ABSTRACT

Title of Dissertation: BORCHERDS FORMS AND GENERALIZATIONS OF SINGULAR MODULI

Jarad John Schofer, Doctor of Philosophy, 2005

Dissertation directed by: Professor Stephen S. Kudla
Department of Mathematics

In the first part of this thesis, we prove an explicit formula for the average of a Borcherds form over CM points associated to a quadratic form of signature $(n, 2)$. One step in the proof extends a theorem of Kudla to the case $n = 0$. The formula we obtain involves the negative Fourier coefficients of a modular form F, and the second terms in the Laurent expansions (at $s = 0$) of the Fourier coefficients of an Eisenstein series of weight one. These Laurent expansion terms were calculated by Kudla, Rapoport and Yang in a special case. We extend their results to a more general case.

In the second part of this thesis, we consider examples of our main theorem for $n = 0$ and $n = 1$ in more detail. When $n = 0$, we let k be an imaginary quadratic field and we obtain a function on the product of the ideal class group of k with the squares of the ideal class group of k. The example for $n = 1$ allows

us to reproduce the well-known singular moduli result of Gross and Zagier. This result gives an explicit factorization of the function $J(D, d)$, defined as a product of $j(z) - j(w)$ over points z and w of discriminant D and d, respectively, where D and d are negative relatively prime fundamental discriminants.

BORCHERDS FORMS AND GENERALIZATIONS OF SINGULAR MODULI

by

Jarad John Schofer

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Advisory Committee:

Professor Stephen S. Kudla, Chairman/Advisor
Professor Jeffrey D. Adams
Professor Niranjan Ramachandran
Professor Lawrence C. Washington
Professor V.S. Subrahmanian, Computer Science

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Chapter 1

Introduction, Notation and Preliminaries

1.1 Introduction

Modular forms of weight 0 for $\Gamma = SL_2(\mathbb{Z})$ are all given as polynomials in $j(\tau)$, the elliptic modular function on the upper half plane $\mathfrak{H}$, whose Fourier expansion at the cusp at ∞ is

$$j(\tau) = \frac{1}{\mathbf{q}} + 744 + 196884\mathbf{q} + \cdots ,$$

where $\mathbf{q} = e^{2\pi i \tau}$. Values of $j(\tau)$ on imaginary quadratic irrationals $\tau \in \mathfrak{H}$ are called *singular moduli* and they are algebraic integers.

Let d_1 and d_2 be two negative fundamental discriminants which are relatively prime. Let w_i denote the number of roots of unity in the imaginary quadratic field of discriminant d_i. Let $[\tau]$ be the equivalence class modulo Γ of $\tau \in \mathfrak{H}$. Define

$$J(d_1, d_2) = \prod_{\substack{[\tau_1],[\tau_2] \\ \mathrm{disc}(\tau_i)=d_i}} \Big(j(\tau_1) - j(\tau_2)\Big)^{\frac{4}{w_1 w_2}}.$$

When $d_1, d_2 < -4$, so that $w_1 w_2 = 4$, this product is the norm of the algebraic integer $j(\tau_1) - j(\tau_2)$ of degree $h_1 h_2$, where h_i is the class number of the order of discriminant d_i. In 1984, Gross and Zagier proved a formula which gives the factorization of $J(d_1, d_2)^2$.

Theorem 1.1 (Gross-Zagier, [7]).

$$J(d_1, d_2)^2 = \pm \prod_{\substack{x,n,n' \in \mathbb{Z} \\ n,n' > 0 \\ x^2 + 4nn' = d_1 d_2}} n^{\epsilon(n')}.$$

The exponent $\epsilon(n')$ is multiplicative and for a prime l, $\epsilon(l)$ is defined via the local Hilbert symbol at l. One example of this theorem is

$$\begin{aligned} J(-67, -163) =& j\left(\frac{1+\sqrt{-67}}{2}\right) - j\left(\frac{1+\sqrt{-163}}{2}\right) \\ =& 2^{15} 3^7 5^3 7^2 (13)(139)(331), \end{aligned}$$

where there is no product since $\mathbb{Q}(\sqrt{-67})$ and $\mathbb{Q}(\sqrt{-163})$ have class number 1. In this thesis, we prove a generalization of Theorem 1.1, which gives a factorization of values of Borcherds forms at CM points on higher dimensional bounded symmetric domains. We work in an adelic setting and recover Theorem 1.1 as a special case.

Let V be a rational vector space with quadratic form Q of signature $(n, 2), n \geq 0$. Let D be the space of oriented negative-definite 2-planes in $V(\mathbb{R})$, and let $H = \mathrm{GSpin}(V)$ be the spinor similitude group of V. We denote the finite adeles of $\mathbb{Q}$ by $\mathbb{A}_f$ and let $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. Associated to $z \in D$, $\tau \in \mathfrak{H}$ and $h \in H$ is a theta function, $\theta(\tau, z, h)$, which is a linear functional on $S(V(\mathbb{A}_f))$, the Schwartz space of $V(\mathbb{A}_f)$. Given a meromorphic modular form $F : \mathfrak{H} \to S(V(\mathbb{A}_f))$ of weight $1 - \frac{n}{2}$ for $Mp_2(\mathbb{Z})$, evaluation of $\theta(\tau, z, h)$ on F gives a Γ-invariant function $\theta(\tau, z, h; F)$

on $\mathfrak{H}$. This function increases rapidly at the cusp, and so is not integrable over $\Gamma\backslash\mathfrak{H}$. However, Borcherds defines a regularized theta integral

$$\Phi(z, h; F) = \int^{\bullet}_{\Gamma\backslash\mathfrak{H}} \theta(\tau, z, h; F) v^{-2} du dv,$$

where $\tau = u + iv$. Then for certain $z \in D$ we have

$$\Phi(z, h; F) = -2 \log ||\Psi(z, h; F)||^2 + C, \tag{1.1}$$

where $\Psi(F)$ is a meromorphic modular form on $D \times H(\mathbb{A}_f)$, $||\ ||$ is the Petersson norm and C is a constant. These functions $\Psi(F)$ are referred to as Borcherds forms.

Let $L \subset V$ be a lattice with dual $L^\vee$ and let $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \subset V(\mathbb{A}_f)$ be its closure in $V(\mathbb{A}_f)$. Assume the meromorphic form F is valued in S_L, the space of functions with support in $\hat{L^\vee}$ and constant on cosets of $\hat{L}$. Then letting φ range over the characteristic functions of cosets of $L^\vee/L$, we can write the Fourier expansion of F as

$$F(\tau) = \sum_{\varphi} \sum_{m} c_\varphi(m) \mathbf{q}^m \varphi.$$

Assuming that $c_\varphi(m) \in \mathbb{Z}$ for $m \leq 0$, Borcherds constructs $\Psi(F)$ of weight $c_0(0)/2$ and explicitly gives its divisor in terms of the $c_\varphi(m)$ for $m < 0$.

To obtain CM points, we take a splitting of our vector space

$$V = V_+ \oplus U$$

into rational subspaces with $\mathrm{sig}(V_+) = (n, 0)$ and $\mathrm{sig}(U) = (0, 2)$. This splitting determines a two-point subset, $D_0 \subset D$, consisting of the rational negative 2-plane $U(\mathbb{R})$ with its two orientations. Let $z_0 \in D_0$. For the introduction, we assume that our lattice splits, i.e., $L = L_+ + L_-$ for $L_+ = V_+ \cap L, L_- = U \cap L$.

Then the Fourier expansion of F can be written in the form

$$F(\tau) = \sum_{\varphi_+,\varphi_-} \sum_m c_{\varphi_+\otimes\varphi_-}(m)\mathbf{q}^m(\varphi_+ \otimes \varphi_-), \tag{1.2}$$

where the sum on $\varphi_\pm$ runs over the coset bases for $L_\pm^\vee/L_\pm$. We can also factor the restriction of the the theta function to the point $z_0 \in D$ as

$$\theta(\tau, z_0, h) = \theta_+(\tau, h_+) \otimes \theta_-(\tau, h_-).$$

The Siegel-Weil formula implies that, for $\tau \in \mathfrak{H}$ and $s \in \mathbb{C}$, there is an Eisenstein series $E(\tau, s; -1)$ of weight -1 such that, for $\varphi_- \in S(U(\mathbb{A}_f))$,

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \theta_-(\tau, h_-)dh_- = E(\tau, 0; \varphi_-, -1).$$

Using Maass operators, $E(\tau, s; \varphi_-, -1)$ can then be related to another Eisenstein series $E(\tau, s; \varphi_-, +1)$, which is “incoherent” in the sense of Kudla, so that $E(\tau, 0; \varphi_-, +1) = 0$. We write

$$E(\tau, s; \varphi_-, +1) = \sum_{m\in\mathbb{Q}} A_{\varphi_-}(s, m, v)\mathbf{q}^m,$$

and

$$A_{\varphi_-}(s, m, v) = b_{\varphi_-}(m, v)s + O(s^2).$$

Then we define

$$\kappa_{\varphi_-}(m) = \begin{cases} \lim_{t\to\infty} b_{\varphi_-}(m, t) & \text{if } m > 0, \\ k_0(0)\varphi_-(0) & \text{if } m = 0, \end{cases}$$

where $k_0(0)$ is a specific constant. Thus, for $m \neq 0$, $\kappa_{\varphi_-}(m)$ is the value at the cusp of the second term in the Laurent expansion of the mth Fourier coefficient of $E(\tau, s; \varphi_-, +1)$. Our main result is the following

Theorem 1.2. *For $F(\tau)$ given by the Fourier expansion (1.2), assume $c_{\varphi_+\otimes\varphi_-}(m) \in \mathbb{Z}$ for $m \le 0$. Let*

$$\kappa_{\varphi_+\otimes\varphi_-}(m) = \sum_{x_1 \in \lambda_{\varphi_+} + L_+} \kappa_{\varphi_-}(m - Q(x_1)),$$

where $\varphi_+ = \mathrm{char}(\lambda_{\varphi_+} + L_+)$. Then for $z_0 \in D_0$,

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh = 2 \sum_{\varphi_+,\varphi_-} \sum_{m \ge 0} c_{\varphi_+\otimes\varphi_-}(-m)\kappa_{\varphi_+\otimes\varphi_-}(m). \quad (1.3)$$

Let $T = \mathrm{GSpin}(U)$ and let $K \subset H(\mathbb{A}_f)$ be a compact open subgroup. Since

$$\mathrm{GSpin}(U) \to SO(U),$$

the above integral can be written as a finite sum over

$$h \in T(\mathbb{Q})\backslash T(\mathbb{A}_f)/(K \cap T(\mathbb{A}_f)). \quad (1.4)$$

Since U is a negative-definite space of signature $(0, 2)$, there is an isomorphism $U \simeq k$ for an imaginary quadratic field k with quadratic form given by a negative multiple of the norm-form. Then the double coset space in (1.4) is essentially an ideal class group. Using (1.1), we see that (1.3) gives a formula for the sum

$$\sum_h \log ||\Psi(z_0, h; F)||^2.$$

To give a geometric interpretation, we consider the quasi-projective variety

$$X_K = H(\mathbb{Q})\backslash\Big(D \times H(\mathbb{A}_f)/K\Big),$$

and, for K large enough, $X_K \simeq \Gamma_K\backslash D^+$ for some group $\Gamma_K \subset H(\mathbb{Q})$. Here, $D^+ \subset D$ is the subset of positively oriented 2-planes. We view the zero cycle

$$T(\mathbb{Q})\backslash\Big(D_0 \times T(\mathbb{A}_f)/(K \cap T(\mathbb{A}_f))\Big) \hookrightarrow X_K$$

as the set $Z(U,K)$ of CM points inside of X_K. Then (1.3) gives the value of $\log ||\Psi(F)^2||$ on $Z(U,K)$. When $U \simeq k$, an imaginary quadratic field with odd discriminant, and $c_0(0) = 0$, the values $\kappa_{\varphi_-}(m)$ for $m \neq 0$ are given as the logarithm of a rational number, which tells us Theorem 1.2 gives a factorization for

$$\prod_{z \in Z(U,K)} ||\Psi(z;F)||^2.$$

The proof of Theorem 1.2 is done in two stages. First, we let $n = 0$ and prove a preliminary version of the theorem. In this case, $V = U$ and $D = D_0$. This is essentially the $n = 0$ version of the main theorem proved by Kudla in [12]. In that paper, this case was not included and some differences do arise. For example, a factor of 2 appears in the Siegel-Weil formula.

The key step in the proof of our main theorem is the *Schwartz space contraction map*. For a factorizable $\varphi = \varphi_+ \otimes \varphi_- \in S(V_+(\mathbb{A}_f)) \otimes S(U(\mathbb{A}_f))$, this is defined as

$$\langle \varphi, \theta_+(\tau, h_+) \rangle_U := \theta_+(\tau, h_+; \varphi_+)\varphi_- \in S(U(\mathbb{A}_f)),$$

and then is extended linearly. We apply the contraction map to the modular form F of weight $1 - \frac{n}{2}$, and obtain a modular form $\langle F, \theta_+ \rangle_U$ of weight 1. Then we can apply the theorem for $n = 0$ to $\langle F, \theta_+ \rangle_U$.

In chapter 4, we give explicit formulas for the values $\kappa_{\varphi_-}(m)$. This is done by viewing $U \simeq k = \mathbb{Q}(\sqrt{-m_0})$ for $m_0 > 0$ and letting $L = \mathfrak{A} \subseteq \mathcal{O}_k, Q(x) = -\frac{Nx}{N\mathfrak{A}}$. We assume $m_0 > 3$ is square-free and $m_0 \equiv 3 \pmod 4$. This extends results of Kudla, Rapoport and Yang in [14], where in that paper $m_0 = q$ is a prime bigger than 3. The reader can compare the positive Fourier coefficients found in Theorem 1 of [14] with Theorem 4.1 of this thesis.

The remainder of the thesis is devoted to looking at explicit examples of the main theorem. For $n = 0$, we obtain input functions, $F(\tau, \mathfrak{A})$, via Hecke's theta functions (cf. [8]) associated to an ideal $\mathfrak{A}$ in an imaginary quadratic field. If I_k is the ideal class group, then the regularized integral $\Phi(z_0, h; F(\tau, \mathfrak{A}))$ can be viewed as a function on $I_k \times I_k^2$, and our theorem computes averages of this function. It is not clear what these functions represent, but they are interesting nonetheless.

The example for $n = 1$ allows us to reproduce Gross-Zagier (Theorem 1.1). In chapter 6, we first look at the general setup and prove many useful facts related to this example. We consider the vector space

$$V = \{x \in M_2(\mathbb{Q}) \mid \operatorname{tr}(x) = 0\}$$

with quadratic form $Q(x) = \det(x)$ of signature $(1, 2)$. For the lattice L we take

$$L = M_2(\mathbb{Z}) \cap V.$$

Using scalar-valued modular forms of weight $\frac{1}{2}$ for $\Gamma_0(4)$, we follow ideas laid out in [1] to obtain appropriate input functions.

For the rational splitting of V, we choose a primitive vector $x_0 \in L^\vee$ such that $Q(x_0) = r$ for some $r > 0$. Then

$$V = \mathbb{Q}x_0 + x_0^\perp.$$

Here we find that the lattice L does not split, and in section 6.2 we compute bases for $L_\pm$ and coset representatives for $L/(L_+ + L_-)$ and $L_\pm^\vee/L_\pm$. Then, in order to interpret Theorem 1.2 in classical language, we describe the double coset space

$$T(\mathbb{Q})\backslash\Big(D_0^+ \times T(\mathbb{A}_f)/(K \cap T(\mathbb{A}_f))\Big)$$

as a certain zero cycle $Z_\mu(r, K) \subset \Gamma\backslash D^+$. We give a formula for these points as elements of $\Gamma\backslash\mathfrak{H} \simeq \Gamma\backslash D^+$.

In the final chapter, we recover the result of Gross and Zagier. Here we briefly sketch the argument. Assume, without loss of generality, that d_1 is odd. Recall that the function we are trying to factor is $J(d_1, d_2)$ from Theorem 1.1. With V, Q and L as above, we choose $x_0 \in L^\vee - L$ such that x_0 is primitive and $Q(x_0) = -\frac{d_1}{4}$. Then we apply the main theorem and get an expression for

$$\sum_h \Phi(z_0, h; F)$$

in terms of the negative Fourier coefficients of F and the values $\kappa_{\varphi_-}(m)$ for $m > 0$ (see chapter 7 for details). Assuming $c_0(0) = 0$ forces the constant in (1.1) to be zero and the Petersson norm becomes the usual absolute value. This gives us a formula for

$$\sum_h \log |\Psi(z_0, h; F)|^2. \tag{1.5}$$

Define

$$J_{d_2}(\tau) = \prod_{\substack{[\tau_2] \\ \mathrm{disc}(\tau_2) = d_2}} j(\tau) - j(\tau_2).$$

Using the explicit divisor of $\Psi(F)^2$ given by Borcherds, we choose the input function F (with $c_0(0) = 0$) so that

$$\mathrm{div}(\Psi(F)^2) = \mathrm{div}(J_{d_2}(\tau)^2).$$

By our choice of x_0, the set of CM points we sum over becomes

$$\{[\tau_1] \in \Gamma \backslash \mathfrak{H} \mid \mathrm{disc}(\tau_1) = d_1\},$$

and (1.5) is $4 \log |J(d_1, d_2)|$. Theorem 4.1, which gives an explicit formula for $\kappa_{\varphi_-}(m)$, implies we have

$$4 \log |J(d_1, d_2)| = \log(r_0) \tag{1.6}$$

for some $r_0 \in \mathbb{Z}_{>0}$. This turns out to be

$$4 \log |J(d_1, d_2)| =$$

$$\sum_{s \in \mathbb{Z}} \left[\sum_{q | d_1} \beta_q(s) \log(q) \rho \left(\frac{d_1 d_2 - s^2}{4} \right) + \sum_{p \text{ inert}} \beta_p(s) \log(p) \rho \left(\frac{d_1 d_2 - s^2}{4p} \right) \right], \quad (1.7)$$

where $\rho(t)$ counts the number of integral ideals in $\mathbb{Q}(\sqrt{d_1})$ of norm t, and $\beta_q(s)$ and $\beta_p(s)$ are some specified integers. The factorization in (1.7) looks very different from the one given by Gross and Zagier. It does, however, resemble the following theorem of Dorman.

Theorem 1.3 (Dorman, [6]). *Let l be a rational prime and e its ramification index in $\mathbb{Q}(\sqrt{d_1})$. Then*

$$\operatorname{ord}_l(J(d_1, d_2)) = \frac{1}{2e} \sum_{s \in \mathbb{Z}} \sum_{n \geq 1} \varrho_l(s) \rho \left(\frac{d_1 d_2 - s^2}{4 l^n} \right),$$

where

$$\varrho_l(s) = \begin{cases} 0 & \text{if } \exists q \mid d_1; q \neq l \text{ such that } \chi_q(s^2 - d_1 d_2) = -1, \\ 2^{a(s)} & \text{otherwise, where } a(s) = \#\{q \mid (s, d_1)\}. \end{cases}$$

Here $\chi_q(\alpha) = (\alpha, d_1)_q$. Dorman's Theorem is equivalent to Gross-Zagier and, to finish the proof, we compare it with (1.7) and see that they agree.

1.2 Notation and Preliminaries

The metaplectic group $Mp_2(\mathbb{R})$ is a double cover of $SL_2(\mathbb{R})$. Elements in this group are given as pairs $(\gamma, \sqrt{c\tau + d})$, where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

Multiplication in $Mp_2(\mathbb{R})$ is defined as follows. Let

$$\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},\ \gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in SL_2(\mathbb{R}),$$

and let $\phi_1(\tau)^2 = c_1\tau + d_1, \phi_2(\tau)^2 = c_2\tau + d_2$. Then the product of (γ_1, ϕ_1) and (γ_2, ϕ_2) is

$$(\gamma_1, \phi_1(\cdot))(\gamma_2, \phi_2(\cdot)) = (\gamma_1\gamma_2, \phi_1(\gamma_2(\cdot))\phi_2(\cdot)).$$

The covering map is $(\gamma, \phi) \mapsto \gamma$ and the inverse image of $SL_2(\mathbb{Z})$ is the metaplectic group $Mp_2(\mathbb{Z})$. This group is generated by the elements

$$T = \left(\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, 1 \right),\ S = \left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \sqrt{\tau} \right),$$

with relations $S^2 = (ST)^3 = Z$ for

$$Z = \left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, i \right).$$

Let $G = SL_2$ and $G'_{\mathbb{A}}$ be the metaplectic cover of $G_{\mathbb{A}}$. We let ω be the Weil representation of $G'_{\mathbb{A}}$ in the Schwartz space $S(V(\mathbb{A}))$. This representation is determined by a fixed additive character ψ of $\mathbb{A}/\mathbb{Q}$ such that $\psi_\infty(x) = e^{2\pi i x}$. Let $L \subset V$ be a lattice, and $S_L \subset S(V(\mathbb{A}_f))$ be the space of functions with basis $\{\mathrm{char}(\lambda + L) \mid \lambda \in L^\vee/L\}$.

We now describe how ω acts as a representation of $Mp_2(\mathbb{Z})$ on S_L and how it is related to the representation ρ_L defined by Borcherds on vector-valued modular forms. Denote the inverse image of $SL_2(\hat{\mathbb{Z}}) \subset G(\mathbb{A}_f)$ by $K' \subset G'_{\mathbb{A}_f}$. Then

$$G'_{\mathbb{A}} = G'_{\mathbb{Q}} G'_{\mathbb{R}} K'.$$

View $\Gamma' = Mp_2(\mathbb{Z})$ as a subset of $G'_{\mathbb{R}}$. If $\gamma' \in G'_{\mathbb{R}}$ has image γ, then under the map

$$G'_{\mathbb{R}} \times G'_{\mathbb{A}_f} \twoheadrightarrow G'_{\mathbb{A}}$$

we have $(\gamma', k',) \mapsto \gamma$ for some element $k' \in G'_{\mathbb{A}_f}$. The kernel of the above map is $\{\pm 1\}$. So once γ' is chosen, this specifies a choice of sign and, hence, specifies k' uniquely. If $\gamma \in \Gamma$, then $\gamma' \in \Gamma'$ and the corresponding element $k'(\gamma') \in K'$. Writing $\omega = \omega_\infty \omega_f$, we define

$$\omega(\gamma') := \omega_f(k'(\gamma')).$$

Now let $\varphi_\mu = \mathrm{char}(\mu + L)$ for a coset $\mu + L$. For the generator $T \in Mp_2(\mathbb{Z})$, the Weil representation acts by

$$\omega(T)(\varphi_\mu)(x) = e(-Q(\mu))\varphi_\mu(x),$$

where $e(y) := e^{2\pi i y}$ and $x \in V(\mathbb{A}_f)$. For S we have

$$\omega(S)(\varphi_\mu)(x) = \frac{\sqrt{i}^{2-n}(-i)^{2-n}}{\sqrt{|L^\vee/L|}} \sum_{\eta \in L^\vee/L} e(-(\mu, \eta))\varphi_\eta(x).$$

In Borcherds' language, $S_L \simeq \mathbb{C}[L^\vee/L]$, the group algebra of $L^\vee/L$. In [2], he defines a representation ρ_L of $Mp_2(\mathbb{Z})$ on $\mathbb{C}[L^\vee/L]$. If we write the elements in the group algebra as $\mathbf{e}_\mu$ for $\mu \in L^\vee/L$ and identify $\mathbf{e}_\mu \leftrightarrow \varphi_\mu$, then the Weil representation defined above agrees with $\rho_L^\vee$, the representation on the dual algebra $\mathbb{C}[L^\vee/L]^\vee$. Borcherds takes the convention that

$$\mathbf{e}_\mu^\vee(\mathbf{e}_\eta) = \begin{cases} 1 & \text{if } \mu + \eta = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We also mention that in the case of n even, the representation ω is actually a representation of $G_{\mathbb{A}}$. When this is the case, we will often write $\omega(g)$ for $g \in G_{\mathbb{A}}$.

Let Q be the quadratic form on U, the space of signature $(0,2)$, and let Δ be the discriminant of Q. Then we may view $U \simeq k = \mathbb{Q}(\sqrt{\Delta})$ and assume $Q(\cdot) = -\frac{N(\cdot)}{|N\mathfrak{A}|}$, where N is the norm on k and $\mathfrak{A}$ is some ideal in k. We will take this point of view when it is convenient. For details on the correspondence between quadratic forms and ideals see [3].

