences do not transmit at those particular channel states. This means that there are no users in the system which are allocated GWBE sequences and are yet transmitting with nonzero powers. Thus, in contrast to signature sequence optimization for non-fading channels, GWBE sequences are never used in transmissions; more precisely, they are used only with zero probability.

Our solution resembles [6] in the sense that there is an ordering of channel states that determines which users will transmit, but it also resembles the solution in [5] in that once we know which users will transmit at each channel state, all users will choose their powers as if they are alone in the system, i.e., they will perform single user waterfilling over favorable regions of the channel state space. This result shows that, when we have the opportunity to control both the signature sequences and the powers of the users, the users completely avoid each other, i.e., certain groups of users transmit on disjoint sets of channel states, and within each group of users that transmit at the same channel state, users place themselves orthogonal to each other in the signature sequence space, thus avoiding any possible interference.

We also provide an iterative algorithm that is guaranteed to converge to the optimal power and signature sequence allocation. The algorithm performs a one-user-at-a-time waterfilling, and converges to the optimum solution described above.

Throughout this thesis, we have been assuming symbol synchronism for the CDMA system under consideration. In Section 4.4, we relax this assumption to consider symbol asynchronous but chip synchronous systems. We show that the asynchrony does not lead to a loss in the sum capacity, nor does it lead to a change in the optimal power allocation policy from the synchronous case. The signature sequence adaptation policy jointly optimal with this power allocation however is updated, using the results from [24].

Finally, we investigate the improvements in sum capacity provided by the use of multiple antennas at the receiver. We provide an algorithm that iterates between what is so called the eigen-update for sequence optimization [53] and the optimal one-user-at-a-time power update for fixed sequences of Chapter 2. The spatial diversity provided by the antennas at the receiver together with the flexibility to choose the transmit directions by adjusting the signature sequences and powers leads to remarkable gains in the capacity.

4.2 Joint Signature Sequence and Power Allocation

For a given set of signature sequences and a fixed set of channel gains, $\mathbf{h}$, the sum capacity $C_{\text{sum}}(\mathbf{h}, \bar{\mathbf{p}}, \mathbf{S})$ is [9],

$$C_{\text{sum}}(\mathbf{h}, \bar{\mathbf{p}}, \mathbf{S}) = \frac{1}{2} \log \left| \mathbf{I}_N + \sigma^{-2} \sum_{i=1}^{K} h_i \bar{p}_i \mathbf{s}_i \mathbf{s}_i^\top \right| \tag{4.1}$$

where $\bar{p}_i$ is the average power of user i, $\bar{\mathbf{p}} = [\bar{p}_1, \cdots, \bar{p}_K]$, and $\mathbf{S} = [\mathbf{s}_1, \cdots, \mathbf{s}_K]$. To maximize the above capacity for that particular $\mathbf{h}$, one can choose the signature sequences of the users for a given set of power constraints. An equivalent problem is solved in [20], in the no-fading case, i.e., $h_i = 1$, for all i.

In the presence of fading, if the channel state is modelled as a random vector, the quantity $C_{\text{sum}}(\mathbf{h}, \bar{\mathbf{p}}, \mathbf{S})$ is random as well, and the ergodic sum capacity is found as the expected value of $C_{\text{sum}}(\mathbf{h}, \bar{\mathbf{p}}, \mathbf{S})$. Instead of keeping the transmit power of user i fixed to $\bar{p}_i$ as in (4.1), we can choose the transmit powers of the users $p_i(\mathbf{h})$, $i = 1, \cdots, K$, as a function of the channel state with the aim of maximizing the ergodic sum capacity of the system subject to average transmit power constraints for all users. Similarly, we can choose the signature sequences $\mathbf{S}$ to be a function of the channel state as well; let us denote it by $\mathbf{S}(\mathbf{h})$ to show the dependence on the channel state. Therefore our problem is to solve for the jointly optimum transmit powers and signature sequences as a function of the channel state in order to maximize the ergodic sum capacity of the system in the presence of fading. The problem can be stated as,

$$\max_{\mathbf{p}(\mathbf{h}),\mathbf{S}(\mathbf{h})} E_{\mathbf{h}}\left[\log\left|\mathbf{I}_N+\sum_{i=1}^{K}\frac{h_i p_i(\mathbf{h})}{\sigma^2}\mathbf{s}_i(\mathbf{h})\mathbf{s}_i(\mathbf{h})^\top\right|\right]$$

$$\text{s.t.} E_{\mathbf{h}}\left[p_i(\mathbf{h})\right| = \bar{p}_i, \quad p_i(\mathbf{h}) \geq 0 \tag{4.2}$$

where the expectation is taken with respect to the joint probability density function $f(\mathbf{h})$ of the channel states.

In order to jointly optimize the powers and signature sequences, we first fix power distributions of all users over all fading states, and find the set of oversized users at each channel state according to the rule [20]

$$p_k(\mathbf{h}) > \frac{\sum_{i=1}^{K} p_i(\mathbf{h}) 1_{p_k(\mathbf{h})>p_i(\mathbf{h})}}{\left(N-\sum_{i=1}^{K} 1_{p_k(\mathbf{h})\leq p_i(\mathbf{h})}\right)} \tag{4.3}$$

Then, the corresponding optimal signature sequence set at every channel state will consist of a combination of orthogonal and GWBE sequences [20]. This is due to the fact that, the signature sequences at a fading state $\mathbf{h}$ can be chosen independently of the signature sequences at any other state, since once the powers are fixed, there are no constraints relating $\mathbf{S}(\mathbf{h})$ to $\mathbf{S}(\bar{\mathbf{h}})$ for $\mathbf{h} \neq \bar{\mathbf{h}}$. That is, we can freely choose a sequence set at a given state $\mathbf{h}$ without changing the contribution to the sum capacity of another state $\bar{\mathbf{h}}$; this clearly is not true for the power allocation, since once we allocate a power level for a given state $\mathbf{h}$, we have less power left to allocate to other states, and overall capacity is affected. Since the optimum signature sequences at each channel state depend only on the powers $\mathbf{p}(\mathbf{h})$ and the channel state $\mathbf{h}$, we can express the capacity at each channel state only as a function of the powers, and optimize the ergodic capacity in terms of the power allocation. Let us define the *signature sequence optimized sum capacity* at channel state $\mathbf{h}$ for a given power control policy $\mathbf{p}(\mathbf{h})$ by

$$C_{\text{opt}}(\mathbf{h}, \mathbf{p}(\mathbf{h})) \triangleq \max_{\mathbf{S}(\mathbf{h})} C_{\text{sum}}(\mathbf{h}, \mathbf{p}(\mathbf{h}), \mathbf{S}(\mathbf{h})) \tag{4.4}$$

where $C_{\text{sum}}(\mathbf{h}, \mathbf{p}(\mathbf{h}), \mathbf{S}(\mathbf{h}))$ is the argument of the expectation in the objective function of (4.2), i.e., it is the function in (4.1) where $\bar{\mathbf{p}}$ is replaced by $\mathbf{p}(\mathbf{h})$ and $\mathbf{S}$ is replaced by $\mathbf{S}(\mathbf{h})$. For a fixed $\mathbf{h}$, it can be shown using majorization theory that $C_{\text{opt}}(\mathbf{h}, \mathbf{p}(\mathbf{h}))$ is a concave function of the power vector at channel state $\mathbf{h}$, $\mathbf{p}(\mathbf{h})$ [21, Proposition 2.2]. Then, the problem in (4.2) can be written only in terms of the powers as

$$\begin{aligned} &\max_{\mathbf{p}(\mathbf{h})} E_{\mathbf{h}} \left[C_{\text{opt}}(\mathbf{h}, \mathbf{p}(\mathbf{h})) \right] \\ &\text{s.t.} E_{\mathbf{h}} \left[p_i(\mathbf{h}) \right] = \bar{p}_i, \quad p_i(\mathbf{h}) \geq 0 \end{aligned} \tag{4.5}$$

First consider the case when $K \leq N$. For any fixed channel state, the optimal choice of signature sequences for a given power control policy $\mathbf{p}(\mathbf{h})$ is an orthogonal set [18, 20]. Noting that the received power levels are $p_i(\mathbf{h})h_i$, (4.5) takes the form,

$$\begin{aligned} &\max_{\mathbf{p}(\mathbf{h})} E_{\mathbf{h}} \left[\sum_{i=1}^{K} \log \left(1 + \frac{p_i(\mathbf{h})h_i}{\sigma^2} \right) \right] \\ &\text{s.t.} E_{\mathbf{h}} \left[p_i(\mathbf{h}) \right] = \bar{p}_i, \end{aligned} \tag{4.6}$$

which is equivalent to solving K independent Goldsmith-Varaiya problems [5] (see also [13]), the solution to which is a single user waterfilling for each user. More precisely, the problem in (4.6) is a concave maximization over affine sets of constraints, therefore the optimal solution $\mathbf{p}^*(\mathbf{h})$ is the unique solution satisfying the Karush-Kuhn-Tucker (KKT) conditions, and is given by,

$$p_i^*(\mathbf{h}) = \left(\frac{1}{\lambda_i} - \frac{\sigma^2}{h_i}\right)^+, \qquad i = 1, \cdots, K \tag{4.7}$$

where λ_i is solved by plugging (4.7) into (4.6).

One remarkable observation is that in obtaining $C_{\text{opt}}(\mathbf{h}, \mathbf{p}(\mathbf{h}))$, it is possible to adopt a channel non-adaptive signature sequence allocation policy, i.e., each user can be assigned a designated signature sequence, which it can use at all channel states, as long as the signature sequences in this set are orthogonal. A channel adaptive scheme will also perform equally well as long as the signature sequences we choose at each $\mathbf{h}$ are from an orthogonal set.

When $K > N$, it has been shown in [20], for a non-fading channel, that given the power constraints of all users, one can group the users into two sets L and $\bar{L}$, of oversized and non-oversized users, respectively. Users $i \in L$ are assigned orthogonal sequences, and users $i \in \bar{L}$ are assigned GWBE sequences. For a channel with fading, at a certain channel state $\mathbf{h}$, and for a certain arbitrary power distribution of users which assigns powers $p_1, \cdots, p_K$ to channel state $\mathbf{h}$, let us define the matrix $\mathbf{D} \triangleq \text{diag}(p_1 h_1, \cdots, p_K h_K)$, and define μ_i to be the eigenvalues of the matrix $\mathbf{SDS}^\top$. Then the signature sequences that maximize the sum capacity for any fixed $\mathbf{h}$ satisfy

[54],

$$\mathbf{S}\mathbf{D}\mathbf{S}^\top \mathbf{s}_i = \mu_i \mathbf{s}_i, \qquad i = 1, \cdots, K \tag{4.8}$$

clearly with repetitions of some of the μ_is (since there are only N eigenvalues of $\mathbf{S}\mathbf{D}\mathbf{S}^\top$), where the optimal μ_is are given by [20],

$$\mu_i(\mathbf{h}) = \begin{cases} \dfrac{\sum_{j \in \bar{L}(\mathbf{h})} p_j h_j}{N - |L(\mathbf{h})|}, & i \in \bar{L}(\mathbf{h}) \\ p_i h_i, & i \in L(\mathbf{h}) \end{cases} \tag{4.9}$$

In the fading case with channel adaptive powers, as suggested by the results in [5,6,13], it is likely that some users will have powers equal to zero at some channel states, and they will not contribute to C_{sum} at those channel states. Although the concept of oversized users is defined for users with nonzero average power constraints, since users which are allocated zero power at state $\mathbf{h}$ will not contribute to the sum capacity, we can add them to the set of non-oversized users at channel state $\mathbf{h}$, $\bar{L}(\mathbf{h})$, and we can assume that we assign arbitrary sequences for those users without changing the solution. Note however that, while finding the set of oversized users, we will disregard the users with zero power. Using the optimum eigenvalue assignment in (4.9) at each state, the objective function of the problem (4.5) can be expressed in the alternative form,

$$E_{\mathbf{h}} \left[\sum_{i \in L(\mathbf{h})} \log \left(1 + \frac{p_i(\mathbf{h}) h_i}{\sigma^2} \right) + (N - |L(\mathbf{h})|) \log \left(1 + \frac{\sum_{i \in \bar{L}(\mathbf{h})} p_i(\mathbf{h}) h_i}{\sigma^2 (N - |L(\mathbf{h})|)} \right) \right] \tag{4.10}$$

For a given channel state $\mathbf{h}$, let the set of users that will transmit with non-zero powers be $\bar{K}(\mathbf{h})$. Then the number of users in $\bar{K}(\mathbf{h})$ cannot exceed N, as stated by the following theorem.

