ABSTRACT

Title: TRAVELING WAVE SOLUTIONS FOR A COMBUSTION MODEL.

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From the compressible Navier-Stokes equations for a reacting mixture, we reduce the system to obtain a one-dimensional 2-species polytropic gas combustion model. We examine the equilibria and determine their stability as well as identify the conditions that provide for the existence and uniqueness of a traveling wave/shock layer solution.

TRAVELING WAVE SOLUTIONS FOR A COMBUSTION MODEL

By

Brian M. Davis

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Acknowledgements

Having served on active duty in the US Army since receiving my undergraduate degree in mathematics in 1997, I have done little more than basic addition and subtraction. Nevertheless, I was fortunate enough to be selected by the US Army in 2003 to begin graduate school in order to complete a masters of science in applied mathematics – allotted two years from beginning to end. Needless to say, this has been the most difficult and humbling experience of my life and I would not have been able to complete this task without the aid of two people in particular.

Professor Konstantina Trivisa has shouldered more than her share of my questions and shortcomings. I am completely honest in stating that in no way could I have succeeded in this effort without her incessant motivation and positive morale. She is truly dedicated as a professor and unflappable in her support. Thank you.

To my wife, Ronnie Park, I owe much as well. Her continual support and words of encouragement helped … even if I did not say they did. Despite our separation as she waits in New York to deliver our first baby, she continues to assist me while I finish here in Maryland. A true Ranger. Thank you.

Sincerely,

Brian Davis

Table of Contents

List of Figures

Chapter 1: Introduction

Traveling Waves

As we consider the solution of a partial differential equation (PDE) of two variables, $x \in \mathrm{R}$ and $t \in \mathrm{R}$, we define the solution *u* to be a traveling wave solution if it has the special form $u(x,t) = v(x - \sigma t)$, for some function *v*. (In this instance, the solution *v* represents a 'wave' traveling to the right with wave speed σ.) The motivation for seeking traveling wave solutions is the desire to identify at least some special solutions for often complicated sets of equations. The substitution of the relationship between *u* and *v* into the original PDE allows us to reduce the PDE into an ordinary differential equation (ODE) that should be easier to solve. We illustrate the idea in the following simple examples.

Let us examine a nonlinear wave equation in R:

$$u_{tt} - c^2 u_{xx} = u^2 \text{ in } \mathrm{R} \times (0, \infty).$$

Assuming that there exists a traveling wave solution such that $u(x,t) = v(x - \sigma t)$, we substitute accordingly:

$$u_{tt} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} u(x,t)\right] = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} v(x - \sigma t)\right] = \frac{\partial}{\partial t}\left(v'(x - \sigma t) \cdot (-\sigma)\right) = \sigma^2 v^2 (x - \sigma t),$$

$$u_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} u(x,t)\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} v(x - \sigma t)\right] = \frac{\partial}{\partial x} v'(x - \sigma t) = v''(x - \sigma t).$$

These relationships allow us to rewrite the original PDE as an ODE:

$$\sigma^2 v'' - c^2 v'' = v^2.$$

As a final example of traveling wave solutions, we will look to the Korteweg-deVries (KdV) equation:

$$u_t + u_{xxx} + 6uu_x = 0 \text{ in } \mathrm{R} \times (0, \infty).$$

Again using the relation: $u(x,t) = v(x - \sigma t)$, we can transform this intimidating PDE into a manageable ODE. The substitutions yield: $-\sigma v' + v''' + 6vv' = 0$. The solution of which can be shown to be $v(x) = \frac{1}{2}\sigma \operatorname{sech}^2\left[\frac{1}{2}\sqrt{\sigma}\left(x - x_0\right)\right]$ [Strauss].

Traveling Wave Behavior

Following the description outlined by VOLPERT et al, the physical representation of a traveling wave is usually one of a transitional process. In our case this will manifest as a transition from a state of equilibrium prior to burning through a rapid irreversible burning reaction to a final state of equilibrium once burnout (complete combustion) is attained. We refer to **Figure 1** as an example of this process.

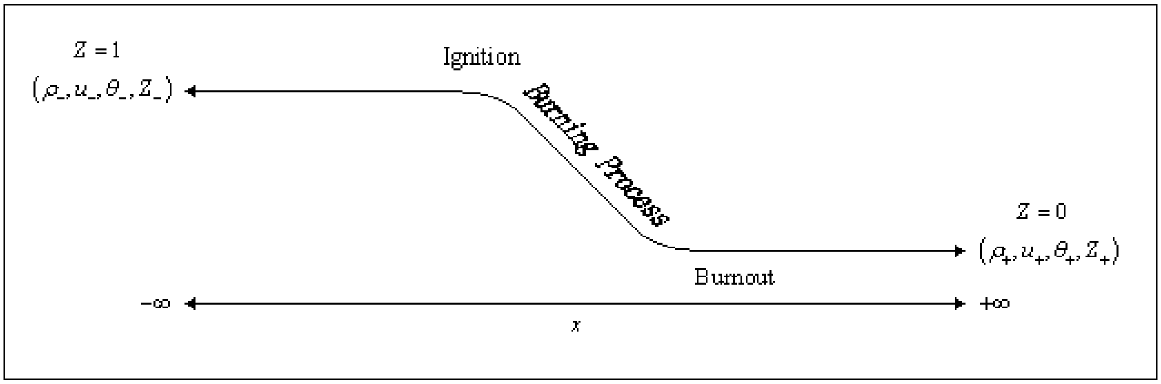

Figure 1. Transitions in Equilibrium.

For completeness we briefly identify the variables listed in Figure 1. We will return to these again once we present our combustion model in greater detail. The four variables listed above are:

$\rho =$ density, $u =$ velocity, $\theta =$ temperature, and $Z =$ remaining fraction of reactant.

As this illustration clearly depicts, the time derivates of these four variables are zero as we approach $\pm\infty$.

Combustion Model

The model described herein follows the presentations by CHEN [1], CHEN, HOFF, & TRIVISA [2] and TRIVISA [3] for the dynamic combustion of compressible reacting fluids (specifically, polytropic gases). The system expresses the conservation of mass, the balance of momentum and energy, and the two-species chemical kinetics as the Navier-Stokes equations for one-dimensional compressible reacting gases (in Euler coordinates):

$$\rho_t + (\rho u)_x = 0, \qquad \textit{mass}$$

$$(\rho u)_t + (\rho u^2 + p)_x = \mu u_{xx}, \qquad \textit{momentum}$$

$$(\rho E)_t + (u(\rho E + p))_x = \mu (u u_x)_x + \lambda \theta_{xx} + (q d \rho Z_x)_x, \qquad \textit{energy}$$

$$(\rho Z)_t + (\rho u Z)_x = (d \rho Z_x)_x - k \varphi(\theta) \rho Z, \qquad \textit{2-species chem. kinetics}$$

where the litany of variables is defined:

$\rho =$ density,

$u =$ velocity,

$R =$ Universal gas constant,

$\theta =$ temperature,

$p =$ pressure (of ideal gas) $= \rho R \theta$,

$\mu, \lambda =$ coefficients of bulk viscosity and heat conduction,

$c_v =$ specific heat at constant volume,

$q =$ heat release parameter (difference in heat between reactant and product),

$Z =$ remaining fraction of reactant (Z=1 $\Rightarrow$ all reactant, Z=0 $\Rightarrow$ all product),

$E =$ total specific energy $= \dfrac{u^2}{2} + c_v \theta + qZ$,

$d =$ species diffusion parameter,

$k =$ reaction rate parameter,

$\varphi(\theta) =$ rate function.

As we peruse the list of variables above, we observe that $\rho, u, \theta,$ and Z are the basic functions of x and t. The rate function, $\varphi(\theta),$ is a Lipschitz function usually determined by the Arrhenius law. In 1889, Svante Arrhenius proposed that an activation energy level was required before chemicals would begin to react. This typically takes the form today as:

$$\varphi(\theta)=\begin{cases} 0, & \theta \leq \theta_{\mathrm{i}} \\ e^{\frac{-A}{\theta}}, & \theta >> \theta_{\mathrm{i}} \end{cases},$$

where θ_i is the ignition temperature and A the activation energy.