Chapter 2

The Adelic $(0, 2)$-Theorem

2.1 Basic Setup

Let V be a vector space over $\mathbb{Q}$ of dimension $n+2$ with quadratic form Q, of signature $(n, 2)$, on V. Let D be the space of oriented negative-definite 2-planes in $V(\mathbb{R})$. For $z \in D$, let $\mathrm{pr}_z : V(\mathbb{R}) \to z$ be the projection map and, for $x \in V(\mathbb{R})$, let $R(x, z) = -(\mathrm{pr}_z(x), \mathrm{pr}_z(x))$. Then we define

$$(x, x)_z = (x, x) + 2R(x, z),$$

and our Gaussian for V is the function

$$\varphi_\infty(x, z) = e^{-\pi(x,x)_z}.$$

For $\tau \in \mathfrak{H}, \tau = u + iv$, let

$$g_\tau = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & \\ & v^{-\frac{1}{2}} \end{pmatrix},$$

and $g'_\tau = (g_\tau, 1) \in Mp_2(\mathbb{R})$. Let $l = \frac{n}{2} - 1$, $G = SL_2$ and ω be the Weil representation of the metaplectic group $G'_\mathbb{A}$ on $S(V(\mathbb{A}))$, the Schwartz space of

$V(\mathbb{A})$. If $H = \mathrm{GSpin}(V)$, then for the linear action of $H(\mathbb{A}_f)$ we write $\omega(h)\varphi(x) = \varphi(h^{-1}x)$ for $\varphi \in S(V(\mathbb{A}_f))$. If $z \in D$ and $h \in H(\mathbb{A}_f)$, we have the linear functional on $S(V(\mathbb{A}_f))$ given by

$$\varphi \longmapsto \theta(\tau, z, h; \varphi) = v^{-\frac{l}{2}} \sum_{x \in V(\mathbb{Q})} \omega(g'_\tau)(\varphi_\infty(\cdot, z) \otimes \omega(h)\varphi)(x). \tag{2.1}$$

Let $L \subset V$ be a lattice and let $S_L \subset S(V(\mathbb{A}_f))$ be the space of functions with support in $\hat{L}^\vee$ and constant on cosets of $\hat{L}$. Let $F : \mathfrak{H} \to S_L$ be a meromorphic modular form of weight $1 - \frac{n}{2}$ and type ω for $\Gamma' = Mp_2(\mathbb{Z})$. Let $\Gamma = SL_2(\mathbb{Z})$. We consider the $\mathbb{C}$-bilinear pairing

$$((\ F(\tau), \theta(\tau, z, h) \ )) = \theta(\tau, z, h; F(\tau)),$$

and using this pairing we define

$$\Phi(z, h; F) := \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau), \theta(\tau, z, h) \ )) d\mu(\tau),$$

where $d\mu(\tau) = v^{-2} du dv$ and the integral is regularized as in [2]. The regularization is defined by

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \phi(\tau) d\mu(\tau) = \underset{\sigma=0}{\mathrm{CT}} \left\{ \lim_{t\to\infty} \int_{\mathcal{F}_t} \phi(\tau) v^{-\sigma} d\mu(\tau) \right\},$$

where we take the constant term in the Laurent expansion at $\sigma = 0$ of

$$\lim_{t\to\infty} \int_{\mathcal{F}_t} \phi(\tau) v^{-\sigma} d\mu(\tau),$$

defined initially for $\mathrm{Re}(\sigma)$ sufficiently large. Here $\mathcal{F}$ is the usual fundamental domain for the action of Γ on $\mathfrak{H}$ and

$$\mathcal{F}_t = \{\tau \in \mathcal{F} \mid \mathrm{Im}(\tau) \leq t\}$$

is the truncated fundamental domain.

2.2 Borcherds Forms

The space D is a bounded symmetric domain. It can be viewed as an open subset $\mathcal{Q}_-$ of a quadric in $\mathbb{P}(V(\mathbb{C}))$. Explicitly,

$$D \simeq \mathcal{Q}_- = \{w \in V(\mathbb{C}) \mid (w, w) = 0, (w, \bar{w}) < 0\}/\mathbb{C}^\times,$$

where the explicit isomorphism is $[z_1, z_2] \mapsto w = z_1 + iz_2$ for a properly oriented basis $[z_1, z_2]$. Assume K is a compact open subgroup of $H(\mathbb{A}_f)$ such that $H(\mathbb{A}) = H(\mathbb{Q})H(\mathbb{R})^+K$, where $H(\mathbb{R})^+$ is the identity component of $H(\mathbb{R})$. Define

$$X_K := H(\mathbb{Q})\backslash\big(D \times H(\mathbb{A}_f)/K\big).$$

This is the set of complex points of a quasi-projective variety rational over $\mathbb{Q}$, and if $\Gamma_K = H(\mathbb{Q}) \cap H(\mathbb{R})^+K$, then $X_K \simeq \Gamma_K\backslash D^+$, where $D^+ \subset D$ is the subset of positively oriented 2-planes.

Let $\mathcal{L}_D$ be the restriction to $D \simeq \mathcal{Q}_-$ of the tautological line bundle on $\mathbb{P}(V(\mathbb{C}))$. From this we get a holomorphic line bundle $\mathcal{L}$ on X_K equipped with a natural norm, $||\cdot||$, called the Petersson norm. Assume we have

$$V(\mathbb{R}) = V_0 + \mathbb{R}e + \mathbb{R}f,$$

where e and f are such that $(e, f) = 1, (e, e) = 0 = (f, f)$. Then $\text{sig}(V_0) = (n-1, 1)$ and for the negative cone

$$\mathcal{C} = \{y \in V_0 \mid (y, y) < 0\},$$

we have

$$D \simeq \mathbb{D} := \{z \in V_0(\mathbb{C}) \mid y = \text{Im}(z) \in \mathcal{C}\}.$$

The explicit isomorphism is

$$\mathbb{D} \to V(\mathbb{C}),\ z \mapsto w(z) := z + e - Q(z)f$$

composed with projection to $\mathcal{Q}_-$. The map $z \mapsto w(z)$ can be viewed as a holomorphic section of $\mathcal{L}_D$.

We now define the notion of a modular form on $D \times H(\mathbb{A}_f)$.

Definition 2.1. *A modular form on $D \times H(\mathbb{A}_f)$ of weight $m \in \frac{1}{2}\mathbb{Z}$ is a function $\Psi : D \times H(\mathbb{A}_f) \to \mathbb{C}$ such that*

1. *$\Psi(z, hk) = \Psi(z, h)$ for all $k \in K$,*

2. *$\Psi(\gamma z, \gamma h) = j(\gamma, z)^m \Psi(z, h)$ for all $\gamma \in H(\mathbb{Q})$, where $j(\gamma, z)$ is an automorphy factor.*

Meromorphic modular forms on $D \times H(\mathbb{A}_f)$ of weight $m \in \mathbb{Z}$ can be identified with meromorphic sections of $\mathcal{L}^{\otimes m}$. If Ψ is such a meromorphic modular form, then the Petersson norm of the section $(z, h) \mapsto \Psi(z, h)w(z)^{\otimes m}$ associated to Ψ is

$$||\Psi(z, h)||^2 = |\Psi(z, h)|^2 |y|^{2m}.$$

Borcherds proved that the regularized integral $\Phi(z, h; F)$ satisfies the equation

$$\Phi(z, h; F) = -2 \log ||\Psi(z, h; F)||^2 - c_0(0)(\log(2\pi) + \Gamma'(1)) \tag{2.2}$$

for a meromorphic modular form $\Psi(F)$ on $D \times H(\mathbb{A}_f)$ of weight $m = \frac{1}{2}c_0(0)$.

Definition 2.2. *A Borcherds form $\Psi(F)$ is a meromorphic modular form on $D \times H(\mathbb{A}_f)$ which arises (via (2.2)) from the regularized theta lift of a modular form F.*

2.3 CM Points

Assume that we have a rational splitting

$$V = V_+ \oplus U,$$

where V_+ has signature $(n, 0)$ and U has signature $(0, 2)$. This determines a two-point subset $D_0 \subset D$ given by $U(\mathbb{R})$ with its two orientations. For $z_0 \in D_0$, we are interested in computing the integral

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F) dh. \tag{2.3}$$

Let $T = \mathrm{GSpin}(U)$ and let K be as in section 2.2. Define $K_T = K \cap T(\mathbb{A}_f)$. The above integral can be written as a finite sum over $T(\mathbb{Q})\backslash\, T(\mathbb{A}_f)/K_T$, and we consider the set of CM points

$$T(\mathbb{Q})\backslash\Big(D_0 \times T(\mathbb{A}_f)/K_T\Big) \hookrightarrow X_K.$$

Our main theorem gives a formula for (2.3), which then, via (2.2), gives a formula for the average of a Borcherds form over these CM points.

2.4 Some Useful Observations for $n = 0$

First we consider the case when $n = 0$ and our space $V = U$ is negative-definite. In this case, $D = D_0$, the Gaussian is $\varphi_\infty(x) = e^{\pi(x,x)}$ and the theta function is

$$\theta(\tau, z_0, h; \varphi) = v^{\frac{1}{2}} \sum_{x \in U(\mathbb{Q})} \omega(g'_\tau) e^{\pi(x,x)} \varphi(h^{-1}x). \tag{2.4}$$

Let $F(\tau)$ be a meromorphic modular form of weight 1 valued in S_L, and let

$$F(\tau) = \sum_{\varphi} f_\varphi(\tau)\varphi = \sum_{\varphi} \sum_{m \in \mathbb{Q}} c_\varphi(m) \mathbf{q}^m \varphi, \tag{2.5}$$

where φ runs over the characteristic functions of cosets of L in $L^\vee$. We assume $c_\varphi(m) \in \mathbb{Z}$ for $m \leq 0$. The functions f_φ are meromorphic modular forms with some real multiplier for a congruence subgroup of $SL_2(\mathbb{Z})$, and it will be very useful to know how large their Fourier coefficients can be.

Lemma 2.3. *Assume $m_\varphi \in \mathbb{Z}$ is such that $c_\varphi(m_\varphi) \neq 0$ and $c_\varphi(m) = 0$ for all $m < m_\varphi$. Then there are constants C and C' such that, for $m > 0$,*

$$|c_\varphi(m)| \leq C' \left((-m_\varphi + 2)(m - m_\varphi)^6 + m^6 e^{C\sqrt{m}} \right),$$

where C depends on m_φ and on the multiplier and C' depends on the polar part of f_φ.

Proof. The cusp form of weight 12, $(2\pi)^{-12}\Delta(\tau) = \mathbf{q}\prod_{n=1}^{\infty}(1-\mathbf{q}^n)^{24}$, has Fourier expansion

$$(2\pi)^{-12}\Delta(\tau) = \sum_{N=1}^{\infty} \tau(N)\mathbf{q}^N,$$

where $|\tau(N)| \leq C_1 N^6$ for some constant C_1. Let $\tilde{\Delta}(\tau) = (2\pi)^{-12}\Delta(\tau)$. We can look at $f_\varphi/\tilde{\Delta}$, which has weight $-11 = 1 - \frac{24}{2}$. If

$$f_\varphi/\tilde{\Delta} = \sum_{m=m_\varphi-1}^{\infty} a_\varphi(m)\mathbf{q}^m,$$

then for $m > 0$, (3.38) of [12] tells us there are constants C_2 and C such that

$$|a_\varphi(m)| \leq C_2 m^{-\frac{25}{4}} e^{C\sqrt{m}},$$

where C depends on m_φ and on the multiplier. We have

$$\begin{aligned} f_\varphi(\tau) &= \Big(\sum_{N=1}^{\infty} \tau(N)\mathbf{q}^N\Big)\Big(\sum_{m=m_\varphi-1}^{\infty} a_\varphi(m)\mathbf{q}^m\Big) \\ &= \sum_{N=1}^{\infty}\sum_{m=m_\varphi-1}^{\infty} \tau(N)a_\varphi(m)\mathbf{q}^{N+m} \\ &= \sum_{m=m_\varphi}^{\infty}\left[\sum_{N=1}^{m-m_\varphi+1} \tau(N)a_\varphi(m-N)\right]\mathbf{q}^m. \end{aligned}$$

Then

$$\begin{aligned}|c_\varphi(m)| &= \left|\sum_{N=1}^{m-m_\varphi+1} \tau(N)a_\varphi(m-N)\right| \\ &= \left|\sum_{N\geq m} \tau(N)a_\varphi(m-N) + \sum_{0<N<m} \tau(N)a_\varphi(m-N)\right| \\ &\leq C_1 \sum_{N=m}^{m-m_\varphi+1} N^6|a_\varphi(m-N)| + C_1C_2 \sum_{0<N<m} N^6(m-N)^{-\frac{25}{4}}e^{C\sqrt{m-N}}.\end{aligned}$$

We know there is a constant C_3 such that $|a_\varphi(m)| \leq C_3$ for $m \in \{m_\varphi, \ldots, 0\}$, and thus

$$\begin{aligned}|c_\varphi(m)| &\leq C_1C_3(-m_\varphi+2)(m-m_\varphi)^6 + C_1C_2m^6e^{C\sqrt{m}} \\ &\leq C'\left((-m_\varphi+2)(m-m_\varphi)^6 + m^6e^{C\sqrt{m}}\right),\end{aligned}$$

for some constant C'. □

In the $n = 0$ case, it turns out that the regularized integral is always finite.

Proposition 2.4. *For $h \in H(\mathbb{A}_f)$,*

$$\Phi(z_0, h; F) = \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau), \theta(\tau, z_0, h)\ ))d\mu(\tau)$$

is always finite.

Proof. This case corresponds to signature $(2, 0)$ in [2]. In Theorem 6.2 of [2], Borcherds points out that Φ is nonsingular except along a locally finite set of codimension 2 sub-Grassmannians $\lambda^\perp$, for some negative norm vectors $\lambda \in L$. No such vectors exist in signature $(2, 0)$. For ease of the reader, we give the proof in our notation. We have

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau), \theta(\tau, z_0, h)\ ))d\mu(\tau) = \underset{\sigma=0}{\mathrm{CT}}\left\{\lim_{t\to\infty}\int_{\mathcal{F}_t} \theta(\tau, z_0, h; F)v^{-\sigma}d\mu(\tau)\right\}, \quad (2.6)$$

and we can write the integral on the right hand side of (2.6) as

$$\int_{1}^{t}\int_{-\frac{1}{2}}^{\frac{1}{2}} \theta(\tau, z_0, h; F) v^{-\sigma} d\mu(\tau) + \int_{\mathcal{F}_1} \theta(\tau, z_0, h; F) v^{-\sigma} d\mu(\tau).$$

The integral over the compact set $\mathcal{F}_1$ is finite and independent of t, so we just look at the first part. By [15], we have

$$\omega(g'_\tau) e^{\pi(x,x)} = v^{\frac{1}{2}} e(uQ(x)) e^{2\pi v Q(x)},$$

where $e(y) = e^{2\pi i y}$. Then (2.4) is

$$\theta(\tau, z_0, h; \varphi) = v \sum_{x \in U(\mathbb{Q})} e(uQ(x)) e^{2\pi v Q(x)} \varphi(h^{-1}x),$$

and so the integral over $\mathcal{F}_t - \mathcal{F}_1$ is

$$\sum_{\varphi} \sum_{m \in \mathbb{Q}} \sum_{x \in U(\mathbb{Q})} c_\varphi(m) \varphi(h^{-1}x) \int_{1}^{t}\int_{-\frac{1}{2}}^{\frac{1}{2}} e(um) e(uQ(x)) e^{-2\pi v m} e^{2\pi v Q(x)} v^{-\sigma-1} du dv. \tag{2.7}$$

Lemma 2.5. *If $m + Q(x) \notin \mathbb{Z}$, then $c_\varphi(m) = 0$.*

Proof. When we consider the transformation law for F, we have $F(\tau + 1) = \omega(T)(F(\tau))$. That is, for any $x \in U(\mathbb{A}_f)$,

$$\begin{aligned}
\sum_{\varphi} \sum_{m} c_\varphi(m) \mathbf{q}^m e(m) \varphi(x) &= \omega(T) \left(\sum_{\varphi} \sum_{m} c_\varphi(m) \mathbf{q}^m \varphi \right)(x) \\
&= \sum_{\varphi} \sum_{m} c_\varphi(m) \mathbf{q}^m \omega(T)(\varphi)(x) \\
&= \sum_{\varphi} \sum_{m} c_\varphi(m) \mathbf{q}^m e(-Q(x)) \varphi(x).
\end{aligned}$$

We see $m + Q(x) \notin \mathbb{Z}$ implies $c_\varphi(m) = 0$. ∎

For $m + Q(x) \in \mathbb{Z}$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e(um)e(uQ(x))du = \begin{cases} 1 & \text{if } m + Q(x) = 0, \\ 0 & \text{otherwise.} \end{cases}.$$

Integrating with respect to u in (2.7) and letting $t \to \infty$ gives

$$\sum_{\varphi} \sum_{\substack{m \in \mathbb{Q} \\ m \geq 0}} \sum_{\substack{x \in U(\mathbb{Q}) \\ Q(x)+m=0}} c_{\varphi}(m)\varphi(h^{-1}x) \int_{1}^{\infty} e^{-4\pi m v} v^{-\sigma-1} dv. \tag{2.8}$$

We have $m \geq 0$ since $Q(x) \leq 0$. When $m = 0$, we get

$$\sum_{\varphi} c_{\varphi}(0)\varphi(0) \int_{1}^{t} v^{-\sigma-1} dv = c_0(0)\frac{1}{\sigma}(1 - t^{-\sigma}),$$

which equals zero when we take the limit as $t \to \infty$ followed by the constant term at $\sigma = 0$. For $m > 0$, (3.35) of [12] says

$$\int_{0}^{\infty} e^{-4\pi m v} v^{-\sigma-1} dv \leq C(\epsilon, \sigma) e^{-4\pi m}$$

for any ϵ with $0 < \epsilon < 4\pi m$, where the constant $C(\epsilon, \sigma)$ is uniform in any σ-halfplane and independent of m. Using this in (2.8), we have

$$C(\epsilon, \sigma) \sum_{\varphi} \sum_{m > 0} c_{\varphi}(m) e^{-4\pi m} \sum_{\substack{x \in U(\mathbb{Q}) \\ Q(x)+m=0}} \varphi(h^{-1}x),$$

which is finite by Lemma 2.3. □

2.5 Eisenstein Series

Here we give the basic definition of an Eisenstein series and some related theory when V has signature $(n, 2)$ for n even. What follows is a summary of the explanations given in [12] for n even, and we refer the reader to that paper for the more general theory. Inside of $G_{\mathbb{A}}$, we have the subgroups

$$N_{\mathbb{A}} = \{n(b) \mid b \in \mathbb{A}\}, \; n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix},$$

and

$$M_{\mathbb{A}} = \{m(a) \mid a \in \mathbb{A}^{\times}\}, \; m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}.$$

Define the quadratic character $\chi = \chi_V$ of $\mathbb{A}^{\times}/\mathbb{Q}^{\times}$ by

$$\chi(x) = (x, -\det(V)),$$

where $\det(V) \in \mathbb{Q}^{\times}/(\mathbb{Q}^{\times})^2$ is the determinant of the matrix for the quadratic form Q on V. For $s \in \mathbb{C}$, let $I(s, \chi)$ be the principal series representation of $G_{\mathbb{A}}$. This space consists of smooth functions $\Phi(s)$ on $G_{\mathbb{A}}$ such that

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|^{s+1}\Phi(g, s).$$

We have a $G_{\mathbb{A}}$-intertwining map

$$\lambda = \lambda_V : S(V(\mathbb{A})) \to I\left(\frac{n}{2}, \chi\right), \tag{2.9}$$

where $\lambda(\varphi)(g) = (\omega(g)\varphi)(0)$. If $K_{\infty} = SO(2)$ and $K_f = SL_2(\hat{\mathbb{Z}})$, then a section $\Phi(s) \in I(s, \chi)$ is called standard if its restriction to $K_{\infty}K_f$ is independent of s. The function $\lambda(\varphi)$ has a unique extension to a standard section $\Phi(s) \in I(s, \chi)$ such that $\Phi\left(\frac{n}{2}\right) = \lambda(\varphi)$. We let $P = MN$ and define the Eisenstein series associated to $\Phi(s)$ by

$$E(g, s; \Phi) = \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \Phi(\gamma g, s),$$

where $G_{\mathbb{Q}}$ is identified with its image in $G_{\mathbb{A}}$. This series converges for $\mathrm{Re}(s) > 1$ and has a meromorphic analytic continuation to the whole s-plane.

One step in proving the $(0,2)$-Theorem is to apply Maass operators to obtain a relation between two Eisenstein series. Let

$$X_{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}).$$

For $r \in \mathbb{Z}$, let χ_r be the character of K_∞ defined by

$$\chi_r(k_\theta) = e^{ir\theta},\ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_\infty.$$

Let $\phi : G_{\mathbb{R}} \to \mathbb{C}$ be a smooth function of weight l, meaning $\phi(gk_\theta) = \chi_l(k_\theta)\phi(g)$, and let $\xi(\tau) = v^{-\frac{l}{2}}\phi(g_\tau)$ be the corresponding function on $\mathfrak{H}$. Then $X_\pm\phi$ has weight $l \pm 2$, and the corresponding function on $\mathfrak{H}$ is

$$v^{-\frac{l\pm 2}{2}} X_\pm\phi(g_\tau) = \begin{cases} \left(2i\frac{\partial\xi}{\partial\tau} + \frac{l}{v}\xi\right)(\tau) & \text{for } +, \\ -2iv^2\frac{\partial\xi}{\partial\bar\tau}(\tau) & \text{for } -. \end{cases}$$

Lemma 2.6 (Lemma 2.7 of [12]). *Let $\Phi^r_\infty(s) \in I_\infty(s,\chi)$ be the normalized eigenvector of weight r for the action of K_∞. Then*

$$X_\pm\Phi^r_\infty(s) = \frac{1}{2}(s+1\pm r)\Phi^{r\pm 2}_\infty(s).$$

For $\varphi \in S(V(\mathbb{A}_f))$, let $E(g,s;\Phi^r_\infty \otimes \lambda(\varphi))$ be the Eisenstein series of weight r on $G_{\mathbb{A}}$ associated to φ. For the Gaussian, $\varphi_\infty(x,z)$, we have $\lambda(\varphi_\infty) = \Phi^l_\infty\left(\frac{n}{2}\right)$, where $l = \frac{n}{2} - 1$. This means we have

$$X_- E(g,s;\Phi^{l+2}_\infty \otimes \lambda(\varphi)) = \frac{1}{2}(s-l-1)E(g,s;\Phi^l_\infty \otimes \lambda(\varphi)).$$

On $\mathfrak{H}$, this translates to

$$-2iv^2\frac{\partial}{\partial\bar\tau}\Big\{E(\tau,s;\varphi,l+2)\Big\} = \frac{1}{2}\left(s-\frac{n}{2}\right)E(\tau,s;\varphi,l), \tag{2.10}$$

where we write $E(\tau, s; \varphi, l) = v^{-\frac{l}{2}} E(g_\tau, s; \Phi^l_\infty \otimes \lambda(\varphi))$. One main result we need is the Siegel-Weil formula.

Theorem 2.7 (Siegel-Weil formula). *Let V be a vector space of signature $(n, 2)$. Assume V is anisotropic or that $\dim(V) - r_0 > 2$, where r_0 is the Witt index of V. Then $E(g, s; \varphi)$ is holomorphic at $s = \frac{n}{2}$ and*

$$E\left(g, \frac{n}{2}; \varphi\right) = \frac{\alpha}{2} \int_{SO(V)(\mathbb{Q})\backslash SO(V)(\mathbb{A})} \theta(g, h; \varphi) dh,$$

where dh is Tamagawa measure on $SO(V(\mathbb{A}))$, and α is 2 if $n = 0$ and is 1 otherwise.

Here $\theta(g, h; \varphi)$ is defined as in (2.1) without $v^{-\frac{l}{2}}$ and with g replacing g'_τ. The integration for $SO(U)(\mathbb{R})$ is with respect to the action $h_\infty^{-1} x$ in the argument of φ_∞.

Let us now consider the situation $V = U$, $\mathrm{sig}(U) = (0, 2)$. The representation we are interested in is $I(0, \chi)$. This global principal series is a restricted tensor product of local ones,

$$I(0, \chi) = \otimes'_v I_v(0, \chi_v).$$

For the local space $U_v = U(\mathbb{Q}_v)$, define the quadratic character χ_v of $\mathbb{Q}_v^\times$ by

$$\chi_v(x) = (x, -\det(U_v))_v.$$

Let $R_v(U)$ be the maximal quotient of $S(U_v)$ on which $O(U_v)$ acts trivially. The following proposition is a special case of Proposition 1.1 of [11].

Proposition 2.8. *(i) If $v \neq \infty$, then*

$$I_v(0, \chi_v) = R_v(U^+) \oplus R_v(U^-),$$

where $U^{\pm}$ has Hasse invariant $\epsilon_v(U^{\pm}) = \pm 1$.

(ii) If $v = \infty$, then

$$I_\infty(0, \chi_\infty) = R_\infty(U(0,2)) \oplus R_\infty(U(2,0)),$$

and the spaces have opposite Hasse invariants.

Now we define the notion of an incoherent collection.