Theorem 4.1 *Let $\bar{K}(\mathbf{h})$ be a subset of $\{1, \cdots, K\}$, such that $\forall i \in \bar{K}(\mathbf{h})$, $p_i^*(\mathbf{h}) > 0$, where $\mathbf{p}^*(\mathbf{h})$ is the maximizer of (4.10). Then, $|\bar{K}(\mathbf{h})| \leq N$, almost surely.*

Proof: By concavity of $C_{\text{opt}}(\mathbf{h}, \mathbf{p}(\mathbf{h}))$, it is clear that the function in (4.10) is concave, and the maximization in (4.5) is over an affine set of constraints. Therefore, a power vector $\mathbf{p}^*(\mathbf{h})$ achieves the global optimum of the maximization problem if and only if it satisfies the KKT conditions. Then, writing the KKT conditions for the objective function in (4.10), it is easy to show that

$$\frac{h_i}{\mu_i(\mathbf{h}) + \sigma^2} \leq \lambda_i, \qquad \forall \mathbf{h} \tag{4.11}$$

where $\mu_i(\mathbf{h})$ is given by (4.9), and equality holds if $p_i(\mathbf{h}) > 0$. Now, let us assume that the number of non-zero components in $\mathbf{p}^*(\mathbf{h})$ is $|\bar{K}(\mathbf{h})| > N$, for a given $\mathbf{h}$. Then, some users must share some of the available dimensions, i.e., not all users can be made orthogonal to each other. In fact, we can find at most $N-1$ sequences that are orthogonal to all other sequences in the system, or equivalently, at least $|\bar{K}(\mathbf{h})| - N + 1$ users will have the same $\mu_i = \sum_{j \in \bar{L}(\mathbf{h})} h_j p_j / (N - |L(\mathbf{h})|)$. Then, substituting this into (4.11), we get $h_i/\lambda_i = h_j/\lambda_j$ for $i \neq j$, $i, j \in \bar{K}(\mathbf{h})$ for at least $|\bar{K}(\mathbf{h})| - N + 1$ users. Note that as the channel fading is assumed to be a continuous random variable, this event has zero probability, and at most one user with GWBE sequences (one with highest h_i/λ_i ratio, as in [6]) may transmit, with probability 1. But this contradicts the assumption that $|\bar{K}(\mathbf{h})| > N$, which establishes our main result, i.e., $|\bar{K}(\mathbf{h})| \leq N$ almost surely. $\Box$

This result may be viewed as a generalization of [6] to a vector channel with a unit rank constraint on the covariance matrices of the inputs; [6] showed that in scalar MAC (i.e., when $N = 1$), at most one user may transmit at a channel state with probability 1. An important implication of Theorem 4.1 is that, since the optimal power allocation dictates that at most N users transmit with positive powers at any given channel state, orthogonal sequences should be assigned to those users that are transmitting with positive powers. That is, although we allowed for allocating GWBE sequences to some of the users, the solution implies that there is at most one such user, and the problem reduces to the orthogonal case. The optimal power allocation is again single user waterfilling, similar to the solution given in (4.7), i.e.,

$$p_i^*(\mathbf{h}) = \begin{cases} \left(\frac{1}{\lambda_i} - \frac{\sigma^2}{h_i}\right), & i \in \bar{K}(\mathbf{h}) \\ 0, & \text{otherwise} \end{cases} \tag{4.12}$$

Here, one needs to be careful about the transmit regions. Unlike the case where the actual number of users $K \leq N$, the users in the set $\bar{K}(\mathbf{h})$ change with $\mathbf{h}$, thus a channel adaptive allocation of the orthogonal sequences is necessary. Our convention is, we assign a sequence from an orthogonal set to a user, wherever its power is positive.

To specify the optimal power allocation completely, let us define $\gamma_i = h_i/\lambda_i$. Then, the probability that $\gamma_i = \gamma_j$, for $i \neq j$ is zero. Therefore, we can always find a unique order statistics $\{\gamma_{[i]}\}_{i=1}^{K}$ such that $\gamma_{[1]} > \cdots > \gamma_{[K]}$, for each given $\mathbf{h}$. Let us now place

σ^2 in that ordering, assuming that at least one of the $\gamma_{[i]}$s is larger than σ^2. Define $\gamma_{[K+1]} = 0$. Then, for some $n \in \{1, \cdots, K\}$, let

$$\gamma_{[1]} \geq \cdots \geq \gamma_{[n]} > \sigma^2 \geq \gamma_{[n+1]} \geq \cdots \geq \gamma_{[K+1]} \tag{4.13}$$

where the equalities are included for the sake of consistency of the indices, and do not affect the solution (note the strict inequality just before σ^2).

First, let $n \leq N$. Then, we see that (4.12) gives positive powers for all n users, and thus all n users with highest γ_is will transmit with the non-zero powers given in (4.12). When $n > N$, there are more than N users satisfying the positivity constraints $\gamma_i > \sigma^2$. However, we know from our derivation that only the user with the highest γ_i from the set we intend to assign GWBE sequences may transmit. Therefore, a total of N users with the highest γ_is transmit at this channel state.

Finally, we can summarize the jointly optimal power and signature sequence allocation policy as,

$$\begin{aligned} p_i^*(\mathbf{h}) &= \begin{cases} \left(\frac{1}{\lambda_i} - \frac{\sigma^2}{h_i}\right), & \text{iff } \ i \in \Omega \\ 0, & \text{otherwise} \end{cases} \\ \mathbf{s}_i^*(\mathbf{h})^\top \mathbf{s}_j^*(\mathbf{h}) &= \quad 0, \qquad i \neq j, \qquad \forall\, i, j \in \Omega \\ \Omega &= \left\{ i : \ \gamma_{[i]} > \sigma^2, \qquad i \leq \min\{K, N\} \right\} \end{aligned} \tag{4.14}$$

4.3 Iterative Power and Sequence Optimization

We found in the previous section that the optimal power control strategy is a water-filling over some favorable channel states for each user. However, in order to obtain the optimal power levels one should also compute the Lagrange multipliers λ_i, from the average power constraints. It turns out that the power allocation of each user still depends in a complicated fashion to those of the other users through these λ_i. In this section we provide an iterative method to obtain the jointly optimal power and signature sequence allocation, and hence the λ_i.

We have already shown in Section 2.2 that for fixed signature sequences $\mathbf{S}$, the optimal single-user update that maximizes the sum capacity as a function of $p_k(\mathbf{h})$ is given by,

$$p_k(\mathbf{h}, \mathbf{S}) = \left(\frac{1}{\lambda_k} - \frac{1}{h_k \mathbf{s}_k^\top \mathbf{A}_k^{-1} \mathbf{s}_k} \right)^+ \tag{4.15}$$

where the interference covariance matrix $\mathbf{A}_k$, defined previously in (2.10), can be rewritten as,

$$\begin{aligned} \mathbf{A}_k &= \sigma^2 \mathbf{I}_N + \sum_{i \neq k} h_i p_i(\mathbf{h}) \mathbf{s}_i \mathbf{s}_i^\top \\ &= \sigma^2 \mathbf{I} + \mathbf{S}\mathbf{D}\mathbf{S}^\top - h_k p_k(\mathbf{h}) \mathbf{s}_k \mathbf{s}_k^\top \end{aligned} \tag{4.16}$$

We can find and fix the optimal signature sequences at each state for a given power allocation using results of [20]. Then, plugging these sequences in (4.16), multiplying both sides by the optimal signature sequence $\mathbf{s}_k^*$, and noting that the signature sequences that maximize the sum capacity for a fixed set of power constraints satisfy

(4.8), we get

$$\mathbf{A}_k \mathbf{s}_k^* = (\sigma^2 + \mu_k - h_k p_k)\mathbf{s}_k^* \tag{4.17}$$

where μ_k are given by (4.9). Therefore,

$$\mathbf{s}_k^{*\top} \mathbf{A}_k^{-1} \mathbf{s}_k^* = \frac{1}{\sigma^2 + \mu_k - h_k p_k} \tag{4.18}$$

This shows that, we can represent the base level for the waterfilling in (4.15) as a function of the power levels in the previous iteration. Substituting this in (4.15), we get the optimal power allocation at the $n+1$st step, $p_k^{n+1}(\mathbf{h})$ for user k, with optimal sequences and fixed powers $\{p_i(\mathbf{h})\}_{i \neq k}$ from the previous iteration

$$p_k^{n+1}(\mathbf{h}) = \left(\frac{1}{\lambda_k^{n+1}} - \frac{\sigma^2 + \mu_k^n(\mathbf{h}) - h_k p_k^n(\mathbf{h})}{h_k} \right)^+, \quad \forall \mathbf{h} \tag{4.19}$$

where we use $\{p_1^{n+1}(\mathbf{h}), \cdots, p_{k-1}^{n+1}(\mathbf{h}), p_k^n(\mathbf{h}), \cdots, p_K^n(\mathbf{h})\}$ to compute $\mu_k^n(\mathbf{h})$. Combining this with (4.9) gives us the power update at each step. It is easy to observe that, once the eigenvalues $\mu_k^n(\mathbf{h})$ are determined using the power levels from the previous iteration, we can use (4.19) to solve for kth user's power by waterfilling. Note that, the Lagrange multiplier λ_k^{n+1} is chosen to satisfy the average power constraint of user k at each iteration, and can be obtained by plugging (4.19) into the constraint in (2.5). The waterfilling algorithm automatically obtains the value of λ_k^{n+1} as it is the inverse of the "water level".

The proposed algorithm may be interpreted in two ways. First, it may be seen as an iteration from a set of powers to another set of powers as given by (4.19).

Therefore, one may run this algorithm starting with an arbitrary power distribution, to obtain the capacity maximizing power distribution when the algorithm converges. The signature sequences may then be assigned to the users after the algorithm converges: at each channel state, the users that have non-zero powers (there will be at most N such users) are assigned signature sequences from an orthogonal set. Second, the algorithm may be seen as an iteration from powers to signature sequences, and then back to powers again. Specifically, for a given set of powers, the optimal sequences may be found using (4.8) and (4.9), i.e., as in [20]; corresponding to these sequences, base levels for the waterfilling in (4.15) can be computed using (4.17) and (2.18), and new powers may be found using (4.15) as in Section 2.2.

