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Ensembles of affine-control systems with applications to Deep Learning

Candidate:
Alessandro Scagliotti

Advisor:
Prof. Andrei Agrachev

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Alessandro Scagliotti

Abstract

This thesis is devoted to the study of optimal control problems of ensembles of dynamical systems, where the dynamics has an affine dependence in the controls. By means of Γ-convergence arguments, we manage to approximate infinite ensembles with a sequence of growing-in-size finite ensembles. The advantage of this approach is that, under a suitable change of the states space, finite ensembles of control systems can be treated as a *single* control system. Motivated by this fact, in the first part of the thesis we formulate a gradient flow equation on the space of admissible controls related to *single* optimal control problems with end-point cost. Then, this is applied to the case of finite ensembles, where it is used to derive an implementable algorithm for the numerical resolution of ensemble optimal control problems. We also consider an iterative method based on the Pontryagin Maximum Principle. Finally, in the last part of the thesis, we formulate the task of the interpolation of a diffeomorphism with a Deep Neural Network as an ensemble optimal control problem. Therefore, we can take advantage the algorithms developed before to *train* the network.

Contents

Introduction

The central topic of this thesis is the optimal control of ensembles of dynamical systems. An ensemble of control systems is a parametrized family of controlled ODEs of the form

$$\begin{cases} \dot{x}^\theta(s) = G^\theta(x^\theta(s), u(s)) & \text{a.e. in } [0,T], \\ x^\theta(0) = x_0^\theta, \end{cases} \tag{I.1}$$

where $\theta \in \Theta \subset \mathbb{R}^d$ is the parameter of the ensemble, $u : [0,T] \to \mathbb{R}^k$ is the control, and, for every $\theta \in \Theta$, $G^\theta : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is the function that prescribes the dynamics of the corresponding system. The peculiarity of this kind of problems is that the elements of the ensemble are *simultaneously* driven by the same control u. This framework is particularly suitable for modeling real-world control systems affected by data uncertainty (see, e.g., [**44**]), or the problem of controlling a large number of particles by means of a signal (see [**16**]). Also from the theoretical viewpoint there is currently an active research interest on this topic. For instance, the problem of the controllability of ensembles of linear equations has been recently investigated in [**25**]. In [**6**] it was proved a generalization of the Chow–Rashevskii theorem for ensembles of linear-control systems. In [**36, 37**] ensembles were studied in the framework of nuclear magnetic resonance spectroscopy. Moreover, as regards ensembles in quantum control, we report the contributions [**12, 13**], and we recall the recent works [**10, 20**].

In the present thesis we focus on a particular instance of (I.1), corresponding to the case in which the dynamics has an affine dependence on the controls. More precisely, we consider ensembles with the following expression:

$$\begin{cases} \dot{x}^\theta(t) = F_0^\theta(x^\theta(s)) + F^\theta(x^\theta(s))u(s) & \text{a.e. in } [0,1], \\ x^\theta(0) = x_0^\theta, \end{cases} \tag{I.2}$$

where $\theta \in \Theta \subset \mathbb{R}^d$ varies in a compact set, and, for every $\theta \in \Theta$, the vector field $F_0^\theta : \mathbb{R}^n \to \mathbb{R}^n$ represents the drift, while the matrix-valued application $F^\theta = (F_1^\theta, \ldots, F_k^\theta) : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ collects the controlled fields. We set $\mathcal{U} := L^2([0,1], \mathbb{R}^k)$ as the space of admissible controls, and, for every $\theta \in \Theta$, the curve $x_u^\theta : [0,1] \to \mathbb{R}^n$ denotes the trajectory of (I.2) corresponding to the parameter θ and to the control $u \in \mathcal{U}$. We are interested in the optimal control problem related to the minimization of a functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ of the form

$$\mathcal{F}^\infty(u) := \int_\Theta a(x_u^\theta(1), \theta)\, d\mu(\theta) + \frac{\beta}{2}||u||_{L^2}^2 \tag{I.3}$$

for every $u \in \mathcal{U}$, where $a : \mathbb{R}^n \times \Theta \to \mathbb{R}_+$ is a non-negative continuous function, while μ is a Borel probability measures on Θ, and finally $\beta > 0$ is a constant that tunes the L^2-squared regularization. When the support of the probability measure μ is not reduced to a finite set of points, the minimization of the functional $\mathcal{F}^\infty$ is often intractable in practical situations, since a single evaluation of $\mathcal{F}^\infty$ potentially requires the resolution of an infinite number of Cauchy problems (I.2). Therefore, it is natural to try to replace μ with a sequence of probability measures $(\mu_N)_{N\in\mathbb{N}}$ such that each of them charges a subset of Θ of cardinality N, and such that $\mu_N \rightharpoonup^* \mu$ as $N \to \infty$. Then, we can consider the sequence of functionals $(\mathcal{F}^N)_{N\in\mathbb{N}}$ defined as

$$\mathcal{F}^N(u) := \int_\Theta a(x_u^\theta(1), \theta)\, d\mu_N(\theta) + \frac{\beta}{2}||u||_{L^2}^2 \tag{I.4}$$

for every $u \in \mathcal{U}$ and for every $N \in \mathbb{N}$. One of the goals of the present thesis is to study in which sense the functionals defined in (I.4) approximate the cost $\mathcal{F}^\infty$. It turns out that, when considering the restrictions to bounded subsets of $\mathcal{U}$, the sequence $(\mathcal{F}^N)_{N\in\mathbb{N}}$ is Γ-convergent to $\mathcal{F}^\infty$ with respect to the *weak* topology of L^2. Moreover, if for every $N \in \mathbb{N}$ we choose $\hat{u}_N \in \arg\min_{\mathcal{U}} \mathcal{F}^N$, standard facts in the theory of Γ-convergence ensure that the sequence $(\hat{u}_N)_{N\in\mathbb{N}}$ is *weakly* pre-compact and that each of its limiting points is a minimizer of the original functional $\mathcal{F}^\infty$ defined in (I.3). What is more surprising is that, owing to the peculiar form of the cost (I.3), it turns out that $(\hat{u}_N)_{N\in\mathbb{N}}$ is also pre-compact with respect to the L^2-*strong* topology. All these facts concerning ensembles of affine-control system are discussed in the paper [**47**].

We observe that the problem of minimizing the functional $\mathcal{F}^N$ defined in (I.4) is equivalent to the resolution of a *single* optimal control problem in $\mathbb{R}^{nN}$. For this reason, in the first part of the thesis we first study the gradient flow associated to a *single* optimal control problem with end-point cost. More precisely, given a base-point $x_0 \in \mathbb{R}^n$, for every $u \in \mathcal{U}$ we consider the absolutely continuous trajectory $x_u : [0, 1] \to \mathbb{R}^n$ that solves the linear-control system

$$\begin{cases} \dot{x}_u(s) = F(x_u(s))u(s) & \text{for a.e. } s \in [0, 1], \\ x_u(0) = x_0, \end{cases} \tag{I.5}$$

where $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ prescribes the controlled fields. For every $\beta > 0$ and $x_0 \in \mathbb{R}^n$, we define the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ as follows:

$$\mathcal{F}(u) := a(x_u(1)) + \frac{\beta}{2}||u||_{L^2}^2, \tag{I.6}$$

where $a : \mathbb{R}^n \to \mathbb{R}_+$ is a non-negative C^1-regular function, and $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of (I.5) corresponding to the control $u \in \mathcal{U}$. Here we investigate the gradient flow induced by the functional $\mathcal{F}$ on the Hilbert space $\mathcal{U}$, i.e., the evolution equation

$$\partial_t U_t = -\mathcal{G}[U_t], \tag{I.7}$$

where $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ is the vector field on the Hilbert space $\mathcal{U}$ that represents the differential $d\mathcal{F} : \mathcal{U} \to \mathcal{U}^*$ through the Riesz's isometry, i.e., $\mathcal{G}[u]$ is defined as the only element of $\mathcal{U}$ such that the identity

$$\langle \mathcal{G}[u], v \rangle_{L^2} = d_u\mathcal{F}(v) \tag{I.8}$$

holds for every $v \in \mathcal{U}$. It turns out that (I.7) can be treated as an infinite-dimensional ODE, and we prove that, for every initial datum $U_0 = u_0$, it admits a unique continuously differentiable solution $U : [0, +\infty) \to \mathcal{U}$. Then we focus on the asymptotic behavior of the curves that solve (I.7). The main result of this part states that, if the application $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ that defines the linear-control system (I.5) is real-analytic as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ that provides the end-point term in (I.6), then, for every $u_0 \in H^1([0, 1], \mathbb{R}^k) \subset \mathcal{U}$, the curve $t \mapsto U_t$ that solves the gradient flow equation (I.7) with initial datum $U_0 = u_0$ satisfies

$$\lim_{t\to+\infty} ||U_t - \hat{u}||_{L^2} = 0, \tag{I.9}$$

where $\hat{u} \in \mathcal{U}$ is a critical point for $\mathcal{F}$. The key-ingredient for this convergence is the establishment of the Lojasiewicz-Simon inequality for the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$, under the assumption that $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ and $a : \mathbb{R}^n \to \mathbb{R}_+$ are real-analytic. This family of inequalities was first introduced by Lojasiewicz in [**38**] for real-analytic functions defined on a finite-dimensional domain. The generalization of this result to real-analytic functionals defined on a Hilbert space was proposed by Simon in [**49**], and since then it has revealed to be an invaluable tool to study convergence properties of evolution equations (see the survey paper [**22**]). Following this approach, the Lojasiewicz-Simon inequality for the functional $\mathcal{F}^\beta$ is the cornerstone for the convergence result (I.9). In this regards, another important observation lies in the fact that the Sobolev space $H^m([0, 1], \mathbb{R}^k)$ is invariant for the gradient flow (I.7). Moreover, we obtain that, when the Cauchy datum belongs to $H^m([0, 1], \mathbb{R}^k)$, the curve $t \mapsto U_t$ that solves (I.7) is bounded in the H^m-norm. The results of this part are contained in [**46**]. We stress the fact that the case of an affine-control system can be easily reconducted to a linear-control system by artificially adding a new component u_0 in the control variable $u = (u_1, \ldots, u_k)$ with constant value $u_0 \equiv 1$. Finally, we report that the gradient flow equation (I.7) is used to formulate an algorithm for the numerical minimization of functionals $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$, related to the optimal control of finite ensembles.

In the last part of the thesis we formulate the problem of training a Deep Neural Network as an ensemble optimal control problem, and we apply the results obtained in the previous parts. Indeed, in [**26**] and [**29**] it was independently observed that some particular Deep Learning architectures (called *ResNets*) can be interpreted as discretizations of control systems. Since then, there has been an increasing interest in the interplay between control theory and Deep Learning, with the aim of providing a solid mathematical explanation to the success of such algorithms in solving practical tasks. For more details, see [**14, 18, 35, 51**]. The problem that we study is related to an observations-driven reconstruction of a diffeomorphism $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ diffeotopic to the identity. The starting points are some recent theoretical results obtained in [**7, 8**] that guarantee that under suitable assumptions the flows generated by linear-control systems can approximate in C^0-norm on compact sets any diffeomorphism obtained as a flow of a non-autonomous vector field. Our task is to derive an implementable procedure to provide such an approximation. Namely, we consider an ensemble of evaluation points $\{x_0^j\}_{j=1,\ldots,N}$ sampled from a probability measure μ supported on a compact set of $\mathbb{R}^n$, and we record the action of Ψ on such *training* points, i.e., $(x_0^j, \Psi(x_0^j))_{j=1,\ldots,N}$. Then, we consider the following ensemble of linear-control systems, namely

$$\begin{cases} \dot{x}_u^j(s) = F(x_u^j(s))u(s) \quad \text{for a.e. } s \in [0,1], \\ x_u^j(0) = x_0^j, \end{cases} \qquad j = 1, \ldots, N, \tag{I.10}$$

with the cost $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ defined as

$$\mathcal{F}^N(u) := \frac{1}{N}\sum_{j=1}^{N} a(x_u^j(1) - \Psi(x_0^j)) + \frac{\beta}{2}||u||_{L^2}^2, \tag{I.11}$$

where $a : \mathbb{R}^n \to \mathbb{R}_+$ is a non-negative *loss* function such that $a(0) = 0$ and $a(y) > 0$ if $y \neq 0$. The first term in (I.11) aims at achieving a good interpolation of the observations, while the regularization term penalizes the L^2-norm of the controls. If $\hat{u}_N \in \arg\min_{\mathcal{U}} \mathcal{F}^N$, then we use $\Phi_{\hat{u}^N} : \mathbb{R}^n \to \mathbb{R}^n$ as approximation of Ψ, where $\Phi_u : x_0 \mapsto x_u(1)$ is the flow associated to the linear-control system (I.10) and corresponding to the admissible control $u \in \mathcal{U}$. We observe that (I.10) is a particular instance of (I.2), where the drift vector field $F_0 \equiv 0$ and the controlled fields are the same for every $j = 1, \ldots, N$. For this reason, in virtue of the Γ-convergence result for general ensembles of affine-control systems, we deduce that the minimization of (I.11) is converging as $N \to \infty$ to the limiting problem of minimizing the functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ given by

$$\mathcal{F}^\infty(u) := \int_K a(\Phi_u(x) - \Psi(x))\, d\mu(x) + \frac{\beta}{2}||u||_{L^2}^2, \tag{I.12}$$

where μ is the probability measure used to sample the observation points. Taking advantage of two algorithms developed in the thesis for the numerical resolution of generic ensemble optimal control problems, we deduce two *training procedures* for the ResNet obtained by discretizing (I.10). The results discussed in this part are detailed in [**48**].

We now briefly describe the content of each capter.

In Chapter 1 we establish some preliminary results for *single* linear-control systems. In particular, we focus on the properties of the trajectories and of the end-point mapping, for which we investigate the first and second-order variation. In Chapter 2 we study the gradient flow induced by the functional (I.6) on the Hilbert space $\mathcal{U}$. We establish existence and uniqueness of the solutions of the evolution equation (I.7), and we prove the convergence of the trajectories with Sobolev-regular initial datum by means of the Lojasiewicz-Simon inequality.

In Chapter 3 we consider optimal control problems involving ensembles of affine-control systems. The main result is the reduction to finite ensembles via Γ-convergence. This fact holds with cost functionals that have a slightly more general form than (I.3), namely

$$\mathcal{F}^{\infty}(u) := \int_{\Theta} \int_{0}^{1} a(s, x_u^{\theta}(s), \theta)\, d\nu(s) d\mu(\theta) + \frac{\beta}{2} ||u||_{L^2}^2,$$

where ν is a Borel probability measure on $[0,1]$. We obtain (I.3) when we set $\nu = \delta_{s=1}$. When the cost has the form (I.3), after using a simple argument to reduce an affine-control system to a linear-control one, we derive the gradient flow equation for finite ensembles. Finally we propose and test two numerical methods for the optimal control of finite ensembles. The former is based on a finite elements projection of the gradient flow (I.7), the latter on the Pontryagin Maximum Principle.

In Chapter 4 we model the Deep Learning reconstruction of a diffeomorphism as an ensemble optimal control problem, and we apply the results obtained in Chapter 3. As a matter of fact, from the discretization of the linear-control system we obtain a ResNet, that we can train by means of the algorithms developed before. Finally, we test the methods on a practical example.

CHAPTER 1

General results for linear-control systems

In this chapter we establish some preliminary technical results for linear-control systems that will be of use throughout the thesis. In particular, given a base-point $x_0 \in \mathbb{R}^n$, we study the first and the second variation of the mapping $P_s : u \mapsto x_u(s)$, where $x_u : [0,1] \to \mathbb{R}^n$ is the trajectory of a linear-control system corresponding to an admissible control u and with initial point x_0.

1.1. Setting and framework

In this chapter we consider control systems on $\mathbb{R}^n$ with linear dependence in the control variable $u \in \mathbb{R}^k$, i.e., of the form

$$\dot{x} = F(x)u, \tag{1.1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ is a Lipschitz-continuous function. We use the notation F_i for $i = 1, \dots, k$ to indicate the vector fields on $\mathbb{R}^n$ obtained by taking the columns of F, and we denote by $L > 0$ the Lipschitz constant of these vector fields, i.e., we set

$$L := \sup_{i=1,\dots,k} \sup_{x,y\in\mathbb{R}^n} \frac{|F_i(x) - F_i(y)|_2}{|x-y|_2}. \tag{1.1.2}$$

We immediately observe that (1.1.2) implies that the vector fields $F_1, \dots, F_k$ have sub-linear growth, i.e., there exists $C > 0$ such that

$$\sup_{i=1,\dots,k} |F_i(x)| \le C(|x|_2 + 1) \tag{1.1.3}$$

for every $x \in \mathbb{R}^n$. Moreover, for every $i = 1, \dots, k$, if F_i is differentiable at $y \in \mathbb{R}^n$, then from (1.1.2) we deduce that

$$\left| \frac{\partial F_i(y)}{\partial x} \right|_2 \le L. \tag{1.1.4}$$

We define $\mathcal{U} := L^2([0,1], \mathbb{R}^k)$ as the space of admissible controls, and we endow $\mathcal{U}$ with the usual Hilbert space structure, induced by the scalar product

$$\langle u, v\rangle_{L^2} = \int_0^1 \langle u(s), v(s)\rangle_{\mathbb{R}^k}\, ds. \tag{1.1.5}$$

Given $x_0 \in \mathbb{R}^n$, for every $u \in \mathcal{U}$, let $x_u : [0,1] \to \mathbb{R}^n$ be the absolutely continuous curve that solves the following Cauchy problem:

$$\begin{cases} \dot{x}_u(s) = F(x_u(s))u(s) & \text{for a.e. } s \in [0,1], \\ x_u(0) = x_0. \end{cases} \tag{1.1.6}$$

We recall that, under the condition (1.1.2), the existence and uniqueness of the solution of (1.1.6) is guaranteed by Carathéodory Theorem (see, e.g, [**30**, Theorem 5.3]). We insist on the fact that in Section 1.2 and Section 1.3 the Cauchy datum $x_0 \in \mathbb{R}^n$ is assumed to be assigned. On the other hand, in Section 1.4 we study some properties of the flows induced by the control system (1.1.1), and therefore we shall consider different solutions of (1.1.6) as the Cauchy datum x_0 varies in $\mathbb{R}^n$.

Before proceeding, in Subsection 1.1.1 we recall some results concerning Sobolev spaces in one-dimensional domains. Then, in Section 1.2 and Section 1.3 we investigate the properties of the solutions of (1.1.6).

1.1.1. Sobolev spaces in one dimension. In this subsection we recall some results for one-dimensional Sobolev spaces. Since in this Thesis we work only in Hilbert spaces, we shall restrict our attention to the Sobolev exponent $p = 2$, i.e., we shall state the results for the Sobolev spaces $H^m := W^{m,2}$ with $m \geq 1$. For a complete discussion on the topic, the reader is referred to [**15**, Chapter 8]. Throughout the Thesis we use the convention $H^0 := L^2$.

For every integer $d \geq 1$, given a compact interval $[a,b] \subset \mathbb{R}$, let $C_c^\infty([a,b],\mathbb{R}^d)$ be the set of the C^∞-regular functions with compact support in $[a,b]$. For every $\phi \in C_c^\infty([a,b],\mathbb{R}^d)$, we use the symbol $\phi^{(\ell)}$ to denote the ℓ-th derivative of the function $\phi : [a,b] \to \mathbb{R}^d$. For every $m \geq 1$, the function $u \in L^2([a,b],\mathbb{R}^d)$ belongs to the Sobolev space $H^m([a,b],\mathbb{R}^d)$ if and only if, for every integer $1 \leq \ell \leq m$ there exists $u^{(\ell)} \in L^2([a,b],\mathbb{R}^d)$ such that the following identity holds

$$\int_a^b \langle u(s), \phi^{(\ell)}(s) \rangle_{\mathbb{R}^d} \, ds = (-1)^\ell \int_a^b \langle u^{(\ell)}(s), \phi(s) \rangle_{\mathbb{R}^d} \, ds$$

for every $\phi \in C_c^\infty([a,b],\mathbb{R}^d)$. If $u \in H^m([a,b],\mathbb{R}^d)$, then for every integer $1 \leq \ell \leq m$ $u^{(\ell)}$ denotes the ℓ-th Sobolev derivative of u. We recall that, for every $m \geq 1$, $H^m([a,b],\mathbb{R}^d)$ is a Hilbert space (see, e.g., [**15**, Proposition 8.1]) when it is equipped with the norm $||\cdot||_{H^m}$ induced by the scalar product

$$\langle u, v \rangle_{H^m} := \langle u, v \rangle_{L^2} + \sum_{\ell=1}^m \int_a^b \langle u^{(\ell)}(s), v^{(\ell)}(s) \rangle_{\mathbb{R}^d} \, ds.$$

We observe that, for every $m_2 > m_1 \geq 0$, we have

$$||u||_{H^{m_1}} \leq ||u||_{H^{m_2}} \tag{1.1.7}$$

for every $u \in H^{m_2}([a,b],\mathbb{R}^l)$, i.e., the inclusion $H^{m_2}([a,b],\mathbb{R}^d) \hookrightarrow H^{m_1}([a,b],\mathbb{R}^d)$ is continuous. We recall that a linear and continuous application $T : E_1 \to E_2$

between two Banach spaces E_1, E_2 is *compact* if, for every bounded set $B \subset E_1$, the image $T(B)$ is pre-compact with respect to the strong topology of E_2. In the following result we list three compact inclusions.

THEOREM 1.1.1. *For every $m \geq 1$, the following inclusions are compact:*

$$H^m([a,b],\mathbb{R}^d) \hookrightarrow L^2([a,b],\mathbb{R}^d), \tag{1.1.8}$$

$$H^m([a,b],\mathbb{R}^d) \hookrightarrow C^0([a,b],\mathbb{R}^d), \tag{1.1.9}$$

$$H^m([a,b],\mathbb{R}^d) \hookrightarrow H^{m-1}([a,b],\mathbb{R}^d), \tag{1.1.10}$$

PROOF. When $m = 1$, (1.1.8)-(1.1.9) descend directly from [**15**, Theorem 8.8]. In the case $m \geq 2$, in virtue of (1.1.7), the inclusion $H^m([a,b],\mathbb{R}^d) \hookrightarrow H^1([a,b],\mathbb{R}^d)$ is continuous. Recalling that the composition of a linear continuous operator with a linear compact one is still compact (see, e.g., [**15**, Proposition 6.3]), we deduce that (1.1.8)-(1.1.9) holds also for $m \geq 2$.

When $m = 1$, (1.1.10) reduces to (1.1.8). For $m \geq 2$, (1.1.10) is proved by induction on m, using (1.1.8) and observing that $u \in H^m([a,b],\mathbb{R}^d)$ implies that $u^{(1)} \in H^{m-1}([a,b],\mathbb{R}^d)$. □

Finally, we recall the notion of *weak convergence*. For every $m \geq 0$ (we set $H^0 := L^2$), if $(u_n)_{n\geq 1}$ is a sequence in $H^m([a,b],\mathbb{R}^d)$ and $u \in H^m([a,b],\mathbb{R}^d)$, then the sequence $(u_n)_{n\geq 1}$ weakly converges to u if and only if

$$\lim_{n\to\infty} \langle v, u_n \rangle_{H^m} = \langle v, u \rangle_{H^m}$$

for every $v \in H^m([a,b],\mathbb{R}^d)$, and we write $u_n \rightharpoonup_{H^m} u$ as $n \to \infty$. For every $m \geq 1$, if $u_n \rightharpoonup_{H^m} u$ as $n \to \infty$, then we have

$$||u||_{H^m} \leq \liminf_{m\to\infty} ||u_n||_{H^m}. \tag{1.1.11}$$

Finally, in view of the compact inclusion (1.1.10) and of [**15**, Remark 6.2], for every $m \geq 1$, if a sequence $(u_n)_{n\geq 1}$ in $H^m([a,b],\mathbb{R}^d)$ satisfies $u_n \rightharpoonup_{H^m} u$ as $n \to \infty$, then

$$\lim_{n\to\infty} ||u_n - u||_{H^{m-1}} = 0.$$

We conclude this part with the following fact concerning the space $H^1([a,b],\mathbb{R}^d)$.

PROPOSITION 1.1.2. *Let $u : [a,b] \to \mathbb{R}^d$ be a function in $H^1([a,b],\mathbb{R}^d)$. Then, u is Hölder-continuous with exponent $\frac{1}{2}$, namely*

$$|u(t_1) - u(t_2)|_2 \leq ||u^{(1)}||_{L^2} |t_1 - t_2|^{\frac{1}{2}}$$

for every $t_1, t_2 \in [a,b]$.

PROOF. The fact that $u \in H^1([a,b],\mathbb{R}^d)$ implies that it is absolutely continuous (see, e.g., [**15**, Theorem 8.2]). Thus, using the Cauchy-Schwartz inequality, we deduce that

$$|u(t_1) - u(t_2)|_2 \leq \int_{t_1}^{t_2} |u^{(1)}(\tau)|_2 \, d\tau \leq \left(\int_{t_1}^{t_2} |u^{(1)}(\tau)|_2^2 \, d\tau \right)^{\frac{1}{2}} |t_1 - t_2|^{\frac{1}{2}}$$

for every $t_1, t_2 \in [a,b]$, and this implies the thesis. □

1.2. First-order properties of trajectories of linear-control systems

In this subsection we investigate basic properties of the solutions of (1.1.6), with a particular focus on the relation between the admissible control $u \in \mathcal{U}$ and the corresponding trajectory x_u. We start by stating a version of the Grönwall-Bellman inequality, that will be widely used later.

LEMMA 1.2.1 (Grönwall-Bellman Inequality). *Let $f : [a,b] \to \mathbb{R}_+$ be a non-negative continuous function and let us assume that there exists a constant $\alpha > 0$ and a non-negative function $\beta \in L^1([a,b],\mathbb{R}_+)$ such that*

$$f(s) \leq \alpha + \int_a^s \beta(\tau) f(\tau) \, d\tau$$

for every $s \in [a,b]$. Then, for every $s \in [a,b]$ the following inequality holds:

$$f(s) \leq \alpha e^{||\beta||_{L^1}}. \tag{1.2.1}$$

PROOF. This statement follows as a particular case of [**27**, Theorem 5.1]. □

We recall that, for every $u \in \mathcal{U} := L^2([0,1],\mathbb{R}^k)$ the following inequality holds:

$$||u||_{L^1} = \int_0^1 \sum_{i=1}^k |u^i(s)| \, ds \leq \sqrt{k} \sqrt{\int_0^1 \sum_{i=1}^k |u^i(s)|^2 \, ds} = \sqrt{k} ||u||_{L^2}. \tag{1.2.2}$$

We first show that, for every admissible control $u \in \mathcal{U}$, the corresponding solution of (1.1.6) is bounded in the C^0-norm. In our framework, given a continuous function $f : [0,1] \to \mathbb{R}^n$, we set

$$||f||_{C^0} := \sup_{s \in [0,1]} |f(s)|_2.$$

LEMMA 1.2.2. *Let $u \in \mathcal{U}$ be an admissible control, and let $x_u : [0,1] \to \mathbb{R}^n$ be the solution of the Cauchy problem* (1.1.6) *corresponding to the control u. Then, the following inequality holds:*

$$||x_u||_{C^0} \leq \left(|x_0|_2 + \sqrt{k} C ||u||_{L^2} \right) e^{\sqrt{k} C ||u||_{L^2}}, \tag{1.2.3}$$

where $C > 0$ is the constant of sub-linear growth prescribed by (1.1.3).

PROOF. Rewriting (1.1.6) in the integral form, we obtain the following inequality

$$|x_u(s)|_2 \le |x_0|_2 + \int_0^s \sum_{i=1}^k \Big(|F_i(x_u(\tau))|_2 |u^i(\tau)| \Big)\, d\tau$$

for every $s \in [0,1]$. Then, using (1.1.3), we deduce that

$$|x_u(s)|_2 \le |x_0|_2 + C||u||_{L^1} + C\int_0^s |u(\tau)|_1 |x_u(\tau)|_2 \, d\tau.$$

Finally, the thesis follows from Lemma 1.2.1 and (1.2.2). □

In the following proposition we prove that the solution of the Cauchy problem (1.1.6) has a continuous dependence on the admissible control.

PROPOSITION 1.2.3. *Let us consider $u, v \in \mathcal{U}$ and let $x_u, x_{u+v} : [0,1] \to \mathbb{R}^n$ be the solutions of the Cauchy problem* (1.1.6) *corresponding, respectively, to the controls u and $u+v$. Then, for every $R > 0$ there exists $L_R > 0$ such that the inequality*

$$||x_{u+v} - x_u||_{C^0} \le L_R ||v||_{L^2} \tag{1.2.4}$$

holds for every $u, v \in \mathcal{U}$ such that $||u||_{L^2}, ||v||_{L^2} \le R$.

PROOF. Using the fact that x_u and x_{u+v} are solutions of (1.1.6), for every $s \in [0,1]$ we have that

$$|x_{u+v}(s) - x_u(s)|_2 \le \int_0^s \sum_{i=1}^k \Big(|F_i(x_{u+v}(\tau))|_2 |v^i(\tau)| \Big)\, d\tau$$
$$+ \int_0^s \sum_{i=1}^k \Big(|F_i(x_{u+v}(\tau)) - F_i(x_u(\tau)|_2 |u^i(\tau)| \Big)\, d\tau.$$

Recalling that $||v||_{L^2} \le R$, in virtue of Lemma 1.2.2, we obtain that there exists $C_R > 0$ such that

$$\sup_{\tau\in[0,1]} \sup_{i=1,\dots,k} |F_i(x_{u+v}(\tau))|_2 \le C_R.$$

Hence, using (1.2.2), we deduce that

$$\int_0^s \sum_{i=1}^k \Big(|F_i(x_{u+v}(\tau))|_2 |v^i(\tau)| \Big)\, d\tau \le C_R \sqrt{k} ||v||_{L^2}. \tag{1.2.5}$$

On the other hand, from the Lipschitz-continuity condition (1.1.2) it follows that

$$|F_i(x_{u+v}(\tau)) - F_i(x_u(\tau)|_2 \le L|x_{u+v}(\tau) - x_u(\tau)|_2 \tag{1.2.6}$$

for every $i = 1, \dots, k$ and for every $\tau \in [0,1]$. Using (1.2.5) and (1.2.6), we deduce that

$$|x_{u+v}(s) - x_u(s)|_2 \le C_R\sqrt{k}||v||_{L^2} + L\int_0^s |u(\tau)|_1 |x_{u+v}(\tau) - x_u(\tau)|_2 \, d\tau, \tag{1.2.7}$$

for every $s \in [0,1]$. By applying Lemma 1.2.1 to (1.2.7), we obtain that

$$|x_{u+v}(s) - x_u(s)|_2 \leq e^{L||u||_{L^1}} C_R \sqrt{k} ||v||_{L^2},$$

for every $s \in [0,1]$. Recalling (1.2.2) and setting

$$L_R := e^{L\sqrt{k}R} C_R \sqrt{k},$$

we prove (1.2.4). □

The previous result shows that the map $u \mapsto x_u$ is Lipschitz-continuous when restricted to any bounded set of the space of admissible controls $\mathcal{U}$. We remark that Proposition 1.2.3 holds under the sole assumption that the controlled vector fields $F_1, \ldots, F_k : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz-continuous. In the next result, by requiring that the controlled vector fields are C^1-regular, we compute the first order variation of the solution of (1.1.6) resulting from a perturbation in the control.

PROPOSITION 1.2.4. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^1-regular. For every $u, v \in \mathcal{U}$, for every $\varepsilon \in (0,1]$, let $x_u, x_{u+\varepsilon v} : [0,1] \to \mathbb{R}^n$ be the solutions of* (1.1.6) *corresponding, respectively, to the admissible controls u and $u + \varepsilon v$. Then, we have that*

$$||x_{u+\varepsilon v} - x_u - \varepsilon y_u^v||_{C^0} = o(\varepsilon) \text{ as } \varepsilon \to 0, \tag{1.2.8}$$

where $y_u^v : [0,1] \to \mathbb{R}^n$ is the solution of the following affine system:

$$\dot{y}_u^v(s) = F(x_u(s))v(s) + \left(\sum_{i=1}^k u^i(s) \frac{\partial F_i(x_u(s))}{\partial x} \right) y_u^v(s) \tag{1.2.9}$$

for a.e. $s \in [0,1]$, and with $y_u^v(0) = 0$.

PROOF. Setting $R := ||u||_{L^2} + ||v||_{L^2}$, we observe that $||u + \varepsilon v||_{L^2} \leq R$ for every $\varepsilon \in (0,1]$. Owing to Lemma 1.2.2, we deduce that there exists a compact $K_R \subset \mathbb{R}^n$ such that $x_u(s), x_{u+\varepsilon v}(s) \in K_R$ for every $s \in [0,1]$ and for every $\varepsilon \in (0,1]$. Using the fact that $F_1, \ldots, F_k$ are assumed to be C^1-regular, we deduce that there exists a non-decreasing function $\delta : [0, +\infty) \to [0, +\infty)$ with $\delta(0) = \lim_{r \to 0} \delta(r) = 0$ and a constant $C > 0$ such that the following inequality is satisfied

$$\left| F_i(x_2) - F_i(x_1) - \frac{\partial F_i(x_1)}{\partial x}(x_2 - x_1) \right|_2 \leq C\delta(|x_1 - x_2|_2)|x_1 - x_2|_2 \tag{1.2.10}$$

for every $i = 1, \ldots, k$ and for every $x_1, x_2 \in K_R$. Let us consider the non-autonomous affine system (1.2.9). In virtue of Carathéodory Theorem (see [**30**, Theorem 5.3]), we deduce that the system (1.2.9) admits a unique absolutely continuous solution $y_u^v : [0,1] \to \mathbb{R}^n$. For every $s \in [0,1]$, let us define

$$\xi(s) := x_{u+\varepsilon v}(s) - x_u(s) - \varepsilon y_u^v(s). \tag{1.2.11}$$

Therefore, in view of (1.1.6) and (1.2.9), for a.e. $s \in [0,1]$ we compute

$$\begin{aligned}|\dot{\xi}(s)|_2 \leq & \varepsilon \sum_{i=1}^{k} |F_i(x_{u+\varepsilon v}(s)) - F_i(x_u(s))|_2 |v^i(s)| \\ & + \sum_{i=1}^{k} \left| F_i(x_{u+\varepsilon v}(s)) - F_i(x_u(s)) - \varepsilon \frac{\partial F_i(x_u(s))}{\partial x} y_u^v(s) \right|_2 |u^i(s)|\end{aligned}$$

On one hand, using Proposition 1.2.3 and the Lipschitz-continuity assumption (1.1.2), we deduce that there exists $L' > 0$ such that

$$\varepsilon \sum_{i=1}^{k} |F_i(x_{u+\varepsilon v}(s)) - F_i(x_u(s))|_2 \leq L' ||v||_{L^2} \varepsilon^2 \tag{1.2.12}$$

for every $s \in [0,1]$ and for every $\varepsilon \in (0,1]$. On the other hand, for every $i = 1, \ldots, n$, combining Proposition 1.2.3, the inequality (1.2.10) and the estimate of the norm of the Jacobian (1.1.4), we obtain that there exists $L'' > 0$ such that

$$\begin{aligned}&\left| F_i(x_{u+\varepsilon v}(s)) - F_i(x_u(s)) - \varepsilon \frac{\partial F_i(x_u(s))}{\partial x} y_u^v(s) \right|_2 \\ &\qquad \leq \left| F_i(x_{u+\varepsilon v}(s)) - F_i(x_u(s)) - \frac{\partial F_i(x_u(s))}{\partial x} \big(x_{u+\varepsilon v}(s) - x_u(s) \big) \right|_2 \\ &\qquad\quad + \left| \frac{\partial F_i(x_u(s))}{\partial x} \big(x_{u+\varepsilon v}(s) - x_u(s) - \varepsilon y_u^v(s) \big) \right|_2 \\ &\qquad \leq C \Big[\delta(L'' ||v||_{L^2} \varepsilon) L'' ||v||_{L^2} \varepsilon \Big] + L |\xi(s)|_2.\end{aligned}$$

for every $s \in [0,1]$ and for every $\varepsilon \in (0,1]$. Combining the last inequality and (1.2.12), it follows that

$$|\dot{\xi}(s)|_2 \leq L_R \varepsilon^2 + L_R |u(s)|_1 \delta(L_R \varepsilon) \varepsilon + L |u(s)|_1 |\xi(s)|_2 \tag{1.2.13}$$

for a.e. $s \in [0,1]$ and for every $\varepsilon \in (0,1]$, where we set $L_R := \max\{L', L''\} ||v||_{L^2}$. Finally, recalling that $|\xi(0)|_2 = |x_{u+\varepsilon v}(0) - x_u(0) - \varepsilon y_u^v(0)|_2 = 0$ for every $\varepsilon \in (0,1]$, we have that

$$|\xi(s)|_2 \leq \int_0^s |\dot{\xi}(\tau)|_2 \, d\tau \leq L_R \varepsilon^2 + L_R ||u||_{L^1} \delta(L_R \varepsilon) \varepsilon + L \int_0^s |u(\tau)|_1 |\xi(\tau)|_2 \, d\tau,$$

for every $s \in [0,1]$ and for every $\varepsilon \in (0,1]$. Using Lemma 1.2.1 and (1.2.11), we deduce (1.2.8). □

Let us assume that $F_1, \ldots, F_k$ are C^1-regular. For every admissible control $u \in \mathcal{U}$, let us define $A_u \in L^2([0,1], \mathbb{R}^{n\times n})$ as

$$A_u(s) := \sum_{i=1}^{k} \left(u^i(s) \frac{\partial F_i(x_u(s))}{\partial x} \right) \tag{1.2.14}$$

for a.e. $s \in [0,1]$. For every $u \in \mathcal{U}$, let us introduce the absolutely continuous curve $M_u : [0,1] \to \mathbb{R}^{n\times n}$, defined as the solution of the following linear Cauchy problem:

$$\begin{cases} \dot{M}_u(s) = A_u(s) M_u(s) & \text{for a.e. } s \in [0,1], \\ M_u(0) = \mathrm{Id}. \end{cases} \tag{1.2.15}$$

The existence and uniqueness of the solution of (1.2.15) descends once again from the Carathéodory Theorem. We can prove the following result.

LEMMA 1.2.5. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^1-regular. For every admissible control $u \in \mathcal{U}$, let $M_u : [0,1] \to \mathbb{R}^{n\times n}$ be the solution of the Cauchy problem* (1.2.15)*. Then, for every $s \in [0,1]$, $M_u(s)$ is invertible, and the following estimates hold:*

$$|M_u(s)|_2 \le C_u, \quad |M_u^{-1}(s)|_2 \le C_u, \tag{1.2.16}$$

where

$$C_u = e^{\sqrt{k}L||u||_{L^2}}.$$

PROOF. Let us consider the absolutely continuous curve $N_u : [0,1] \to \mathbb{R}^{n\times n}$ that solves

$$\begin{cases} \dot{N}_u(s) = -N_u(s) A_u(s) & \text{for a.e. } s \in [0,1], \\ N_u(0) = \mathrm{Id}. \end{cases} \tag{1.2.17}$$

The existence and uniqueness of the solution of (1.2.17) is guaranteed by Carathéodory Theorem. Recalling the Leibniz rule for Sobolev functions (see, e.g., [**15**, Corollary 8.10]), a simple computation shows that the identity $N_u(s)M_u(s) = \mathrm{Id}$ holds for every $s \in [0,1]$. This proves that $M_u(s)$ is invertible and that $N_u(s) = M_u^{-1}(s)$ for every $s \in [0,1]$. In order to prove the bound on the norm of the matrix $M_u(s)$, we shall study $|M_u(s)z|_2$, for $z \in \mathbb{R}^n$. Using (1.2.15), we deduce that

$$\begin{aligned} |M_u(s)z|_2 &\le |z|_2 + \int_0^s |A_u(\tau)|_2 |M_u(\tau)z|_2 \, d\tau \\ &\le |z|_2 + L \int_0^s |u(s)|_1 |M_u(\tau)z|_2 \, d\tau, \end{aligned}$$

where we used (1.1.4). Using Lemma 1.2.1, and recalling (1.2.2), we obtain that the inequality (1.2.16) holds for $M_u(s)$, for every $s \in [0,1]$. Using (1.2.17) and applying the same argument, it is possible to prove that (1.2.16) holds as well for $N_u(s) = M_u^{-1}(s)$, for every $s \in [0,1]$. □

Using the curve $M_u : [0,1] \to \mathbb{R}^{n\times n}$ defined by (1.2.15), we can rewrite the solution of the affine system (1.2.9) for the first-order variation of the trajectory. Indeed, for every $u, v \in \mathcal{U}$, a direct computation shows that the function $y_u^v : [0,1] \to \mathbb{R}^n$ that solves (1.2.9) can be expressed as

$$y_u^v(s) = \int_0^s M_u(s)M_u^{-1}(\tau)F(x_u(\tau))v(\tau)\,d\tau \tag{1.2.18}$$

for every $s \in [0,1]$. Using (1.2.18) we can prove an estimate of the norm of y_u^v.

LEMMA 1.2.6. *Let us assume that the vector fields $F_1,\ldots,F_k$ defining the control system* (1.1.6) *are C^1-regular. Let us consider $u, v \in \mathcal{U}$, and let $y_u^v : [0,1] \to \mathbb{R}^n$ be the solution of the affine system* (1.2.9) *with $y_u^v(0) = 0$. Then, for every $R > 0$ there exists $C_R > 0$ such that the following inequality holds*

$$|y_u^v(s)|_2 \le C_R||v||_{L^2} \tag{1.2.19}$$

for every $s \in [0,1]$ and for every $u \in \mathcal{U}$ satisfying $||u||_{L^2} \le R$.

PROOF. In virtue of (1.2.18), we have that

$$|y_u^v(s)|_2 \le \int_0^s \left|M_u(s)M_u^{-1}(\tau)F(x_u(\tau))v(\tau)\right|\,d\tau.$$

Using (1.2.16), (1.2.3) and (1.1.3), we deduce that there exists $C_R' > 0$ such that

$$|y_u^v(s)|_2 \le C_R' \int_0^s |v(s)|_1\,d\tau,$$

for every $s \in [0,1]$. Combining this with (1.2.2), we deduce the thesis. □

Let us introduce the end-point map associated to the control system (1.1.6). For every $s \in [0,1]$, let us consider the map $P_s : \mathcal{U} \to \mathbb{R}^n$ defined as

$$P_s : u \mapsto P_s(u) := x_u(s), \tag{1.2.20}$$

where $x_u : [0,1] \to \mathbb{R}^n$ is the solution of (1.1.6) corresponding to the admissible control $u \in \mathcal{U}$. Using the results obtained before, it follows that the end-point map is differentiable.

PROPOSITION 1.2.7. *Let us assume that the vector fields $F_1,\ldots,F_k$ defining the control system* (1.1.6) *are C^1-regular. For every $s \in [0,1]$, let $P_s : \mathcal{U} \to \mathbb{R}^n$ be the end-point map defined by* (1.2.20)*. Then, for every $u \in \mathcal{U}$, P_s is Gateaux differentiable at u, and the differential $D_uP_s = (D_uP_s^1,\ldots,D_uP_s^n) : \mathcal{U} \to \mathbb{R}^n$ is a linear and continuous operator. Moreover, using the Riesz's isometry, for every $u \in \mathcal{U}$ and for every $s \in [0,1]$, every component of the differential D_uP_s can be represented as follows:*

$$D_uP_s^j(v) = \int_0^1 \left\langle g_{s,u}^j(\tau), v(\tau)\right\rangle_{\mathbb{R}^k}\,d\tau, \tag{1.2.21}$$

where, for every $j = 1, \ldots, n$, the function $g^j_{s,u} : [0,1] \to \mathbb{R}^k$ is defined as

$$g^j_{s,u}(\tau) = \begin{cases} \Big((\mathbf{e}^j)^T M_u(s) M_u^{-1}(\tau) F(x_u(\tau)) \Big)^T & \tau \in [0,s], \\ 0 & \tau \in (s,1], \end{cases} \tag{1.2.22}$$

where the column vector $\mathbf{e}^j$ is the j-th element of the standard basis $\{\mathbf{e}^1, \ldots, \mathbf{e}^n\}$ of $\mathbb{R}^n$.

Proof. For every $s \in [0,1]$, Proposition 1.2.4 guarantees that the end-point map $P_s : \mathcal{U} \to \mathbb{R}^n$ is Gateaux differentiable at every point $u \in \mathcal{U}$. In particular, for every $u, v \in \mathcal{U}$ and for every $s \in [0,1]$ the following identity holds:

$$D_u P_s(v) = y^v_u(s). \tag{1.2.23}$$

Moreover, (1.2.18) shows that the differential $D_u P_s : \mathcal{U} \to \mathbb{R}^n$ is linear, and Lemma 1.2.6 implies that it is continuous. The representation follows as well from (1.2.18). □

Remark 1.2.1. In the previous proof we used Lemma 1.2.6 to deduce for every $u \in \mathcal{U}$ the continuity of the linear operator $D_u P_s : \mathcal{U} \to \mathbb{R}^n$. Actually, Lemma 1.2.6 is slightly more informative, since it implies that for every $R > 0$ there exists $C_R > 0$ such that

$$|D_u P_s(v)|_2 \leq C_R ||v||_{L^2} \tag{1.2.24}$$

for every $v \in \mathcal{U}$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. As a matter of fact, we deduce that

$$||g^j_{s,u}||_{L^2} \leq C_R \tag{1.2.25}$$

for every $j = 1, \ldots, n$, for every $s \in [0,1]$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$.