For polytropic gases, ideal gases with constant specific heats, the internal energy is related to the pressure in the following manner:

$$e=\frac{p}{\rho(\gamma-1)}=c_v\theta,$$

where $c_v=\dfrac{RM_w}{\gamma-1}>0,$ with c_v the specific heat at constant volume and M_w the molecular weight and γ is the adiabatic constant given by the heat capacity ratio i.e. $\gamma=\dfrac{c_p}{c_v}>1$ where c_p is the specific heat at constant pressure and $c_p=c_v+R$. Again making use of ideal gas laws, we arrive at the speed of sound in an ideal gas, $s=\sqrt{\gamma RT}$. We will use this fact later in our substitutions as we introduce the Mach number of our reaction, M, such that

$M = \frac{u}{\sqrt{\gamma RT}} < 1$. This implies that our reaction is subsonic and consequently that

$$M^2 = \frac{u^2}{\gamma RT} < 1.$$

Combustion Model Derivation

Before proceeding, we briefly develop the first three equations from which we originated our model: continuity, conservation of momentum and energy. The fourth equation, 2-species chemical kinetics, will not be developed other than to note that it resembles the continuity equation with the addition of two terms and basically states that the reactant mass fraction of a fixed fluid particle will decrease according to standard chemical kinetics. The change is due to the diffusion and loss of reactant in the reaction. We present a simplified derivation of the others and attempt to provide a coherent realization of the factors involved in our system.

For the continuity equation's development we focus on the Law of Conservation of Mass. Excluding any nuclear reactions, the mass of the reactants equals the mass of the products. Another way to visualize this concept is to suppose that we have an elemental box within our closed system of dimensions: $\Delta x, \Delta y,$ and Δz. So, the volume (a constant) of this box is $\Delta V = \Delta x \Delta y \Delta z$. Additionally, we know that any change in mass in the box must equal any mass coming into the box minus any mass that exits the box.

$$\frac{\partial}{\partial t}(\text{mass inside box}) = \frac{\partial}{\partial t}(\text{mass entering box}) - \frac{\partial}{\partial t}(\text{mass leaving box}).$$

By first using $\rho = \frac{m}{V}$, where *m* is mass, ρ is density, and *V* is volume; and then using the relationship that $\bar{m} = \rho u$ where *u* is velocity, and where $\bar{m}$ represents the mass flow rate per unit area.

$$\frac{\partial}{\partial t}(\rho V) = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = \frac{\partial}{\partial t}(\bar{m})_{\text{IN}} - \frac{\partial}{\partial t}(\bar{m})_{\text{OUT}} = \frac{\partial}{\partial t}(\rho u)_{\text{IN}} - \frac{\partial}{\partial t}(\rho u)_{\text{OUT}}.$$

We may further expand by separating velocity in its three components and consolidating.

$$\frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z = \Delta y \Delta z\left[(\rho u_x)\big|_x - (\rho u_x)\big|_{x+\Delta x}\right] + \Delta x \Delta z\left[(\rho u_y)\big|_y - (\rho u_y)\big|_{y+\Delta y}\right] + \Delta x \Delta y\left[(\rho u_z)\big|_z - (\rho u_z)\big|_{z+\Delta z}\right].$$

If we now divide by ΔV and run the limit as $\Delta V \to 0$ then we have:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z)\right].$$

Which, in one-dimension, reduces to our continuity equation:

$$\rho_t + (\rho u)_x = 0.$$

For our second equation, conservation of momentum, we look to the Law of Conservation of Momentum. The initial momentum of a system is equal to the final momentum of the system plus any additional momentum added by external forces.

Momentum is determined by multiplying the mass of an object and its velocity i.e. $\bar{\rho} = mu$, where $\bar{\rho}$ is momentum. Using this we may write the change in momentum with respect to time as:

$$\frac{\partial}{\partial t}\bar{\rho} = \frac{\partial}{\partial t}(mu) = \frac{\partial}{\partial t}(\rho V u) = \frac{\partial}{\partial t}(\rho u)V = \frac{\partial}{\partial t}(\rho u)\Delta x \Delta y \Delta z.$$

In this case we will also consider only velocity in the direction of the x-axis (**Figure 2**).

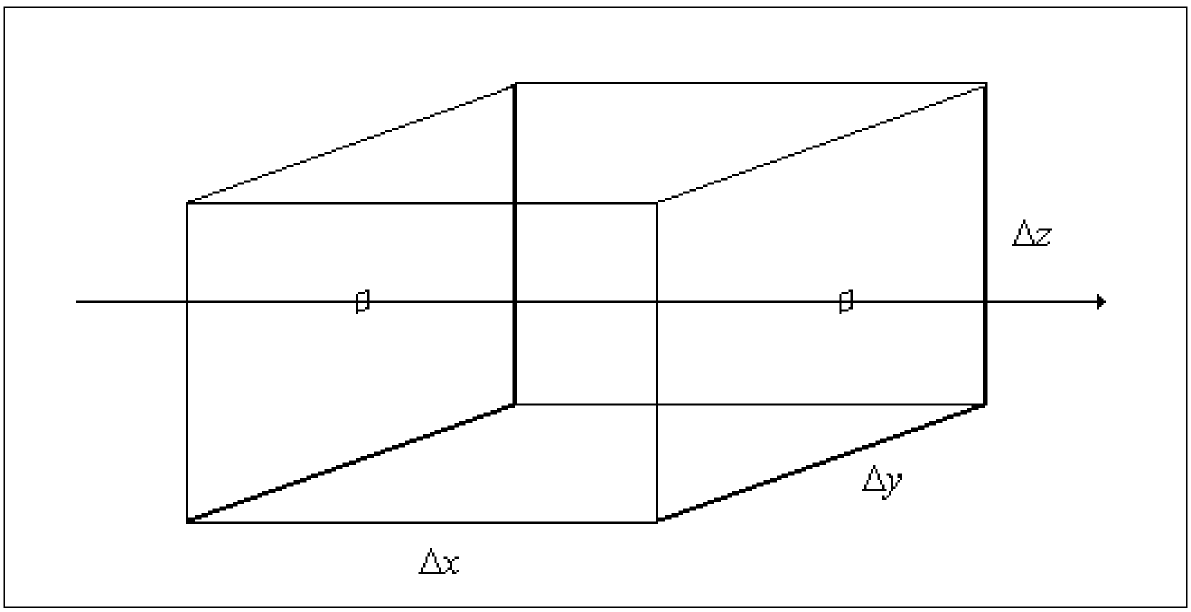

Figure 2. Momentum in one direction.

The rate of flow of the momentum, also known as the flux of the momentum or momentum flux, is the product of the mass flux and velocity. The mass flux can be found using taking $\frac{m}{V}u$ (which gives mass per area per second) and multiplying by the surface area traveled through which yields: $\rho u \Delta y \Delta z$. If we now multiply by velocity again we obtain the momentum flux through the leftmost edge of our box: $\rho u u \Delta y \Delta z$. Similarly, the momentum flux exiting the rightmost edge of our box is equal to the amount coming in the left and any changes that occur inside the box i.e. $\left(\rho uu+\frac{\partial}{\partial x}(\rho uu)\Delta x\right)\Delta y\Delta z.$

Assimilating what we have developed so far we have a rather general equation.

$$\frac{\partial}{\partial t}(\rho u)\Delta x\Delta y\Delta z=\rho uu\Delta y\Delta z-\left(\rho uu+\frac{\partial}{\partial x}(\rho uu)\Delta x\right)\Delta y\Delta z+\sum \text{Forces}_{\text{EXT}}.$$

The external forces in our basic model neglect any body forces imposed by gravity and assume only the surface forces associated with pressure and the viscous stress associated with Newtonian fluids i.e. μu_{xx} where μ is the viscosity. In other words, our external forces are $\left(\frac{\partial}{\partial x}(-\text{p})+\mu u_{xx}\right)\Delta x\Delta y\Delta z.$ Through some simplification and rearranging we arrive at the conservation of momentum equation:

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho uu)+\frac{\partial}{\partial x}\text{p}=\mu u_{xx}.$$

This equation is equivalent to the equation previously listed: $(\rho u)_t+(\rho u^2+p)_x=\mu u_{xx}$.