We will now show that (4.19) and equivalently the sequential signature sequence and power update algorithm indeed converges to the global optimum of the sum capacity function. To see this, first observe that for fixed signature sequences, the update (4.15) is the best one-user-at-a-time power update and is guaranteed to give a non-decreasing sequence of sum capacity values. Similarly, for fixed powers, the signature sequence update will increase (or keep constant) the value of the sum capacity. The sum capacity is upper bounded, therefore it is guaranteed that the sequence of non-decreasing sum capacity values obtained through these iterations have a limit. Moreover, the algorithm terminates if and only if the update (4.19) yields a fixed point $\mathbf{p}(\mathbf{h})$. Since the fixed point is characterized by $\mathbf{p}^{n+1} = \mathbf{p}^n$, it is easy to see that the fixed point of the update (4.19) actually satisfies the KKT conditions for our original problem. Since the convergence point $\mathbf{p}(\mathbf{h})$ satisfies the KKT conditions, it achieves the global optimum of the sum capacity, proving the convergence of the sequential algorithm.

Note that we have incorporated the eigenvalues of $\mathbf{SDS}^\top$ in the power iteration (4.19) rather than including the signature sequences explicitly. This implementation is very useful, since it does not require us to compute the signature sequences at intermediate steps. On the other hand, the development in (4.15)-(4.19) makes a subtle point transparent, namely the problem of what sequences to assign to users with zero powers at each iteration, if we were to compute the signature sequences at each step. We obtain (4.17) from (4.16) by replacing $\mathbf{s}_k^*$ by an eigenvector of the matrix $\mathbf{SDS}^\top$ corresponding to the eigenvalue μ_k, even for the users with zero powers. Clearly, for each round of iterations (say n), it is not important what signature sequences are assigned to users with zero powers, as this choice will not reflect on the sum capacity value. Therefore, any update for the signature sequences of users with zero powers would maintain the non-decreasing nature of the sum capacity with each sequence update. Although this choice of sequences might affect the power levels in the next round of iterations, the convergence of the algorithm is still maintained, even if we allocate arbitrary signature sequences to users with zero powers, instead of the eigenvectors of $\mathbf{SDS}^\top$. This is due to the fact that even though the sequence of sum capacity values will follow a different path, at the fixed point they will still satisfy the KKT conditions.

Lastly, it is useful to point out that, although the power allocation policy that maximizes the sum capacity is unique, the signature sequence selection that is jointly optimal with this power allocation is not, for two reasons: first, because of the arbitrariness of the optimal sequences for users with zero powers; and second, because of the fact that even for users with non-zero powers, there are infinitely many sets of orthogonal sequences.

4.4 Joint Power and Sequence Optimization for Asynchronous CDMA

For the CDMA systems of consideration in this thesis, we have so far made the assumption of symbol synchrony, meaning that the symbols, which are modulated by the corresponding signature waveforms at each of the transmitters, are aligned at the receiver. This is a common simplifying assumption for information theoretic analysis of wireless systems. On the other hand, even if the transmitters' clocks are synchronized so as to provide synchronous transmissions, the presence of various delays on different paths in a typical wireless channel is bound to shift in time the transmitted signals from each user by different amounts. Therefore, the analysis of asynchronous systems is of practical importance.

In this section, we consider a symbol asynchronous, but chip synchronous CDMA channel, where the chip waveform is identical for all users. This special type of communication scheme is named direct sequence CDMA (DS-CDMA), which is widely employed in practice [12]. The continuous time model for the signal received as a result of the transmission of M symbols is given by

$$r(t) = \sum_{k=1}^{M} \sum_{i=1}^{K} \sqrt{p_i h_i[k]} b_i[k] s_i(t - c_k T_c - kT) + n(t) \tag{4.20}$$

where, as before, we assume that the channel state $h_i[k]$ remains constant over the kth symbol interval, and the delay between the users are integer multiples of the chip length T_c, i.e., c_k are integers. Note that, as in other related work, i.e., [24, 55], this assumption is made in order to make the analysis tractable; in practice, the delays between the users are in general arbitrary. The systems with arbitrary delay profiles are called totally asynchronous systems, and in fact, for such systems, even the signature sequence optimization alone is still an open problem. Moreover, in those cases, the choice of the chip waveform as well as the modulating signature sequence is a factor in determining the capacity.

For the sake of simplicity of the presentation, let us assume that we have rearranged the order of occurrence of the fading values in time, so that each fading level $\mathbf{h}$ is observed in a single block of $m_{\mathbf{h}}$ symbols. By the stationarity and ergodicity assumption, $\lim_{M\to\infty} m_{\mathbf{h}}/M = f(\mathbf{h})$, and time averages converge to statistical averages. Upon projection of the received signal onto shifted versions of the chip waveform common to all the users, the received signal at each channel state $\mathbf{h}$ can be expressed as an $m_{\mathbf{h}} \times N$ vector [24],

$$\mathbf{r} = \sum_{i=1}^{K} \sqrt{p_i h_i}\mathbf{S}_i\mathbf{b}_i + \mathbf{n} \tag{4.21}$$

where $\mathbf{S}_i$ is the signature sequence matrix of user i defined as [24]

$$\begin{pmatrix} \mathbf{s}_i^L & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{s}_i^R \\ \mathbf{s}_i^R & \mathbf{s}_i^L & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} & \mathbf{s}_i^R & \mathbf{s}_i^L \end{pmatrix}_{m_{\mathbf{h}}N\times m_{\mathbf{h}}} \tag{4.22}$$

Here $\mathbf{s}_i^L$ and $\mathbf{s}_i^R$ denote the left and right signature sequences for user i padded with zeros, forming N-vectors, i.e.,

$$\begin{aligned}
\mathbf{s}_i^L &= [0, \cdots, 0, \mathbf{s}_{i1}, \mathbf{s}_{i2}, \cdots, \mathbf{s}_{N-c_k}]^\top \\
\mathbf{s}_i^R &= [\mathbf{s}_{N-c_k+1}, \mathbf{s}_{N-c_k+2}, \cdots, \mathbf{s}_N, 0, \cdots, 0]^\top
\end{aligned} \tag{4.23}$$

and, $\mathbf{b}_i$ is an $m_{\mathbf{h}} \times 1$ vector of unit energy input symbols. For a given channel state, and fixed power levels, the problem of optimally selecting the signature sequences has been addressed in [24], where it was shown that the sum capacity of the sequence optimized symbol asynchronous CDMA channel is exactly the same as that of the symbol synchronous channel, which is given by,

$$\sum_{i \in L(\mathbf{h})} \log\left(1 + \frac{p_i h_i}{\sigma^2}\right) + (N - |L(\mathbf{h})|) \log\left(1 + \frac{\sum_{i \in \bar{L}(\mathbf{h})} p_i h_i}{\sigma^2 (N - |L(\mathbf{h})|)}\right) \tag{4.24}$$

where $L(\mathbf{h})$ and $\bar{L}(\mathbf{h})$ are the sets of oversized and non-oversized users respectively, defined exactly as in [20] for the symbol synchronous case, only as a function of the powers. The signature sequences that achieve (4.24) are shown to satisfy [24]

1. For all $k \in L(\mathbf{h})$ and $j \neq k$

$$\begin{aligned}
{\mathbf{s}_k^L}^\top \mathbf{s}_j^L + {\mathbf{s}_k^R}^\top \mathbf{s}_j^R &= 0 \\
{\mathbf{s}_k^L}^\top \mathbf{s}_j^R &= 0 \\
{\mathbf{s}_k^R}^\top \mathbf{s}_j^L &= 0
\end{aligned} \tag{4.25}$$

2.

$$\sum_{i \notin L} p_k \mathbf{s}_k^L \mathbf{s}_k^{R^\top} = \mathbf{0} \tag{4.26}$$

3.

$$\left\| \sum_{i \notin L} p_k \left(\mathbf{s}_k^L \mathbf{s}_k^{L^\top} + \mathbf{s}_k^R \mathbf{s}_k^{R^\top} \right) \right\|_F^2 = \frac{\sum_{i \in \bar{L}(\mathbf{h})} p_i h_i}{\sigma^2 (N - |L(\mathbf{h})|)} \tag{4.27}$$

The first condition above can be seen as an equivalent to the orthogonality condition for the oversized users in the synchronous case, and second and third conditions together can be seen an equivalent to the condition for the non-oversized users.

We are now ready to characterize the jointly optimal power and sequence allocation for the asynchronous system in (4.21). Note that, if the powers are chosen as a function of the channel states, the sequence optimized ergodic sum capacity is the expectation of (4.24), and is exactly equivalent to (4.10). Thus, the power optimization can be carried out exactly as in Section 4.2, and the optimal power allocation dictates that only up to the best N users transmit simultaneously, according to a single user waterfilling policy, as in (4.19). Also, following from the synchronous case, there will not be any non-oversized users in the system, and the sequences jointly optimal with this power allocation policy need only satisfy condition 1 above. Therefore, the analysis in this chapter covers symbol asynchronous chip synchronous systems as well as symbol synchronous systems.

4.5 Joint Signature Sequence and Power Allocation for Fading CDMA Systems with Multiple Receive Antennas

Inspired by the promising recent results regarding the capacity of MIMO systems, in this section we investigate the possible sum capacity gain that may be achieved by using multiple antennas at the receiving end of the uplink of fading CDMA, where the transmitters have perfect CSI, and they choose the powers and the signature sequences as a function of the CSI. This model is well suited for the uplink of a CDMA system, since usually the transmitters are mobile devices, each of which would have a single antenna, whereas the base station may make use of multiple antennas to provide spatial diversity.

Denoting the channel gain from user i to receiver antenna j by h_{ij}, letting $\mathbf{H}_{ij} \triangleq h_{ij}$, and defining the "super signature sequences" as $\mathbf{q}_i(\mathbf{H}) \triangleq [h_{i1} \ \cdots \ h_{iM}]^\top \otimes \mathbf{s}_i(\mathbf{H})$, where $\otimes$ represents the Kronecker product, the problem can be stated as,

$$
\begin{aligned}
&\max_{\mathbf{p}(\mathbf{H}),\mathbf{S}(\mathbf{H})} E_{\mathbf{H}}\left[\log\left|\mathbf{I}_{NM} + \sum_{i=1}^{K} \frac{p_i(\mathbf{H})}{\sigma^2}\mathbf{q}_i(\mathbf{H})\mathbf{q}_i(\mathbf{H})^\top\right|\right] \\
&\text{s.t.} E_{\mathbf{H}}\left[p_i(\mathbf{H})\right] = \bar{p}_i, \quad p_i(\mathbf{H}) \geq 0
\end{aligned}
\tag{4.28}
$$

Note that, since the joint selection of the signature sequences and the powers enables us to choose the amount of power we allocate to each dimension, this problem is equivalent to choosing the covariance matrices of the transmitted vectors; however, care must be taken since there is an additional constraint on the covariance matrices, which is a unit rank constraint. Therefore, the waterfilling solution for the multiple antenna case in [10] is not applicable.

The super sequences $\mathbf{q}_i(\mathbf{H})$ have an effective processing gain of NM, while not causing any increase in the required bandwidth, thanks to the spatial diversity introduced by the channel. This fact is likely to translate into a higher sum capacity, as with higher processing gain the users can be placed further apart from each other, and the interference is reduced. However, due to the fact that the channel evolution is not under the control of the designer, the super sequences cannot be arbitrarily chosen from the entire NM dimensional space, and the problem of finding the N-dimensional sequences, which when translated by the channel will have the best placement in the signal space is rather difficult. In this section, we provide an algorithmic solution to the joint sequence and power optimization problem. Our approach is guaranteed to give increasing values of sum capacity at each iteration, but we do not provide a proof for the global optimality of the fixed point.