Remark 1.2.2. It is interesting to observe that, for every $s \in (0,1]$ and for every $u \in \mathcal{U}$, the function $g^j_{s,u} : [0,1] \to \mathbb{R}^k$ that provides the representation the j-th component of $D_u P_s$ is absolutely continuous on the interval $[0,s]$, being the product of absolutely continuous matrix-valued curves. Indeed, on one hand, the application $\tau \mapsto F(x_u(\tau))$ is absolutely continuous, being the composition of a C^1-regular function with the absolutely continuous curve $\tau \mapsto x_u(\tau)$ (see, e.g., [**15**, Corollary 8.11]). On the other hand, $\tau \mapsto M_u^{-1}(\tau)$ is absolutely continuous as well, since it solves (1.2.17).

We now prove that for every $s \in [0,1]$ the differential of the end-point map, i.e. $u \mapsto D_u P_s$, is Lipschitz-continuous on the bounded subsets of $\mathcal{U}$. This result requires further regularity assumptions on the controlled vector fields. We first establish an auxiliary result concerning the matrix-valued curve that solves (1.2.15).

Lemma 1.2.8. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^2-regular. For every $u, w \in \mathcal{U}$, let $M_u, M_{u+w} : [0,1] \to$*

$\mathbb{R}^{n\times n}$ *be the solutions of* (1.2.15) *corresponding to the admissible controls* u *and* $u+w$, *respectively. Then, for every* $R>0$ *there exists* $L_R>0$ *such that, for every* $u,w\in\mathcal{U}$ *satisfying* $||u||_{L^2},||w||_{L^2}\le R$, *we have*

$$|M_{u+w}(s)-M_u(s)|_2 \le L_R||w||_{L^2}, \tag{1.2.26}$$

and

$$|M_{u+w}^{-1}(s)-M_u^{-1}(s)|_2 \le L_R||w||_{L^2} \tag{1.2.27}$$

for every $s\in[0,1]$.

Proof. Let us consider $R>0$, and let $u,w\in\mathcal{U}$ be with $||u||_{L^2},||w||_{L^2}\le R$. We observe that Lemma 1.2.2 implies that there exists a compact set $K_R\subset\mathbb{R}^n$ such that $x_u(s),x_{u+w}(s)\in K_R$ for every $s\in[0,1]$. The hypothesis that $F_1,\dots,F_k$ are C^2-regular implies that there exists $L'_R>0$ such that $\frac{\partial F_1}{\partial x},\dots,\frac{\partial F_k}{\partial x}$ are Lipschitz-continuous in K_R with constant L'_R. From (1.2.15), we have that

$$|\dot M_{u+w}(s)-\dot M_u(s)|_2 = |A_{u+w}(s)M_{u+w}(s)-A_u(s)M_u(s)|_2, \tag{1.2.28}$$

for a.e. $s\in[0,1]$. In particular, for a.e. $s\in[0,1]$, we can compute

$$\begin{aligned}|A_{u+w}(s)-A_u(s)|_2 \le &\sum_{i=1}^k \left|\frac{\partial F_i(x_{u+w}(s))}{\partial x}-\frac{\partial F_i(x_u(s))}{\partial x}\right|_2 |u^i(s)|\\ &+\sum_{i=1}^k\left|\frac{\partial F_i(x_{u+w}(s))}{\partial x}\right|_2 |w^i(s)|,\end{aligned}$$

and using Proposition 1.2.3, the Lipschitz continuity of $\frac{\partial F_1}{\partial x},\dots,\frac{\partial F_k}{\partial x}$ and (1.1.4), we obtain that there exists $L''_R>0$ such that

$$|A_{u+w}(s)-A_u(s)|_2 \le L''_R||w||_{L^2}|u(s)|_1 + L|w(s)|_1, \tag{1.2.29}$$

for a.e. $s\in[0,1]$. Using once again (1.1.4), we have that

$$|A_u(s)|_2 \le L|u(s)|_1, \tag{1.2.30}$$

for a.e. $s\in[0,1]$. Combining (1.2.29)-(1.2.30) with the triangular inequality at the right-hand side of (1.2.28), we deduce that

$$\begin{aligned}|\dot M_{u+w}(s)-\dot M_u(s)|_2 \le & C'_R\big(L''_R||w||_{L^2}|u(s)|_1+L|w(s)|_1\big)\\ &+L|u(s)|_1|M_{u+w}(s)-M_u(s)|_2,\end{aligned}$$

for a.e. $s\in[0,1]$, where we used Lemma 1.2.5 to deduce that there exists $C'_R>0$ such that $|M_{u+w}(s)|\le C'_R$ for every $s\in[0,1]$. Recalling that the Cauchy datum of (1.2.15) prescribes $M_{u+w}(0)=M_u(0)=\mathrm{Id}$, the last inequality yields

$$\begin{aligned}|M_{u+w}(s)-M_u(s)|_2 &\le \int_0^s |\dot M_{u+w}(\tau)-\dot M_u(\tau)|_2\,d\tau\\ &\le C''_R||w||_{L^2} + L\int_0^s |u(s)|_1|M_{u+w}(\tau)-M_u(\tau)|_2\,d\tau,\end{aligned}$$

for every $s \in [0,1]$, where we used (1.2.2) and where $C''_R > 0$ is a constant depending only on R. Finally, Lemma 1.2.1 implies the first inequality of the thesis. Recalling that $s \mapsto M_u^{-1}(s)$ and $s \mapsto M_{u+w}^{-1}(s)$ are absolutely continuous curves that solve (1.2.17), repeating *verbatim* the same argument as above, we deduce the second inequality of the thesis. □

We are now in position to prove the regularity result on the differential of the end-point map.

Proposition 1.2.9. *Let us assume that the vector fields $F_1, \dots, F_k$ defining the control system* (1.1.6) *are C^2-regular. Then, for every $R > 0$ there exists $L_R > 0$ such that, for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$, the following inequality holds*

$$|D_{u+w}P_s(v) - D_uP_s(v)|_2 \leq L_R||w||_{L^2}||v||_{L^2} \tag{1.2.31}$$

for every $s \in [0,1]$ and for every $v \in \mathcal{U}$.

Proof. In virtue of Proposition 1.2.7, it is sufficient to prove that there exists $L_R > 0$ such that

$$||g^j_{s,u+w} - g^j_{s,u}||_{L^2} \leq L_R||w||_{L^2} \tag{1.2.32}$$

for every $j = 1, \dots, n$ and for every $u, w \in \mathcal{U}$ such that $||u||_{L^2}, ||w||_{L^2} \leq R$, where $g^j_{s,u+w}, g^j_{s,u}$ are defined as in (1.2.22). Let us consider $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. The inequality (1.2.32) will in turn follow if we show that there exists a constant $L_R > 0$ such that

$$|M_{u+w}(s)M_{u+w}^{-1}(\tau)F(x_{u+w}(\tau)) - M_u(s)M_u^{-1}(\tau)F(x_u(\tau))|_2 \leq L_R||w||_{L^2}, \tag{1.2.33}$$

for every $s \in [0,1]$, for every $\tau \in [0,s]$ and for every $u, w \in \mathcal{U}$ that satisfy $||u||_{L^2}, ||w||_{L^2} \leq R$. Owing to Proposition 1.2.3 and (1.1.2), it follows that there exists $L'_R > 0$ such that

$$|F(x_{u+w}(s)) - F(x_u(s))|_2 \leq L'_R||w||_{L^2}, \tag{1.2.34}$$

for every $s \in [0,1]$ and for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. Using the triangular inequality in (1.2.33), we compute

$$\begin{aligned}
&|M_{u+w}(s)M_{u+w}^{-1}(\tau)F(x_{u+w}(\tau)) - M_u(s)M_u^{-1}(\tau)F(x_u(\tau))|_2 \\
&\qquad \leq |M_{u+w}(s) - M_u(s)|_2|M_{u+w}^{-1}(\tau)|_2|F(x_{u+w}(\tau))|_2 \\
&\qquad\quad + |M_u(s)|_2|M_{u+w}^{-1}(\tau) - M_u^{-1}(\tau)|_2|F(x_{u+w}(\tau))|_2 \\
&\qquad\quad + |M_u(s)|_2|M_u^{-1}(\tau)|_2|F(x_{u+w}(\tau)) - F(x_u(\tau))|_2
\end{aligned}$$

for every $s \in [0,1]$ and for every $\tau \in [0,s]$. Using (1.2.34), Lemma 1.2.5 and Lemma 1.2.8 in the last inequality, we deduce that (1.2.33) holds. This concludes the proof. □

1.3. Second-order differential of the end-point map

In this subsection we study the second-order variation of the end-point map $P_s : \mathcal{U} \to \mathbb{R}^n$ defined in (1.2.20). The main results reported here will be stated in the case $s = 1$, which corresponds to the final evolution instant of the control system (1.1.6). However, they can be extended (with minor adjustments) also in the case $s \in (0, 1)$. Similarly as done in Section 1.2, we show that, under proper regularity assumptions on the controlled vector fields $F_1, \ldots, F_k$, the end-point map $P_1 : \mathcal{U} \to \mathbb{R}^n$ is C^2-regular. Therefore, for every $u \in \mathcal{U}$ we can consider the second differential $D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$, which turns out to be a bilinear and symmetric operator. For every $\nu \in \mathbb{R}^n$, we provide a representation of the bilinear form $\nu \cdot D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$, and we prove that it is a compact self-adjoint operator.

Before proceeding, we introduce some notations. We define $\mathcal{V} := L^2([0, 1], \mathbb{R}^n)$, and we equip it with the usual Hilbert space structure. In order to avoid confusion, in the present section we denote with $|| \cdot ||_{\mathcal{U}}$ and $|| \cdot ||_{\mathcal{V}}$ the norms of the Hilbert spaces $\mathcal{U}$ and $\mathcal{V}$, respectively. We use a similar convention for the respective scalar products, too. Moreover, given an application $\mathcal{R} : \mathcal{U} \to \mathcal{V}$, for every $u \in \mathcal{U}$ we use the notation $\mathcal{R}[u] \in \mathcal{V}$ to denote the image of u through $\mathcal{R}$. Then, for $s \in [0, 1]$, we write $\mathcal{R}[u](s) \in \mathbb{R}^n$ to refer to the value of (a representative of) the function $\mathcal{R}[u]$ at the point s. More generally, we adopt this convention for every function-valued operator.

It is convenient to introduce a linear operator that will be useful to derive the expression of the second differential of the end-point map. Assuming that the controlled fields $F_1, \ldots, F_k$ are C^1-regular, for every $u \in \mathcal{U}$ we define $\mathcal{L}_u : \mathcal{U} \to \mathcal{V}$ as follows:

$$\mathcal{L}_u[v](s) := y_u^v(s) \tag{1.3.1}$$

for every $s \in [0, 1]$, where $y_u^v : [0, 1] \to \mathbb{R}^n$ is the curve introduced in Proposition 1.2.4 that solves the affine system (1.2.9). Recalling (1.2.18), we have that the identity

$$\mathcal{L}_u[v](s) = \int_0^s M_u(s) M_u^{-1}(\tau) F(x_u(\tau)) v(\tau) \, d\tau \tag{1.3.2}$$

holds for every $s \in [0, 1]$ and for every $v \in \mathcal{U}$, and this shows that $\mathcal{L}_u$ is a linear operator. Moreover, using the standard Hilbert space structure of $\mathcal{U}$ and of $\mathcal{V}$, for every $u \in \mathcal{U}$ we can introduce the adjoint of $\mathcal{L}_u$, namely the linear operator $\mathcal{L}_u^* : \mathcal{V} \to \mathcal{U}$ that satisfies

$$\langle \mathcal{L}_u^*[w], v \rangle_{\mathcal{U}} = \langle \mathcal{L}_u[v], w \rangle_{\mathcal{V}} \tag{1.3.3}$$

for every $v \in \mathcal{U}$ and $w \in \mathcal{V}$.

Remark 1.3.1. We recall a result in functional analysis concerning the norm of the adjoint of a bounded linear operator. For further details, see [**15**, Remark 2.16]. Given two Banach spaces E_1, E_2, let $\mathscr{L}(E_1, E_2)$ be the Banach space of the bounded linear operators from E_1 to E_2, equipped with the norm induced by E_1 and E_2. Let E_1^*, E_2^* be the dual spaces of E_1, E_2, respectively, and let $\mathscr{L}(E_2^*, E_1^*)$ be defined as above. Therefore, if $A \in \mathscr{L}(E_1, E_2)$, then the adjoint operator satisfies $A^* \in \mathscr{L}(E_2^*, E_1^*)$, and the following identity holds:

$$||A^*||_{\mathscr{L}(E_2^*, E_1^*)} = ||A||_{\mathscr{L}(E_1, E_2)}.$$

If E_1, E_2 are Hilbert spaces, using the Riesz's isometry it is possible to write A^* as an element of $\mathscr{L}(E_2, E_1)$, and the identity of the norms is still satisfied.

We now show that $\mathcal{L}_u$ and $\mathcal{L}_u^*$ are bounded and compact operators.

LEMMA 1.3.1. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^1-regular. Then, for every $u \in \mathcal{U}$, the linear operators $\mathcal{L}_u : \mathcal{U} \to \mathcal{V}$ and $\mathcal{L}_u^* : \mathcal{V} \to \mathcal{U}$ defined, respectively, by* (1.3.1) *and* (1.3.3) *are bounded and compact.*

PROOF. It is sufficient to prove the statement for the operator $\mathcal{L}_u : \mathcal{U} \to \mathcal{V}$. Indeed, if $\mathcal{L}_u$ is bounded and compact, then $\mathcal{L}_u^* : \mathcal{V} \to \mathcal{U}$ is as well. Indeed, the boundedness of the adjoint descends from Remark 1.3.1, while the compactness from [**15**, Theorem 6.4]). Using Lemma 1.2.6 we obtain that, for every $u \in \mathcal{U}$, there exists $C > 0$ such that the following inequality holds

$$||\mathcal{L}_u[v]||_{C^0} \leq C||v||_{\mathcal{U}}, \tag{1.3.4}$$

for every $v \in \mathcal{U}$. Recalling the continuous inclusion $C^0([0,1], \mathbb{R}^n) \hookrightarrow \mathcal{V}$, we deduce that $\mathcal{L}_u$ is a continuous linear operator. In view of Theorem 1.1.1, in order to prove that $\mathcal{L}_u$ is compact, it is sufficient to prove that, for every $u \in \mathcal{U}$, there exists $C' > 0$ such that

$$||\mathcal{L}_u[v]||_{H^1} \leq C'||v||_{\mathcal{U}} \tag{1.3.5}$$

for every $v \in \mathcal{U}$. However, from the definition of $\mathcal{L}_u[v]$ given in (1.3.1), it follows that

$$\frac{d}{ds}\mathcal{L}_u[v](s) = \dot{y}_u^v(s)$$

for a.e. $s \in [0,1]$. Therefore, from (1.2.9) and Lemma 1.2.6, we deduce that (1.3.5) holds. □

In the next result we study the local Lipschitz-continuity of the correspondence $u \mapsto \mathcal{L}_u$.

LEMMA 1.3.2. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^2-regular. Then, for every $R > 0$ there exists $L_R > 0$ such that*

$$||\mathcal{L}_{u+w}[v] - \mathcal{L}_u[v]||_{\mathcal{V}} \leq L_R||w||_{\mathcal{U}}||v||_{\mathcal{U}} \tag{1.3.6}$$

for every $v \in \mathcal{U}$ and for every $u, w \in \mathcal{U}$ such that $||u||_{\mathcal{U}}, ||w||_{\mathcal{U}} \leq R$.

PROOF. Recalling the continuous inclusion $C^0([0,1],\mathbb{R}^n) \hookrightarrow \mathcal{V}$, it is sufficient to prove that for every $R > 0$ there exists $L_R > 0$ such that, for every $s \in [0,1]$, the following inequality is satisfied

$$|\mathcal{L}_{u+w}[v](s) - \mathcal{L}_u[v](s)|_2 \leq L_R ||w||_{\mathcal{U}} ||v||_{\mathcal{U}} \tag{1.3.7}$$

for every $v \in \mathcal{U}$ and for every $u, w \in \mathcal{U}$ such that $||u||_{\mathcal{U}}, ||w||_{\mathcal{U}} \leq R$. On the other hand, (1.3.2) implies that

$$|\mathcal{L}_{u+w}[v](s) - \mathcal{L}_u[v](s)|_2 \leq \int_0^s |M_{u+w}(s)M_{u+w}^{-1}(\tau)F(x_{u+w}(\tau)) - M_u(s)M_u^{-1}(\tau)F(x_u(\tau))|_2 |v(\tau)|_2 \, d\tau.$$

However, using Proposition 1.2.3, Lemma 1.2.5 and Lemma 1.2.8, we obtain that there exists $L'_R > 0$ such that

$$|M_{u+w}(s)M_{u+w}^{-1}(\tau)F(x_{u+w}(\tau)) - M_u(s)M_u^{-1}(\tau)F(x_u(\tau))|_2 \leq L'_R ||w||_{\mathcal{U}}$$

for every $s, \tau \in [0,1]$ and for every $u, w \in \mathcal{U}$ such that $||u||_{\mathcal{U}}, ||w||_{\mathcal{U}} \leq R$. Combining the last two inequalities, we deduce that (1.3.7) holds. □

REMARK 1.3.2. From Lemma 1.3.2 and Remark 1.3.1 it follows that the correspondence $u \mapsto \mathcal{L}_u^*$ is as well Lipschitz-continuous on the bounded sets of $\mathcal{U}$.

If the vector fields $F_1, \ldots, F_k$ are C^2-regular, we write $\frac{\partial^2 F_1}{\partial x^2}, \ldots, \frac{\partial^2 F_k}{\partial x^2}$ to denote their second differential. In the next result we investigate the second-order variation of the solutions produced by the control system (1.1.6).

PROPOSITION 1.3.3. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^2-regular. For every $u, v, w \in \mathcal{U}$, for every $\varepsilon \in (0,1]$, let $y_u^v, y_{u+\varepsilon w}^v : [0,1] \to \mathbb{R}^n$ be the solutions of* (1.2.9) *corresponding to the first-order variation v and to the admissible controls u and $u + \varepsilon w$, respectively. Therefore, we have that*

$$\sup_{||v||_{L^2} \leq 1} ||y_{u+\varepsilon w}^v - y_u^v - \varepsilon z_u^{v,w}||_{C^0} = o(\varepsilon) \text{ as } \varepsilon \to 0, \tag{1.3.8}$$

where $z_u^{v,w} : [0,1] \to \mathbb{R}^n$ is the solution of the following affine system:

$$\dot{z}_u^{v,w}(s) = \sum_{i=1}^k \left[v^i(s) \frac{\partial F_i(x_u(s))}{\partial x} y_u^w(s) + w^i(s) \frac{\partial F_i(x_u(s))}{\partial x} y_u^v(s) \right] \tag{1.3.9}$$

$$+ \sum_{i=1}^k u^i(s) \frac{\partial^2 F_i(x_u(s))}{\partial x^2} (y_u^v(s), y_u^w(s)) \tag{1.3.10}$$

$$+ \sum_{i=1}^k u^i(s) \frac{\partial F_i(x_u(s))}{\partial x} z_u^{v,w}(s) \tag{1.3.11}$$

with $z_u^{v,w}(0) = 0$*, and where* $y_u^v, y_u^w : [0,1] \to \mathbb{R}^n$ *are the solutions of* (1.2.9) *corresponding to the admissible control* u *and to the first-order variations* v *and* w*, respectively.*

PROOF. The proof of this result follows using the same kind of techniques and computations as in the proof of Proposition 1.2.4. □

REMARK 1.3.3. Similarly as done in (1.2.18) for the first-order variation, we can express the solution of the affine system (1.3.9)-(1.3.11) through an integral formula. Indeed, for every $u, v, w \in \mathcal{U}$, for every $s \in [0,1]$ we have that

$$z_u^{v,w}(s) = \int_0^s M_u(s)M_u^{-1}(\tau)\left(\sum_{i=1}^k v^i(\tau)\frac{\partial F_i(x_u(\tau))}{\partial x}\mathcal{L}_u[w](\tau)\right. \tag{1.3.12}$$

$$+\sum_{i=1}^k w^i(\tau)\frac{\partial F_i(x_u(\tau))}{\partial x}\mathcal{L}_u[v](\tau) \tag{1.3.13}$$

$$\left.+\sum_{i=1}^k u^i(\tau)\frac{\partial^2 F_i(x_u(\tau))}{\partial x^2}(\mathcal{L}_u[v](\tau), \mathcal{L}_u[w](\tau))\right) d\tau, \tag{1.3.14}$$

where we used the linear operator $\mathcal{L}_u : \mathcal{U} \to \mathcal{V}$ defined in (1.3.1). From the previous expression it follows that, for every $u, v, w \in \mathcal{U}$, the roles of v and w are interchangeable, i.e., for every $s \in [0,1]$ we have that $z_u^{v,w}(s) = z_u^{w,v}(s)$. Moreover, we observe that, for every $s \in [0,1]$ and for every $u \in \mathcal{U}$, $z_u^{v,w}(s)$ is bilinear with respect to v and w.

We are now in position to introduce the second differential of the end-point map $P_s : \mathcal{U} \to \mathbb{R}^n$ defined in (1.2.20). In view of the applications in the forthcoming sections, we shall focus on the case $s = 1$, i.e., we consider the map $P_1 : \mathcal{U} \to \mathbb{R}^n$. Before proceeding, for every $u \in \mathcal{U}$ we define the symmetric and bilinear map $\mathcal{B}_u : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$ as follows

$$\mathcal{B}_u(v, w) := z_u^{v,w}(1). \tag{1.3.15}$$

PROPOSITION 1.3.4. *Let us assume that the vector fields* $F_1, \ldots, F_k$ *defining the control system* (1.1.6) *are* C^2*-regular. Let* $P_1 : \mathcal{U} \to \mathbb{R}^n$ *be the end-point map defined by* (1.2.20)*, and, for every* $u \in \mathcal{U}$*, let* $D_uP_1 : \mathcal{U} \to \mathbb{R}^n$ *be its differential. Then, the correspondence* $u \mapsto D_uP_1$ *is Gateaux differentiable at every* $u \in \mathcal{U}$*, namely*

$$\lim_{\varepsilon \to 0} \sup_{||v||_{L^2} \le 1} \left| \frac{D_{u+\varepsilon w}P_1(v) - D_uP_1(v)}{\varepsilon} - \mathcal{B}_u(v,w) \right|_2 = 0, \tag{1.3.16}$$

where $B_u : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$ *is the bilinear, symmetric and bounded operator defined in* (1.3.15)*.*

Proof. In view of (1.2.23), for every $u, v, w \in \mathcal{U}$ and for every $\varepsilon \in (0,1]$, we have that $D_u P_1(v) = y_u^v(1)$ and $D_{u+\varepsilon w} P_1(v) = y_{u+\varepsilon w}^v(1)$. Therefore, (1.3.16) follows directly from (1.3.8) and from (1.3.15). The symmetry and the bilinearity of $\mathcal{B}_u : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$ descend from the observations in Remark 1.3.3. Finally, we have to show that, for every $u \in \mathcal{U}$, there exists $C > 0$ such that

$$|\mathcal{B}_u(v,w)|_2 \leq C||v||_{L^2}||w||_{L^2}$$

for every $v, w \in \mathcal{U}$. Recalling (1.3.15) and the integral expression (1.3.12)-(1.3.14), the last inequality follows from the estimate (1.3.4), from Lemma 1.2.5, from Proposition 1.2.2 and from the C^2-regularity of $F_1, \ldots, F_k$. □

In view of the previous result, for every $u \in \mathcal{U}$, we use $D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$ to denote the second differential of the end-point map $P_1 : \mathcal{U} \to \mathbb{R}^n$. Moreover, for every $u, v, w \in \mathcal{U}$ we have the following identities:

$$D_u^2 P_1(v,w) = \mathcal{B}_u(v,w) = z_u^{v,w}(1). \tag{1.3.17}$$

Remark 1.3.4. It is possible to prove that the correspondence $u \mapsto D_u^2 P_1$ is continuous. In particular, under the further assumption that the controlled vector fields $F_1, \ldots, F_k$ are C^3-regular, the application $u \mapsto D_u^2 P_1$ is Lipschitz-continuous on the bounded subsets of $\mathcal{U}$. Indeed, taking into account (1.3.17) and (1.3.12)-(1.3.14), this fact follows from Lemma 1.2.8, from Lemma 1.3.2 and from the regularity of $F_1, \ldots, F_k$.

For every $\nu \in \mathbb{R}^n$ and for every $u \in \mathcal{U}$, we can consider the bilinear form $\nu \cdot D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$, which is defined as

$$\nu \cdot D_u^2 P_1(v,w) := \langle \nu, D_u^2 P_1(v,w) \rangle_{\mathbb{R}^n}. \tag{1.3.18}$$

We conclude this section by showing that, using the scalar product of $\mathcal{U}$, the bilinear form defined in (1.3.18) can be represented as a self-adjoint compact operator. Before proceeding, it is convenient to introduce two auxiliary linear operators. In this part we assume that the vector fields $F_1, \ldots, F_k$ are C^2-regular. For every $u \in \mathcal{U}$ let us consider the application $\mathcal{M}_u^\nu : \mathcal{U} \to \mathcal{V}$ defined as follows:

$$\mathcal{M}_u^\nu[v](\tau) := \left(M_u(1) M_u^{-1}(\tau) \sum_{i=1}^k v^i(\tau) \frac{\partial F_i(x_u(\tau))}{\partial x} \right)^T \nu \tag{1.3.19}$$

for a.e. $\tau \in [0,1]$, where $x_u : [0,1] \to \mathbb{R}^n$ is the solution of (1.1.6) and $M_u : [0,1] \to \mathbb{R}^{n \times n}$ is defined in (1.2.15). We recall that, for every $i = 1, \ldots, k$ and for every $y \in \mathbb{R}^n$, $\frac{\partial^2 F_i(y)}{\partial x^2} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric and bilinear function. Hence, for every $i = 1, \ldots, k$, for every $u \in \mathcal{U}$ and for every $\tau \in [0,1]$, we have that the application

$$(\eta_1, \eta_2) \mapsto \nu^T M_u(1) M_u^{-1}(\tau) \frac{\partial^2 F_i(x_u(\tau))}{\partial x^2}(\eta_1, \eta_2)$$

is a symmetric and bilinear form on $\mathbb{R}^n$. Therefore, for every $i = 1, \dots, k$, for every $u \in \mathcal{U}$ and for every $\tau \in [0,1]$, we introduce the symmetric matrix $S_u^{\nu,i}(\tau) \in \mathbb{R}^{n\times n}$ that satisfies the identity

$$\langle S_u^{\nu,i}(\tau)\eta_1, \eta_2\rangle_{\mathbb{R}^n} = \nu^T M_u(1) M_u^{-1}(\tau) \frac{\partial^2 F_i(x_u(\tau))}{\partial x^2}(\eta_1, \eta_2)$$

for every $\eta_1, \eta_2 \in \mathbb{R}^n$. We define the linear operator $\mathcal{S}_u^\nu : C^0([0,1], \mathbb{R}^n) \to \mathcal{V}$ as follows:

$$\mathcal{S}_u^\nu[v](\tau) := \sum_{i=1}^k u^i(\tau) S_u^{\nu,i}(\tau) v(\tau) \tag{1.3.20}$$

for every $v \in C^0([0,1], \mathbb{R}^n)$ and for a.e. $\tau \in [0,1]$.

In the next result we prove that the linear operators introduced above are both continuous.

Lemma 1.3.5. *Let us assume that the vector fields $F_1, \dots, F_k$ defining the control system* (1.1.6) *are C^2-regular. Therefore, for every $u \in \mathcal{U}$ and for every $\nu \in \mathbb{R}^n$ the linear operators $\mathcal{M}_u^\nu : \mathcal{U} \to \mathcal{V}$ and $\mathcal{S}_u^\nu : C^0([0,1], \mathbb{R}^n) \to \mathcal{V}$ defined, respectively, in* (1.3.19) *and* (1.3.20) *are continuous.*

Proof. Let us start with $\mathcal{M}_u^\nu : \mathcal{U} \to \mathcal{V}$. Using Lemma 1.2.5 and (1.1.4), we immediately deduce that there exists $C_1 > 0$ such that

$$||\mathcal{M}_u^\nu[v]||_\mathcal{V} \le C_1 ||v||_\mathcal{U}$$

for every $v \in \mathcal{U}$. As regards $\mathcal{S}^\nu : C^0([0,1], \mathbb{R}^n) \to \mathcal{V}$, from (1.3.20) we deduce that

$$\left|\mathcal{S}_u^\nu[v](\tau)\right|_2 \le \left(\sum_{i=1}^k |u^i(\tau)| |S_u^{\nu,i}(\tau)|_2 \right) ||v||_{C^0}$$

for every $v \in \mathcal{U}$ and for a.e. $\tau \in [0,1]$. Moreover, from Lemma 1.2.5, from Lemma 1.2.2 and the regularity of $F_1, \dots, F_k$, we deduce that there exists $C' > 0$ such that

$$|S_u^{\nu,i}(\tau)|_2 \le C'$$

for every $\tau \in [0,1]$. Combining the last two inequalities and recalling that $u \in \mathcal{U} = L^2([0,1], \mathbb{R}^k)$, we deduce that the linear operator $\mathcal{S}_u^\nu : C^0([0,1], \mathbb{R}^n) \to \mathcal{V}$ is continuous. □

We are now in position to represent the bilinear form $\nu \cdot D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ through the scalar product of $\mathcal{U}$. Indeed, recalling (1.3.18) and (1.3.17), from (1.3.12)-(1.3.14) for every $u \in \mathcal{U}$ we obtain that

$$\begin{aligned} \nu \cdot D_u^2 P_1(v,w) &= \langle M_u^\nu[v], \mathcal{L}_u[w]\rangle_\mathcal{V} + \langle M_u^\nu[w], \mathcal{L}_u[v]\rangle_\mathcal{V} + \langle \mathcal{S}_u^\nu \mathcal{L}_u[v], \mathcal{L}_u[w]\rangle_\mathcal{V} \\ &= \langle \mathcal{L}_u^* M_u^\nu[v], w\rangle_\mathcal{U} + \langle (\mathcal{M}_u^\nu)^* \mathcal{L}_u[v], w\rangle_\mathcal{U} + \langle \mathcal{L}_u^* \mathcal{S}_u^\nu \mathcal{L}_u[v], w\rangle_\mathcal{U} \end{aligned}$$

for every $v, w \in \mathcal{U}$, where $(\mathcal{M}_u^\nu)^* : \mathcal{V} \to \mathcal{U}$ is the adjoint of the linear operator $\mathcal{M}_u^\nu : \mathcal{U} \to \mathcal{V}$. Recalling Remark 1.3.1, we have that $(\mathcal{M}_u^\nu)^*$ is a bounded linear operator. This shows that the bilinear form $\nu \cdot D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ can be represented by the linear operator $\mathcal{N}_u^\nu : \mathcal{U} \to \mathcal{U}$, i.e.,

$$\nu \cdot D_u^2 P_1(v, w) = \langle \mathcal{N}_u^\nu[v], w \rangle_{\mathcal{U}} \tag{1.3.21}$$

for every $v, w \in \mathcal{U}$, where

$$\mathcal{N}_u^\nu := \mathcal{L}_u^* M_u^\nu + (\mathcal{M}_u^\nu)^* \mathcal{L}_u + \mathcal{L}_u^* \mathcal{S}_u^\nu \mathcal{L}_u. \tag{1.3.22}$$

We conclude this section by proving that $\mathcal{N}_u^\nu : \mathcal{U} \to \mathcal{U}$ is a bounded, compact and self-adjoint operator.

PROPOSITION 1.3.6. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^2-regular. For every $u \in \mathcal{U}$ and for every $\nu \in \mathbb{R}^n$, let $\mathcal{N}_u^\nu : \mathcal{U} \to \mathcal{U}$ be the linear operator that represents the bilinear form $\nu \cdot D_u^2 P_1 : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ through the identity* (1.3.21)*. Then $\mathcal{N}_u^\nu$ is continuous, compact and self-adjoint.*

PROOF. We observe that the term $\mathcal{L}_u^* M_u^\nu + (\mathcal{M}_u^\nu)^* \mathcal{L}_u$ at the right-hand side of (1.3.22) is continuous, since it is obtained as the sum and the composition of continuous linear operators, as shown in Lemma 1.3.1 and Lemma 1.3.5. Moreover, it is also compact, since both $\mathcal{L}_u$ and $\mathcal{L}_u^*$ are, in virtue of Lemma 1.3.1. Finally, the fact that $\mathcal{L}_u^* M_u^\nu + (\mathcal{M}_u^\nu)^* \mathcal{L}_u$ is self-adjoint is immediate. Let us consider the last term at the right-hand side of (1.3.22), i.e., $\mathcal{L}_u^* \mathcal{S}_u^\nu \mathcal{L}_u$. We first observe that $\mathcal{S}_u^\nu \mathcal{L}_u : \mathcal{U} \to \mathcal{V}$ is continuous, owing to Lemma 1.3.5 and the inequality (1.3.4). Recalling that $\mathcal{L}_u^* : \mathcal{V} \to \mathcal{U}$ is compact, the composition $\mathcal{L}_u^* \mathcal{S}_u^\nu \mathcal{L}_u : \mathcal{U} \to \mathcal{U}$ is compact as well. Once again, the operator is clearly self-adjoint. □

1.4. Stability of trajectories with weakly convergent controls

We conclude this chapter by proving that the solutions of (1.1.6) corresponding to weakly convergent controls are C^0-convergent. Namely, we consider $x_0 \in \mathbb{R}^n$ and a L^2-weakly convergent sequence of controls $(u_m)_{m \geq 1} \subset \mathcal{U}$, and we study the convergence of the trajectories $(x_m)_{m \geq 1}$, where, for every $m \geq 1$, the curve $x_m : [0, 1] \to \mathbb{R}^n$ is the solution of the Cauchy problem (1.1.6) corresponding to the admissible control u_m and satisfying $x_m(0) = x_0$.

PROPOSITION 1.4.1. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *satisfy the Lipschitz-continuity condition* (1.1.2)*. Let $(u_m)_{m \geq 1} \subset \mathcal{U}$ be a sequence such that $u_m \rightharpoonup_{L^2} u_\infty$ as $m \to \infty$. For every $m \in \mathbb{N} \cup \{\infty\}$, let $x_m : [0, 1] \to \mathbb{R}^n$ be the solution of* (1.1.6) *corresponding to the control u_m. Then, we have that*

$$\lim_{m \to \infty} ||x_m - x_\infty||_{C^0} = 0.$$

PROOF. Being the sequence $(u_m)_{m \geq 1}$ weakly convergent, we deduce that there exists $R > 0$ such that $||u_m||_{L^2} \leq R$ for every $m \geq 1$. The estimate established in

Lemma 1.2.2 implies that there exists $C_R > 0$ such that

$$||x_m||_{C^0} \leq C_R, \tag{1.4.1}$$

for every $m \geq 1$. Moreover, using the sub-linear growth inequality (1.1.3), we have that there exists $C > 0$ such that

$$|\dot{x}_m(s)|_2 \leq \sum_{i=1}^{k} |F_i(x_m(s)|_2 |u_m^j(s)| \leq C(1 + C_R) \sum_{i=1}^{k} |u_m^i(s)|,$$

for a.e. $s \in [0,1]$. Then, recalling that $||u_m||_{L^2} \leq R$ for every $m \geq 1$, we deduce that

$$||\dot{x}_m||_{L^2} \leq C(1 + C_R)kR \tag{1.4.2}$$

for every $m \geq 1$. Combining (1.4.1) and (1.4.2), we obtain that the sequence $(x_m)_{m\geq 1}$ is pre-compact with respect to the weak topology of $H^1([0,1],\mathbb{R}^n)$. Our goal is to prove that the set of the H^1-weak limiting points of the sequence $(x_m)_{m\geq 1}$ coincides with $\{x_\infty\}$, i.e., that the whole sequence $x_m \rightharpoonup_{H^1} x_\infty$ as $m \to \infty$. Let $\hat{x} \in H^1([0,1],\mathbb{R}^n)$ be any H^1-weak limiting point of the sequence $(x_m)_{m\geq 1}$, and let $(x_{m_\ell})_{\ell \geq 1}$ be a sub-sequence such that $x_{m_\ell} \rightharpoonup_{H^1} \hat{x}$ as $\ell \to \infty$. Recalling (1.1.9) in Theorem 1.1.1, we have that the inclusion $H^1([0,1],\mathbb{R}^n) \hookrightarrow C^0([0,1],\mathbb{R}^n)$ is compact, and this implies that

$$x_{m_\ell} \to_{C^0} \hat{x} \tag{1.4.3}$$

as $\ell \to \infty$. From (1.4.3) and the assumption (1.1.2), for every $i = 1,\ldots,k$ it follows that

$$||F_i(x_{m_l}) - F_i(\hat{x})||_{C^0} \to 0 \tag{1.4.4}$$

as $\ell \to \infty$. Let us consider a smooth and compactly supported test function $\phi \in C_c^\infty([0,1],\mathbb{R}^n)$. Therefore, recalling that x_{m_ℓ} is the solution of the Cauchy problem (1.1.6) corresponding to the control $u_{m_\ell} \in \mathcal{U}$, we have that

$$\int_0^1 x_{m_\ell}(s) \cdot \dot{\phi}(s)\, ds = -\sum_{i=1}^{k} \int_0^1 \left(F_i(x_{m_\ell}(s)) \cdot \phi(s)\right) u_{m_\ell}^j(s)\, ds$$

for every $\ell \geq 1$. Thus, passing to the limit as $\ell \to \infty$ in the previous identity, we obtain

$$\int_0^1 \hat{x}(s) \cdot \dot{\phi}(s)\, ds = -\sum_{i=1}^{k} \int_0^1 \left(F_i(\hat{x}(s)) \cdot \phi(s)\right) u_\infty^j(s)\, ds. \tag{1.4.5}$$

Indeed, the convergence of the right-hand side is guaranteed by (1.4.3). On the other hand, for every $j = 1,\ldots,k$, from (1.4.4) we deduce the strong convergence $F_i(x_{m_\ell}) \cdot \phi \to_{L^2} F_i(\hat{x}) \cdot \phi$ as $\ell \to \infty$, while $u_{m_\ell}^j \rightharpoonup_{L^2} u_\infty^j$ as $\ell \to \infty$ by the hypothesis. Finally, observing that (1.4.3) gives $\hat{x}(0) = x_0$, we deduce that

$$\begin{cases} \dot{\hat{x}}(s) = F(\hat{x}(s))u_\infty(s), & \text{for a.e. } s \in [0,1], \\ \hat{x}(0) = x_0, \end{cases}$$

that implies $\hat{x} \equiv x_\infty$. This argument shows that $x_m \rightharpoonup_{H^1} x_\infty$ as $m \to \infty$. Finally, the thesis follows using again the compact inclusion (1.1.9). $\square$

CHAPTER 2

Gradient flow for optimal control problems with end-point cost

In this chapter we study the gradient flows associated to optimal control problems of linear-control systems with end-point cost and with a L^2-squared penalization on the controls. In particular, we shall derive the gradient field induced by $\mathcal{F}$ on the its domain, i.e., the Hilbert space $\mathcal{U}$, and we show that the gradient flow equation is well posed. The main result of the Chapter is Theorem 2.5.3, where we prove a convergence result for the trajectories of the gradient flow with Sobolev-regular initial datum.

2.1. Existence of minimizers

Using the same notations as in Chapter 1, let $x_u : [0,1] \to \mathbb{R}^n$ be the solution of the Cauchy problem

$$\begin{cases} \dot{x}_u(s) = F(x_u(s))u(s) & \text{for a.e. } s \in [0,1], \\ x_u(0) = x_0, \end{cases} \tag{2.1.1}$$

where $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^{n \times k}$ satisfies the Lipschitz-continuity condition (1.1.2), $x_0 \in \mathbb{R}^n$ is prescribed, and $u \in \mathcal{U} := L^2([0,1], \mathbb{R}^k)$. In the present chapter we study the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ defined for every $u \in \mathcal{U}$ as follows:

$$\mathcal{F}(u) := a(x_u(1)) + \frac{\beta}{2}||u||_{L^2}^2, \tag{2.1.2}$$

where $a : \mathbb{R}^n \to \mathbb{R}_+$ is a non-negative C^1-regular function, $\beta > 0$ is a positive constant, and $x_u : [0,1] \to \mathbb{R}^n$ is the solution of (2.1.1) corresponding to the control u.

We first address the question of the existence of minimizers of the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ defined in (2.1.2).

Proposition 2.1.1. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (2.1.1) *satisfy the Lipschitz-continuity condition* (1.1.2)*, and let $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ be the functional defined in* (2.1.2)*. Then there exists $\hat{u} \in \mathcal{U}$ such that*

$$\mathcal{F}(\hat{u}) = \inf_{u \in \mathcal{U}} \mathcal{F}(u).$$

PROOF. We prove the existence of minimizers by means of the direct method of calculus of variations (see, e.g., [**24**, Theorem 3.15]). Equipping $\mathcal{U}$ with the weak topology, it is sufficient to prove that the sub-levels of $\mathcal{F}$ are pre-compact, and that it is sequentially lower semi-continuous. For every $M \geq 0$ we have

$$\{u \in \mathcal{U} \mid \mathcal{F}(u) \leq M\} \subset \{u \in \mathcal{U} \mid ||u||_{L^2}^2 \leq 2M/\beta\},$$

where we used the fact that the end-point cost $a : \mathbb{R}^n \to \mathbb{R}_+$ is non-negative. The last inclusion shows the pre-compactness of the sub-levels of $\mathcal{F}$. As regards the lower semi-continuity, let us consider a sequence $(u_m)_{m\in\mathbb{N}} \subset \mathcal{U}$ such that $u_m \rightharpoonup_{L^2} u_\infty$. For every $m \in \mathbb{N} \cup \{\infty\}$, let $x_m : [0,1] \to \mathbb{R}^n$ be the solution of (2.1.1) corresponding to the admissible control u_m. Using Proposition 1.4.1, we deduce the convergence of the terminal points of the trajectories, namely that $x_m(1) \to x_\infty(1)$ as $m \to \infty$. In virtue
of the end-point cost $a : \mathbb{R}^n \to \mathbb{R}_+$, we obtain that

$$\lim_{m\to\infty} a(x_m(1)) = a(x_\infty(1)). \tag{2.1.3}$$

Recalling that the L^2-norm is lower semi-continuous with respect to the weak convergence, we have

$$\mathcal{F}(u_\infty) \leq \liminf_{m\to\infty} \mathcal{F}(u_m),$$

and this concludes the proof. □

As well as in the finite-dimensional case, the study of the gradient flow induced by $\mathcal{F}$ on its domain is motivated by the problem of finding a minimizer. In the next section we detail the derivation of the gradient field, and we prove that the gradient flow equation is well-posed.

2.2. Gradient flow: well-posedness and global definition

We consider the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ defined as in (2.1.2). In this section we want to study the gradient flow induced by the functional $\mathcal{F}$ on the Hilbert space $\mathcal{U}$. In particular, we establish a result that guarantees existence, uniqueness and global definition of the solutions of the gradient flow equation associated to $\mathcal{F}$. In this section we adopt the approach of the monograph [**33**], where the theory of ODEs in Banach spaces is developed.

We start from the notion of differentiable curve in $\mathcal{U}$. We stress that in the present Thesis the time variable t is exclusively employed for curves taking values in $\mathcal{U}$. In particular, we recall that we use $s \in [0,1]$ to denote the time variable only in the control system (1.1.6) and in the related objects (e.g., admissible controls, controlled trajectories, etc.). Given a curve $U : (a,b) \to \mathcal{U}$, we say that it is (strongly) differentiable at $t_0 \in (a,b)$ if there exists $u \in \mathcal{U}$ such that

$$\lim_{t\to t_0} \left|\left| \frac{U_t - U_{t_0}}{t - t_0} - u \right|\right|_{L^2} = 0. \tag{2.2.1}$$

In this case, we use the notation $\partial_t U_{t_0} := u$. In the present section we study the well-posedness in $\mathcal{U}$ of the evolution equation

$$\begin{cases} \partial_t U_t = -\mathcal{G}[U_t], \\ U_0 = u_0, \end{cases} \tag{2.2.2}$$

where $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ is the representation of the differential $d\mathcal{F} : \mathcal{U} \to \mathcal{U}^*$ through the Riesz isomorphism, i.e.,

$$\langle \mathcal{G}[u], v \rangle_{L^2} = d_u \mathcal{F}(v) \tag{2.2.3}$$

for every $u, v \in \mathcal{U}$. More precisely, for every initial datum $u_0 \in \mathcal{U}$ we prove that there exists a curve $t \mapsto U_t$ that solves (2.2.2), that it is unique and that it is defined for every $t \geq 0$.