Switching over more or less to the First Law of Thermodynamics, we find that the change in the amount of energy in our box is equal to the amount of heat entering it plus the net sum of the amount of work (negative if being done therein and positive if being done upon the box). Usually, when dealing with a single fluid, the amount of energy contained in this derivation is simply the sum of the internal energy, e, and the kinetic energy per unit mass, $\frac{u^2}{2}$. This allows us to write our total energy per unit volume: $\bar{E}=\frac{u^2}{2}+e.$

Subsequently, we arrive at the change of our total energy per unit volume in the box:

$\frac{\partial}{\partial t}\left(\rho \bar{E}\right).$

As we determined for momentum above, so should we conclude that the total transfer of energy per unit volume through our box is given by $\frac{\partial}{\partial x}\left(\rho u \bar{E}\right)$ where we replaced an occurrence of velocity with energy. Here at least we have identified the total change in energy for which our equation should account i.e. $\Delta \text{ Energy} = \frac{\partial}{\partial t}\left(\rho \bar{E}\right) + \frac{\partial}{\partial x}\left(\rho u \bar{E}\right).$ However, there remain other factors, as with momentum that affect our derivation.

These factors typically include the work of gravitational forces, heat transferred into the box, and the work of surface stresses. We will continue to neglect the forces associated with gravity. Utilizing Fourier's Law of Heat Conduction that relates heat flow and temperature to the coefficient of heat conduction in a steady-state we find a contribution of $\lambda \theta_{xx}$ where λ is the coefficient of heat conduction and θ the temperature. Additionally, we again assume that pressure and viscosity are our only surface forces i.e. $\left(\frac{\partial}{\partial x}\left(-\mathrm{p}u\right) + \mu\left(uu_x\right)_x\right)$, where we have an additional velocity term than in the momentum since we are calculating the rate of work.

Splicing these terms together we arrive at a standardized conservation of energy formula for a single fluid:

$$\frac{\partial}{\partial t}\left(\rho \bar{E}\right) + \frac{\partial}{\partial x}\left(\rho u \bar{E}\right) = \frac{\partial}{\partial x}\left(-\mathrm{p}u\right) + \mu\left(uu_x\right)_x + \lambda \theta_{xx}.$$

Notwithstanding, this equation may be slightly prettied-up to resemble:

$$\left(\rho\bar{E}\right)_t + \left(u\left(\rho\bar{E} + \mathrm{p}\right)\right)_x = \mu\left(uu_x\right)_x + \lambda\theta_{xx}.$$

However, we are considering two fluids in this model and this is accounted for by adding an additional term to account for the energy provided when the reactant is converted into product.

$$\sum_j\left[\left(\rho_j\bar{E}_j\right)_t + \left(u\left(\rho_j\bar{E}_j + p_j\right)\right)_x\right] = \sum_j\left[\mu_j\left(uu_x\right)_x + \lambda_j\theta_{xx} + (-1)^j q_j k\varphi\left(\theta\right)\rho Z\right]$$

$$j = \begin{Bmatrix} 1, & \text{reactant} \\ 2, & \text{product} \end{Bmatrix}$$

In combustion reactions of two species we append the term on the far right to account for the differences in energy from the burning of the reactant and production of the product. As we sum over the species we arrive at:

$$\left(\rho\bar{E}\right)_t + \left(u\left(\rho\bar{E} + p\right)\right)_x = \mu\left(uu_x\right)_x + \lambda\theta_{xx} + qk\varphi\left(\theta\right)\rho Z$$

From a slight manipulation of the 2-species chemical kinetics equation we find that

$$qk\varphi\left(\theta\right)\rho Z = \left(qd\,\rho Z_x\right)_x - \left(q\rho Z\right)_t - \left(q\rho uZ\right)_x$$

So, if we adjust our definition of the total energy to become $E = \frac{u^2}{2} + c_v\theta + qZ$ ($e = c_v\theta$ for a polytropic gas), we account for the two additional terms at the far right of the above equation and explain the inclusion of $\left(qd\rho Z_x\right)_x$ in our conservation of energy equation.

Chapter 2: System Reduction

Initial Assumptions

As we begin to grapple with this cumbersome system of four equations, we attempt to reduce the variables with a few initial assumptions. Previously we stated that we assume:

$$E = \frac{u^2}{2} + c_v\theta + qZ,$$

$$p = \rho R\theta.$$

Using these relations we can eliminate the variables *E* and *p* from the system. Our system is also invariant under the transformation:

$$\begin{aligned} \bar{x} &= x - \sigma t, \\ \bar{t} &= t, \\ \bar{u} &= u - \sigma. \end{aligned}$$

We can therefore without a loss of generality concentrate on traveling waves with speed $\sigma = 0$. In some sense, this is equivalent to searching for steady-state solutions, i.e. $(\cdot)_t = 0$.

Consolidating these initial results yields the following system:

$$(\rho u)_x = 0,$$

$$\left(\rho u^2 + \rho R\theta\right)_x = \mu u_{xx},$$

$$\left(\rho u\left(\frac{u^2}{2}+c_v\theta+qZ+R\theta\right)\right)_x=\mu\left(uu_x\right)_x+\lambda\theta_{xx}+\left(qd\rho Z_x\right)_x,$$

$$\left(\rho uZ\right)_x=\left(d\rho Z_x\right)_x-k\varphi(\theta)\rho Z.$$

We also reiterate earlier assumptions that we will substitute in the next section:

$$\gamma=\frac{c_p}{c_v}>1,\ c_p=c_v+R,\ \text{and}\ M^2=\frac{u^2}{\gamma RT}.$$

Initial Substitutions

From the first equation above we notice that the quantity ρu does not change under spatial translation; therefore, $\rho u=\left(\rho u\right)_-$. If we introduce a new variable m, the mass flow rate, such that $m=\left(\rho u\right)_-$ then we may substitute $\rho=\frac{m}{u}$ into the last three equations. Our modified system of three equations is now:

$$\left(mu+mR\frac{\theta}{u}\right)_x=\mu u_{xx},$$

$$\left(m\left(\frac{u^2}{2}+c_v\theta+qZ+R\theta\right)\right)_x=\mu\left(uu_x\right)_x+\lambda\theta_{xx}+d\left(q\frac{m}{u}Z_x\right)_x,$$

$$\left(mZ\right)_x=d\left(\frac{m}{u}Z_x\right)_x-k\varphi(\theta)\frac{m}{u}Z.$$

From here we see that the last equation above is a second-order equation that cannot be integrated. We let $\bar{Z} = Z - \frac{d}{u} Z_x$, and replace the last equation while adding a new first-order equation:

$$dZ_x = u\left(Z - \bar{Z}\right),$$

$$\bar{Z}_x = -k\varphi\left(\theta\right)\frac{Z}{u}.$$

We are now ready to integrate our first two equations.

$$\int_x^\infty \left(mu + mR\frac{\theta}{u} \right)_x dx = \int_x^\infty \mu u_{xx} dx.$$

$$\int_x^\infty \left[\left(m\left(\frac{u^2}{2} + c_v\theta + qZ + R\theta \right) \right)_x \right] dx = \int_x^\infty \left[\mu\left(uu_x\right)_x + \lambda\theta_{xx} + \left(qd\frac{m}{u}Z_x \right)_x \right] dx.$$

If we let $c_p = c_v + R$, the evaluation yields:

$$m\left(u - u_+\right) + mR\left(\frac{\theta}{u} - \frac{\theta_+}{u_+} \right) = \mu u_x,$$

$$\frac{m}{2}\left(u^2 - u_+^2\right) + mc_p\left(\theta - \theta_+\right) + mq\left(Z - Z_+\right) = \mu uu_x + \lambda\theta_x + qd\left(\frac{m}{u}\right)Z_x.$$

If we now multiply the first evaluation by *u* and substitute it into the second evaluation:

$$\frac{m}{2}\left(u^2-u_+^2\right)+mc_p\left(\theta-\theta_+\right)+mq\left(Z-Z_+\right)=mu\left(u-u_+\right)+mRu\left(\frac{\theta}{u}-\frac{\theta_+}{u_+}\right)+\lambda\theta_x+qd\left(\frac{m}{u}\right)Z_x.$$