Definition 2.9. *An incoherent collection $\mathcal{C} = \{\mathcal{C}_v\}$ of quadratic spaces is a set of quadratic spaces $\mathcal{C}_v$ such that*

1. *For all v, $\dim_{\mathbb{Q}_v}(\mathcal{C}_v) = 2$, and $\chi_{\mathcal{C}_v} = \chi$.*

2. *For almost all v, $\mathcal{C}_v \simeq U_v$.*

3. *(Incoherence condition) The product formula fails for the Hasse invariants:*

$$\prod_v \epsilon_v(\mathcal{C}_v) = -1.$$

Then we have, cf. (2.10) in [11],

$$I(0, \chi) \simeq \left(\bigoplus_{U'} \Pi(U') \right) \oplus \left(\bigoplus_{\mathcal{C}} \Pi(\mathcal{C}) \right)$$

as a sum of two irreducible pieces defined as follows. U' runs over all global quadratic spaces of dimension 2 with $\chi_{U'} = \chi$, while $\mathcal{C}$ runs over all incoherent collections of dimension 2 and character χ, and

$$\Pi(U') = \otimes'_v R_v(U'),\ \Pi(\mathcal{C}) = \otimes'_v R_v(\mathcal{C}).$$

For $\lambda = \lambda_U$ as in (2.9), we have $\lambda(\varphi_\infty) = \Phi_\infty^{-1}(0)$, where Φ_∞^{-1} is the normalized eigenvector of weight -1 for the action of K_∞. From the theory of principal series representations, we have $\Phi_\infty^{-1}(0) \in R_\infty(U(0,2))$ and $\Phi_\infty^{1}(0) \in R_\infty(U(2,0))$. Then Lemma 2.6 implies

$$X_+\Phi_\infty^{-1}(s) = \frac{1}{2}s\Phi_\infty^{1}(s),$$

so we see that the Maass operator X_+ shifts the coherent Eisenstein series $E(g,s;\Phi_\infty^{-1}\otimes\lambda(\varphi))$ to the *incoherent* Eisenstein series $E(g,s;\Phi_\infty^{1}\otimes\lambda(\varphi))$. Theorem 2.2 of [11] then tells us that

$$E(g,0;\Phi_\infty^{1}\otimes\lambda(\varphi)) = 0.$$

2.6 The $(0,2)$-Theorem

The integral we want to compute is

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0,h;F)dh, \tag{2.11}$$

which is equal to

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau),\theta(\tau,z_0,h)\ ))d\mu(\tau)dh. \tag{2.12}$$

As in [12], we would like to be able to switch the order of integration, where the inside integral is regularized. That is, we want (2.12) to equal

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau),\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \theta(\tau,z_0,h)dh\ ))d\mu(\tau).$$

Note that $F:\mathfrak{H}\to S_L$ implies $F(\tau)\in S(U(\mathbb{A}_f))^K$, where

$$K = \{h\in H(\mathbb{A}_f) \mid h(\lambda+L) = \lambda+L, \forall\lambda\in L^\vee/L\}$$

is an open subset of $H(\mathbb{A}_f)$. Before we justify the interchange of integrals, we need to make some remarks about our specific case. For a reference on Clifford algebras, see [4] or [9]. The Clifford algebra $C(U)$ can be written as $C(U) = C^0(U)\oplus C^1(U)$, where $C^0(U)$ and $C^1(U)$ are the even and odd parts, respectively. $C^0(U)^\times$ acts on $C^1(U)$ by conjugation. Assume U has basis $\{u, v\}$ with $Q(u) = a, Q(v) = b$ and $(u, v) = 0$. Then $C(U)$ is spanned by $\{1, u, v, uv\}$ with $C^0(U) = \text{span}\{1, uv\}$ and $C^1(U) = \text{span}\{u, v\}$. By definition,

$$H = \{g \in C^0(U)^\times \mid gUg^{-1} = U\}.$$

Since $C^1(U) = U$, $H = C^0(U)^\times$. In $C^0(U)$ we have $(uv)^2 = -ab$, so if $k = \mathbb{Q}\left(\sqrt{-ab}\right)$, then $H \simeq k^\times$. This means $SO(U) \simeq k^1$ and $k^\times \to k^1$ is the map

$$x \mapsto \frac{x}{x^\sigma}$$

by Hilbert's Theorem 90. We have the exact sequence

$$1 \to Z \to H \to SO(U) \to 1,$$

where $H(\mathbb{A}_f) \simeq k^\times_{\mathbb{A}_f}, H(\mathbb{Q}) \simeq k^\times, Z(\mathbb{A}_f) \simeq \mathbb{Q}^\times_{\mathbb{A}_f}$ and $Z(\mathbb{Q}) \simeq \mathbb{Q}^\times$. If $B(h)$ is a function on $H(\mathbb{A}_f)$ which only depends on the image of h in $SO(U)(\mathbb{A}_f)$, then we can view B as a function on $SO(U)(\mathbb{A}_f)$ as well.

Lemma 2.10. *Let $B(h)$ be a function on $H(\mathbb{A}_f)$ depending only on the image of h in $SO(U)(\mathbb{A}_f)$. Assume B is invariant under K and $H(\mathbb{Q})$. Then*

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} B(h)dh = \text{vol}(K) \sum_{h\in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K} B(h),$$

and the sum is finite.

Proof. We have the exact sequence

$$1 \to k^1_{\mathbb{A}} \to k^\times_{\mathbb{A}} \to \mathbb{R}^\times_+ \to 1,$$

where the map to $\mathbb{R}^\times_+$ is the absolute value map. By the product formula, $k^\times \subset k^1_{\mathbb{A}}$ and we know $k^\times\backslash k^1_{\mathbb{A}}$ is compact.

Lemma 2.11. $k_{\mathbb{A}}^{\times} = \mathbb{Q}_{\mathbb{A}}^{\times} k_{\mathbb{A}}^{1}$.

Proof. $\mathbb{Q}_{\mathbb{A}}^{\times}$ injects into $k_{\mathbb{A}}^{\times}$ and also maps onto $\mathbb{R}_{+}^{\times}$. So if $(a) \in k_{\mathbb{A}}^{\times}$ then $\exists (b) \in \mathbb{Q}_{\mathbb{A}}^{\times}$ with $|(b)| = |(a)|$. Then $(b) \in \mathbb{Q}_{\mathbb{A}}^{\times} \subset k_{\mathbb{A}}^{\times}$ implies $k_{\mathbb{A}}^{1}(b) = k_{\mathbb{A}}^{1}(a)$, so $(a) \in \mathbb{Q}_{\mathbb{A}}^{\times} k_{\mathbb{A}}^{1}$. $\square$

Lemma 2.11 implies

$$k^{\times} \backslash k_{\mathbb{A}}^{1} \twoheadrightarrow k^{\times} \mathbb{Q}_{\mathbb{A}}^{\times} \backslash k_{\mathbb{A}}^{\times},$$

and so $k^{\times} \mathbb{Q}_{\mathbb{A}}^{\times} \backslash k_{\mathbb{A}}^{\times}$ is also compact. The set we integrate over is

$$SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A}_f) = H(\mathbb{Q}) \backslash H(\mathbb{A}_f) / Z(\mathbb{A}_f) \simeq k^{\times} \mathbb{Q}_{\mathbb{A}_f}^{\times} \backslash k_{\mathbb{A}_f}^{\times}.$$

This is compact since $k^{\times} \mathbb{Q}_{\mathbb{A}}^{\times} \backslash k_{\mathbb{A}}^{\times}$ maps onto it. Then K is open and $K \supset Z(\mathbb{A}_f)$ so $H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K$ is finite. The volume term appears since B is K-invariant. $\square$

Proposition 2.12.

$$\begin{aligned} &\int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A}_f)} \int_{\Gamma \backslash \mathfrak{H}}^{\bullet} ((\ F(\tau), \theta(\tau, z_0, h)\ )) d\mu(\tau) dh \\ &= \int_{\Gamma \backslash \mathfrak{H}}^{\bullet} ((\ F(\tau), \int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A}_f)} \theta(\tau, z_0, h) dh\ )) d\mu(\tau). \end{aligned}$$

Proof. The main point is that since $F(\tau) \in S(U(\mathbb{A}_f))^K$, we know

$$\int_{\Gamma \backslash \mathfrak{H}}^{\bullet} ((\ F(\tau), \theta(\tau, z_0, h)\ )) d\mu(\tau)$$

is K-invariant. So if we let

$$B(h) = \int_{\Gamma \backslash \mathfrak{H}}^{\bullet} ((\ F(\tau), \theta(\tau, z_0, h)\ )) d\mu(\tau),$$

then Lemma 2.10 says

$$\begin{aligned} \int_{SO(U)(\mathbb{Q}) \backslash SO(U)(\mathbb{A}_f)} B(h) dh &= \mathrm{vol}(K) \sum_{h \in H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K} B(h) \\ &= \int_{\Gamma \backslash \mathfrak{H}}^{\bullet} \mathrm{vol}(K) \sum_{h \in H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K} \theta(\tau, z_0, h; F(\tau)) d\mu(\tau), \end{aligned} \tag{2.13}$$

since the sum is finite. Now apply Lemma 2.10 again to $\theta(\tau, z_0, h; F(\tau))$ and (2.13) is

$$= \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\, F(\tau), \int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \theta(\tau, z_0, h) dh\,)) d\mu(\tau).$$

□

The quadratic space U is anisotropic, so we can apply Theorem 2.7. This tells us that for any $\varphi \in S(U(\mathbb{A}))$,

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A})} \theta(\tau, z_0, h; \varphi) dh = v^{\frac{1}{2}} E(g_\tau, 0; \varphi, -1), \tag{2.14}$$

where $E(g_\tau, s; \varphi, -1)$ is a coherent Eisenstein series of weight -1.

Lemma 2.13.

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \theta(\tau, z_0, h; \varphi) dh = v^{\frac{1}{2}} E(g_\tau, 0; \varphi, -1).$$

Proof. Since the Gaussian is $\varphi_\infty(x) = e^{\pi(x,x)}$, the theta function is invariant under the action of $SO(U)(\mathbb{R})$. We have that $SO(U)(\mathbb{R})$ acts on $SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A})$, so we can project

$$SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}) \to SO(U)(\mathbb{Q})SO(U)(\mathbb{R})\backslash SO(U)(\mathbb{A}),$$

and

$$SO(U)(\mathbb{Q})SO(U)(\mathbb{R})\backslash SO(U)(\mathbb{A}) \simeq SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f), \tag{2.15}$$

since $SO(U)(\mathbb{A}) = SO(U)(\mathbb{R}) \times SO(U)(\mathbb{A}_f)$. For this factorization, we choose a factorization for the measure $dh = dh_\infty \times dh_f$ such that $\mathrm{vol}(SO(U)(\mathbb{R})) = 1$. We have

$$\mathrm{vol}\big(SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A})\big) = 2$$

and so

$$\mathrm{vol}\big(SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)\big) = 2.$$

Then

$$\begin{aligned}
&\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A})} \theta(\tau, z_0, h; \varphi) dh \qquad (2.16)\\
&= \int_{SO(U)(\mathbb{Q})SO(U)(\mathbb{R})\backslash SO(U)(\mathbb{A})} \int_{SO(U)(\mathbb{R})} \theta(\tau, z_0, h_\infty h_f; \varphi) dh_\infty dh_f.
\end{aligned}$$

Using $\mathrm{vol}(SO(U)(\mathbb{R})) = 1$ and the invariance under $SO(U)(\mathbb{R})$ along with (2.15), we see (2.16) equals

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \theta(\tau, z_0, h_f; \varphi) dh_f.$$

Hence, writing dh instead of dh_f, we have

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \theta(\tau, z_0, h; \varphi) dh = v^{\frac{1}{2}} E(g_\tau, 0; \varphi, -1).$$

□

Now we can compute $\mathrm{vol}(K)$.

Lemma 2.14.

$$\mathrm{vol}(K) = \frac{2}{\#(H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K)}.$$

Proof. Using Lemma 2.10 and the volume assumptions made in the proof of Lemma 2.13, we have

$$2 = \int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} dh = \mathrm{vol}(K)(\#(H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K)).$$

□

We let

$$E(\tau, s; -1) := v^{\frac{1}{2}} E(g_\tau, s; -1).$$

Then for (2.11) we have

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh = \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\, F(\tau), E(\tau, 0; -1)\,))d\mu(\tau). \tag{2.17}$$

For F as in (2.5), the right hand side of (2.17) is

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\, F(\tau), E(\tau, 0; -1)\,))d\mu(\tau) = \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \sum_{\varphi} f_\varphi(\tau)E(\tau, 0; \varphi, -1)d\mu(\tau). \tag{2.18}$$

Let

$$I(s,t) := \int_{\mathcal{F}_t} \sum_{\varphi} f_\varphi(\tau)E(\tau, s; \varphi, -1)v^{-2}dudv.$$

In order to state the main theorem of this chapter, we view $U \simeq k = \mathbb{Q}(\sqrt{-d})$, where $d \in \mathbb{Z}_{>0}$ is square-free, and let χ_d be the character of $\mathbb{Q}_{\mathbb{A}}^{\times}$ defined by $\chi_d(x) = (x, -d)_{\mathbb{A}}$. Let Δ be the absolute value of the discriminant of k. We define the normalized L-series

$$\Lambda(s, \chi_d) = \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s, \chi_d).$$

Theorem 2.15 (The $(0,2)$-Theorem). *For $\varphi \in S(U(\mathbb{A}_f))$, let*

$$E(\tau, s; \varphi, +1) = \sum_m A_\varphi(s, m, v)\mathbf{q}^m,$$

where the Fourier coefficients have Laurent expansions

$$A_\varphi(s, m, v) = b_\varphi(m, v)s + O(s^2)$$

at $s = 0$. For any $\varphi \in S(U(\mathbb{A}_f))$, let

$$\kappa_\varphi(m) := \begin{cases} \lim_{t\to\infty} b_\varphi(m, t) & \text{if } m > 0, \\ k_0(0)\varphi(0) & \text{if } m = 0, \end{cases}$$

where

$$k_0(0) = \log(\Delta) + 2\frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)}.$$

Let $F : \mathfrak{H} \to S_L \subset S(U(\mathbb{A}_f))$ be a meromorphic modular form for $SL_2(\mathbb{Z})$ of weight 1, with Fourier expansion

$$F(\tau) = \sum_{\varphi} f_{\varphi}(\tau)\varphi = \sum_{\varphi}\sum_{m} c_{\varphi}(m)\mathbf{q}^m\varphi,$$

where φ runs over the coset basis with respect to some lattice L. Also, assume $c_{\varphi}(m) \in \mathbb{Z}$ for $m \leq 0$. Let

$$\Phi(z_0, h; F) = \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau), \theta(\tau, z_0, h)\ ))d\mu(\tau).$$

Then

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh = 2\sum_{\varphi}\sum_{m\geq 0} c_{\varphi}(-m)\kappa_{\varphi}(m).$$

Proof. Our proof is similar to that in [12]. The integral we want to compute is given by (2.18). Letting $l = -1$ in (2.10), we have

$$E(\tau, s; \varphi, -1)v^{-2} = \frac{-4i}{s}\frac{\partial}{\partial\bar{\tau}}\left\{E(\tau, s; \varphi, +1)\right\}.$$

This means we can write

$$I(s,t) = \frac{1}{2i}\int_{\mathcal{F}_t} d\Big(\sum_{\varphi} f_{\varphi}(\tau)\frac{-4i}{s}E(\tau, s; \varphi, +1)d\tau\Big).$$

By Stokes' Theorem, this is

$$\begin{aligned} &= \frac{-2}{s}\int_{\partial\mathcal{F}_t}\sum_{\varphi} f_{\varphi}(\tau)E(\tau, s; \varphi, +1)d\tau \\ &= \frac{-2}{s}\int_{\frac{1}{2}+it}^{-\frac{1}{2}+it}\sum_{\varphi} f_{\varphi}(\tau)E(\tau, s; \varphi, +1)du \\ &= \frac{2}{s}\cdot \text{const. term of } \Big(\sum_{\varphi} f_{\varphi}(\tau)E(\tau, s; \varphi, +1)\Big)\Big|_{v=t}. \end{aligned} \tag{2.19}$$

The definition of the regularized integral implies

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau), E(\tau,0)\ ))d\mu(\tau) =$$
$$\underset{\sigma=0}{\mathrm{CT}}\left\{ \lim_{t\to\infty} \int_{\mathcal{F}_t} \sum_{\varphi} f_\varphi(\tau) E(\tau,0;\varphi,-1) v^{-\sigma} d\mu(\tau) \right\}.$$

We need Proposition 2.5 of [12] to hold for $n = 0$. If we use Proposition 2.6 of [12] and the fact that a factor of 2 appears in the Siegel-Weil formula here, then in our notation this is

Proposition 2.16.

$$\underset{\sigma=0}{\mathrm{CT}}\left\{ \lim_{t\to\infty} \int_{\mathcal{F}_t} \sum_{\varphi} f_\varphi(\tau) E(\tau,0;\varphi,-1) v^{-\sigma-2} du dv \right\}$$
$$= \lim_{t\to\infty} \left[\int_{\mathcal{F}_t} \sum_{\varphi} f_\varphi(\tau) E(\tau,0;\varphi,-1) v^{-2} du dv - 2c_0(0)\log(t) \right].$$

Proof. From Lemma 2.10, the left hand side of the desired identity is

$$\mathrm{vol}(K) \sum_h \underset{\sigma=0}{\mathrm{CT}} \left\{ \lim_{t\to\infty} \int_{\mathcal{F}_t} ((F(\tau), \theta(\tau, z_0, h)\ )) v^{-\sigma-2} du dv \right\},$$

where $\mathrm{vol}(K) = \frac{2}{\#(H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K)}$. Fixing h, we have

$$\underset{\sigma=0}{\mathrm{CT}} \left\{ \lim_{t\to\infty} \int_{\mathcal{F}_t - \mathcal{F}_1} ((F(\tau), \theta(\tau, z_0, h)\ )) v^{-\sigma-2} du dv \right\} + \int_{\mathcal{F}_1} ((F(\tau), \theta(\tau, z_0, h)\ )) d\mu(\tau). \tag{2.20}$$

The first term in (2.20) can be written as

$$\underset{\sigma=0}{\mathrm{CT}} \left\{ \lim_{t\to\infty} \int_1^t C(v,h) v^{-\sigma-1} dv \right\}, \tag{2.21}$$

where

$$C(v,h) = v^{-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} ((F(\tau), \theta(\tau, z_0, h)\ ))du$$

$$= \text{const. term of } v^{-1}((F(\tau), \theta(\tau, z_0, h)\ ))$$

$$= \sum_{\varphi} \sum_{\substack{m\in\mathbb{Q} \\ m\geq 0}} c_{\varphi}(m) \sum_{\substack{x\in U(\mathbb{Q}) \\ Q(x)+m=0}} \varphi(h^{-1}x)e^{4\pi v Q(x)}.$$

Then we write (2.21) as

$$\underset{\sigma=0}{\text{CT}} \left\{ \lim_{t\to\infty} \int_1^t [C(v,h) - c_0(0)]v^{-\sigma-1}dv + \lim_{t\to\infty} \int_1^t c_0(0)v^{-\sigma-1}dv \right\}. \tag{2.22}$$

As in [12],

$$\int_1^{\infty} [C(v,h) - c_0(0)]v^{-\sigma-1}dv$$

is a holomorphic function of σ. Note, this fact follows, in part, from Lemma 2.3. For the other piece of (2.22) we have

$$\int_1^t c_0(0)v^{-\sigma-1}dv = c_0(0)\frac{1}{\sigma}(1 - t^{-\sigma}).$$

This term makes no contribution when we take the limit as $t \to \infty$ followed by the constant term at $\sigma = 0$. We are left with

$$\lim_{t\to\infty} \left[\int_1^t C(v,h)v^{-1}dv - \int_1^t c_0(0)v^{-1}dv \right] = \lim_{t\to\infty} \left[\int_1^t C(v,h)v^{-1}dv - c_0(0)\log(t) \right].$$

We have the volume term in front and we sum over $h \in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K$, so this adds on a factor of 2. □

We point out that the value $c_0(0)$ appearing in (2.5) and in Proposition 2.16 is independent of the choice of L. If we view $F(\tau) \in S(U(\mathbb{A}_f))$ as $F(\tau, x)$ for

$x \in U(\mathbb{A}_f)$, then $c_0(0)$ is the zeroth Fourier coefficient of $F(\tau, 0)$. Proposition 2.16 tells us

$$\begin{aligned}
&\underset{\sigma=0}{\mathrm{CT}}\left\{\lim_{t\to\infty}\int_{\mathcal{F}_t}\sum_{\varphi} f_\varphi(\tau)E(\tau,0;\varphi,-1)v^{-\sigma}d\mu(\tau)\right\}\\
&=\lim_{t\to\infty}\left[\int_{\mathcal{F}_t}\sum_{\varphi} f_\varphi(\tau)E(\tau,0;\varphi,-1)v^{-2}dudv\ -\ 2c_0(0)\log(t)\right]\\
&=\lim_{t\to\infty}\Big[I(0,t)-2c_0(0)\log(t)\Big].
\end{aligned}$$

We need to compute $I(0,t)$. We have

$$A_\varphi(s,m,v) = b_\varphi(m,v)s + O(s^2), \tag{2.23}$$

where there is no constant term in $A_\varphi(s,m,v)$ since $E(\tau,s;\varphi,+1)$ vanishes at $s=0$. Then (2.19) implies

$$I(s,t) = \frac{2}{s}\sum_{\varphi}\sum_{m} c_\varphi(-m)A_\varphi(s,m,t),$$

so using (2.23) we have

$$I(0,t) = 2\sum_{\varphi}\sum_{m} c_\varphi(-m)b_\varphi(m,t). \tag{2.24}$$

Now we show that parts (i) and (ii) of Proposition 2.11 of [12] hold for $n=0$.

Proposition 2.17. *(i) For $m<0$, $b_\varphi(m,t)$ decays exponentially as $t\to\infty$.*

(ii)

$$\lim_{t\to\infty}\left(2\sum_{\varphi}\sum_{m<0} c_\varphi(-m)b_\varphi(m,t)\right) = 0.$$

Proof. If $\varphi = \otimes_p\varphi_p \in S(U(\mathbb{A}_f))$ and

$$E(\tau,s;\varphi,+1) = \sum_{m} E_m(\tau,s;\varphi,+1),$$

then for $m \neq 0$ we have the product formula

$$E_m(\tau, s; \varphi, +1) = A_\varphi(s, m, v)\mathbf{q}^m = W_{m,\infty}(\tau, s; +1) \prod_p W_{m,p}(s, \varphi_p).$$

Proposition 2.6 (iii) of [14] tells us that for $m < 0$,

$$W_{m,\infty}(\tau, 0; +1) = 0,$$

and

$$W'_{m,\infty}(\tau, 0; +1) = \pi i \mathbf{q}^m \int_1^\infty r^{-1} e^{-4\pi|m|vr} dr.$$

For the finite primes we have

$$C(m) := \left(\prod_p W_{m,p}(s, \varphi_p) \right)\Big|_{s=0} = O(1).$$

Then

$$\begin{aligned} b_\varphi(m, t) &= C(m) W'_{m,\infty}(\tau, 0; +1) \\ &= C(m) \pi i \mathbf{q}^m \int_1^\infty r^{-1} e^{-4\pi|m|vr} dr, \end{aligned}$$

and we have

$$|b_\varphi(m, t)| = O\big(v^{-1}|m|^{-1} e^{-4\pi|m|v}\big).$$

This proves (i). Part (ii) then follows from Lemma 2.3. □

Part (ii) of Proposition 2.17 tells us that we may ignore the sum on $m < 0$ in (2.24). This means our formula for the integral is

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F) dh = \lim_{t\to\infty} \left[2 \sum_\varphi \sum_{m \geq 0} c_\varphi(-m) b_\varphi(m, t) - 2c_0(0) \log(t) \right].$$

We can improve this by looking at the $m = 0$ part. The analogue of Proposition 2.11 (iii) of [12] is

Lemma 2.18. *For $m = 0$,*

$$b_0(0,t) - \log(t) = \log(\Delta) + 2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)},$$

and for $\varphi \neq \varphi_0, b_\varphi(0,t) = 0$.

Proof. By Theorem 3.1 of [16], we have

$$\begin{aligned} E_0(\tau, s; \varphi, +1) &= v^{\frac{s}{2}}\varphi(0) + W_{0,\infty}(\tau, s; +1)\prod_p W_{0,p}(s,\varphi_p) \\ &= v^{\frac{s}{2}}\varphi(0) - 2\pi i\frac{2^{-s}\Gamma(s)v^{-\frac{s}{2}}}{\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}\right)}\prod_p W_{0,p}(s,\varphi_p), \end{aligned}$$

which by the duplication formula is

$$= v^{\frac{s}{2}}\varphi(0) - \sqrt{\pi} i v^{-\frac{s}{2}}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}\prod_p W_{0,p}(s,\varphi_p).$$

Theorem 5.2 of [16] implies $W_{0,p}(s,\varphi_p) = 0$ if φ_p is not the characteristic function of the local lattice. So $b_\varphi(0,t) = 0$ for $\varphi \neq \varphi_0$. Now let $\varphi = \varphi_0$. Propositions 2.1 and 6.3 of [16] imply

$$E_0(\tau, s; \varphi_0, +1) = v^{\frac{s}{2}} - \sqrt{\pi}v^{-\frac{s}{2}}\frac{\Gamma\left(\frac{s+1}{2}\right)L(s,\chi_d)}{\Gamma\left(\frac{s}{2}+1\right)L(s+1,\chi_d)}c_0,$$

where

$$c_0 = 2^{\alpha_2}\prod_{\substack{q|d \\ q=\text{odd prime}}} q^{-\frac{1}{2}}$$

and

$$\alpha_2 = \begin{cases} 0 & \text{if 2 is unramified,} \\ -1 & \text{if } 2 \nmid d \text{ and } d \equiv 1 (\text{mod } 4), \\ -\frac{3}{2} & \text{if } 2 \mid d. \end{cases}$$

Then $c_0 = \Delta^{-\frac{1}{2}}$, where Δ is the absolute value of the discriminant of $\mathbb{Q}(\sqrt{-d})$. We have

$$E_0(\tau, s; \varphi_0, +1) = v^{\frac{s}{2}} - v^{-\frac{s}{2}} \frac{\pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi_d)}{\pi^{-\frac{s}{2}-1} \Gamma\left(\frac{s}{2}+1\right) L(s+1, \chi_d)} \Delta^{-\frac{1}{2}}$$
$$= v^{\frac{s}{2}} - v^{-\frac{s}{2}} \frac{\Lambda(s, \chi_d)}{\Lambda(s+1, \chi_d)} \Delta^{-\frac{1}{2}}.$$

The functional equation for $\Lambda(s, \chi_d)$ (cf. [5]) is

$$\Lambda(s, \chi_d) = \Delta^{\frac{1}{2}-s} \Lambda(1-s, \chi_d).$$

We normalize $E_0(\tau, s; \varphi_0, +1)$ by $\Delta^{\frac{s+1}{2}} \Lambda(s+1, \chi_d)$ giving

$$E_0^*(\tau, s; \varphi_0, +1) = \Delta^{\frac{s+1}{2}} v^{\frac{s}{2}} \Lambda(1+s, \chi_d) - \Delta^{\frac{s+1}{2}} v^{-\frac{s}{2}} \Delta^{-s} \Lambda(1-s, \chi_d)$$
$$= \Delta^{\frac{s+1}{2}} v^{\frac{s}{2}} \Lambda(1+s, \chi_d) - \Delta^{\frac{1-s}{2}} v^{-\frac{s}{2}} \Lambda(1-s, \chi_d).$$

Hence,

$$E_0^{*,\prime}(\tau, 0; \varphi_0, +1) = 2 \frac{\partial}{\partial s} \left\{ \Delta^{\frac{s+1}{2}} v^{\frac{s}{2}} \Lambda(1+s, \chi_d) \right\} \Big|_{s=0}$$
$$= \Delta^{\frac{1}{2}} \Lambda(1, \chi_d) \left\{ \log(\Delta) + \log(v) + 2 \frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)} \right\}$$
$$= h_k \left\{ \log(\Delta) + \log(v) + 2 \frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)} \right\},$$

by the residue formula. Then since $E^{*,\prime}(\tau, 0; \varphi_0, +1) = h_k E'(\tau, 0; \varphi_0, +1)$, we have

$$b_0(0, t) - \log(t) = \log(\Delta) + 2 \frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)}.$$