We first show that $d_u\mathcal{F}$ can be actually represented as an element of $\mathcal{U}$, for every $u \in \mathcal{U}$. We immediately observe that this problem reduces to study the differential of the end-point cost, i.e., the functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$, defined as

$$\mathcal{E}(u) := a(x_u(1)), \tag{2.2.4}$$

where $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of (1.1.6) corresponding to the admissible control $u \in \mathcal{U}$.

Lemma 2.2.1. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^1-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. Then the functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$ is Gateaux differentiable at every $u \in \mathcal{U}$. Moreover, using the Riesz's isomorphism, for every $u \in \mathcal{U}$ the differential $d_u\mathcal{E} : \mathcal{U} \to \mathbb{R}$ can be represented as follows:*

$$d_u\mathcal{E}(v) = \int_0^1 \sum_{j=1}^n \left(\frac{\partial a(x_u(1))}{\partial x^j} \langle g^j_{1,u}(\tau), v(\tau) \rangle_{\mathbb{R}^k} \right) d\tau \tag{2.2.5}$$

for every $v \in \mathcal{U}$, where, for every $j = 1, \ldots, n$, the function $g^j_{1,u} \in \mathcal{U}$ is defined as in (1.2.22).

Proof. We observe that the functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$ is defined as the composition

$$\mathcal{E} = a \circ P_1,$$

where $P_1 : \mathcal{U} \to \mathbb{R}^n$ is the end-point map defined in (1.2.20). Proposition 1.2.4 guarantees that the end-point map P_1 is Gateaux differentiable at every $u \in \mathcal{U}$. Recalling that $a : \mathbb{R}^n \to \mathbb{R}_+$ is assumed to be C^1, we deduce that, for every $u \in \mathcal{U}$, $\mathcal{E}$ is Gateaux differentiable at u and that, for every $v \in \mathcal{U}$, the following identity holds:

$$d_u\mathcal{E}(v) = \sum_{j=1}^n \frac{\partial a(x_u(1))}{\partial x^j} D_u P_1^j(v), \tag{2.2.6}$$

where $x_u : [0, 1] \to \mathbb{R}^n$ is the solution of (1.1.6) corresponding to the control $u \in \mathcal{U}$. Recalling that $D_u P_1^1, \ldots, D_u P_1^n : \mathcal{U} \to \mathbb{R}$ are linear and continuous functionals for every $u \in \mathcal{U}$ (see Proposition 1.2.7), from (2.2.6) we deduce that $d_u\mathcal{E} : \mathcal{U} \to \mathbb{R}$ is as well. Finally, from (1.2.21) we obtain (2.2.5). □

REMARK 2.2.1. Similarly as done in Remark 1.2.1, we can provide a uniform estimate of the norm of $d_u\mathcal{E}$ when u varies on a bounded set. Indeed, recalling Lemma 1.2.2 and the fact that $a : \mathbb{R}^n \to \mathbb{R}_+$ is C^1-regular, for every $R > 0$ there exists $C'_R > 0$ such that

$$\left| \frac{\partial a(x_u(1))}{\partial x^j} \right| \leq C'_R$$

for every $j = 1, \ldots, n$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. Combining the last inequality with (2.2.6) and (1.2.24), we deduce that there exists $C_R > 0$ such that for every $||u||_{L^2} \leq R$ the estimate

$$|d_u\mathcal{E}(v)|_2 \leq C_R ||v||_{L^2} \tag{2.2.7}$$

holds for every $v \in \mathcal{U}$.

REMARK 2.2.2. We observe that, for every $u, v \in \mathcal{U}$, we can rewrite (2.2.5) as follows

$$d_u\mathcal{E}(v) = \int_0^1 \left\langle F^T(x_u(\tau))\lambda_u^T(\tau), v(\tau) \right\rangle_{\mathbb{R}^k} d\tau, \tag{2.2.8}$$

where $\lambda_u : [0,1] \to (\mathbb{R}^n)^*$ is an absolutely continuous curve defined for every $s \in [0,1]$ by the relation

$$\lambda_u(s) := \nabla a(x_u(1)) \cdot M_u(1) M_u^{-1}(s), \tag{2.2.9}$$

where $M_u : [0,1] \to \mathbb{R}^{n\times n}$ is defined as in (1.2.15), and $\nabla a(x_u(1))$ is understood as a row vector. Recalling that $s \mapsto M_u^{-1}(s)$ solves (1.2.17), it turns out that $s \mapsto \lambda_u(s)$ is the solution of the following linear Cauchy problem:

$$\begin{cases} \dot\lambda_u(s) = -\lambda_u(s) \sum_{i=1}^k \left(u^i(s) \frac{\partial F_i(x_u(s))}{\partial x} \right) & \text{for a.e. } s \in [0,1], \\ \lambda_u(1) = \nabla a(x_u(1)). \end{cases} \tag{2.2.10}$$

Finally, (2.2.8) implies that, for every $u \in \mathcal{U}$, we can represent $d_u\mathcal{E}$ with the function $h_u : [0,1] \to \mathbb{R}^k$ defined as

$$h_u(s) := F^T(x_u(s))\lambda_u^T(s) \tag{2.2.11}$$

for a.e. $s \in [0,1]$. We observe that (2.2.7) and the Riesz's isometry imply that for every $R > 0$ there exists $C_R > 0$ such that

$$||h_u||_{L^2} \leq C_R \tag{2.2.12}$$

for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. We further underline that the representation $h_u : [0,1] \to \mathbb{R}^k$ of the differential $d_u\mathcal{E}$ is actually absolutely continuous, similarly as observed in Remark 1.2.2 for the representations of the components of the differential of the end-point map.

Under the assumption that the controlled vector fields $F_1, \ldots, F_k$ and the function $a : \mathbb{R}^n \to \mathbb{R}_+$ are C^2-regular, we can show that the differential $u \mapsto d_u\mathcal{E}$ is Lipschitz-continuous on bounded sets.

LEMMA 2.2.2. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^2-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. Then, for every $R > 0$ there exists $L_R > 0$ such that*

$$||h_{u+w} - h_u||_{L^2} \leq L_R||w||_{L^2} \tag{2.2.13}$$

for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$, where h_{u+w}, h_u are the representations, respectively, of $d_{u+w}\mathcal{E}$ and $d_u\mathcal{E}$ provided by (2.2.11).

PROOF. Let us consider $R > 0$. In virtue of (2.2.5), it is sufficient to prove that there exists $L_R > 0$ such that

$$\left|\left| \frac{\partial a(x_{u+w}(1))}{\partial x^j} g^j_{1,u+w} - \frac{\partial a(x_u(1))}{\partial x^j} g^j_{1,u} \right|\right|_{L^2} \leq L_R||w||_{L^2} \tag{2.2.14}$$

for every $j = 1, \ldots, n$ and for every $u, w \in \mathcal{U}$ such that $||u||_{L^2}, ||w||_{L^2} \leq R$. Lemma 1.2.2 implies that there exists a compact set $K_R \subset \mathbb{R}^n$ depending only on R such that $x_u(1), x_{u+w}(1) \in K_R$ for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. Recalling that $a : \mathbb{R}^n \to \mathbb{R}_+$ is assumed to be C^2-regular, we deduce that there exists $L'_R > 0$ such that

$$\left| \frac{\partial a(y_1)}{\partial x^j} - \frac{\partial a(y_2)}{\partial x^j} \right|_2 \leq L'_R|y_1 - y_2|_2$$

for every $y_1, y_2 \in K_R$. Moreover, combining the previous inequality with (1.2.4), we deduce that there exists $L^1_R > 0$ such that

$$\left| \frac{\partial a(x_{u+w}(1))}{\partial x^j} - \frac{\partial a(x_u(1))}{\partial x^j} \right|_2 \leq L^1_R||w||_{L^2} \tag{2.2.15}$$

for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. On the other hand, using (1.2.32), we have that there exists $L^2_R > 0$ such that

$$\left|\left| g^j_{1,u+w} - g^j_{1,u} \right|\right|_{L^2} \leq L^2_R||w||_{L^2} \tag{2.2.16}$$

for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. Combining (2.2.15) and (2.2.16), and recalling (1.2.25), the triangular inequality yields (2.2.14). □

REMARK 2.2.3. In Lemma 2.2.1 we have computed the Gateaux differential $d_u\mathcal{E}$ of the functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}$. The continuity of the map $u \mapsto d_u\mathcal{E}$ implies that the Gateaux differential coincides with the Fréchet differential (see, e.g., [**9**, Theorem 1.9]).

Using Lemma 2.2.1 and Remark 2.2.2, we can provide an expression for the representation map $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ defined in (2.2.3). Indeed, we have that

$$\mathcal{G}[u] = h_u + \beta u, \tag{2.2.17}$$

where $h_u : [0,1] \to \mathbb{R}^k$ is defined in (2.2.11). Before proving that the solution of the gradient flow (2.2.2) exists and is globally defined, we report the statement of a local existence and uniqueness result for the solution of ODEs in infinite-dimensional spaces.

THEOREM 2.2.3. *Let $(E, ||\cdot||_E)$ be a Banach space, and, for every $u_0 \in E$ and $R > 0$, let $B_R(u_0)$ be the set*

$$B_R(u_0) := \{u \in E : ||u - u_0||_E \le R\}.$$

Let $\mathcal{K} : E \to E$ be a continuous map such that

(i) $||\mathcal{K}[u]||_E \le M$ *for every* $u \in B_R(u_0)$*;*
(ii) $||\mathcal{K}[u_1] - \mathcal{K}[u_2]||_E \le L||u_1 - u_2||_E$ *for every* $u_1, u_2 \in B_R(u_0)$.

For every $t_0 \in \mathbb{R}$, let us consider the following Cauchy problem:

$$\begin{cases} \partial_t U_t = \mathcal{K}[U_t], \\ U_{t_0} = u_0. \end{cases} \tag{2.2.18}$$

Then, setting $\alpha := \frac{R}{M}$, the equation (2.2.18) *admits a unique and continuously differentiable solution $t \mapsto U_t$, which is defined for every $t \in \mathcal{I} := [t_0 - \alpha, t_0 + \alpha]$ and satisfies $U_t \in B_R(u_0)$ for every $t \in \mathcal{I}$.*

PROOF. This result descends directly from [**33**, Theorem 5.1.1]. □

In the following result we show that, whenever it exists, any solution of (2.2.2) is bounded with respect to the L^2-norm.

LEMMA 2.2.4. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^2-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. For every initial datum $u_0 \in \mathcal{U}$, let $U : [0, \alpha) \to \mathcal{U}$ be a continuously differentiable solution of the Cauchy problem* (2.2.2)*. Therefore, for every $R > 0$ there exists $C_R > 0$ such that, if $||u_0||_{L^2} \le R$, then*

$$||U_t||_{L^2} \le C_R$$

for every $t \in [0, \alpha)$.

PROOF. Recalling (2.2.2) and using the fact that both $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ and $t \mapsto U_t$ are differentiable, we observe that

$$\frac{d}{dt}\mathcal{F}(U_t) = d_{U_t}\mathcal{F}(\partial_t U_t) = \langle \mathcal{G}[U_t], \partial_t U_t \rangle_{L^2} = -||\partial_t U_t||^2_{L^2} \le 0 \tag{2.2.19}$$

for every $t \in [0, \alpha)$, and this immediately implies that

$$\mathcal{F}(U_t) \le \mathcal{F}(U_0)$$

for every $t \in [0, \alpha)$. Moreover, from the definition of the functional $\mathcal{F}$ given in (2.1.2) and recalling that the end-point term is non-negative, it follows that

$\frac{1}{2}||u||_{L^2}^2 \leq \mathcal{F}(u)/\beta$ for every $u \in \mathcal{U}$. Therefore, combining these facts, if $||u_0||_{L^2} \leq R$, we deduce that

$$\frac{1}{2}||U_t||_{L^2}^2 \leq \frac{1}{\beta} \sup_{||u_0||_{L^2} \leq R} \mathcal{F}(u_0) \leq \frac{1}{2\beta}R^2 + \frac{1}{\beta} \sup_{||u_0||_{L^2} \leq R} a(x_{u_0}(1))$$

for every $t \in [0, \alpha)$. Finally, using Lemma 1.2.2 and the continuity of $a : \mathbb{R}^n \to \mathbb{R}_+$, we deduce the thesis. □

We are now in position to prove that the gradient flow equation (2.2.2) admits a unique and globally defined solution.

THEOREM 2.2.5. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^2-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. For every $u_0 \in \mathcal{U}$, let us consider the Cauchy problem* (2.2.2) *with initial datum $U_0 = u_0$. Then,* (2.2.2) *admits a unique, globally defined and continuously differentiable solution $U : [0, +\infty) \to \mathcal{U}$.*

PROOF. Let us fix the initial datum $u_0 \in \mathcal{U}$, and let us set $R := ||u_0||_{L^2}$. Let $C_R > 0$ be the constant provided by Lemma 2.2.4. Let us introduce $R' := C_R + 1$ and let us consider

$$B_{R'}(0) := \{u \in \mathcal{U} : ||u||_{L^2} \leq R'\}.$$

We observe that, for every $\bar{u} \in \mathcal{U}$ such that $||\bar{u}||_{L^2} \leq C_R$, we have that

$$B_1(\bar{u}) \subset B_{R'}(0), \tag{2.2.20}$$

where $B_1(\bar{u}) := \{u \in \mathcal{U} : ||u - \bar{u}||_{L^2} \leq 1\}$. Recalling that the vector field that generates the gradient flow (2.2.2) has the form $\mathcal{G}[u] = h_u + \beta u$ for every $u \in \mathcal{U}$, from (2.2.12) we deduce that there exists $M_{R'} > 0$ such that

$$||\mathcal{G}[u]||_{L^2} \leq M_{R'} \tag{2.2.21}$$

for every $u \in B_{R'}(0)$. On the other hand, Lemma 2.2.2 implies that there exists $L_{R'} > 0$ such that

$$||\mathcal{G}[u_1] - \mathcal{G}[u_2]||_{L^2} \leq L_{R'}||u_1 - u_2||_{L^2} \tag{2.2.22}$$

for every $u_1, u_2 \in B_{R'}(0)$. Recalling the inclusion (2.2.20), (2.2.21)-(2.2.22) guarantee that the hypotheses of Theorem 2.2.3 are satisfied in the ball $B_1(\bar{u})$, for every $\bar{u}$ satisfying $||\bar{u}||_{L^2} \leq C_R$. This implies that, for every $t_0 \in \mathbb{R}$, the evolution equation

$$\begin{cases} \partial_t U_t = -\mathcal{G}[U_t], \\ U_{t_0} = \bar{u}, \end{cases} \tag{2.2.23}$$

admits a unique and continuously differentiable solution defined in the interval $[t_0 - \alpha, t_0 + \alpha]$, where we set $\alpha := \frac{1}{M_{R'}}$. In particular, if we choose $t_0 = 0$ and $\bar{u} = u_0$ in (2.2.23), we deduce that the gradient flow equation (2.2.2) with initial datum $U_0 = u_0$ admits a unique and continuously differentiable solution $t \mapsto U_t$ defined in the interval $[0, \alpha]$. We shall now prove that we can extend this local solution to every positive time. In virtue of Lemma 2.2.4, we obtain that the local solution $t \mapsto U_t$ satisfies

$$||U_t||_{L^2} \leq C_R \tag{2.2.24}$$

for every $t \in [0, \alpha]$. Therefore, if we set $t_0 = \frac{\alpha}{2}$ and $\bar{u} = U_{\frac{\alpha}{2}}$ in (2.2.23), recalling that, if $||\bar{u}||_{L^2} \leq C_R$, then (2.2.23) admits a unique solution defined in $[t_0 - \alpha, t_0 + \alpha]$, it turns out that the curve $t \mapsto U_t$ that solves (2.2.2) with Cauchy datum $U_0 = u_0$ can be uniquely defined for every $t \in [0, \frac{3}{2}\alpha]$. Since Lemma 2.2.4 guarantees that (2.2.24) holds whenever the solution $t \mapsto U_t$ exists, we can repeat recursively the argument and we can extend the domain of the solution to the whole half-line $[0, +\infty)$. □

We observe that Theorem 2.2.3 suggests that the solution of the gradient flow equation (2.2.2) could be defined also for negative times. In the following result we investigate this fact.

COROLLARY 2.2.6. *Under the same assumptions of Theorem 2.2.5, for every $R_2 > R_1 > 0$, there exists $\alpha > 0$ such that, if $||u_0||_{L^2} \leq R_1$, then the solution $t \mapsto U_t$ of the Cauchy problem* (2.2.2) *with initial datum $U_0 = u_0$ is defined for every $t \in [-\alpha, +\infty)$. Moreover, $||U_t||_{L^2} \leq R_2$ for every $t \in [-\alpha, 0]$.*

PROOF. The fact that the solutions are defined for every positive time descends from Theorem 2.2.5. Recalling the expression of $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ provided by (2.2.17), from (2.2.12) it follows that, for every $R_2 > 0$, there exists M_{R_2} such that

$$||\mathcal{G}[u]||_{L^2} \leq M_{R_2}$$

for every $u \in B_{R_2}(0) := \{u \in \mathcal{U} : ||u||_{L^2} \leq R_2\}$. On the other hand, in virtue of Lemma 2.2.2, we deduce that there exists L_{R_2} such that

$$||\mathcal{G}[u_1] - \mathcal{G}[u_2]||_{L^2} \leq L_{R_2} ||u_1 - u_2||_{L^2}$$

for every $u_1, u_2 \in B_{R_2}(0)$. We further observe that, for every $u_0 \in \mathcal{U}$ such that $||u_0||_{L^2} \leq R_1$, we have the inclusion $B_R(u_0) := \{u \in \mathcal{U} : ||u - u_0|| \leq R\} \subset B_{R_2}(0)$, where we set $R := R_2 - R_1$. Therefore, the previous inequalities guarantee that the hypotheses of Theorem 2.2.3 are satisfied in $B_R(u_0)$, whenever $||u_0||_{L^2} \leq R_1$. Finally, in virtue of Theorem 2.2.3 and the inclusion $B_R(u_0) \subset B_{R_2}(0)$, we obtain the thesis with

$$\alpha = \frac{R_2 - R_1}{M_{R_2}}.$$

□

2.3. Pre-compactness of gradient flow trajectories

In Section 2.2 we considered the $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ defined in (2.1.2) and we proved that the gradient flow equation (2.2.2) induced on $\mathcal{U}$ by $\mathcal{F}$ admits a unique solution $U : [0, +\infty) \to \mathcal{U}$, for every Cauchy datum $U_0 = u_0 \in \mathcal{U}$. The aim of the present section is to investigate the pre-compactness in $\mathcal{U}$ of the gradient flow trajectories

$t \mapsto U_t$. In order to do that, we first show that, under suitable regularity assumptions on the vector fields $F_1, \ldots, F_k$ and on the function $a : \mathbb{R}^n \to \mathbb{R}_+$, for every $t \geq 0$ the value of the solution $U_t \in \mathcal{U}$ has the same Sobolev regularity as the initial datum u_0. The key-fact is that, when $F_1, \ldots, F_k$ are C^r-regular with $r \geq 2$ and $a : \mathbb{R}^n \to \mathbb{R}_+$ is of class C^2, the map $\mathcal{G} : H^m([0,1], \mathbb{R}^k) \to H^m([0,1], \mathbb{R}^k)$ is locally Lipschitz continuous, for every non-negative integer $m \leq r - 1$. This implies that the gradient flow equation (2.2.2) can be studied as an evolution equation in the Hilbert space $H^m([0,1], \mathbb{R}^k)$.

The following result concerns the curve $\lambda_u : [0,1] \to (\mathbb{R}^n)^*$ defined in (2.2.9).

LEMMA 2.3.1. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^2-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. For every $R > 0$, there exists $C_R > 0$ such that, for every $u \in \mathcal{U}$ satisfying $||u||_{L^2} \leq R$, the following inequality holds*

$$||\lambda_u||_{C^0} \leq C_R, \tag{2.3.1}$$

where the curve $\lambda_u : [0,1] \to (\mathbb{R}^n)^*$ is defined as in (2.2.9). *Moreover, for every $R > 0$, there exists $L_R > 0$ such that, for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$, for the corresponding curves $\lambda_u, \lambda_{u+w} : [0,1] \to (\mathbb{R}^n)^*$ the following inequality holds:*

$$||\lambda_{u+w} - \lambda_u||_{C^0} \leq L_R ||w||_{L^2}. \tag{2.3.2}$$

PROOF. Recalling the definition of λ_u given in (2.2.9), we have that

$$|\lambda_u(s)|_2 \leq |\nabla a(x_u(1))|_2 |M_u(1)|_2 |M_u^{-1}(s)|_2$$

for every $s \in [0,1]$, where $x_u : [0,1] \to \mathbb{R}^n$ is solution of (1.1.6) corresponding to the control $u \in \mathcal{U}$. Lemma 1.2.2 implies that there exists $C'_R > 0$ such that $|\nabla a(x_u(1))|_2 \leq C'_R$ for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. Combining this with (1.2.16), we deduce (2.3.1).

To prove (2.3.2) we first observe that the C^2-regularity of $a : \mathbb{R}^n \to \mathbb{R}_+$ and Proposition 1.2.3 imply that, for every $R > 0$, there exists $L'_R > 0$ such that

$$|\nabla a(x_{u+w}(1)) - \nabla a(x_u(1))|_2 \leq L'_R ||w||_{L^2}$$

for every $u, w \in \mathcal{U}$ such that $||u||_{L^2}, ||w||_{L^2} \leq R$. Therefore, recalling (1.2.16) and (1.2.26)-(1.2.27), we deduce (2.3.2) by applying the triangular inequality to the identity

$$|\lambda_{u+w}(s) - \lambda_u(s)|_2 = |\nabla a(x_{u+w}(1)) \cdot M_{u+w}(1) M_{u+w}^{-1}(s) - \nabla a(x_u(1)) \cdot M_u(1) M_u^{-1}(s)|_2$$

for every $s \in [0,1]$. □

We recall the notion of *Lie bracket* of vector fields. Let $G^1, G^2 : \mathbb{R}^n \to \mathbb{R}^n$ be two vector fields such that $G^1 \in C^{r_1}(\mathbb{R}^n, \mathbb{R}^n)$ and $G^2 \in C^{r_2}(\mathbb{R}^n, \mathbb{R}^n)$, with

$r_1, r_2 \geq 1$, and let us set $r := \min(r_1, r_2)$. Then the *Lie bracket of G^1 and G^2* is the vector field $[G^1, G^2] : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows:

$$[G^1, G^2](y) = \frac{\partial G^2(y)}{\partial x} G^1(y) - \frac{\partial G^1(y)}{\partial x} G^2(y).$$

We observe that $[G^1, G^2] \in C^{r-1}(\mathbb{R}^n, \mathbb{R}^n)$. In the following result we establish some estimates for vector fields obtained via iterated Lie brackets.

LEMMA 2.3.2. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system (1.1.6) are C^m-regular, with $m \geq 2$. For every compact $K \subset \mathbb{R}^n$, there exist $C > 0$ and $L > 0$ such that, for every $j_1, \ldots, j_m = 1, \ldots, k$, the vector field*

$$G := [F_{j_m}, [\ldots, [F_{j_3}, [F_{j_2}, F_{j_1}]] \ldots] : \mathbb{R}^n \to \mathbb{R}^n$$

satisfies the following inequalities:

$$|G(x)|_2 \leq C \tag{2.3.3}$$

for every $x \in K$, and

$$|G(x) - G(y)|_2 \leq L|x - y|_2 \tag{2.3.4}$$

for every $x, y \in K$.

PROOF. The thesis follows immediately from the fact that the vector field G is C^1-regular. □

The next result is the cornerstone this section. It concerns the regularity of the function $h_u : [0, 1] \to \mathbb{R}^k$ introduced in (2.2.11). We recall that, for every $u \in \mathcal{U}$, h_u is the representation of the differential $d_u\mathcal{E}$ through the scalar product of $\mathcal{U}$, where the functional $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$ is defined as in (2.2.4). We recall the convention $H^0([0, 1], \mathbb{R}^k) = L^2([0, 1], \mathbb{R}^k) = \mathcal{U}$.

LEMMA 2.3.3. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system (1.1.6) are C^r-regular with $r \geq 2$, and that the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost is C^2-regular. For every $u \in \mathcal{U}$, let $h_u : [0, 1] \to \mathbb{R}^k$ be the representation of the differential $d_u\mathcal{E} : \mathcal{U} \to \mathbb{R}$ provided by (2.2.11). For every integer $1 \leq m \leq r - 1$, if $u \in H^{m-1}([0, 1], \mathbb{R}^k) \subset \mathcal{U}$, then $h_u \in H^m([0, 1], \mathbb{R}^k)$.*

Moreover, for every integer $1 \leq m \leq r-1$, for every $R > 0$ there exist $C_R^m > 0$ and $L_R^m > 0$ such that

$$||h_u||_{H^m} \leq C_R^m \tag{2.3.5}$$

for every $u \in H^{m-1}([0, 1], \mathbb{R}^k)$ such that $||u||_{H^{m-1}} \leq R$, and

$$||h_{u+w} - h_u||_{H^m} \leq L_R^m ||w||_{H^{m-1}} \tag{2.3.6}$$

for every $u, w \in H^{m-1}([0, 1], \mathbb{R}^k)$ such that $||u||_{H^{m-1}}, ||w||_{H^{m-1}} \leq R$.

PROOF. It is sufficient to prove the thesis in the case $m = r - 1$, for every integer $r \geq 2$. When $r = 2, m = 1$, we have to prove that, for every $u \in \mathcal{U}$, the function $h_u : [0,1] \to \mathbb{R}^k$ is in H^1. Recalling (2.2.11), we have that, for every $j = 1, \ldots, k$, the j-th component of h_u is given by the product

$$h_u^j(s) = \lambda_u(s) \cdot F_j(x_u(s))$$

for every $s \in [0,1]$, where $\lambda_u : [0,1] \to (\mathbb{R}^n)^*$ was defined in (2.2.9). Since both $s \mapsto \lambda_u(s)$ and $s \mapsto F_j(x_u(s))$ are in H^1, then their product is in H^1 as well (see, e.g., [**15**, Corollary 8.10]). Therefore, since $\lambda_u : [0,1] \to (\mathbb{R}^n)^*$ solves (2.2.10), we can compute

$$\dot{h}_u^j(s) = \lambda_u(s) \cdot \sum_{i=1}^{k} [F_i, F_j]_{x_u(s)} u^i(s) \tag{2.3.7}$$

for every $j = 1, \ldots, k$ and for a.e. $s \in [0,1]$. In virtue of (2.3.1), (1.2.3) and (2.3.3), for every $R > 0$, there exists $C'_R > 0$ such that

$$|\dot{h}_u^j(s)| \leq C'_R |u(s)|_1$$

for a.e. $s \in [0,1]$, for every $j = 1, \ldots, k$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. Recalling (1.2.2), we deduce that

$$||\dot{h}_u^j||_{L^2} \leq \sqrt{k} C'_R ||u||_{L^2} \tag{2.3.8}$$

for every $j = 1, \ldots, k$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. Finally, using (2.2.12), we obtain that (2.3.5) holds for $r = 2, m = 1$. To prove (2.3.6), we observe that, for every $j = 1, \ldots, k$ and for every $u, w \in \mathcal{U}$ we have

$$\begin{aligned}
|\dot{h}_{u+w}^j(s) - \dot{h}_u^j(s)| \leq{}& |\lambda_{u+w}(s) - \lambda_u(s)|_2 \sum_{i=1}^{k} \Big| [F_i, F_j]_{x_{u+w}(s)} \Big|_2 |u^i(s) + w^i(s)| \\
&+ |\lambda_u(s)|_2 \sum_{i=1}^{k} \Big| [F_i, F_j]_{x_{u+w}(s)} - [F_i, F_j]_{x_u(s)} \Big|_2 |u^i(s) + w^i(s)| \\
&+ |\lambda_u(s)|_2 \sum_{i=1}^{k} \Big| [F_i, F_j]_{x_u(s)} \Big|_2 |w^i(s)|
\end{aligned}$$

for a.e. $s \in [0,1]$. In virtue of Lemma 2.3.1, Lemma 1.2.2, Proposition 1.2.3 and Lemma 2.3.2, for every $R > 0$ there exist $L'_R > 0$ and $C''_R > 0$ such that for every $j = 1, \ldots, k$ the inequality

$$|\dot{h}_{u+w}^j(s) - \dot{h}_u^j(s)| \leq L'_R ||w||_{L^2} |u(s) + w(s)|_1 + C''_R |w(s)|_1$$

holds for a.e. $s \in [0,1]$ and for every $u, w \in \mathcal{U}$ satisfying $||u||_{L^2}, ||w||_{L^2} \leq R$. Using (1.2.2), the previous inequality implies that there exists $L''_R > 0$ such that

$$||\dot{h}_{u+w}^j - \dot{h}_u^j||_{L^2} \leq L''_R ||w||_{L^2} \tag{2.3.9}$$

for every $u, w \in \mathcal{U}$ such that $||u||_{L^2}, ||w||_{L^2} \leq R$. Recalling (2.2.13), we conclude that (2.3.6) holds for $r = 2, m = 1$.

For $r = 3, m = 2$, we have to prove that, for every $u \in H^1([0,1], \mathbb{R}^k)$, the function h_u belongs to $H^2([0,1], \mathbb{R}^k)$. This follows if we show that $\dot{h}_u \in H^1([0,1], \mathbb{R}^k)$ for for every $u \in H^1([0,1], \mathbb{R}^k)$. Using the identity (2.3.7), we deduce that, whenever $u \in H^1([0,1], \mathbb{R}^k)$, $\dot{h}_u^j$ is the product of three H^1-regular functions, for every $j = 1, \ldots, k$. Therefore, using again [**15**, Corollary 8.10], we deduce that $\dot{h}_u^j$ is H^1-regular as well. From (2.3.7), for every $j = 1, \ldots, k$ we have that

$$\ddot{h}_u^j(s) = \lambda_u(s) \cdot \sum_{i_1,i_2=1}^{k} [F_{i_2}, [F_{i_1}, F_j]]_{x_u(s)} u^{i_1}(s) u^{i_2}(s) + \lambda_u(s) \cdot \sum_{i_1=1}^{k} [F_{i_1}, F_j]_{x_u(s)} \dot{u}^{i_1}(s)$$

for a.e. $s \in [0,1]$. Using Lemma 2.3.1, Lemma 1.2.2, Lemma 2.3.2, and recalling Theorem 1.1.1, we obtain that, for every $R > 0$ there exist $C'_R, C''_R > 0$ such that

$$||\ddot{h}_u^j(s)||_{L^2} \leq C'_R + C''_R ||\dot{u}(s)||_{L^2} \tag{2.3.10}$$

for a.e. $s \in [0,1]$, for every $j = 1, \ldots, k$ and for every $u \in H^1([0,1], \mathbb{R}^k)$ such that $||u||_{H^1} \leq R$. Therefore, combining (2.2.12), (2.3.8) and (2.3.10), the inequality (2.3.5) follows for the case $r = 3, m = 2$. In view of (2.2.13) and (2.3.9), in order to prove (2.3.6) for $r = 3, m = 2$ it is sufficient to show that, for every $R > 0$ there exists $L'_R > 0$ such that

$$||\ddot{h}_{u+w}^j - \ddot{h}_u^j||_{L^2} \leq L'_R ||w||_{H^1} \tag{2.3.11}$$

for every $u, w \in H^1([0,1], \mathbb{R}^k)$ such that $||u||_{H^1}, ||w||_{H^1} \leq R$. The inequality (2.3.11) can be deduced with an argument based on the triangular inequality, similarly as done in the case $r = 2, m = 1$.

The same strategy works for every $r \geq 4$. □

The main consequence of Lemma 2.3.3 is that, when the map $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ defined in (2.2.17) is restricted to $H^m([0,1], \mathbb{R}^k)$, the restriction $\mathcal{G} : H^m([0,1], \mathbb{R}^k) \to H^m([0,1], \mathbb{R}^k)$ is bounded and Lipschitz continuous on bounded sets.

PROPOSITION 2.3.4. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^r-regular with $r \geq 2$, and that the function $a : \mathbb{R}^n \to \mathbb{R}$ designing the end-point cost is C^2-regular. For every $\beta > 0$, let $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ be the representation map defined in* (2.2.3). *Then, for every integer $1 \leq m \leq r - 1$, we have that*

$$\mathcal{G}(H^m([0,1], \mathbb{R}^k)) \subset H^m([0,1], \mathbb{R}^k).$$

Moreover, for every integer $1 \leq m \leq r-1$ and for every $R > 0$ there exists $C_R^m > 0$ such that

$$||\mathcal{G}[u]||_{H^m} \leq C_R^m \tag{2.3.12}$$

for every $u \in H^m([0,1], \mathbb{R}^k)$ such that $||u||_{H^m} \leq R$, and there exists $L_R^m > 0$ such that

$$||\mathcal{G}[u + w] - \mathcal{G}[u]||_{H^m} \leq L_R^m ||w||_{H^m} \tag{2.3.13}$$

for every $u, w \in H^m([0,1], \mathbb{R}^k)$ such that $||u||_{H^m}, ||w||_{H^m} \leq R$.

PROOF. Recalling that for every $u \in \mathcal{U}$ we have

$$\mathcal{G}[u] = h_u + \beta u,$$

the thesis follows directly from Lemma 2.3.3. □

Proposition 2.3.4 suggests that, when the vector fields $F_1, \ldots, F_k$ are C^r-regular with $r \geq 2$, we can restrict the gradient flow equation (2.2.2) to the Hilbert spaces $H^m([0,1], \mathbb{R}^k)$, for every integer $1 \leq m \leq r-1$. Namely, for every integer $1 \leq m \leq r-1$, we shall introduce the application $\mathcal{G}_m : H^m([0,1], \mathbb{R}^k) \to H^m([0,1], \mathbb{R}^k)$ defined as the restriction of $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ to H^m, i.e.,

$$\mathcal{G}_m := \mathcal{G}|_{H^m}. \tag{2.3.14}$$

For every integer $m \geq 1$, given a curve $U : (a,b) \to H^m([0,1], \mathbb{R}^k)$, we say that it is (strongly) differentiable at $t_0 \in (a,b)$ if there exists $u \in H^m([0,1], \mathbb{R}^k)$ such that

$$\lim_{t \to t_0} \left|\left| \frac{U_t - U_{t_0}}{t - t_0} - u \right|\right|_{H^m} = 0. \tag{2.3.15}$$

In this case, we use the notation $\partial_t U_{t_0} := u$. For every $\ell = 1, \ldots, m$ and for every $t \in (a,b)$, we shall write $U_t^{(\ell)} \in H^{m-\ell}([0,1], \mathbb{R}^k)$ to denote the ℓ-th Sobolev derivative of the function $U_t : s \mapsto U_t(s)$, i.e.,

$$\int_0^1 \langle U_t(s), \phi^{(\ell)}(s) \rangle_{\mathbb{R}^k} \, ds = (-1)^\ell \int_0^1 \langle U_t^{(\ell)}(s), \phi(s) \rangle_{\mathbb{R}^k} \, ds$$

for every $\phi \in C_c^\infty([0,1], \mathbb{R}^k)$. It is important to observe that, for every order of derivation $\ell = 1, \ldots, m$, (2.3.15) implies that

$$\lim_{t \to t_0} \left|\left| \frac{U_t^{(\ell)} - U_{t_0}^{(\ell)}}{t - t_0} - u^{(\ell)} \right|\right|_{L^2} = 0,$$

and we use the notation $\partial_t U_{t_0}^{(\ell)} := u^{(\ell)}$. In particular, for every $\ell = 1, \ldots, m$, it follows that

$$\frac{d}{dt} ||U_t^{(\ell)}||_{L^2}^2 = 2 \int_0^1 \langle \partial_t U_t^{(\ell)}(s), U_t^{(\ell)}(s) \rangle_{\mathbb{R}^k} \, ds = 2 \langle \partial_t U_t^{(\ell)}, U_t^{(\ell)} \rangle_{L^2}. \tag{2.3.16}$$

In the next result we study the following evolution equation

$$\begin{cases} \partial_t U_t = -\mathcal{G}_m[U_t], \\ U_0 = u_0, \end{cases} \tag{2.3.17}$$

with $u_0 \in H^m([0,1], \mathbb{R}^k)$, and where $\mathcal{G}_m : H^m([0,1], \mathbb{R}^k) \to H^m([0,1], \mathbb{R}^k)$ is defined as in (2.3.14). Before establishing the existence, uniqueness and global definition result for the Cauchy problem (2.3.17), we study the evolution of the semi-norms $||U_t^{(\ell)}||_{L^2}$ for $\ell = 1, \ldots, m$ along its solutions.

LEMMA 2.3.5. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^r-regular with $r \geq 2$, and that the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost is C^2-regular. For every integer $1 \leq m \leq r-1$ and for every inital datum $u_0 \in H^m([0,1], \mathbb{R}^k)$, let $U : [0, \alpha) \to H^m([0,1], \mathbb{R}^k)$ be a continuously differentiable solution of the Cauchy problem* (2.3.17)*. Therefore, for every $R > 0$ there exists $C_R > 0$ such that, if $||u_0||_{H^m} \leq R$, then*

$$||U_t||_{H^m} \leq C_R \tag{2.3.18}$$

for every $t \in [0, \alpha)$.

PROOF. It is sufficient to prove the statement in the case $r \geq 2, m = r-1$. We shall use an induction argument on r.

Let us consider the case $r = 2, m = 1$. We observe that if $U : [0, \alpha) \to H^1([0,1], \mathbb{R}^k)$ is a solution of (2.3.17) with $m = 1$, then it solves as well the Cauchy problem (2.2.2) in $\mathcal{U}$. Therefore, recalling that $||u_0||_{L^2} \leq ||u_0||_{H^1}$, in virtue of Lemma 2.2.4, for every $R > 0$ there exists $C'_R > 0$ such that, if $||u_0||_{H^1} \leq R$, we have that

$$||U_t||_{L^2} \leq C'_R \tag{2.3.19}$$

for every $t \in [0, \alpha)$. Hence it is sufficient to provide an upper bound to the seminorm $||U_t^{(1)}||_{L^2}$. From (2.3.16) and from the fact that $t \mapsto U_t$ solves (2.3.17) for $m = 1$, it follows that

$$\begin{aligned}
\frac{d}{dt}||U_t^{(1)}||^2_{L^2} = 2\langle \partial_t U_t^{(1)}, U_t^{(1)} \rangle_{L^2} &= -2 \int_0^1 \left\langle \beta U_t^{(1)}(s) + h_{U_t}^{(1)}(s), U_t^{(1)}(s) \right\rangle_{\mathbb{R}^k} ds \\
&\leq -2\beta ||U_t^{(1)}||^2_{L^2} + 2||h_{U_t}^{(1)}||_{L^2} ||U_t^{(1)}||_{L^2} \\
&\leq -\beta ||U_t^{(1)}||^2_{L^2} + \frac{1}{\beta} ||h_{U_t}^{(1)}||^2_{L^2}
\end{aligned}$$

for every $t \in [0, \alpha)$, where $h_{U_t} : [0,1] \to \mathbb{R}^k$ is the absolutely continuous curve defined in (2.2.11), and $h_{U_t}^{(1)}$ is its Sobolev derivative. Combining (2.3.19) with (2.3.5), we obtain that there exists $C_R^1 > 0$ such that

$$\frac{d}{dt}||U_t^{(1)}||^2_{L^2} \leq -\beta ||U_t^{(1)}||^2_{L^2} + \frac{1}{\beta} C_R^1$$

for every $t \in [0, \alpha)$. This implies that

$$||U_t^{(1)}||_{L^2} \leq \max \left\{ ||U_0^{(1)}||_{L^2}, \frac{1}{\beta} \sqrt{C_R^1} \right\}$$

for every $t \in [0, \alpha)$. This proves the thesis in the case $r = 2, m = 1$.

Let us prove the induction step. We shall prove the thesis in the case $r, m = r-1$. Let $U : [0, \alpha) \to H^m([0,1], \mathbb{R}^k)$ be a solution of (2.3.17) with $m = r-1$.

We observe that $t \mapsto U_t$ solves as well

$$\begin{cases} \partial_t U_t = -\mathcal{G}_{m-1}[U_t], \\ U_0 = u_0. \end{cases}$$

Using the inductive hypothesis and that $||u_0||_{H^{m-1}} \leq ||u_0||_{H^m}$, for every $R > 0$ there exists $C'_R > 0$ such that, if $||u_0||_{H^m} \leq R$, we have that

$$||U_t||_{H^{m-1}} \leq C'_R \tag{2.3.20}$$

for every $t \in [0, \alpha)$. Hence it is sufficient to provide an upper bound to the semi-norm $||U_t^{(m)}||_{L^2}$. Recalling (2.3.16) the same computation as before yields

$$\frac{d}{dt}||U_t^{(m)}||_{L^2}^2 \leq -\beta||U_t^{(m)}||_{L^2}^2 + \frac{1}{\beta}||h_{U_t}^{(m)}||_{L^2}^2$$

for every $t \in [0, \alpha)$. Combining (2.3.20) with (2.3.5), we obtain that there exists $C_R^1 > 0$ such that

$$\frac{d}{dt}||U_t^{(m)}||_{L^2}^2 \leq -\beta||U_t^{(m)}||_{L^2}^2 + \frac{1}{\beta}C_R^1$$

for every $t \in [0, \alpha)$. This yields (2.3.18) for the inductive case $r, m = r - 1$. □

We are now in position to prove that the Cauchy problem (2.3.17) admits a unique and globally defined solution. The proof of the following result follows the lines of the proof of Theorem 2.2.5.

THEOREM 2.3.6. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^r-regular with $r \geq 2$, and that the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost is C^2-regular. Then, for every integer $1 \leq m \leq r - 1$ and for every inital datum $u_0 \in H^m([0, 1], \mathbb{R}^k)$, the evolution equation* (2.3.17) *admits a unique, globally defined and continuously differentiable solution $U : [0, +\infty) \to H^m([0, 1], \mathbb{R}^k)$. Moreover, there exists $C_{u_0} > 0$ such that*

$$||U_t||_{H^m} \leq C_{u_0} \tag{2.3.21}$$

for every $t \in [0, +\infty)$.

PROOF. It is sufficient to prove the statement in the case $r \geq 2, m = r - 1$. In virtue of Lemma 2.3.5 and Proposition 2.3.4, the global existence of the solution of (2.3.17) follows from a *verbatim* repetition of the argument of the proof of Theorem 2.2.5. Finally, (2.3.21) descends directly from Lemma 2.3.5. □

REMARK 2.3.1. We insist on the fact that, under the regularity assumptions of Theorem 2.3.6, if the initial datum u_0 is H^m-Sobolev regular with $m \leq r - 1$, then the solution $U : [0, +\infty) \to \mathcal{U}$ of (2.2.2) does coincide with the solution of

(2.3.17). In other words, let us assume that the hypotheses of Theorem 2.3.6 are met, and let us consider the evolution equation

$$\begin{cases} \partial_t U_t = -\mathcal{G}[U_t], \\ U_0 = u_0, \end{cases} \tag{2.3.22}$$

where $u_0 \in H^m([0,1],\mathbb{R}^k)$, with $m \leq r-1$. Owing to Theorem 2.2.5, it follows that (2.3.22) admits a unique solution $U : [0,+\infty) \to \mathcal{U}$. We claim that $t \mapsto U_t$ solves as well the evolution equation

$$\begin{cases} \partial_t U_t = -\mathcal{G}_m[U_t], \\ U_0 = u_0. \end{cases} \tag{2.3.23}$$

Indeed, Theorem 2.3.6 implies that (2.3.23) admits a unique solution $\tilde{U} : [0,+\infty) \to H^m([0,1],\mathbb{R}^k)$. Moreover, any solution of (2.3.23) is also a solution of (2.3.22), therefore we must have $U_t = \tilde{U}_t$ for every $t \geq 0$ by the uniqueness of the solution of (2.3.22). Hence, it follows that, if the controlled vector fields $F_1, \ldots, F_k$ and the function $a : \mathbb{R}^n \to \mathbb{R}_+$ are regular enough, then for every $t \in [0,+\infty)$ each point of the gradient flow trajectory U_t solving (2.3.22) has the same Sobolev regularity as the initial datum.

We now prove a pre-compactness result for the gradient flow trajectories. We recall that we use the convention $H^0 = L^2$.

Corollary 2.3.7. *Under the same assumptions of Theorem 2.3.6, let us consider* $u_0 \in H^m([0,1],\mathbb{R}^k)$ *with the integer* m *satisfying* $1 \leq m \leq r-1$. *Let* $U : [0,+\infty) \to \mathcal{U}$ *be the solution of the Cauchy problem* (2.2.2) *with initial condition* $U_0 = u_0$. *Then the trajectory* $\{U_t : t \geq 0\}$ *is pre-compact in* $H^{m-1}([0,1],\mathbb{R}^k)$.