Further simplification allows:

$$\lambda\theta_x=mc_p\left(\theta-\theta_+\right)+mq\left(Z-\left(\frac{d}{u}\right)Z_x-Z_+\right)-mRu\left(\frac{\theta}{u}-\frac{\theta_+}{u_+}\right)-\frac{m}{2}\left(u^2-2uu_++u_+^2\right).$$

Using previous assignments, we know that $Z_+=0$ and $\bar{Z}=Z-\frac{d}{u}Z_x$:

$$\lambda\theta_x=mc_p\left(\theta-\theta_+\right)+mq\bar{Z}-mRu\left(\frac{\theta}{u}-\frac{\theta_+}{u_+}\right)-\frac{m}{2}\left(u-u_+\right)^2.$$

Finally, our system of four equations has become:

$$\mu u_x=m\left(u-u_+\right)+mR\left(\frac{\theta}{u}-\frac{\theta_+}{u_+}\right),$$

$$\lambda\theta_x=mc_p\left(\theta-\theta_+\right)+mq\bar{Z}-mRu\left(\frac{\theta}{u}-\frac{\theta_+}{u_+}\right)-\frac{m}{2}\left(u-u_+\right)^2,$$

$$dZ_x=u\left(Z-\bar{Z}\right),$$

$$\bar{Z}_x=-k\varphi\left(\theta\right)\frac{Z}{u}.$$

Nondimensionalize

Using dimensionless forms will make our derivations and results more transparent. If we nondimensionalize following procedures similar to WILLIAMS [9] we utilize:

$$\tilde{u} = \frac{u}{u_+},$$

$$\tilde{\theta} = \frac{\theta}{\theta_+},$$

$$\tilde{x} = \frac{k}{u_+}x,$$

$$\tilde{\varphi}\left(\tilde{\theta}\right) = \varphi\left(\theta_+\tilde{\theta}\right) = \varphi\left(\theta\right),$$

$$\tilde{d} = \frac{k}{u_+}d,$$

$$\tilde{\lambda} = \frac{k}{mc_p u_+}\lambda,$$

$$\tilde{\mu} = \frac{k}{mu_+}\mu,$$

$$\tilde{q} = \frac{k}{c_p\theta_+}q.$$

We then obtain:

$$\tilde{\mu}\tilde{u}_{\tilde{x}} = \tilde{u} - 1 + \frac{R\theta_+}{u_+^2}\left(\frac{\tilde{\theta}}{\tilde{u}} - 1\right),$$

$$\tilde{\lambda}\tilde{\theta}_{\tilde{x}} = \tilde{\theta} - 1 - \frac{R}{c_p}\left(\tilde{\theta} - \tilde{u}\right) + \tilde{q}\overline{Z} - \frac{u_+^2}{2c_p\theta_+}\left(\tilde{u} - 1\right)^2,$$

$$\tilde{d}Z_{\tilde{x}} = \tilde{u}\left(Z - \overline{Z}\right),$$

$$\overline{Z}_{\tilde{x}} = -\frac{Z}{\tilde{u}}\tilde{\varphi}\left(\tilde{\theta}\right).$$

For the next simplification we will let $\gamma = \dfrac{c_p}{c_v} > 1$ and $M^2 = \dfrac{u^2}{\gamma R\theta}$. This substitution generates:

$$\tilde{\mu}\tilde{u}_{\tilde{x}} = \tilde{u} - 1 + \frac{1}{\gamma M_+^2}\left(\frac{\tilde{\theta}}{\tilde{u}} - 1\right),$$

$$\tilde{\lambda}\tilde{\theta}_{\tilde{x}} = \tilde{\theta} - 1 - \frac{\gamma - 1}{\gamma}\left(\tilde{\theta} - \tilde{u}\right) + \tilde{q}\overline{Z} - \frac{\gamma - 1}{2}M_+^2\left(\tilde{u} - 1\right)^2,$$

$$\tilde{d}Z_{\tilde{x}} = \tilde{u}\left(Z - \overline{Z}\right),$$

$$\overline{Z}_{\tilde{x}} = -\frac{Z}{\tilde{u}}\tilde{\varphi}\left(\tilde{\theta}\right).$$

Setting $\tilde{\mu} = \varepsilon\mu$, $\tilde{\lambda} = \varepsilon\lambda$, $\tilde{d} = \varepsilon d$, and dropping the remaining tildes, we have:

$$\varepsilon\mu u_x = u - 1 + \frac{1}{\gamma M_+^2}\left(\frac{\theta}{u} - 1\right),$$

$$\varepsilon\lambda\theta_x = \theta - 1 - \frac{\gamma - 1}{\gamma}\left(\theta - u\right) + q\overline{Z} - \frac{\gamma - 1}{2}M_+^2\left(u - 1\right)^2,$$

$$\varepsilon dZ_x = u\left(Z - \overline{Z}\right),$$

$$\bar{Z}_x = -\frac{Z}{u}\varphi(\theta).$$

Finalized System

Our last substitution is to set $\xi = \varepsilon x$ so that $\frac{\partial}{\partial x} = \varepsilon \frac{\partial}{\partial \xi}$ and let $\left(\dot{\ }\right) = \frac{\partial}{\partial \xi}$:

$$\mu \dot{u} = u - 1 + \frac{1}{\gamma M_+^2}\left(\frac{\theta}{u} - 1\right),$$

$$\lambda \dot{\theta} = \theta - 1 - \frac{\gamma - 1}{\gamma}(\theta - u) + q\bar{Z} - \frac{\gamma - 1}{2} M_+^2 (u-1)^2,$$

$$d\dot{Z} = u\left(Z - \bar{Z}\right),$$

$$\dot{\bar{Z}} = -\varepsilon \frac{Z}{u}\varphi(\theta).$$

Chapter 3: Analysis

Equilibrium

In this analysis we let $\varepsilon = 0$, this allows us to reduce our system significantly and aids in the following development. We may rewrite our system at equilibrium as:

$$\theta = -\gamma M_+^2 u^2 + \left(1 + \gamma M_+^2\right) u,$$

$$\theta = \frac{\gamma(\gamma - 1)}{2} M_+^2 (u-1)^2 - (\gamma - 1)(u - 1) + 1 - \gamma q \bar{Z},$$

$$Z = \bar{Z}.$$

The first equation describes a parabola in $u\theta$-space. This parabola intersects the u-axis at two points: $u = 0$ and $u = 1 + \frac{1}{\gamma M_+^2}$. Additionally, the second equation also describes a parabola in the same space. However, the height of this parabola is a function of Z and its minimum is attained at $u = 1 + \frac{1}{\gamma M_+^2}$. We refer to **Figure 3** for a representation of these two equations (in this graph we utilized $\gamma = 1.5, M_+^2 = 0.3, q = 0.2,$ and $Z = 0$).

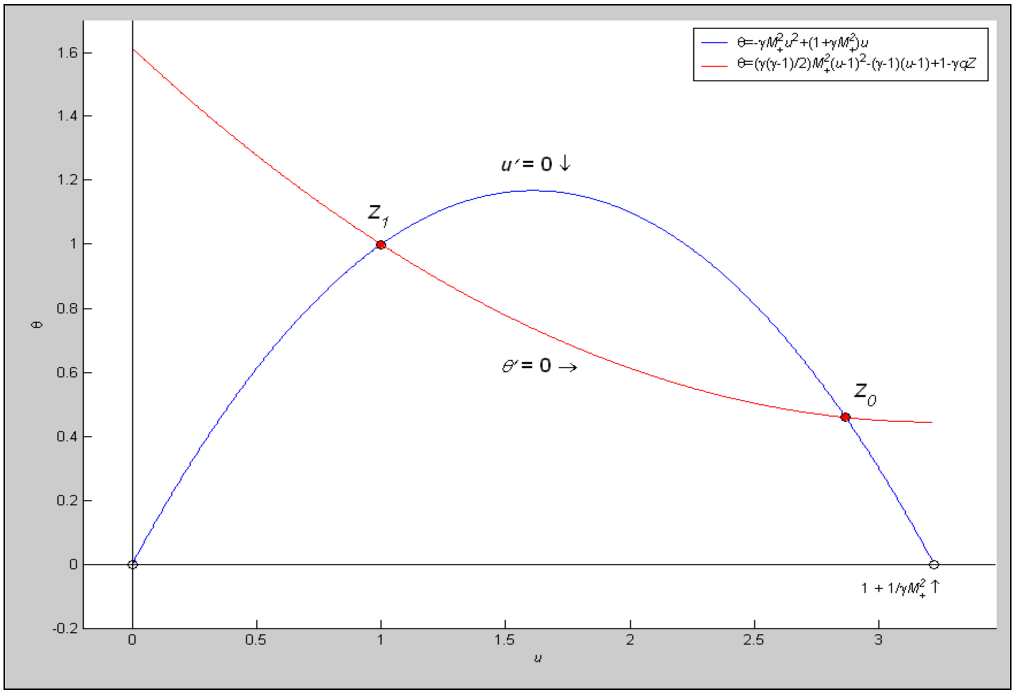

Figure 3. Intersections for Equilibria.