Inspired by the findings in Section 4.3, we propose a one-user-at-a-time iterative algorithm, which iterates between the powers and signature sequences. We first start by the power update for a given set of sequences. Then, the super signature sequences $\mathbf{q}_k$ are fixed, and the problem reduces to the one in Chapter 2, where the signature sequence $\mathbf{s}_k$ is replaced by $\mathbf{q}_k$, and the channel state is also absorbed into $\mathbf{q}_k$. Then, from (2.6),

$$C_{\text{sum}} = \frac{1}{2} E_{\mathbf{H}} \left[\log |\mathbf{B}_k| + \log \left(1 + p_k(\mathbf{H}) \mathbf{q}_k^\top \mathbf{B}_k^{-1} \mathbf{q}_k \right) \right\} \tag{4.29}$$

where

$$\mathbf{B}_k = \sigma^2 \mathbf{I}_N + \sum_{i \neq k} p_i(\mathbf{H}) \mathbf{q}_i \mathbf{q}_i^\top \tag{4.30}$$

Therefore, proceeding as in Chapter 2, for fixed sequences, the optimal power update for user k is given by

$$p_k(\mathbf{H}) = \left(\frac{1}{\tilde{\lambda}_k} - \frac{1}{\mathbf{q}_k^\top \mathbf{B}_k^{-1} \mathbf{q}_k} \right)^+ \tag{4.31}$$

Once the power of user k is updated, it is fixed for the signature sequence update. Unlike the single antenna case, the solution to the signature sequence optimization problem, even for the case of fixed fading and powers, is not known. However, there are algorithms which are known to improve the sum capacity value at each iteration for the case of a single antenna at the receiver, and these algorithms generalize to the multiple antenna case easily, as follows.

We will use the eigen-update [53] for the signature sequence of user k. Note that, in (4.29), the contribution of the signature sequence, as well as the power of user k is isolated. Defining

$$\tilde{\mathbf{Q}}_k = [h_{i1} \ \cdots \ h_{iM}]^\top \otimes \mathbf{I}_N \tag{4.32}$$

we can rewrite (4.29) explicitly as a function of the signature sequence $\mathbf{s}_k$, i.e.,

$$C_{\text{sum}} = \bar{C}_k + \frac{1}{2} E_{\mathbf{H}} \left[\log \left(1 + p_k(\mathbf{H}) \mathbf{s}_k^\top \tilde{\mathbf{Q}}_k^\top \mathbf{B}_k^{-1} \tilde{\mathbf{Q}}_k \mathbf{s}_k \right) \right] \tag{4.33}$$

where, as in Chapter 2, $\bar{C}_k$ denotes the part of the sum capacity that is independent of $\mathbf{s}_k$. Therefore, the sum capacity is maximized if the signature sequence $\mathbf{s}_k$ is chosen in the direction of the eigenvector of $\tilde{\mathbf{Q}}_k^\top \mathbf{B}_k^{-1} \tilde{\mathbf{Q}}_k$, which corresponds to the largest eigenvalue [53].

To summarize, the proposed algorithm is stated as follows: for given fixed initial signature sequences, we use the optimal single user power update (4.31) to compute the power of user k. Then for the new set of fixed powers, we find the new signature sequence of user k using the eigen-update. We iterate over the users until the algorithm converges.

Clearly, this algorithm is not guaranteed to converge to a global optimum for the problem. However, it is guaranteed to give increasing values of C_{sum}. This uncertainty exists even for the sequence only update, as its convergence to the global optimum is yet to be proved. Nevertheless, through simulations, and by direct comparison to exhaustive searches for small systems (i.e., $K = 3, N = 2, M = 2$) we have observed that the eigen-update for signature sequences always converges to the optimal sequence set, for a fixed level of fading. We observed, as in [56] that, a very high percentage of the time, the resulting sequence set consisted of orthogonal partitions, while for a small portion of channel gain values, we also clearly observed a deviation from this behavior. Based on the numerical observation that the fixed point of the algorithm always converges to the optimum, we conjecture that some recent results regarding the convergence of iterative sequence design algorithms [57, 58] would generalize to systems with multiple antennas at the receiver, and that such algorithms indeed converge to the optimum signature sequence set. Since it is a formidable task to compute and compare the channel adaptive sequence and power allocations exhaustively, we are not able to provide a similar conjecture for the iterative joint power and sequence optimization algorithm. The convergence of the iterative algorithm proposed in this section is illustrated in the numerical results section.

4.6 Simulation Results

Firstly, we simulate a system where the number of users is equal to the processing gain: $K = N = 3$. In all of our simulations, we pick $\sigma^2 = 1$, the average power of each user to be 1, the initial power distribution uniformly, and the probability distribution of the channel to be uniform on the intervals shown in figures. In this case, by our arguments in Section 4.2, we expect the optimal signature sequences to be three orthogonal sequences. Figure 4.1 shows the convergence of our algorithm, together with the convergence of the iterative waterfilling algorithm we provided for fixed sequences in Section 4.3. When we optimize the powers and signature sequences jointly, we see that the sum capacity achieved is identical to that of a system with fixed orthogonal sequences, meaning channel adaptive and non-adaptive sequence selections give us the same capacity value. The power allocation strategy corresponding to the orthogonal signature sequences found by the algorithm is independent one-user-at-a-time waterfilling for each user. Our algorithm in this case converges to the optimum in one round of iterations (one iteration for each user). The capacity achieved by a randomly generated signature sequence matrix $\mathbf{S}$ containing unit-norm sequences is also given for comparison; as expected the sum capacity for that matrix $\mathbf{S}$ is inferior to the orthogonal sequences case.

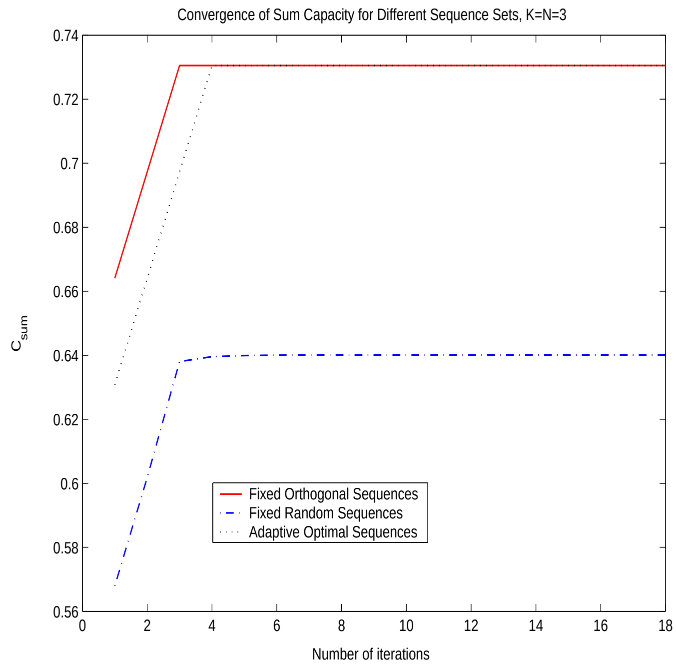


Figure 4.1: Convergence of the sum capacity for $K = N = 3$.

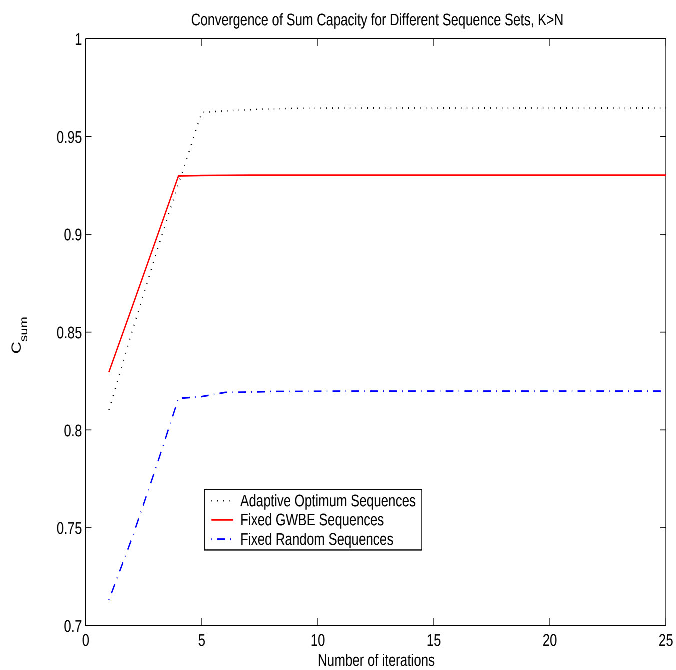
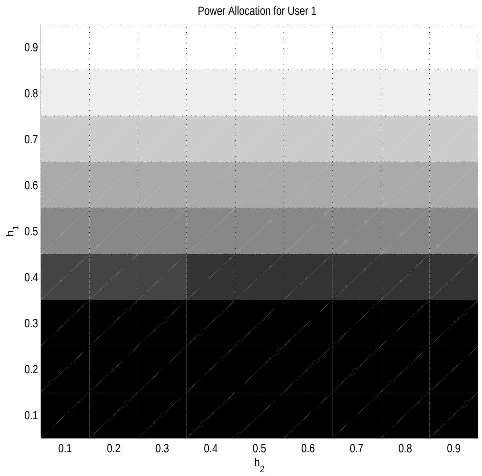


Figure 4.2: Convergence of the sum capacity for $K = 4$, $N = 3$.

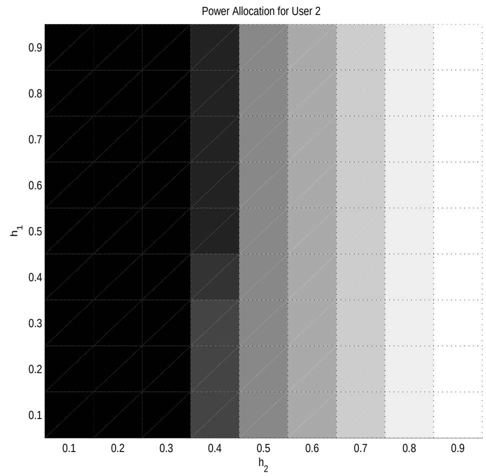
The convergence plots for a more interesting case where $K = 4$, $N = 3$ are given in Figure 4.2. Here, we again compare the capacity achieved by our algorithm to some fixed random sequences, and we see that we get a higher capacity. We also compare our result to a fixed set of WBE sequences, which are the optimum sequences for a fixed channel state and equal average received powers. The iterative waterfilling with sequence optimization again achieves a better sum capacity. Also remarkably, the transmit strategy is such that at most 3 of the 4 users transmit together (on a region with non-zero probability, after eliminating the states where the channel states of any two users are equal), and they are allocated orthogonal sequences. Figures 4.3(a)-4.3(d) further illustrate the details of the power and signature sequence allocation.

Figures 4.3(a) and 4.3(b) both pertain to a plane in 4 dimensional channel state space, where we pick $h_3 = h_4 = 0.4$, and observe the power distribution of users 1 and 2 as a function of their fading states. The gray levels correspond to the amount of power allocated, lighter colors indicating more power. Clearly, the users perform single user waterfilling for the chosen channel states, and their powers do not depend on fading states and powers of each other. As $h_3 = h_4 = 0.4$, from (4.14) we expect that users 1 and 2 would transmit when their channels are better than 0.4, with orthogonal sequences, and hence the single user waterfilling, which is what we observe. Note that, according to the notion in [20], users 1 and 2 are oversized whenever their channel gains are better than 0.4.