Proof. As observed in Remark 2.3.1, we have that the solution $U : [0,+\infty) \to \mathcal{U}$ of (2.2.2) satisfies $U_t \in H^m([0,1],\mathbb{R}^k)$ for every $t \geq 0$, and that it solves (2.3.17) as well. In virtue of Theorem 1.1.1, the inclusion $H^m([0,1],\mathbb{R}^k) \hookrightarrow H^{m-1}([0,1],\mathbb{R}^k)$ is compact for every integer $m \geq 1$, therefore from (2.3.21) we deduce the thesis. □

2.4. Lojasiewicz-Simon inequality

In this section we show that, when the controlled vector fields $F_1, \ldots, F_k$ and the function $a : \mathbb{R}^n \to \mathbb{R}_+$ are real-analytic, then the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ satisfies the Lojasiewicz-Simon inequality. This fact will be of crucial importance for the convergence proof of the next section.

The first result on the Lojasiewicz inequality dates back to 1963, when in [**38**] Lojasiewicz proved that, if $f : \mathbb{R}^d \to \mathbb{R}$ is a real-analytic function, then for every $x \in \mathbb{R}^d$ there exist $\gamma \in (1,2]$, $C > 0$ and $r > 0$ such that

$$|f(y) - f(x)| \leq C|\nabla f(y)|_2^\gamma \tag{2.4.1}$$

for every $y \in \mathbb{R}^d$ satisfying $|y - r|_2 < r$. This kind of inequalities are ubiquitous in several branches of Mathematics. For example, as suggested by Lojasiewicz in [**38**], (2.4.1) can be employed to study the convergence of the solutions of

$$\dot{x} = -\nabla f(x).$$

Another important application can be found in [**42**], where Polyak studied the convergence of the gradient descent algorithm for strongly convex functions using a particular instance of (2.4.1), which is sometimes called Polyak-Lojasiewicz inequality. In [**49**], Simon extended (2.4.1) to real-analytic functionals defined on Hilbert spaces, and he employed it to establish convergence results for evolution equations. For further details, see also the lecture notes [**50**]. The infinite-dimensional version of (2.4.1) is often called Lojasiewicz-Simon inequality. For a complete survey on the topic, we refer the reader to the paper [**22**].

In this section we prove that for every $\beta > 0$ the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ defined in (2.1.2) satisfies the Lojasiewicz-Simon inequality. We first show that, when the function $a : \mathbb{R}^n \to \mathbb{R}_+$ involved in the definition of the end-point cost (2.2.4) and the controlled vector fields $F_1, \ldots, F_k$ are real-analytic, the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ is real-analytic as well, for every $\beta > 0$. We recall the notion of real-analytic application defined on a Banach space. For an introduction to the subject, see, for example, [**54**].

Definition 1. Let E_1, E_2 be Banach spaces, and let us consider an application $\mathcal{T} : E_1 \to E_2$. The function $\mathcal{T}$ is said to be *real-analytic at* $e_0 \in E_1$ if for every $N \geq 1$ there exists a continuous and symmetric multi-linear application $l_N \in \mathscr{L}((E_1)^N, E_2)$ and if there exists $r > 0$ such that, for every $e \in E_1$ satisfying $||e - e_0||_{E_1} < r$, we have

$$\sum_{N=1}^{\infty} ||l_N||_{\mathscr{L}((E_1)^N, E_2)} \, ||e - e_0||_{E_1}^N < +\infty$$

and

$$\mathcal{T}(e) - \mathcal{T}(e_0) = \sum_{N=1}^{\infty} l_N(e - e_0)^N,$$

where, for every $N \geq 1$, we set $l_N(e - e_0)^N := l_N(e - e_0, \ldots, e - e_0)$. Finally, $\mathcal{T} : E_1 \to E_2$ is *real-analytic on* E_1 if it is real-analytic at every $e_0 \in E_1$.

In the next result we provide the conditions that guarantee that $\mathcal{F} : \mathcal{U} \to \mathbb{R}$ is real-analytic.

Proposition 2.4.1. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are real-analytic, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost* (2.2.4). *Therefore, for every $\beta > 0$, the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ defined in* (2.1.2) *is real-analytic.*

PROOF. Since $\mathcal{F}(u) = \mathcal{E}(u) + \frac{\beta}{2}||u||_{L^2}^2$ for every $u \in \mathcal{U}$, the proof reduces to show that the end-point cost $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$ is real-analytic. Recalling the definition of $\mathcal{E}$ given in (2.2.4) and the end-point map $P_1 : \mathcal{U} \to \mathbb{R}^n$ introduced in (1.2.20), we have that the former can be expressed as the composition

$$\mathcal{E} = a \circ P_1.$$

In the proof of [**5**, Proposition 8.5] it is shown that P_1 is smooth as soon as $F_1, \ldots, F_k$ are C^∞-regular, and the expression of the Taylor expansion of P_1 at every $u \in \mathcal{U}$ is provided. In [**3**, Proposition 2.1] it is proved that, when $a : \mathbb{R}^n \to \mathbb{R}_+$ and the controlled vector fields are real-analytic, the Taylor series of $a \circ P_1$ is actually convergent. □

The previous result implies that the differential $d\mathcal{F} : \mathcal{U} \to \mathcal{U}^*$ is real-analytic.

COROLLARY 2.4.2. *Under the same assumptions as in Proposition 2.4.1, for every $\beta > 0$ the differential $d\mathcal{F} : \mathcal{U} \to \mathcal{U}^*$ is real-analytic.*

PROOF. Owing to Proposition 2.4.1, the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ is real-analytic. Using this fact, the thesis follows from [**54**, Theorem 2, p.1078]. □

Another key-step in view of the Lojasiewicz-Simon inequality is the study of the Hessian of the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$. In our framework, the Hessian of $\mathcal{F}$ at a point $u \in \mathcal{U}$ is the bounded linear operator $\mathrm{Hess}_u\mathcal{F} : \mathcal{U} \to \mathcal{U}$ that satisfies the identity:

$$\langle \mathrm{Hess}_u\mathcal{F}[v], w\rangle_{L^2} = d_u^2\mathcal{F}(v, w) \tag{2.4.2}$$

for every $v, w \in \mathcal{U}$, where $d_u^2\mathcal{F} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ is the second differential of $\mathcal{F}$ at the point u. In the next proposition we prove that, for every $u \in \mathcal{U}$, $\mathrm{Hess}_u\mathcal{F}$ has finite-dimensional kernel. We stress the fact that, unlike the other results of the present section, we do not have to assume that $F_1, \ldots, F_k$ and $a : \mathbb{R}^n \to \mathbb{R}_+$ are real-analytic to study the kernel of $\mathrm{Hess}_u\mathcal{F}$.

PROPOSITION 2.4.3. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are C^2-regular, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ defining the end-point cost* (2.2.4)*. For every $u \in \mathcal{U}$, let $\mathrm{Hess}_u\mathcal{F} : \mathcal{U} \to \mathcal{U}$ be the linear operator that represents the second differential $d_u^2\mathcal{F} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ through the identity* (2.4.2)*. Then, the kernel of $\mathrm{Hess}_u\mathcal{F}$ is finite-dimensional.*

PROOF. For every $u \in \mathcal{U}$ we have that

$$d_u^2\mathcal{F}(v, w) = d_u^2\mathcal{E}(v, w) + \beta\langle v, w\rangle_{L^2}$$

for every $v, w \in \mathcal{U}$. Therefore, we are reduced to study the second differential of the end-point cost $\mathcal{E} : \mathcal{U} \to \mathbb{R}_+$. Recalling its definition in (2.2.4) and applying the chain-rule, we obtain that

$$d_u^2\mathcal{E}(v, w) = \big[D_uP_1(v)\big]^T \nabla^2 a(x_u(1))\big[D_uP_1(w)\big] + \big(\nabla a(x_u(1))\big)^T \cdot D_u^2P_1(v, w), \tag{2.4.3}$$

where $P_1 : \mathcal{U} \to \mathbb{R}^n$ is the end-point map defined in (1.2.20), and where the curve $x_u : [0,1] \to \mathbb{R}^n$ is the solution of (1.1.6) corresponding to the control $u \in \mathcal{U}$. We recall that, for every $y \in \mathbb{R}^n$, we understand $\nabla a(y)$ as a row vector. Let us set $\nu_u := \big(\nabla a(x_u(1))\big)^T$ and $H_u := \nabla^2 a(x_u(1))$, where $H_u : \mathbb{R}^n \to \mathbb{R}^n$ is the self-adjoint linear operator associated to the Hessian of $a : \mathbb{R}^n \to \mathbb{R}_+$ at the point $x_u(1)$. Therefore we can write

$$d^2_u \mathcal{E}(v,w) = \langle \big(D_u P_1^* \circ H_u \circ D_u P_1\big)[v], w\rangle_{L^2} + \nu_u \cdot D^2_u P_1(v,w) \tag{2.4.4}$$

for every $v, w \in \mathcal{U}$, where $D_u P_1^* : \mathbb{R}^n \to \mathcal{U}$ is the adjoint of the differential $D_u P_1 : \mathcal{U} \to \mathbb{R}^n$. Moreover, recalling the definition of the linear operator $\mathcal{N}_u^\nu : \mathcal{U} \to \mathcal{U}$ given in (1.3.21), we have that

$$\nu_u \cdot D^2_u P_1(v,w) = \langle \mathcal{N}_u^{\nu_u}[v], w\rangle_{L^2}$$

for every $v, w \in \mathcal{U}$. Therefore, we obtain

$$d^2_u \mathcal{E}(v,w) = \langle \mathrm{Hess}_u \mathcal{E}[v], w\rangle_{L^2} \tag{2.4.5}$$

for every $v, w \in \mathcal{U}$, where $\mathrm{Hess}_u \mathcal{E} : \mathcal{U} \to \mathcal{U}$ is the linear operator that satisfies the identity:

$$\mathrm{Hess}_u \mathcal{E} = D_u P_1^* \circ H_u \circ D_u P_1 + \mathcal{N}_u^{\nu_u}.$$

We observe that $\mathrm{Hess}_u \mathcal{E}$ is a self-adjoint compact operator. Indeed, $N_u^{\nu_u}$ is self-adjoint and compact in virtue of Proposition 1.3.6, while $D_u P_1^* \circ H_u \circ D_u P_1$ has finite-rank and it self-adjoint as well. Combining (2.4.3) and (2.4.5), we deduce that

$$\mathrm{Hess}_u \mathcal{F} = \mathrm{Hess}_u \mathcal{E} + \beta \mathrm{Id}, \tag{2.4.6}$$

where $\mathrm{Id} : \mathcal{U} \to \mathcal{U}$ is the identity. Finally, using the Fredholm alternative (see, e.g., [**15**, Theorem 6.6]), we deduce that the kernel of $\mathrm{Hess}_u \mathcal{F}$ is finite-dimensional. □

We are now in position to prove that the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ satisfies the Lojasiewicz-Simon inequality.

THEOREM 2.4.4. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are real-analytic, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ defining end-point cost* (2.2.4)*. For every $\beta > 0$ and for every $u \in \mathcal{U}$, there exist $r > 0$, $C > 0$ and $\gamma \in (1,2]$ such that*

$$|\mathcal{F}(v) - \mathcal{F}(u)| \leq C ||d_v \mathcal{F}||^\gamma_{\mathcal{U}^*} \tag{2.4.7}$$

for every $v \in \mathcal{U}$ such that $||v-u||_{L^2} < r$.

PROOF. If $u \in \mathcal{U}$ is not a critical point for $\mathcal{F}$, i.e., $d_u \mathcal{F} \neq 0$, then there exists $r_1 > 0$ and $\kappa > 0$ such that

$$||d_v \mathcal{F}||^2_{\mathcal{U}^*} \geq \kappa$$

for every $v \in \mathcal{U}$ satisfying $||v-u||_{L^2} < r_1$. On the other hand, by the continuity of $\mathcal{F}$, we deduce that there exists $r_2 > 0$ such that

$$|\mathcal{F}(v) - \mathcal{F}(u)| \leq \kappa$$

for every $v \in \mathcal{U}$ satisfying $||v - u||_{L^2} < r_2$. Combining the previous inequalities and taking $r := \min\{r_1, r_2\}$, we deduce that, when $d_u\mathcal{F} \neq 0$, (2.4.7) holds with $\gamma = 2$.

The inequality (2.4.7) in the case $d_u\mathcal{F} = 0$ follows from [**22**, Corollary 3.11]. We shall now verify the assumptions of this result. First of all, [**22**, Hypothesis 3.2] is satisfied, being $\mathcal{U}$ an Hilbert space. Moreover, [**22**, Hypothesis 3.4] follows by choosing $W = \mathcal{U}^*$. In addition, we recall that $d\mathcal{F} : \mathcal{U} \to \mathcal{U}^*$ is real-analytic in virtue of Corollary 2.4.2, and that $\mathrm{Hess}_u\mathcal{F}$ has finite-dimensional kernel owing to Proposition 2.4.3. These facts imply that the conditions (1)–(4) of [**22**, Corollary 3.11] are verified if we set $X = \mathcal{U}$ and $Y = \mathcal{U}^*$. □

2.5. Convergence of the gradient flow

In this section we show that the gradient flow trajectory $U : [0 + \infty) \to \mathcal{U}$ that solves (2.2.2) is convergent to a critical point of the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}$, provided that the Cauchy datum $U_0 = u_0$ satisfies $u_0 \in H^1([0,1], \mathbb{R}^k) \subset \mathcal{U}$. The Lojasiewicz-Simon inequality established in Theorem 2.4.4 will play a crucial role in the proof of the convergence result. Indeed, we use this inequality to show that the trajectories with Sobolev-regular initial datum have finite length. This approach was first proposed in [**38**] in the finite-dimensional framework, and in [**49**] for evolution PDEs. In order to satisfy the assumptions of Theorem 2.4.4, we need to assume throughout the section that the controlled vector fields $F_1, \ldots, F_k$ and the function $a : \mathbb{R}^n \to \mathbb{R}_+$ are real-analytic.

We first recall the notion of the Riemann integral of a curve that takes values in $\mathcal{U}$. For general statements and further details, we refer the reader to [**33**, Section 1.3]. Let us consider a continuous curve $V : [a, b] \to \mathcal{U}$. Therefore, using [**33**, Theorem 1.3.1], we can define

$$\int_a^b V_t \, dt := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} V_{\frac{b-a}{n}k}.$$

We immediately observe that the following inequality holds:

$$\left|\left| \int_a^b V_t \, dt \right|\right|_{L^2} \leq \int_a^b ||V_t||_{L^2} \, dt. \tag{2.5.1}$$

Moreover, [**33**, Theorem 1.3.4] guarantees that, if the curve $V : [a, b] \to \mathcal{U}$ is continuously differentiable, then we have:

$$V_b - V_a = \int_a^b \partial_t V_\theta \, d\theta, \tag{2.5.2}$$

where $\partial_t V_\theta$ is the derivative of the curve $t \mapsto V_t$ defined as in (2.2.1) and computed at the instant $\theta \in [a, b]$. Finally, combining (2.5.2) and (2.5.1), we deduce that

$$||V_b - V_a||_{L^2} \leq \int_a^b ||\partial_t V_\theta||_{L^2} \, d\theta. \tag{2.5.3}$$

We refer to the quantity at the right-hand side of (2.5.3) as *the length of the continuously differentiable curve* $V : [a, b] \to \mathcal{U}$.

Let $U : [0, +\infty) \to \mathcal{U}$ be the solution of the gradient flow equation (2.2.2) with initial datum $u_0 \in \mathcal{U}$. We say that $u_\infty \in \mathcal{U}$ is a *limiting point* for the curve $t \mapsto U_t$ if there exists a sequence $(t_j)_{j \geq 1}$ such that $t_j \to +\infty$ and $||U_{t_j} - u_\infty||_{L^2} \to 0$ as $j \to \infty$. In the next result we study the length of $t \mapsto U_t$ in a neighborhood of a limiting point.

Proposition 2.5.1. *Let us assume that the vector fields* $F_1, \ldots, F_k$ *defining the control system* (1.1.6) *are real-analytic, as well as the function* $a : \mathbb{R}^n \to \mathbb{R}_+$ *designing the end-point cost. Let* $U : [0, +\infty) \to \mathcal{U}$ *be the solution of the Cauchy problem* (2.2.2) *with initial datum* $U_0 = u_0$*, and let* $u_\infty \in \mathcal{U}$ *be any of its limiting points. Then there exists* $r > 0$ *such that the portion of the curve that lies in* $B_r(u_\infty)$ *has finite length, i.e.,*

$$\int_{\mathcal{I}} ||\partial_t U_\theta||_{L^2} \, d\theta < \infty, \tag{2.5.4}$$

where $\mathcal{I} := \{t \geq 0 : U_t \in B_r(u_\infty)\}$*, and* $B_r(u_\infty) := \{u \in \mathcal{U} : ||u - u_\infty||_{L^2} < r\}$*.*

Proof. Let $u_\infty \in \mathcal{U}$ be a limiting point of $t \mapsto U_t$, and let $(\bar{t}_j)_{j \geq 1}$ be a sequence such that $\bar{t}_j \to +\infty$ and $||U_{\bar{t}_j} - u_\infty||_{L^2} \to 0$ as $j \to \infty$. The same computation as in (2.2.19) implies that the functional $\mathcal{F} : \mathcal{U} \to \mathbb{R}_+$ is decreasing along the trajectory $t \mapsto U_t$, i.e.,

$$\mathcal{F}(U_{t'}) \leq \mathcal{F}(U_t) \tag{2.5.5}$$

for every $t' \geq t \geq 0$. In addition, using the continuity of $\mathcal{F}$, it follows that $\mathcal{F}(U_{\bar{t}_j}) \to \mathcal{F}(u_\infty)$ as $j \to \infty$. Combining these facts, we have that

$$\mathcal{F}(U_t) - \mathcal{F}(u_\infty) \geq 0 \tag{2.5.6}$$

for every $t \geq 0$. Moreover, owing to Theorem 2.4.4, we deduce that there exist $C > 0$, $\gamma \in (1, 2]$ and $r > 0$ such that

$$|\mathcal{F}(v) - \mathcal{F}(u_\infty)| \leq \frac{1}{C} ||d_v \mathcal{F}||_{\mathcal{U}^*}^\gamma \tag{2.5.7}$$

for every $v \in B_r(u_\infty)$. Let $t_1 \geq 0$ be the infimum of the instants such that $U_t \in B_r(u_\infty)$, i.e.,

$$t_1 := \inf_{t \geq 0} \{U_t \in B_r(u_\infty)\}.$$

We observe that the set where we take the infimum is nonempty, in virtue of the convergence $||U_{\bar{t}_j} - u_\infty||_{L^2} \to 0$ as $j \to \infty$. Then, there exists $t_1' \in (t_1, +\infty]$ such that $U_t \in B_r(u_\infty)$ for every $t \in (t_1, t_1')$, and we take the supremum $t_1' > t_1$ such that the previous condition is satisfied, i.e.,

$$t_1' := \sup_{t' > t_1} \{U_t \in B_r(u_\infty), \forall t \in (t_1, t')\}.$$

If $t_1' < \infty$, we set

$$t_2 := \inf_{t \geq t_1'} \{U_t \in B_r(u_\infty)\},$$

and

$$t_2' := \sup_{t' > t_2} \{U_t \in B_r(u_\infty), \forall t \in (t_2, t')\}.$$

We repeat this procedure (which terminates in a finite number of steps if and only if there exits $\bar{t} > 0$ such that $U_t \in B_r(u_\infty)$ for every $t \geq \bar{t}$), and we obtain a family of intervals $\{(t_j, t_j')\}_{j=1,\ldots,N}$, where $N \in \mathbb{N} \cup \{\infty\}$. We observe that $\bigcup_{j=1}^N (t_j, t_j') = \mathcal{I}$, where we set $\mathcal{I} := \{t \geq 0 : U_t \in B_r(u_\infty)\}$.

Without loss of generality, we may assume that $\mathcal{I}$ is a set of infinite Lebesgue measure. Indeed, if this is not the case, we would have the thesis:

$$\int_{\mathcal{I}} ||\partial_t U_\theta||_{L^2} \, d\theta = \int_{\mathcal{I}} ||\mathcal{G}[U_\theta]||_{L^2} \, d\theta < \infty,$$

since $||\mathcal{G}[u]||_{L^2}$ is bounded on the bounded subsets of $\mathcal{U}$, as shown in (2.2.21). Therefore, we focus on the case when the Lebesgue measure of $\mathcal{I}$ is infinite. Let us introduce the following sequence:

$$\tau_0 = t_1, \;\; \tau_1 = t_1', \;\; \tau_2 = \tau_1 + (t_2' - t_2), \;\; \ldots, \;\; \tau_j = \tau_{j-1} + (t_j' - t_j), \;\; \ldots, \tag{2.5.8}$$

where $t_1, t_1', \ldots$ are the extremes of the intervals $\{(t_j, t_j')\}_{j=1,\ldots,N}$ constructed above. Finally, we define the function $\sigma : [\tau_0, +\infty) \to [\tau_0, +\infty)$ as follows:

$$\sigma(t) := \begin{cases} t & \text{if } \tau_0 \leq t < \tau_1, \\ t - \tau_1 + t_2 & \text{if } \tau_1 \leq t < \tau_2, \\ t - \tau_2 + t_3 & \text{if } \tau_2 \leq t < \tau_3, \\ \ldots & \ldots \end{cases} \tag{2.5.9}$$

We observe that $\sigma : [\tau_0, +\infty) \to [\tau_0, +\infty)$ is piecewise affine and it is monotone increasing. In particular, we have that

$$\sigma(\tau_j) = t_{j+1} \geq t_j' = \lim_{t \to \tau_j^-} \sigma(t). \tag{2.5.10}$$

Moreover, from (2.5.8) and from the definition of the intervals $\{(t_j, t_j')\}_{j \geq 1}$, it follows that

$$U_{\sigma(t)} \in B_r(u_\infty) \tag{2.5.11}$$

for every $t \in [\tau_0, +\infty)$. Let us define the function $g : [\tau_0, +\infty) \to \mathbb{R}_+$ as follows:

$$g(t) := \mathcal{F}(U_{\sigma(t)}) - \mathcal{F}(u_\infty), \tag{2.5.12}$$

where we used (2.5.6) to deduce that g is always non-negative. From (2.5.9), we obtain that the restriction $g|_{(\tau_j,\tau_{j+1})}$ is C^1-regular, for every $j \geq 0$. Therefore, using the fact that $\dot{\sigma}|_{(\tau_j,\tau_{j+1})} \equiv 1$, we compute

$$\dot{g}(t) = \frac{d}{dt}\big(\mathcal{F}(U_{\sigma(t)}) - \mathcal{F}(u_\infty)\big) = -d_{U_{\sigma(t)}}\mathcal{F}\big(\mathcal{G}[U_{\sigma(t)}]\big)$$

for every $t \in (\tau_j, \tau_{j+1})$ and for every $j \geq 0$. Recalling that $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ is the Riesz's representation of the differential $d\mathcal{F} : \mathcal{U} \to \mathcal{U}^*$, it follows that

$$\dot{g}(t) = -||d_{U_{\sigma(t)}}\mathcal{F}||^2_{\mathcal{U}^*} \tag{2.5.13}$$

for every $t \in (\tau_j, \tau_{j+1})$ and for every $j \geq 0$. Moreover, owing to the Lojasiewicz-Simon inequality (2.5.7), from (2.5.11) we deduce that

$$\dot{g}(t) \leq -Cg^{\frac{2}{\gamma}}(t) \tag{2.5.14}$$

for every $t \in (\tau_j, \tau_{j+1})$ and for every $j \geq 0$. Let $h : [\tau_0, \infty) \to [0, +\infty)$ be the solution of the Cauchy problem

$$\dot{h} = -Ch^{\frac{2}{\gamma}}, \quad h(\tau_0) = g(\tau_0), \tag{2.5.15}$$

whose expression is

$$h(t) = \begin{cases} \Big(h(\tau_0)^{1-\frac{2}{\gamma}} + \frac{(2-\gamma)C}{\gamma}(t-\tau_0)\Big)^{-1-\frac{2\gamma-2}{2-\gamma}} & \text{if } \gamma \in (1,2), \\ h(\tau_0)e^{-Ct} & \text{if } \gamma = 2, \end{cases}$$

for every $t \in [\tau_0, \infty)$. Using the fact that $g|_{(\tau_0,\tau_1)}$ is C^1-regular, in view of (2.5.14), we deduce that

$$g(t) \leq h(t), \tag{2.5.16}$$

for every $t \in [\tau_0, \tau_1)$. We shall now prove that the previous inequality holds for every $t \in [\tau_0, +\infty)$ using an inductive argument. Let us assume that (2.5.16) holds in the interval $[\tau_0, \tau_j)$, with $j \geq 1$. From the definition of g, combining (2.5.5) and (2.5.10), we obtain that

$$g(\tau_j) \leq \lim_{t \to \tau_j^-} g(t) \leq \lim_{t \to \tau_j^-} h(t) = h(\tau_j). \tag{2.5.17}$$

Using that the restriction $g|_{(\tau_j,\tau_{j+1})}$ is C^1-regular, in virtue of (2.5.14), (2.5.15) and (2.5.17) , we extend the the inequality (2.5.16) to the interval $[\tau_0, \tau_{j+1})$. This shows that (2.5.16) is satisfied for every $t \in [\tau_0, +\infty)$.

We now prove that the portion of the trajectory that lies in $B_r(u_\infty)$ is finite. We observe that

$$\int_{\mathcal{I}} ||\partial_t U_\theta||_{L^2}\, d\theta = \int_{\mathcal{I}} ||\mathcal{G}(U_\theta)||_{L^2}\, d\theta = \int_{\mathcal{I}} ||d_{U_\theta}\mathcal{F}||_{\mathcal{U}^*}\, d\theta, \tag{2.5.18}$$

where we recall that $\mathcal{I} = \bigcup_{j=1}^N (t_j, t_j')$. For every $j \geq 1$, in the interval (t_j, t_j') we use the change of variable $\theta = \sigma(\vartheta)$, where σ is defined in (2.5.9). Using (2.5.8) and

(2.5.9), we observe that $\sigma^{-1}\{(t_j, t'_j)\} = (\tau_{j-1}, \tau_j)$ and that $\dot\sigma|_{(\tau_{j-1},\tau_j)} \equiv 1$. These facts yield

$$\int_{t_j}^{t'_j} ||d_{U_\theta}\mathcal{F}||_{\mathcal{U}^*}\, d\theta = \int_{\tau_{j-1}}^{\tau_j} ||d_{U_{\sigma(\vartheta)}}\mathcal{F}||_{\mathcal{U}^*}\, d\vartheta = \int_{\tau_{j-1}}^{\tau_j} \sqrt{-\dot g(\vartheta)}\, d\vartheta \tag{2.5.19}$$

for every $j \geq 1$, where we used (2.5.13) in the last identity. Therefore, combining (2.5.18) and (2.5.19), we deduce that

$$\int_{\mathcal{I}} ||\partial_t U_\theta||_{L^2}\, d\theta = \int_{\tau_0}^{+\infty} \sqrt{-\dot g(\vartheta)}\, d\vartheta. \tag{2.5.20}$$

Then the thesis reduces to prove that the quantity at the right-hand side of (2.5.20) is finite. Let $\delta > 0$ be a positive quantity whose value will be specified later. From the Cauchy-Schwarz inequality, it follows that

$$\int_{\tau_0}^{+\infty} \sqrt{-\dot g(\vartheta)}\, d\vartheta \leq \left(\int_{\tau_0}^{\infty} -\dot g(\vartheta)\vartheta^{1+\delta}\, d\vartheta\right)^{\frac{1}{2}} \left(\int_{\tau_0}^{\infty} \vartheta^{-1-\delta}\, d\vartheta\right)^{\frac{1}{2}}. \tag{2.5.21}$$

On the other hand, for every $j \geq 1$, using the integration by parts on each interval $(\tau_0, \tau_1), \ldots, (\tau_{j-1}, \tau_j)$, we have that

$$\begin{aligned}
\int_{\tau_0}^{\tau_j} -\dot g(\vartheta)\vartheta^{1+\delta}\, d\vartheta &= \sum_{i=1}^{j} \left(\tau_{i-1}^{1+\delta} g(\tau_{i-1}) - \tau_i^{1+\delta} g(\tau_i^-) + (1+\delta)\int_{\tau_{i-1}}^{\tau_i} g(\vartheta)\vartheta^\delta\, d\vartheta\right) \\
&\leq \tau_0^{1+\delta} g(\tau_0) - \tau_j^{1+\delta} g(\tau_j^-) + (1+\delta)\int_{\tau_0}^{\tau_j} h(\vartheta)\vartheta^\delta\, d\vartheta \\
&\leq \tau_0^{1+\delta} g(\tau_0) + (1+\delta)\int_{\tau_0}^{\tau_j} h(\vartheta)\vartheta^\delta\, d\vartheta,
\end{aligned}$$

where we introduced the notation $g(\tau_i^-) := \lim_{\vartheta\to\tau_i^-} g(\vartheta)$, and we used the first inequality of (2.5.17) and the fact that g is always non-negative. Finally, if the exponent γ in (2.5.7) satisfies $\gamma = 2$, we can choose any positive $\delta > 0$. On the other hand, if $\gamma \in (1, 2)$, we choose δ such that $0 < \delta < \frac{2\gamma-2}{2-\gamma}$. This choice guarantees that that

$$\lim_{j\to\infty} \int_{\tau_0}^{\tau_j} -\dot g(\vartheta)\vartheta^{1+\delta}\, d\vartheta = \int_{\tau_0}^{\infty} -\dot g(\vartheta)\vartheta^{1+\delta}\, d\vartheta < \infty,$$

and therefore, in virtue of (2.5.21) and (2.5.20), we deduce the thesis. □

In the following corollary we state an immediate (but important) consequence of Proposition 2.5.1.

Corollary 2.5.2. *Under the same assumptions as in Proposition 2.5.1, let the curve* $U : [0, +\infty) \to \mathcal{U}$ *be the solution of the Cauchy problem* (2.2.2) *with* *initial datum* $U_0 = u_0$. *If* $u_\infty \in \mathcal{U}$ *is a limiting point for the curve* $t \mapsto U_t$, *then the whole solution converges to* u_∞ *as* $t \to \infty$, *i.e.,*

$$\lim_{t\to\infty} ||U_t - u_\infty||_{L^2} = 0.$$

Moreover, the length of the whole solution is finite.

PROOF. We prove the statement by contradiction. Let us assume that $t \mapsto U_t$ is not converging to u_∞ as $t \to \infty$. Let $B_r(u_\infty)$ be the neighborhood of u_∞ given by Proposition 2.5.1. Diminishing $r > 0$ if necessary, we can find two sequences $\{t_j\}_{j\geq 0}$ and $\{t'_j\}_{j\geq 0}$ such that for every $j \geq 0$ the following conditions hold:

- $t_j < t'_j < t_{j+1}$;
- $||U_{t_j} - u_\infty||_{L^2} \leq \frac{r}{4}$;
- $\frac{r}{2} \leq ||U_{t'_j} - u_\infty||_{L^2} \leq r$;
- $U_t \in B_r(u_\infty)$ for every $t \in (t_j, t'_j)$.

We observe that $\bigcup_{j=1}^\infty (t_j, t'_j) \subset \mathcal{I}$, where $\mathcal{I} := \{t \geq 0 : U_t \in B_r(u_\infty)\}$. Moreover the inequality (2.5.3) and the previous conditions imply that

$$\int_{t_j}^{t'_j} ||\partial_t U_\theta||_{\mathcal{U}} \, d\theta \geq ||U_{t'_k} - U_{t_k}||_{\mathcal{U}} \geq \frac{r}{4}$$

for every $j \geq 0$. However, this contradicts (2.5.4). Therefore, we deduce that $||U_t - u_\infty||_{\mathcal{U}} \to 0$ as $t \to \infty$. In particular, this means that there exists $\bar{t} \geq 0$ such that $U_t \in B_r(u_\infty)$ for every $t \geq \bar{t}$. This in turn implies that the whole trajectory has finite length, since

$$\int_0^{\bar{t}} ||\partial_t U_\theta||_{L^2} \, d\theta < +\infty.$$

□

We observe that in Corollary 2.5.2 we need to assume *a priori* that the solution of the Cauchy problem (2.2.2) admits a limiting point. However, for a general initial datum $u_0 \in \mathcal{U}$ we cannot prove that this is actually the case. On the other hand, if we assume more regularity on the Cauchy datum u_0, we can use the compactness results proved in Section 2.3. We recall the notation $H^0([0,1], \mathbb{R}^k) := \mathcal{U}$.

THEOREM 2.5.3. *Let us assume that the vector fields* $F_1, \ldots, F_k$ *defining the control system* (1.1.6) *are real-analytic, as well as the function* $a : \mathbb{R}^n \to \mathbb{R}_+$ *designing the end-point cost. Let* $U : [0, +\infty) \to \mathcal{U}$ *be the solution of the Cauchy problem* (2.2.2) *with initial datum* $U_0 = u_0$, *and let* $m \geq 1$ *be an integer such that* u_0 *belongs to* $H^m([0,1], \mathbb{R}^k)$. *Then there exists* $u_\infty \in H^m([0,1], \mathbb{R}^k)$ *such that*

$$\lim_{t\to\infty} ||U_t - u_\infty||_{H^{m-1}} = 0. \quad (2.5.22)$$

PROOF. Let us consider $u_0 \in H^m([0,1],\mathbb{R}^k)$ and let $U : [0,+\infty) \to \mathcal{U}$ be the solution of (2.2.2) satisfying $U_0 = u_0$. Owing to Theorem 2.3.6, we have that $U_t \in H^m([0,1],\mathbb{R}^k)$ for every $t \geq 0$, and that the trajectory $\{U_t : t \geq 0\}$ is bounded in $H^m([0,1],\mathbb{R}^k)$. In addition, from Corollary 2.3.7, we deduce that $\{U_t : t \geq 0\}$ is pre-compact with respect to the strong topology of $H^{m-1}([0,1],\mathbb{R}^k)$. Therefore, there exist $u_\infty \in H^{m-1}([0,1],\mathbb{R}^k)$ and a sequence $(t_j)_{j\geq 1}$ such that we have $t_j \to +\infty$ and $||U_{t_j} - u_\infty||_{H^{m-1}} \to 0$ as $j \to \infty$. In particular, this implies that $||U_{t_j} - u_\infty||_{L^2} \to 0$ as $j \to \infty$. In virtue of Corollary 2.5.2, we deduce that $||U_t - u_\infty||_{L^2} \to 0$ as $t \to +\infty$. Using again the pre-compactness of the trajectory $\{U_t : t \geq 0\}$ with respect to the strong topology of $H^{m-1}([0,1],\mathbb{R}^k)$, the previous convergence implies that $||U_t - u_\infty||_{H^{m-1}} \to 0$ as $t \to +\infty$.

To conclude, we have to show that $u_\infty \in H^m([0,1],\mathbb{R}^k)$. Owing to the compact inclusion (1.1.10) in Theorem 1.1.1, and recalling that the trajectory $\{U_t : t \geq 0\}$ is pre-compact with respect to the weak topology of $H^m([0,1],\mathbb{R}^k)$, the convergence (2.5.22) guarantees that $u_\infty \in H^m([0,1],\mathbb{R}^k)$ and that $U_t \rightharpoonup_{H^m} u_\infty$ as $t \to +\infty$. □

In the next result we study the regularity of the limiting points of the gradient flow trajectories.

THEOREM 2.5.4. *Let us assume that the vector fields $F_1, \ldots, F_k$ defining the control system* (1.1.6) *are real-analytic, as well as the function $a : \mathbb{R}^n \to \mathbb{R}_+$ designing the end-point cost. Let $U : [0,+\infty) \to \mathcal{U}$ be the solution of the Cauchy problem* (2.2.2) *with initial datum $U_0 = u_0$, and let $u_\infty \in \mathcal{U}$ be any of its limiting points. Then u_∞ is a critical point for the functional $\mathcal{F}$, i.e., $d_{u_\infty}\mathcal{F} = 0$. Moreover, $u_\infty \in H^m([0,1],\mathbb{R}^k)$ for every integer $m \geq 1$.*

PROOF. By Corollary 2.5.2, we have that the solution $t \mapsto U_t$ converges to u_∞ as $t \to +\infty$ with respect to the strong topology of $\mathcal{U}$. Let us consider the radius $r > 0$ prescribed by Proposition 2.5.1. If $d_{u_\infty}\mathcal{F} \neq 0$, taking a smaller $r > 0$ if necessary, we have that there exists $\varepsilon > 0$ such that $||d_u\mathcal{F}||_{\mathcal{U}^*} \geq \varepsilon$ for every $u \in B_r(u_\infty)$. Recalling that $||U_t - u_\infty||_{\mathcal{U}} \to 0$ as $t \to +\infty$, then there exists $\bar{t} \geq 0$ such that $U_t \in B_r(u_\infty)$ and for every $t \geq \bar{t}$. On the other hand, this fact implies that $||\partial_t U_t||_{\mathcal{U}} = ||d_{U_t}\mathcal{F}||_{\mathcal{U}^*} \geq \varepsilon$ for every $t \geq \bar{t}$, but this contradicts (2.5.4), i.e., the fact that the length of the trajectory is finite. Therefore, we deduce that $d_{u_\infty}\mathcal{F} = 0$. As regards the regularity of u_∞, we observe that $d_{u_\infty}\mathcal{F} = 0$ implies that $\mathcal{G}[u_\infty] = 0$, which in turn gives

$$u_\infty = -\frac{1}{\beta} h_{u_\infty},$$

where the function $h_{u_\infty} : [0,1] \to \mathbb{R}^k$ is defined as in (2.2.11). Owing to Lemma 2.3.3, we deduce that the right-hand side of the previous equality has regularity H^{m+1} whenever $u_\infty \in H^m$, for every integer $m \geq 0$. Using a bootstrapping argument, this implies that $u_\infty \in H^m([0,1],\mathbb{R}^k)$, for every integer $m \geq 1$. □

REMARK 2.5.1. We can give a further characterization of the critical points of the functional $\mathcal{F}$. Let $\hat{u}$ be such that $d_{\hat{u}}\mathcal{F} = 0$. Therefore, as seen in the proof of Theorem 2.5.4, we have that the identity

$$\hat{u}(s) = -\frac{1}{\beta} h_{\hat{u}}(s)$$

is satisfied for every $s \in [0,1]$. Recalling the definition of $h_{\hat{u}} : [0,1] \to \mathbb{R}^k$ given in (2.2.11), we observe that the previous relation yields

$$\hat{u}(s) = \arg\max_{u \in \mathbb{R}^k} \left\{ -\lambda_{\hat{u}}(s) F(x_{\hat{u}}(s)) u - \frac{\beta}{2} |u|_2^2 \right\}, \tag{2.5.23}$$

where $x_{\hat{u}} : [0,1] \to \mathbb{R}^n$ solves

$$\begin{cases} \dot{x}_{\hat{u}}(s) = F(x_{\hat{u}}(s))\hat{u}(s) & \text{for a.e. } s \in [0,1], \\ x_{\hat{u}}(0) = x_0, \end{cases} \tag{2.5.24}$$

and $\lambda_{\hat{u}} : [0,1] \to (\mathbb{R}^n)^*$ satisfies

$$\begin{cases} \dot{\lambda}_{\hat{u}}(s) = -\lambda_{\hat{u}}(s) \sum_{i=1}^{k} \left(\hat{u}^i(s) \frac{\partial F_i(x_{\hat{u}}(s))}{\partial x} \right) & \text{for a.e. } s \in [0,1], \\ \lambda_{\hat{u}}(1) = \nabla a(x_{\hat{u}}(1)). \end{cases} \tag{2.5.25}$$

Recalling the Pontryagin Maximum Principle (see, e.g., [**4**, Theorem 12.10]), from (2.5.23)-(2.5.25) we deduce that the curve $x_{\hat{u}} : [0,1] \to \mathbb{R}^n$ is a normal Pontryagin extremal for the following optimal control problem:

$$\begin{cases} \min_{u \in \mathcal{U}} \left\{ a(x_u(1)) + \frac{\beta}{2} ||u||_{L^2}^2 \right\}, \\ \text{subject to } \begin{cases} \dot{x}_u = F(x_u)u, \\ x_u(0) = x_0. \end{cases} \end{cases}$$

CHAPTER 3

Ensembles of affine-control systems

In this chapter we consider the problem of the optimal control of an ensemble of affine-control systems. After introducing the notations and the framework, we study the properties of the trajectories of the controlled ensembles, and we prove the well-posedness of the minimization problem in exam. Then we establish a Γ-convergence result that allows us to substitute the original (and usually infinite) ensemble with a sequence of finite increasing-in-size sub-ensembles of control systems. The solutions of the optimal control problems involving these sub-ensembles provide approximations in the L^2-strong topology of the minimizers of the original problem. Using the results of Chapter 2, we derive the gradient field induced by the optimal control problems related to the sub-ensembles. Moreover, in the case of finite sub-ensembles, we can address the minimization of the related cost by means of some numerical schemes. In particular, we propose an algorithm that consists in a subspace projection of the induced gradient field, and we consider an iterative method based on the Pontryagin Maximum Principle. Finally, we perform some numerical experiments.

3.1. Framework and Assumptions

In this chapter we study ensembles of control systems in $\mathbb{R}^n$ with affine dependence in the control variable $u \in \mathbb{R}^k$. More precisely, given a compact set Θ embedded into a finite-dimensional Euclidean space, for every $\theta \in \Theta$ we are assigned an affine-control system of the form

$$\begin{cases} \dot{x}^\theta(s) = F_0^\theta(x^\theta(s)) + F^\theta(x^\theta(s))u(s), \\ x^\theta(0) = x_0^\theta, \end{cases} \tag{3.1.1}$$

where for every $\theta \in \Theta$ we require that $F_0^\theta : \mathbb{R}^n \to \mathbb{R}^n$ and $F^\theta : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ are Lipschitz-continuous applications. We stress the fact that the control $u : [0,1] \to \mathbb{R}^k$ does not depend on θ, so that it is the same for every control system of the ensemble. Let us introduce $F_0 : \mathbb{R}^n \times \Theta \to \mathbb{R}^n$ and $F : \mathbb{R}^n \times \Theta \to \mathbb{R}^{n\times k}$ defined respectively as

$$F_0(x,\theta) := F_0^\theta(x) \qquad \text{and} \qquad F(x,\theta) := F^\theta(x)$$

for every $(x, \theta) \in \mathbb{R}^n \times \Theta$. We assume that F_0 and F are Lipschitz-continuous mappings, i.e., that there exists a constant $L > 0$ such that

$$|F_0(x_1, \theta_1) - F_0(x_2, \theta_2)|_2 \leq L\big(|x_1 - x_2|_2 + |\theta_1 - \theta_2|_2\big) \tag{3.1.2}$$

and

$$\sup_{i=1,\ldots,k} |F_i(x_1, \theta_1) - F_i(x_2, \theta_2)|_2 \leq L\big(|x_1 - x_2|_2 + |\theta_1 - \theta_2|_2\big) \tag{3.1.3}$$

for every $(x_1, \theta_1), (x_2, \theta_2) \in \mathbb{R}^n \times \Theta$. In (3.1.3) we used $F_i(x, \theta)$ to denote the vector obtained by taking the i^{th} column of the matrix $F(x, \theta)$, for every $i = 1, \ldots, k$. Similarly, for every $\theta \in \Theta$ we shall use $F_i^\theta : \mathbb{R}^n \to \mathbb{R}^n$ to denote the vector field corresponding to the i^{th} column of the matrix-valued application $F^\theta : \mathbb{R}^n \to \mathbb{R}^{n \times k}$. We observe that (3.1.2)-(3.1.3) imply that the vector fields $F_0^\theta, F_1^\theta, \ldots, F_k^\theta$ are uniformly Lipschitz-continuous as θ varies in Θ. Another consequence of the Lipschitz-continuity conditions (3.1.2)-(3.1.3) is that the vector fields constituting the affine-control system (3.1.1) have sub-linear growth, uniformly with respect to the dependence on θ. Namely, we have that there exists a constant $C > 0$ such that

$$\sup_{\theta \in \Theta} |F_0^\theta(x)|_2 \leq C\big(|x|_2 + 1\big) \tag{3.1.4}$$

and

$$\sup_{\theta \in \Theta} \sup_{i=1,\ldots,k} |F_i^\theta(x)|_2 \leq C\big(|x|_2 + 1\big) \tag{3.1.5}$$

for every $x \in \mathbb{R}^n$. Finally, let us consider the application $x_0 : \Theta \to \mathbb{R}^n$ that prescribes the initial state of (3.1.1), i.e.,

$$x_0(\theta) := x_0^\theta \tag{3.1.6}$$

for every $\theta \in \Theta$. We assume that x_0 is continuous. As a matter of fact, we deduce that there exists a constant $C' > 0$ such that

$$\sup_{\theta \in \Theta} |x_0(\theta)|_2 \leq C'. \tag{3.1.7}$$

Using the same notations as in the previous chapters, we set $\mathcal{U} := L^2([0, 1], \mathbb{R}^k)$ as the space of admissible controls, and we equip it with the usual Hilbert space structure given by the scalar product (1.1.5). For every $u \in \mathcal{U}$ and $\theta \in \Theta$, the curve $x_u^\theta : [0, 1] \to \mathbb{R}^n$ denotes the solution of the Cauchy problem (3.1.1) corresponding to the system identified by θ and to the admissible control u. We recall that, for every $u \in \mathcal{U}$ and $\theta \in \Theta$, the existence and uniqueness of the solution of (3.1.1) is guaranteed by the Carathéodory Theorem (see, e.g., [**30**, Theorem 5.3]). Given $u \in \mathcal{U}$, we describe the evolution of the ensemble of control systems (3.1.1) through the mapping $X_u : [0, 1] \times \Theta \to \mathbb{R}^n$ defined as follows:

$$X_u(s, \theta) := x_u^\theta(s) \tag{3.1.8}$$

for every $(s, \theta) \in [0, 1] \times \Theta$. In other words, for every $u \in \mathcal{U}$ the application X_u collects the trajectories of the ensemble of control systems (3.1.1). We study the properties of the mapping X_u in the next section.