If we were to view these equations as representing a surface in the $u\theta Z$-space we would obtain:

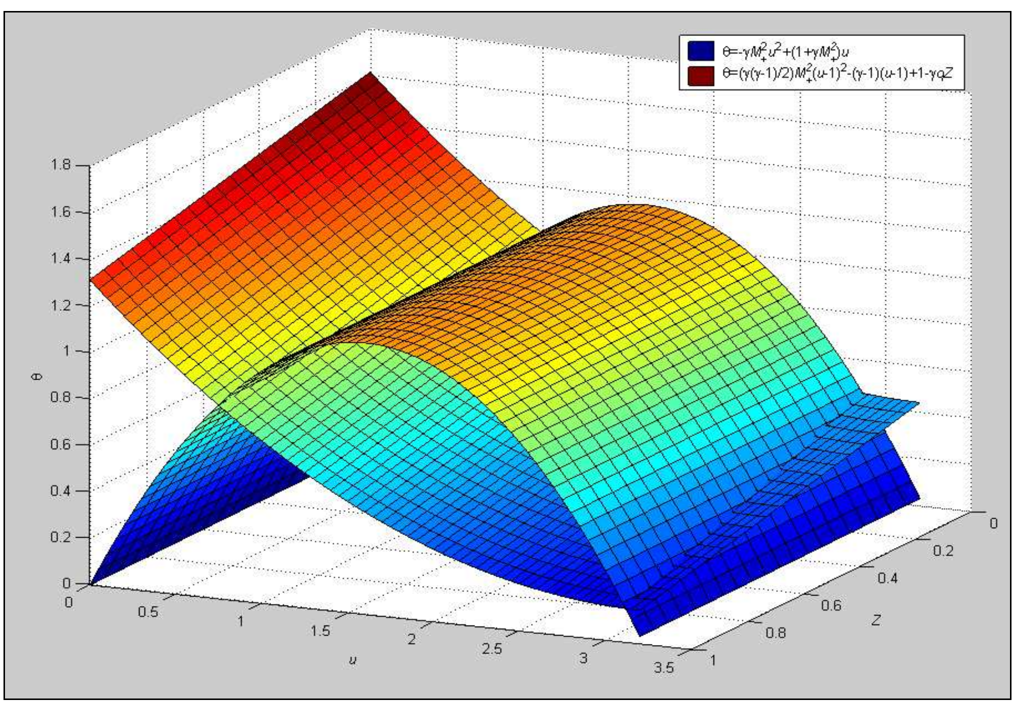

Figure 4. $u\theta Z$-space **Plot at Equilibrium.**

We also note that for each $Z > -\frac{\left(1-M_+^2\right)^2}{\left(\gamma+1\right)M_+^2}$, the two equations' intersection contains two points such that: $u = \frac{1+\gamma M_+^2 \pm \sqrt{\left(M_+^2-1\right)^2 + 2\left(\gamma+1\right)M_+^2 qZ}}{\left(\gamma+1\right)M_+^2}$. In fact when $Z = 0$, we obtain these two points: $u = 1$ and $u = 1 + \frac{2\left(1-M_+^2\right)}{\left(\gamma+1\right)M_+^2}$ (We shall assume the *u*=1 is the smaller root and therefore $M_+^2 < 1$). As is also noticeable from our graphs, we are only interested in the points where $0 \le Z \le 1$ as this is the only range relevant for our combustion model. An overlay of the intersection points generated by the above formula is provided in **Figure 5** below.

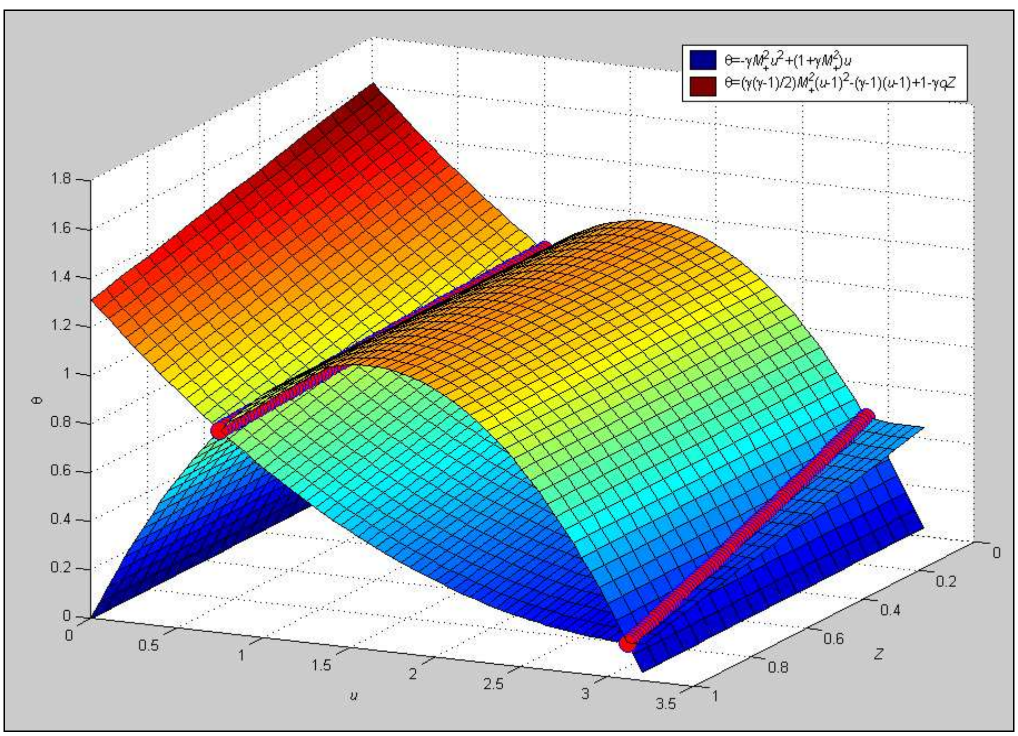

Figure 5. Overlay of Intersection Points.

Phase Portrait

We will now enumerate a set of conditions on our two functions that will assist our determination of the stability of our equilibrium points. For a fixed Z , let:

$$\mu\dot{u} = u - 1 + \frac{1}{\gamma M_+^2}\left(\frac{\theta}{u} - 1\right) \equiv L(u,\theta),$$

$$\lambda\dot{\theta} = \theta - 1 - \frac{\gamma - 1}{\gamma}(\theta - u) + qZ - \frac{\gamma - 1}{2} M_+^2 (u-1)^2 \equiv M(u,\theta).$$

Our conditions are:

(1) There are two curves, L and M , on which $L(u,\theta) = 0$ and $M(u,\theta) = 0$ and two

points, $Z_0 = (u_0, \theta_0)$ and $Z_1 = (u_1, \theta_1)$, where $u_0 > u_1$; such that these points are the only simultaneous solutions of $L(u,\theta) = 0$ and $M(u,\theta) = 0$ (see **Figure 3** and **Figure 5**). We will hitherto call that portion of $L(u,\theta)$ and $M(u,\theta)$ between these points of intersection $\bar{L}(u,\theta)$ and $\bar{M}(u,\theta)$, respectively, and we label the region within the interior of this intersection $\Re$. Additionally, we note that from condition (2) below that $L(u,\theta) < 0$ and $M(u,\theta) > 0$ within the region $\Re$.