□

Now the $m = 0$ part is

$$2 \sum_{\varphi} c_\varphi(0) b_\varphi(0, t) - 2c_0(0) \log(t) = 2 \sum_{\varphi \neq \varphi_0} c_\varphi(0) b_\varphi(0, t) + 2c_0(0)(b_0(0, t) - \log(t)),$$

and Lemma 2.18 tells us that this expression is $2c_0(0)k_0(0)$. This finishes the proof of Theorem 2.15. □

Chapter 3

The Adelic $(n, 2)$-Theorem

3.1 The Rational Splitting $V = V_+ \oplus U$

Now we consider the general case. Assume we have the decomposition $V = V_+ \oplus U$ where V_+ has signature $(n, 0)$ and U has signature $(0, 2)$. For $x \in V$, write $x = x_1 + x_2$, $x_1 \in V_+, x_2 \in U$. Let $z_0 \in D_0$. Then $R(x, z_0) = -(x_2, x_2)$ so we see

$$\varphi_\infty(x, z_0) = e^{-\pi(x,x)_{z_0}} = e^{-\pi[(x_1,x_1)-(x_2,x_2)]} = e^{-\pi(x_1,x_1)} e^{\pi(x_2,x_2)},$$

which is equal to $\varphi_{\infty,+}(x_1)\varphi_{\infty,-}(x_2)$ for the Gaussians on V_+ and U, respectively. We also have $\omega(g'_\tau)\varphi_\infty = \omega_+(g'_\tau)\varphi_{\infty,+} \otimes \omega_-(g'_\tau)\varphi_{\infty,-}$ for the corresponding Weil representations. For this decomposition of V, we can write the theta function on $S(V(\mathbb{A}_f))$ as a tensor product of two distributions, one on $S(V_+(\mathbb{A}_f))$ and one on $S(U(\mathbb{A}_f))$. To see this, let $\varphi \in S(V(\mathbb{A}_f))$. The theta functions are linear, so it suffices to look at a factorizable Schwartz function $\varphi = \varphi_+ \otimes \varphi_-$. This gives

$$\begin{aligned}\theta(\tau, z_0, h; \varphi) =& v^{-\frac{l}{2}} \sum_{x \in V(\mathbb{Q})} \omega(g'_\tau)(\varphi_\infty(\cdot, z_0) \otimes \omega(h)\varphi)(x) \\ =& v^{-\frac{l}{2}} \sum_{x_1, x_2} (\omega_+(g'_\tau)\varphi_{\infty,+}(x_1)\varphi_+(h_+^{-1}x_1))(\omega_-(g'_\tau)\varphi_{\infty,-}(x_2)\varphi_-(h_-^{-1}x_2)) \\ =& v^{-\frac{n}{4}} \left(\sum_{x_1} \omega_+(g'_\tau)\varphi_{\infty,+}(x_1)\varphi_+(h_+^{-1}x_1) \right) \times \\ & v^{\frac{1}{2}} \left(\sum_{x_2} \omega_-(g'_\tau)\varphi_{\infty,-}(x_2)\varphi_-(h_-^{-1}x_2) \right) \\ =& \theta_+(\tau, z_0, h_+; \varphi_+)\theta_-(\tau, z_0, h_-; \varphi_-).\end{aligned}$$

Hence,

$$\theta(\tau, z_0, h) = \theta_+(\tau, z_0, h_+) \otimes \theta_-(\tau, z_0, h_-),$$

where their respective weights are $\frac{n}{2}$ and -1. Since z_0 is fixed, we write

$$\theta_\pm(\tau, h_\pm) = \theta_\pm(\tau, z_0, h_\pm).$$

3.2 The Contraction Map

Now we describe the main way in which we use the above factorization of the theta function. Let $\varphi \in S(V(\mathbb{A}_f))$. Then we can write $\varphi = \sum_j \varphi_+^j \otimes \varphi_-^j$, where $\varphi_+^j \in S(V_+(\mathbb{A}_f)), \varphi_-^j \in S(U(\mathbb{A}_f))$ and the sum is finite. We define the *Schwartz space contraction map*

$$\langle \cdot, \theta_+(\tau, h_+) \rangle_U : S(V(\mathbb{A}_f)) \to S(U(\mathbb{A}_f))$$

by

$$\langle \varphi, \theta_+(\tau, h_+) \rangle_U := \sum_j \theta_+(\tau, h_+; \varphi_+^j)\varphi_-^j.$$

It is clear that

$$((\varphi, \theta(\tau, z_0, h))) = ((\langle \varphi, \theta_+(\tau, h_+) \rangle_U, \theta_-(\tau, h_-))).$$

The expression on the right hand side is nice because it is the pairing of a function in $S(U(\mathbb{A}_f))$ and the theta function for U. This is just as in the $n = 0$ case. The value of the contraction map that we are interested in is $\langle F(\tau), \theta_+(\tau, h_+) \rangle_U$.

Lemma 3.1. *$\langle F(\tau), \theta_+(\tau, h_+) \rangle_U$ is a modular form of weight 1 and type ω_- for Γ'.*

Proof. By definition,

$$\langle F(\gamma'\tau), \theta_+(\gamma'\tau, h_+) \rangle_U = (c\tau + d) \left\langle \omega(\gamma')(F(\tau)), \omega_+^\vee(\gamma')(\theta_+(\tau, h_+)) \right\rangle_U. \tag{3.1}$$

Assume that $F(\tau) = \sum_j \varphi_+^j \otimes \varphi_-^j$. We have

$$\omega_+^\vee(\gamma')(\theta_+(\tau, h_+)) = \theta_+(\tau, h_+; \omega_+(\gamma')^{-1} \circ \cdot),$$

so (3.1) is

$$\begin{aligned} &= (c\tau + d) \left\langle \sum_j \omega_+(\gamma')(\varphi_+^j) \otimes \omega_-(\gamma')(\varphi_-^j), \theta_+(\tau, h_+; \omega_+(\gamma')^{-1} \circ \cdot) \right\rangle_U \\ &= (c\tau + d) \sum_j \theta_+(\tau, h_+; \omega_+(\gamma')^{-1}\omega_+(\gamma')(\varphi_+^j))\omega_-(\gamma')(\varphi_-^j) \\ &= (c\tau + d) \sum_j \theta_+(\tau, h_+; \varphi_+^j)\omega_-(\gamma')(\varphi_-^j) \\ &= (c\tau + d)\omega_-(\gamma') \left(\langle F(\tau), \theta_+(\tau, h_+) \rangle_U\right). \end{aligned}$$

□

We will also see that assuming the non-positive Fourier coefficients of F lie in $\mathbb{Z}$ implies the same is true for $\langle F(\tau), \theta_+(\tau, h_+) \rangle_U$. In order to compute this

Fourier expansion, we need the expansion of $\theta_+(\tau, h_+; \varphi_+)$ for $\varphi_+ \in S(V_+(\mathbb{A}_f))$. We take $h_+ = 1$ since the integral we are interested in is

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F) dh.$$

The explicit $\mathbf{q}$-expansion of $\theta_+(\tau, 1; \varphi_+)$ follows from how the Weil representation acts in $S(V_+(\mathbb{R}))$. In our particular case,

$$\begin{aligned}\theta_+(\tau, 1; \varphi_+) &= v^{-\frac{n}{4}} \sum_{x_1 \in V_+(\mathbb{Q})} \omega_+(g'_\tau)\varphi_{\infty,+}(x_1)\varphi_+(x_1) \\ &= v^{-\frac{n}{4}} \sum_{x_1} \omega_+(g'_\tau) e^{-\pi(x_1,x_1)}\varphi_+(x_1),\end{aligned}$$

which by [15] is

$$\begin{aligned}&= v^{-\frac{n}{4}} \sum_{x_1} v^{\frac{n}{4}} e^{2\pi i u Q(x_1)} e^{-\pi v(x_1,x_1)}\varphi_+(x_1) \\ &= \sum_{x_1} e^{2\pi i \tau Q(x_1)}\varphi_+(x_1) \\ &= \sum_{m\in\mathbb{Q}} \Big(\sum_{\substack{x_1 \\ Q(x_1)=m}} \varphi_+(x_1)\Big)\mathbf{q}^m. \qquad (3.2)\end{aligned}$$

Define

$$d_{\varphi_+}(m) := \sum_{\substack{x_1 \\ Q(x_1)=m}} \varphi_+(x_1).$$

Let $L_+ \subset V_+$ be a lattice. Note that if φ_+ is the characteristic function of a coset $\lambda_+ + L_+$, then $d_{\varphi_+}(m)$ is an integer which counts the number of vectors $x_1 \in \lambda_+ + L_+$ such that $Q(x_1) = m$. Also, $V_+(\mathbb{Q})$ is positive definite so $m \geq 0$ in (3.2).

Now we compute the Fourier expansion of $\langle F(\tau), \theta_+(\tau, 1)\rangle_U$. We know $F(\tau) \in S_L$ for some lattice $L \subset V$. If we let $L_+ = V_+ \cap L$ and $L_- = U \cap L$, then generally the lattice L does not split. We have

$$L_+ + L_- \subset L \subset L^\vee \subset L_+^\vee + L_-^\vee.$$

Let

$$L^\vee = \bigcup_\eta \eta + L,\ L = \bigcup_\lambda \lambda + (L_+ + L_-),$$

where η and λ range over $L^\vee/L$ and $L/(L_+ + L_-)$, respectively. If we write $\eta = \eta_+ + \eta_-$ and $\lambda = \lambda_+ + \lambda_-$, then

$$L^\vee = \bigcup_\eta \bigcup_\lambda (\eta_+ + \lambda_+ + L_+) + (\eta_- + \lambda_- + L_-)\,.$$

Let $F(\tau) = \sum_\eta f_\eta(\tau)\varphi_{\eta+L}$ for $\varphi_{\eta+L} = \mathrm{char}(\eta + L)$. Then

$$\varphi_{\eta+L} = \sum_\lambda \varphi_{\eta_++\lambda_++L_+} \otimes \varphi_{\eta_-+\lambda_-+L_-},$$

and we have

$$F(\tau) = \sum_\eta \sum_\lambda f_\eta(\tau) \left(\varphi_{\eta_++\lambda_++L_+} \otimes \varphi_{\eta_-+\lambda_-+L_-}\right).$$

By definition of the contraction map, this gives

$$\langle F(\tau), \theta_+(\tau,1)\rangle_U = \sum_\eta \sum_\lambda f_\eta(\tau)\theta_+\left(\tau, 1; \varphi_{\eta_++\lambda_++L_+}\right)\varphi_{\eta_-+\lambda_-+L_-}. \tag{3.3}$$

In order to apply the $(0,2)$-Theorem (Theorem 2.15), we want to have

$$\langle F(\tau), \theta_+(\tau,1)\rangle_U \in S_{L_-}.$$

From (3.3), we see this is in fact the case, but we point out that the cosets $\eta_- + \lambda_- + L_-$ need not be incongruent mod L_-. Let $c_\eta(m) = c_{\varphi_{\eta+L}}(m)$ and $d_{\eta_++\lambda_+}(m) = d_{\varphi_{\eta_++\lambda_++L_+}}(m)$. Then the Fourier expansion of $\langle F(\tau), \theta_+(\tau,1)\rangle_U$ is

$$\begin{aligned}
\langle F(\tau), \theta_+(\tau,1)\rangle_U &= \sum_\eta \sum_\lambda \Big(\sum_m c_\eta(m)\mathbf{q}^m\Big)\Big(\sum_m d_{\eta_++\lambda_+}(m)\mathbf{q}^m\Big)\varphi_{\eta_-+\lambda_-+L_-} \\
&= \sum_\eta \sum_\lambda \sum_m \Big(\sum_{m_1+m_2=m} c_\eta(m_1)d_{\eta_++\lambda_+}(m_2)\Big)\mathbf{q}^m \varphi_{\eta_-+\lambda_-+L_-} \\
&= \sum_\eta \sum_\lambda \sum_m C_{\eta,\lambda_+}(m)\mathbf{q}^m \varphi_{\eta_-+\lambda_-+L_-},
\end{aligned}$$

where we define

$$C_{\eta,\lambda_+}(m) := \sum_{m_1+m_2=m} c_\eta(m_1) d_{\eta_++\lambda_+}(m_2).$$

The coefficients $d_{\eta_++\lambda_+}(m) \in \mathbb{Z}_{\geq 0}$ for $m \geq 0$ and $d_{\eta_++\lambda_+}(m) = 0$ if $m < 0$. So assuming $c_\eta(m) \in \mathbb{Z}$ for $m \leq 0$ implies $C_{\eta,\lambda_+}(m) \in \mathbb{Z}$ for $m \leq 0$. We have pointed out several facts about the function $\langle F(\tau), \theta_+(\tau, 1)\rangle_U$ which we summarize in the following proposition.

Proposition 3.2. *If $F : \mathfrak{H} \to S_L$ is a meromorphic modular form of weight $1 - \frac{n}{2}$ and type ω for Γ', then*

(i) $\langle F(\tau), \theta_+(\tau, 1)\rangle_U$ is a meromorphic modular form of weight 1 and type ω_- for Γ',

(ii) $\langle F(\tau), \theta_+(\tau, 1)\rangle_U \in S_{L_-}$ for $L_- = U \cap L$,

(iii) The non-positive Fourier coefficients of $\langle F(\tau), \theta_+(\tau, 1)\rangle_U$ lie in $\mathbb{Z}$.

3.3 The $(n, 2)$-Theorem

As in chapter 2, there is a coherent Eisenstein series of weight -1 such that, for any $\varphi_- \in S(U(\mathbb{A}_f))$, we have

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \theta_-(\tau, h_-; \varphi_-) dh_- = v^{\frac{1}{2}} E(g'_\tau, 0; \varphi_-, -1),$$

and we let $E(\tau, s) := v^{\frac{1}{2}} E(g'_\tau, s)$. Now we can state and prove the main theorem.

Theorem 3.3 (The $(n, 2)$-Theorem). *For $\varphi_- \in S(U(\mathbb{A}_f))$, let*

$$E(\tau, s; \varphi_-, +1) = \sum_m A_{\varphi_-}(s, m, v)\mathbf{q}^m,$$

where the Fourier coefficients have Laurent expansions

$$A_{\varphi_-}(s, m, v) = b_{\varphi_-}(m, v)s + O(s^2)$$

at $s = 0$. Let $F : \mathfrak{H} \to S_L \subset S(V(\mathbb{A}_f))$ be a meromorphic modular form for Γ' of weight $1 - \frac{n}{2}$, with Fourier expansion

$$F(\tau) = \sum_{\eta} f_\eta(\tau)\varphi_{\eta+L} = \sum_{\eta}\sum_{m} c_\eta(m)\mathbf{q}^m \varphi_{\eta+L},$$

where $\varphi_{\eta+L} = \mathrm{char}(\eta + L)$ and η runs over $L^\vee/L$. Also, assume $c_\eta(m) \in \mathbb{Z}$ for $m \leq 0$. For $\lambda \in L/(L_+ + L_-)$, we have

$$\theta_+(\tau, 1; \varphi_{\eta_+ + \lambda_+ + L_+}) = \sum_m d_{\eta_+ + \lambda_+}(m)\mathbf{q}^m,$$

where $d_{\eta_+ + \lambda_+}(m) = \#\{x_1 \in \eta_+ + \lambda_+ + L_+ \mid Q(x_1) = m\}$. Let

$$\kappa_{\eta,\lambda}(m_1) := \sum_{0 \leq m \leq m_1} d_{\eta_+ + \lambda_+}(m_1 - m)\kappa_{\eta_- + \lambda_-}(m),$$

where

$$\kappa_{\eta_- + \lambda_-}(m) := \kappa_{\varphi_{\eta_- + \lambda_- + L_-}}(m) = \begin{cases} \lim_{t\to\infty} b_{\varphi_{\eta_- + \lambda_- + L_-}}(m, t) & \text{if } m > 0, \\ k_0(0)\varphi_{\eta_- + \lambda_- + L_-}(0) & \text{if } m = 0, \end{cases}$$

and $k_0(0)$ is as in Lemma 2.18. Define

$$\Phi(z, h; F) := \int^{\bullet}_{\Gamma\backslash\mathfrak{H}} ((\ F(\tau), \theta(\tau, z, h)\ ))d\mu(\tau).$$

Then

$$\kappa_{\eta,\lambda}(m_1) = \sum_{x_1 \in \eta_+ + \lambda_+ + L_+} \kappa_{\eta_- + \lambda_-}(m_1 - Q(x_1)),$$

and for $z_0 \in D_0$ we have

$$\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh = 2\sum_{\eta}\sum_{\lambda}\sum_{m\geq 0} c_\eta(-m)\kappa_{\eta,\lambda}(m).$$

Proof. The desired integral can be written

$$\begin{aligned}
&\int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h; F)dh \\
&= \int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ F(\tau), \theta(\tau, z_0, h)\ ))d\mu(\tau)dh \\
&= \int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} ((\ \langle F(\tau), \theta_+(\tau, 1)\rangle_U, \theta_-(\tau, h_-)\ ))d\mu(\tau)dh_- \\
&= \int_{SO(U)(\mathbb{Q})\backslash SO(U)(\mathbb{A}_f)} \Phi(z_0, h_-; \langle F(\tau), \theta_+(\tau, 1)\rangle_U)dh_-. \qquad (3.4)
\end{aligned}$$

Proposition 3.2 tells us we may apply the $(0, 2)$-Theorem to (3.4). Doing this we see

$$\begin{aligned}
(3.4) &= 2\sum_{\eta}\sum_{\lambda}\sum_{m\geq 0} C_{\eta,\lambda_+}(-m)\kappa_{\eta_-+\lambda_-}(m) \\
&= 2\sum_{\eta}\sum_{\lambda}\sum_{m\geq 0}\Big(\sum_{m_1+m_2=-m} c_\eta(m_1)d_{\eta_++\lambda_+}(m_2)\Big)\kappa_{\eta_-+\lambda_-}(m) \\
&= 2\sum_{\eta}\sum_{\lambda}\sum_{m\geq 0}\Big(\sum_{m_1\leq 0} c_\eta(m_1)d_{\eta_++\lambda_+}(-m-m_1)\Big)\kappa_{\eta_-+\lambda_-}(m) \\
&= 2\sum_{\eta}\sum_{\lambda}\sum_{m\geq 0}\Big(\sum_{m_1\geq 0} c_\eta(-m_1)d_{\eta_++\lambda_+}(m_1-m)\Big)\kappa_{\eta_-+\lambda_-}(m). \qquad (3.5)
\end{aligned}$$

If $m > m_1$, then $d_{\eta_++\lambda_+}(m_1 - m) = 0$, so

$$\begin{aligned}
(3.5) &= 2\sum_{\eta}\sum_{\lambda}\sum_{m_1\geq 0} c_\eta(-m_1)\Big(\sum_{0\leq m\leq m_1} d_{\eta_++\lambda_+}(m_1-m)\kappa_{\eta_-+\lambda_-}(m)\Big) \\
&= 2\sum_{\eta}\sum_{\lambda}\sum_{m_1\geq 0} c_\eta(-m_1)\kappa_{\eta,\lambda}(m_1).
\end{aligned}$$

Then

$$\begin{aligned}
\kappa_{\eta,\lambda}(m_1) &= \sum_{0\leq m\leq m_1} \left(\#\{x_1 \in \eta_+ + \lambda_+ + L_+ \mid Q(x_1) = m_1 - m\}\right)\kappa_{\eta_-+\lambda_-}(m) \\
&= \sum_{\substack{x_1\in\eta_++\lambda_++L_+ \\ 0\leq Q(x_1)\leq m_1}} \kappa_{\eta_-+\lambda_-}(m_1 - Q(x_1)). \qquad (3.6)
\end{aligned}$$

We know that $Q(x_1) \geq 0$ and if we define $\kappa_{\varphi_-}(m) = 0$ for $m < 0$, then we see (3.6) is

$$\sum_{x_1 \in \eta_+ + \lambda_+ + L_+} \kappa_{\eta_- + \lambda_-}(m_1 - Q(x_1)),$$

which gives the desired formula for $\kappa_{\eta,\lambda}(m_1)$. Note, defining $\kappa_{\varphi_-}(m) = 0$ for $m < 0$ is very natural according to Proposition 2.17(i). □

We now state an important corollary of Theorem 3.3, which gives the average value of the logarithm of a Borcherds form over CM points. As in chapter 2, let $T = \mathrm{GSpin}(U)$ and let $K \subset H(\mathbb{A}_f)$ be a compact open subgroup such that $F : \mathfrak{H} \to S_L^K$. Write $K_T = K \cap T(\mathbb{A}_f)$.

Corollary 3.4. *(i) When (2.2) holds, the result of Theorem 3.3 can be stated as*

$$\sum_{t \in T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K_T} \log ||\Psi(z_0, t; F)||^2 = \frac{-1}{\mathrm{vol}(K_T)} \Big(\sum_{\eta} \sum_{\lambda} \sum_{m \geq 0} c_\eta(-m) \kappa_{\eta,\lambda}(m) - C \Big),$$

where $\Psi(F)$ is a Borcherds form and $C = -c_0(0)(\log(2\pi) + \Gamma'(1))$.

(ii) If $U \simeq k = \mathbb{Q}(\sqrt{-d})$ where d is an odd fundamental discriminant, then we have the factorization

$$\prod_t ||\Psi(z_0, t; F)||^4 = r \left(2de^{-3\gamma} e^{2\frac{L'(1,\chi_d)}{L(1,\chi_d)}} \right)^{-h_k c_0(0)},$$

where $\gamma = -\Gamma'(1)$ is Euler's constant and $r \in \mathbb{Q}$. This can also be written as

$$\prod_t ||\Psi(z_0, t; F)||^4 = r \left[\left(\frac{e^\gamma}{8d\pi^2} \right)^{h_k} \prod_{a=1}^{d-1} \Gamma\left(\frac{a}{d}\right)^{w_k \chi_d(a)} \right]^{c_0(0)},$$

where w_k is the number of roots of unity in k.

Proof. (i) follows from (2.2). For (ii), we have $\mathrm{vol}(K_T) = \frac{2}{h_k}$, where h_k is the class number of k, and we will see from Theorem 4.1 of the next chapter that

$$h_k \sum_{\eta} \sum_{\lambda} \sum_{m > 0} c_\eta(-m) \kappa_{\eta,\lambda}(m) \tag{3.7}$$

is the logarithm of a rational number. From $\Lambda(s,\chi_d) = \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi_d)$, we see

$$\frac{\Lambda'(s,\chi_d)}{\Lambda(s,\chi_d)} = -\frac{1}{2}\log(\pi) + \Gamma'(1) + \frac{L'(1,\chi_d)}{L(1,\chi_d)}.$$

So for the corresponding $m = 0$ part of (3.7) we have

$$-h_k c_0(0)\left(\log(d) - \log(\pi) + 2\Gamma'(1) + 2\frac{L'(1,\chi_d)}{L(1,\chi_d)} + \log(2\pi) + \Gamma'(1)\right)$$

$$= -h_k c_0(0)\left(\log(2d) - 3\gamma + 2\frac{L'(1,\chi_d)}{L(1,\chi_d)}\right).$$

The last identity follows from the Chowla-Selberg formula, which says

$$\frac{L'(1,\chi_d)}{L(1,\chi_d)} = \log(2\pi) + \gamma - \frac{w_k}{2h_k}\sum_{a=1}^{d-1}\chi_d(a)\log\Gamma\left(\frac{a}{d}\right).$$

□

Chapter 4

Explicit Computation of κ_φ

In order to compute examples of our main theorem, we need to derive explicit formulas for $\kappa_{\varphi_\mu}(t)$. We assume $U = k$, an imaginary quadratic field, and write $\kappa_{\varphi_\mu}(t)$ as $\kappa(t, \mu, \mathfrak{A})$, where our lattice $L = \mathfrak{A} \subset \mathcal{O}_k$ is an ideal and $\varphi_\mu = \mathrm{char}(\mu + \mathfrak{A})$ for $\mu \in \mathfrak{A}^\vee/\mathfrak{A}$. For $t > 0$, $\kappa(t, \mu, \mathfrak{A})$ is given by the second term in the Laurent expansion of a certain Eisenstein series. These Eisenstein series have factorizations in terms of local Whittaker functions, and we use these factorizations to derive formulas for $\kappa(t, \mu, \mathfrak{A})$. For simplicity, we assume that $k = \mathbb{Q}(\sqrt{-d})$, where $d > 3, d \equiv 3 \pmod 4$ and is square-free. Let $\mathfrak{A} \subseteq \mathcal{O}_k$ be any integral ideal and let Q be the quadratic form given by $Q(x) = -\frac{Nx}{N\mathfrak{A}}$. Let χ be the character of $\mathbb{Q}_\mathbb{A}^\times$ associated to k, which is defined via the global Hilbert symbol so that $\chi(t) = (t, -d)_\mathbb{A}$. Then for a prime $p \leq \infty$, the local character is $\chi_p(t) = (t, -d)_p$ where $(\ ,\ )_p$ is the local Hilbert symbol.

Throughout this section we let p denote an unramified prime and q denote a ramified prime. Let μ be a coset in $\mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}$, where $\mathcal{D}$ is the different, and let $t \in \mathbb{Q}_{>0}$. Write μ_q for the image of μ under the map

$$\mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A} \to \mathcal{D}_q^{-1}\mathfrak{A}_q/\mathfrak{A}_q,$$

where $\mathfrak{A}_q = \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_q$, and similarly for $\mathcal{D}^{-1}$. For $t \in \mathbb{N}$, we introduce the function

$$\rho(t) = \#\{\mathfrak{A} \subseteq \mathcal{O}_k \mid N\mathfrak{A} = t\}.$$

This function factors as

$$\rho(t) = \prod_p \rho_p(t), \tag{4.1}$$

where $\rho_p(t) = \rho(p^{\mathrm{ord}_p(t)})$. The explicit formula for $\kappa(t, \mu, \mathfrak{A})$ is given by the following theorem.