3.2. Trajectories of the controlled ensemble

We devote the present subsection to establish some auxiliary properties of the mapping $X_u : [0, 1] \times \Theta \to \mathbb{R}^n$, which has been defined in (3.1.8) for every $u \in \mathcal{U}$. We first prove that for every $u \in \mathcal{U}$ the mapping $X_u : [0, 1] \times \Theta \to \mathbb{R}^n$ is bounded.

LEMMA 3.2.1. *For every $u \in \mathcal{U}$, let $X_u : [0, 1] \times \Theta \to \mathbb{R}^n$ be the application defined in* (3.1.8) *collecting the trajectories of the ensemble of control systems* (3.1.1)*. Therefore, for every $R > 0$ there exists $C_R > 0$ such that, if $||u||_{L^2} \leq R$, then*

$$|X_u(s, \theta)|_2 \leq C_R, \tag{3.2.1}$$

for every $(s, \theta) \in [0, 1] \times \Theta$.

PROOF. The thesis follows from a *verbatim* repetition of the argument in the proof of Lemma 1.2.2 □

In the next result we show that the trajectories of the ensemble are Hölder-continuous, uniformly with respect to the parameter $\theta \in \Theta$.

LEMMA 3.2.2. *For every $u \in \mathcal{U}$, let $X_u : [0, 1] \times \Theta \to \mathbb{R}^n$ be the application defined in* (3.1.8) *collecting the trajectories of the ensemble of control systems* (3.1.1)*. Therefore, for every $R > 0$ there exists $L_R > 0$ such that, if $||u||_{L^2} \leq R$, then*

$$|X_u(s_1, \theta) - X_u(s_2, \theta)|_2 \leq L_R |s_1 - s_2|^{\frac{1}{2}} \tag{3.2.2}$$

for every $s_1, s_2 \in [0, 1]$ and for every $\theta \in \Theta$.

PROOF. Owing to Proposition 1.1.2 and recalling that $X_u(s, \theta) = x_u^\theta(s)$ for every $(s, \theta) \in [0, 1] \times \Theta$ by (3.1.8), we observe that the thesis will follow if we prove that there exists a bounded subset of H^1 that includes the trajectories $\{x_u^\theta : [0, 1] \to \mathbb{R}^n\}_{\theta \in \Theta}$ of (3.1.1) for every admissible control $u \in \mathcal{U}$ satisfying $||u||_{L^2} \leq R$. From Lemma 3.2.1 we obtain that for every $R > 0$ there exists $C_R > 0$ such that

$$|x_u^\theta(s)| \leq C_R \tag{3.2.3}$$

for every $s \in [0, 1]$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. In virtue of Lemma 3.2.1 and the sub-linear inequalities (3.1.4)-(3.1.5), we deduce that for every $R > 0$ there exists $C'_R > 0$ such that

$$\sup_{\theta \in \Theta} |F_0^\theta(x_u^\theta(s))| \leq C'_R, \qquad \sup_{\theta \in \Theta} \sup_{i=1,\ldots,k} |F_i^\theta(x_u^\theta(s))| \leq C'_R$$

for every $s \in [0, 1]$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. Therefore, we have that

$$|\dot{x}_u^\theta(s)|_2 \leq C'_R(1 + |u(s)|_1) \tag{3.2.4}$$

for every $s \in [0,1]$, for every $\theta \in \Theta$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. Combining (3.2.4) and (3.2.3), we deduce that there exists $C''_R > 0$ such that

$$||x_u^\theta||_{H^1} \leq C''_R$$

for every $\theta \in \Theta$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$. The last inequality and Proposition 1.1.2 imply that

$$|x_u^\theta(s_1) - x_u^\theta(s_2)|_2 \leq L_R |s_1 - s_2|^{\frac{1}{2}}$$

for every $s_1, s_2 \in [0,1]$, for every $\theta \in \Theta$ and for every $u \in \mathcal{U}$ such that $||u||_{L^2} \leq R$, where we set $L_R := \sqrt{C''_R}$. This establishes (3.2.2). □

We shall now prove that, when the control u varies in a bounded subset of $\mathcal{U}$, the corresponding functions $X_u : [0,1] \times \Theta \to \mathbb{R}^n$ that captures the evolution of the ensemble of control systems (3.1.1) are uniformly equi-continuous on their domain. Before proceeding, we introduce the modulus of continuity of the function $x_0 : \Theta \to \mathbb{R}^n$ defined in (3.1.6). Indeed, since $x_0 : \Theta \to \mathbb{R}^n$ is a continuous function defined on a compact domain, it is uniformly continuous, i.e., there exists a non-decreasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $0 = \omega(0) = \lim_{r \to 0^+} \omega(r)$ and such that

$$|x_0(\theta_1) - x_0(\theta_1)|_2 \leq \omega(|\theta_1 - \theta_2|_2) \tag{3.2.5}$$

for every $\theta_1, \theta_2 \in \Theta$.

PROPOSITION 3.2.3. *For every $u \in \mathcal{U}$, let $X_u : [0,1] \times \Theta \to \mathbb{R}^n$ be the application defined in* (3.1.8) *collecting the trajectories of the ensemble of control systems* (3.1.1)*. Therefore, for every $R > 0$ there exists $L_R > 0$ and $\omega_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that, if $||u||_{L^2} \leq R$, then*

$$|X_u(s_1, \theta_1) - X_u(s_2, \theta_2)|_2 \leq L_R |s_1 - s_2|^{\frac{1}{2}} + \omega_R(|\theta_1 - \theta_2|_2) \tag{3.2.6}$$

for every $(s_1, \theta_1), (s_2, \theta_2) \in [0,1] \times \Theta$, where ω_R is a non-decreasing function that satisfies $\omega(0) = \lim_{r \to 0^+} \omega_R(r) = 0$.

PROOF. We observe that by the triangular inequality we have

$$|X_u(s_1, \theta_1) - X_u(s_2, \theta_2)|_2 \leq |X_u(s_1, \theta_1) - X_u(s_2, \theta_1)|_2 + |X_u(s_2, \theta_1) - X_u(s_2, \theta_2)|_2 \tag{3.2.7}$$

for every $(s_1, \theta_1), (s_2, \theta_2) \in [0,1] \times \Theta$ and for every $u \in \mathcal{U}$. Moreover, from Lemma 3.2.2 it follows that there exists $L_R > 0$ such that

$$|X_u(s_1, \theta_1) - X_u(s_2, \theta_1)|_2 \leq L_R |s_1 - s_2|^{\frac{1}{2}} \tag{3.2.8}$$

for every $(s_1, \theta_1), (s_2, \theta_2) \in [0,1] \times \Theta$ and for every $u \in \mathcal{U}$ satisfying $||u||_{L^2} \leq R$. Thus, we are left to study the second term at the right-hand side of (3.2.7). First, we introduce the function $\omega_R : \mathbb{R}_+ \to \mathbb{R}_+$ defined as follows:

$$\omega_R(r) := e^{L\left(1+\sqrt{k}R\right)} \left(\omega(r) + L(1 + \sqrt{k}R)r\right),$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity for the mapping $x_0 : \Theta \to \mathbb{R}^n$ (see (3.2.5)). Using Grönwall Lemma and similar computations as in the proof of Proposition 1.2.3, we deduce that

$$|X_u(s,\theta_1) - X_u(s,\theta_2)|_2 = |x_u^{\theta_1}(s) - x_u^{\theta_2}(s)|_2 \leq \omega_R(|\theta_1 - \theta_2|_2), \tag{3.2.9}$$

for every $s \in [0,1]$, for every $\theta_1, \theta_2 \in \Theta$ and for every $u \in \mathcal{U}$ with $||u||_{L^2} \leq R$. Finally, combining (3.2.7), (3.2.8) and (3.2.9), we obtain the thesis (3.2.6). □

We now investigate the evolution of the ensemble of control systems (3.1.1) when we consider a sequence of L^2-weakly convergent admissible controls. In view of the next auxiliary result, we introduce some notations. For every $\theta \in \Theta$, we define $\tilde{F}^\theta : \mathbb{R}^n \to \mathbb{R}^{n\times(k+1)}$ as follows:

$$\tilde{F}^\theta(x) := (F_0^\theta(x), F^\theta(x)), \tag{3.2.10}$$

for every $x \in \mathbb{R}^n$, i.e., we add the column $F_0^\theta(x)$ to the $n \times k$ matrix $F^\theta(x)$. Similarly, for every $u \in \mathcal{U} = L^2([0,1], \mathbb{R}^k)$, we consider the extended control $\tilde{u} \in \tilde{\mathcal{U}} := L^2([0,1], \mathbb{R}^{k+1})$ defined as

$$\tilde{u}(s) = (1, u(s))^T \tag{3.2.11}$$

for every $s \in [0,1]$, i.e., we add the component $u_0 = 1$ to the column-vector $u(s)$.

Lemma 3.2.4. *Let us consider a sequence of admissible controls $(u_m)_{m\in\mathbb{N}} \subset \mathcal{U}$ such that $u_m \rightharpoonup_{L^2} u_\infty$ as $m \to \infty$. For every $m \in \mathbb{N}\cup\{\infty\}$ and for every $\theta \in \Theta$, let $x_m^\theta : [0,1] \to \mathbb{R}^n$ be the solution of* (3.1.1) *corresponding to the ensemble parameter θ and to the admissible control u_m. Therefore, for every $s \in [0,1]$ and for every $\theta \in \Theta$, we have*

$$\lim_{m\to\infty} x_m^\theta(s) = x_\infty^\theta(s). \tag{3.2.12}$$

Proof. Let us fix $\theta \in \Theta$. By means of the matrix-valued function $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^{n\times(k+1)}$ and the extended control $\tilde{u} : [0,1] \to \mathbb{R}^{k+1}$ defined in (3.2.10) and (3.2.11) respectively, we can equivalently rewrite the affine-control system (3.1.1) corresponding to θ as follows:

$$\begin{cases} \dot{x}^\theta = \tilde{F}^\theta(x^\theta)\tilde{u}, \\ x^\theta(0) = x_0^\theta, \end{cases} \tag{3.2.13}$$

for every $u \in \mathcal{U}$. In other words, any solution $x_u^\theta : [0,1] \to \mathbb{R}^n$ of (3.1.1) corresponding to the admissible control $u \in \mathcal{U}$ is in turn a solution of the linear-control system (3.2.13) corresponding to the extended control $\tilde{u} \in \tilde{\mathcal{U}}$. On the other hand, the convergence $u_m \rightharpoonup_{L^2} u_\infty$ as $m \to \infty$ implies the convergence of the respective extended controls, i.e., $\tilde{u}_m \rightharpoonup_{L^2} \tilde{u}_\infty$ as $m \to \infty$. Therefore, $(x_m^\theta)_{m\in\mathbb{N}}$ is the sequence of solutions of the linear-control system (3.2.13) corresponding to the L^2-weakly convergent sequence of controls $(\tilde{u}_m)_{m\in\mathbb{N}}$. Moreover, x_∞^θ is the solution of (3.2.13)

associated to the weak-limiting control $\tilde{u}_\infty$. Using Proposition 1.4.1, we deduce that
$$\lim_{m\to\infty} ||x_m^\theta - x_\infty^\theta||_{C^0} = 0,$$
which, in particular, implies (3.2.12). □

We are now in position to prove the main result of the present subsection.

THEOREM 3.2.5. *Let us consider a sequence of admissible controls* $(u_m)_{m\in\mathbb{N}} \subset \mathcal{U}$ *such that* $u_m \rightharpoonup_{L^2} u_\infty$ *as* $m \to \infty$. *For every* $m \in \mathbb{N}\cup\{\infty\}$, *let* $X_m : [0,1]\times\Theta \to \mathbb{R}^n$ *be the application defined in* (3.1.8) *that collects the trajectories of the ensemble of control systems* (3.1.1) *corresponding to the admissible control* u_m. *Therefore, we have that*
$$\lim_{m\to\infty} \sup_{(s,\theta)\in[0,1]\times\Theta} |X_m(s,\theta) - X_\infty(s,\theta)|_2 = 0. \tag{3.2.14}$$

PROOF. Let us consider a L^2-weakly convergent sequence $(u_m)_{m\in\mathbb{N}} \subset \mathcal{U}$ such that $u_m \rightharpoonup_{L^2} u_\infty$ as $m\to\infty$. We immediately deduce that there exists $R > 0$ such that
$$||u_m||_{L^2} \leq R, \quad \forall m \in \mathbb{N}\cup\{\infty\}.$$
Thus, in virtue of Proposition 3.2.3, we deduce that the sequence of mappings $\{X_m : [0,1]\times\Theta \to \mathbb{R}^n\}_{m\in\mathbb{N}}$ is uniformly equi-continuous, while Lemma 3.2.1 guarantees that it is uniformly equi-bounded. Therefore, applying the Ascoli-Arzelà Theorem (see, e.g., [**15**, Theorem 4.25]), we deduce that the family $(X_m)_{m\in\mathbb{N}}$ is pre-compact with respect to the strong topology of the Banach space $C^0([0,1]\times\Theta,\mathbb{R}^n)$. Finally, Lemma 3.2.4 implies that
$$\lim_{m\to\infty} X_m(s,\theta) = X_\infty(s,\theta)$$
for every $(s,\theta)\in[0,1]\times\Theta$. In particular, we deduce that the set of limiting points of the pre-compact sequence $(X_m)_{m\in\mathbb{N}}$ is reduced to the single-element set $\{X_\infty\}$. This proves (3.2.14). □

REMARK 3.2.1. Theorem 3.2.5 is the cornerstone of the theoretical results presented in this chapter. Indeed, the fact fact that the trajectories of the ensemble (3.1.1) are uniformly convergent when the corresponding controls are L^2-weakly convergent is used both to prove the existence of optimal controls (see Theorem 3.4.2) and to establish the Γ-convergence result (see Theorem 3.5.3). We stress that the fact that the systems in the ensemble (3.1) have affine dependence in the controls is crucial for the proof of Theorem 3.2.5.

3.3. Gradient field for affine-control systems with end-point cost

In this section we generalize to the case of affine-control systems some of the results obtained in Chapter 2 in the framework of linear-control systems with end-point cost. As we shall see, the strategy that we pursue consists in embedding the affine-control system into a larger linear-control system, similarly as done in the proof of Lemma 3.2.4. Therefore, we can exploit a consistent part of the machinery developed in Chapter 2 to cover the present case. Let us consider a *single* affine-control system on $\mathbb{R}^n$ of the form

$$\begin{cases} \dot{x}(s) = F_0(x(s)) + F(x(s))u(s), & \text{for a.e. } s \in [0,1], \\ x(0) = x_0, \end{cases} \tag{3.3.1}$$

where $F_0 : \mathbb{R}^n \to \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ are C^2-regular applications that design the affine-control system, and $u \in \mathcal{U} = L^2([0,1],\mathbb{R}^k)$ is the control. We introduce the functional $\mathcal{J} : \mathcal{U} \to \mathbb{R}_+$ defined on the space of admissible controls as follows:

$$\mathcal{J}(u) := a(x_u(1)) + \frac{\beta}{2}||u||_{L^2}^2 \tag{3.3.2}$$

for every $u \in \mathcal{U}$, where $a : \mathbb{R}^n \to \mathbb{R}_+$ is a C^2-regular function and $\beta > 0$ a positive parameter. After proving that the functional $\mathcal{J}$ is differentiable, we provide the expression of the mapping $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ that, for every $u \in \mathcal{U}$ represents the differential $d_u\mathcal{J} : \mathcal{U} \to \mathbb{R}$.

Before proceeding, it is convenient to introduce the linear-control system in which we embed (3.3.1). Similarly as done in (3.2.10), let $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^{n\times(k+1)}$ be the function defined as

$$\tilde{F}(x) := (F_0(x), F(x)) \tag{3.3.3}$$

for every $x \in \mathbb{R}^n$. If we define the extended space of admissible controls as $\tilde{\mathcal{U}} := L^2([0,1],\mathbb{R}^{k+1})$, we may consider the following linear-control system

$$\begin{cases} \dot{x}(s) = \tilde{F}(x(s))\tilde{u}(s) & \text{for a.e. } s \in [0,1], \\ x(0) = x_0, \end{cases} \tag{3.3.4}$$

where $\tilde{u} \in \tilde{\mathcal{U}}$. We observe that we can recover the affine system (3.3.1) by restricting the set of admissible controls in (3.3.4) to the image of the affine embedding $i : \mathcal{U} \to \tilde{\mathcal{U}}$ defined as

$$i[u] := \begin{pmatrix} 1 \\ u \end{pmatrix}. \tag{3.3.5}$$

We introduce the extended cost functional $\tilde{\mathcal{J}} : \tilde{\mathcal{U}} \to \mathbb{R}_+$ as

$$\tilde{\mathcal{J}}(\tilde{u}) := a(x_{\tilde{u}}(1)) + \frac{\beta}{2}||\tilde{u}||_{L^2}^2 \tag{3.3.6}$$

for every $\tilde{u} \in \tilde{\mathcal{U}}$, where $x_{\tilde{u}} : [0,1] \to \mathbb{R}^n$ is the absolutely continuous solution of (3.3.4) corresponding to the control $\tilde{u}$. To avoid confusion, in the present subsection we denote by $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ and $\langle\cdot,\cdot\rangle_{\tilde{\mathcal{U}}}$ the scalar products in $\mathcal{U}$ and $\tilde{\mathcal{U}}$, respectively. In the next result we prove that the functional $\mathcal{J} : \mathcal{U} \to \mathbb{R}_+$ defined in (3.3.2) is differentiable.

PROPOSITION 3.3.1. *Let us assume that* $F_0 : \mathbb{R}^n \to \mathbb{R}^n$ *and* $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ *are* C^1*-regular, as well as the function* $a : \mathbb{R}^n \to \mathbb{R}_+$ *designing the end-point cost. Then the functionals* $\mathcal{J} : \mathcal{U} \to \mathbb{R}_+$ *and* $\tilde{\mathcal{J}} : \tilde{\mathcal{U}} \to \mathbb{R}_+$ *defined, respectively, in* (3.3.2) *and in* (3.3.6) *are Gateaux differentiable at every point of their respective domains.*

PROOF. We observe that the functional $\mathcal{J} : \mathcal{U} \to \mathbb{R}^+$ satisfies the following identity:

$$\mathcal{J}(u) = \tilde{\mathcal{J}}(i(u)) - \frac{\beta}{2} \tag{3.3.7}$$

for every $u \in \mathcal{U}$, where $i : \mathcal{U} \to \tilde{\mathcal{U}}$ is the affine embedding reported in (3.3.5). Since $i : \mathcal{U} \to \tilde{\mathcal{U}}$ is analytic, the proof reduces to show that the functional $\tilde{\mathcal{J}} : \tilde{\mathcal{U}} \to \mathbb{R}_+$ is Gateaux differentiable. This is actually the case, since $\tilde{u} \mapsto \frac{\beta}{2}||\tilde{u}||_{L^2}$ is smooth, while the first term at the right-hand side of (3.3.6) (i.e., the end-point cost) is Gateaux differentiable owing to Lemma 2.2.1. □

By differentiation of the identity (3.3.7) we deduce that

$$d_u\mathcal{J}(v) = d_{i[u]}\tilde{\mathcal{J}}\big(i_\#[v]\big) \tag{3.3.8}$$

for every $u, v \in \mathcal{U}$, where we have introduced the linear inclusion $i_\# : \mathcal{U} \to \tilde{\mathcal{U}}$ defined as

$$i_\#[v] := \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{3.3.9}$$

for every $v \in \mathcal{U}$. In virtue of Proposition 3.3.1, we can consider the vector field $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ that represents the differential of the functional $\mathcal{J} : \mathcal{U} \to \mathbb{R}_+$. Namely, for every $u \in \mathcal{U}$, let $\mathcal{G}[u]$ be the unique element of $\mathcal{U}$ such that

$$\langle \mathcal{G}[u], v\rangle_{\mathcal{U}} = d_u\mathcal{J}(v) \tag{3.3.10}$$

for every $v \in \mathcal{U}$. Similarly, let us denote with $\tilde{\mathcal{G}} : \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ the vector field such that

$$\langle \tilde{\mathcal{G}}[\tilde{u}], \tilde{v}\rangle_{\tilde{\mathcal{U}}} = d_{\tilde{u}}\tilde{\mathcal{J}}(\tilde{v}) \tag{3.3.11}$$

for every $\tilde{u}, \tilde{v} \in \tilde{\mathcal{U}}$. In Chapter 2 it was derived the expression of the vector field $\tilde{\mathcal{G}}$ associated to the linear-control system (3.3.4) and to the cost (3.3.6). In the next result we use it in order to obtain the expression of $\mathcal{G}$. We use the notation $F(x)^T$ to denote the matrix in $\mathbb{R}^{k\times n}$ obtained by the transposition of the matrix $F(x) \in \mathbb{R}^{n\times k}$, for every $x \in \mathbb{R}^n$. The analogue convention holds for $\tilde{F}(x)^T$, for every $x \in \mathbb{R}^n$.

THEOREM 3.3.2. *Let us assume that* $F_0 : \mathbb{R}^n \to \mathbb{R}^n$ *and* $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ *are* C^1*-regular, as well as the function* $a : \mathbb{R}^n \to \mathbb{R}_+$ *designing the end-point cost. Let* $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ *be the gradient vector field on* $\mathcal{U}$ *that satisfies* (3.3.10)*. Then, for every* $u \in \mathcal{U}$ *we have*

$$\mathcal{G}[u](s) = F(x_u(s))^T \lambda_u^T(s) + \beta u(s) \tag{3.3.12}$$

for a.e. $s \in [0,1]$, *where* $x_u : [0,1] \to \mathbb{R}^n$ *is the solution of* (3.3.1) *corresponding to the control* u, *and* $\lambda_u : [0,1] \to (\mathbb{R}^n)^*$ *is the absolutely continuous curve of covectors that solves*

$$\begin{cases} \dot{\lambda}_u(s) = -\lambda_u(s)\left(\frac{\partial F_0(x_u(s))}{\partial x} + \sum_{i=1}^k u_i(s)\frac{\partial F_i(x_u(s))}{\partial x}\right) & a.e.\ in\ [0,1], \\ \lambda_u(1) = \nabla a(x_u(1)). \end{cases} \tag{3.3.13}$$

REMARK 3.3.1. As done in Chapter 2, we understand the elements of $(\mathbb{R}^n)^*$ as row-vectors. Therefore, for every $s \in [0,1]$, $\lambda_u(s)$ should be read as a row-vector. This should be considered to give sense to (3.3.13).

PROOF OF THEOREM 3.3.2. In virtue of (3.3.8), from the definitions (3.3.10) and (3.3.11) we deduce that

$$\langle \mathcal{G}[u], v\rangle_{\mathcal{U}} = \langle \tilde{\mathcal{G}}[i[u]], i_\#[v]\rangle_{\tilde{\mathcal{U}}} = \langle \pi\tilde{\mathcal{G}}[i[u]], v\rangle_{\mathcal{U}} \tag{3.3.14}$$

for every $u, v \in \mathcal{U}$, where $\tilde{\mathcal{G}} : \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ is the gradient vector field corresponding to the functional $\tilde{\mathcal{J}} : \tilde{\mathcal{U}} \to \mathbb{R}_+$, and $\pi : \tilde{\mathcal{U}} \to \mathcal{U}$ is the linear application

$$\pi : \begin{pmatrix} \tilde{v}_0 \\ \vdots \\ \tilde{v}_k \end{pmatrix} \mapsto \begin{pmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_k \end{pmatrix} \tag{3.3.15}$$

for every $\tilde{v} \in \tilde{\mathcal{U}}$. Therefore, we can rewrite (3.3.14) as

$$\mathcal{G} = \pi \circ \tilde{\mathcal{G}} \circ i, \tag{3.3.16}$$

where i and π are defined, respectively, in (3.3.5) and in (3.3.15). This implies that we can deduce the expression of $\mathcal{G}$ from the one of $\tilde{\mathcal{G}}$. In particular, from Remark 2.2.2 it follows that for every $\tilde{u} \in \tilde{\mathcal{U}}$ we have

$$\tilde{\mathcal{G}}[\tilde{u}](s) = \tilde{F}(x_{\tilde{u}}(s))^T \lambda_{\tilde{u}}^T(s) + \beta\tilde{u}(s) \tag{3.3.17}$$

for a.e. $s \in [0,1]$, where $x_{\tilde{u}} : [0,1] \to \mathbb{R}^n$ is the solution of (3.3.4) corresponding to the control $\tilde{u}$, and $\lambda_{\tilde{u}} : [0,1] \to (\mathbb{R}^n)^*$ is the absolutely continuous curve of covectors that solves

$$\begin{cases} \dot{\lambda}_{\tilde{u}}(s) = -\lambda_{\tilde{u}}(s)\sum_{i=0}^k \left(\tilde{u}_i(s)\frac{\partial \tilde{F}_i(x_{\tilde{u}}(s))}{\partial x}\right) & \text{for a.e. } s \in [0,1], \\ \lambda_{\tilde{u}}(1) = \nabla a(x_{\tilde{u}}(1)). \end{cases} \tag{3.3.18}$$

We stress the fact that the summation index in (3.3.18) starts from 0. Then, the thesis follows immediately from (3.3.16)-(3.3.18). □

REMARK 3.3.2. The identity (3.3.16) implies that the gradient field $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ is at least as regular as $\tilde{\mathcal{G}} : \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$. In particular, under the further assumption that $F_0 : \mathbb{R}^n \to \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ and $a : \mathbb{R}^n \to \mathbb{R}_+$ are C^2-regular, from Lemma 2.2.2 it follows that $\tilde{\mathcal{G}} : \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$ is Lipschitz-continuous on the bounded sets of $\tilde{\mathcal{U}}$. As a matter of fact, we deduce that, under the same regularity hypotheses, $\mathcal{G} : \mathcal{U} \to \mathcal{U}$ is Lipschitz-continuous on the bounded sets of $\mathcal{U}$.

3.4. Optimal control of ensembles

In this section we formulate a minimization problem for the ensemble of affine-control systems (3.1.1). Namely, let us consider a non-negative continuous mapping $a : [0,1]\times\mathbb{R}^n\times\Theta \to \mathbb{R}_+$, a positive real number $\beta > 0$ and a Borel probability measure ν on the time interval $[0,1]$. Therefore, for every $\theta \in \Theta$ we can study the following optimal control problem:

$$\int_0^1 a(s, x_u^\theta(s), \theta)\, d\nu(s) + \frac{\beta}{2}||u||_{L^2}^2 \to \min, \tag{3.4.1}$$

where the curve $x_u^\theta : [0,1] \to \mathbb{R}^n$ is the solution of (3.1.1) corresponding to the parameter $\theta \in \Theta$ and to the admissible control $u \in \mathcal{U}$. We recall that the ensemble of control systems (3.1.1) is aimed at modeling our partial knowledge of the data of the controlled dynamical system. Therefore, it is natural to assume that the space of parameters Θ is endowed with a Borel probability measure μ that quantifies this uncertainty. In view of this fact, we can formulate an optimal control problem for the ensemble of control systems (3.1.1) as follows:

$$\int_\Theta \int_0^1 a(s, x_u^\theta(s), \theta)\, d\nu(s)\, d\mu(\theta) + \frac{\beta}{2}||u||_{L^2}^2 \to \min. \tag{3.4.2}$$

The minimization problem (3.4.2) is obtained by averaging out the parameters $\theta \in \Theta$ in the optimal control problem (3.4.1) by means of the probability measure μ.

In this section we study the variational problem (3.4.2), and we prove that it admits a solution. Before proceeding, we introduce the functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ associated to the minimization problem (3.4.2). For every admissible control $u \in \mathcal{U}$, we set

$$\mathcal{F}^\infty(u) := \int_\Theta \int_0^1 a(s, x_u^\theta(s), \theta)\, d\nu(s)\, d\mu(\theta) + \frac{\beta}{2}||u||_{L^2}^2. \tag{3.4.3}$$

We first prove an auxiliary lemma regarding the integral cost in (3.4.2).

LEMMA 3.4.1. *Let us consider a sequence of admissible controls $(u_m)_{m\in\mathbb{N}} \subset \mathcal{U}$ such that $u_m \rightharpoonup_{L^2} u_\infty$ as $m \to \infty$. For every $m \in \mathbb{N}\cup\{\infty\}$, let $Y_m : [0,1]\times\Theta \to \mathbb{R}_+$ be defined as follows:*

$$Y_m(s,\theta) := a(s, X_m(s,\theta), \theta), \tag{3.4.4}$$

where $X_m : [0,1]\times\Theta \to \mathbb{R}^n$ is the application defined in (3.1.8) *corresponding to the admissible control u_m. Then, we have that*

$$\lim_{m\to\infty} \sup_{(s,\theta)\in[0,1]\times\Theta} |Y_m(s,\theta) - Y_\infty(s,\theta)| = 0. \tag{3.4.5}$$

PROOF. Since the sequence $(u_m)_{m\in\mathbb{N}}$ is weakly convergent, it follows that there exists $R > 0$ such that $||u_m||_{L^2} \le R$ for every $m \in \mathbb{N}\cup\{\infty\}$. For every $m \in \mathbb{N}\cup\{\infty\}$, let $X_m : [0,1]\times\Theta \to \mathbb{R}^n$ be the application defined in (3.1.8) corresponding to the control u_m. In virtue of Lemma 3.2.1, we deduce that there exists a compact set $K \subset \mathbb{R}^n$ such that

$$X_m(s,\theta) \in K$$

for every $(s,\theta) \in [0,1] \times \Theta$ and for every $m \in \mathbb{N} \cup \{\infty\}$. Recalling that the function $a : [0,1] \times \mathbb{R}^n \times \Theta \to \mathbb{R}_+$ that defines the integral term in (3.4.3) is assumed to be continuous, it follows that the restriction

$$\tilde{a} := a|_{[0,1]\times K \times \Theta}$$

is uniformly continuous. In addition, Theorem 3.2.5 guarantees that $X_m \to_{C^0} X_\infty$ as $m \to \infty$. Therefore, observing that

$$Y_m(s,\theta) = \tilde{a}(s, X_m(s,\theta), \theta) \tag{3.4.6}$$

for every $(s,\theta) \in [0,1] \times \Theta$ and for every $m \in \mathbb{N} \cup \{\infty\}$, we deduce that (3.4.5) holds. □

We are now in position to prove that (3.4.2) admits a solution.

Theorem 3.4.2. *Let $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ be the functional defined in* (3.4.3). *Then, there exists $\hat{u} \in \mathcal{U}$ such that*

$$\mathcal{F}^\infty(\hat{u}) = \inf_{\mathcal{U}} \mathcal{F}^\infty.$$

Proof. We establish the thesis by means of the direct method of calculus of variations (see, e.g., [**24**, Theorem 1.15]). Namely, we show that the functional $\mathcal{F}^\infty$ is coercive and lower semi-continuous with respect to the weak topology of L^2. We first address the coercivity, i.e., we prove that the sub-level sets of the functional $\mathcal{F}^\infty$ are L^2-weakly pre-compact. To see that, it is sufficient to observe that for every $M \geq 0$ we have

$$\{u \in \mathcal{U} : \mathcal{F}^\infty(u) \leq M\} \subset \{u \in \mathcal{U} : ||u||^2_{L^2} \leq 2M/\beta\},$$

where we used the fact that the first term at the right-hand side of (3.4.3) is non-negative. To study the lower semi-continuity, let us consider a sequence of admissible controls $(u_m)_{m\in\mathbb{N}} \subset \mathcal{U}$ such that $u_m \rightharpoonup_{L^2} u_\infty$ as $m \to \infty$. Using the family of applications $(Y_m)_{m\in\mathbb{N}\cup\{\infty\}}$ defined as in (3.4.4), we observe that the integral term at the right-hand side of (3.4.3) can be rewritten as follows

$$\int_\Theta \int_0^1 a(s, x^\theta_{u_m}(s), \theta)\, d\nu(s)\, d\mu(\theta) = \int_\Theta \int_0^1 Y_m(s,\theta)\, d\nu(s)\, d\mu(\theta)$$

for every $m \in \mathbb{N} \cup \{\infty\}$. Moreover, the uniform convergence $Y_m \to_{C^0} Y_\infty$ as $m \to \infty$ provided by Lemma 3.4.1 implies in particular the convergence of the integral term at the right-hand side of (3.4.3):

$$\lim_{m\to\infty} \int_\Theta \int_0^1 a(s, x_{u_m}(s)^\theta, \theta)\, d\nu(s)\, d\mu(\theta) = \int_\Theta \int_0^1 a(s, x_{u_\infty}(s)^\theta, \theta)\, d\nu(s)\, d\mu(\theta). \tag{3.4.7}$$

Finally, combining (1.1.11) with (3.4.7), we deduce that

$$\mathcal{F}^\infty(u_\infty) \leq \liminf_{m\to\infty} \mathcal{F}^\infty(u_m).$$

This proves that the functional $\mathcal{F}^\infty$ is lower semi-continuous, and therefore we obtain the thesis. □

REMARK 3.4.1. The constant $\beta > 0$ in (3.4.3) is aimed at balancing the effect of the squared L^2-norm regularization and of the integral term. This fact can be crucial in some cases, relevant for applications. Indeed, let us assume that, for every $\varepsilon > 0$, there exists $u_\varepsilon \in \mathcal{U}$ such that

$$\int_\Theta \int_0^1 a(s, x^\theta_{u_\varepsilon}(s), \theta)\, d\nu(s)\, d\mu(\theta) \leq \frac{\varepsilon}{2}.$$

Then, let us set

$$\beta = \frac{\varepsilon}{||u_\varepsilon||^2_{L^2}},$$

and let $\hat{u} \in \mathcal{U}$ be a minimizer for the functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ defined as in (3.4.3). Therefore, we have that

$$\int_\Theta \int_0^1 a(s, x^\theta_{\hat{u}}(s), \theta)\, d\nu(s)\, d\mu(\theta) \leq \mathcal{F}^\infty(\hat{u}) \leq \mathcal{F}^\infty(u_\varepsilon) \leq \varepsilon.$$

In particular, this means that, when the constant $\beta > 0$ is chosen small enough, the integral cost achieved by the minimizers of $\mathcal{F}^\infty$ can be made arbitrarily small.

3.5. Reduction to finite ensembles via Γ-convergence

The existence result in Theorem 3.4.2 guarantees that the set of the solutions of the minimization problem (3.4.2) is nonempty. However, if the support of the probability measure μ is not contained in a fine subset of the space of parameters Θ, the problem (3.4.2) involves the resolution of an infinite number of Cauchy problems. Therefore, a natural attempt to make the ensemble optimal control problem (3.4.2) tractable consists in approximating μ with a probability measure $\bar{\mu}$ that charges a finite number of elements of Θ. Therefore, if μ and $\bar{\mu}$ are close in some appropriate sense, we may expect that the solutions of the minimization problem involving $\bar{\mu}$ provide approximations of the minimizers of the original ensemble optimal control problem (3.4.2). This argument can be made rigorous by means of the tools of Γ-convergence. We briefly recall below this notion. For a thorough introduction to this topic we refer the reader to the textbook [**24**].

DEFINITION 2. Let $(\mathcal{X}, d)$ be a metric space, and for every $N \geq 1$ let $\mathcal{F}^N : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a functional defined over X. The sequence $(\mathcal{F}^N)_{N\geq 1}$ is said to Γ-converge to a functional $\mathcal{F}^\infty : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ if the following conditions holds:

- *liminf condition*: for every sequence $(u_N)_{N\geq 1} \subset \mathcal{X}$ such that $u_N \to_{\mathcal{X}} u$ as $N \to \infty$ the following inequality holds

$$\mathcal{F}^\infty(u) \leq \liminf_{N\to\infty} \mathcal{F}^N(u_N); \tag{3.5.1}$$

- *limsup condition*: for every $x \in \mathcal{X}$ there exists a sequence $(u_N)_{N\geq 1} \subset \mathcal{X}$ such that $u_N \to_{\mathcal{X}} u$ as $N \to \infty$ and such that the following inequality holds:

$$\mathcal{F}^\infty(u) \geq \limsup_{N\to\infty} \mathcal{F}^N(u_N). \tag{3.5.2}$$

If the conditions listed above are satisfied, then we write $\mathcal{F}^N \to_\Gamma \mathcal{F}^\infty$ as $N \to \infty$.

The importance of the Γ-convergence is due to the fact that it relates the minimizers of the functionals $(\mathcal{F}^N)_{N\geq 1}$ to the minimizers of the limiting functional $\mathcal{F}^\infty$. Namely, under the hypothesis that the functionals of the sequence $(\mathcal{F}^N)_{N\geq 1}$ are equi-coercive, if $\hat{u}_N \in \arg\min \mathcal{F}^N$ for every $N \geq 1$, then the sequence $(\hat{u}_N)_{N\geq 1}$ is pre-compact in $\mathcal{X}$, and any of its limiting points is a minimizer for $\mathcal{F}^\infty$ (see [**24**, Corollary 7.20]). In other words, the problem of minimizing $\mathcal{F}^\infty$ can be approximated by the minimization of $\mathcal{F}^N$, when N is sufficiently large. We report that a similar approach was undertaken in [**41**], where the authors considered general ensembles of control systems (not only affine in the controls), and it was proved that the *averaged approximations* of the cost functional in exam are Γ-convergent to the original objective with respect to the *strong* topology of L^2. We insist on the fact that our result is not reduced to a particular case of the one studied in [**41**]. Indeed, on one hand, by means of the *strong* topology in [**41**] it was possible to establish Γ-convergence for more general ensembles of control systems, and not only under the affine-control dynamics (3.1.1). On the other hand, in the general situation considered in [**41**] the functionals of the approximating sequence are not equi-coercive (often neither coercive) in the L^2-*strong* topology, and proving that the minimizers of the approximating functionals are (up to subsequences) convergent could be a challenging task. However, in the case of affine-control systems we manage to prove Γ-convergence even if the space of admissible controls $\mathcal{U}$ is equipped with the *weak* topology.

We now focus on the ensemble optimal control problem (3.4.2) studied in Section 3.4 and on the functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ defined in (3.4.3). As done in the proof of Theorem 3.4.2, it is convenient to equip the space of admissible controls $\mathcal{U} := L^2([0,1],\mathbb{R}^k)$ with the weak topology. However, Definition 2 requires the domain $\mathcal{X}$ where the limiting and the approximating functionals are defined to be a metric space. Unfortunately, the weak topology of L^2 is metrizable only when restricted to bounded sets (see, e.g., [**15**, Remark 3.3 and Theorem 3.29]). In the next lemma we see how we should choose the restriction without losing any of the minimizers of $\mathcal{F}^\infty$.

LEMMA 3.5.1. *Let* $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ *be the functional defined in* (3.4.3). *Therefore, there exists* $\rho > 0$ *such that, if* $\hat{u} \in \mathcal{U}$ *satisfies* $\mathcal{F}^\infty(\hat{u}) = \inf_{\mathcal{U}} \mathcal{F}^\infty$, *then*

$$||\hat{u}||_{L^2} \leq \rho. \tag{3.5.3}$$

PROOF. Let us consider the control $\bar{u} \equiv 0$. If $\hat{u} \in \mathcal{U}$ is a minimizer for the functional $\mathcal{F}^\infty$, then we have $\mathcal{F}^\infty(\hat{u}) \leq \mathcal{F}^\infty(\bar{u})$. Moreover, recalling that the function $a : [0,1] \times \mathbb{R}^n \times \Theta \to \mathbb{R}_+$ that designs the integral cost in (3.4.2) is non-negative, we deduce that

$$\frac{\beta}{2}||\hat{u}||^2_{L^2} \leq \mathcal{F}^\infty(\hat{u}) \leq \mathcal{F}^\infty(\bar{u}).$$

Thus, to prove (3.5.3) it is sufficient to set $\rho := \sqrt{2\mathcal{F}^\infty(\bar{u})/\beta}$. □

The previous result implies that the following inclusion holds

$$\arg\min \mathcal{F}^\infty \subset \mathcal{X},$$

where we set

$$\mathcal{X} := \{u \in \mathcal{U} : ||u||_{L^2} \leq \rho\}, \tag{3.5.4}$$

and where $\rho > 0$ is provided by Lemma 3.5.1. Since $\mathcal{X}$ is a closed ball of L^2, the weak topology induced on $\mathcal{X}$ is metrizable. Hence, we can restrict the functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ to $\mathcal{X}$ in order to construct an approximation in the sense of Γ-convergence. With a slight abuse of notations, we shall continue to denote by $\mathcal{F}^\infty$ the functional restricted to $\mathcal{X}$. As anticipated at the beginning of the present section, the construction of the functionals $(\mathcal{F}^N)_{N\geq 1}$ relies on the introduction of a proper sequence of probability measures $(\mu_N)_{N\geq 1}$ on Θ that approximate the probability measure μ prescribing the integral cost in (3.4.2). We first recall the notion of weak convergence of probability measures. For further details, see, e.g., the textbook [**21**, Definition 3.5.1].

DEFINITION 3. Let $(\mu_N)_{N\geq 1}$ be a sequence of Borel probability measures on the compact set Θ. The sequence $(\mu_N)_{N\geq 1}$ is weakly convergent to the probability measure μ as $N \to \infty$ if the following identity holds

$$\lim_{N\to\infty} \int_\Theta f(\theta)\, d\mu_N(\theta) = \int_\Theta f(\theta)\, d\mu(\theta), \tag{3.5.5}$$

for every function $f \in C^0(\Theta, \mathbb{R})$. If the previous condition is satisfied, we write $\mu_N \rightharpoonup^* \mu$ as $N \to \infty$.

For every $N \geq 1$ we consider a subset $\{\theta_1, \ldots, \theta_N\} \subset \Theta$ and the probability measure that charges uniformly these elements:

$$\mu_N := \frac{1}{N}\sum_{j=1}^{N} \delta_{\theta_j}. \tag{3.5.6}$$

We assume that the sequence $(\mu_N)_{N\geq 1}$ approximates the probability measure μ in the weak sense, i.e., $\mu_N \rightharpoonup^* \mu$ as $N \to \infty$. This can be achieved if, for example,

$(\mu_N)_{N\geq 1}$ are empirical measures associated to independent samples of the probability measure μ. We are now in position to introduce the family of functionals $(\mathcal{F}^N)_{N\geq 1}$. For every $N \geq 1$, let $\mathcal{F}^N : \mathcal{X} \to \mathbb{R}_+$ be defined as follows

$$\mathcal{F}^N(u) := \int_\Theta \int_0^1 a(s, x_u^\theta(s), \theta)\, d\nu(s) d\mu_N(\theta) + \frac{\beta}{2}||u||_{L^2}^2, \tag{3.5.7}$$

where $x_u^\theta : [0,1] \to \mathbb{R}^n$ denotes the solution on (3.1.1) corresponding to the parameter $\theta \in \Theta$ and to the control $u \in \mathcal{X}$. We observe that $\mathcal{F}^N$ and $\mathcal{F}^\infty$ have essentially the same structure: the only difference is that the integral term of (3.4.3) involves the measure μ, while (3.5.7) the measure μ_N. Before proceeding to the main result of the section, we prove an auxiliary result.