(2) $L_\theta > 0$ and $M_\theta > 0$ (Immediate from:

$L_\theta = \dfrac{1}{\gamma M_+^2 u} > 0$ and $M_\theta = 1 - \dfrac{\gamma - 1}{\gamma} > 0$).

(3) $M_u > 0$ (Immediate from: $M_u > 0 \Leftrightarrow \frac{1}{\gamma M_+^2} + 1 > u$; see **Figure 3**). Another consequence of this is that $M\left(u,\theta\right)$ is monotonically decreasing on the interval $u_1 \le u \le u_0$.

(4) $\frac{L_u}{L_\theta} > \frac{M_u}{M_\theta}$ at Z_0 and $\frac{L_u}{L_\theta} < \frac{M_u}{M_\theta}$ at Z_1 (Again, from **Figure 3** and **Figure 5**; L lies above M between their points of intersection).

To help assess the stability of our points of intersection, we examine the characteristic equation of our system at Z_0 and Z_1,

$$\begin{vmatrix} \frac{L_u}{\mu} - \kappa & \frac{L_\theta}{\mu} \\ \frac{M_u}{\lambda} & \frac{M_\theta}{\lambda} - \kappa \end{vmatrix} = 0.$$

We expand this and then manipulate the terms a little to see that:

$$\kappa^2 - \left(\frac{L_u}{\mu} + \frac{M_\theta}{\lambda}\right)\kappa + \left(\frac{L_\theta M_\theta}{\lambda\mu}\right)\left(\frac{L_u}{L_\theta} - \frac{M_u}{M_\theta}\right) = 0.$$

Following further manipulation of the coefficients of our quadratic equation in κ, our discriminant is: $\left(\frac{L_u}{\mu} - \frac{M_\theta}{\lambda}\right)^2 + 4\left(\frac{L_\theta M_u}{\lambda\mu}\right)$. From previous conditions (1) and (2), we know that our discriminant is always positive and therefore the roots of the characteristic equation are real numbers. Lastly, the constant term of our quadratic equation is positive at Z_0 and negative at Z_1. Therefore, we know that Z_0 is a node (unstable) and that Z_1 is a saddle.

These results are readily apparent by referring to the vector fields in **Figure 6** and **Figure 7** below.

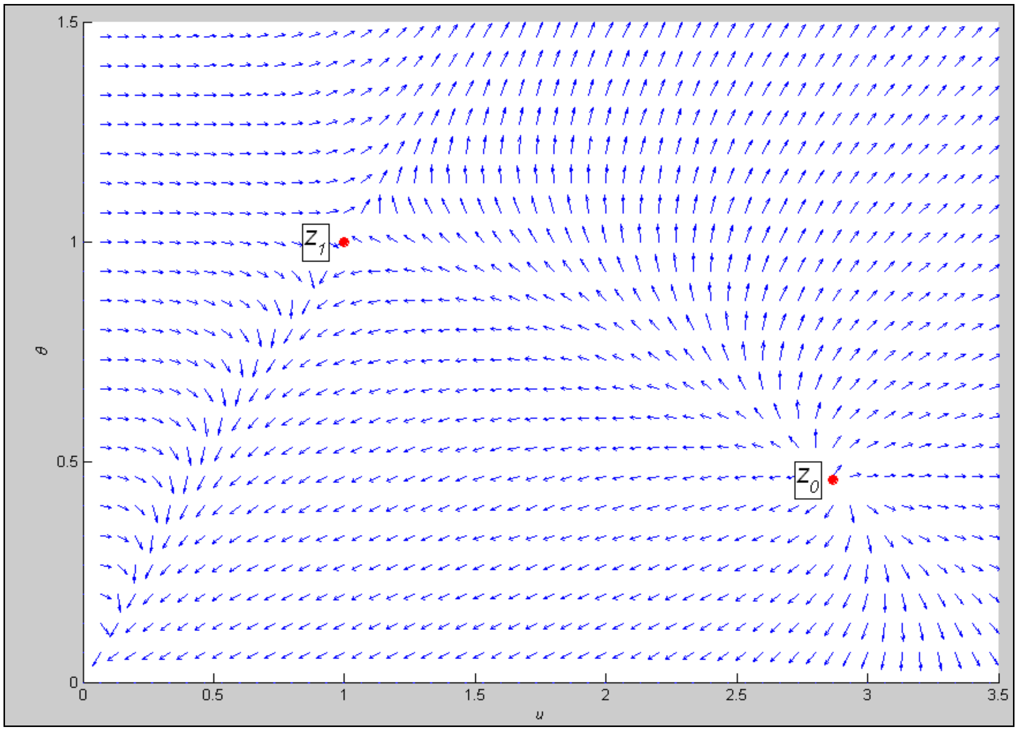

Figure 6. Vector Field and Nodes.

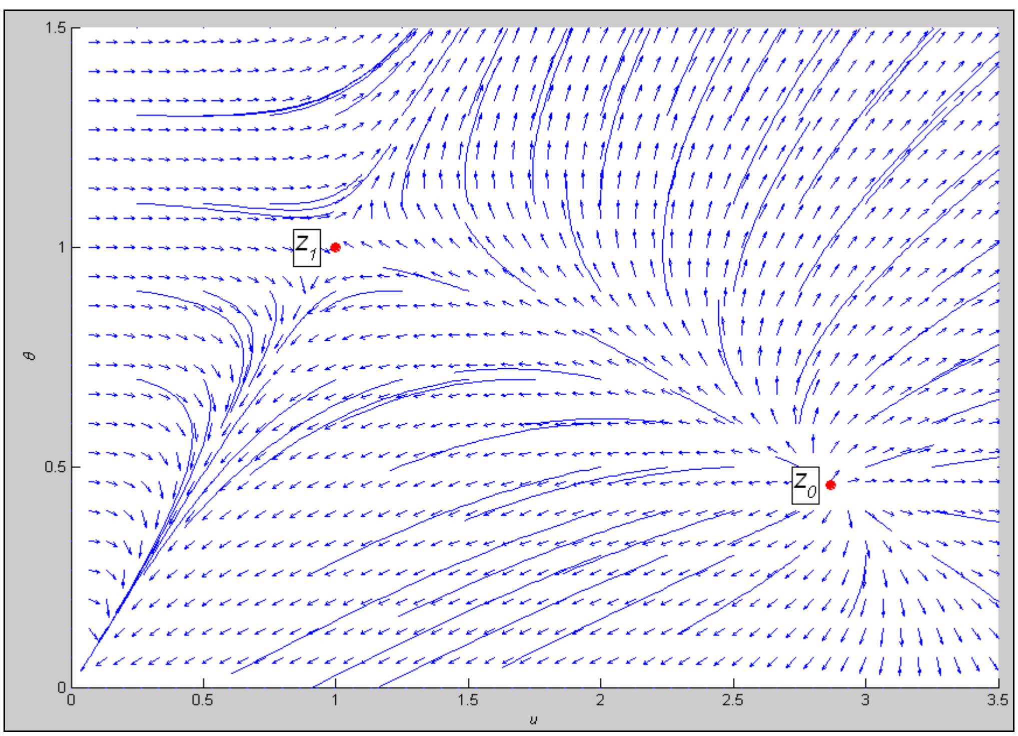

Figure 7. Vector Field with Trajectories and Nodes.

Shock Layer

Following the presentation of GILBARG [5], we define the characteristics of a shock wave with small viscosity and heat conductivity as a shock layer: “such flows display the character of a shock wave … in that they differ sensibly from their end states at $x = \pm\infty$ only in a small interval of rapid transition.” From **Figure 7** we believe it is relatively clear that our points Z_0 and Z_1 depict possible initial and final states, respectively, of a normal shock layer. We will continue with the definitions of GILBARG; a solution, $S(x) = (u(x), \theta(x))$, $(-\infty < x < +\infty)$, of equations $L(u,\theta)$ and $M(u,\theta)$ will be called a shock layer if

$$\lim_{x \to -\infty} S(x) = Z_0 \text{ and } \lim_{x \to +\infty} S(x) = Z_1.$$

As such, the solution that we look to as a traveling wave is equivalent to the shock layer in that we have connected our two equilibrium points through the transition of our combustion process. We present an example of such a solution in **Figure 8** (indeed, this is a numerically approximated shock layer).