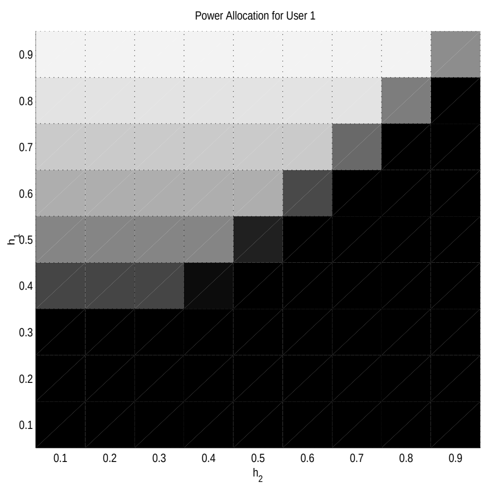
Figures 4.3(c) and 4.3(d) correspond to a case where we pick the maximum possible values for the channel states h_3 and h_4, i.e., $h_3 = h_4 = 0.9$, so that except for the degenerate equality cases, users 3 and 4 will always be oversized on the plane of channel states we consider. Then, the remaining user, according to our results, should transmit if and only if it has the next best channel (note that since channels are all



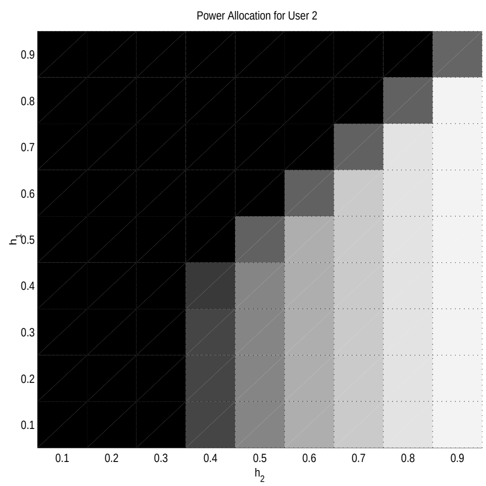
(a) Power distribution of user 1 with $h_3 = h_4 = 0.4$.



(b) Power distribution of user 2 with $h_3 = h_4 = 0.4$.



(c) Power distribution of user 1 with $h_3 = h_4 = 0.9$.



(d) Power distribution of user 2 with $h_3 = h_4 = 0.9$.

Figure 4.3: Cross sections of power distributions for users 1 and 2.

taken to be identically distributed, the λ_is are the same for all users and they do not effect the ordering). This is what is observed in Figures 4.3(c) and 4.3(d), the stronger of users 1 and 2 perform single user waterfilling, and the weaker one remains silent, as in the Knopp-Humblet [6] solution. The arbitrariness in powers in equal channels case is again observed, and is consistent with our previous arguments.

Finally, we simulate the iterative algorithm presented in Section 4.5. Figure 4.4

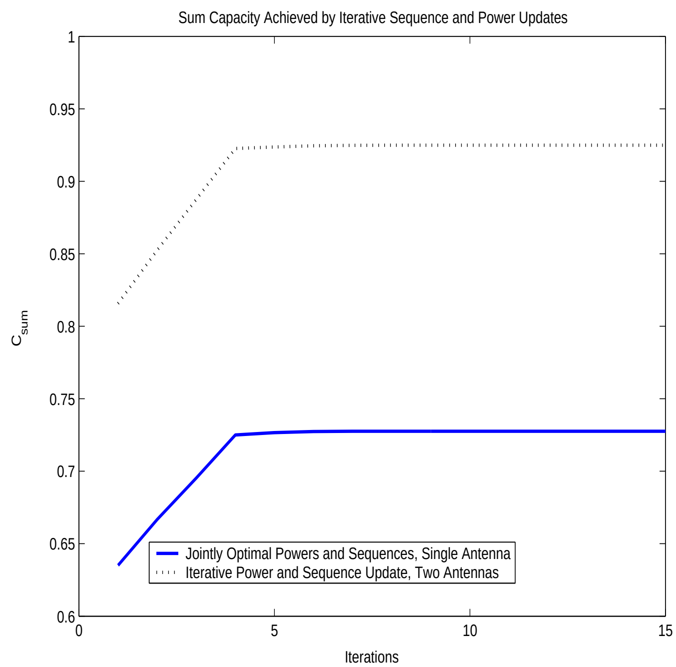


Figure 4.4: Convergence of sum capacity for a fading CDMA system equipped with multiple antennas at the receiver, $K = 3$, $N = 2$, $M = 2$.

illustrates the convergence of the iterative algorithm for a system with $K = 3$, $N = 2$, $M = 2$. The fading is assumed to be uniformly distributed in $\{0.3,0.6,0.9\}$ in both single and two antenna cases. The use of multiple antennas improves the capacity significantly, due to the extra spatial dimension provided.

4.7 Summary and Conclusions

For a CDMA system subject to fading, we showed that the ergodic sum capacity is maximized by allocating orthogonal signature sequences to $\min(N, K)$ of the users with *favorable* channel states, and allocating powers to those users by a single user waterfilling strategy over some partitions of channel state space. In each partition, a group of users perform orthogonal transmissions, thus the users avoid any interference from each other in order to maximize the sum capacity.

A significant property of the jointly optimal power and signature sequence allocation policy is that although we have started with assuming all users have perfect CSI, including the channel states of other users, the solution shows that it is in fact sufficient if a particular user knows its own channel state, and whether its channel state is one of the top N channel states. Therefore, the optimum policy can be easily implemented with a minor feedback from the base station, indicating each transmitter whether it should transmit, and if so, with what signature sequence. The power allocation is also significantly simpler than the case of power control only, since by the virtue of orthogonal transmissions, there is no multiaccess interference in the system, and the users do not need to have any SIR information.

We also proposed an iterative signature-update/power-waterfilling algorithm to find the optimal allocation of signature sequences and powers, and proved its convergence to the globally optimum solution.

We have extended the results for the symbol synchronous CDMA to the asynchronous case. The sum capacity of the asynchronous CDMA is equivalent to that of its synchronous counterpart when the signature sequences are optimized, and the power allocation policy that achieves this capacity is also the same for both systems. The sequence allocation policy that is jointly optimum with the established power allocation policy is also obtained, and involves orthogonality like conditions involving left and right signature sequences.

Finally we have investigated the problem of maximizing the sum capacity of a fading CDMA channel with multiple antennas at the receiver, and we developed an iterative algorithm that is a combination of the iterative waterfillng algorithm of Chapter 2, and the eigen-update for sequence optimization.

The results of this chapter have been published in part in [22, 23].

Chapter 5

Power Control for Fading Multiple Access Channels with User Cooperation

5.1 Introduction

Increasing demand for higher rates in wireless communication systems have recently triggered major research efforts to characterize the capacities of such systems. The wireless medium brings along its unique challenges such as fading and multiuser interference, which make the analysis of the communication systems more complicated. On the other hand, the same challenging properties of such systems are what give rise to the concepts such as diversity, over-heard information, etc., which can be carefully exploited to the advantage of the network capacity.

In the early 1980s, several problems which form a basis for the idea of user cooperation in wireless networks were solved. First, the case of a two user MAC where both users have access to the channel output was considered by Cover and Leung [28], and an achievable rate region was obtained for this channel. Willems and van der Meulen then demonstrated [29] that the same rate region is achievable if there is a feedback link to only one of the tansmitters from the channel output.

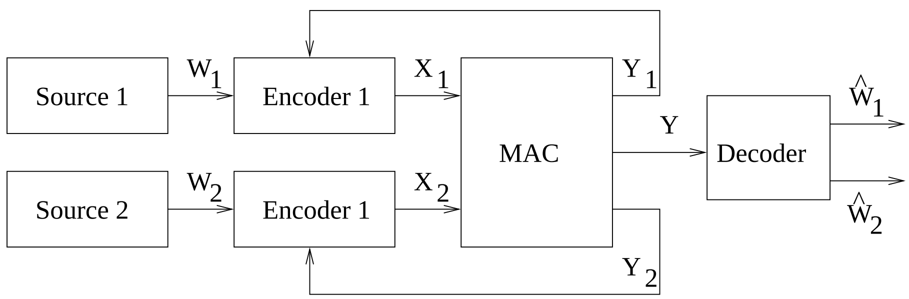


Figure 5.1: Multiple access channel with generalized feedback.

The capacity region of the MAC with partially cooperating encoders was obtained by Willems in [30]. In this setting, the encoders are assumed to be connected by finite capacity communication links, which allow the cooperation. Willems and van der Meulen also considered a limiting case of cooperation where the encoders "crib" from each other, that is, they learn each others' codewords before the next transmission [31]. Several scenarios regarding which encoder(s) crib, and how much of the codewords the encoders learn are treated, and the capacity region for each case is obtained in [31]. The capacity of such channels are an upper bound to the rates achievable by cooperative schemes, since in the case of cribbing encoders, the sharing of information comes for free, i.e., the transmitters do not allocate any resources such as powers, to establish a common information.

An achievable rate region for a MAC with generalized feedback was found in [32]. This channel model, which is illustrated in Figure 5.1 [32] is worth special attention as far as the wireless channels are concerned, since it models the over-heard information by the transmitters. In particular, for a two user discrete memoryless MAC with generalized feedback described by $(\mathcal{X}_1 \times \mathcal{X}_2, P(y, y_1, y_2|x_1, x_2), \mathcal{Y} \times \mathcal{Y}_1 \times \mathcal{Y}_2)$, where user 1 has access to channel output Y_1 and user 2 has access to channel output Y_2, the encoding functions were defined in [32] as

$$
\begin{aligned}
X_{1n} =& f_{1n}(W_1, Y_1^{n-1}), \\
X_{2n} =& f_{2n}(W_2, Y_2^{n-1}), \quad n = 1, \cdots, N
\end{aligned}
\tag{5.1}
$$

and the decoding function was defined as

$$
(\hat{W}_1, \hat{W}_2) = g(Y^N) \tag{5.2}
$$

where N denotes the total number of channel uses, and Y_1^n denotes the vector $[Y_{11}, Y_{12}, \cdots, Y_{1n}]$, etc. Then, an achievable rate region was obtained by using a superposition block Markov encoding scheme, together with backward decoding, where the receiver waits to receive all B blocks of codewords before decoding.

Recently, Sendonaris, Erkip and Aazhang have successfully employed the results of these rather general problems, particularly that of generalized feedback, to a Gaussian MAC in the presence of fading, leading to user cooperation diversity and higher rates [33]. In this setting, both the receiver and the transmitters receive noisy versions of the messages of each other, and slightly modifying the basic relay channel case, the transmitters form their codewords not only based on their own information, but also on the information they have received from each other. It is assumed in [33] that channel state information for each link is known to the corresponding receiver on that link, and also phase of the channel state needs to be known at the transmitters in order to obtain a coherent combining gain. The achievable rate region is shown to improve significantly over the capacity region of MAC with non-cooperating transmitters, especially when the channel between the two users is relatively good on average.

There have also been some recent work on user cooperation systems under various assumptions on the available channel state information, and the level of cooperation among the users. Laneman, Tse and Wornell [59] have characterized the outage probability behavior for a system where the users are allowed to cooperate only in half-duplex mode, and where no CSI is available at the transmitters. For the relay channel, which is a special one-sided case of user cooperation, Host-Madsen and Zhang [60] have solved for power allocation policies that optimize some upper and lower bounds on the ergodic capacity when perfect channel state information is available at the transmitters and the receiver. For a user cooperation system with finite capacity cooperation links, Erkip [61] has proposed a suboptimal solution to the problem of maximizing the sum rate in the presence of full channel state information, where it was also noted that the resulting optimization problem is non-convex.