LEMMA 3.5.2. *Let $(\mu_N)_{N\geq 1}$ be a sequence of probability measures on Θ such that $\mu_N \rightharpoonup^* \mu$ as $N \to \infty$, and let ν be a probability measure on $[0,1]$. Then, the sequence of the product probability measures $(\nu \otimes \mu_N)_{N\geq 1}$ on the product space $[0,1] \times \Theta$ satisfies $\nu \otimes \mu_N \rightharpoonup^* \nu \otimes \mu$ as $N \to \infty$.*

PROOF. It is sufficient to prove that

$$\lim_{N\to\infty} \int_{[0,1]\times\Theta} f(s,\theta)\, d(\nu \otimes \mu_N)(s,\theta) = \int_{[0,1]\times\Theta} f(s,\theta)\, d(\nu \otimes \mu)(s,\theta) \tag{3.5.8}$$

for every $f \in C^0([0,1] \times \Theta, \mathbb{R})$. In virtue of Fubini Theorem (see, e.g., [**15**, Theorem 4.5]), for every $f \in C^0([0,1] \times \Theta, \mathbb{R})$ we have

$$\int_{[0,1]\times\Theta} f(s,\theta)\, d(\nu \otimes \mu_N)(s,\theta) = \int_\Theta \bar{f}(\theta)\, d\mu_N(\theta) \tag{3.5.9}$$

for every $N \geq 1$, where $\bar{f} : \Theta \to \mathbb{R}$ is the continuous function defined as $\bar{f}(\theta) := \int_0^1 f(s,\theta)\, d\nu(s)$ for every $\theta \in \Theta$. Moreover, the hypothesis $\mu_N \rightharpoonup^* \mu$ as $N \to \infty$ implies that

$$\lim_{N\to\infty} \int_\Theta \bar{f}(\theta) d\mu_N(\theta) = \int_\Theta \bar{f}(\theta) d\mu(\theta) = \int_{[0,1]\times\Theta} f(s,\theta)\, d(\nu \otimes \mu)(s,\theta) \tag{3.5.10}$$

for every $f \in C^0([0,1] \times \Theta, \mathbb{R})$, where we used again Fubini Theorem in the last identity. Finally, combining (3.5.9) and (3.5.10) we obtain (3.5.8), and this concludes the proof. □

We now show that the sequence of functionals $(\mathcal{F}^N)_{N\geq 1}$ introduced in (3.5.7) is Γ-convergent to the functional that defines the ensemble optimal control problem (3.4.2).

THEOREM 3.5.3. *Let $\mathcal{X} \subset \mathcal{U}$ be the set defined in* (3.5.4)*, equipped with the weak topology of L^2. For every $N \geq 1$, let $\mathcal{F}^N : \mathcal{X} \to \mathbb{R}_+$ be the functional introduced in* (3.5.7)*, and let $\mathcal{F}^\infty : \mathcal{X} \to \mathbb{R}_+$ be the restriction to $\mathcal{X}$ of the application defined in* (3.4.3)*. Then, we have $\mathcal{F}^N \to_\Gamma \mathcal{F}^\infty$ as $N \to \infty$.*

PROOF. We first establish the *liminf condition*. Let us consider a sequence of controls $(u_N)_{N\geq 1} \subset \mathcal{X}$ such that $u_N \rightharpoonup_{L^2} u_\infty$ as $N \to \infty$. As done in Lemma 3.4.1, for every $N \in \mathbb{N} \cup \{\infty\}$ let us define the functions $Y_N : [0,1] \times \Theta \to \mathbb{R}_+$ as follows:

$$Y_N(s,\theta) := a(s, X_N(s,\theta), \theta) \tag{3.5.11}$$

for every $(s,\theta) \in [0,1] \times \Theta$, where, for every $N \in \mathbb{N} \cup \{\infty\}$, $X_N : [0,1] \times \Theta \to \mathbb{R}^n$ is the mapping introduced in (3.1.8) that describes the evolution of the ensemble in correspondence of the admissible control u_N. From (3.5.11) and the definition of the functionals $(\mathcal{F}^N)_{N\geq 1}$ in (3.5.7), we obtain that

$$\mathcal{F}^N(u_N) = \int_\Theta \int_0^1 Y_N(s,\theta)\, d\nu(s) d\mu_N(\theta) + \frac{\beta}{2} ||u_N||_{L^2}^2 \tag{3.5.12}$$

for every $N \in \mathbb{N}$. Moreover, we observe that the uniform convergence $Y_N \to_{C^0} Y_\infty$ as $N \to \infty$ guaranteed by Lemma 3.4.1 implies that

$$\lim_{N\to\infty} \int_\Theta \int_0^1 |Y_N(s,\theta) - Y_\infty(s,\theta)|\, d\nu(s) d\mu_N(\theta) = 0. \tag{3.5.13}$$

Therefore, using the triangular inequality and Lemma 3.5.2, from (3.5.13) we deduce that

$$\lim_{N\to\infty} \int_\Theta \int_0^1 Y_N(s,\theta)\, d\nu(s) d\mu_N(\theta) = \int_\Theta \int_0^1 Y_\infty(s,\theta)\, d\nu(s) d\mu(\theta). \tag{3.5.14}$$

Combining (3.5.12) with (3.5.14) and (1.1.11), we have that

$$\mathcal{F}^\infty(u_\infty) \leq \liminf_{N\to\infty} \mathcal{F}^N(u_N),$$

which concludes the first part of the proof.

We now establish the *limsup condition*. For every $u \in \mathcal{X}$, let us consider the constant sequence $u_N = u$ for every $N \geq 1$. In virtue of Lemma 3.5.2, we have that

$$\lim_{N\to\infty} \int_\Theta \int_0^1 a(s, X_u(s,\theta), \theta)\, d\nu(s) d\mu_N(\theta) = \int_\Theta \int_0^1 a(s, X_u(s,\theta), \theta)\, d\nu(s) d\mu(\theta) \tag{3.5.15}$$

for every $u \in \mathcal{X}$, where $X_u : [0,1] \times \Theta \to \mathbb{R}^n$ is defined as in (3.1.8). This fact gives

$$\mathcal{F}^\infty(u) = \lim_{N\to\infty} \mathcal{F}^N(u)$$

for every $u \in \mathcal{X}$, and this shows that the *limsup condition* holds. □

REMARK 3.5.1. We observe that Theorem 3.4.2 holds also for $\mathcal{F}^N : \mathcal{X} \to \mathbb{R}_+$ for every $N \in \mathbb{N}$. Indeed, the domain $\mathcal{X}$ is itself sequentially weakly compact, and the convergence (3.4.7) occurs also with the probability measure μ_N in place of μ. Therefore, being the functional $\mathcal{F}^N$ coercive and sequentially lower semi-continuous with respect to the weak topology of L^2, it admits a minimizer.

The next result is a direct consequence of the Γ-convergence result established in Theorem 3.5.3. Indeed, as anticipated before, the fact that the minimizers of the functionals $(\mathcal{F}^N)_{N\in\mathbb{N}}$ provide approximations of the minimizers of the limiting functional $\mathcal{F}^\infty$ is a well-established fact, as well as the convergence $\inf_\mathcal{X} \mathcal{F}^N \to \inf_\mathcal{X} \mathcal{F}^\infty$ as $N \to \infty$ (see [**24**, Corollary 7.20]). We stress the fact that, usually, the approximation of the minimizers occurs in the topology that underlies the Γ-convergence result. However, we can actually prove that, in this case, the approximation is provided with respect to the *strong* topology of L^2, and not just in the weak sense.

COROLLARY 3.5.4. *Let $\mathcal{X} \subset \mathcal{U}$ be the set defined in* (3.5.4)*. For every $N \geq 1$, let $\mathcal{F}^N : \mathcal{X} \to \mathbb{R}_+$ be the functional introduced in* (3.5.7) *and let $\hat{u}_N \in \mathcal{X}$ be any of its minimizers. Finally, let $\mathcal{F}^\infty : \mathcal{X} \to \mathbb{R}_+$ be the restriction to $\mathcal{X}$ of the application defined in* (3.4.3)*. Therefore, we have*

$$\inf_\mathcal{X} \mathcal{F}^\infty = \lim_{N\to\infty} \inf_\mathcal{X} \mathcal{F}^N. \tag{3.5.16}$$

Moreover, the sequence $(\hat{u}_N)_{N\in\mathbb{N}}$ is pre-compact with respect to the strong topology of L^2, and any limiting point of this sequence is a minimizer of $\mathcal{F}^\infty$.

PROOF. Owing to Theorem 3.5.3, we have that $\mathcal{F}^N \to_\Gamma \mathcal{F}^\infty$ as $N \to \infty$ with respect to the weak topology of L^2. Therefore, from [**24**, Corollary 7.20] it follows that (3.5.16) holds and that the sequence of minimizers $(\hat{u}_N)_{N\in\mathbb{N}}$ is pre-compact with respect to the weak topology of L^2, and its limiting points are minimizers of $\mathcal{F}^\infty$. To conclude we have to prove that it is pre-compact with respect to the strong topology, too. Let us consider a subsequence $(\hat{u}_{N_j})_{j\in\mathbb{N}}$ such that $\hat{u}_{N_j} \rightharpoonup_{L^2} \hat{u}_\infty$ as $j \to \infty$. Using the fact that $\hat{u}_\infty$ is a minimizer for $\mathcal{F}^\infty$, as well as $\hat{u}_{N_j}$ is for $\mathcal{F}^{N_j}$ for every $j \in \mathbb{N}$, from (3.5.16) it follows that

$$\mathcal{F}^\infty(\hat{u}_\infty) = \lim_{j\to\infty} \mathcal{F}^{N_j}(\hat{u}_{N_j}). \tag{3.5.17}$$

Moreover, with the same argument used in the proof of Theorem 3.5.3 to deduce the identity (3.5.14), we obtain that

$$\int_\Theta \int_0^1 a(s, x^\theta_{\hat{u}_\infty}(s), \theta)\, d\nu(s) d\mu(\theta) = \lim_{j\to\infty} \int_\Theta \int_0^1 a(s, x^\theta_{\hat{u}_{N_j}}(s), \theta)\, d\nu(s) d\mu_{N_j}(\theta). \tag{3.5.18}$$

Combining (3.5.17) and (3.5.18), and recalling the definitions (3.5.7) and (3.4.3) of the functionals $\mathcal{F}^N : \mathcal{X} \to \mathbb{R}_+$ and $\mathcal{F}^\infty : \mathcal{X} \to \mathbb{R}_+$, we have that

$$\frac{\beta}{2}||\hat{u}_\infty||^2_{L^2} = \lim_{j\to\infty} \frac{\beta}{2}||\hat{u}_{N_j}||^2_{L^2},$$

which implies that $\hat{u}_{N_j} \to_{L^2} \hat{u}_\infty$ as $j \to \infty$. Since the argument holds for every L^2-weakly convergent subsequence of the sequence of minimizers $(\hat{u}_N)_{N\in\mathbb{N}}$, this concludes the proof. □

3.6. Gradient field and Maximum Principle for finite ensembles

In Section 3.4 we formulated the minimization problem (3.4.2) for the ensemble of control systems (3.1.1), and in Theorem 3.4.2 we showed that it admits solutions by proving that the corresponding cost functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ has minimizers. However, the minimization of $\mathcal{F}^\infty$ could potentially result in handling simultaneously an infinite number of controlled dynamical systems. In this regards, the Γ-convergence result obtained in Section 3.5 suggests that we can replace the original functional cost $\mathcal{F}^\infty$ with a sequence of functionals $(\mathcal{F}^N)_{N\in\mathbb{N}}$, each of them involving a finite sample of the ensemble of control systems (3.1.1). Moreover, Corollary 3.5.4 guarantees that the minimizers of $(\mathcal{F}^N)_{N\in\mathbb{N}}$ are actually convergent to minimizers of $\mathcal{F}^\infty$, i.e., to solutions of (3.4.2). In the present section we address the question of actually finding the minimizers of the approximating functionals $(\mathcal{F}^N)_{N\in\mathbb{N}}$. Namely, starting from the result stated in Theorem 3.3.2 for a *single* affine-control system with end-point cost, we obtain the expression of the gradient fields that the functionals $(\mathcal{F}^N)_{N\in\mathbb{N}}$ induce on their domain. Moreover, we state the Pontryagin Maximum Principle for the optimal control problems corresponding to the minimization of the functionals $(\mathcal{F}^N)_{N\in\mathbb{N}}$. Both the gradient fields and the Maximum Principle will be used for the construction of the numerical algorithms presented in the next section.

From now on, we specialize on the following particular form of the cost associated to the ensemble optimal control problem (3.4.2):

$$\mathcal{F}^\infty(u) = \int_\Theta a(x_u^\theta(1),\theta)\,d\mu(\theta) + \frac{\beta}{2}||u||_{L^2}^2 \tag{3.6.1}$$

for every $u \in \mathcal{U}$, where $a : \mathbb{R}^n \times \Theta \to \mathbb{R}_+$ is a C^1-regular function, and $\beta > 0$ is a positive parameter that tunes the L^2-regularization. We observe that (3.6.1) is a particular instance of (3.4.3). Indeed, it corresponds to the case $\nu = \delta_{t=1}$, where ν is the probability measure on the time interval $[0, 1]$ that appears in the first term at the right-hand side of (3.4.2). In other words, we assume that the integral cost in (3.4.2) depends only on the final state of the trajectories of the ensemble. For every $N \in \mathbb{N}$, let the probability measure μ_N have the same expression as in (3.5.6), i.e., it is a finite uniform convex combination of Dirac deltas centered at $\{\theta_1, \dots, \theta_N\} \subset \Theta$. Therefore, for every $N \in \mathbb{N}$, the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ that we consider in place of (3.6.1) has the form

$$\mathcal{F}^N(u) = \int_\Theta a(x_u^\theta(1),\theta)\,d\mu_N(\theta) + \frac{\beta}{2}||u||_{L^2}^2 = \frac{1}{N}\sum_{j=1}^N a(x_u^{\theta_j}(1),\theta_j) + \frac{\beta}{2}||u||_{L^2}^2 \tag{3.6.2}$$

for every $u \in \mathcal{U}$.

Remark 3.6.1. In Section 3.5 for technical reasons we defined the functionals $(\mathcal{F}^N)_{N\in\mathbb{N}}$ on the domain $\mathcal{X} \subset \mathcal{U}$ introduced in (3.5.4). However, the functionals

$(\mathcal{F}^N)_{N\in\mathbb{N}}$ and the corresponding gradient fields can be actually defined over the whole space of admissible controls $\mathcal{U}$.

At this point, it is convenient to approach the minimization of the functional $\mathcal{F}^N$ in the framework of optimal control problem in finite-dimensional Euclidean spaces. For this purpose, we introduce some notations. For every $N \in \mathbb{N}$, let $\{\theta_1, \ldots, \theta_N\} \subset \Theta$ be the set of parameters charged by the discrete probability measure μ_N. Then, we study the finite sub-ensemble of (3.1.1) corresponding to the parameters $\{\theta_1, \ldots, \theta_N\}$. Namely, we consider the following affine-control system on $\mathbb{R}^{nN}$:

$$\begin{cases} \dot{\mathbf{x}}_u(s) = \mathbf{F}_0^N(\mathbf{x}_u) + \mathbf{F}^N(\mathbf{x}_u)u(s), & \text{for a.e. } t \in [0,1], \\ \mathbf{x}_u(0) = \mathbf{x}_0, \end{cases} \tag{3.6.3}$$

where $\mathbf{x} = (x^1, \ldots, x^N)^T \in \mathbb{R}^{nN}$, and $\mathbf{F}_0^N : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ and $\mathbf{F}^N : \mathbb{R}^{nN} \to \mathbb{R}^{nN\times k}$ are applications defined as follows:

$$\mathbf{F}_0^N(\mathbf{x}) := \begin{pmatrix} F_0^{\theta_1}(x^1) \\ \vdots \\ F_0^{\theta_N}(x^N) \end{pmatrix} \tag{3.6.4}$$

and

$$\mathbf{F}^N(\mathbf{x}) := \begin{pmatrix} F^{\theta_1}(x^1) \\ \vdots \\ F^{\theta_N}(x^N) \end{pmatrix} = \begin{pmatrix} F_1^{\theta_1}(x^1) & \ldots & F_k^{\theta_1}(x^1) \\ \vdots & & \vdots \\ F_1^{\theta_N}(x^N) & \ldots & F_k^{\theta_N}(x^N) \end{pmatrix} \tag{3.6.5}$$

for every $\mathbf{x} \in \mathbb{R}^{nN}$. Finally, the initial value is set as $\mathbf{x}_0 := (x_0(\theta_1), \ldots, x_0(\theta_N))$, where $x_0 : \Theta \to \mathbb{R}^n$ is the mapping defined (3.1.6) that prescribes the initial data of the Cauchy problems of the ensemble (3.1.1). Moreover, we can introduce the function $\mathbf{a}^N : \mathbb{R}^{nN} \to \mathbb{R}_+$ defined as

$$\mathbf{a}^N(\mathbf{x}) = \mathbf{a}^N((x^1, \ldots, x^N)) := \frac{1}{N}\sum_{j=1}^N a(x^j, \theta_j), \tag{3.6.6}$$

where $a : \mathbb{R}^n \times \Theta \to \mathbb{R}_+$ is the function that designs the integral cost in (3.6.1). In this framework, the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ can be rewritten as follows:

$$\mathcal{F}^N(u) = \mathbf{a}(\mathbf{x}_u(1)) + \frac{\beta}{2}||u||_{L^2}^2 \tag{3.6.7}$$

for every $u \in \mathcal{U}$, where $\mathbf{x}_u^N : [0,1] \to \mathbb{R}^{nN}$ is the solution of (3.6.3) corresponding to the admissible control u. In the next result we derive the expression of the vector field $\mathcal{G}^N : \mathcal{U} \to \mathcal{U}$ that represents the differential of the functional $\mathcal{F}^N$, namely

$$\langle \mathcal{G}^N[u], v\rangle_{\mathcal{U}} = d_u\mathcal{F}^N(v) \tag{3.6.8}$$

for every $u, v \in \mathcal{U}$.

THEOREM 3.6.1. *Let us assume that for every $\theta \in \Theta$ the functions $x \mapsto F_0(x,\theta)$ and $x \mapsto F(x,\theta)$ are C^1-regular, as well as the function $x \mapsto a(x,\theta)$ that defines the end-point cost in* (3.6.1). *Let $\{\theta_1,\ldots,\theta_N\} \subset \Theta$ be the subset of parameters charged by the measure μ^N that designs the integral cost in* (3.6.2). *Let $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ be the functional defined in* (3.6.2). *Then $\mathcal{F}^N$ is Gateaux differentiable at every $u \in \mathcal{U}$, and we define $\mathcal{G}^N : \mathcal{U} \to \mathcal{U}$ as the gradient vector field on $\mathcal{U}$ that satisfies* (3.6.8). *Then, for every $u \in \mathcal{U}$ we have*

$$\mathcal{G}^N[u](s) = \sum_{j=1}^{N} F^{\theta_j}(x_u^{\theta_j}(s))^T \lambda_u^{j\,T}(s) + \beta u(s) \tag{3.6.9}$$

for a.e. $s \in [0,1]$, where for every $j = 1,\ldots,N$ the curve $x_u^{\theta_j} : [0,1] \to \mathbb{R}^n$ is the solution of (3.1.1) *corresponding to the parameter θ_j and to the admissible control u, and $\lambda_u^j : [0,1] \to (\mathbb{R}^n)^*$ is the absolutely continuous curve of covectors that solves*

$$\begin{cases} \dot\lambda_u^j(s) = -\lambda_u^j(s)\left(\frac{\partial F_0^{\theta_j}(x_u^{\theta_j}(s))}{\partial x} + \sum_{i=1}^k u_i(s)\frac{\partial F_i^{\theta_j}(x_u^{\theta_j}(s))}{\partial x}\right) & a.e.\ in\ [0,1], \\ \lambda_u^j(1) = \frac{1}{N}\nabla a(x_u^{\theta_j}(1),\theta_j). \end{cases} \tag{3.6.10}$$

REMARK 3.6.2. We use the convention that the elements of $(\mathbb{R}^n)^*$ are row-vectors. Therefore, for every $j = 1,\ldots,N$ and $s \in [0,1]$, $\lambda_u^j(s)$ should be read as a row-vector. This should be considered to give sense to (3.6.9) and (3.6.10). The same observation holds for Theorem 3.6.2.

PROOF OF THEOREM 3.6.1. As done in (3.6.3), we can equivalently rewrite the sub-ensemble of control systems corresponding to the parameters $\{\theta_1,\ldots,\theta_N\} \subset \Theta$ as a *single* affine-control system in $\mathbb{R}^{nN}$. Moreover, the regularity hypotheses guarantee that the functions $\mathbf{F}_0^N : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ and $\mathbf{F}^N : \mathbb{R}^{nN} \to \mathbb{R}^{nN\times k}$ defined in (3.6.5) are C^1-regular, as well as the function $\mathbf{a} : \mathbb{R}^{nN} \to \mathbb{R}_+$ introduced in (3.6.6). Therefore, owing to Theorem 3.3.2, we obtain the expression for the gradient field induced by the functional $\mathcal{F}^N$ written in (3.6.7). Indeed, we deduce that

$$\mathcal{G}^N[u](s) = \mathbf{F}^N(\mathbf{x}_u(s))^T \mathbf{\Lambda}_u(s) + \beta u(s) \tag{3.6.11}$$

for a.e. $s \in [0,1]$ and for every $u \in \mathcal{U}$, where $\mathbf{x}_u : [0,1] \to \mathbb{R}^{nN}$ is the solution of (3.6.3) corresponding to the control u, and $\mathbf{\Lambda}_u : [0,1] \to (\mathbb{R}^{nN})^*$ is the curve of covectors that solves

$$\begin{cases} \dot{\mathbf{\Lambda}}_u(s) = -\mathbf{\Lambda}_u(s)\left(\frac{\partial \mathbf{F}_0^N(\mathbf{x}_u(s))}{\partial \mathbf{x}} + \sum_{i=1}^k u_i(s)\frac{\partial \mathbf{F}_i^N(\mathbf{x}_u(s))}{\partial \mathbf{x}}\right) & \text{for a.e. } s \in [0,1], \\ \mathbf{\Lambda}_u(1) = \nabla_{\mathbf{x}}\mathbf{a}(\mathbf{x}_u(1)), \end{cases} \tag{3.6.12}$$

where $\mathbf{F}_1^N,\ldots,\mathbf{F}_k^N : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ denote the vector fields obtained by taking the columns of the matrix-valued application $\mathbf{F}^N : \mathbb{R}^{nN} \to \mathbb{R}^{nN\times k}$. Moreover, if we consider the curves of covectors $\lambda_u^1,\ldots,\lambda_u^N : [0,1] \to (\mathbb{R}^n)^*$ that solve (3.6.10) for $j = 1,\ldots,N$, it turns out that the solution of (3.6.12) can be written as

$\mathbf{\Lambda}_u(s) = (\lambda_u^1(s), \dots, \lambda_u^N(s))$ for every $s \in [0,1]$. Finally, using this decoupling of $\mathbf{\Lambda}_u$, the identity (3.6.10) can be deduced from (3.6.11) using the expression of $\mathbf{F}_0^N, \dots, \mathbf{F}_k^N$. □

In the previous result we obtained the gradient field on $\mathcal{U}$ that represents the differential of the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$. We now establish the necessary condition for an admissible control $\hat{u}_N \in \mathcal{U}$ to be a minimizer of $\mathcal{F}^N$. This essentially descends as a standard application of Pontryagin Maximum Principle. For a complete survey on the topic, the reader is referred to the textbook [**4**].

THEOREM 3.6.2. *Under the same assumptions and notations of Theorem 3.6.1, let* $\hat{u}_N = (\hat{u}_{N,1}, \dots, \hat{u}_{N,k}) \in \mathcal{U}$ *be a minimizer of the functional* $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ *defined as in* (3.6.2). *For every* $j = 1, \dots, N$, *let* $x_{\hat{u}_N}^{\theta_j} : [0,1] \to \mathbb{R}^n$ *be the solution of* (3.1.1) *corresponding to the parameter* $\theta_j \in \Theta$ *and to the optimal control* $\hat{u}_N$. *Then, for every* $j = 1, \dots, N$ *there exists a curve of covectors* $\lambda_{\hat{u}_N}^j : [0,1] \to (\mathbb{R}^n)^*$ *such that*

$$\begin{cases} \dot{\lambda}_{\hat{u}_N}^j(s) = -\lambda_{\hat{u}_N}^j(s) \left(\frac{\partial F_0^{\theta_j}(x_{\hat{u}_N}^{\theta_j}(s))}{\partial x} + \sum_{i=1}^k \hat{u}_{N,i}(s) \frac{\partial F_i^{\theta_j}(x_{\hat{u}_N}^{\theta_j}(s))}{\partial x} \right) & \text{a.e. in } [0,1], \\ \lambda_{\hat{u}_N}^j(1) = \frac{1}{N} \nabla a(x_{\hat{u}_N}^{\theta_j}(1), \theta_j), \end{cases} \tag{3.6.13}$$

and such that

$$\hat{u}_N(s) \in \arg\max_{v \in \mathbb{R}^k} \left\{ \sum_{j=1}^N \left(-\lambda_{\hat{u}_N}^j(s) F(x_{\hat{u}_N}^{\theta_j}(s)) v \right) - \frac{\beta}{2} |v|_2^2 \right\} \tag{3.6.14}$$

for a.e. $s \in [0,1]$.

PROOF. As done in the proof of Theorem 3.6.1, we observe that we can equivalently consider the *single* affine-control system (3.6.3) in place of the sub-ensemble of affine-control systems corresponding to the parameters $\{\theta_1, \dots, \theta_N\} \subset \Theta$. Moreover, if we rewrite the cost functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ as in (3.6.7), we reduce to a standard optimal control problem in $\mathbb{R}^{nN}$. Let $\hat{u}_N \in \mathcal{U}$ be an optimal control for this problem, and let $\mathbf{x}_{\hat{u}_N} : [0,1] \to \mathbb{R}^{nN}$ be the solution of (3.6.3) corresponding to $\hat{u}_N$. Then, from the Pontryagin Maximum Principle (see, e.g., [**4**, Chapter 12]), it follows that there exists $\alpha \in \{0, -1\}$ and $\mathbf{\Lambda}_{\hat{u}_N} : [0,1] \to (\mathbb{R}^{nN})^*$ such that $(\alpha, \mathbf{\Lambda}_{\hat{u}_N}(s)) \neq 0$ for every $s \in [0,1]$ and such that

$$\begin{cases} \dot{\mathbf{\Lambda}}_{\hat{u}_N}(s) = -\mathbf{\Lambda}_{\hat{u}_N}(s) \left(\frac{\partial \mathbf{F}_0^N(\mathbf{x}_{\hat{u}_N}(s))}{\partial \mathbf{x}} + \sum_{i=1}^k \hat{u}_{N,i}(s) \frac{\partial \mathbf{F}_i^N(\mathbf{x}_{\hat{u}_N}(s))}{\partial \mathbf{x}} \right) & \text{a.e. in } [0,1], \\ \mathbf{\Lambda}_{\hat{u}_N}(1) = \alpha \nabla_{\mathbf{x}} \mathbf{a}(\mathbf{x}_{\hat{u}_N}(1)). \end{cases} \tag{3.6.15}$$

Moreover, for a.e. $s \in [0,1]$ the following condition holds

$$\hat{u}_N(s) \in \arg\max_{v \in \mathbb{R}^k} \left\{ \mathbf{\Lambda}_{\hat{u}_N}(s) \big(\mathbf{F}_0^N(\mathbf{x}_{\hat{u}_N}(s)) + \mathbf{F}^N(\mathbf{x}_{\hat{u}_N}(s)) v \big) + \alpha \frac{\beta}{2} |v|_2^2 \right\}. \tag{3.6.16}$$

Since the differential equation (3.6.15) is linear, if $\alpha = 0$ we have $\mathbf{\Lambda}_{\hat{u}_N}(s) \equiv 0$, and this violates the condition $(\alpha, \mathbf{\Lambda}_{\hat{u}_N}(s)) \neq 0$ for every $s \in [0,1]$. Therefore we deduce that $\alpha = -1$. This shows that the optimal control problem in consideration has no abnormal extremals. Moreover, if we consider the curves of covectors $\lambda^1_{\hat{u}_N}, \dots, \lambda^N_{\hat{u}_N} : [0,1] \to (\mathbb{R}^n)^*$ that solve (3.6.13) for $j = 1, \dots, N$, it turns out that the solution of (3.6.15) corresponding to $\alpha = -1$ can be written as $\mathbf{\Lambda}_u(s) = (-\lambda^1_{\hat{u}_N}(s), \dots, -\lambda^N_{\hat{u}_N}(s))$ for every $s \in [0,1]$. Finally, using this decoupling of $\mathbf{\Lambda}_u$, the condition (3.6.14) can be deduced from (3.6.16) using the expression of $\mathbf{F}^N_0, \dots, \mathbf{F}^N_k$, and observing that the term $\mathbf{\Lambda}_{\hat{u}_N}(s)\mathbf{F}^N_0(\mathbf{x}_{\hat{u}_N}(s))$ in (3.6.16) does not affect the minimizer. □

We recall that the Pontryagin Maximum Principle provides *necessary* condition for minimality. An admissible control $\bar{u} \in \mathcal{U}$ is a (normal) *Pontryagin extremal* for the optimal control problem related to the minimization of $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ if there exist $\lambda^1_{\bar{u}}, \dots, \lambda^N_{\bar{u}} : [0,1] \to (\mathbb{R}^n)^*$ satisfying (3.6.13) and such that the relation (3.6.14) holds.

REMARK 3.6.3. Let $\bar{u} \in \mathcal{U}$ be a critical point for the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$, i.e., $\mathcal{G}^N[\bar{u}] = 0$. Therefore, from (3.6.9) it turns out that

$$\bar{u}(s) = -\frac{1}{\beta} \sum_{j=1}^{N} F^{\theta_j}(x^{\theta_j}_{\bar{u}}(s))^T \lambda^j_{\bar{u}}(s)^T$$

for a.e. $s \in [0,1]$, where for every $j = 1, \dots, N$ the curve $x^{\theta_j}_{\bar{u}} : [0,1] \to \mathbb{R}^n$ is the trajectory of (3.1.1) corresponding to the parameter θ_j and to the control $\bar{u}$, and $\lambda^j_{\bar{u}} : [0,1] \to (\mathbb{R}^n)^*$ is the solution of (3.6.10). We observe that, for every $j = 1, \dots, N$, $\lambda^j_{\bar{u}} : [0,1] \to (\mathbb{R}^n)^*$ solves as well (3.6.13), and that $\bar{u}(s)$ satisfies

$$\bar{u}^N(s) \in \arg\max_{v \in \mathbb{R}^k} \left\{ \sum_{j=1}^{N} \left(-\lambda^j_{\bar{u}^N}(s) F(x^{\theta_j}_{\bar{u}^N}(s)) v \right) - \frac{\beta}{2} |v|_2^2 \right\}$$

for a.e. $s \in [0,1]$. This shows that any critical point of $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ is a (normal) Pontryagin extremal for the corresponding optimal control problem. Conversely, an analogue argument shows that any Pontryagin extremal is a critical point for the functional $\mathcal{F}^N$.

3.7. Numerical schemes for optimal control of ensembles

In this section we present two algorithms for the numerical resolution of the problem of optimal control of ensembles. In Section 3.4 we formulated it as the minimization of a proper functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ introduced in (3.4.3) and defined over the space of admissible controls. The Γ-convergence result obtained in Section 3.5 allows us to consider the functionals $(\mathcal{F}^N)_{N\in\mathbb{N}}$ to approximate $\mathcal{F}^\infty$. The advantage of this approach lies in the fact that any of the functionals $(\mathcal{F}^N)_{N\in\mathbb{N}}$ deals with a *finite* sub-ensemble of the original (in general, infinite) ensemble of control systems. Finally, in Section 3.6 we wrote for every $N \in \mathbb{N}$ the gradient field induced on $\mathcal{U}$ by the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$, and we derived the Pontryagin Maximum Principle for the optimal control problem related to the minimization of $\mathcal{F}^N$. We recall that in Section 3.6 we focused on the case of end-point costs, i.e., when the measure ν that appears in the first term at the right-hand side of (3.4.3) and (3.5.7) satisfies $\nu = \delta_{s=1}$. In the present section we introduce two numerical schemes starting from the results of Section 3.6. The first method consists in the projection of the field $\mathcal{G}^N : \mathcal{U} \to \mathcal{U}$ induced by $\mathcal{F}^N$ onto a finite-dimensional subspace $\mathcal{U}_M \subset \mathcal{U}$. The second one is based on the Pontryagin Maximum Principle and it was first proposed in [**45**].

Before proceeding, we introduce the notations and the framework that are shared by the two methods. Let us consider the interval $[0, 1]$, i.e., the evolution time horizon of the ensemble of controlled dynamical systems (3.1.1), and for $M \geq 2$ let us take the equi-spaced nodes $\{0, \frac{1}{M}, \ldots, \frac{M-1}{M}, 1\}$. Recalling that $\mathcal{U} := L^2([0, 1], \mathbb{R}^k)$, let us define the subspace $\mathcal{U}_M \subset \mathcal{U}$ as follows:

$$u \in \mathcal{U}_M \iff u(s) = \begin{cases} u_1 & \text{if } 0 \leq s < \frac{1}{M} \\ \vdots & \\ u_M & \text{if } \frac{M-1}{M} \leq s \leq 1, \end{cases} \tag{3.7.1}$$

where $u_1, \ldots, u_M \in \mathbb{R}^k$. For every $l = 1, \ldots, M$, we shall write $u_l = (u_{1,l}, \ldots, u_{k,l})$ to denote the components of $u_l \in \mathbb{R}^k$. Then, any element $u \in \mathcal{U}_M$ will be represented by the following array:

$$u = (u_{i,l})_{l=1,\ldots,M}^{i=1,\ldots,k}. \tag{3.7.2}$$

For every $N \geq 1$, let μ^N be the discrete probability measure (3.5.6) on Θ that approximates the probability measure μ involved in the definition of the functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ in (3.6.1). Let $\{\theta_1, \ldots, \theta_N\} \subset \Theta$ be the points charged by μ^N, and, for every $j = 1, \ldots, N$, let $x_u^{\theta_j} : [0, 1] \to \mathbb{R}^n$ be the solution of (3.1.1) corresponding to the parameter θ_j and to the control u. Then, for every $j = 1, \ldots, N$ and $l = 0, \ldots, M$ we define the array that collects the evaluation of the trajectories at the time nodes:

$$(x_l^j)_{l=0,\ldots,M}^{j=1,\ldots,N}, \qquad x_l^j := x_u^{\theta_j}\left(\frac{l}{M}\right). \tag{3.7.3}$$

We observe that in (3.7.3) we dropped the reference to the control that generates the trajectories. This is done to avoid hard notations, and in the following we hope that it will be clear from the context the correspondence between trajectories and control. Similarly, for every $j = 1, \ldots, N$, let $\lambda_u^j : [0, 1] \to (\mathbb{R}^n)^*$ be the solution of (3.6.10), and let us introduce the corresponding array of the evaluations:

$$(\lambda_l^j)_{l=0,\ldots,M}^{j=1,\ldots,N}, \qquad \lambda_l^j := \lambda_u^j\left(\frac{l}{M}\right). \tag{3.7.4}$$

3.7.1. Projected gradient field. In this subsection we describe a method for the numerical minimization of the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ defined as in (3.6.2). This algorithm consists in the projection of the gradient field $\mathcal{G}^N : \mathcal{U} \to \mathcal{U}$ derived in (3.6.9) onto the finite-dimensional subspace $\mathcal{U}_M \subset \mathcal{U}$ defined as in (3.7.1). We observe that we can explicitly compute the expression of the orthogonal projector $P_M : \mathcal{U} \to \mathcal{U}_M$. Indeed, we have

$$P_M[u](s) = \begin{cases} M \int_0^{\frac{1}{M}} u(s)\, ds & \text{if } 0 \leq s < \frac{1}{M}, \\ \vdots \\ M \int_{\frac{M-1}{M}}^1 u(s)\, ds & \text{if } \frac{M-1}{M} \leq s \leq 1, \end{cases} \tag{3.7.5}$$

for every $u \in \mathcal{U}$. Thus, we can can define the projected field $\mathcal{G}_M^N : \mathcal{U}_M \to \mathcal{U}_M$ as

$$\mathcal{G}_M^N[u] := P_M[\mathcal{G}^N[u]] \tag{3.7.6}$$

for every $u \in \mathcal{U}_M$, and we end up with vector field on a finite-dimensional space. At this point, in view of a numerical implementation of the method, it is relevant to observe that the computation of $\mathcal{G}^N[u]$ requires the knowledge of the trajectories $x_u^{\theta_1}, \ldots, x_u^{\theta_N} : [0,1] \to \mathbb{R}^n$ and of the curves $\lambda_u^1, \ldots, \lambda_u^N : [0,1] \to (\mathbb{R}^n)^*$. However, during the execution of the algorithm, we have access only to the (approximated) values of these functions at the time nodes $\{0, \frac{1}{M}, \ldots, 1\}$. Therefore, we need to adapt (3.7.6) in order to meet our needs. For every $u \in \mathcal{U}_M$, let us consider the corresponding arrays $(x_l^j)_{l=0,\ldots,M}^{j=1,\ldots,N}$ and $(\lambda_l^j)_{l=0,\ldots,M}^{j=1,\ldots,N}$ defined as in (3.7.3) and (3.7.4), respectively. In practice, they can be computed using standard numerical schemes for the approximation of ODEs. For every $l = 1, \ldots, M$, we use the approximation

$$M \int_{\frac{l-1}{M}}^{\frac{l}{M}} \sum_{j=1}^N \left(F^{\theta_j}(x_u^{\theta_j}(s))^T \lambda_u^j(s)^T \right) + \beta u(s)\, dt$$
$$\simeq \frac{1}{2} \sum_{j=1}^N \left(F^{\theta_j}(x_{l-1}^j)^T \lambda_{l-1}^{j\,T} + F^{\theta_j}(x_l^j)^T \lambda_l^{j\,T} \right) + \beta u_l.$$

Then, for every $u \in \mathcal{U}_M$, after computing the corresponding arrays $(x_l^j)_{l=0,\ldots,M}^{j=1,\ldots,N}$ and $(\lambda_l^j)_{l=0,\ldots,M}^{j=1,\ldots,N}$ with a proper ODEs integrator scheme, we use the quantity $\Delta u = (\Delta u_1, \ldots, \Delta u_M) \in \mathcal{U}_M$ to approximate $\mathcal{G}_M^N[u]$, where we set

$$\Delta u_l := \frac{1}{2} \sum_{j=1}^N \left(F^{\theta_j}(x_{l-1}^j)^T \lambda_{l-1}^{j\,T} + F^{\theta_j}(x_l^j)^T \lambda_l^{j\,T} \right) + \beta u_l \tag{3.7.7}$$

for every $l = 1, \ldots, M$. We are now in position to describe the Projected Gradient Field algorithm. We report it in Algorithm 1.

REMARK 3.7.1. We observe that the *for loops* at the lines 9–12 and 18–21 (corresponding, respectively, to the update of the curves of covectors and of the

Algorithm 1: Projected Gradient Field

Data:

- $\{\theta_1, \dots \theta_N\} \subset \Theta$ subset of parameters;
- $F_0^{\theta_1}, \dots, F_0^{\theta_N} : \mathbb{R}^n \to \mathbb{R}^n$ drift fields;
- $F^{\theta_1}, \dots, F^{\theta_N} : \mathbb{R}^n \to \mathbb{R}^{n \times k}$ controlled fields;
- $(x_0^j)^{j=1,\dots,N} = (x_0^{\theta_1}, \dots, x_0^{\theta_N})$ initial states of trajectories;
- $a(\cdot, \theta_1), \dots, a(\cdot, \theta_N) : \mathbb{R}^n \to \mathbb{R}_+$ end-point costs, and $\beta > 0$.

Algorithm setting: $M = \dim \mathcal{U}_M$, $\tau \in (0,1)$, $c \in (0,1)$, $\gamma > 0$, $\max_{\text{iter}} \geq 1$, $u \in \mathcal{U}_M$.

$h \leftarrow \frac{1}{M}$;
for $j = 1, \dots, N$ **do** `// First computation of trajectories`
  Compute $(x_l^j)_{l=1,\dots,M}$ using $(u_l)_{l=1,\dots,M}$ and x_0^j;
end
Cost $\leftarrow \frac{1}{N}\sum_{j=1}^N a(x_M^j, \theta_j) + \frac{\beta}{2}||u||_{L^2}^2$;
flag $\leftarrow 1$;
for $r = 1, \dots, \max_{\text{iter}}$ **do** `// Iterations of Projected Gradient Field`
  if flag $= 1$ **then** `// Update covectors only if necessary`
    for $j = 1, \dots, N$ **do** `// Backward computation of covectors`
      $\lambda_M^j \leftarrow \frac{1}{N}\nabla a(x_M^j, \theta_j)$;
      Compute $(\lambda_l^j)_{l=0,\dots,M-1}$ using $(u_l)_{l=1,\dots,M}$, $(x_l^j)_{l=0,\dots,M}$ and λ_M^j;
    end
  end
  for $l = 1, \dots, M$ **do** `// Compute` Δu `using (3.7.7)`
    $\Delta u_l \leftarrow \frac{1}{2}\sum_{j=1}^N \left(F^{\theta_j}(x_{l-1}^j)^T \lambda_{l-1}^{j\,T} + F^{\theta_j}(x_l^j)^T \lambda_l^{j\,T} \right) + \beta u_l$;
  end
  $u^{\text{new}} \leftarrow u - \gamma \Delta u$;
  for $j = 1, \dots, N$ **do** `// Forward computation of trajectories`
    $x_0^{j,\text{new}} \leftarrow x_0^j$;
    Compute $(x_l^{j,\text{new}})_{l=1,\dots,M}$ using $(u_l^{\text{new}})_{l=1,\dots,M}$ and $x_0^{j,\text{new}}$;
  end
  Cost$^{\text{new}} \leftarrow \frac{1}{N}\sum_{j=1}^N a(x_M^{j,\text{new}}, \theta_j) + \frac{\beta}{2}||u^{\text{new}}||_{L^2}^2$;
  if Cost $\geq$ Cost$^{\text{new}} + c\gamma||\Delta u||_{L^2}^2$ **then** `// Backtracking for` γ
    $u \leftarrow u^{\text{new}}$, $x \leftarrow x^{\text{new}}$;
    Cost $\leftarrow$ Cost$^{\text{new}}$;
    flag $\leftarrow 1$;
  else
    $\gamma \leftarrow \tau\gamma$;
    flag $\leftarrow 0$;
  end
end

trajectories) can be carried out in parallel with respect to the index $j = 1, \dots, N$. This can be considered when dealing with large sub-ensembles of parameters.

REMARK 3.7.2. The step-size $\gamma > 0$ for Algorithm 1 is set during the initialization of the method, and it is adaptively adjusted through the *if clause* at the lines 23–30 via the classical Armijo-Goldstein condition (see, e.g., [**40**, Section 1.2.3]). We observe that, if the update of the control at the r-th iteration is rejected, at the $r+1$-th iteration it is not necessary to re-compute the array of covectors $(\lambda_l^j)_{l=0,\dots,M}^{j=1,\dots,N}$. In this regards, the *if clause* at the line 8 prevents this computation in the case of rejection at the previous passage.

3.7.2. Iterative Maximum Principle. In this subsection we present a second numerical method for the minimization of the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$, based on the Pontryagin Maximum Principle. The idea of using the Maximum Principle to design approximation schemes for optimal control problems was well established in the Russian literature (see [**19**] for a survey paper in English). In this subsection we adapt to our problem the method proposed in [**45**], which is in turn a stabilization of one of the algorithms reported in [**19**].

The key-idea relies in iterative updates of the control through the resolution of a maximization problem related to the condition (3.6.14). However, the substantial difference from Algorithm 1 consists in the fact that the controls and the trajectories are computed simultaneously. More precisely, let us consider $M \geq 1$ and let $\mathcal{U}_M \subset \mathcal{U}$ be the finite-dimensional subspace introduced in (3.7.1). Given an initial guess $u = (u_l)_{l=1,\dots,M} \in \mathcal{U}_M$, let $(x_l^j)_{l=0,\dots,M}^{j=1,\dots,N}$ and $(\lambda_l^j)_{l=0,\dots,M}^{j=1,\dots,N}$ be the corresponding arrays, defined as in (3.7.3) and (3.7.4), respectively. For $l = 1$, the value of u_1^{new} (i.e., the updated value of control in the time interval $[0, 1/M]$) is computed using $(x_0^j)^{j=1,\dots,N}$ and $(\lambda_0^j)^{j=1,\dots,N}$ as follows:

$$u_1^{\text{new}} = \arg\max_{v \in \mathbb{R}^k} \left\{ \sum_{j=1}^{N} \left(-\lambda_0^j \, F^{\theta_j}(x_0^j) v \right) - \frac{\beta}{2} |v|_2^2 - \frac{1}{2\gamma} |v - u_1|_2^2 \right\}, \tag{3.7.8}$$

where $\gamma > 0$ plays the role of the step-size of the update. From the value u_1^{new} just obtained and the initial conditions $(x_0^j)^{j=1,\dots,N}$, we compute $(x_1^j)^{j=1,\dots,N}$, i.e., the approximation of the trajectories at the time-node $1/M$. At this point, using $(x_1^j)^{j=1,\dots,N}$ and $(\lambda_1^j)^{j=1,\dots,N}$, we calculate u_2^{new} with a maximization problem analogue to (3.7.8). Finally, we sequentially repeat the same procedure for every $l = 2, \dots, M$. We report the scheme in Algorithm 2.