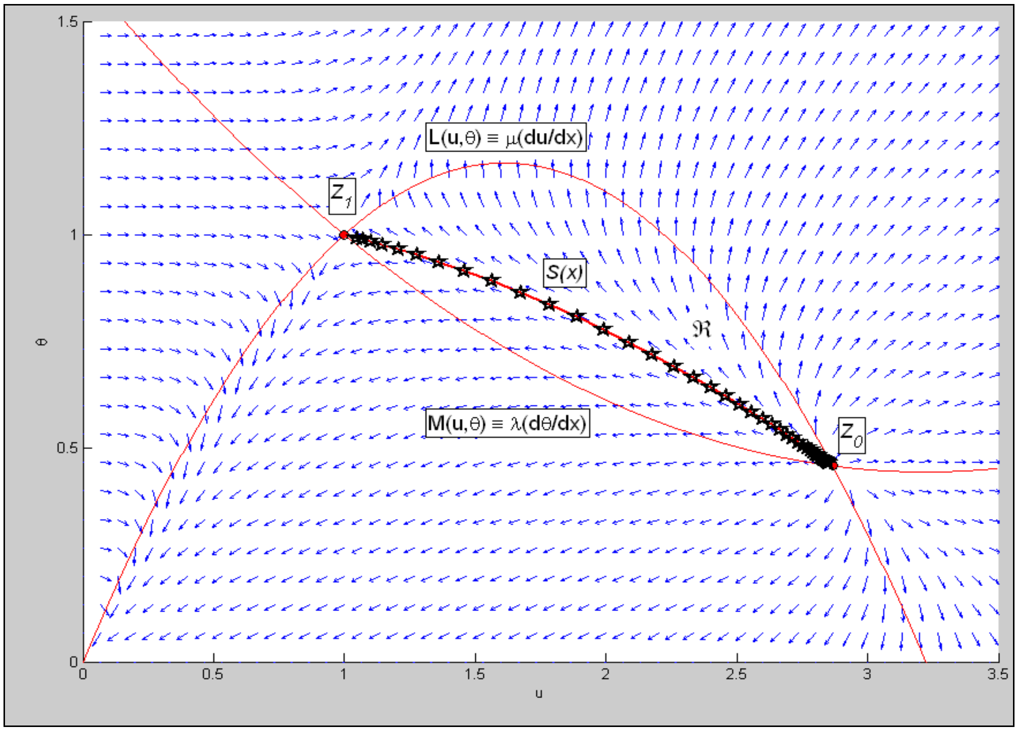

Figure 8. Shock Layer.

To prove that there in fact exists a shock layer we must consolidate a few more details. Since the equilibrium point Z_1 is a saddle point, we know that there are exactly two integral curves of our system that approach it as $x \to +\infty$ and exactly two integral curves that approach it as $x \to -\infty$. Since the two pairs of curves line on the same line, they have the same slope; yet, they approach Z_1 from different directions (see **Figure 9**).

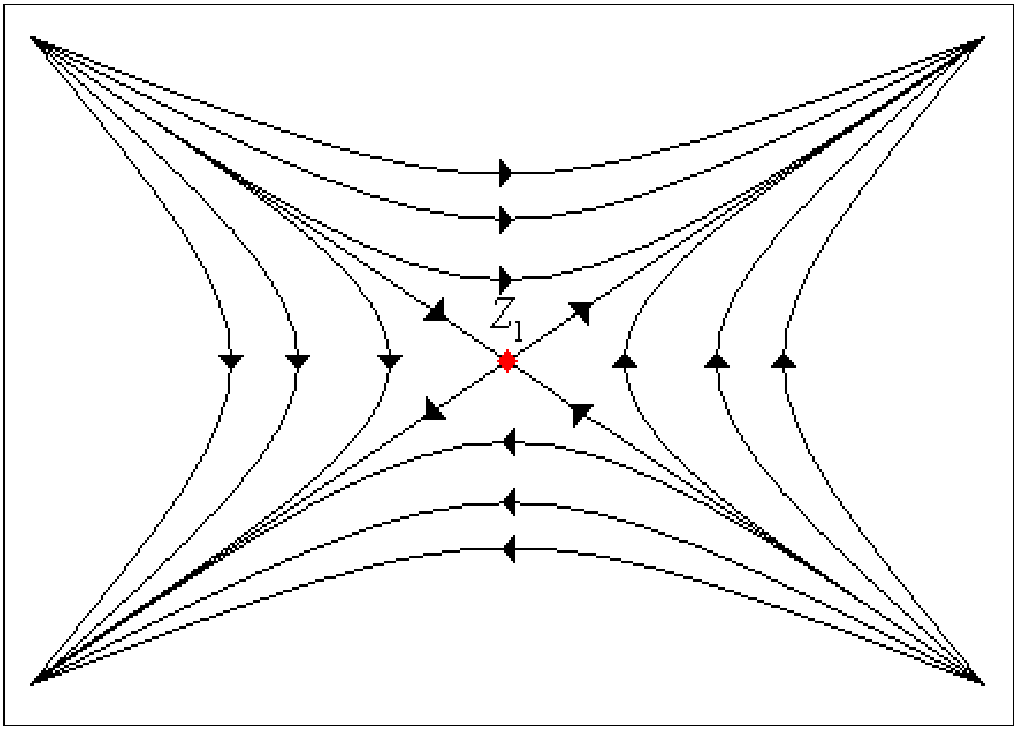

Figure 9. Saddle Point and Integral Curves.

The slopes of the lines are given by the characteristic equation that we previously completed. It may be rewritten as: $-\frac{L_\theta}{L_u - \kappa\mu} = -\frac{M_\theta - \kappa\lambda}{M_u}$. From this we note that when $\kappa < 0$, we obtain a negative slope as well. We also know that within the area between our intersecting curves that the slope of our integral curves, $\frac{M(u,\theta)}{L(u,\theta)}$, is negative around Z_1 and with this we observe that one of our solutions that converges to Z_1 as $x \to +\infty$ must approach from within $\Re$, our region of intersection, where $\frac{M(u,\theta)}{L(u,\theta)} < 0$. This is our solution, $S(x)$.

In order to show that $S(x)$ is a shock layer we begin by observing the tangent vectors of $L(u,\theta)$ and $M(u,\theta)$ as $x \to +\infty$. When on the curve $L(u,\theta) = 0$, we know that $M(u,\theta) > 0$ and it follows that all of our tangent vectors are vertical and pointing out of $\Re$ when on $\bar{L}(u,\theta)$. Similarly, when we are on the curve $M(u,\theta)=0$, then $L(u,\theta)>0$ and our tangent vectors are horizontal and pointing out of $\Re$ when on $\bar{M}(u,\theta)$. These characteristics are presented in **Figure 10**.