Theorem 4.1. *For $\mu \in \mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}$ and $t \in \mathbb{Q}_{>0}$,*

$$\kappa(t, \mu, \mathfrak{A}) = -\frac{2^{k(\mu)}}{h_k} \prod_{q|d} \mathrm{char}(Q(\mu_q) + \mathbb{Z}_q)(t) \times$$

$$\left(\sum_{q|d} \eta_q(t,\mu) \log(q)(\mathrm{ord}_q(t)+1)\rho(dt) + \sum_{p \text{ inert}} \eta_p(t,\mu) \log(p)(\mathrm{ord}_p(t)+1)\rho\left(\frac{dt}{p}\right) \right),$$

where

$$k(\mu) = \#\{q \text{ ramified} \mid \mu_q = 0\},$$

$$\eta_q(t,\mu) = \begin{cases} 0 & \text{if } \mu_q \neq 0, \text{ or } \mu_q = 0 \text{ and } \chi_q(-t) = 1, \text{ or } \chi_q(-t) = -1 = \chi_{q'}(-t) \\ & \text{for some ramified prime } q' \neq q \text{ with } \mu_{q'} = 0, \\ 1 & \text{if } \mu_q = 0, \chi_q(-t) = -1, \text{ and } \chi_{q'}(-t) = 1 \text{ for all ramified} \\ & \text{primes } q' \neq q \text{ with } \mu_{q'} = 0, \end{cases}$$

and

$$\eta_p(t,\mu) = \begin{cases} 0 & \text{if } \chi_q(-t) = -1 \text{ and } \mu_q = 0 \text{ for some ramified prime } q, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let φ_{μ_q} be the characteristic function of the coset μ_q, $X = p^{-s}$, and $\tau = u + iv \in \mathfrak{H}$. Using [16] and [14], we have the following formulas for the normalized local Whittaker functions. For $\mu = 0$, Lemma 2.3 of [14] tells us we only need to consider $t \in \mathbb{Z}$, and for $t > 0$ we have,

$$W^*_{t,\infty}(\tau, s) = \gamma_\infty v^{\frac{1-s}{2}} e(tu) \frac{2i\pi^{\frac{s}{2}} e^{2\pi tv}}{\Gamma(\frac{s}{2})} \int_{u>2tv} e^{-2\pi u} u^{\frac{s}{2}} (u - 2tv)^{\frac{s}{2}-1} du, \tag{4.2}$$

$$W^*_{t,p}(s, \varphi_0) = \sum_{r=0}^{\operatorname{ord}_p(t)} (\chi_p(p)X)^r, \tag{4.3}$$

$$W^*_{t,q}(s, \varphi_0) = \gamma_q q^{-\frac{1}{2}} \begin{cases} 1 + (q, -t)_q X^{\operatorname{ord}_q(t)+1} & \text{if } \operatorname{ord}_q(t) \text{ is even,} \\ 1 + (-1)^{\frac{q-1}{2}} (q, dt)_q X^{\operatorname{ord}_q(t)+1} & \text{if } \operatorname{ord}_q(t) \text{ is odd.} \end{cases} \tag{4.4}$$

Here γ_∞ and γ_q are local factors which do not affect our global computations since $\gamma_\infty \prod_q \gamma_q = 1$, where the product is over all ramified primes. For an unramified prime p, the local lattice $\mathfrak{A}_p = \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is unimodular. Hence, we only need to consider the Whittaker functions at nonzero cosets for ramified primes. For $\mu_q \neq 0$ we have,

$$W^*_{t,q}(s, \varphi_{\mu_q}) = \gamma_q q^{-\frac{1}{2}} \operatorname{char}(Q(\mu_q) + \mathbb{Z}_q)(t). \tag{4.5}$$

Note that in (4.5), $W^*_{t,q}(s, \varphi_{\mu_q})$ is either a nonzero constant or is identically zero. Following [14], the normalized Eisenstein series has Fourier coefficients given by

$$E^*_t\left(\tau, s, \Phi^{1,\mu}\right) = v^{-\frac{1}{2}} d^{\frac{s+1}{2}} W^*_{t,\infty}(\tau, s) \prod_{q|d} W^*_{t,q}(s, \varphi_\mu) \prod_{p\nmid d} W^*_{t,p}(s, \varphi_0). \tag{4.6}$$

Write $t = q^{\alpha_q} u$ where $\alpha_q = \operatorname{ord}_q(t)$. We now show that (4.4) can be combined into one nice formula.

Lemma 4.2. $W^*_{t,q}(s, \varphi_0) = \gamma_q q^{-\frac{1}{2}} \left(1 + \chi_q(-t) X^{\alpha_q+1}\right)$.

Proof. For α_q even, we have

$$(q, -t)_q = (-t, q)_q = (-t, -1)_q (-t, -q)_q = (-t, -1)_q (-t, dq)_q \chi_q(-t),$$

and

$$(-t,-1)_q(-t,dq)_q = (-t,-dq^{-1})_q = \left(\frac{-dq^{-1}}{q}\right)^{\alpha_q} = 1.$$

For α_q odd,

$$\begin{aligned}(-1)^{\frac{q-1}{2}}(q,dt)_q &= (-1)^{\frac{q-1}{2}}(q,d)_q(q,t)_q \\ &= (-1,q)_q(q,d)_q(-t,-q)_q(-1,q)_q(-t,-1)_q \\ &= (q,d)_q(-t,-1)_q(-t,dq)_q\chi_q(-t),\end{aligned}$$

and

$$\begin{aligned}(q,d)_q(-t,-1)_q(-t,dq)_q &= (q,d)_q(-t,-dq^{-1})_q \\ &= (-1)^{\frac{q-1}{2}}\left(\frac{dq^{-1}}{q}\right)^{\alpha_q}\left(\frac{-dq^{-1}}{q}\right)^{\alpha_q} \\ &= 1.\end{aligned}$$

So (4.4) can be rewritten as

$$W^*_{t,q}(s,\varphi_0) = \gamma_q q^{-\frac{1}{2}}\left(1+\chi_q(-t)X^{\alpha_q+1}\right). \tag{4.7}$$

□

Let us first compute $\kappa(t,\mu,\mathfrak{A})$ for $\mu = 0$ and $t \in \mathbb{N}$. To do this, we need the following special values for the local Whittaker functions.

Lemma 4.3. *At $s = 0$ we have*

(i) $W^*_{t,\infty}(\tau,0) = -\gamma_\infty 2v^{\frac{1}{2}}e(t\tau)$.

(ii) $W^*_{t,p}(0,\varphi_0) = \rho_p(t)$, *and if* $\rho_p(t) = 0$ *then*

$$W^{*,\prime}_{t,p}(0,\varphi_0) = \log(p)\frac{1}{2}(\mathrm{ord}_p(t)+1)\rho_p\left(\frac{t}{p}\right).$$

(iii) $W^*_{t,q}(0,\varphi_0) = \gamma_q q^{-\frac{1}{2}}(1+\chi_q(-t))$, *and if* $\chi_q(-t) = -1$ *then*

$$W^{*,\prime}_{t,q}(0,\varphi_0) = \gamma_q q^{-\frac{1}{2}}\log(q)(\mathrm{ord}_q(t)+1)\rho_q(t).$$

Proof. See Lemma 2.5 and Propositions 2.6 and 2.7 of [14]. □

Given (4.6), we consider different cases based on when one and only one local Whittaker function vanishes at $s = 0$. Since $W^*_{t,\infty}(\tau, 0) \neq 0$ for $t \in \mathbb{N}$, there are two cases.

<u>Case 1</u>: $W^*_{t,p}(0, \varphi_0) = 0$ for p unramified, $W^*_{t,p'}(0, \varphi_0) \neq 0 \ \forall p' \neq p$.

$W^*_{t,p}(0, \varphi_0) = 0$ implies that p is inert and $\text{ord}_p(t)$ is odd. Since $W^*_{t,q}(0, \varphi_0) \neq 0$ for q ramified, we have $\chi_q(-t) = 1$ and $W^*_{t,q}(0, \varphi_0) = \gamma_q 2 q^{-\frac{1}{2}}$. Computing the derivative of the Fourier coefficient we get

$$E^{*,\prime}_t(\tau, 0, \Phi^{1,0}) = W^{*,\prime}_{t,p}(0, \varphi_0)\left[v^{-\frac{1}{2}} d^{\frac{1}{2}} W^*_{t,\infty}(\tau, 0) \prod_{q|d} W^*_{t,q}(0, \varphi_0) \prod_{\substack{p' \nmid d \\ p' \neq p}} W^*_{t,p'}(0, \varphi_0)\right]$$

$$= \log(p)\frac{1}{2}(\text{ord}_p(t) + 1)\rho_p\left(\frac{t}{p}\right)\left[-v^{-\frac{1}{2}} d^{\frac{1}{2}} \gamma_\infty 2 v^{\frac{1}{2}} e(t\tau) 2^{k(0)} \prod_{q|d} \gamma_q q^{-\frac{1}{2}} \prod_{\substack{p' \nmid d \\ p' \neq p}} \rho_{p'}(t)\right]$$

$$= -\log(p)(\text{ord}_p(t) + 1)\rho_p\left(\frac{t}{p}\right) e(t\tau) 2^{k(0)} \prod_{q|d} \rho_q\left(\frac{t}{p}\right) \prod_{\substack{p' \nmid d \\ p' \neq p}} \rho_{p'}\left(\frac{t}{p}\right),$$

since $\rho_q\left(\frac{t}{p}\right) = 1$ and $\rho_{p'}(t) = \rho_{p'}\left(\frac{t}{p}\right)$. So we see

$$E^{*,\prime}_t(\tau, 0, \Phi^{1,0}) = -\log(p)(\text{ord}_p(t) + 1) 2^{k(0)} \rho\left(\frac{t}{p}\right) e(t\tau). \tag{4.8}$$

<u>Case 2</u>: $W^*_{t,q}(0, \varphi_0) = 0$ for q ramified, $W^*_{t,p}(0, \varphi_0) \neq 0 \ \forall p \neq q$.

$W^*_{t,q}(0, \varphi_0) = 0$ implies $\chi_q(-t) = -1$ while for any ramified prime $q' \neq q$ we have

$\chi_{q'}(-t) = 1$ and $W^*_{t,q'}(0, \varphi_0) = \gamma_{q'} 2(q')^{-\frac{1}{2}}$. In this case, we see

$$\begin{aligned} E_t^{*,\prime}\big(\tau, 0, \Phi^{1,0}\big) =& W^{*,\prime}_{t,q}(0, \varphi_0) \left[v^{-\frac{1}{2}} d^{\frac{1}{2}} W^*_{t,\infty}(\tau, 0) \prod_{\substack{q' \mid d \\ q' \neq q}} W^*_{t,q'}(0, \varphi_0) \prod_{p \nmid d} W^*_{t,p}(0, \varphi_0) \right] \\ =& \gamma_q q^{-\frac{1}{2}} \log(q)(\mathrm{ord}_q(t) + 1)\rho_q(t) \times \\ & \left[- v^{-\frac{1}{2}} d^{\frac{1}{2}} \gamma_\infty 2 v^{\frac{1}{2}} e(t\tau) 2^{k(0)-1} \prod_{\substack{q' \mid d \\ q' \neq q}} \gamma_{q'} (q')^{-\frac{1}{2}} \prod_{p \nmid d} \rho_p(t) \right] \\ =& - \log(q)(\mathrm{ord}_q(t) + 1) 2^{k(0)} \rho(t) e(t\tau). \end{aligned} \tag{4.9}$$

Recall that the definition of $\kappa(t, \mu, \mathfrak{A})$ involves the non-normalized Eisenstein series, and at $s = 0$ we have $E^{*,\prime}(\tau, 0, \Phi^{1,\mu}) = h_k E'(\tau, 0, \Phi^{1,\mu})$. This fact and the above analysis, particularly (4.8) and (4.9), give

$$\kappa(t, 0, \mathfrak{A}) =$$

$$-\frac{2^{k(0)}}{h_k} \left(\sum_{q \mid d} \eta_q(t) \log(q)(\mathrm{ord}_q(t) + 1)\rho(t) + \sum_{p \text{ inert}} \eta_p(t) \log(p)(\mathrm{ord}_p(t) + 1)\rho\left(\frac{t}{p}\right) \right),$$

where

$$\eta_q(t) = \begin{cases} 0 & \text{if } \chi_q(-t) = 1 \text{ or } \chi_q(-t) = -1 = \chi_{q'}(-t), \text{ for some ramified prime} \\ & q' \neq q, \\ 1 & \text{if } \chi_q(-t) = -1 \text{ and } \chi_{q'}(-t) = 1 \text{ for all ramified primes } q' \neq q, \end{cases}$$

and

$$\eta_p(t) = \begin{cases} 0 & \text{if } \chi_q(-t) = -1 \text{ for some ramified prime } q, \\ 1 & \text{otherwise.} \end{cases}$$

Now we compute $\kappa(t, \mu, \mathfrak{A})$ for $\mu \neq 0$. One main thing to keep in mind is that there is at least one ramified prime q such that $\mu_q \neq 0$, but the coset can be zero locally at other ramified primes. Write $\mu = (\mu_p)$, where if p is unramified then

$\mu_p = 0$. Again, we consider two cases.

Case 1: $W^*_{t,p}(0,\varphi_0) = 0$ for p unramified, $W^*_{t,p'}(0,\varphi_{\mu_{p'}}) \neq 0 \; \forall p' \neq p$.

The formula for the derivative of the Fourier coefficient is

$$E^{*,\prime}_t\big(\tau, 0, \Phi^{1,\mu}\big) = W^{*,\prime}_{t,p}(0,\varphi_0)\left[v^{-\frac{1}{2}} d^{\frac{1}{2}} W^*_{t,\infty}(\tau,0) \prod_{q|d} W^*_{t,q}(0,\varphi_{\mu_q}) \prod_{\substack{p'\nmid d \\ p'\neq p}} W^*_{t,p'}(0,\varphi_0)\right].$$

Then after cancelling some terms and using Lemma 4.3 and (4.5), we get

$$= \log(p)\frac{1}{2}(\mathrm{ord}_p(t)+1)\rho_p\left(\frac{t}{p}\right)\left[-2e(t\tau)2^{k(\mu)} \prod_{\substack{q|d \\ \mu_q\neq 0}} \mathrm{char}(Q(\mu_q)+\mathbb{Z}_q)(t) \prod_{\substack{p'\nmid d \\ p'\neq p}} \rho_{p'}(t)\right].$$

If q is a ramified prime with $\mu_q \neq 0$, then $W^*_{t,q}(0,\varphi_{\mu_q}) \neq 0$ implies $\mathrm{ord}_q(t) = -1$. This means $\rho_q(qt) = 1$ and this also equals $\rho_q(dt)$. If $\mu_q = 0$, then $\rho_q(t) = 1 = \rho_q(dt)$. Similarly, $\rho_p\left(\frac{t}{p}\right) = \rho_p\left(\frac{dt}{p}\right)$ and $\rho_{p'}(t) = \rho_{p'}(dt) = \rho_{p'}\left(\frac{dt}{p}\right)$. We also see that if $\mu_q = 0$, then $\mathrm{char}(Q(\mu_q)+\mathbb{Z}_q)(t) = \mathrm{char}(\mathbb{Z}_q)(t) = 1$. So the above formula is

$$= -2^{k(\mu)}\log(p)(\mathrm{ord}_p(t)+1)\rho\left(\frac{dt}{p}\right)e(t\tau)\prod_{q|d}\mathrm{char}(Q(\mu_q)+\mathbb{Z}_q)(t). \tag{4.10}$$

Case 2: $W^*_{t,q}(0,\varphi_0) = 0$ for q ramified, $W^*_{t,p}(0,\varphi_{\mu_p}) \neq 0 \; \forall p \neq q$.

Here the derivative is given by

$$E^{*,\prime}_t\big(\tau, 0, \Phi^{1,\mu}\big) = W^{*,\prime}_{t,q}(0,\varphi_0)\left[v^{-\frac{1}{2}} d^{\frac{1}{2}} W^*_{t,\infty}(\tau,0) \prod_{\substack{q'|d \\ q'\neq q}} W^*_{t,q'}(0,\varphi_{\mu_{q'}}) \prod_{p\nmid d} W^*_{t,p}(0,\varphi_0)\right]$$

$$= \log(q)(\mathrm{ord}_q(t)+1)\rho_q(t)\left[-2e(t\tau)2^{k(\mu)-1} \prod_{\substack{q|d \\ \mu_q\neq 0}} \mathrm{char}(Q(\mu_q)+\mathbb{Z}_q)(t) \prod_{p\nmid d}\rho_p(t)\right]$$

$$= -2^{k(\mu)}\log(q)(\mathrm{ord}_q(t)+1)\rho(dt)e(t\tau)\prod_{q|d}\mathrm{char}(Q(\mu_q)+\mathbb{Z}_q)(t). \tag{4.11}$$

Note that we do not consider the case where $W^*_{t,q}(0,\varphi_{\mu_q}) = 0$ for $\mu_q \neq 0$, since then the Whittaker function is identically zero and there is no contribution to the derivative. Formulas (4.10) and (4.11) imply that for $\mu \neq 0$,

$$\kappa(t,\mu,\mathfrak{A}) = -\frac{2^{k(\mu)}}{h_k}\prod_{q|d}\text{char}(Q(\mu_q)+\mathbb{Z}_q)(t) \times$$

$$\left(\sum_{q|d}\eta_q(t,\mu)\log(q)(\text{ord}_q(t)+1)\rho(dt) + \sum_{p \text{ inert}}\eta_p(t,\mu)\log(p)(\text{ord}_p(t)+1)\rho\left(\frac{dt}{p}\right)\right), \tag{4.12}$$

where

$$\eta_q(t,\mu) = \begin{cases} 0 & \text{if } \mu_q \neq 0, \text{ or } \mu_q = 0 \text{ and } \chi_q(-t) = 1, \text{ or } \chi_q(-t) = -1 = \chi_{q'}(-t) \\ & \text{for some ramified prime } q' \neq q \text{ with } \mu_{q'} = 0, \\ 1 & \text{if } \mu_q = 0, \chi_q(-t) = -1, \text{ and } \chi_{q'}(-t) = 1 \text{ for all ramified} \\ & \text{primes } q' \neq q \text{ with } \mu_{q'} = 0, \end{cases}$$

and

$$\eta_p(t,\mu) = \begin{cases} 0 & \text{if } \chi_q(-t) = -1 \text{ and } \mu_q = 0 \text{ for some ramified prime } q, \\ 1 & \text{otherwise.} \end{cases}$$

If we take $\mu = 0$ in the above equations, we see that $\eta_q(t,0) = \eta_q(t)$ and $\eta_p(t,0) = \eta_p(t)$. Also, when $\mu = 0$ then $t \in \mathbb{N}$ so $\rho(dt) = \rho(t), \rho\left(\frac{dt}{p}\right) = \rho\left(\frac{t}{p}\right)$ and the characteristic functions can be ignored. This means (4.12) holds when $\mu = 0$ as well. This finishes the proof of Theorem 4.1. $\square$

Chapter 5

The Example $n = 0$

The simplest case of our main theorem is taking $n = 0$. Then $V = U$ is a quadratic space of signature $(0, 2)$. Assume $U = k = \mathbb{Q}\left(\sqrt{-d}\right)$ where $d > 0$ and is square-free. Then letting $K = \hat{\mathcal{O}}_k^\times$, the set we average over in Corollary 3.4 is isomorphic to I_k, the ideal class group of k. In order to obtain input functions, we make Schwartz functions out of Hecke's theta functions. Multiplying by certain weight-zero modular forms leads to an even richer supply. These input functions depend on an ideal in the ring of integers, $\mathcal{O} = \mathcal{O}_k$, and we will see the main theorem produces a function on $I_k \times I_k^2$.

5.1 The Ideal Class Group

Let $T = \mathrm{GSpin}(U)$. Since $U = k$, $SO(U) = k^1$ and $T = k^\times$. We have the exact sequence

$$1 \to \mathbb{Q}^\times \to k^\times \to k^1 \to 1,$$

and the map $k^\times \to k^1$ is $x \mapsto \frac{x}{x^\sigma}$ by Hilbert's Theorem 90. The space we sum over in Corollary 3.4 is

$$T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K, \tag{5.1}$$

where $K \subset T(\mathbb{A}_f)$ is an open subgroup. The double coset space (5.1) is isomorphic to $k_\mathbb{A}^\times/k_\infty^\times k^\times K$. We define our lattice to be $L = \mathfrak{A} \subset \mathcal{O}$, where $\mathfrak{A}$ is an ideal with $N\mathfrak{A} = A$, and define the quadratic form $Q(x) = -\frac{Nx}{A}$. Then the dual lattice is

$$L^\vee = \left\{ x \in k \mid \frac{\operatorname{tr}(x\mathfrak{A})}{A} \in \mathbb{Z} \right\} = \mathcal{D}^{-1}\mathfrak{A}.$$

$T(\mathbb{A}_f)$ acts on lattices by

$$t \cdot L = tt^{-\sigma}\hat{L} \cap U(\mathbb{Q}).$$

Take K to be the subset which acts trivially on $L^\vee/L$.

Lemma 5.1. *$K = \hat{\mathcal{O}}^\times$.*

Proof. $\hat{\mathcal{O}}^\times$ fixes L so $K \subseteq \hat{\mathcal{O}}^\times$. We have the map $\hat{\mathcal{O}}^\times \to \operatorname{Aut}(L^\vee/L)$ with K as the kernel. Then $L^\vee/L \simeq \mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A} \simeq \mathcal{D}^{-1}/\mathcal{O}$, which locally is either isomorphic to $\mathbb{F}_q$ at a ramified prime q or is trivial. If $\mathfrak{q}$ is the prime ideal lying above q, then the inertia group of $\mathfrak{q}$ equals the decomposition group of $\mathfrak{q}$, so $\frac{x}{x^\sigma} \equiv 1 \pmod{q}$ for any $x \in \mathcal{O}^\times$. This tells us that $K = \hat{\mathcal{O}}^\times$. □

Corollary 5.2. *$T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K = k_\mathbb{A}^\times/k_\infty^\times k^\times \hat{\mathcal{O}}^\times \simeq I_k$.*

5.2 Input Functions

Next we consider the input functions that we plug into the regularized integral Φ. These must be meromorphic modular forms for $SL_2(\mathbb{Z})$ of weight 1 valued in

$S_L \subset S(U(\mathbb{A}_f))$. We also require that their non-positive Fourier coefficients lie in $\mathbb{Z}$. We take a variation of Hecke's theta function, which we write as

$$\vartheta(\tau, \mu, \mathfrak{A}) = \sum_{x \in \mu + \mathfrak{A}} e\left(\tau \frac{Nx}{A}\right), \tag{5.2}$$

and make it into a Schwartz function lying in $S(U(\mathbb{A}_f))$. To do this, view $\mathfrak{A} \subset \mathcal{O}$ as a lattice inside of U and let $\varphi_\mu = \mathrm{char}(\mu + \hat{\mathfrak{A}})$ for $\mu \in \mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}$. Then the Schwartz function we consider is

$$F(\tau, \mathfrak{A}) = \sum_{\mu \in \mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} \vartheta(\tau, \mu, \mathfrak{A}) \varphi_\mu, \tag{5.3}$$

which is valued in S_L.