REMARK 3.7.3. The maximization at line 17 can be solved directly at a very low computational cost. Indeed, we have that

$$u_l^{\text{new}} \leftarrow \frac{1}{1+\gamma\beta} \left(u_l - \sum_{j=1}^{N} \left(\lambda_l^{j,\text{corr}} F^{\theta_j}(x_{l-1}^{j,\text{new}}) \right)^T \right)$$

Algorithm 2: Iterative Maximum Principle

Data:

- $\{\theta_1, \dots \theta_N\} \subset \Theta$ subset of parameters;
- $F_0^{\theta_1}, \dots, F_0^{\theta_N} : \mathbb{R}^n \to \mathbb{R}^n$ drift fields;
- $F^{\theta_1}, \dots, F^{\theta_N} : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ controlled fields;
- $(x_0^j)^{j=1,\dots,N} = (x_0^{\theta_1}, \dots, x_0^{\theta_N})$ initial states of trajectories;
- $a(\cdot,\theta_1), \dots, a(\cdot,\theta_N) : \mathbb{R}^n \to \mathbb{R}_+$ end-point costs, and $\beta > 0$.

Algorithm setting: $M = \dim \mathcal{U}_M$, $\tau \in (0,1)$, $\gamma > 0$, $\max_{\text{iter}} \geq 1$, $u \in \mathcal{U}_M$.

$h \leftarrow \frac{1}{M}$;
for $j = 1, \dots, N$ **do** `// First computation of trajectories`
    Compute $(x_l^j)_{l=1,\dots,M}$ using $(u_l)_{l=1,\dots,M}$ and x_0^j;
end
Cost $\leftarrow \frac{1}{N}\sum_{j=1}^N a(x_M^j, \theta_j) + \frac{\beta}{2}||u||_{L^2}^2$;
flag $\leftarrow 1$;
for $r = 1, \dots, \max_{\text{iter}}$ **do** `// Iterations of Iterative Maximum Principle`
    if flag $= 1$ **then** `// Update covectors only if necessary`
        for $j = 1, \dots, N$ **do** `// Backward computation of covectors`
            $\lambda_M^j \leftarrow \frac{1}{N}\nabla a(x_M^j, \theta_j)$;
            Compute $(\lambda_l^j)_{l=0,\dots,M-1}$ using $(u_l)_{l=1,\dots,M}$, $(x_l^j)_{l=0,\dots,M}$ and λ_M^j;
        end
    end
    $(x_0^{j,\text{new}})^{j=1,\dots,N} \leftarrow (x_0^j)^{j=1,\dots,N}$;
    $(\lambda_0^{j,\text{corr}})^{j=1,\dots,N} \leftarrow (\lambda_0^j)^{j=1,\dots,N}$;
    for $l = 1, \dots, M$ **do** `// Update of controls and trajectories`
        $u_l^{\text{new}} \leftarrow$
        $\arg\max_{v\in\mathbb{R}^k}\left\{\sum_{j=1}^N \left(-\lambda_{l-1}^{j,\text{corr}} F^{\theta_j}(x_{l-1}^{j,\text{new}})v\right) - \frac{\beta}{2}|v|_2^2 - \frac{1}{2\gamma}|v-u_l|_2^2\right\}$;
        for $j = 1, \dots, N$ **do**
            Compute $x_l^{j,\text{new}}$ using $x_{l-1}^{j,\text{new}}$ and u_l^{new};
            $\lambda_l^{j,\text{corr}} \leftarrow \lambda_l^j - \frac{1}{N}\nabla a(x_l^j, \theta_j) + \frac{1}{N}\nabla a(x_l^{j,\text{new}}, \theta_j)$;
        end
    end
    $\text{Cost}^{\text{new}} \leftarrow \frac{1}{N}\sum_{j=1}^N a(x_M^{j,\text{new}}, \theta_j) + \frac{\beta}{2}||u^{\text{new}}||_{L^2}^2$;
    if Cost $>$ Cost^{new} **then** `// Backtracking for` γ
        $u \leftarrow u^{\text{new}}$, $x \leftarrow x^{\text{new}}$;
        Cost $\leftarrow$ Cost^{new};
        flag $\leftarrow 1$;
    else
        $\gamma \leftarrow \tau\gamma$;
        flag $\leftarrow 0$;
    end
end

for every $l = 1, \dots, M$. This is essentially due to the fact that the systems of the ensemble (3.1.1) have an affine dependence on the control.

REMARK 3.7.4. As well as in Algorithm 1, in this case the computation of $(\lambda_l^j)_{l=0,\dots,M-1}^{j=1,\dots,N}$ can be carried out in parallel (see the *for loop* at the lines 9–12). Unfortunately, this is no more true for the update of the trajectories, since in Algorithm 2 the computation of $(x_l^{j,\mathrm{new}})^{j=1,\dots,N}$ takes place immediately after obtaining u_l^{new}, for every $l = 1, \dots, M$ (see lines 17–21).

REMARK 3.7.5. At the line 20 of Algorithm 2 we introduce a correction for the value of the covector. This feature is not present in the original scheme proposed in [**45**], where the authors considered optimal control problems without end-point cost.

REMARK 3.7.6. Also in Algorithm 2 the step-size is adaptively adjust, and it is reduced if, after the iteration, the value of the functional has not decreased. In case of rejection of the update, it is not necessary to recompute $(\lambda_l^j)_{l=0,\dots,M}^{j=1,\dots,N}$. This is a common feature with Algorithm 1, as observed in Remark 3.7.2.

3.8. Numerical experiments

In this section we test the algorithms described in Section 3.7 on an optimal control problem involving an ensemble of linear dynamical systems in $\mathbb{R}^2$. Namely, given $\theta_{\min} < \theta_{\max} \in \mathbb{R}$, let us set $\Theta := [\theta_{\min}, \theta_{\max}] \subset \mathbb{R}$, and let us consider the ensemble of control systems

$$\begin{cases} \dot{x}_u^\theta(s) = A^\theta x_u^\theta(s) + b_1 u_1(s) + b_2 u_2(s) & \text{a.e. in } [0,1], \\ x_u^\theta(0) = x_0^\theta, \end{cases} \tag{3.8.1}$$

where $\theta \mapsto x_0^\theta$ is a continuous function that prescribes the initial states, $u = (u_1, u_2)^T \in \mathcal{U} := L^2([0,1], \mathbb{R}^2)$, and, for every $\theta \in \Theta$, we have

$$A^\theta := \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix}, \quad b_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.8.2}$$

For every $N \geq 1$ and for every subset of parameters $\{\theta_1, \dots, \theta_N\} \subset \Theta$, we represent the corresponding sub-ensemble of (3.8.1) as an affine-control system on $\mathbb{R}^{2N}$, as done in Section 3.6. More precisely, we consider

$$\begin{cases} \dot{\mathbf{x}}_u(s) = \mathbf{A}^N \mathbf{x}_u(s) + \mathbf{b}_1 u_1(s) + \mathbf{b}_2 u_2(s) & \text{a.e. in } [0,1], \\ \mathbf{x}_u(0) = \mathbf{x}_0, \end{cases} \tag{3.8.3}$$

where $\mathbf{A}^N \in \mathbb{R}^{2N \times 2N}$ and $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^{2N}$ are defined as follows:

$$\mathbf{A}^N := \begin{pmatrix} A^{\theta_1} & \mathbf{0}_{2\times 2} & \ddots \\ \mathbf{0}_{2\times 2} & \ddots & \mathbf{0}_{2\times 2} \\ \ddots & \mathbf{0}_{2\times 2} & A^{\theta_N} \end{pmatrix}, \ \mathbf{b}_1 := \begin{pmatrix} b_1 \\ \vdots \\ b_1 \end{pmatrix}, \ \mathbf{b}_2 := \begin{pmatrix} b_2 \\ \vdots \\ b_2 \end{pmatrix}. \tag{3.8.4}$$

Moreover, we observe that (3.8.1) can be interpreted as a control system in the space $C^0(\Theta,\mathbb{R}^2)$. Indeed, we can consider the control system

$$X_{u,t} = X_0 + \int_0^t \mathcal{A}[X_{u,\tau}]\,d\tau + \int_0^t \mathfrak{b}_1 u_1(\tau) + \mathfrak{b}_2 u_2(\tau)\,d\tau, \quad t\in[0,1], \tag{3.8.5}$$

where $\mathcal{A}: C^0(\Theta,\mathbb{R}^2)\to C^0(\Theta,\mathbb{R}^2)$ is the bounded linear operator defined as

$$\mathcal{A}[Y](\theta) := A^\theta Y(\theta)$$

for every $\theta\in\Theta$ and for every $Y\in C^0([0,1],\mathbb{R}^2)$, and $\mathfrak{b}_1,\mathfrak{b}_1:\Theta\to\mathbb{R}^2$ are defined as

$$\mathfrak{b}_1(\theta) := b_1, \quad \mathfrak{b}_2(\theta) := b_2$$

for every $\theta\in\Theta$, and finally $X_0:\Theta\to\mathbb{R}^2$ satisfies $X_0(\theta) := x_0^\theta$ for every $\theta\in\Theta$. The integrals in (3.8.5) should be understood in the Bochner sense, and, for every $u\in\mathcal{U}$, the existence and uniqueness of a continuous curve $t\mapsto X_{u,t}$ in $C^0(\Theta,\mathbb{R}^2)$ solving (3.8.5) descends from classical results in linear inhomogeneous ODEs in Banach spaces (see, e.g., [**23**, Chapter 3]). In particular, from the uniqueness we deduce that

$$X_{u,s}(\theta) = x_u^\theta(s) \tag{3.8.6}$$

for every $u\in\mathcal{U}$, $t\in[0,1]$ and $\theta\in\Theta$, where $x_u^\theta:[0,1]\to\mathbb{R}^2$ is the solution of (3.8.1) corresponding to the parameter θ and to the control u. We now prove some controllability results for the control systems (3.8.3) and (3.8.5).

PROPOSITION 3.8.1. *For every $N\geq 1$ and for every subset $\{\theta_1,\ldots,\theta_N\}\subset\Theta$, let us consider $\mathbf{y}_{\mathrm{tar}}\in\mathbb{R}^{2N}$. Then, there exists a control $\bar u\in\mathcal{U}$ such that the corresponding solution $\mathbf{x}_{\bar u}:[0,1]\to\mathbb{R}^{2N}$ of (3.8.3) satisfies $\mathbf{x}_{\bar u}(1)=\mathbf{y}_{\mathrm{tar}}$.*
Moreover, for every $Y_{\mathrm{tar}}\in C^0(\Theta,\mathbb{R}^2)$ and for every $\varepsilon>0$, there exists a control $u_\varepsilon\in\mathcal{U}$ such that the curve $s\mapsto X_{u_\varepsilon,s}$ that solves (3.8.5) satisfies

$$||Y - X_{u_\varepsilon,1}||_{C^0}\leq\varepsilon.$$

PROOF. We observe that the first part of the thesis follows if we prove the exact controllability of the system (3.8.3). An elementary result in control theorey (see, e.g., [**4**, Theorem 3.3]) ensures that the last condition is implied by the identity

$$\mathrm{span}\left\{(\mathbf{A}^N)^r\,\mathbf{b_1}, (\mathbf{A}^N)^r\,\mathbf{b_2}|\ 0\leq r\leq 2N-1\right\} = \mathbb{R}^{2N}.$$

A direct computation shows that this is actually the case.

As regards the second part of the thesis, owing to [**53**, Theorem 3.1.1] we have that it is sufficient to prove that

$$\overline{\mathrm{span}\,\{\mathcal{A}^r[\mathfrak{b}_1],\mathcal{A}^r[\mathfrak{b}_2]|\ r\geq 0\}}^{C^0} = C^0(\Theta,\mathbb{R}^2). \tag{3.8.7}$$

We observe that

$$\mathrm{span}\,\{\mathcal{A}^r[\mathfrak{b}_1],\mathcal{A}^r[\mathfrak{b}_2]|\ r\geq 0\} = \mathrm{span}\left\{\begin{pmatrix}\theta^r\\0\end{pmatrix},\begin{pmatrix}0\\\theta^r\end{pmatrix}\mid r\geq 0\right\},$$

therefore the identity (3.8.7) follows from the Weierstrass Theorem on polynomial approximation. □

We now introduce the problem that we studied in the numerical simulations. We set $\theta_{\min} = -\frac{1}{2}, \theta_{\max} = \frac{1}{2}$, and we consider on $\Theta = [-\frac{1}{2}, \frac{1}{2}]$ the probability measure μ, distributed as a Beta(4, 4) centered at 0. Let us assume that the initial data in (3.8.1) is not affected by the parameter θ, i.e, there exists $x_0 \in \mathbb{R}^2$ such that $x_0^\theta = x_0$ for every $\theta \in \Theta$. We imagine that we want to steer the end-points of the trajectories of (3.8.1) as close as possible to a target point $y_{\text{tar}} \in \mathbb{R}^2$. Therefore, we consider the functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}_+$ defined as

$$\mathcal{F}^\infty(u) := \int_\Theta |x_u^\theta(1) - y_{\text{tar}}|_2^2 \, d\mu(\theta) + \frac{\beta}{2}||u||_{L^2}^2 \tag{3.8.8}$$

for every $u \in \mathcal{U}$. We observe that the second part of Proposition 3.8.1 implies that we are in the situation described in Remark 3.4.1. Indeed, if we set $Y_{\text{tar}}(\theta) := y_{\text{tar}}$ for every $\theta \in \Theta$, we have that for every $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{U}$ such that

$$\int_\Theta |x_{u_\varepsilon}^\theta(1) - y_{\text{tar}}|_2^2 \, d\mu(\theta) \leq ||X_{u_\varepsilon,1} - Y_{\text{tar}}||_{C^0} \leq \frac{\varepsilon}{2},$$

where we used the identity (3.8.6). Therefore, in correspondence of small values of β, we expect that the minimizers of (3.8.8) drive the end-point of the controlled trajectories very close to y_{tar}. In the simulations we considered $\beta = 10^{-3}$. Finally, we approximated the probability measure μ with the empirical distribution μ^N, obtained with N independent samplings of μ, using $N = 300$. Moreover, we chose $x_0 = (0, 0)^T$ and $y_{\text{tar}} = (-1, -1)^T$. We report below the results obtained with Algorithm 1 and Algorithm 2, where we set $M = 64$. We observe that performances of the two numerical methods are very similar, as regards both the qualitative aspect of the controlled trajectories and the decay of the cost during the execution.

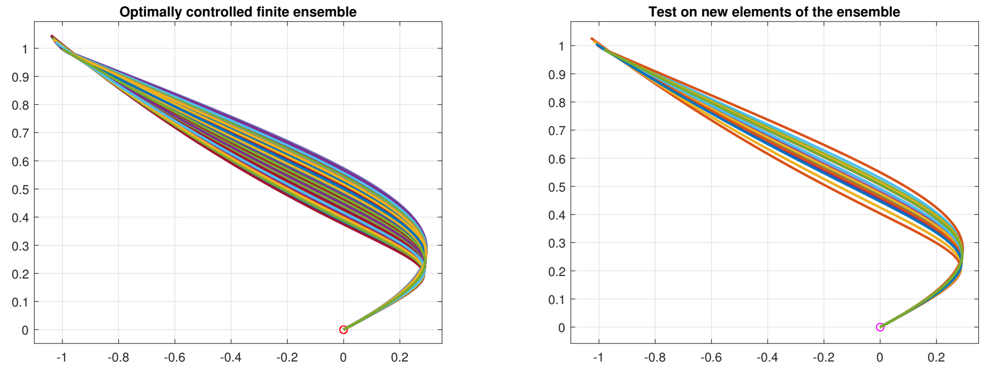

FIGURE 1. Controlled ensemble. On the left, we reported the optimally controlled trajectories of the sub-ensemble of Θ obtained by sampling $N = 300$ parameters. On the right, we tested the controls obtained before on a new sub-ensemble of Θ, obtained by sampling 20 new parameters. As we can see, the trajectories belonging to the testing sub-ensemble are correctly steered to the target point $y_{\text{tar}} = (-1, 1)^T$.

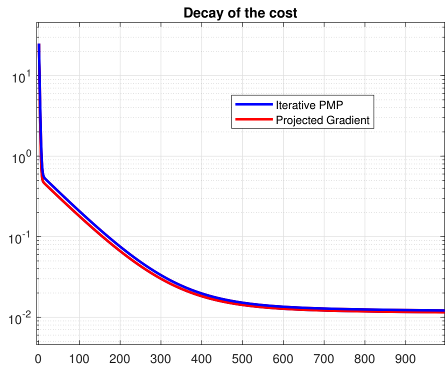

FIGURE 2. In the graph we reported the decay of the discrete cost achieved by Algorithm 1 (Projected Gradient) and Algorithm 2 (Iterative PMP). As we can see, the performances on this problem are very similar.

CHAPTER 4

Linear-control systems and Deep Learning

In this chapter we propose a Deep Learning architecture to approximate diffeomorphisms diffeotopic to the identity. We consider a linear-control system and we use the corresponding flow to approximate the action of a diffeomorphism on a compact ensemble of points. Despite the simplicity of the control system, it has been recently shown that a Universal Approximation Property holds (see Theorem 4.3.3). We apply the tools developed in Chapter 3 to formulate the approximation task as an ensemble optimal control problem. The discretization of the problem naturally leads to a ResNet, i.e., a specific Deep Learning architecture. Finally, we use Γ-convergence to provide an estimate of the expected generalization error, and we perform some numerical experiments.

4.1. ResNets and control theory

Residual Neural Networks (ResNets) are particular instances of Deep Learning architectures and they were originally introduced in [**32**] in order to overcome some issues related to the training process of traditional Deep Learning networks. Indeed, it had been observed that, as the number of the layers in non-residual architectures is increased, the learning of the parameters is affected by the vanishing of the gradients (see, e.g., [**11**]) and the accuracy of the network gets rapidly saturated (see [**31**]).

ResNets can be represented as the composition of non-linear mappings

$$\Phi = \Phi_M \circ \ldots \circ \Phi_1,$$

where M represents the *depth* of the Neural Network and, for every $l = 1 \ldots M$, the *building blocks* $\Phi_l : \mathbb{R}^n \to \mathbb{R}^n$ are of the form

$$\Phi_l(x) = x + \sigma(W_l x + b_l), \tag{4.1.1}$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is a non-linear activation function that acts component-wise, and $W_l \in \mathbb{R}^{n\times n}$ and $b_l \in \mathbb{R}^n$ are the parameters to be learned. In contrast, we recall that in non-residual architectures $\bar\Phi = \bar\Phi_M \circ \ldots \circ \bar\Phi_1$, the building blocks have usually the form

$$\bar\Phi_l(x) = \sigma(W_l x + b_l)$$

for $l = 1, \ldots, N$. In some recent contributions [**26, 35, 29**], ResNets have been studied in the framework of mathematical control theory. The bridge between ResNets and control theory was independently discovered in [**26**] and [**29**], where it was observed that each function $\Phi_1, \dots, \Phi_M$ defined as in (4.1.1) can be interpreted as an Explicit Euler discretization of the control system

$$\dot{x} = \sigma(Wx + b), \tag{4.1.2}$$

where W and b are the control variables. Since then, control theory has been fruitfully applied to the theoretical understanding of ResNets. In [**51**] a Universal Approximation result for the flow generated by (4.1.2) was established under suitable assumptions on the activation function σ. In [**35**] and [**14**] the problem of learning an unknown mapping was formulated as an Optimal Control problem, and the Pontryagin Maximum Principle was employed in order to learn the optimal parameters. In [**18**] it was consider the mean-field limit of (4.1.2), and it was proposed a training algorithm based on the discretization of the necessary optimality conditions for an optimal control problem in the space of probability measures. The Maximum Principle for optimal control of probability measures was first introduced in [**43**], and recently it has been independently rediscovered in [**17**]. In this chapter, rather than using tools from control theory to study properties of existing ResNets, we propose an architecture inspired by theoretical observations on control systems with *linear* dependence in the control variables. As a matter of fact, the building blocks of the ResNets that we shall construct depend linearly in the parameters, namely they have the form

$$\Phi_l(x) = x + G(x)u_l,$$

where $G : \mathbb{R}^n \to \mathbb{R}^{n \times k}$ is a non-linear function of the input, and $u_l \in \mathbb{R}^k$ is the vector of the parameters at the l-th layer. The starting points of our analysis are the controllability results proved in [**7, 8**], where the authors considered a control system of the form

$$\dot{x} = F(x)u = \sum_{i=1}^{k} F_i(x)u_i, \tag{4.1.3}$$

where $F_1, \dots, F_k$ are smooth and bounded vector fields on $\mathbb{R}^n$, and $u \in \mathcal{U} := L^2([0,1], \mathbb{R}^k)$ is the control. We immediately observe that (4.1.3) has a simpler structure than (4.1.2), having linear dependence with respect to the control variables. Despite this apparent simplicity, the flows associated to (4.1.3) are capable of interesting approximating results. Given a control $u \in \mathcal{U}$, let $\Phi_u : \mathbb{R}^n \to \mathbb{R}^n$ be the flow obtained by evolving (4.1.3) on the time interval $[0,1]$ using the control u. In [**8**] it was formulated a condition for the controlled vector fields $F_1, \dots, F_k$ called *Lie Algebra Strong Approximating Property*. On one hand, this condition guarantees exact controllability on finite ensembles, i.e., for every $N \geq 1$, for every $\{x_0^j\}_{j=1,\dots,N} \subset \mathbb{R}^n$ such that $j_1 \neq j_2 \implies x_0^{j_1} \neq x_0^{j_2}$, and for every injective mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, there exists a control $u \in \mathcal{U}$ such that $\Phi_u(x_0^j) = \Psi(x_0^j)$ for every $j = 1, \dots, N$. On the other hand, this property is also a sufficient condition for a C^0-approximate controllability result in the space of diffeomorphisms. More precisely, given a diffeomorphism $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ diffeotopic to the identity, and given a compact set $K \subset \mathbb{R}^n$, for every $\varepsilon > 0$ there exists a control $u_\varepsilon \in \mathcal{U}$ such that $\sup_K |\Phi_{u_\varepsilon}(x) - \Psi(x)|_2 \leq \varepsilon$. The aim of this chapter is to use the results obtained in Chapter 3 to provide implementable algorithms for the approximation of diffeomorphisms diffeotopic to the identity. More precisely, we discretize the control system (4.1.3) on the evolution interval $[0, 1]$ using the Explicit Euler scheme and the uniformly distributed time-nodes $\{0, \frac{1}{M}, \dots, \frac{M-1}{M}, 1\}$, and we obtain the ResNet represented by the composition $\Phi = \Phi_M \circ \ldots \circ \Phi_1$, where, for every $l = 1, \dots, M$, Φ_l is of the form

$$x_l = \Phi_l(x_{l-1}) = x_{l-1} + h \sum_{i=1}^{k} F_i(x_{l-1}) u_{i,l}, \quad h = \frac{1}{M}, \tag{4.1.4}$$

where $x_0 \in \mathbb{R}^n$ represents an initial input of the network. In this construction, the points $(x_l)_{l=0,\dots,M}$ represent the approximations at the time-nodes $\{\frac{l}{M}\}_{l=0,\dots,M}$ of the trajectory $x_u : [0, 1] \to \mathbb{R}^n$ that solves (4.1.3) with Cauchy datum $x_u(0) = x_0$. We insist on the fact that in our discrete-time model we assume that the controls are piecewise constant on the time intervals $\{[\frac{l-1}{M}, \frac{l}{M})\}_{l=1,\dots,M}$. For this reason, when we derive a ResNet with M hidden layers, we deal with M building blocks (i.e., $\Phi_1, \dots, \Phi_M$) and with M k-dimensional parameters $(u_l)_{l=1,\dots,N} = (u_{i,l})_{l=1,\dots,M}^{i=1,\dots,k}$. Moreover, when we evaluate the ResNet at a point $x_0 \in \mathbb{R}^n$, we end up with $M+1$ input/output variables $(x_l)_{l=0,\dots,M} \subset \mathbb{R}^n$.

The chapter is organized as follows. In Section 4.2 we establish the notations and we prove preliminary results regarding the flow generated by control system (4.1.3). In Section 4.3 we recall some results contained in [**7**] and [**8**] concerning exact and approximate controllability of ensembles. In Section 4.4 we explain why the "null training error strategy" is not suitable for the approximation purpose, and we outline the alternative strategy based on the resolution of an optimal control problem. In Section 4.5 we prove a Γ-convergence result that holds when the size M of the training dataset tends to infinity. As a byproduct, we obtain the upper bound on the so called *expected generalization error*. In Section 4.6 we discretize the linear-control system (4.1.3) and we obtain the corresponding ResNet, and we explain how the algorithms presented in Chapter 3 can be used for the training of the network. Finally, in Section 4.7 we test numerically the algorithms by approximating a diffeomorphism in the plane.

4.2. Notations and preliminary results

In this chapter we consider control systems of the form

$$\dot{x} = F(x)u = \sum_{i=1}^{k} F_i(x) u_i, \tag{4.2.1}$$

where the controlled vector fields $(F_i)_{i=1,\dots,k}$ satisfy the following assumption.

Assumption 1. The vector fields $F_1, \dots, F_k$ are smooth and Lipschitz-continuous, i.e., there exists $C_1 > 0$ such that

$$\sup_{i=1,\dots,k} \sup_{x \neq y} \frac{|F_i(x) - F_i(y)|_2}{|x-y|_2} \leq C_1. \tag{4.2.2}$$

The space of admissible controls is $\mathcal{U} := L^2([0,1], \mathbb{R}^k)$, endowed with the usual Hilbert space structure. Using Assumption 1, the classical Carathéodory Theorem guarantees that, for every $u \in \mathcal{U}$ and for every $x_0 \in \mathbb{R}^n$, the Cauchy problem

$$\begin{cases} \dot{x}(s) = \sum_{i=1}^{l} F_i(x(s))u_i(s), \\ x(0) = x_0, \end{cases} \tag{4.2.3}$$

has a unique solution $x_{u,x_0} : [0,1] \to \mathbb{R}^n$. Hence, for every $u \in \mathcal{U}$, we can define the flow $\Phi_u : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\Phi_u : x_0 \mapsto x_{u,x_0}(1), \tag{4.2.4}$$

where x_{u,x_0} solves (4.2.3). We recall that Φ_u is a diffeomorphism, since it is smooth and invertible. We now prove an estimate of the Lipschitz constant of the flow $\Phi_u : \mathbb{R}^n \to \mathbb{R}^n$ for $u \in \mathcal{U}$.

Lemma 4.2.1. *For every admissible control $u \in \mathcal{U}$, let $\Phi_u : \mathbb{R}^n \to \mathbb{R}^n$ be corresponding flow defined in* (4.2.4)*. Then Φ_u is Lipschitz-continuous with constant*

$$L_{\Phi_u} \leq e^{C_1 \sqrt{k} ||u||_{L^2}}, \tag{4.2.5}$$

where C_1 is the Lipschitz constant of the controlled fields $F_1, \dots, F_k$.

Proof. The thesis follows from Grönwall Lemma and similar computations as in the proof of Proposition 1.2.3. □

The next result regards the convergence of the flows $(\Phi_{u_m})_{m \geq 1}$ corresponding to a weakly convergent sequence $(u_m)_{m \geq 1} \subset \mathcal{U}$.

Proposition 4.2.2. *Given a sequence $(u_m)_{m \geq 1} \subset \mathcal{U}$ such that $u_m \rightharpoonup_{L^2} u$ as $m \to \infty$ with respect to the weak topology of $\mathcal{U}$, then the flows $(\Phi_{u_m})_{m \geq 1}$ converge to Φ_u uniformly on compact sets.*

Proof. The thesis follows from the same argument as in the proof of Theorem 3.2.5. Namely, Proposition 1.4.1 implies that $(\Phi_{u_m})_{m \geq 1}$ is point-wise convergent to Φ_u in $\mathbb{R}^n$, while Lemma 4.2.1 and Ascoli-Arzelà Theorem guarantee the C^0 convergence on compact sets. □

4.3. Ensemble controllability

In this section we recall the most important results regarding the issue of ensemble controllability. For the proofs and the statements in full generality we refer the reader to [**7**] and [**8**]. We begin with the definition of ensemble in $\mathbb{R}^n$. In this section we will further assume that $n > 1$, which is the most interesting case.

DEFINITION 4. Given a compact set $\Theta \subset \mathbb{R}^n$, an *ensemble of points* in $\mathbb{R}^n$ is an injective and continuous map $\gamma : \Theta \to \mathbb{R}^n$. We denote by $\mathcal{E}_\Theta(\mathbb{R}^n)$ the space of ensembles.

REMARK 4.3.1. If $|\Theta| = N < \infty$, then an ensemble can be simply thought as an injective map from $\{1, \dots, N\}$ to $\mathbb{R}^n$, or, equivalently, as an element of $(\mathbb{R}^n)^N \setminus \Delta^{(N)}$, where

$$\Delta^{(N)} := \{(x^1, \dots, x^N) \in (\mathbb{R}^n)^N : \exists j_1 \neq j_2 \text{ s.t. } x^{j_1} = x^{j_2}\}.$$

We define $(\mathbb{R}^n)^{(N)} := (\mathbb{R}^n)^N \setminus \Delta^{(N)}$. Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, we define its N*-fold* vector field $F^{(N)} : (\mathbb{R}^n)^{(N)} \to (\mathbb{R}^n)^{(N)}$ as $F^{(N)}(x^1, \dots, x^N) = (F(x^1), \dots, F(x^N))$, for every $(x^1, \dots, x^N) \in (\mathbb{R}^n)^{(N)}$. Finally, we introduce the notation $\mathcal{E}_N(\mathbb{R}^n)$ to denote the space of ensembles of $\mathbb{R}^n$ with N elements.

We give the notion of *reachable* ensemble.

DEFINITION 5. The ensemble $\gamma(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$ is *reachable* from the ensemble $\alpha(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$ if there exists an admissible control $u \in \mathcal{U}$ such that its corresponding flow Φ_u defined in (4.2.4) satisfies:

$$\Phi_u(\alpha(\cdot)) = \gamma(\cdot).$$

We can equivalently say that $\alpha(\cdot)$ can be steered to $\gamma(\cdot)$.

DEFINITION 6. Control system (4.2.1) is *exactly controllable* from $\alpha(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$ if every $\gamma(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$ is reachable from $\alpha(\cdot)$. The system is *globally exactly controllable* if it is exactly controllable from every $\alpha(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$.

We recall the definition of Lie algebra generated by a system of vector fields. Given the vector fields $F_1, \dots, F_l$, the linear space $\mathrm{Lie}(F_1, \dots, F_k)$ is defined as

$$\mathrm{Lie}(F_1, \dots, F_k) := \mathrm{span}\{[F_{i_s}, [\dots, [F_{i_2}, F_{i_1}] \dots]] : s \geq 1, i_1, \dots, i_s \in \{1, \dots, k\}\}, \tag{4.3.1}$$

where $[F, F']$ denotes the Lie bracket between F, F', smooth vector fields of $\mathbb{R}^n$. In the case of finite ensembles, i.e., when $|\Theta| = N < \infty$, we can provide sufficient condition for controllability. The proof reduces to the classical Chow-Rashevsky theorem (see, e.g., the textbook [**4**]).

THEOREM 4.3.1. *Let $F_1, \dots, F_k$ be a system of vector fields on $\mathbb{R}^n$. Given $N \geq 1$, if for every $x^{(N)} = (x^1, \dots, x^N) \in (\mathbb{R}^n)^{(N)}$ the system of M-fold vector fields $F_1^{(N)}, \dots, F_k^{(N)}$ is bracket generating at $x^{(N)}$, i.e.,*

$$\mathrm{Lie}(F_1^{(N)}, \dots, F_k^{(N)})|_{x^{(N)}} = (\mathbb{R}^n)^N, \tag{4.3.2}$$

then the control system (4.2.1) *is globally exactly controllable on $\mathcal{E}_N(\mathbb{R}^n)$.*

For a finite ensemble, the global exact controllability holds for a generic system of vector fields.

PROPOSITION 4.3.2. *For every $k \geq 2$, $N \geq 1$ and m sufficiently large, then the k-tuples of vector fields $(F_1, \ldots, F_k) \in (\mathrm{Vect}(\mathbb{R}^n))^k$ such that system (4.2.1) is globally exactly controllable on $\mathcal{E}_N(\mathbb{R}^n)$ is residual with respect to the Whitney C^m-topology.*

PROOF. See [**7**, Theorem 3.2]. □

We recall that a set is said *residual* if it is the complement of a countable union of nowhere dense sets. Proposition 4.3.2 means that, given any k-tuple $(F_1, \ldots, F_k)$ of vector fields, the corresponding control system (4.2.1) can be made globally exactly controllable in $\mathcal{E}_N(\mathbb{R}^n)$ by means of an arbitrary small perturbation of the fields $F_1, \ldots, F_k$ in the C^m-topology.

When dealing with infinite ensembles, the notions of "exact reachable" and "exact controllable" are too strong. However, they can be replaced by their respective C^0-approximate versions.

DEFINITION 7. The ensemble $\gamma(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$ is C^0-*approximately reachable* from the ensemble $\alpha(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$ if for every $\varepsilon > 0$ there exists an admissible control $u \in \mathcal{U}$ such that its corresponding flow Φ_u defined in (4.2.4) satisfies:

$$\sup_{\theta \in \Theta} |\Phi_u(\alpha(\theta)) - \gamma(\theta)|_2 < \varepsilon. \tag{4.3.3}$$

We can equivalently say that $\alpha(\cdot)$ can be C^0-approximately steered to $\gamma(\cdot)$.

DEFINITION 8. Control system (4.2.1) is C^0-*approximately controllable* from $\alpha(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$ if every $\gamma(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$ is C^0-approximately reachable from $\alpha(\cdot)$. The system is *globally C^0-approximately controllable* if it is C^0-approximately controllable from every $\alpha(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$.

REMARK 4.3.2. Let us further assume that the compact set $\Theta \subset \mathbb{R}^n$ has positive Lebesgue measure, and that it is equipped with a finite and positive measure μ, absolutely continuous w.r.t. the Lebesgue measure. Then, the distance between the target ensemble $\gamma(\cdot)$ and the approximating ensemble $\Phi_u(\alpha(\cdot))$ can be quantified using the L^p_μ-norm:

$$||\Phi_u(\alpha(\cdot)) - \gamma(\cdot)||_{L^p_\mu} = \left(\int_\Theta |\Phi_u(\alpha(\theta)) - \gamma(\theta)|_2^p \, d\mu(\theta) \right)^{\frac{1}{p}},$$

and we can equivalently formulate the notion of L^p_μ-approximate controllability. In general, given a non-negative continuous function $a : \mathbb{R}^n \to \mathbb{R}$ such that $a(0) = 0$, we can express the approximation error as

$$\int_\Theta a(\Phi_u(\alpha(\theta)) - \gamma(\theta)) \, d\mu(\theta). \tag{4.3.4}$$

In Section 4.5 we will consider an integral penalization term of this form. It is important to observe that, if $\gamma(\cdot)$ is C^0-approximately reachable from $\alpha(\cdot)$, then (4.3.4) can be made arbitrarily small.

Before stating the next result we introduce some notations. Given a vector field $Z : \mathbb{R}^n \to \mathbb{R}^n$ and a compact set $K \subset \mathbb{R}^n$, we define

$$||Z||_{1,K} := \sup_{x\in K} \left(|Z(x)|_2 + \sum_{i=1}^{n} |D_{x_i} Z(x)|_2 \right).$$

Then we set

$$\mathrm{Lie}^{\delta}_{1,K}(F_1, \dots, F_k) := \{Z \in \mathrm{Lie}(F_1, \dots, F_k) : ||Z||_{1,K} \le \delta\}.$$

We now formulate the assumption required for the approximability result.

Assumption 2. The system of vector fields $F_1, \dots, F_k$ satisfies the *Lie algebra strong approximating property*, i.e., there exists $m \ge 1$ such that, for every C^m-regular vector field $Y : \mathbb{R}^n \to \mathbb{R}^n$ and for each compact set $K \subset \mathbb{R}^n$ there exists $\delta > 0$ such that

$$\inf \left\{ \sup_{x\in K} |X(x) - Y(x)|_2 \,\middle|\, X \in \mathrm{Lie}^{\delta}_{1,K}(F_1, \dots, F_k) \right\} = 0. \tag{4.3.5}$$

The next result establishes a Universal Approximating Property for flows.

Theorem 4.3.3. *Let $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism diffeotopic to the identity. Let $F_1, \dots, F_k$ be a system of vector fields satisfying Assumptions 1 and 2. Then for each compact set $K \subset \mathbb{R}^n$ and each $\varepsilon > 0$ there exists an admissible control $u \in \mathcal{U}$ such that*

$$\sup_{x\in K} |\Psi(x) - \Phi_u(x)|_2 \le \varepsilon, \tag{4.3.6}$$

where Φ_u is the flow corresponding to the control u defined in (4.2.4).

Proof. See [**8**, Theorem 5.1]. □

We recall that $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is diffeotopic to the identity if and only if there exists a family of diffeomorphisms $(\Psi_s)_{s\in[0,1]}$ smoothly depending on s such that $\Psi_0 = \mathrm{Id}$ and $\Psi_1 = \Psi$. In this case, Ψ can be seen as the flow generated by the non-autonomous vector field $(s, x) \mapsto Y_s(x)$, where

$$Y_s(x) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Psi_{s+\varepsilon}(x).$$

From Theorem 4.3.3 we can deduce a C^0-approximate reachability result for infinite ensembles.

Corollary 4.3.4. *Let $\alpha(\cdot), \gamma(\cdot) \in \mathcal{E}_\Theta(\mathbb{R}^n)$ be diffeotopic, i.e., there exists a diffeomorphism $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ diffeotopic to the identity such that $\gamma = \Psi \circ \alpha$. Then $\gamma(\cdot)$ is C^0-approximate reachable from $\alpha(\cdot)$.*

Remark 4.3.3. If a system of vector fields satisfies Assumption 2, then, for every $N \ge 1$ and for every $x^{(N)} \in (\mathbb{R}^n)^{(N)}$ Lie bracket generating condition (4.3.2) is automatically satisfied (see [**8**, Theorem 4.3]). This means that Assumption 2 guarantees global exact controllability in $\mathcal{E}_N(\mathbb{R}^n)$, and C^0-approximate reachability for infinite diffeotopic ensembles.

We conclude this section with the exhibition of a system of vector fields in $\mathbb{R}^n$ meeting Assumptions 1 and 2.

THEOREM 4.3.5. *For every $n > 1$ and $\nu > 0$, consider the vector fields in $\mathbb{R}^n$*

$$\bar{F}_i(x) := \frac{\partial}{\partial x_i}, \quad \bar{F}_i'(x) := e^{-\frac{1}{2\nu}|x|^2}\frac{\partial}{\partial x_i}, \quad i = 1, \dots, n. \tag{4.3.7}$$

Then the system $\bar{F}_1, \dots, \bar{F}_n, \bar{F}_1', \dots, \bar{F}_n'$ satisfies Assumptions 1 and 2.

PROOF. See [**8**, Proposition 6.1]. □

REMARK 4.3.4. If we consider the vector fields $\bar{F}_1, \dots, \bar{F}_n, \bar{F}_1', \dots, \bar{F}_n'$ defined in (4.3.7), then Theorem 4.3.3 and Theorem 4.3.5 guarantee that the flows generated by the corresponding linear-control system can approximate on compact sets diffeorphisms diffeotopic to the identity. From a theoretical viewpoint, this approximation result cannot be strengthened by enlarging the family of controlled vector fields, since the flows produced by any controlled dynamical system are themselves diffeotopic to the identity. On the other hand, in view of the discretization of the dynamics and the consequent construction of the ResNet, it could be useful to enrich the system of the controlled fields. As suggested by Assumption 2, the expressivity of the linear-control system is more directly related to the space $\mathrm{Lie}(F_1, \dots, F_k)$, rather than to the family $F_1, \dots, F_k$ itself. However, as we are going to see, "reproducing" the flow of a field that belongs to $\mathrm{Lie}(F_1, \dots, F_k) \setminus \mathrm{span}\{F_1, \dots, F_k\}$ can be expensive. Let us consider an evolution step-size $h \in (0, 1/4)$ and let us choose two of the controlled vector fields, say F_1, F_2, and let us assume that $[F_1, F_2] \in \mathrm{Lie}(F_1, \dots, F_k) \setminus \mathrm{span}\{F_1, \dots, F_k\}$. Let us denote by $e^{\pm hF_i} : \mathbb{R}^n \to \mathbb{R}^n, i = 1, 2$ the flows obtained by evolving $\pm F_i, i = 1, 2$ for an amount of time equal to h. Then, using for instance the computations in [**4**, Subsection 2.4.7], for every $x \in K$ compact we obtain that

$$\left(e^{-hF_2} \circ e^{-hF_1} \circ e^{hF_2} \circ e^{hF_1}\right)(x) = e^{h^2[F_1,F_2]}(x) + o(h^2)$$

as $h \to 0$. The previous computation shows that, in order to approximate the effect of evolving the vector field $[F_1, F_2]$ for an amount of time equal to h^2, we need tho evolve the fields $\pm F_i, i = 1, 2$ for a total amount $4h$. If h represents the discretization step-size used to derive the ResNet (4.1.4) from the linear-control system (4.2.1) on the interval $[0, 1]$, then we have that $h = \frac{1}{M}$, where N is the number of layers of the ResNet. The argument above suggests that we need to use 4 layers of the network to "replicate" the effect of evolving $[F_1, F_2]$ for the amount of time $h^2 = \frac{1}{M^2}$ (note that $h^2 \ll h$ when $M \gg 1$). This observation provides an insight for the practical choice of the system of controlled fields. In first place, the system $F_1, \dots, F_k$ should meet Assumption 2. If the ResNet obtained from the discretization of the system does not seem to be expressive enough, it should be considered to enlarge the family of the controlled fields, for example by including some elements of $\mathrm{span}\{[F_{i_1}, F_{i_2}] : i_1, i_2 \in \{1, \ldots, k\}\}$ (or, more generally, of $\mathrm{Lie}(F_1, \ldots, F_k)$). We insist on the fact that this procedure increases the width of the network, since the larger is the number of fields in the control system, the larger is the number of parameters per layer in the ResNet.

4.4. Approximation of diffeomorphisms: robust strategy

In this section we introduce the central problem of the chapter, i.e., the *training* of control system (4.2.1) in order to approximate an unknown diffeomorphism $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ diffeotopic to the identity. A typical situation that may arise in applications is that we want to approximate Ψ on a compact set $K \subset \mathbb{R}^n$, having observed the action of Ψ on a finite number of *training* points $\{x_0^1, \ldots, x_0^N\} \subset K$. Our aim is to formulate a strategy that is *robust* with respect to the size N of the training dataset, and for which we can give upper bounds for the *generalization* error. In order to obtain higher and higher degree of approximation, we may think to triangulate the compact set K with an increasing number of nodes where we can evaluate the unknown map Ψ. Using the language introduced in Section 4.3, we have that, for every $N \geq 1$, we may for instance understand a triangulation of K with N nodes as an ensemble $\alpha^N(\cdot) \in \mathcal{E}_N(\mathbb{R}^n)$. After evaluating Ψ in the nodes, we obtain the target ensemble $\gamma^N(\cdot) \in \mathcal{E}_N(\mathbb{R}^n)$ as $\gamma^N(\cdot) := \Psi(\alpha^N(\cdot))$.