In this chapter, we consider a two user fading cooperative Gaussian MAC with complete CSI at the transmitters and the receiver, and average power constraints on the transmit powers. Note that, this requires only a small quantity of additional feedback, namely the amplitude information on the forward links, over the systems requiring coherent combining [33]. In this case, the transmitters can adapt their coding strategies as a function of the channel states, by adjusting their transmit powers [5, 7]. We characterize the optimal power allocation policies which maximize the set of ergodic rates achievable by block Markov superposition coding. To this end, we first prove that the seemingly non-concave optimization problem of maximizing the achievable rates can be reduced to a concave problem, by noting that some of the

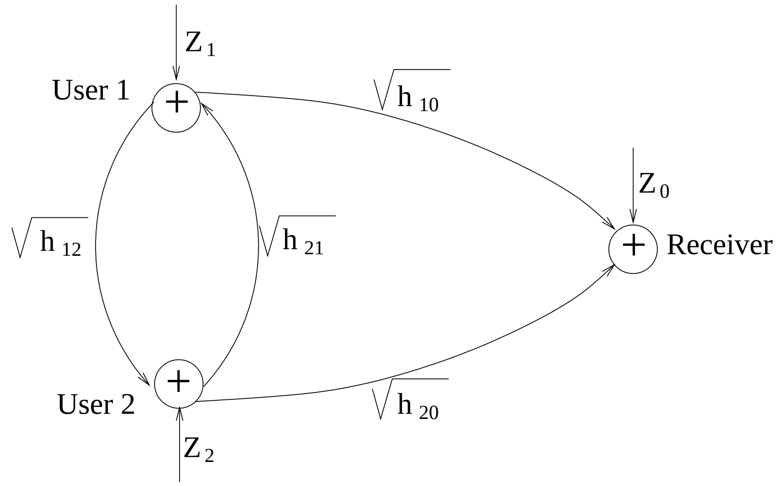


Figure 5.2: Two user fading cooperative MAC.

transmit power levels are essentially zero at every channel state, which reduces the dimensionality of the problem. By this, we also show that the block Markov superposition coding strategy proposed in [32] and employed in [33] for a Gaussian channel can be simplified considerably by making use of the CSI. Due to the non-differentiable nature of the objective function, we use sub-gradient methods to obtain the optimal power distributions that maximize the achievable rates, and we provide the corresponding achievable rate regions for various fading distributions. We demonstrate that controlling the transmit powers in conjunction with user cooperation provides significant gains over the existing rate regions for cooperative systems.

5.2 System Model

We consider a two user fading Gaussian MAC, where both the receiver and the transmitters receive noisy versions of the transmitted messages, as illustrated in Figure 5.2. The system is modelled by,

$$Y_0 = \sqrt{h_{10}} X_1 + \sqrt{h_{20}} X_2 + Z_0 \tag{5.3}$$

$$Y_1 = \sqrt{h_{21}} X_2 + Z_1 \tag{5.4}$$

$$Y_2 = \sqrt{h_{12}} X_1 + Z_2 \tag{5.5}$$

where X_i is the symbol transmitted by node i, Y_i is the symbol received at node i, and the receiver is denoted by $i = 0$. Z_i is the zero-mean additive white Gaussian noise at node i, having variance σ_i^2, $\sqrt{h_{ij}}$ are the random fading coefficients, the instantaneous realizations of which are assumed to be known by both the transmitters and the receiver. We assume that the channel variation is slow enough so that the fading parameters can be tracked accurately at the transmitters, yet fast enough to ensure that the long term ergodic properties of the channel are observed within the blocks of transmission [35].

The transmitters are capable of making decoding decisions based on the signals they receive and thus can form their transmitted codewords not only based on their own information, but also based on the information they have received from each other. This channel model is a special case of the MAC with generalized feedback [32]. The achievable rate region is obtained by using a superposition block Markov encoding scheme, together with backward decoding, where the receiver waits to receive all B blocks of codewords before decoding. For the Gaussian case, the superposition block Markov encoding is realized as follows [33]: the transmitters allocate some of their powers to establish some common information in every block, and in the next block, they coherently combine part of their transmitted codewords. In the presence of channel state information, by suitably modifying the coding scheme given by [33] to accommodate for channel adaptive coding strategies, the encoding is performed by

$$X_i = \sqrt{p_{i0}(\mathbf{h})} X_{i0} + \sqrt{p_{ij}(\mathbf{h})} X_{ij} + \sqrt{p_{U_i}(\mathbf{h})} U_i \tag{5.6}$$

for $i, j \in \{1, 2\}$, $i \neq j$, where X_{i0} carries the fresh information intended for the receiver, X_{ij} carries the information intended for transmitter j for cooperation in the next block, and U_i is the common information sent by both transmitters for resolution of the remaining uncertainty from the previous block, all chosen from unit power Gaussian distributions. All the transmit power is therefore captured by the power levels associated with each component, i.e., $p_{i0}(\mathbf{h})$, $p_{ij}(\mathbf{h})$ and $p_{U_i}(\mathbf{h})$, which are required to satisfy average power constraints,

$$E\left[p_{i0}(\mathbf{h}) + p_{ij}(\mathbf{h}) + p_{U_i}(\mathbf{h})\right] = E[p_i(\mathbf{h})] \leq \bar{p}_i, \quad i = 1, 2. \tag{5.7}$$

Following the results in [33], it can be shown that the achievable rate region is given by the convex hull of all rate pairs satisfying,

$$R_1 < E\left[\log\left(1 + \frac{h_{12}p_{12}(\mathbf{h})}{h_{12}p_{10}(\mathbf{h}) + \sigma_2^2}\right) + \log\left(1 + \frac{h_{10}p_{10}(\mathbf{h})}{\sigma_0^2}\right)\right] \tag{5.8}$$

$$R_2 < E\left[\log\left(1 + \frac{h_{21}p_{21}(\mathbf{h})}{h_{21}p_{20}(\mathbf{h}) + \sigma_1^2}\right) + \log\left(1 + \frac{h_{20}p_{20}(\mathbf{h})}{\sigma_0^2}\right)\right] \tag{5.9}$$

$$\begin{aligned} R_1 + R_2 < \min \Bigg\{ & E\left[\log\left(1 + \frac{h_{10}p_1(\mathbf{h}) + h_{20}p_2(\mathbf{h}) + 2\sqrt{h_{10}h_{20}p_{U_1}(\mathbf{h})p_{U_2}(\mathbf{h})}}{\sigma_0^2}\right)\right], \\ & E\Bigg[\log\left(1 + \frac{h_{10}p_{10}(\mathbf{h}) + h_{20}p_{20}(\mathbf{h})}{\sigma_0^2}\right) + \log\left(1 + \frac{h_{12}p_{12}(\mathbf{h})}{h_{12}p_{10}(\mathbf{h}) + \sigma_2^2}\right) \\ & + \log\left(1 + \frac{h_{21}p_{21}(\mathbf{h})}{h_{21}p_{20}(\mathbf{h}) + \sigma_1^2}\right)\Bigg]\Bigg\} \end{aligned} \tag{5.10}$$

where the convex hull is taken over all power allocation policies that satisfy (5.7).

For a given power allocation, the rate region in (5.8)-(5.10) is either a pentagon or a triangle, since, unlike the traditional MAC, the sum rate constraint in (5.10) may dominate the individual rate constraints completely. The achievable rate region may alternatively be represented as the convex hull of the union of all such regions. Our goal is to find the power allocation policies that maximize the rate tuples on the rate region boundary.

5.3 Structure of the Sum Rate and the Optimal Policies

We first consider the problem of optimizing the sum rate of the system, as it will also shed some light onto the optimization of an arbitrary point on the rate region boundary. The sum rate (5.10) is not a concave function of the vector of variables $\mathbf{p}(\mathbf{h}) = [p_{10}(\mathbf{h})\ p_{12}(\mathbf{h})\ p_{U_1}(\mathbf{h})\ p_{20}(\mathbf{h})\ p_{21}(\mathbf{h})\ p_{U_2}(\mathbf{h})]$, due to the variables in the denominators. In what follows, we show that for the sum rate to be maximized, for every given $\mathbf{h}$, at least two of the four components of $[p_{10}(\mathbf{h})\ p_{12}(\mathbf{h})\ p_{20}(\mathbf{h})\ p_{21}(\mathbf{h})]$ should be equal to zero, which reduces the dimensionality of the problem and yields a concave optimization problem.

Theorem 5.1 *Let the effective channel gains normalized by the noise powers be defined as $s_{ij} = h_{ij}/\sigma_j^2$. Then, for the power control policy $\mathbf{p}^*(\mathbf{h})$ that maximizes (5.10), we need*

1. $p_{10}^*(\mathbf{h}) = p_{20}^*(\mathbf{h}) = 0$, *if* $s_{12} > s_{10}$ *and* $s_{21} > s_{20}$

2. $p_{10}^*(\mathbf{h}) = p_{21}^*(\mathbf{h}) = 0$, *if* $s_{12} > s_{10}$ *and* $s_{21} \leq s_{20}$

3. $p_{12}^*(\mathbf{h}) = p_{20}^*(\mathbf{h}) = 0,\ \textit{if}\ s_{12} \leq s_{10}\ \textit{and}\ s_{21} > s_{20}$

4. $$\left.\begin{array}{c} p_{12}^*(\mathbf{h}) = p_{21}^*(\mathbf{h}) = 0 \\ \textit{OR} \\ p_{10}^*(\mathbf{h}) = p_{21}^*(\mathbf{h}) = 0 \\ \textit{OR} \\ p_{12}^*(\mathbf{h}) = p_{20}^*(\mathbf{h}) = 0 \end{array}\right\} \textit{if}\ s_{12} \leq s_{10}\ \textit{and}\ s_{21} \leq s_{20}$$

Proof: To simplify the notation, let us drop the dependence of the powers on the channel states, whenever such dependence is obvious from the context. Let $p_i = p_{i0} + p_{ij} + p_{U_i}$ be the total power allocated to a given channel state. Let us define

$$A = 1 + s_{10}p_1 + s_{20}p_2 + 2\sqrt{s_{10}s_{20}p_{U_1}p_{U_2}} \tag{5.9}$$

$$B = \frac{1 + s_{10}p_{10+}s_{20}p_{20}}{(1 + s_{12}p_{10})(1 + s_{21}p_{20})} \tag{5.10}$$

$$C = (1 + s_{12}(p_{10} + p_{12}))\,(1 + s_{21}(p_{20} + p_{21})) \tag{5.11}$$

Then, an equivalent representation of the sum rate (5.10) is

$$R_{\text{sum}} = \min\left\{E\left[\log(A)\right], E\left[\log(BC)\right]\right\} \tag{5.12}$$

Now, let us arbitrarily fix the total power level, p_i, as well as the power level used for cooperation signals, $p_{U_i} < p_i$, allocated to a given state for each user. For each such allocation, the quantities A and C appearing in the sum-rate expression are fixed, i.e., allocating the remaining available power $p_i - p_{U_i}$ among p_{i0} and p_{ij} will not alter these quantities. Note that, such allocation also does not alter the total power consumption at the given state, so we may limit our attention to the maximization,

$$\max_{\{p_{10},p_{20}\}} \quad B\left(p_{10},p_{20}\right)$$

$$\text{s.t.} \quad p_{10}+p_{12}=p_1-p_{U_1}$$

$$p_{20}+p_{21}=p_2-p_{U_2} \tag{5.13}$$

The partial derivatives of B with respect to p_{10} and p_{20} are

$$\frac{\partial B}{\partial p_{10}} = \frac{s_{10}-s_{12}(1+s_{20}p_{20})}{(1+s_{12}p_{10})^2(1+s_{21}p_{20})} \tag{5.14}$$

$$\frac{\partial B}{\partial p_{20}} = \frac{s_{20}-s_{21}(1+s_{10}p_{10})}{(1+s_{21}p_{20})^2(1+s_{12}p_{10})} \tag{5.15}$$

1. $s_{12} > s_{10}$, $s_{21} > s_{20}$. Then, $\frac{\partial B}{\partial p_{10}} < 0$ and $\frac{\partial B}{\partial p_{20}} < 0$, i.e., $B(p_{10},p_{20})$ is monotonically decreasing in both p_{10} and p_{20}, therefore the sum rate is maximized at $p_{10} = p_{20} = 0$.