Lemma 5.3. *$F(\tau, \mathfrak{A})$ is an appropriate input function.*

Proof. For the matrices

$$T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},$$

we have

$$\omega(T)(\varphi_\mu) = e^{-2\pi i Q(\mu)} \varphi_\mu,$$

and

$$\omega(S)(\varphi_\mu) = \frac{-i}{\sqrt{|L^\vee/L|}} \sum_{\delta \in L^\vee/L} e^{2\pi i (\mu, \delta)} \varphi_\delta.$$

Note, in [8], Hecke uses theta functions of the form

$$\vartheta\left(\tau, \mu, \mathfrak{A}, \sqrt{D_k}\right) = \sum_{x \in \mu + \mathfrak{A}\sqrt{D_k}} e\left(\tau \frac{Nx}{A\,|D_k|}\right),$$

where D_k is the discriminant of k. These functions are related to (5.2) by

$$\vartheta(\tau, \mu, \mathfrak{A}) = \vartheta\left(\tau, \sqrt{D_k}\mu, \mathfrak{A}, \sqrt{D_k}\right).$$

The transformation laws (12) and (14) on pp. 222-223 of [8] imply

$$\vartheta(\tau+1,\mu,\mathfrak{A}) = e^{2\pi i \frac{N\mu}{A}} \vartheta(\tau,\mu,\mathfrak{A}), \tag{5.4}$$

and

$$\vartheta\left(-\frac{1}{\tau},\mu,\mathfrak{A}\right) = \frac{-i\tau}{\left|\sqrt{D_k}\right|} \sum_{\delta\in\mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} e^{-2\pi i \mathrm{tr}\left(\frac{\mu\delta^\sigma}{A}\right)} \vartheta(\tau,\delta,\mathfrak{A}). \tag{5.5}$$

For T, (5.4) gives

$$\begin{aligned} F(\tau+1,\mathfrak{A}) &= \sum_\mu \vartheta(\tau+1,\mu,\mathfrak{A})\varphi_\mu \\ &= \sum_\mu e^{2\pi i \frac{N\mu}{A}} \vartheta(\tau,\mu,\mathfrak{A})\varphi_\mu \\ &= \sum_\mu \vartheta(\tau,\mu,\mathfrak{A})\omega(T)(\varphi_\mu) \\ &= \omega(T)(F(\tau,\mathfrak{A})). \end{aligned}$$

For S, (5.5) implies

$$\begin{aligned} F\left(-\frac{1}{\tau},\mathfrak{A}\right) &= \sum_\mu \vartheta\left(-\frac{1}{\tau},\mu,\mathfrak{A}\right)\varphi_\mu \\ &= \sum_\mu \left(\frac{-i\tau}{\left|\sqrt{D_k}\right|} \sum_{\delta\in\mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} e^{-2\pi i \mathrm{tr}\left(\frac{\mu\delta^\sigma}{A}\right)} \vartheta(\tau,\delta,\mathfrak{A})\right)\varphi_\mu, \end{aligned}$$

while

$$\begin{aligned} \omega(S)(F(\tau,\mathfrak{A})) &= \sum_\mu \vartheta(\tau,\mu,\mathfrak{A})\omega(S)(\varphi_\mu) \\ &= \sum_\mu \vartheta(\tau,\mu,\mathfrak{A}) \left[\frac{-i}{\left|\sqrt{D_k}\right|} \sum_{\delta\in\mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} e^{-2\pi i \mathrm{tr}\left(\frac{\mu\delta^\sigma}{A}\right)} \varphi_\delta\right] \\ &= \frac{-i}{\left|\sqrt{D_k}\right|} \sum_\mu \sum_\delta e^{-2\pi i \mathrm{tr}\left(\frac{\mu\delta^\sigma}{A}\right)} \vartheta(\tau,\mu,\mathfrak{A})\varphi_\delta. \end{aligned} \tag{5.6}$$

If we interchange what we call δ and μ, then $\mathrm{tr}\left(\frac{\mu\delta^\sigma}{A}\right)$ is unchanged and (5.6) is

$$\frac{-i}{\left|\sqrt{D_k}\right|} \sum_\delta \sum_\mu e^{-2\pi i \mathrm{tr}\left(\frac{\mu\delta^\sigma}{A}\right)} \vartheta(\tau,\mu,\mathfrak{A})\varphi_\mu.$$

Hence, $F(S\tau,\mathfrak{A}) = \tau\omega(S)(F(\tau,\mathfrak{A})) = j(S,\tau)\omega(S)(F(\tau,\mathfrak{A}))$. ∎

We can obtain more input functions by letting $k^1_{\mathbb{A}_f}$ act on $F(\tau,\mathfrak{A})$. Let $h \in k^1_{\mathbb{A}_f} = SO(U(\mathbb{A}_f))$. Define the Schwartz function $\omega(h)(F(\tau,\mathfrak{A}))$ by

$$\omega(h)(F(\tau,\mathfrak{A}))(x) = F(\tau,\mathfrak{A})(h^{-1}x)$$

for $x \in U(\mathbb{A}_f)$. Recall $k^1_{\mathbb{A}_f}$ acts on ideals by $h \cdot \mathfrak{A} = h\hat{\mathfrak{A}} \cap k$. Then we have

Lemma 5.4. *For $h \in k^1_{\mathbb{A}_f}$, $\omega(h)(F(\tau,\mathfrak{A})) = F(\tau, h\cdot\mathfrak{A})$ and this is an appropriate input function valued in $S_{h\cdot L}$, where the quadratic form is $Q(x) = -\frac{Nx}{A}$.*

Proof. For a coset $\mu \in L^\vee/L$, define $h\cdot\mu = h(\mu+\hat{\mathfrak{A}})\cap k$. Then $\varphi_\mu(h^{-1}x) = \varphi_{h\cdot\mu}(x)$. So we have

$$\begin{aligned}\omega(h)(F(\tau,\mathfrak{A}))(x) &= \sum_{\mu\in\mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} \vartheta(\tau,\mu,\mathfrak{A})\varphi_{h\cdot\mu}(x)\\ &= \sum_{\mu\in\mathcal{D}^{-1}(h\cdot\mathfrak{A})/h\cdot\mathfrak{A}} \vartheta\left(\tau, h^{-1}\cdot\mu,\mathfrak{A}\right)\varphi_\mu(x).\end{aligned}$$

Note that $N(h\cdot\mathfrak{A}) = (Nh)N\mathfrak{A} = N\mathfrak{A}$ since $h \in k^1_{\mathbb{A}_f}$. Then

$$\begin{aligned}\vartheta\left(\tau,h^{-1}\cdot\mu,\mathfrak{A}\right) &= \sum_{y\in h^{-1}\cdot\mu+\mathfrak{A}} e\left(\tau\frac{Ny}{A}\right) = \sum_{y\in\mu+h\cdot\mathfrak{A}} e\left(\tau\frac{N(h^{-1}y)}{A}\right)\\ &= \sum_{y\in\mu+h\cdot\mathfrak{A}} e\left(\tau\frac{Ny}{N\mathfrak{A}}\right) = \sum_{y\in\mu+h\cdot\mathfrak{A}} e\left(\tau\frac{Ny}{N(h\cdot\mathfrak{A})}\right)\\ &= \vartheta(\tau,\mu,h\cdot\mathfrak{A}).\end{aligned}$$

So

$$\omega(h)(F(\tau,\mathfrak{A})) = \sum_{\mu\in\mathcal{D}^{-1}(h\cdot\mathfrak{A})/h\cdot\mathfrak{A}} \vartheta(\tau,\mu,h\cdot\mathfrak{A})\varphi_\mu = F(\tau,h\cdot\mathfrak{A}),$$

which is an input function by Lemma 5.3. ∎

We note that, for any $t \in k_{\mathbb{A}_f}^\times$, $t\hat{\mathcal{O}} \cap k = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(t_{\mathfrak{p}})}$. This is the ideal given by the usual mapping from ideles to ideals, which implies that for $t, t' \in k_{\mathbb{A}_f}^\times$ we have

$$(t\hat{\mathcal{O}} \cap k)(t'\hat{\mathcal{O}} \cap k) = tt'\hat{\mathcal{O}} \cap k.$$

It is also clear that $(t\hat{\mathcal{O}} \cap k)^{-\sigma} = t^{-\sigma}\hat{\mathcal{O}} \cap k$. Then for $h \in k_{\mathbb{A}_f}^1$, we have $h = tt^{-\sigma}$ for some $t \in k_{\mathbb{A}_f}^\times$, which tells us that in the ideal class group

$$[h \cdot \mathfrak{A}] = [tt^{-\sigma}\hat{\mathfrak{A}} \cap k] = [\mathcal{C}][\mathcal{C}]^{-\sigma}[\mathfrak{A}], \tag{5.7}$$

where $\mathcal{C} = t\hat{\mathcal{O}} \cap k$. Raising to the power $-\sigma$ is trivial in I_k, which means (5.7) is $[\mathcal{C}]^2[\mathfrak{A}]$. So for different h, applying $\omega(h)$ to $F(\tau, \mathfrak{A})$ gives input functions that cycle over the coset of $\mathfrak{A}$ modulo the principal genus I_k^2.

Using the function $F(\tau, \mathfrak{A})$, we can produce even more input functions by considering the weight-zero modular forms $J_r(\tau)$ for $r \in \mathbb{N}$. These are defined to be the unique modular function whose Fourier expansion is

$$J_r(\tau) = \mathbf{q}^{-r} + c_r(1)\mathbf{q} + \cdots .$$

That is, $c_r(-r) = 1$ and all other non-positive coefficients are zero. These are given as monic polynomials of degree r in $j(\tau)$. The first two are (cf. [17])

$$J_1(\tau) = j(\tau) - 744 = \mathbf{q}^{-1} + 196884\mathbf{q} + \cdots$$

and

$$J_2(\tau) = j(\tau)^2 - 1488j(\tau) + 159768 = \mathbf{q}^{-2} + 42987520\mathbf{q} + 40491909396\mathbf{q}^2 + \cdots .$$

Then we get more input functions by letting

$$F_r(\tau, \mathfrak{A}) = F(\tau, \mathfrak{A})J_r(\tau).$$

5.3 The Function $F(\tau, \mathfrak{A})$

We now look more closely at the $(0,2)$-Theorem for the input function $F(\tau, \mathfrak{A})$. Let

$$B(\mathfrak{A}, t) := \Phi(t; F(\tau, \mathfrak{A})).$$

Proposition 5.5. *$B(\mathfrak{A}, t)$ satisfies the following properties:*

1. $$B(\mathfrak{A}, t) = \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \sum_{\mu} \vartheta(\tau, \mu, \mathfrak{A}) \overline{\vartheta(\tau, tt^{-\sigma}\mu, tt^{-\sigma}\mathfrak{A})} v^{-1} du dv.$$

2. *As a function of $\mathfrak{A}$, $B(\mathfrak{A}, t)$ only depends on $[\mathfrak{A}] \in I_k$.*

3. *As a function of t, $B(\mathfrak{A}, t)$ only depends on the double coset of t in $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$. So $B(\mathfrak{A}, t)$ can be viewed as a function on $I_k \times I_k$.*

4. *If $[t\hat{\mathcal{O}} \cap k]^2 = [t'\hat{\mathcal{O}} \cap k]^2$, then $B(\mathfrak{A}, t) = B(\mathfrak{A}, t')$. This means $B(\mathfrak{A}, t)$ can actually be viewed as a function on $I_k \times I_k^2$.*

5. $\overline{B(\mathfrak{A}, t)} = B(tt^{-\sigma}\mathfrak{A}, t^{-1})$.

Proof. $B(\mathfrak{A}, t)$ is defined as

$$B(\mathfrak{A}, t) = \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \theta(\tau, t; F(\tau, \mathfrak{A})) d\mu(\tau).$$

For $h \in k^1_{\mathbb{A}_f}$, we have

$$\theta(\tau, h; F(\tau, \mathfrak{A})) = \sum_{\mu} \vartheta(\tau, \mu, \mathfrak{A}) \sum_{x \in U(\mathbb{Q})} \varphi_\infty(\tau, x) \varphi_\mu(h^{-1}x), \tag{5.8}$$

where $\varphi_\infty(\tau, x) = \omega(g'_\tau)(\varphi_\infty(x))$. Then $U(\mathbb{Q}) = k$ and $h = tt^{-\sigma}, t \in T(\mathbb{A}_f)$, so (5.8) is

$$= \sum_{\mu} \vartheta(\tau, \mu, \mathfrak{A}) \sum_{x \in k} \varphi_\infty(\tau, x) \varphi_\mu(t^{-1}t^{\sigma}x). \tag{5.9}$$

For φ_∞ we have, by [15],

$$\varphi_\infty(\tau, x) = ve^{2\pi i Q(x)\overline{\tau}} = ve^{-2\pi i\overline{\tau}\frac{Nx}{A}} = v\overline{\mathbf{q}}^{\frac{Nx}{A}}. \tag{5.10}$$

This, along with $\varphi_\mu(t^{-1}t^\sigma x) = \varphi_{tt^{-\sigma}\mu}(x)$, implies (5.9) becomes

$$= \sum_\mu \vartheta(\tau, \mu, \mathfrak{A}) \sum_{x\in(tt^{-\sigma}\mu+tt^{-\sigma}\mathfrak{A})\cap k} v\overline{\mathbf{q}}^{\frac{Nx}{A}} = \sum_\mu \vartheta(\tau, \mu, \mathfrak{A})\overline{\vartheta(\tau, tt^{-\sigma}\mu, tt^{-\sigma}\mathfrak{A})}v,$$

since $N(tt^{-\sigma}\mathfrak{A}) = N\mathfrak{A} = A$. This proves (1). For (2), we view the function $F(\tau, \mathfrak{A})$ as $F(\tau, \mathfrak{A}, x)$ for $x \in U(\mathbb{A}_f)$. Then for $\alpha \in k^\times$, we have

$$F(\tau, \alpha\mathfrak{A}, x) = \sum_{\mu\in\mathcal{D}^{-1}(\alpha\mathfrak{A})/\alpha\mathfrak{A}} \vartheta(\tau, \mu, \alpha\mathfrak{A})\varphi_\mu(x) = \sum_{\mu\in\mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} \vartheta(\tau, \alpha\mu, \alpha\mathfrak{A})\varphi_{\alpha\mu}(x),$$

and

$$\vartheta(\tau, \alpha\mu, \alpha\mathfrak{A}) = \sum_{x\in\alpha\mu+\alpha\mathfrak{A}} e\left(\tau\frac{Nx}{N\alpha A}\right) = \sum_{x\in\mu+\mathfrak{A}} e\left(\tau\frac{Nx}{A}\right) = \vartheta(\tau, \mu, \mathfrak{A}).$$

So

$$F(\tau, \alpha\mathfrak{A}, x) = \sum_{\mu\in\mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} \vartheta(\tau, \mu, \mathfrak{A})\varphi_{\alpha\mu}(x) = F(\tau, \mathfrak{A}, \alpha^{-1}x), \tag{5.11}$$

which shows $F(\tau, \mathfrak{A})$ is not a function of $[\mathfrak{A}]$. However, when we compute $\theta(\tau, h; F(\tau, \alpha\mathfrak{A}))$, using (5.11), we get

$$\begin{aligned} \theta(\tau, h; F(\tau, \alpha\mathfrak{A})) &= \sum_{\mu\in\mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} \vartheta(\tau, \mu, \mathfrak{A}) \sum_{x\in U(\mathbb{Q})} \varphi_\infty(\tau, x)\varphi_{\alpha\mu}(t^{-1}t^\sigma x) \\ &= \sum_{\mu\in\mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} \vartheta(\tau, \mu, \mathfrak{A}) \sum_{x\in U(\mathbb{Q})} \varphi_\infty(\tau, \alpha x)\varphi_\mu(t^{-1}t^\sigma x). \end{aligned} \tag{5.12}$$

By (5.10), the Gaussian for the lattice $\alpha\mathfrak{A}$ is $\varphi_\infty(\tau, x) = v\overline{\mathbf{q}}^{\frac{Nx}{N(\alpha A)}}$, so (5.12) is

$$= \sum_\mu \vartheta(\tau, \mu, \mathfrak{A}) \sum_{x\in(tt^{-\sigma}\mu+tt^{-\sigma}\mathfrak{A})\cap k} v\overline{\mathbf{q}}^{\frac{Nx}{A}} = \theta(\tau, h; F(\tau, \mathfrak{A})),$$

which tells us $B(\mathfrak{A}, t)$ only depends on $[\mathfrak{A}]$. Now we need to see how $B(\mathfrak{A}, t)$ depends on t. To do this, part (1) implies we can look at $\vartheta(\tau, tt^{-\sigma}\mu, tt^{-\sigma}\mathfrak{A})$. Fix

μ and $\mathfrak{A}$ and let $t \in K = \hat{\mathcal{O}}^\times$. Then $tt^{-\sigma}$ acts trivially on $L^\vee/L \simeq \hat{L}^\vee/\hat{L}$, so $tt^{-\sigma}(\mu + \hat{\mathfrak{A}}) = \mu + \hat{\mathfrak{A}}$ and $\vartheta(\tau, tt^{-\sigma}\mu, tt^{-\sigma}\mathfrak{A}) = \vartheta(\tau, \mu, \mathfrak{A})$. If we let $t \in T(\mathbb{Q})$, then we have

$$\begin{aligned}\theta(\tau, tt^{-\sigma}; F(\tau, \mathfrak{A})) &= \sum_{\mu} \vartheta(\tau, \mu, \mathfrak{A}) \sum_{x \in k} \varphi_\infty(\tau, x)\varphi_\mu(t^{-1}t^\sigma x) \\ &= \sum_{\mu} \vartheta(\tau, \mu, \mathfrak{A}) \sum_{x \in k} \varphi_\infty(\tau, tt^{-\sigma}x)\varphi_\mu(x). \qquad (5.13)\end{aligned}$$

So $tt^{-\sigma} \in SO(U(\mathbb{Q})) \subset SO(U(\mathbb{R}))$ implies (5.13) is

$$= \sum_{\mu} \vartheta(\tau, \mu, \mathfrak{A}) \sum_{x \in k} \varphi_\infty(\tau, x)\varphi_\mu(x) = \theta(\tau, 1; F(\tau, \mathfrak{A})).$$

Hence, $\theta(\tau, t; F(\tau, \mathfrak{A}))$ is a function of $t \in T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K \simeq I_k$. Statement (4) is stronger than (3). We have

$$[t\hat{\mathcal{O}} \cap k]^2 = [t\hat{\mathcal{O}} \cap k][t\hat{\mathcal{O}} \cap k] = [t\hat{\mathcal{O}} \cap k][t\hat{\mathcal{O}} \cap k]^{-\sigma} = [tt^{-\sigma}\hat{\mathcal{O}} \cap k],$$

and by assumption this is $[t'(t')^{-\sigma}\hat{\mathcal{O}} \cap k]$. So (3) and the fact that $B(\mathfrak{A}, t)$ is really a function of $tt^{-\sigma}$ imply $B(\mathfrak{A}, t) = B(\mathfrak{A}, t')$. For the last statement, let $\mathfrak{A}_t = tt^{-\sigma}\mathfrak{A}$. Then

$$\begin{aligned}B(\mathfrak{A}_t, t^{-1}) &= \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \sum_{\mu \in \mathcal{D}^{-1}\mathfrak{A}_t/\mathfrak{A}_t} \vartheta(\tau, \mu, \mathfrak{A}_t)\overline{\vartheta(\tau, t^{-1}t^\sigma\mu, t^{-1}t^\sigma\mathfrak{A}_t)}v^{-1}dudv \\ &= \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \sum_{\mu \in \mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} \vartheta(\tau, tt^{-\sigma}\mu, tt^{-\sigma}\mathfrak{A})\overline{\vartheta(\tau, \mu, \mathfrak{A})}v^{-1}dudv = \overline{B(\mathfrak{A}, t)}.\end{aligned}$$

□

We mention that $B(\mathfrak{A}, t)$ is like the Petersson inner product of $\vartheta(\tau, \mu, \mathfrak{A})$ and $\vartheta(\tau, tt^{-\sigma}\mu, tt^{-\sigma}\mathfrak{A})$ summed over μ. The Fourier expansion of $F(\tau, \mathfrak{A})$ is

$$F(\tau, \mathfrak{A}) = \sum_{\mu \in \mathcal{D}^{-1}\mathfrak{A}/\mathfrak{A}} \sum_{n \in \mathbb{Q}_{\geq 0}} d(n, \mu, \mathfrak{A})\mathbf{q}^n\varphi_\mu,$$

where

$$d(n, \mu, \mathfrak{A}) = \#\left\{x \in \mu + \mathfrak{A} \mid \frac{Nx}{A} = n\right\}.$$

Then the (0,2)-Theorem gives

$$\frac{1}{h_k} \sum_{t \in T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K} B(\mathfrak{A}, t) = \sum_{\mu} \sum_{n \geq 0} d(-n, \mu, \mathfrak{A}) \kappa(n, \mu, \mathfrak{A}). \tag{5.14}$$

We have

$$d(0, \mu, \mathfrak{A}) = \begin{cases} 1 & \text{if } \mu = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So (5.14) becomes

$$\begin{aligned} \frac{1}{h_k} \sum_{t \in T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K} B(\mathfrak{A}, t) &= \kappa(0, 0, \mathfrak{A}) \\ &= \log(\Delta_k) + 2\frac{\Lambda'(1, \chi_d)}{\Lambda(1, \chi_d)} \end{aligned}$$

by Lemma 2.18, where Δ_k is the absolute value of the discriminant of k and $\chi_d(\alpha) = (\alpha, -d)_{\mathbb{A}}$.

Chapter 6

The Example $n = 1$

6.1 Input Functions

Let $V = \{x \in M_2(\mathbb{Q}) \mid \operatorname{tr}(x) = 0\}$ with quadratic form $Q(x) = \det(x)$ and bilinear form $(x, y) = \operatorname{tr}(xy^\iota)$, where ι is the involution $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\iota = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. This is a quadratic form of signature $(1, 2)$. We define our lattice L to be $L = V \cap M_2(\mathbb{Z})$. The dual lattice is

$$L^\vee = \left\{ \begin{pmatrix} \frac{a}{2} & b \\ c & -\frac{a}{2} \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\},$$

and so $\mid L^\vee/L \mid = 2$. Write L_0 and L_1 for the trivial and nontrivial cosets, respectively. The input functions we need are modular forms of weight $\frac{1}{2}$ and type ω for $Mp_2(\mathbb{Z})$. In this section, we show that such an input function can be obtained from a scalar-valued meromorphic modular form of weight $\frac{1}{2}$ for $\Gamma_0(4)$ whose Fourier coefficients satisfy certain congruence conditions. We also mention a result of Borcherds on the existence of these scalar-valued modular forms.

Let f_0 be a scalar-valued modular form of weight $\frac{1}{2}$ for $\Gamma_0(4)$. Assume f_0 has

Fourier expansion

$$f_0(\tau) = \sum_n c_0(n)\mathbf{q}^n.$$

Definition 6.1. *We say f_0 lies in the Kohnen "plus space," denoted $M_{\frac{1}{2}}(\Gamma_0(4))^+$, if $c_0(n) \in \mathbb{Z}$ for all n and $c_0(n) = 0$ unless $n \equiv 0, 1 \pmod 4$.*

Given $f_0 \in M_{\frac{1}{2}}(\Gamma_0(4))^+$, let

$$h_0(\tau) = \sum_n c_0(4n)\mathbf{q}^n,$$

and

$$h_1(\tau) = \sum_n c_0(4n+1)\mathbf{q}^{n+\frac{1}{4}}.$$

Then $f_0(\tau) = h_0(4\tau) + h_1(4\tau)$.

Proposition 6.2. *Let $\varphi_\mu = \mathrm{char}(L_\mu)$ for $\mu = 0, 1$. For $f_0(\tau), h_0(\tau)$ and $h_1(\tau)$ as above,*

$$F(\tau) = h_0(\tau)\varphi_0 + h_1(\tau)\varphi_1$$

is a meromorphic modular form for $Mp_2(\mathbb{Z})$ of weight $\frac{1}{2}$ and type ω.

Proof. We need to show that

$$F(\tau+1) = \omega(T)(F(\tau)) \tag{6.1}$$

and

$$F\left(-\frac{1}{\tau}\right) = \sqrt{\tau}\omega(S)(F(\tau)). \tag{6.2}$$

Here we follow the ideas in the proof of Lemma 14.2 of [1]. For any $\gamma \in \Gamma_0(4)$, we have $f_0(\gamma\tau) = j(\gamma,\tau)^{\frac{1}{2}} f_0(\tau)$. If we let $\sigma \in \mathfrak{H}$, then

$$f_0\left(\frac{\sigma}{4\sigma+1}\right) = f_0\left(\begin{pmatrix} 1 & \\ 4 & 1 \end{pmatrix}\sigma\right) = \sqrt{(4\sigma+1)}f_0(\sigma). \tag{6.3}$$

Let $\tau = 4\sigma + 1$. Then (6.3) becomes

$$f_0\left(\frac{\tau-1}{4\tau}\right) = \sqrt{\tau} f_0\left(\frac{\tau-1}{4}\right),$$

which says

$$h_0\left(\frac{\tau-1}{\tau}\right) + h_1\left(\frac{\tau-1}{\tau}\right) = \sqrt{\tau}\big(h_0(\tau-1) + h_1(\tau-1)\big),$$

or

$$h_0\left(-\frac{1}{\tau}+1\right) + h_1\left(-\frac{1}{\tau}+1\right) = \sqrt{\tau}\big(h_0(\tau-1) + h_1(\tau-1)\big). \tag{6.4}$$

If $z \in \mathfrak{H}$, the definitions of h_0 and h_1 imply $h_0(z \pm 1) = h_0(z)$ and $h_1(z \pm 1) = \pm i h_1(z)$. This means (6.4) becomes

$$h_0\left(-\frac{1}{\tau}\right) + i h_1\left(-\frac{1}{\tau}\right) = \sqrt{\tau}(h_0(\tau) - i h_1(\tau)). \tag{6.5}$$

If we let $\tau = ix$ be purely imaginary, plugging this into (6.5) gives

$$h_0\left(-\frac{1}{ix}\right) + i h_1\left(-\frac{1}{ix}\right) = \sqrt{ix}(h_0(ix) - i h_1(ix)). \tag{6.6}$$

Using $\sqrt{i} = \frac{\sqrt{2}}{2}(1+i)$ and equating real and imaginary parts in (6.6) gives

$$h_0\left(-\frac{1}{ix}\right) = \frac{\sqrt{2x}}{2}(h_0(ix) + h_1(ix)),$$

and

$$h_1\left(-\frac{1}{ix}\right) = \frac{\sqrt{2x}}{2}(h_0(ix) - h_1(ix)).$$

Since these two identities hold for all $x \in \mathbb{R}_{>0}$, we have

$$h_0\left(-\frac{1}{\tau}\right) = \left(\frac{1-i}{2}\right)\sqrt{\tau}(h_0(\tau) + h_1(\tau)), \tag{6.7}$$

and

$$h_1\left(-\frac{1}{\tau}\right) = \left(\frac{1-i}{2}\right)\sqrt{\tau}(h_0(\tau) - h_1(\tau)). \tag{6.8}$$

Note that in the proof of Lemma 14.2 in [1], the equation resembling (6.8) is off by a sign. We now go back to the modular form F and prove formulas (6.1) and (6.2). Take $\xi = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}$ as a coset representative for L_1. For (6.1), $\omega(T)(\varphi_0) = \varphi_0$ and $\omega(T)(\varphi_1) = e^{-2\pi i Q(\xi)}\varphi_1 = i\varphi_1$, so

$$\omega(T)(F(\tau)) = h_0(\tau)\varphi_0 + ih_1(\tau)\varphi_1 = h_0(\tau+1)\varphi_0 + h_1(\tau+1)\varphi_1 = F(\tau+1).$$

If $\mu \in L^\vee/L$, then for signature $(1,2)$ we have

$$\omega(S)(\varphi_\mu) = \frac{\sqrt{i}(-i)}{\sqrt{2}}\left(\varphi_0 + e^{2\pi i(\xi,\mu)}\varphi_1\right) = \frac{1-i}{2}\left(\varphi_0 + e^{2\pi i(\xi,\mu)}\varphi_1\right),$$

where in the superscript we mean $\mu = 0$ or ξ. This implies

$$\begin{aligned}\omega(S)(F(\tau)) &= h_0(\tau)\omega(S)(\varphi_0) + h_1(\tau)\omega(S)(\varphi_1)\\ &= h_0(\tau)\left(\frac{1-i}{2}(\varphi_0+\varphi_1)\right) + h_1(\tau)\left(\frac{1-i}{2}\left(\varphi_0 + e^{2\pi i(\xi,\xi)}\varphi_1\right)\right)\\ &= \frac{1-i}{2}\Big(h_0(\tau)(\varphi_0+\varphi_1) + h_1(\tau)(\varphi_0-\varphi_1)\Big).\end{aligned}$$

Using (6.7) and (6.8),

$$\begin{aligned}F\left(-\frac{1}{\tau}\right) &= h_0\left(-\frac{1}{\tau}\right)\varphi_0 + h_1\left(-\frac{1}{\tau}\right)\varphi_1\\ &= \left(\frac{1-i}{2}\right)\sqrt{\tau}\Big((h_0(\tau)+h_1(\tau))\varphi_0 + (h_0(\tau)-h_1(\tau))\varphi_1\Big)\\ &= \left(\frac{1-i}{2}\right)\sqrt{\tau}\Big(h_0(\tau)(\varphi_0+\varphi_1) + h_1(\tau)(\varphi_0-\varphi_1)\Big),\end{aligned}$$

which proves (6.2). □

Proposition 6.2 tells us we can construct input functions from modular forms $f_0 \in M_{\frac{1}{2}}(\Gamma_0(4))^+$. We would also like to know to what extent these input functions exist. The following lemma is about the existence of modular forms in $M_{\frac{1}{2}}(\Gamma_0(4))^+$, and, therefore, tells us something about the existence of our input functions.