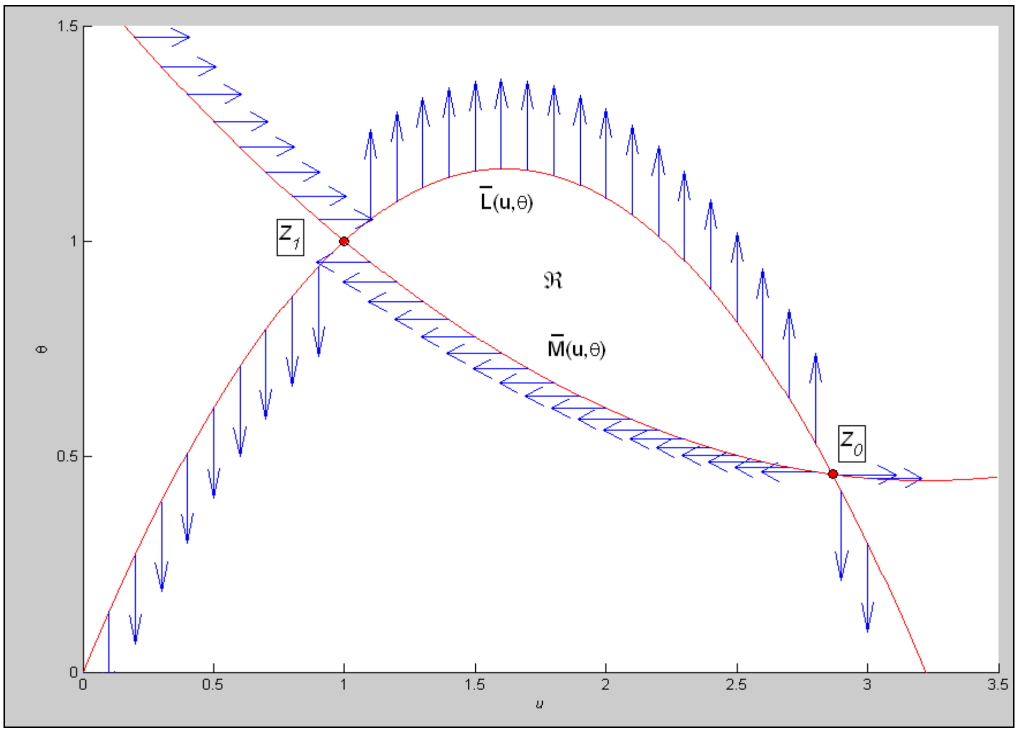

Figure 10. Tangent Vectors to Equilibrium Curves as $x \to +\infty$.

Consequently, as $x \to -\infty$ these flows are reversed and therefore pointing into $\Re$. Therefore, as x decreases our solution, $S(x)$, cannot intersect $\bar{L}(u,\theta)$ or $\bar{M}(u,\theta)$. As there are no other equilibrium points inside of $\Re$, we know that $S(x)$ has no other course than to approach Z_0 as $x \to -\infty$. Consequently, we know that $S(x)$ is a shock layer.

In order to show that this shock layer is unique, we assume that it is not unique. Therefore, there must be a second integral curve, $\bar{S}(x)$, that also joins our points of equilibrium, Z_0 and Z_1. This second solution must also enter Z_0 as $x \to -\infty$ and enter Z_1

as $x \to +\infty$. As we noted previously, there are two integral curves that enter Z_1 as $x \to -\infty$ (**Figure 9**). If we have two solutions that connect Z_0 and Z_1 then one of these integral curves entering Z_1 as $x \to -\infty$ must also cross one of our solutions (see **Figure 11**). Since integral curves are unique they cannot cross. This is our contradiction; the original solution is unique.

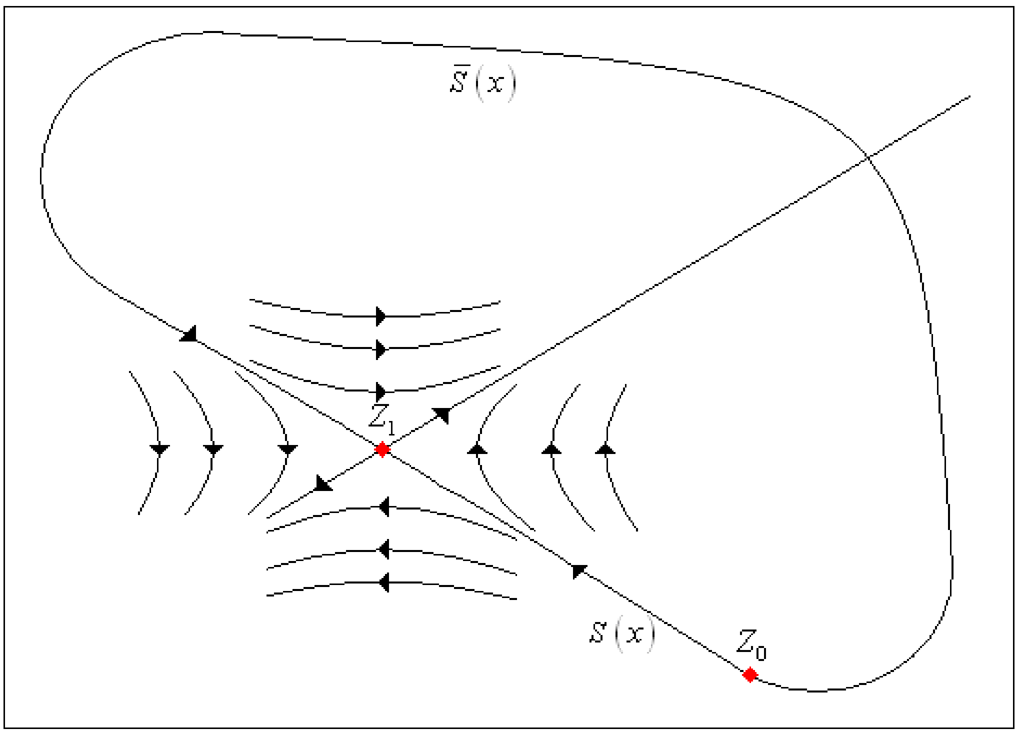

Figure 11. Contradiction from Two Shock Layers.

We now state our only theorem given the conditions cited earlier.

THEOREM. Given $Z_0 = (u_0, \theta_0)$ and $Z_1 = (u_1, \theta_1)$, where $u_0 > u_1$, such that these points are the only simultaneous solutions of $L(u,\theta) = 0$ and $M(u,\theta) = 0$, then there exists a unique shock layer connecting Z_0 to Z_1.

We conclude with, what we may now label, *the* shock layer/traveling wave for our system (**Figure 12**).

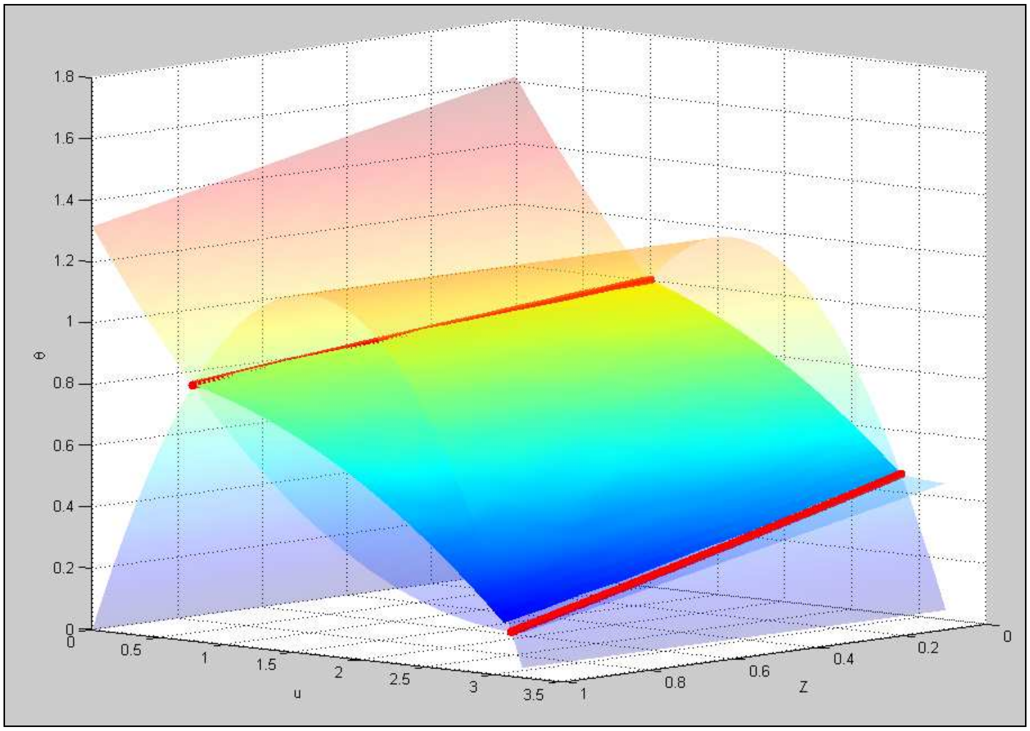

Figure 12. Shock Layer Overlay.

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