4.4.1. Approximation via ensemble controllability. If the vector fields $F_1, \ldots, F_k$ that define control system (4.2.1) meet Assumptions 1 and 2, then Theorem 4.3.1 and Remark 4.3.3 may suggest a first natural attempt to design an approximation strategy. Indeed, for every N, we can *exactly* steer the ensemble $\alpha^N(\cdot)$ to the ensemble $\gamma^N(\cdot)$ with an admissible control $u^N \in \mathcal{U}$. Hence, we can choose the flow Φ_{u^N} defined in (4.2.4) as an approximation of Ψ on K, achieving a *null training error*. Assume that, for every $N \geq 1$, the corresponding triangulation is a ϵ^N-approximation of the set K, i.e., for every $y \in K$ there exists $\bar{j} \in \{1, \ldots, N\}$ such that $|y - x_0^{\bar{j},M}|_2 \leq \epsilon^N$, where $x_0^{\bar{j},N} := \alpha^N(\bar{j})$. Then, for every $y \in K$, we can give the following estimate for the generalization error:

$$
\begin{aligned}
|\Psi(y) - \Phi_{u^N}(y)|_2 &\leq |\Psi(y) - \Psi(x_0^{\bar{j},N})|_2 + |\Phi_{u^N}(x_0^{\bar{j},N}) - \Phi_{u^N}(y)|_2 \\
&\leq L_\Psi \epsilon^N + L_{\Phi_{u^N}} \epsilon^N,
\end{aligned}
$$

where $L_\Psi, L_{\Phi_{u^N}}$ are respectively the Lipschitz constants of Ψ and Φ_{u^N}. Assuming (as it is natural to do) that $\epsilon^N \to 0$ as $N \to \infty$, the strategy of achieving zero training error works if, for example, the Lipschitz constants of the approximating flows $(\Phi_{u^N})_{N\geq 1}$ are bounded from above. This in turn would follow if the sequence of controls $(u^N)_{N\geq 1}$ were bounded in L^2-norm. However, as we are going to see in the following proposition, in general this is not the case. Let us define

$$
\mathrm{Flows}_K(F_1, \ldots, F_k) := \{\Phi_u : u \in \mathcal{U}\},
$$

the space of flows restricted to K obtained via (4.2.4) with admissible controls, and let $\mathrm{Diff}^0_K(\mathbb{R}^n)$ be the space of diffeomorphisms diffeotopic to the identity restricted to K. Theorem 4.3.3 guarantees that, for every $K \subset \mathbb{R}^n$,

$$\overline{\mathrm{Flows}_K(F_1, \ldots, F_k)}^{C^0} = \mathrm{Diff}^0_K(\mathbb{R}^n).$$

PROPOSITION 4.4.1. *Given a diffeomorphism diffeotopic to the identity* $\Psi \in \mathrm{Diff}^0_K(\mathbb{R}^n) \setminus \mathrm{Flows}_K(F_1, \ldots, F_k)$ *and an approximating sequence of flows* $(\Phi_{u_m})_{m \geq 1} \subset \mathrm{Flows}_K(F_1, \ldots, F_k)$ *such that* $\Phi_{u_m} \to_{C^0} \Psi$ *on* K *as* $m \to \infty$, *then the sequence of controls* $(u_m)_{m \geq 1} \subset \mathcal{U}$ *is unbounded in the* L^2*-norm.*

PROOF. By contradiction, let $(u_m)_{m \geq 1}$ be a bounded sequence in $\mathcal{U}$. Then, we can extract a subsequence $(u_{m_\ell})_{\ell \geq 1}$ weakly convergent to $u \in \mathcal{U}$. In virtue of Proposition 4.2.2, we have that $\Phi_{u_{m_\ell}} \to_{C^0} \Phi_u$ on K as $\ell \to \infty$. However, since $\Phi_{u_m} \to_{C^0} \Psi$ on K, we deduce that $\Psi = \Phi_u$ on K, but this contradicts the hypothesis $\Psi \in \mathrm{Diff}^0_K(\mathbb{R}^n) \setminus \mathrm{Flows}_K(F_1, \ldots, F_k)$. □

The previous result sheds light on a weakness of the approximation strategy described above. Indeed, the main drawback is that we have no bounds on the norm of the controls $(u^N)_{N \geq 1}$, and therefore, even though the triangulation of K is fine, we cannot give an *a priori* estimate of the testing error. We point out that, in the different framework of simultaneous control of several systems, a similar situation was described in [**6**].

4.4.2. Approximation via optimal control. In order to avoid the issues described above, we propose a training strategy based on the solution of an ensemble optimal control problem with a regularization term penalizing the L^2-norm of the control. Namely, given a set of training points $\{x_0^1, \ldots, x_0^N\} \subset K$, we consider the nonlinear functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}$ defined as follows:

$$\mathcal{F}^N(u) := \frac{1}{N} \sum_{j=1}^{N} a(\Phi_u(x_0^j) - \Psi(x_0^j)) + \frac{\beta}{2} ||u||^2_{L^2}, \tag{4.4.1}$$

where $a : \mathbb{R}^n \to \mathbb{R}$ is a smooth loss function such that $a \geq 0$ and $a(0) = 0$, and $\beta > 0$ is a fixed parameter. The functional $\mathcal{F}^N$ is composed by two competing terms: the first aims at achieving a low *mean* training error, while the second aims at keeping bounded the L^2-norm of the control. In this framework, it is worth assuming that the compact set K is equipped with a Borel probability measure μ. In this way, we can give higher weight to the regions in K where we need more accuracy in the approximation. As done before, for every $N \geq 1$ we understand the training dataset as an ensemble $\alpha^N(\cdot) \in \mathcal{E}_N(\mathbb{R}^n)$. Moreover, we associate to it the discrete probability measure μ_N defined as

$$\mu_N := \frac{1}{N} \sum_{j=1}^{N} \delta_{\alpha(j)}, \tag{4.4.2}$$

and we can equivalently express the mean training error as

$$\frac{1}{N}\sum_{j=1}^{N} a(\Phi_u(x_0^j) - \Psi(x_0^j)) = \int_{\mathbb{R}^n} a(\Phi_u(x) - \Psi(x))\, d\mu_N(x).$$

From now on, when considering datasets growing in size, we make the following assumption on the sequence of probability measures $(\mu_N)_{N\geq 1}$.

ASSUMPTION 3. There exists a Borel probability measure μ supported in the compact set $K \subset \mathbb{R}^n$ such that the sequence $(\mu_N)_{N\geq 1}$ is weakly convergent to μ, i.e.,

$$\lim_{N\to\infty} \int_{\mathbb{R}^n} f(x)\, d\mu_N = \int_{\mathbb{R}^n} f(x)\, d\mu, \tag{4.4.3}$$

for every bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$. Moreover, we ask that μ_N is supported in K for every $N \geq 1$.

REMARK 4.4.1. The request of Assumption 3 is rather natural. Indeed, if the elements of the ensembles $\alpha^N(\cdot) \in \mathcal{E}_N(\mathbb{R}^n)$ are sampled using the probability distribution μ associated to the compact set K, it follows from the law of large numbers that (3.5.5) holds. On the other hand, since we ask that all the ensembles are contained in the compact set K, we have that the sequence of probability measures $(\mu_N)_{N\geq 1}$ is tight. Therefore, in virtue of Prokhorov Theorem, $(\mu_N)_{N\geq 1}$ is sequentially weakly pre-compact (for details see, e.g., [**21**]).

REMARK 4.4.2. When $K = \overline{\mathrm{int}(K)}$, if for every N $\alpha^N(\cdot)$ is a ϵ^N-approximation of K such that $\epsilon^N \to 0$ as $N \to \infty$, then the corresponding sequence of probability measures $(\mu_N)_{N\geq 1}$ is weakly convergent to $\mu = \frac{1}{\mathcal{L}(K)}\mathcal{L}|_K$, where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^n$.

We observe that the problem of minimizing the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ defined in (4.4.1) is strictly related to the resolution of an ensemble optimal control problem, of the type studied in Chapter 3. More precisely, using the same notations as in the previous chapter, we can consider the following ensemble of linear-control systems in $\mathbb{R}^n$:

$$\begin{cases} \dot{x}^\theta(s) = \sum_{i=1}^k F_i(x^\theta(s))u_i(s), & s \in [0,1] \\ x^\theta(0) = \iota(\theta), \end{cases} \tag{4.4.4}$$

where $\iota : K \to \mathbb{R}^n$ is the inclusion of K into $\mathbb{R}^n$. We observe that in (4.4.4) the parameter $\theta \in K$ affects only the value of the Cauchy datum, but not the controlled vector fields. In analogy with the discussion in the previous chapter, we can rewrite the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}_+$ as follows:

$$\mathcal{F}^N(u) = \int_K \tilde{a}(x_u^\theta(1), \theta) d\mu_N(\theta) + \frac{\beta}{2}||u||_{L^2}^2, \tag{4.4.5}$$

where $\tilde{a}(x,\theta) := a(x - \psi(\iota(\theta)))$ for every $x \in \mathbb{R}^n$ and $\theta \in K$. Therefore, the results proved in Chapter 3 for general ensembles of affine-control systems imply immediately the following proposition.

PROPOSITION 4.4.2. *For every $N \geq 1$ the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}$ defined in* (4.4.1) *admits a minimizer. Moreover, if Assumption 3 is met, then there exists $C_\beta > 0$ such that, for every $N \geq 1$, any minimizer $\tilde{u}^N$ of $\mathcal{F}^N$ satisfies the following inequality:*

$$||\tilde{u}_N||_{L^2} \leq C_\beta. \tag{4.4.6}$$

PROOF. The thesis follows as a particular case from Theorem 3.4.2 and from Lemma 3.5.1. □

The previous result suggests as a training strategy to look for a minimizer of the functional $\mathcal{F}^N$. In the next section we investigate the properties of the functionals $(\mathcal{F}^N)_{N\geq 1}$ using the tools of Γ-convergence.

4.5. Ensembles growing in size and Γ-convergence

In this section we study the limiting problem when the size of the training dataset tends to infinity. The main fact is that a Γ-convergence result holds. Roughly speaking, this means that *increasing the size of the training dataset does not make the problem harder, at least from a theoretical viewpoint.* Even though the problems studied in the present chapter are a direct applications of the tools developed in Chapter 3, it is interesting to observe that the interplay between finite and infinite ensembles is done in the opposite direction. Indeed, in Chapter 3 we were assigned a problem involving an infinite ensemble of control systems, and we used Γ-convergence to approximate them with optimal control problems of easier-to-handle finite ensembles. On the other hand, in this case the problem of the diffeomorphism approximation naturally involves a finite number of observations, and we employ Γ-convergence to study the limiting case when the size of the dataset goes to infinity.

For every $N \geq 1$, let $\alpha^N(\cdot) \in \mathcal{E}_N(\mathbb{R}^n)$ be an ensemble of points in the compact set $K \subset \mathbb{R}^n$, and let us consider the discrete probability measure μ_N defined in (4.4.2). For every $N \geq 1$ we consider the functional $\mathcal{F}^N : \mathcal{U} \to \mathbb{R}$ defined as follows:

$$\mathcal{F}^N(u) := \int_{\mathbb{R}^n} a(\Phi_u(x) - \Psi(x))\, d\mu_N(x) + \frac{\beta}{2}||u||^2_{L^2}. \tag{4.5.1}$$

The tools of Γ-convergence requires the domain where the functionals are defined to be equipped with a metrizable topology. Recalling that the weak topology of L^2 is metrizable only on bounded sets, we need to properly restrict the functionals. For every $\rho > 0$, we set

$$\mathcal{U}_\rho := \{u \in \mathcal{U} : ||u||_{L^2} \leq \rho\}.$$

Using Proposition 4.4.2 we can choose $\rho = C_\beta$, so that

$$\arg\min_{\mathcal{U}} \mathcal{F}^N = \arg\min_{\mathcal{U}_\rho} \mathcal{F}^N,$$

for every $N \geq 1$. With this choice we restrict the minimization problem to a bounded subset of $\mathcal{U}$, without losing any minimizer. As done in the previous chapter, with a slight abuse of notations we still denote by $\mathcal{F}^N$ the restriction of $\mathcal{F}^N$ to $\mathcal{U}_\rho$. Let us define the functional $\mathcal{F}^\infty : \mathcal{U} \to \mathbb{R}$ as follows:

$$\mathcal{F}^\infty(u) := \int_{\mathbb{R}^n} a(\Phi_u(x) - \Psi(x))\, d\mu(x) + \frac{\beta}{2}||u||_{L^2}^2, \tag{4.5.2}$$

where the probability measure μ is the weak limit of the sequence $(\mu_N)_{N\geq 1}$. Using the same argument as in the proof of Proposition 4.4.2, we can prove that $\mathcal{F}^\infty$ attains minimum and that

$$\arg\min_{\mathcal{U}} \mathcal{F}^\infty = \arg\min_{\mathcal{U}_\rho} \mathcal{F}^\infty,$$

with $\rho = C_\beta$. As before, we use $\mathcal{F}^\infty$ to denote as well the restriction of $\mathcal{F}^\infty$ to $\mathcal{U}_\rho$. The following result follows as a particular case of the Theorem 3.5.3.

THEOREM 4.5.1. *Given $\rho > 0$, let us consider $\mathcal{F}^N : \mathcal{U}_\rho \to \mathbb{R}$ with $N \geq 1$. Let $\mathcal{F}^\infty : \mathcal{U}_\rho \to \mathbb{R}$ be the restriction to $\mathcal{U}_\rho$ of the functional defined in* (4.5.2). *If Assumption 3 holds, then the functionals $(\mathcal{F}^N)_{N\geq 1}$ Γ-converge to $\mathcal{F}^\infty$ as $N \to \infty$ with respect to the weak topology of $\mathcal{U}$.*

PROOF. The Γ-convergence descends as a particular case of Theorem 3.5.3. □

REMARK 4.5.1. Using the equi-coercivity of the functionals $(\mathcal{F}^N)_{N\geq 1}$ and [**24**, Corollary 7.20], we deduce that

$$\lim_{N\to\infty} \min_{\mathcal{U}_\rho} \mathcal{F}^N = \min_{\mathcal{U}_\rho} \mathcal{F}^\infty, \tag{4.5.3}$$

and that any cluster point $\tilde{u}$ of a sequence of minimizers $(\tilde{u}_N)_{N\geq 1}$ is a minimizer of $\mathcal{F}^\infty$. Let us assume that a sub-sequence $\tilde{u}_{N_j} \rightharpoonup \tilde{u}$ as $j \to \infty$. Using Proposition 4.2.2 and the Dominated Convergence Theorem we deduce that

$$\lim_{j\to\infty} \int_K a(\Phi_{\tilde{u}_{N_j}}(x) - \Psi(x))\, d\mu_{N_j}(x) = \int_K a(\Phi_{\tilde{u}}(x) - \Psi(x))\, d\mu(x), \tag{4.5.4}$$

where we stress that $\tilde{u}$ is a minimizer of $\mathcal{F}^\infty$. Combining (4.5.3) and (4.5.4) we obtain that

$$\lim_{j\to\infty} \frac{\beta}{2}||\tilde{u}_{N_j}||_{L^2}^2 = \frac{\beta}{2}||\tilde{u}||_{L^2}^2.$$

Since $\tilde{u}_{N_j} \rightharpoonup \tilde{u}$ as $j \to \infty$, the previous equation implies that the subsequence $(\tilde{u}_{N_j})_{j\geq 1}$ converges to $\tilde{u}$ also with respect to the strong topology of L^2. This argument shows that any sequence of minimizers $(\tilde{u}_M)_{N\geq 1}$ is pre-compact with respect to the strong topology of L^2.

We can establish an asymptotic upper bound for the mean training error. Let us define

$$\kappa_\beta := \sup \left\{ \int_K a(\Phi_{\tilde{u}}(x) - \Psi(x)) \, d\mu(x) \middle| \tilde{u} \in \arg\min_{\mathcal{U}} \mathcal{F}^\infty \right\}. \tag{4.5.5}$$

As suggested by the notation, the value of κ_β highly depends on the positive parameter β that tunes the L^2-regularization. Given a sequence of minimizers $(\tilde{u}_N)_{N\geq 1}$ of the functionals $(\mathcal{F}^N)_{N\geq 1}$, from (4.5.4) we deduce that

$$\limsup_{N\to\infty} \int_K a(\Phi_{\tilde{u}_N}(x) - \Psi(x)) \, d\mu_N(x) \leq \kappa_\beta. \tag{4.5.6}$$

In the next result we show that under the hypotheses of Theorem 4.3.3 κ_β can be made arbitrarily small with a proper choice of β.

PROPOSITION 4.5.2. *Let κ_β be defined as in* (4.5.5)*. If the vector fields $F_1, \ldots, F_k$ that define control system* (4.2.1) *satisfy Assumption 1 and 2, then*

$$\lim_{\beta\to 0^+} \kappa_\beta = 0. \tag{4.5.7}$$

PROOF. Let us fix $\varepsilon > 0$. Since $a(0) = 0$, there exists $\rho > 0$ such that

$$\sup_{B_\rho(0)} a \leq \varepsilon.$$

Using Theorem 4.3.3, we deduce that there exists a control $\hat{u} \in \mathcal{U}$ such that

$$\sup_{x\in K} |\Phi_{\hat{u}}(x) - \Psi(x)|_2 < \rho. \tag{4.5.8}$$

This implies that

$$\int_K a(\Phi_{\hat{u}}(x) - \Psi(x)) \, d\mu(x) \leq \varepsilon.$$

Let us set $\hat{\beta} := \frac{2\varepsilon}{||\hat{u}||_{L^2}^2}$. For any $\beta \leq \hat{\beta}$, let $\mathcal{F}^\infty$ be the functional defined in (4.5.2) with tuning parameter β, and let $\tilde{u} \in \mathcal{U}$ be a minimizer of $\mathcal{F}$. Then we have

$$\mathcal{F}^\infty(\tilde{u}) \leq \mathcal{F}^\infty(\hat{u}) \leq \varepsilon + \frac{\beta}{2}||\hat{u}||_{L^2}^2 \leq 2\varepsilon,$$

and this concludes the proof. □

4.5.1. An estimate of the generalization error. We now discuss an estimate of the expected generalization error based on the observation of the mean training error, similar to the one established in [**39**] for the control system (4.1.2). A similar estimate was obtained also in [**18**]. Assumption 3 implies that the Wasserstein distance $W_1(\mu_N, \mu) \to 0$ as $N \to \infty$ (for details, see [**1**, Proposition 7.1.5]). We recall that, if $\nu_1, \nu_2 \in \mathcal{P}(K)$ are Borel probability measures on K, then

$$W_1(\nu_1, \nu_2) := \inf_{\pi\in\mathcal{P}(K\times K)} \left\{ \int_{K\times K} |x - y|_2 \, d\pi(x, y) \middle| \pi(\cdot, K) = \nu_1, \pi(K, \cdot) = \nu_2 \right\}.$$

For every $N \geq 1$ let us introduce $\pi_N \in \mathcal{P}(K \times K)$ such that $\pi_N(\cdot, K) = \mu_N$ and $\pi_N(K, \cdot) = \mu$, and

$$W_1(\mu_N, \mu) = \int_{K\times K} |x - y|_2 \, d\pi_N(x, y).$$

Let us consider an admissible control $u \in \mathcal{U}$, and let $\Phi_u : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding flow. If the testing samples are generated using the probability distribution μ, then the expected generalization error that we commit by approximating $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ with Φ_u is

$$\mathbb{E}_\mu[a(\Phi_u(\cdot) - \Psi(\cdot))] = \int_K a(\Phi_u(y) - \Psi(y)) \, d\mu(y).$$

On the other hand, we recall that the corresponding training error is expressed by

$$\int_K a(\Phi_u(x) - \Psi(x)) \, d\mu_N(x).$$

Hence we can compute

$$\begin{aligned} \left| \mathbb{E}_\mu[a(\Phi_u(\cdot) - \Psi(\cdot))] - \int_K a(\Phi_u(x) - \Psi(x)) \, d\mu_N(x) \right| & \\ \leq \int_{K\times K} |a(\Phi_u(y) - \Psi(y)) - a(\Phi_u(x) - \Psi(x))| \, d\pi_N(x, y) & \\ \leq L_a \int_{K\times K} |\Psi(y) - \Psi(x)| + |\Phi_u(x) - \Phi_u(y)| \, d\pi_N(x, y). & \end{aligned}$$

Then for every $N \geq 1$ we have

$$\left| \mathbb{E}_\mu[a(\Phi_u(\cdot) - \Psi(\cdot))] - \int_K a(\Phi_u(x) - \Psi(x)) \, d\mu_N(x) \right| \leq L_a(L_\Psi + L_{\Phi_u}) W_1(\mu_N, \mu),$$

where L_Ψ, L_{Φ_u} and L_a are the Lipschitz constant, respectively, of Ψ, Φ_u and a. The last inequality finally yields

$$\mathbb{E}_\mu[a(\Phi_u(\cdot)-\Psi(\cdot))] \leq \int_K a(\Phi_u(x)-\Psi(x)) \, d\mu_N(x) + L_a(L_\Psi + L_{\Phi_u}) W_1(\mu_N, \mu) \quad (4.5.9)$$

for every $N \geq 1$.

Remark 4.5.2. We observe that the estimate (4.5.9) does not involve any testing dataset. In other words, in principle we can use (4.5.9) to provide an upper bound to the expected generalization error, without the need of computing the mismatch between Ψ and Φ_u on a testing dataset. In practice, while we can directly measure the first quantity at the right-hand side of (4.5.9), the second term could be challenging to estimate. Indeed, if on one hand we can easily approximate the quantity L_{Φ_u} (for instance by means of (4.2.5)), on the other hand we may have no access to the distance $W_1(\mu_N, \mu)$. This is actually the case when the measure μ used to sample the training dataset is unknown.

In the case we consider the flow $\Phi_{\hat{u}_N}$ corresponding to a minimizer $\hat{u}_N$ of the functional $\mathcal{F}^N$ we can further specify (4.5.9). Indeed, combining Proposition 4.4.2 and Lemma 4.2.1, we deduce that $L_{\Phi_{\tilde{u}_N}}$ is uniformly bounded with respect to N by a constant L_β. Provided that N is large enough, from (4.5.9) and (4.5.6) we obtain that

$$\mathbb{E}_\mu[a(\Phi_{\tilde{u}_N}(\cdot) - \Psi(\cdot))] \leq 2\kappa_\beta + L_a(L_\Psi + L_\beta)W_1(\mu_N, \mu). \tag{4.5.10}$$

The previous inequality shows how we can achieve an arbitrarily small expected generalization error, provided that the vector fields $F_1, \ldots, F_k$ of control system (4.2.1) satisfy Assumption 2, and provided that the size of the training dataset could be chosen arbitrarily large. First, using Proposition 4.5.2 we set the tuning parameter β such that the quantity κ_β is as small as desired. Then, we consider a training dataset large enough to guarantee that the second term at the right-hand side of (4.5.10) is of the same order of κ_β.

Remark 4.5.3. Given $\varepsilon > 0$, Proposition 4.5.2 guarantees the existence of $\hat{\beta} > 0$ such that $\kappa_\beta \leq \varepsilon$ if $\beta \leq \hat{\beta}$. The expression of $\hat{\beta}$ obtained in the proof of Proposition 4.5.2 is given in terms of the norm of a control $\hat{u} \in \mathcal{U}$ such that (4.5.8) holds. In [**7**], where Theorem 4.3.3 is proved, it is explained the construction of an admissible control whose flow approximate the target diffeomorphism with assigned precision. However the control produced with this procedure is, in general, far from having minimal L^2-norm, and as a matter of fact the corresponding $\hat{\beta}$ might be smaller than necessary. Unfortunately, at the moment, we can not provide a more practical rule for the computation of $\hat{\beta}$.

Remark 4.5.4. As observed in Remark 4.5.2 for (4.5.9), the estimate (4.5.10) of the expected generalization error holds as well *a priori* with respect to the choice of a testing dataset. Moreover, if the size M of the training dataset is assigned and it cannot be enlarged, in principle one could choose the regularization parameter β by minimizing the right-hand side of (4.5.10). However, in practice this may be very complicated, since we have no direct access to the function $\beta \mapsto \kappa_\beta$.

4.6. Construction and training of the ResNet

A ResNet with M layers is an application $\Phi = \Phi_M \circ \ldots \Phi_1$ defined as the composition of parametric functions (called *building blocks*) $\Phi_l : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ with $l = 1, \ldots, M$. The building blocks have a precise structure, namely they are of the form:

$$\Phi_l(x) = x + G(x, u_l), \tag{4.6.1}$$

where $G : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is assigned and, together with the number of layers M, constitutes the *architecture* of the network. As observed in [**26, 29**] the building blocks (4.6.1) can be seen as an Explicit Euler discretization of the following control system in $\mathbb{R}^n$

$$\dot{x}(s) = G(x(s), u(s)), \quad s \in [0, M], \tag{4.6.2}$$

with discretization step-size $h = 1$. Of course, it is also possible to proceed in the other way round, i.e., to work out an M-layer ResNet by discretizing a control system on a time interval $[0, T]$, setting the step-size $h = \frac{T}{M}$. The parameters $u_1, \dots, u_M \in \mathbb{R}^k$ that appear in (4.6.1) should be properly adjusted during the so called *training phase* of the ResNet, which consists in the resolution of a non-linear minimization problem. The objective function that is minimized depends on the task we are training the network for. For example, in the case of the problem of diffeomorphisms approximation studied so far, we could consider

$$\min_{u_1,\dots,u_M \in \mathbb{R}^k} \left\{ \frac{1}{N} \sum_{j=1}^{N} a(\Phi_{\mathbf{u}}(x_0^j) - \Psi(x_0^j)) + \frac{\beta}{2} \sum_{l=1}^{M} |u_l|_2^2 \right\}, \tag{4.6.3}$$

where $\Phi_{\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^n$ is the map corresponding to the choices of parameters $\mathbf{u} = (u_1, \dots, u_M)$ in the building blocks (4.6.1), and $\{x_0^1, \dots, x_0^N\}$ is the ensemble of points where $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is observed. The minimization problem (4.6.3) is usually solved by applying the gradient method (or some of its variations) to the parameters $(u_1, \dots, u_M)$. For more details, we refer the reader to the textbook [**28**]. On the other hand, the fact that ResNets are discretizations of control systems paves the way for the use of numerical methods specifically developed for optimal control problems, as done in some recent contributions (see [**14, 18, 35**]). In particular, regarding the problem of the observations-based diffeomorphism approximation, it can be formulated as an ensemble optimal control problem, as explained in Section 4.4. Therefore, since the system that we consider is linear in the controls, we can employ the numerical schemes introduced in Chapter 3. Moreover, the algorithms introduced in Section 3.7 make both use of the Explicit Euler scheme to discretize the control system. Therefore, if we consider the control system (4.2.1) on the time horizon $[0, 1]$ and we use as step-size $h = \frac{1}{M}$, then we obtain a ResNet with building blocks

$$\Phi_l(x) = x + h \sum_{i=1}^{k} F_i(x) u_{l,i}, \quad h = \frac{1}{M}, \tag{4.6.4}$$

for $l = 1, \dots, M$. We insist on the fact that the building blocks (4.6.4) are linear with respect to the parameters. This is an original feature in ResNets architectures, and the main advantage is that it positively affects the amount of computations in the training phase. In the next section we test on an example the training algorithms obtained using Algorithm 1 and Algorithm 2, that were originally developed for the resolution of more general ensemble optimal control problems.

4.7. Numerical experiments: learning a diffeomorphism

In this section we describe the numerical experiments involving the approximation of an unknown diffeomorphism by means of Algorithm 1 and Algorithm 2.

We consider the following diffeomorphism $\tilde{\Psi} : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\tilde{\Psi}(x) := x + \begin{pmatrix} 2x_1 e^{x_1^2-1} \\ 2x_2^3 \end{pmatrix} + \begin{pmatrix} -4 \\ -4.5 \end{pmatrix},$$

the rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ centered at the origin and with angle $\pi/3$, and the translation $T : \mathbb{R}^2 \to \mathbb{R}^2$ prescribed by the vector $(0.3, 0.2)$. Finally, we set

$$\Psi := \tilde{\Psi} \circ T \circ R. \tag{4.7.1}$$

We generate the training dataset $\{x^1, \ldots, x^M\}$ with $M = 900$ points by constructing a uniform grid in the square centered at the origin and with side of length $\ell = 1.5$. In Figure 1 we report the training dataset and its image trough Ψ. We have implemented the codes in Matlab and we have ran them on a laptop with 16 GB of RAM and a 6-core 2.20 GHz processor.

4.7.1. Diffeomorphism approximation: first attempt. Since we consider $\mathbb{R}^2$ as ambient space, Theorem 4.3.3 and Theorem 4.3.5 guarantee that the linear-control system associated to the fields

$$F_1(x) := \frac{\partial}{\partial x_1}, \quad F_2(x) := \frac{\partial}{\partial x_2},$$

$$F_1'(x) := e^{-\frac{1}{2\nu}|x|^2} \frac{\partial}{\partial x_1}, \quad F_2'(x) := e^{-\frac{1}{2\nu}|x|^2} \frac{\partial}{\partial x_2},$$

is capable of approximating on compact sets diffeomorphisms that are diffeotopic to the identity. However, it looks natural to include in the set of the controlled vector fields also the following ones:

$$G_1^1(x) := x_1 \frac{\partial}{\partial x_1}, \quad G_1^2(x) := x_2 \frac{\partial}{\partial x_1},$$

$$G_2^1(x) := x_1 \frac{\partial}{\partial x_2}, \quad G_2^2(x) := x_2 \frac{\partial}{\partial x_2}.$$

Indeed, with this choice, we observe that the corresponding control system

$$\dot{x} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + e^{-\frac{1}{2\nu}|x|^2} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} + \begin{pmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{4.7.2}$$

can reproduce *exactly* non-autonomous vector fields that are linear in the state variables (x_1, x_2). Moreover, the discretization of the control system (4.7.2) on the evolution interval $[0, 1]$ with step-size $h = \frac{1}{N}$ gives rise to a ResNet $\Phi = \Phi_N \circ \ldots \circ \Phi_1$ with N layers, whose building blocks have the form:

$$\Phi_k(x) = x + h \left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + e^{-\frac{1}{2\nu}|x|^2} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} + \begin{pmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right], \tag{4.7.3}$$

and each of them has 8 parameters.

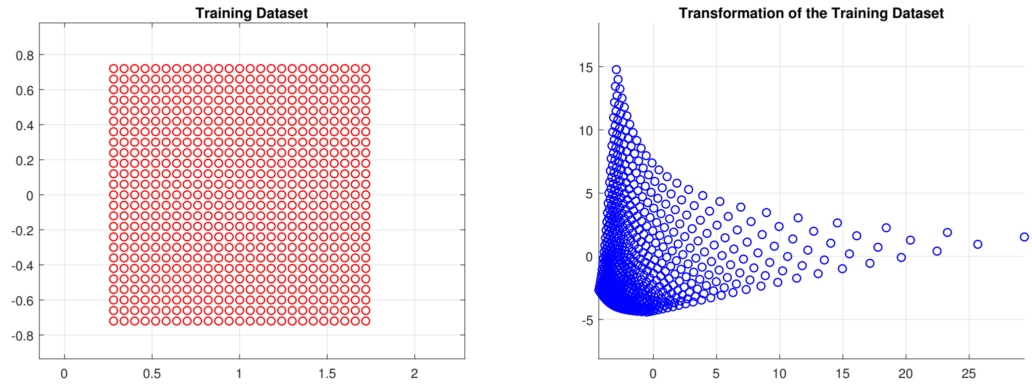

FIGURE 1. On the left we report the grid of points $\{x^1, \dots, x^M\}$ where we have evaluated the diffeomorphism $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined as in (4.7.1). The picture on the right represents the transformation of the training dataset through the diffeomorphism Ψ.

If we denote by μ the probability measure that charges uniformly the square and if we set $\mu_N := \frac{1}{N}\sum_{j=1}^N \delta_{x^j}$, we obtain the following estimate

$$W_1(\mu_N, \mu) \leq \frac{\sqrt{2}\ell}{2\sqrt{N}},$$

that can be used to compute the *a priori* estimate of the generalization error provided by (4.5.9). We use the $1-$Lipschitz loss function

$$a(x-y) := \sqrt{1 + (x_1 - y_1)^2 + (x_2 - y_2)^2} - 1,$$

and we look for a minimizer of

$$\mathcal{F}^N(u) := \frac{1}{900}\sum_{j=1}^{900} a(\Phi_u(x^j) - \Psi(x^j)) + \frac{\beta}{2}||u||_{L^2}^2, \tag{4.7.4}$$

where $\beta > 0$ is the regularization hyper-parameter. In the training phase we use the same dataset for Algorithm 1 and Algorithm 2, and in both cases the initial guess of the control is $u \equiv 0$. Finally, the testing dataset has been generated by randomly sampling 300 points using μ, the uniform probability measure on the square. The value of the hyper-parameter ν is set equal to 20. We first try to approximate the diffeomorphism Ψ using $h = 2^{-4}$, resulting in 16 inner layers. Hence, recalling that each building-block (4.7.3) has 8 parameters, the corresponding ResNet has in total 128 parameters. We have tested different values of β, and we set $\max_{iter} = 500$. The results obtained by Algorithm 1 and Algorithm 2 are reported in Table 1 and Table 2, respectively. We observe that in both algorithms the Lipschitz constant of the produced diffeomorpism grows as the hyper-parameter β gets smaller, consistently with the theoretical intuition. As

β	L_{Φ_u}	Training error	Testing error
10^0	1.19	3.8785	3.8173
10^{-1}	8.40	1.3143	1.2476
10^{-2}	9.32	1.1991	1.1451
10^{-3}	9.37	1.1852	1.1330
10^{-4}	9.37	1.1839	1.1318

Table 1. ResNet 4.7.3, 16 layers, 128 parameters, Algorithm 1. Running time ~ 160 s.

β	L_{Φ_u}	Training error	Testing error
10^0	1.19	3.8749	3.8157
10^{-1}	8.40	1.3084	1.2455
10^{-2}	9.32	1.2014	1.1486
10^{-3}	9.33	1.1898	1.1387
10^{-4}	9.33	1.1898	1.1379

Table 2. ResNet 4.7.3, 16 layers, 128 parameters, Algorithm 2. Running time ~ 130 s.

regards the testing error, we observe that it always remains reasonably close to the corresponding training error. We report in Figure 2 the image of the approximation that achieves the best training and testing errors, namely Algorithm 1 with $\beta = 10^{-4}$. As we may observe, the prediction is quite unsatisfactory, both on the training and on the testing data-sets. Finally, we report that the formula (4.5.9) correctly provides an upper bound to the testing error, even though it is too pessimistic to be of practical use.

In order to improve the quality of the approximation, a natural attempt consists in trying to increase the depth of the ResNet. Therefore, we have repeated the experiments setting $h = 2^{-5}$, that corresponds to 32 layers. Recalling that the ResNet in exam has 8 parameters per layer, the architecture has globally 256 weights. The results are reported in Table 3 and Table 4. Unfortunately, despite doubling the depth of the ResNet, we do not observe any relevant improvement in the training nor in the testing error. Using the idea explained in Remark 4.3.4, instead of further increase the number of the layers, we try to enlarge the family of the controlled vector fields in the control system associated to the ResNet.

4.7.2. Diffeomorphism approximation: enlarged family of controlled fields. Using the ideas expressed in Remark 4.3.4, we enrich the family of the controlled fields. In particular, in addition to the fields considered above, we

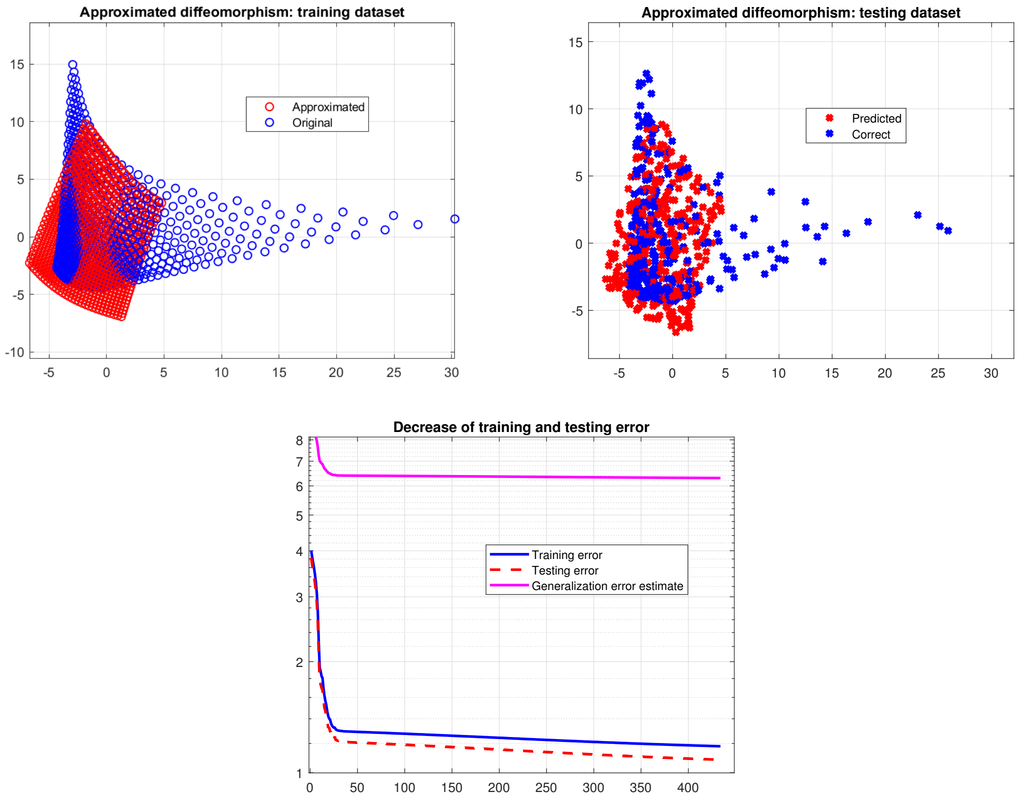

FIGURE 2. ResNet 4.7.3, 16 layers, Algorithm 1, $\beta = 10^{-4}$. On the top-left we reported the transformation of the initial grid through the approximating diffeomorphism (red circles) and through the original one (blue circles). On the top-right, we plotted the prediction on the testing data-set provided by the approximating diffeomorphism (red crosses) and the correct values obtained through the original transformation (blue crosses). In both cases, the approximation obtained is unsatisfactory. At bottom we plotted the decrease of the training error and the testing error versus the number of iterations. Finally, the curve in magenta represents the estimate of the generalization error provided by (4.5.9).

include the following ones:

$$G_1^{1,1} := x_1^2 e^{-\frac{1}{2\nu}|x|^2} \frac{\partial}{\partial x_1}, \quad G_1^{1,2} := x_1 x_2 e^{-\frac{1}{2\nu}|x|^2} \frac{\partial}{\partial x_1}, \quad G_1^{2,2} := x_2^2 e^{-\frac{1}{2\nu}|x|^2} \frac{\partial}{\partial x_1},$$

$$G_2^{1,1} := x_1^2 e^{-\frac{1}{2\nu}|x|^2} \frac{\partial}{\partial x_2}, \quad G_2^{1,2} := x_1 x_2 e^{-\frac{1}{2\nu}|x|^2} \frac{\partial}{\partial x_2}, \quad G_2^{2,2} := x_2^2 e^{-\frac{1}{2\nu}|x|^2} \frac{\partial}{\partial x_2}.$$

β	L_{Φ_u}	Training error	Testing error
10^0	1.19	3.8779	3.8168
10^{-1}	8.40	1.3074	1.2425
10^{-2}	9.26	1.2015	1.1477
10^{-3}	9.34	1.1860	1.1352
10^{-4}	9.34	1.1842	1.1332

TABLE 3. ResNet 4.7.3, 32 layers, 256 parameters, Algorithm 1. Running time $\sim$ 320 s.

β	L_{Φ_u}	Training error	Testing error
10^0	1.19	3.8739	3.8148
10^{-1}	8.35	1.3085	1.2449
10^{-2}	9.23	1.2075	1.1538
10^{-3}	9.26	1.1931	1.1416
10^{-4}	9.26	1.1918	1.1404

TABLE 4. ResNet 4.7.3, 32 layers, 256 parameters, Algorithm 2. Running time $\sim$ 260 s.

Therefore, the resulting linear-control system on the time interval $[0, 1]$ has the form

$$\begin{aligned}\dot{x} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + e^{-\frac{1}{2\nu}|x|^2} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} + \begin{pmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ + e^{-\frac{1}{2\nu}|x|^2} \begin{pmatrix} u_1^{1,1} x_1^2 + u_1^{1,2} x_1 x_2 + u_1^{2,2} x_2^2 \\ u_2^{1,1} x_1^2 + u_2^{1,2} x_1 x_2 + u_2^{2,2} x_2^2 \end{pmatrix},\end{aligned}$$

while the building blocks of the corresponding ResNet have the following expression:

$$\Phi_k(x) = x + h \left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + e^{-\frac{1}{2\nu}|x|^2} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} + \begin{pmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right. \tag{4.7.5}$$

$$\left. + e^{-\frac{1}{2\nu}|x|^2} \begin{pmatrix} u_1^{1,1} x_1^2 + u_1^{1,2} x_1 x_2 + u_1^{2,2} x_2^2 \\ u_2^{1,1} x_1^2 + u_2^{1,2} x_1 x_2 + u_2^{2,2} x_2^2 \end{pmatrix} \right] \tag{4.7.6}$$

for $k = 1, \ldots, N$, where $h = \frac{1}{N}$ is the discretization step-size and N is the number of layers of the ResNet. We observe that each building block has 14 parameters.

As before, we set $\nu = 20$, $\max_{iter} = 500$ and we consider $h = 2^{-4}$, resulting in a ResNet with 16 layers and with total number of weights equal to 224. We use the same training data-set as above, namely the grid of points and the corresponding image trough Ψ depicted in Figure 1. We trained the network using both Algorithm 1 and Algorithm 2. The results are collected in Table 5 and Table 6,

β	L_{Φ_u}	Training error	Testing error
10^0	10.14	2.3791	2.3036
10^{-1}	13.84	0.1809	0.2314
10^{-2}	15.64	0.1290	0.1784
10^{-3}	15.83	0.1254	0.1747
10^{-4}	15.86	0.1257	0.1751

TABLE 5. ResNet 4.7.5-4.7.6, 16 layers, 224 parameters, Algorithm 1. Running time $\sim$ 320 s.

β	L_{Φ_u}	Training error	Testing error
10^0	10.78	2.3638	2.3910
10^{-1}	14.32	0.1921	0.2422
10^{-2}	15.43	0.1887	0.2347
10^{-3}	15.56	0.2260	0.2719
10^{-4}	15.59	0.2127	0.2564

TABLE 6. ResNet 4.7.5-4.7.6, 16 layers, 224 parameters, Algorithm 2. Running time $\sim$ 310 s.

respectively. Once again, we observe that the Lipschitz constant of the approximating diffeomorphisms grows as β is reduced. In this case, with both algorithms, the training and testing errors are much lower if compared with the best case of the ResNet 4.7.3. We insist on the fact that in the present case the ResNet 4.7.5-4.7.6 has in total 224 parameters divided into 16 layers, and it overperforms the ResNet 4.7.3 with 256 parameters divided into 32 layers. We report in Figure 3 the approximation produced by Algorithm 1 with $\beta = 10^{-3}$. In this case the approximation provided is very satisfactory, and we observe that it is better in the area where more observations are available. Finally, also in this case the estimate on the expected generalization error (4.5.9) provides an upper bound for the testing error, but at the current state it is too coarse to be of practical use.

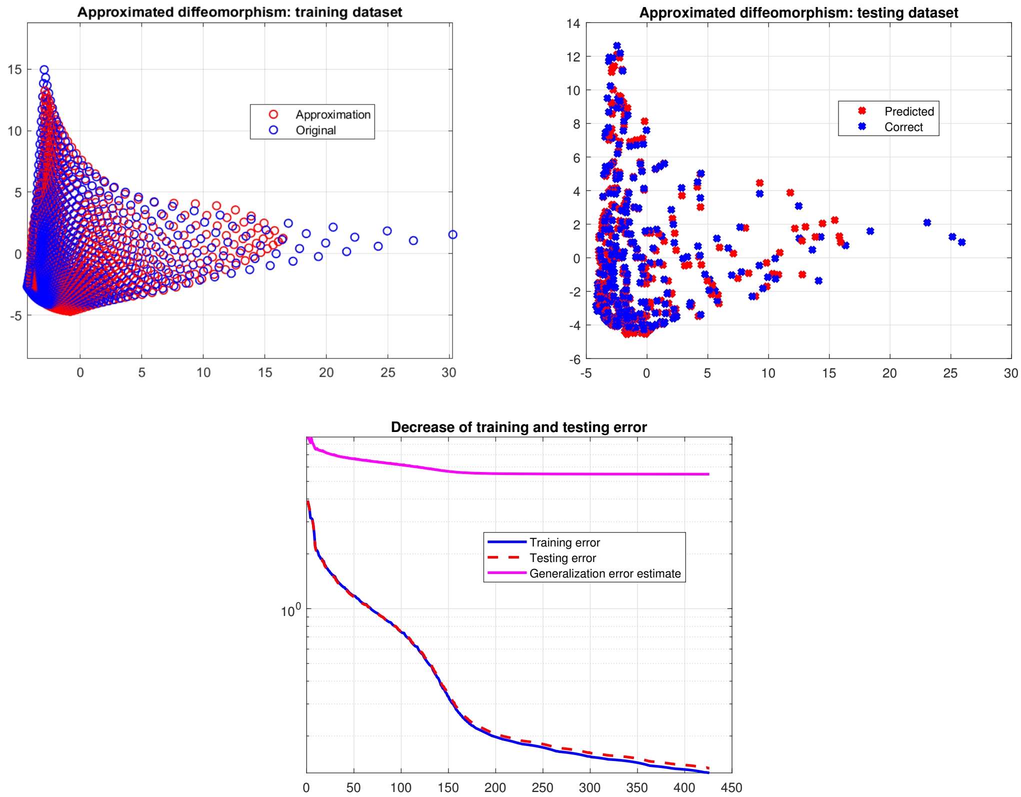

FIGURE 3. ResNet 4.7.5-4.7.6, 16 layers, Algorithm 1, $\beta = 10^{-3}$. On the top-left we reported the transformation of the initial grid through the approximating diffeomorphism (red circles) and through the original one (blue circles). On the top-right, we plotted the prediction on the testing data-set provided by the approximating diffeomorphism (red crosses) and the correct values obtained through the original transformation (blue crosses). In both cases, the approximation obtained is good, and we observe that it is better where we have more data density. At bottom we plotted the decrease of the training error and the testing error versus the number of iterations. Finally, the curve in magenta represents the estimate of the generalization error provided by (4.5.9).

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