2. $s_{12} > s_{10}$, $s_{21} \leq s_{20}$. Then, $\frac{\partial B}{\partial p_{10}} < 0$, and the function is maximized at $p_{10} = 0$ for any p_{20}. But this gives $\frac{\partial B}{\partial p_{20}}|_{p_{10}=0} > 0$, meaning p_{20} should take its maximum possible value, i.e., $p_{21} = 0$.

3. $s_{12} \leq s_{10}$, $s_{21} > s_{20}$. Follows the same lines of case 2) with roles of user 1 and 2 reversed.

4. $s_{12} \leq s_{10}$, $s_{21} \leq s_{20}$. In this case, the partial derivatives of B can be both made equal to zero within the constraint set, yielding a critical point. However, using higher order tests, it is possible to show that this solution corresponds to a saddle point, and B is again maximized at one of the boundaries, $p_{10} = 0$, $p_{20} = 0$, $p_{10} = p_1 - p_{U_1}$, $p_{20} = p_2 - p_{U_2}$. Inspection of the gradient on these boundary points

yields one of the three corner points $\{(p_1 - p_{U_1}, 0), (0, p_2 - p_{U_2}), (p_1 - p_{U_1}, p_2 - p_{U_2})\}$ as candidates, each of which corresponds to one of the solutions in case 4). Although two of the components of the power vector are guaranteed to be equal to zero, which ones will be zero depends on the p_i and p_{U_i} that we fixed, therefore we are not able to completely specify the solution, independent of p_i and p_{U_i}, in this case. On the other hand, the settings of interest to us are those where the channels between the cooperating users are on the average much better than their direct links, since it is in these settings when cooperative diversity yields high capacity gains [33]. In such scenarios, the probability of both users' direct link gains exceeding their corresponding cooperation link gains (case 4) is a very low probability event. Therefore, which of the three possible operating points is chosen is not of practical importance, and we can safely fix the power allocation policy to one of them to carry on with our optimization problem for other variables. Although admittedly this argument is likely to cause some suboptimality in our scheme, as will be seen in the numerical examples, we still obtain a significant gain in the achievable rates.

$\square$

The significance of this result is two-fold. Firstly, given a channel state, it greatly simplifies the well known block Markov coding, in a very intuitive way: if the direct links of both users are inferior to their cooperation links, the users do not transmit direct messages to the receiver as a part of their codewords, and they use each other as relays. If one of the users' direct channel is better than its cooperation channel, and the other user is in the opposite situation, then the user with the strong direct channel chooses to transmit directly to the receiver, while the weaker direct channel user still chooses to relay its information over its partner. Second important implication of this result is that it now makes the problem of solving for the optimal power allocation policy more tractable, since it shrinks the constraint set on the variables, and more importantly, it makes the sum rate a concave function over the reduced set of constraints and variables.

Corollary 5.1 *The sum rate R_{sum} given by (5.10), (5.12) is a concave function of* $\mathbf{p}(\mathbf{h})$*, over the reduced constraint set described by Theorem 5.1.*

Proof: The proof of this result follows from directly substituting the zero power components into the sum rate expression in (5.12). Note that in each of the four cases, the second function in the minimization, i.e., $\log(BC)$ takes either the form $\log(1+a)+\log(1+b)$, or $\log(1+a+b)$, both of which are clearly concave in a and b. Also, $\log(A)$ is clearly a concave function of $\mathbf{p}(\mathbf{h})$ since it is a composition of a concave function with the concave and increasing logarithm. The desired result is obtained by noting that the minimum of two concave functions is concave. □

Thus far we have discussed the structure of the sum rate, as well as some properties of the optimal power allocation that maximizes that rate. We now turn back to the problem of maximizing other rate points on the rate region boundary. To this end, we point out another remarkable property of the solution in Theorem 5.1. Consider maximizing the bound on R_1 in (5.8). For fixed p_{U_1} and p_1, it is easy to verify that all of the available power should be allocated to the channel with the higher gain, i.e., if $s_{12} > s_{10}$, then we need $p_{10} = 0$ and $p_{12} = p_1 - p_{U_1}$. The same result also applies to R_2. But this shows that, the policies described in Theorem 5.1 completely agree with optimal policies for maximizing the individual rate constraints in cases 1)-3), and they also agree if we choose the operating point in case 4) to be $p_{12} = 0$, $p_{21} = 0$. Therefore, the allocation in Theorem 5.1 enlarges the entire rate region in all directions, except for the subtlety in case 4) for the sum rate. This has the benefit that the weighted sum of rates, say $R_{\boldsymbol{\mu}} = \mu_1 R_1 + \mu_2 R_2$ also has the same concavity properties of the sum rate, since for $\mu_i > \mu_j$, the weighted sum of rates can be written as $R_{\boldsymbol{\mu}} = \mu_j R_{\text{sum}} + (\mu_i - \mu_j) R_i$, where both R_{sum} and R_i are concave. Optimum power control policies that achieve the points on the boundary of the achievable rate region can then be obtained by maximizing the weighted sum of rates, which is the goal of the next section.

5.4 Rate Maximization via Subgradient Methods

In this section we focus on maximizing the weighted sum of rates. To illustrate both the results of the preceding section and the problem statement for this section more precisely, let us consider, without loss of generality, the case when $\mu_1 \geq \mu_2$, and write down the optimization problem explicitly:

$$
\begin{aligned}
\max_{\mathbf{p}(\mathbf{h})}\ & (\mu_1-\mu_2)\Big\{E_{1,2}\left[\log(1+p_{12}(\mathbf{h})s_{12})\right] \\
& + E_{3,4}\left[\log(1+p_{10}(\mathbf{h})s_{10})\right]\Big\} + \mu_2 \min\Big\{E[\log(A)], \\
& + E_1\left[\log(1+p_{12}(\mathbf{h})s_{12}) + \log(1+p_{21}(\mathbf{h})s_{21})\right] \\
& + E_2\left[\log(1+p_{12}(\mathbf{h})s_{12}) + \log(1+p_{20}(\mathbf{h})s_{20})\right] \\
& + E_3\left[\log(1+p_{10}(\mathbf{h})s_{10}) + \log(1+p_{21}(\mathbf{h})s_{21})\right] \\
& + E_4\left[\log(1+p_{10}(\mathbf{h})s_{10} + p_{20}(\mathbf{h})s_{20})\right]\Big\} \\
\text{s.t. } & E_{3,4}\left[p_{10}(\mathbf{h})\right] + E_{1,2}\left[p_{12}(\mathbf{h})\right] + E\left[p_{U_1}\right] \le \bar{p}_1 \\
& E_{2,4}\left[p_{20}(\mathbf{h})\right] + E_{1,3}\left[p_{21}(\mathbf{h})\right] + E\left[p_{U_2}\right] \le \bar{p}_2
\end{aligned}
\tag{5.16}
$$

where, E_S denotes the expectation over the event that case $S \subset \{1,2,3,4\}$ from Theorem 5.1 occurs, and A is as given by (5.9). Note that the objective function is concave, and the constraint set is convex, therefore we can conclude that any local optimum for the constrained optimization problem is a global optimum. However, it is not possible to characterize the optimal allocation using standard approaches such as employing Lagrangian optimization and Karush-Kuhn-Tucker conditions, nor it is possible to resort to algorithms such as gradient ascent, because of the non-differentiable nature of the objective function. Although $R_{\boldsymbol{\mu}}$ is differentiable almost everywhere since it is concave, its optimal value is attained along the discontinuity of its gradient, namely when the two arguments of the minimum operation in (5.10) are equal. Hence, we solve the optimization problem using the method of subgradients from non-differentiable optimization theory [62, 63].

The subgradient methods are very similar to gradient ascent methods in that whenever the function is differentiable (in our case almost everywhere), the subgradient is equivalent to the gradient. However, their major difference from gradient ascent methods is that they are not necessarily monotonically non-decreasing. A subgradient for a concave function $f(\mathbf{x})$ is any vector $\boldsymbol{\gamma}$ that satisfies [63],

$$f(\mathbf{x}') \leq f(\mathbf{x}) + (\mathbf{x}' - \mathbf{x})^\top \boldsymbol{\gamma} \tag{5.17}$$

and the subgradient method for constrained maximization uses the update [63]

$$\mathbf{x}(k+1) = [\mathbf{x}(k) + \alpha_k \boldsymbol{\gamma}_k]^+ \tag{5.18}$$

where $[\cdot]^+$ denotes the Euclidian projection on the constraint set, and α_k is the step size at iteration k. There are various ways to choose α_k to guarantee convergence of these methods to the global optimum; for our particular problem, we choose the diminishing stepsize, normalized by the norm of the subgradient to ensure convergence [62]

$$\alpha_k = \frac{a}{b + \sqrt{k}} \frac{1}{\parallel \boldsymbol{\gamma} \parallel} \tag{5.19}$$

5.5 Simulation Results

In this section we provide some numerical examples to illustrate the performance of the proposed joint power allocation and cooperation scheme.

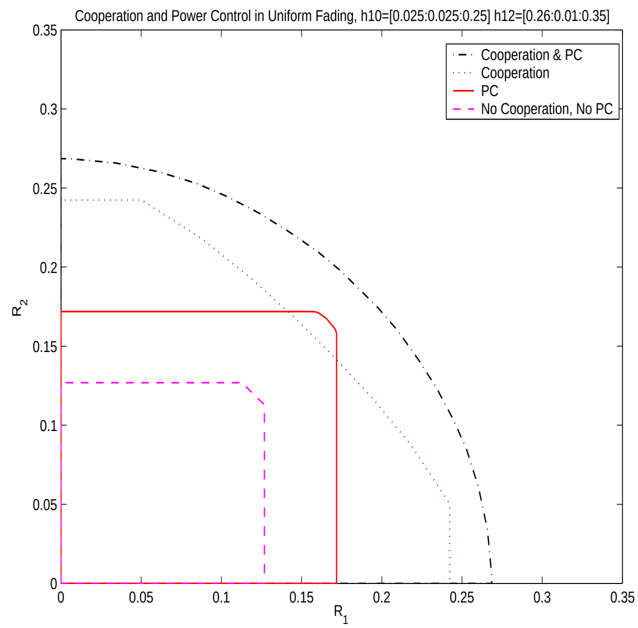


Figure 5.3: Rates achievable by joint power control and user cooperation for uniform fading.

Figure 5.3 illustrates the achievable rate region we obtain for a system with $\bar{p}_i = \sigma_i^2 = 1$, subject to uniform fading, where the links from the transmitters to the receiver are symmetric and take values from the set $\{0.025, 0.050, \cdots, 0.25\}$, each with probability $1/10$, while the link among the transmitters is also symmetric and uniform, and takes the values $\{0.26, 0.27, \cdots, 0.35\}$. Notice that here, we have intentionally chosen the fading coefficients such that the cooperation link is always better than the direct links, therefore, the system operates only in case 1) of Theorem 5.1. Consequently, in this particular case, our power allocation scheme is actually the optimal power allocation policy for the block Markov superposition encoding scheme (i.e., case 4) never happens).