Lemma 6.3 (Lemma 14.2 of [1]). *Every sequence of integers* $c_0(n)$ *for* $n \leq 0, n \equiv 0, 1 \pmod 4$ *which are almost all zero is the set of coefficients of non-positive degree for a unique modular form* $f_0 \in M_{\frac{1}{2}}(\Gamma_0(4))^+$.

6.2 Lattice Computations

Define

$$L_\mu(r) = \{x \in L_\mu \mid Q(x) = r\},$$

for $r \in \mathbb{Q}$ and $\mu = 0, 1$. Let $r > 0$ and $x_0 \in L_\mu(r)$ be given by

$$x_0 = \begin{pmatrix} \frac{a}{1+\mu} & b \\ c & -\frac{a}{1+\mu} \end{pmatrix}, \tag{6.9}$$

where $a, b, c \in \mathbb{Z}$ and if $\mu = 1$, then a is odd. Note that $r \in \mathbb{Z}$ if $\mu = 0$, and $r \in \frac{1}{4}\mathbb{Z} - \frac{1}{2}\mathbb{Z}$ if $\mu = 1$. We also assume x_0 is primitive, i.e., $\gcd(a, b, c) = 1$. Then x_0 determines a splitting of our vector space, $V = V_+ + V_-$, where $V_+ = \mathbb{Q}x_0$ and $V_- = x_0^\perp$. Define the positive and negative lattices, $L_\pm$, as $L_\pm = V_\pm \cap L$. We have the projection maps $\mathrm{pr}_\pm : V \to V_\pm$. When comparing L with the sublattice $L_+ + L_-$, we will see that L does not split. In this section, we give $\mathbb{Z}$-bases for $L_\pm$, and also investigate the structure of various coset spaces, namely, $L/(L_+ + L_-)$, $\mathrm{pr}_\pm(L^\vee)/L_\pm$ and $L_\pm^\vee/L_\pm$. The facts that we prove are very useful for doing explicit computations.

Proposition 6.4. *For* $r > 0$, *let* $x_0 \in L_\mu(r)$ *be given by (6.9) and assume* x_0 *is primitive. We have the decomposition,* $V = V_+ + V_- = \mathbb{Q}x_0 + x_0^\perp$, *and positive and negative lattices* $L_\pm = V_\pm \cap L$. *Choose* $u, v \in \mathbb{Z}$ *such that* $uc - vb = (b, c)$. *Then*

$$L_+ = (1 + \mu)\mathbb{Z}x_0,$$

and

$$L_- = \begin{cases} \frac{1}{2}l_1\mathbb{Z} + l_2\mathbb{Z} & \textit{if } (b,c) \textit{ is even and } \mu = 0, \\ l_1\mathbb{Z} + l_2\mathbb{Z} & \textit{otherwise,} \end{cases}$$

where

$$l_1 = ue_1 + ve_2 = \begin{pmatrix} (b,c) & -\frac{2a}{1+\mu}u \\ \frac{2a}{1+\mu}v & -(b,c) \end{pmatrix},$$

$$l_2 = \frac{(1+\mu)b}{2a(b,c)}e_1 + \frac{(1+\mu)c}{2a(b,c)}e_2 = \begin{pmatrix} & -\frac{b}{(b,c)} \\ \frac{c}{(b,c)} & \end{pmatrix},$$

for

$$e_1 = \begin{pmatrix} c & -\frac{2a}{1+\mu} \\ & -c \end{pmatrix}, \ e_2 = \begin{pmatrix} -b & \\ \frac{2a}{1+\mu} & b \end{pmatrix}.$$

Proof. Since x_0 is primitive, we have

$$L_+ = \mathbb{Q}x_0 \cap L = (1+\mu)\mathbb{Z}x_0.$$

Let

$$Y_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} x_0 = \begin{pmatrix} c & -\frac{a}{1+\mu} \\ \frac{a}{1+\mu} & b \end{pmatrix},$$

and

$$Y_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} x_0 = \begin{pmatrix} c & -\frac{a}{1+\mu} \\ -\frac{a}{1+\mu} & -b \end{pmatrix}.$$

Then

$$\mathrm{tr}(x_0 Y_1^\iota) = \mathrm{tr}\left(x_0 x_0^\iota \begin{pmatrix} & -1 \\ -1 & \end{pmatrix} \right) = 0,$$

and similarly $\mathrm{tr}(x_0 Y_2^\iota) = 0$. Note that Y_1 and Y_2 do not have trace zero. If we modify them into matrices which do have trace zero, we get

$$y_1 = \begin{pmatrix} \frac{c-b}{2} & -\frac{a}{1+\mu} \\ \frac{a}{1+\mu} & \frac{b-c}{2} \end{pmatrix},$$

and

$$y_2 = \begin{pmatrix} \frac{b+c}{2} & -\frac{a}{1+\mu} \\ -\frac{a}{1+\mu} & -\frac{b+c}{2} \end{pmatrix}.$$

Then y_1 and y_2 are perpendicular to x_0 and have trace zero, i.e., they lie in V_-. In fact, they form a basis for V_-, yet they do not lie in L_-. Let $e_1 = y_1 + y_2$ and $e_2 = y_1 - y_2$. That is,

$$e_1 = \begin{pmatrix} c & -\frac{2a}{1+\mu} \\ & -c \end{pmatrix}, \ e_2 = \begin{pmatrix} -b & \\ \frac{2a}{1+\mu} & b \end{pmatrix}.$$

In terms of this basis, L_- is given by

$$L_- = \left\{ B_1 e_1 + B_2 e_2 \mid B_1, B_2 \in \frac{1+\mu}{2a}\mathbb{Z}, B_1 c - B_2 b \in \mathbb{Z} \right\}.$$

Write $B_j = \frac{(1+\mu)C_j}{2a}$ for $j = 1, 2$. There are two cases to consider.

<u>Case 1</u>: (b, c) is odd.

Choose $u, v \in \mathbb{Z}$ such that $uc - vb = (b, c)$. Then $C_1 c - C_2 b \in \frac{2a}{1+\mu}\mathbb{Z}$ implies there exists $r_1 \in \mathbb{Z}$ such that $C_1 c - C_2 b = \frac{2a}{1+\mu}(b, c) r_1 = \frac{2a}{1+\mu}(uc - vb) r_1$. From this we have

$$\left(C_1 - r_1 \frac{2a}{1+\mu} u \right) c = \left(C_2 - r_1 \frac{2a}{1+\mu} v \right) b$$

or

$$\left(C_1 - r_1 \frac{2a}{1+\mu} u \right) \frac{c}{(b, c)} = \left(C_2 - r_1 \frac{2a}{1+\mu} v \right) \frac{b}{(b, c)}.$$

$\frac{c}{(b,c)}$ and $\frac{b}{(b,c)}$ are relatively prime so there exists $r_2 \in \mathbb{Z}$ such that

$$C_1 - r_1 \frac{2a}{1+\mu} u = r_2 \frac{b}{(b, c)}, \ C_2 - r_1 \frac{2a}{1+\mu} v = r_2 \frac{c}{(b, c)}.$$

That is,

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = r_1 \begin{pmatrix} \frac{2a}{1+\mu} u \\ \frac{2a}{1+\mu} v \end{pmatrix} + r_2 \begin{pmatrix} \frac{b}{(b,c)} \\ \frac{c}{(b,c)} \end{pmatrix}.$$

This tells us

$$B_1e_1 + B_2e_2 = \left(r_1u + \frac{(1+\mu)b}{2a(b,c)}r_2\right)e_1 + \left(r_1v + \frac{(1+\mu)c}{2a(b,c)}r_2\right)e_2$$
$$= r_1(ue_1 + ve_2) + r_2\left(\frac{(1+\mu)b}{2a(b,c)}e_1 + \frac{(1+\mu)c}{2a(b,c)}e_2\right),$$

and, hence, $\{l_1, l_2\}$ gives a $\mathbb{Z}$-basis for L_-. Note that if $\mu = 1$, then the above argument works for (b,c) even as well.

Case 2: (b,c) is even, $\mu = 0$.

As in case 1, we have $uc - vb = (b,c)$ for some $u, v \in \mathbb{Z}$. Then b and c are both even, so $C_1c - C_2b \in 2a\mathbb{Z}$ implies there exists $r_1 \in \mathbb{Z}$ such that $C_1c - C_2b = a(b,c)r_1 = a(uc - vb)r_1$. Continuing as in case 1, we get a $\mathbb{Z}$-basis for L_- given by $\left\{\frac{1}{2}l_1, l_2\right\}$. ☐

The remainder of this section deals with looking at the structure of different coset spaces. We begin with $L/(L_+ + L_-)$.

Lemma 6.5.

$$L/(L_+ + L_-) \simeq \mathrm{pr}_\pm(L)/L_\pm. \tag{6.10}$$

Proof. We have $L_\pm \subset \mathrm{pr}_\pm(L)$, so we can map $L \to \mathrm{pr}_\pm(L)/L_\pm$. Then for $l \in L, l = \mathrm{pr}_+(l) + \mathrm{pr}_-(l)$ and $\mathrm{pr}_\pm(l) \in L_\pm$ if and only if $\mathrm{pr}_\mp(l) = l - \mathrm{pr}_\pm(l) \in L_\mp$, i.e., $l \in L_+ + L_-$. ☐

Using the basis $\{x_0, y_1, y_2\}$, we are able to identify $\mathrm{pr}_+(L)$.

Lemma 6.6.

$$\mathrm{pr}_+(L) = \begin{cases} \frac{1}{r}\mathbb{Z}x_0 & \textit{if } (b,c) \textit{ is even and } \mu = 0, \\ \frac{1}{2r}\mathbb{Z}x_0 & \textit{otherwise.} \end{cases}$$

Proof. In terms of x_0, y_1 and y_2, the lattice L is given as

$$L = A^{-1}\mathbb{Z}^3,$$

where the matrix A is

$$A = \begin{pmatrix} \frac{a}{1+\mu} & \frac{c-b}{2} & \frac{c+b}{2} \\ c & \frac{a}{1+\mu} & -\frac{a}{1+\mu} \\ b & -\frac{a}{1+\mu} & -\frac{a}{1+\mu} \end{pmatrix},$$

with inverse

$$A^{-1} = \frac{1+\mu}{2ar} \begin{pmatrix} -\frac{2a^2}{(1+\mu)^2} & -\frac{ab}{1+\mu} & -\frac{ac}{1+\mu} \\ \frac{a}{1+\mu}(c-b) & -\frac{a^2}{(1+\mu)^2} - \frac{b^2+bc}{2} & \frac{a^2}{(1+\mu)^2} + \frac{bc+c^2}{2} \\ -\frac{a}{1+\mu}(c+b) & \frac{a^2}{(1+\mu)^2} - \frac{b^2+bc}{2} & \frac{a^2}{(1+\mu)^2} + \frac{bc-c^2}{2} \end{pmatrix}.$$

From the entries in the first row, we see

$$\mathrm{pr}_+(L) = \frac{1}{2r}\gcd\left(\frac{2a}{1+\mu}, b, c\right)\mathbb{Z}x_0 = \begin{cases} \frac{1}{r}\mathbb{Z}x_0 & \text{if } (b,c) \text{ is even and } \mu = 0, \\ \frac{1}{2r}\mathbb{Z}x_0 & \text{otherwise.} \end{cases}$$

□

Corollary 6.7.

$$|L : L_+ + L_-| = \begin{cases} r & \textit{if } (b,c) \textit{ is even and } \mu = 0, \\ 2r & \textit{if } (b,c) \textit{ is odd and } \mu = 0, \\ 4r & \textit{if } \mu = 1. \end{cases} \tag{6.11}$$

Proof. This follows from Lemmas 6.5 and 6.6 and the fact that $L_+ = (1+\mu)\mathbb{Z}x_0$.

□

For the positive and negative lattices and their respective duals, we have

Lemma 6.8.

$$\left|L_+^\vee : L_+\right| = \begin{cases} 2r & \textit{if } \mu = 0, \\ 8r & \textit{if } \mu = 1, \end{cases} \tag{6.12}$$

and

$$\left|L_-^\vee : L_-\right| = \begin{cases} r & \textit{if } (b,c) \textit{ is even and } \mu = 0, \\ 4r & \textit{otherwise.} \end{cases} \tag{6.13}$$

Proof. $L_+ = (1+\mu)\mathbb{Z}x_0$ so

$$L_+^\vee = \{x \in V_+ \mid (x, L_+) \subseteq \mathbb{Z}\} = \frac{1}{(1+\mu)2r}\mathbb{Z}x_0,$$

which takes care of (6.12). For (6.13), we compare

$$\left|L_+^\vee + L_-^\vee : L_+ + L_-\right| = \left|L_+^\vee : L_+\right| \left|L_-^\vee : L_-\right| \tag{6.14}$$

with

$$\left|L_+^\vee + L_-^\vee : L_+ + L_-\right| = \left|L_+^\vee + L_-^\vee : L^\vee\right| \left|L^\vee : L\right| \left|L : L_+ + L_-\right|.$$

Then $\left|L_+^\vee + L_-^\vee : L^\vee\right| = |L : L_+ + L_-|$ and $|L^\vee : L| = 2$ imply

$$\left|L_+^\vee + L_-^\vee : L_+ + L_-\right| = \begin{cases} 2r^2 & \text{if } (b,c) \text{ is even and } \mu = 0, \\ 8r^2 & \text{if } (b,c) \text{ is odd and } \mu = 0, \\ 32r^2 & \text{if } \mu = 1. \end{cases}$$

Now formulas (6.12) and (6.14) give (6.13). □

We will see that $\mathrm{pr}_\pm(L^\vee)/L_\pm = L_\pm^\vee/L_\pm$, which implies that in some cases $\mathrm{pr}_\pm(L)/L_\pm = \mathrm{pr}_\pm(L^\vee)/L_\pm$, while in other cases they are not equal. We conclude this section by giving explicit coset representatives for $\mathrm{pr}_\pm(L^\vee)/L_\pm$.

Lemma 6.9. $\mathrm{pr}_\pm(L^\vee)/L_\pm = L_\pm^\vee/L_\pm$.

Proof. Map

$$L^\vee \to \mathrm{pr}_\pm(L^\vee)/L_\pm$$

by $l \mapsto \mathrm{pr}_\pm(l) + L_\pm$. If we assume $\mathrm{pr}_\pm(l) \in L_\pm$, then $l - \mathrm{pr}_\pm(l) = \mathrm{pr}_\mp(l) \in L^\vee \cap V_\mp$. So

$$L^\vee / \left(L_+ + (L^\vee \cap V_-)\right) \simeq \mathrm{pr}_+(L^\vee)/L_+,$$

and

$$L^\vee / \left((L^\vee \cap V_+) + L_-\right) \simeq \mathrm{pr}_-(L^\vee)/L_-. \tag{6.15}$$

Now

$$\begin{aligned} L^\vee \cap V_+ &= \left\{ \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} \mid a' \in \frac{1}{2}\mathbb{Z}, b', c' \in \mathbb{Z} \right\} \cap \mathbb{Q} \begin{pmatrix} \frac{a}{1+\mu} & b \\ c & -\frac{a}{1+\mu} \end{pmatrix} \\ &= \begin{cases} \frac{1}{2}\mathbb{Z}x_0 & \text{if } (b,c) \text{ is even and } \mu = 0, \\ \mathbb{Z}x_0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6.16}$$

Using $|L^\vee : (L^\vee \cap V_+) + L_-|\,|(L^\vee \cap V_+) : L_+| = |L^\vee : L_+ + L_-| = 2\,|L : L_+ + L_-|$ together with (6.11), (6.15) and (6.16), we see

$$\left|\mathrm{pr}_-(L^\vee)/L_-\right| = \begin{cases} r & \text{if } (b,c) \text{ is even and } \mu = 0, \\ 4r & \text{otherwise.} \end{cases}$$

Formula (6.13) implies $\mathrm{pr}_-(L^\vee)/L_- = L_-^\vee/L_-$. For $\mathrm{pr}_+(L^\vee)/L_+$, we know $L^\vee \cap V_- \supseteq L_-$ but we need to see when we have equality. Let

$$\gamma_1 = \begin{pmatrix} \frac{a_1}{2} & b_1 \\ c_1 & -\frac{a_1}{2} \end{pmatrix} \in L^\vee \cap V_-.$$

Then

$$\mathrm{tr}\left(\begin{pmatrix} \frac{a_1}{2} & b_1 \\ c_1 & -\frac{a_1}{2} \end{pmatrix} \begin{pmatrix} \frac{a}{1+\mu} & b \\ c & -\frac{a}{1+\mu} \end{pmatrix}^\iota \right) = -\frac{aa_1}{1+\mu} - b_1 c - c_1 b = 0.$$

If (b,c) is even and $\mu = 0$, or if $\mu = 1$, then a is odd so a_1 must be even. This implies $L^\vee \cap V_- = L_-$. Otherwise, a_1 can be odd and we have $|(L^\vee \cap V_-) : L_-| = 2$. As with $\mathrm{pr}_-(L^\vee)/L_-$, we see

$$\left|\mathrm{pr}_+(L^\vee)/L_+\right| = \begin{cases} 2r & \text{if } \mu = 0, \\ 8r & \text{if } \mu = 1, \end{cases}$$

so $\mathrm{pr}_+(L^\vee)/L_+ = L_+^\vee/L_+$ by (6.12). □

Corollary 6.10. *1. If (b,c) is even and $\mu = 0$, then $\mathrm{pr}_+(L)/L_+ \subsetneq \mathrm{pr}_+(L^\vee)/L_+$ while $\mathrm{pr}_-(L)/L_- = \mathrm{pr}_-(L^\vee)/L_-$.*

2. If (b,c) is odd and $\mu = 0$, then $\mathrm{pr}_+(L)/L_+ = \mathrm{pr}_+(L^\vee)/L_+$ while $\mathrm{pr}_-(L)/L_- \subsetneq \mathrm{pr}_-(L^\vee)/L_-$.

3. If $\mu = 1$, then $\mathrm{pr}_+(L)/L_+ \subsetneq \mathrm{pr}_+(L^\vee)/L_+$ while $\mathrm{pr}_-(L)/L_- = \mathrm{pr}_-(L^\vee)/L_-$.

Proof. This follows from (6.10), (6.11), Lemma 6.8 and Lemma 6.9. □

Let $L^\vee/L = \{0, \xi\}$, where $\xi = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}$ represents the nontrivial coset, and assume $\mathrm{pr}_+(\xi) = \eta x_0$ for some $\eta \in \mathbb{Q}^\times$. Then $(\xi, x_0) = (\mathrm{pr}_+(\xi), x_0) = 2r\eta$, and, by definition, $(\xi, x_0) = \mathrm{tr}(\xi x_0^\iota) = -\frac{a}{1+\mu}$. So $\mathrm{pr}_+(\xi) = -\frac{a}{2r(1+\mu)} x_0$. This implies

$$\begin{aligned} \mathrm{pr}_-(\xi) &= \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix} + \frac{a}{2r(1+\mu)} \begin{pmatrix} \frac{a}{1+\mu} & b \\ c & -\frac{a}{1+\mu} \end{pmatrix} \\ &= \frac{1}{2r(1+\mu)^2} \begin{pmatrix} r(1+\mu)^2 + a^2 & (1+\mu)ab \\ (1+\mu)ac & -r(1+\mu)^2 - a^2 \end{pmatrix} \\ &= \frac{1}{2r} \begin{pmatrix} -bc & \frac{ab}{1+\mu} \\ \frac{ac}{1+\mu} & bc \end{pmatrix}, \end{aligned}$$

since $-\frac{a^2}{(1+\mu)^2} - bc = r$. Let $\delta \in L^\vee/L$ be either coset. Then for a given coset $\lambda_+ \in \mathrm{pr}_+(L^\vee)/L_+$, we want to find the corresponding coset $\lambda_- \in \mathrm{pr}_-(L^\vee)/L_-$ such that $\lambda_+ + L_+ + \lambda_- + L_- \subseteq \delta + L$. Define

$$(\lambda_+, \lambda_-) < \delta$$

to mean $\lambda_+ + L_+ + \lambda_- + L_- \subseteq \delta + L$. Let l_1, l_2, u and v be as in Proposition 6.4.

Lemma 6.11. *1. If (b,c) is even and $\mu = 0$, then $\lambda_+ \in \mathrm{pr}_+(L)/L_+$ has the form $\lambda_+ = \frac{y}{r}x_0, 0 \le y < r$, and the λ_- for which $(\lambda_+, \lambda_-) < 0$ is $\lambda_- = y\left(\frac{A}{2}l_1 + Bl_2\right)$, where we choose $(\alpha, \beta, \gamma) \in \mathbb{Z}^3$ such that $2a\alpha + c\beta + b\gamma = -2$ and*

$$A = \frac{2(r\alpha - a)}{r(b,c)},\ B = \frac{(b,c)(b - r\beta) - 2au(r\alpha - a)}{rb}.$$

2. If (b,c) is odd and $\mu = 0$, then $\lambda_+ \in \mathrm{pr}_+(L)/L_+$ has the form $\lambda_+ = \frac{y}{2r}x_0, 0 \le y < 2r$, and the λ_- for which $(\lambda_+, \lambda_-) < 0$ is $\lambda_- = y(Al_1 + Bl_2)$, where we choose $(\alpha, \beta, \gamma) \in \mathbb{Z}^3$ such that $2a\alpha + c\beta + b\gamma = -1$ and

$$A = \frac{2r\alpha - a}{2r(b,c)},\ B = \frac{(b,c)(b - 2r\beta) - 2au(2r\alpha - a)}{2rb}.$$

3. If $\mu = 1$, then $\lambda_+ \in \mathrm{pr}_+(L)/L_+$ has the form $\lambda_+ = \frac{y}{2r}x_0, 0 \le y < 4r$, and the λ_- for which $(\lambda_+, \lambda_-) < 0$ is $\lambda_- = y(Al_1 + Bl_2)$, where we choose $(\alpha, \beta, \gamma) \in \mathbb{Z}^3$ such that $a\alpha + c\beta + b\gamma = -1$ and

$$A = \frac{4r\alpha - a}{4r(b,c)},\ B = \frac{(b,c)(2b - 4r\beta) - au(4r\alpha - a)}{4rb}.$$

Furthermore, in all 3 cases, (b,c) divides the numerator of A and b divides the numerator of B.

Proof. The proof in each case is very similar, so we only prove (1). Clearly we can just work with $\lambda_+ = \frac{1}{r}x_0$.

$$\frac{1}{r}x_0 + \frac{A}{2}l_1 + Bl_2 = \begin{pmatrix} \frac{a}{r} + \frac{A(b,c)}{2} & \frac{b}{r} - Aau - \frac{b}{(b,c)}B \\ \frac{c}{r} + Aav + \frac{c}{(b,c)}B & -\frac{a}{r} - \frac{A(b,c)}{2} \end{pmatrix}.$$

We want this matrix to lie in $M_2(\mathbb{Z})$. If $\frac{a}{r} + \frac{A(b,c)}{2} = \alpha \in \mathbb{Z}$, then $A = \frac{2(r\alpha - a)}{r(b,c)}$. For the second entry in row 1, if this is equal to β for some $\beta \in \mathbb{Z}$, then

$$\frac{b}{r} - Aau - \frac{b}{(b,c)}B = \frac{b(b,c) - 2au(r\alpha - a)}{r(b,c)} - \frac{b}{(b,c)}B = \beta$$

implies

$$B = -\frac{\beta(b,c)}{b} + \frac{b(b,c) - 2au(r\alpha - a)}{rb} = \frac{(b,c)(b - r\beta) - 2au(r\alpha - a)}{rb}. \quad (6.17)$$

For the first entry in row 2, if this equals γ for some $\gamma \in \mathbb{Z}$, then

$$\frac{c}{r} + Aav + \frac{c}{(b,c)}B = \frac{c(b,c) + 2av(r\alpha - a)}{r(b,c)} + \frac{c}{(b,c)}B = \gamma$$

implies

$$B = \frac{\gamma(b,c)}{c} - \frac{c(b,c) + 2av(r\alpha - a)}{rc} = \frac{(b,c)(\gamma r - c) - 2av(r\alpha - a)}{rc}. \quad (6.18)$$

Comparing (6.17) and (6.18) we have

$$b((b,c)(\gamma r - c) - 2av(r\alpha - a)) = c((b,c)(b - r\beta) - 2au(r\alpha - a)),$$

or

$$-2bc(b,c) + 2r\alpha a(uc - vb) + 2a^2(vb - uc) + (b,c)r(b\gamma + c\beta) = 0.$$

Since $uc - vb = (b,c)$, this becomes

$$2(b,c)\left(-a^2 - bc + r\alpha a + \frac{r}{2}(b\gamma + c\beta)\right) = 0.$$

From $-a^2 - bc = r$ we get

$$r(b,c)(2 + 2a\alpha + c\beta + b\gamma) = 0.$$

This means we want to find $(\alpha, \beta, \gamma) \in \mathbb{Z}^3$ such that $2 + 2a\alpha + c\beta + b\gamma = 0$, and we know we can do this since x_0 is primitive and (b,c) is even. Now we prove the last statement. The numerator of A is $2(r\alpha - a) = -2a^2\alpha - 2a - 2bc\alpha$, which modulo (b,c) is $-a(-2a\alpha - 2) \equiv -a(c\beta + b\gamma) \equiv 0 \pmod{(b,c)}$. Similarly, the numerator of $B \pmod b$ is $(b,c)(b - r\beta) - 2au(r\alpha - a) \equiv -(b,c)r\beta - 2aur\alpha + 2a^2u$ and using $a^2 \equiv -r \pmod b$, the numerator is congruent to $a^2(b,c)\beta + 2a^3u\alpha + 2a^2u \equiv a^2(b,c)\beta + ua^2(2a\alpha + 2) \equiv a^2(b,c)\beta - ua^2(c\beta + b\gamma) \equiv a^2\beta((b,c) - uc) \equiv a^2\beta(-vb)$. □

Corollary 6.12. *If $\delta_+ \in \mathrm{pr}_+(L^\vee)/L_+$ and $\delta_+ \notin \mathrm{pr}_+(L)/L_+$, then $\delta_+ = \mathrm{pr}_+(\xi) + \lambda_+$ for some $\lambda_+ \in \mathrm{pr}_+(L)/L_+$ and $(\mathrm{pr}_+(\xi) + \lambda_+, \mathrm{pr}_-(\xi) + \lambda_-) < \xi$, where λ_- is as in Lemma 6.11.*

Now we let $\lambda_\pm$ be the generators of $\mathrm{pr}_\pm(L)/L$ and let y be as in Lemma 6.11. Corollary 6.10 tells us that sometimes the cosets $\{y\lambda_\pm\}$ are disjoint from $\{\mathrm{pr}_\pm(\xi) + y\lambda_\pm\}$, while other times they are the same set of cosets. The following lemma gives an explicit description of $\mathrm{pr}_\pm(L^\vee)/L$, and in the cases where $\{y\lambda_\pm\} = \{\mathrm{pr}_\pm(\xi) + y\lambda_\pm\}$ shows exactly how the two sets agree.