The region for joint power control and cooperation is generated using the subgradient method with parameters $a = 50$ and $b = 5$. We carried out the optimization for

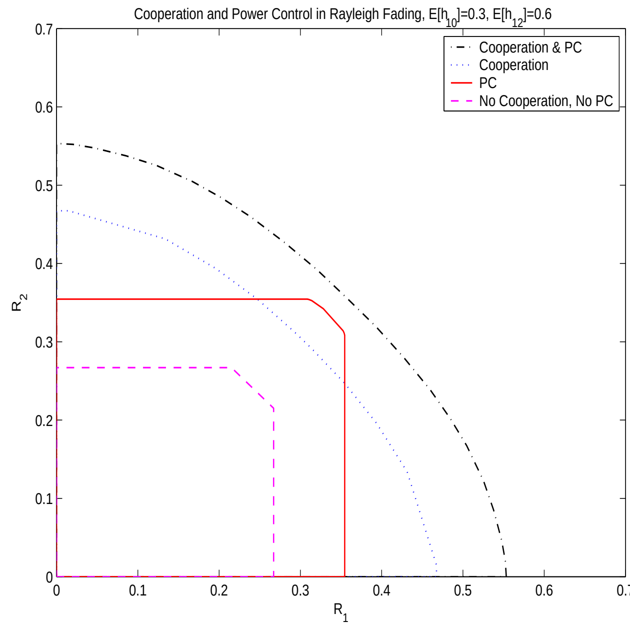


Figure 5.4: Rates achievable by joint power control and user cooperation for Rayleigh fading.

various values of the priorities μ_i of the users, each of which give a point on the rate region boundary, and then we performed a convex hull operation over these points. We observe that power control by itself improves on the rate region of the cooperative system with no power control, for rate pairs close to the sum rate, by utilizing the direct link more efficiently. Joint user cooperation and power control scheme significantly improves on all other schemes, as it takes advantage of both cooperation diversity and time diversity in the system. In fact, we can view this joint diversity utilization as adaptively performing coding, medium accessing and routing, thereby yielding a cross-layer approach for the design of the communication system.

Figure 5.4 also corresponds to a system with unit SNR, but this time subject to Rayleigh fading, i.e., the power gains to the receivers are exponential random variables, with $E[h_{10}] = E[h_{20}] = 0.3$, $E[h_{12}] = E[h_{21}] = 0.6$. In this setting, all

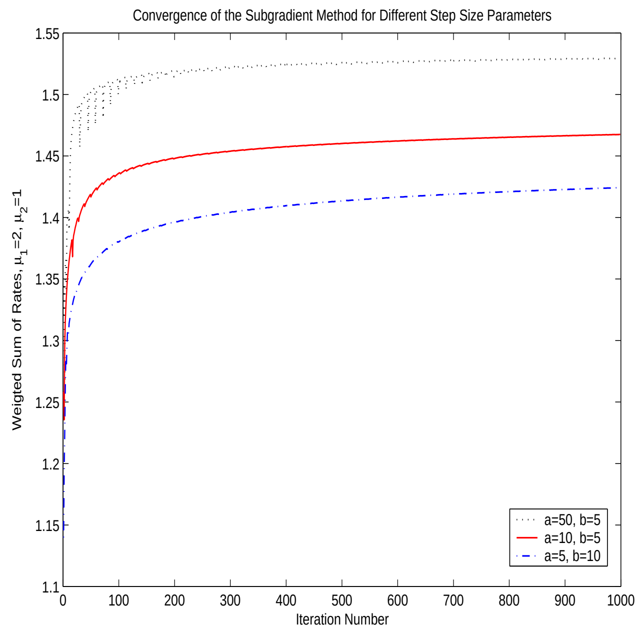


Figure 5.5: Convergence of $R_{\boldsymbol{\mu}}$ using subgradient method for different step size parameters.

four cases in Theorem 5.1 are realized, and there is potentially some loss over the optimally achievable rates. However, we obtain a very similar set of rate regions to the uniform case, indicating that in fact the loss, if any, is very small thanks to the very low probability of both of the direct links outperforming the cooperation link.

It is interesting to note in both Figures 5.3 and 5.4 that cooperation with power control improves relatively less over power control only near the sum capacity. This can be attributed to the fact that, for the traditional MAC, the sum rate is achieved by time division among the users, which does not allow for coherent combining gain [6]. Therefore, it is not surprising to see that in order to attain cooperative diversity gain, users may have to sacrifice some of the gain they obtain from exploiting the time diversity.

In Figure 5.5, we illustrate the convergence of the subgradient method. The objective function is $R_{\boldsymbol{\mu}}$ with $\mu_1 = 2$ and $\mu_2 = 1$, and the step size parameters are varied. We observe that, by choosing larger step sizes, the non-monotonic behavior of the subgradient algorithm becomes more apparent, however the convergence is significantly faster than the smaller step sizes, as the algorithm is more likely to get near the optimal value of the function in the initial iterations. Note that, in our simulations we terminated the algorithm after 1000 iterations, and the three curves would eventually converge after sufficiently large number of iterations.

5.6 Summary and Conclusions

We have addressed the problem of optimal power allocation for a fading cooperative MAC, where the transmitters and the receiver have CSI, and are therefore able to adapt their coding and decoding strategies by allocating their resources. We have characterized the power control policies that maximize the rates achievable by block Markov superposition coding, and proved that, in the presence of CSI, the coding strategy is significantly simplified: given any channel state, for each of the users, one of the three components, i.e., those that are intended for the receiver, for the other transmitter, and for cooperation, should be allocated zero power at that channel state. This result also enabled us to formulate the otherwise non-concave problem of maximizing the achievable rates as a concave optimization problem. The power control policies, which are jointly optimal with block Markov coding, were obtained using subgradient method for non-differentiable optimization. The resulting achievable rate regions for joint power control and cooperation improve significantly on cooperative systems without power control, since our joint approach makes use of both cooperative diversity and time diversity.

The results of this chapter have been published in [34].

Chapter 6

Conclusions

The rapidly growing demand for wireless communication systems that support higher data rates and larger number of users brings along the need for in-depth research that investigates the theoretical performance limits of such systems, and searches for policies and algorithms that can achieve or approach those limits in practice. As in many engineering problems, the resources in a wireless communication system are extremely scarce, due to physical constraints such as limited battery power and available bandwidth.

In this thesis we have addressed the fundamental problem of characterizing the ultimate capacity limits of practical wireless systems, and optimally allocating the available system resources to achieve such limits. The main focus of the thesis has been the information theoretic analysis of vector MACs, and specifically CDMA channels, that are subject to fading due to the several scatterers and multi-paths in the transmission medium. The results in the thesis owe to the synthesis of several methods from information theory, estimation and detection theory, optimization theory, parallel and distributed computing, matrix analysis, probability and statistics. The main contributions can be briefly summarized as follows.

- **Sum capacity and optimum power allocation for fading CDMA**

  The future generation wireless networks are bound to be a transfer medium for high rate data traffic, in addition to the traditional voice traffic. Data traffic (email, file transfers etc.) is less delay sensitive, thereby allowing more flexibility in transmit scheduling based on the quality of the channel, and average performance metrics become more meaningful than instantaneous performance metrics. Motivated by this, part of this dissertation characterizes the optimum power allocation policy that maximizes the *average sum capacity* of a fading CDMA channel, subject to average power constrains on the mobile stations.

  The optimum policy can be viewed as a simultaneous waterfilling of powers in time, as functions of the channel states, where the mobile users spend more power at favorable channel states and less, or even no power, at bad channel states. The power allocation policy also automatically dictates the optimum medium accessing, or scheduling policy. Therefore, the information theoretic framework allows us to come up with a cross layer design combining the physical and MAC layers of wireless systems. In contrast to scalar MACs, where the optimum transmit policy is a TDMA-like policy [6], the usage of multiple signal dimensions in the CDMA system leads to many users transmitting simultaneously while optimizing the system wide sum rate. The number of users accessing the medium can be as high as $N(N+1)/2$, where N is the processing gain, i.e., the dimensionality of the signal space.

  This thesis also develops an iterative one-user-at-a-time waterfilling algorithm, which converges to the optimum solution, and which makes the numerical solution of the otherwise highly non-linear problem practical and feasible.

- **Capacity region of fading CDMA – policies that achieve arbitrary rate points**

  While sum capacity is a very commonly used metric due to the fact that it represents the total throughput flowing in a network, it falls short when we would like to accommodate users with multiple priorities/rate classes in the system. This leads to the need to characterize the set of all achievable rates, i.e., the capacity region. The results presented here have addressed this need by establishing the capacity region of fading CDMA channels, as well as the individual power allocation policies that achieve any given point on the capacity region boundary. The optimal power allocation is obtained by developing a generalized version of the iterative waterfilling algorithm. We have further established some geometric properties such as the non-strict convexity of the capacity region, which translates to the sum capacity being achieved at many different rate tuples, unlike the scalar MAC. This provides more flexibility to the designer to choose a target rate tuple, while still obtaining the highest throughput.

  In practice, due to the bandwidth limitations on the feedback link, the CSI often needs to be quantized. However, the results of this thesis demonstrate that, even at very low feedback rates, resource allocation proves to be very useful in terms of improving the capacity, and the capacity region for a system with limited feedback is very close to the capacity with perfect channel state information. We conclude that, the knowledge about the relative quality of the channels of different users, which does not require a high level of resolution, is what plays the major role in providing the gains obtained by resource allocation.

- **Jointly optimal power and waveform selection**

  Aside from adjusting the transmit power at the mobile units, availability of the CSI at the transmitters can further be used to choose the spreading waveforms (sequences) in spread spectrum systems, such as CDMA systems. The research presented in this dissertation has led to the characterization of the jointly optimal transmit powers and transmit waveforms for a CDMA channel, which maximize the sum capacity. The more complicated problem of joint allocation of all available resources strikingly leads to a very compact and practical solution: only the users with the best N channel states transmit simultaneously using single user waterfilling, and these users are assigned orthogonal waveforms. In other words, the users position themselves orthogonally to each other in either space or time, thus avoiding any interference. The resulting power and waveform allocation can again be obtained by running a one-user-at-a-time algorithm that iterates between power and signature sequence updates. Moreover, from each user's point of view, the optimal policy depends only on the user's own channel state and whether the user has one of the top N channel states, which significantly reduces the amount of required feedback and facilitates a distributed implementation.

- **Joint resource allocation and user cooperation in wireless networks**

  A very important feature arising from the physical nature of the wireless channel is what is traditionally thought of as interference in networks: *overheard information*. The fact that signals from all sources in the network are superposed in the transmit medium can be taken advantage of in the design of wireless networks, by allowing cooperation between the nodes in the network, yielding cooperative diversity.

  Facilitated by the increasing processing power and ability provided by the electronics technology, future generation wireless networks are likely to have a more complex infrastructure, where the individual components of the networks are also more capable. Therefore, user cooperation and its information theoretic analysis will surely become increasingly essential for wireless networks, not only because of the presence of the "free" side information overheard by the nodes, but also because they provide a natural way to encode, transmit, route and decode information in a wireless network, providing a unifying cross-layer design. Such networks are very desirable because of their ability to adapt to changing system conditions, and efficient use of resources. Simple cooperating networks can be viewed as building blocks of larger ad-hoc networks. Moreover, the need for user cooperation naturally arises in many practical situations, i.e., in sensor networks where the nodes have correlated information, or in multiuser settings, where some of the users, which are blocked from their intended receivers by obstacles, can still reach other users with better channel states.

This thesis provides a joint treatment of resource allocation and user cooperation, for a simple fading Gaussian multiple access channel with two users. Power control, when performed optimally in conjunction with block Markov superposition coding, not only achieves much higher rates for both users, but it also simplifies the underlying coding scheme and dictates the optimal multiple accessing, and routing policies. Specifically, either the cooperation component or the fresh information component of the transmitted codeword is shown to have zero power, depending on the channel states. This is evidence that the information theoretic approach is a powerful tool which combines multiple aspects of the communication and provides a way to achieve the aforementioned cross layer design of wireless networks.

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