Lemma 6.13.

1. *If (b,c) is even and $\mu = 0$, then $\mathrm{pr}_+(L^\vee)/L_+ = \{y\lambda_+ \mid 0 \le y < r\} \sqcup \{\mathrm{pr}_+(\xi) + y\lambda_+ \mid 0 \le y < r\}$ while $\mathrm{pr}_-(L^\vee)/L_- = \{y\lambda_+ \mid 0 \le y < r\}$ and $\mathrm{pr}_-(\xi) + y\lambda_- = \left(y - \frac{a+r}{2}\right)\lambda_-$.*

2. *If (b,c) is odd and $\mu = 0$, then $\mathrm{pr}_-(L^\vee)/L_- = \{y\lambda_- \mid 0 \le y < 2r\} \sqcup \{\mathrm{pr}_-(\xi) + y\lambda_- \mid 0 \le y < 2r\}$ while $\mathrm{pr}_+(L^\vee)/L_+ = \{y\lambda_+ \mid 0 \le y < 2r\}$ and $\mathrm{pr}_+(\xi) + y\lambda_+ = (y - a)\lambda_+$.*

3. *If* $\mu = 1$, *then* $\mathrm{pr}_+(L^\vee)/L_+ = \{y\lambda_+ \mid 0 \leq y < 4r\} \sqcup \{\mathrm{pr}_+(\xi) + y\lambda_+ \mid 0 \leq y < 4r\}$ *while* $\mathrm{pr}_-(L^\vee)/L_- = \{y\lambda_+ \mid 0 \leq y < 4r\}$ *and* $\mathrm{pr}_-(\xi) + y\lambda_- = \left(y - \frac{4r+a}{2}\right)\lambda_-$.

Proof. Our notation is as in Lemma 6.11. We just need to show that either the two sets are disjoint or that they agree as above. For case 1, if $\mathrm{pr}_+(\xi)+y\lambda_+ = \tilde{y}\lambda_+$ for some y and $\tilde{y}$, this is equivalent to having $\mathrm{pr}_+(\xi)+y\lambda_+ \in L_+ = \mathbb{Z}x_0$ for some y. Then

$$\mathrm{pr}_+(\xi) + y\lambda_+ = \frac{2y-a}{2r}\begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

but a is odd so $(2y-a)a \notin 2r\mathbb{Z}$. Since (b,c) is even, a and r are both odd so $\frac{a+r}{2} \in \mathbb{Z}$. For $\mathrm{pr}_-(L^\vee)/L_-$, we need to show that for $y = \frac{a+r}{2} \pmod r$, $\mathrm{pr}_-(\xi) + \frac{a+r}{2}\lambda_- = 0$. We have

$$\begin{aligned}
\mathrm{pr}_-(\xi) + \frac{a+r}{2}\lambda_- =& \frac{1}{2r}\begin{pmatrix} -bc & ab \\ ac & bc \end{pmatrix} + \frac{a+r}{2}\left(\frac{A}{2}l_1 + Bl_2\right) \\
=& \frac{1}{2r}\begin{pmatrix} -bc & ab \\ ac & bc \end{pmatrix} + \frac{a+r}{2r}\left[\frac{r\alpha - a}{(b,c)}\begin{pmatrix} (b,c) & -2au \\ 2av & -(b,c) \end{pmatrix} + \right. \\
& \left. \frac{(b,c)(b-r\beta) - 2au(r\alpha - a)}{b(b,c)}\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}\right] \\
=& \frac{1}{2r}\begin{pmatrix} -bc & ab \\ ac & bc \end{pmatrix} + \frac{a+r}{2rb(b,c)} \times \\
& \begin{pmatrix} b(b,c)(r\alpha - a) & b(b,c)(r\beta - b) \\ (r\alpha - a)2a(vb - uc) + c(b,c)(b - r\beta) & b(b,c)(a - r\alpha) \end{pmatrix}
\end{aligned}$$

$$= \frac{1}{2r}\begin{pmatrix} -bc & ab \\ ac & bc \end{pmatrix} + \frac{a+r}{2r}\begin{pmatrix} r\alpha - a & r\beta - b \\ \frac{1}{b}(c(b-r\beta) - (r\alpha - a)2a) & a - r\alpha \end{pmatrix}. \tag{6.19}$$

Then $c(b-r\beta)-(r\alpha-a)2a = cb - cr\beta - 2ar\alpha + 2a^2 = -2r - cr\beta - 2ar\alpha - bc = -r(-b\gamma) - bc$. So (6.19) becomes

$$= \frac{1}{2r}\begin{pmatrix} (a+r)(r\alpha-a)-bc & (a+r)(r\beta-b)+ab \\ (a+r)(r\gamma-c)+ac & bc-(a+r)(r\alpha-a) \end{pmatrix}$$

$$= \frac{1}{2r}\begin{pmatrix} ar\alpha + r^2\alpha + r - ra & ar\beta + r^2\beta - rb \\ ar\gamma + r^2\gamma - rc & -ar\alpha - r^2\alpha - r + ra \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \alpha(a+r)+1-a & \beta(a+r)-b \\ \gamma(a+r)-c & -\alpha(a+r)-1+a \end{pmatrix},$$

which lies in L_- since $a+r, 1-a, c, b \in 2\mathbb{Z}$. For case 2,

$$\mathrm{pr}_+(\xi) + a\lambda_+ = -\frac{a}{2r}x_0 + \frac{a}{2r}x_0 = 0.$$

Then a computation similar to the proof of case 1, with y instead of $\frac{a+r}{2}$, gives

$$\mathrm{pr}_-(\xi) + y\lambda_- = \frac{1}{2r}\begin{pmatrix} y(2r\alpha-a)-bc & y(2r\beta-b)+ab \\ y(2r\gamma-c)+ac & bc-y(2r\alpha-a) \end{pmatrix}.$$

If this matrix lies in L_-, then $-ay-bc, b(a-y), c(a-y) \in 2r\mathbb{Z}$. We know b or c is odd so assume b is. Then x_0 being primitive implies $(b,r)=1$. So $b(a-y) \in 2r\mathbb{Z}$ tells us $a-y \in 2r\mathbb{Z}$. Then $-ay-bc = -ay+r+a^2 = a(a-y)+r \in 2r\mathbb{Z}$, which is a contradiction. For case 3, $L_+ = 2\mathbb{Z}x_0$, while

$$\mathrm{pr}_+(\xi) + y\lambda_+ = \frac{2y-a}{4r}\begin{pmatrix} \frac{a}{2} & b \\ c & -\frac{a}{2} \end{pmatrix},$$

which never lies in $2\mathbb{Z}x_0$ since a is odd. Again, a computation similar to the above gives

$$\mathrm{pr}_-(\xi) + y\lambda_- = \frac{1}{4r}\begin{pmatrix} y(4r\alpha-a)-2bc & y(4r\beta-2b)+ab \\ y(4r\gamma-2c)+ac & 2bc-y(4r\alpha-a) \end{pmatrix}. \tag{6.20}$$

Since $r \in \frac{1}{4}\mathbb{Z} - \frac{1}{2}\mathbb{Z}$, $4r$ is odd. Letting $y = \frac{4r+a}{2} \pmod{4r}$ and using $-a^2 - 4bc = 4r$, (6.20) becomes

$$= \frac{1}{4r}\begin{pmatrix} 8r^2\alpha + 2ra\alpha - 2ra + 2r & 8r^2\beta + 2ra\beta - 4rb \\ 8r^2\gamma + 2ra\gamma - 4rc & -8r^2\alpha - 2ra\alpha + 2ra + 2r \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} \alpha(4r+a) + 1 - a & \beta(4r+a) - 2b \\ \gamma(4r+a) - 2c & -\alpha(4r+a) - 1 + a \end{pmatrix},$$

which is in L_- since $4r + a, 1 - a \in 2\mathbb{Z}$. □

6.3 Classical Interpretation of the $(n, 2)$-Theorem

Let $V(r) = \{x \in V \mid Q(x) = r\}$ for $r \in \mathbb{Q}$. For $\mu = 0, 1$, we have $L_\mu(r) = V(r) \cap L_\mu$. Let $G = GL_2, K = GL_2(\hat{\mathbb{Z}})$ and $\Gamma = GL_2(\mathbb{Z})$. We have $G(\mathbb{A}_f) = G(\mathbb{Q})K$. Then for $x \in L_\mu(r)$, we consider the sequence of maps

$$G_x(\mathbb{Q})\backslash (D_x \times G_x(\mathbb{A}_f)/K_x) \to G(\mathbb{Q})\backslash (D \times G(\mathbb{A}_f)/K) \simeq \Gamma\backslash D, \qquad (6.21)$$

where $K_x = K \cap G_x(\mathbb{A}_f)$. By Lemma 2.1 of [10], we have $G_x \simeq \mathrm{GSpin}\left(x^\perp\right)$ and $x^\perp$ is a negative definite space of signature $(0, 2)$. This tells us that

$$G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/K_x$$

is the space we sum over in Corollary 3.4. We wish to identify the image in $\Gamma\backslash D$ of the first space in (6.21). The isomorphism in (6.21) is given by

$$G(\mathbb{Q})(z, gK) \mapsto \Gamma\left(\gamma^{-1}z\right),$$

where $g = \gamma k_0, \gamma \in G(\mathbb{Q}), k_0 \in K$.

For $x \in V(r)$, we have

$$-x^2 = -\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^2 = -\begin{pmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{pmatrix} = rI.$$

So for $k = \mathbb{Q}\left(\sqrt{-r}\right)$ we get an embedding

$$\phi_x : k \hookrightarrow M_2(\mathbb{Q}),$$

by sending $\sqrt{-r} \mapsto x$. Note, if $A + B\sqrt{-r} \in k$, then $N(A + B\sqrt{-r}) = \det(AI + Bx)$. We define

$$\mathcal{O}_x = \phi_x^{-1}(M_2(\mathbb{Z})),$$

and $\mathcal{O}_x$ is an order in k.

Definition 6.14. *Let $R \subset M_2(\mathbb{Q})$ be an order. Given an order $\mathcal{O} \subset k$, we say $\phi : k \hookrightarrow M_2(\mathbb{Q})$ is $\mathcal{O}$-optimal with respect to R if $\phi(k) \cap R = \phi(\mathcal{O})$ (or $\phi^{-1}(R) = \mathcal{O}$).*

Note that if $\mu = 1$, then $r = \frac{r_0}{4}$ for some odd integer r_0. Then $k = \mathbb{Q}\left(\sqrt{-r}\right) = \mathbb{Q}\left(\sqrt{-r_0}\right)$ and $\phi_x\left(\sqrt{-r_0}\right) = 2x \in M_2(\mathbb{Z})$, while $\phi_x\left(\sqrt{-r}\right) = x \notin M_2(\mathbb{Z})$.

The group $G(\mathbb{A}_f)$ acts on orders in $M_2(\mathbb{Q})$. For $g \in G(\mathbb{A}_f)$ and an order $R \subset M_2(\mathbb{Q})$, the action is given by $gR = g\hat{R}g^{-1} \cap M_2(\mathbb{Q})$, where $\hat{R} = R \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. Let $R_0 = M_2(\mathbb{Z})$ and $T = G_x$, which is isomorphic to $k^\times$.

Lemma 6.15. *Assume ϕ_x is $\mathcal{O}$-optimal with respect to R_0 for some order $\mathcal{O} \subset k$. Then ϕ_x is $\mathcal{O}$-optimal with respect to gR_0 for all $g \in T(\mathbb{A}_f)$.*

Proof. Let $g \in T(\mathbb{A}_f)$ and $R = gR_0$. Then

$$\phi_x(k) \cap R = \phi_x(k) \cap gR_0,$$

and $g \in G_x(\mathbb{A}_f)$ tells us $g\phi_x(k) = \phi_x(k)$, so the above is

$$= g(\phi_x(k) \cap R_0) = g\phi_x(\mathcal{O}) = \phi_x(\mathcal{O}).$$

□

Fix an order $\mathcal{O} \subset k$. Define

$$\mathrm{Opt}(\phi_x, \mathcal{O}) := \{R \subset M_2(\mathbb{Q}) \mid R \text{ is an order and } \phi_x \text{ is } \mathcal{O}\text{-optimal w.r.t. } R\}.$$

We have the following well-known theorem.

Theorem 6.16. *$T(\mathbb{A}_f)$ acts transitively on $\mathrm{Opt}(\phi_x, \mathcal{O})$.*

Inside of $L_\mu(r)$, we define

$$L_\mu(r, \mathcal{O}) := \{x \in L_\mu(r) \mid \phi_x \text{ is } \mathcal{O}\text{-optimal w.r.t. } R_0\}.$$

Since $\sqrt{-r} \mapsto x$ we have $\mathbb{Z}\left[\sqrt{-r}\right] \subseteq \mathcal{O}$ if $\mu = 0$, and $\mathbb{Z}[\sqrt{-r_0}] \subseteq \mathcal{O}$ if $\mu = 1, r = \frac{r_0}{4}$.

Lemma 6.17. *$\Gamma = GL_2(\mathbb{Z})$ acts on $L_\mu(r, \mathcal{O})$.*

Proof. Let $g \in \Gamma, x \in L_\mu(r, \mathcal{O})$. We know that $g \cdot x \in L_\mu(r)$. By definition, $\phi_{g\cdot x}(\sqrt{-r}) = g \cdot x = gxg^{-1}$, so $\phi_{g\cdot x}(k) = g\phi_x(k)g^{-1}$. Then

$$\begin{aligned}
\phi_{g\cdot x}^{-1}(R_0) &= \{y \in k \mid \phi_{g\cdot x}(y) \in R_0\} \\
&= \left\{y \in k \mid \phi_x(y) \in g^{-1}R_0 g\right\} \\
&= \{y \in k \mid \phi_x(y) \in R_0\} \\
&= \phi_x^{-1}(R_0).
\end{aligned}$$

□

Fix $x_0 \in L_\mu(r, \mathcal{O})$. It follows from Witt's Theorem that $V(r) = G(\mathbb{Q})x_0$. So if $x \in L_\mu(r, \mathcal{O})$, then $x = \gamma \cdot x_0$ and $\phi_x = \gamma \cdot \phi_{x_0}$ for some $\gamma \in G(\mathbb{Q})$. Let $T = G_{x_0}$ and $\phi_0 = \phi_{x_0}$. The choice of γ in the expression $x = \gamma \cdot x_0$ is not unique, but is determined up to $\gamma T(\mathbb{Q})$.

Proposition 6.18. $\Gamma \backslash L_\mu(r, \mathcal{O}) \simeq T(\mathbb{Q}) \backslash \mathrm{Opt}(\phi_0, \mathcal{O})$.

Proof. We map

$$L_\mu(r, \mathcal{O}) \to T(\mathbb{Q}) \backslash \mathrm{Opt}(\phi_0, \mathcal{O}) \tag{6.22}$$

by sending $x \mapsto [\gamma^{-1} R_0]$. Note that $\phi_x(k) \cap R_0 = \phi_x(\mathcal{O})$ if and only if $\gamma \cdot \phi_0(k) \cap R_0 = \gamma \cdot \phi_0(\mathcal{O})$, which is equivalent to $\phi_0(k) \cap \gamma^{-1} R_0 = \phi_0(\mathcal{O})$. This tells us ϕ_0 is $\mathcal{O}$-optimal with respect to $\gamma^{-1} R_0$, i.e., (6.22) is well-defined. Now let $x_1, x_2 \in L_\mu(r, \mathcal{O}), x_1 = \gamma_1 x_0, x_2 = \gamma_2 x_0$. Then $\gamma_1^{-1} R_0 = \gamma_2^{-1} R_0$ if and only if $R_0 = \gamma_1 \gamma_2^{-1} R_0$. The action on R_0 is conjugation, so we need the following lemma.

Lemma 6.19. $N_{G(\mathbb{Q})}(M_2(\mathbb{Z})) = \Gamma \cdot Z(\mathbb{Q})$.

Proof. $GL_2(\mathbb{Q})$ acts on $\hat{R}_0$ if and only if it acts on each local piece. This means we need to prove

$$N_{G(\mathbb{Q}_p)}(M_2(\mathbb{Z}_p)) = GL_2(\mathbb{Z}_p) Z(\mathbb{Q}_p).$$

By the theory of elementary divisors, we have

$$GL_2(\mathbb{Q}_p) = \coprod_{a \geq b} GL_2(\mathbb{Z}_p) \begin{pmatrix} p^a & \\ & p^b \end{pmatrix} GL_2(\mathbb{Z}_p).$$

Let $g \in GL_2(\mathbb{Q}_p)$ and assume $g = g_1 \delta(a,b) g_2$ for $g_1, g_2 \in GL_2(\mathbb{Z}_p), \delta(a,b) = \left(\begin{smallmatrix} p^a & \\ & p^b \end{smallmatrix}\right)$. Then $g \in N_{G(\mathbb{Q}_p)}(M_2(\mathbb{Z}_p))$ if and only if

$$\delta(a,b) M_2(\mathbb{Z}_p) \delta(a,b)^{-1} = M_2(\mathbb{Z}_p).$$

The left hand side looks like

$$\begin{pmatrix} p^a & \\ & p^b \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} p^{-a} & \\ & p^{-b} \end{pmatrix} = \begin{pmatrix} r & sp^{a-b} \\ tp^{b-a} & u \end{pmatrix}, \tag{6.23}$$

and (6.23) is in $M_2(\mathbb{Z}_p)$ for all $r, s, t, u \in \mathbb{Z}_p$. This implies $a = b$, i.e., $\delta(a, b) \in Z(\mathbb{Q}_p)$ and $g \in GL_2(\mathbb{Z}_p)Z(\mathbb{Q}_p)$. □

So $R_0 = \gamma_1\gamma_2^{-1}R_0$ if and only if $\gamma_1\gamma_2^{-1} \in \Gamma \cdot Z(\mathbb{Q})$, which says x_1 and x_2 are Γ-equivalent. This means we have

$$\Gamma\backslash L_\mu(r, \mathcal{O}) \hookrightarrow T(\mathbb{Q})\backslash \mathrm{Opt}(\phi_0, \mathcal{O}).$$

It remains to show the map is onto. Let $R \in \mathrm{Opt}(\phi_0, \mathcal{O})$. We know, by Theorem 6.16, there is some element $g \in T(\mathbb{A}_f) \subset G(\mathbb{A}_f)$ such that $g^{-1}R_0 = R$. Then $G(\mathbb{A}_f) = G(\mathbb{Q})K$ and K stabilizes R_0, so writing $g^{-1} = \gamma^{-1}k_0 \in G(\mathbb{Q})K$ we have $\gamma^{-1}R_0 = R$. Let $x = \gamma \cdot x_0$. Then $[x]_\Gamma \mapsto [\gamma^{-1}R_0] = [R]$. □

Corollary 6.20. $\Gamma\backslash L_\mu(r, \mathcal{O}) \simeq T(\mathbb{Q})\backslash T(\mathbb{A}_f)/\hat{\mathcal{O}}^\times$.

Proof. Theorem 6.16 implies $\mathrm{Opt}(\phi_0, \mathcal{O}) = T(\mathbb{A}_f) \cdot R_0$. The stabilizer of R_0 in $G(\mathbb{A}_f)$ is $K \cdot Z(\mathbb{A}_f)$.

Lemma 6.21. *$K_x \simeq \hat{\mathcal{O}}^\times$ for any $x \in L_\mu(r, \mathcal{O})$.*

Proof. We have $\phi_x : k^\times \hookrightarrow GL_2(\mathbb{Q})$. Since $\phi_x(k) \cap M_2(\mathbb{Z}) = \phi_x(\mathcal{O})$, we know $\phi_x(k^\times) \cap GL_2(\mathbb{Z}) = \phi_x(\mathcal{O}^\times)$. Then $G_x \simeq k^\times$ implies $K_x = K \cap G_x(\mathbb{A}_f) \simeq \hat{\mathcal{O}}^\times$. □

So $\mathrm{Opt}(\phi_0, \mathcal{O}) = T(\mathbb{A}_f) \cdot R_0 \simeq T(\mathbb{A}_f)/\hat{\mathcal{O}}^\times Z(\mathbb{A}_f)$, but $Z(\mathbb{A}_f) = \mathbb{Q}^\times\hat{\mathbb{Z}}^\times$ so modding out by $\hat{\mathcal{O}}^\times$ kills the action of $Z(\mathbb{A}_f)$. □

If $D_{x_0} = \{z_0^+, z_0^-\}$, let $D_{x_0}^\pm = \{z_0^\pm\}$. For $x \in L_\mu(r, \mathcal{O})$, write $D_x = \{z_x^+, z_x^-\}$ and let $x = \gamma \cdot x_0, \gamma \in G(\mathbb{Q})$. We know γ is unique up to $T(\mathbb{Q})$ and we have

$T(\mathbb{Q}) \simeq k^\times$. Viewing the imaginary quadratic field inside of $M_2(\mathbb{Q})$, we have $\det(t) > 0$ for $t \in T(\mathbb{Q})$ since the determinant is the same as the norm. This means we can define

$$\operatorname{sgn}(x, x_0) := \operatorname{sgn}(\det(\gamma)).$$

Lemma 6.22. *Let $g \in G(\mathbb{Q})$. Then*

(i) $\operatorname{sgn}(gx, x_0) = \operatorname{sgn}(\det(g))\operatorname{sgn}(x, x_0)$*,*

(ii) $\operatorname{sgn}(-x, x_0) = -\operatorname{sgn}(x, x_0)$*.*

Proof. (i) is clear. For (ii), we know there is some element $\eta \in G(\mathbb{Q})$ such that $\eta \cdot x_0 = -x_0$. This says $\eta x_0 = -x_0\eta$ and so η^2 commutes with each element in the algebra $[1, x_0, \eta, \eta x_0]$. This implies $\eta^2 = sI$, some $s \in \mathbb{Q}^\times$. Then $x_0^2 = -rI$ implies we have the algebra $(s, -r)_{\mathbb{Q}}$, so $r > 0$ implies $s < 0$. Since $\operatorname{tr}(\eta) = 0$, $s = -\det(\eta)$, and hence $\det(\eta) < 0$. Then $\gamma\eta \cdot x_0 = -x$ implies $\operatorname{sgn}(-x, x_0) = \operatorname{sgn}(\det(\gamma\eta)) = -\operatorname{sgn}(x, x_0)$. $\square$

Let

$$L_\mu^\pm(r, \mathcal{O}) = \{x \in L_\mu(r, \mathcal{O}) \mid \operatorname{sgn}(x, x_0) = \pm 1\}.$$

Then $L_\mu(r, \mathcal{O}) = L_\mu^+(r, \mathcal{O}) \sqcup L_\mu^-(r, \mathcal{O})$. Part (i) of Lemma 6.22 implies $\Gamma^+ = SL_2(\mathbb{Z})$ preserves each piece, while $\Gamma - \Gamma^+$ switches the two. Part (ii) tells us that $x \mapsto -x$ gives a bijection between the two. If $x \in L_\mu^+(r, \mathcal{O})$, then $[x]_\Gamma = [x]_{\Gamma^+} \in \Gamma^+\backslash L_\mu^+(r, \mathcal{O})$, while if $x \in L_\mu^-(r, \mathcal{O})$, then $\exists \beta \in \Gamma - \Gamma^+$ such that $\beta x \in L_\mu^+(r, \mathcal{O})$ and so $[x]_\Gamma = [\beta x]_{\Gamma^+}$. This means

$$\Gamma\backslash L_\mu(r, \mathcal{O}) = \Gamma^+\backslash L_\mu^+(r, \mathcal{O}), \tag{6.24}$$

and we also have $\Gamma\backslash D = \Gamma^+\backslash D^+$.

Proposition 6.23. *The classical interpretation of*

$$T(\mathbb{Q})\backslash \left(D_{x_0}^+ \times T(\mathbb{A}_f)/\hat{\mathcal{O}}^\times \right)$$

is

$$Z_\mu(r,\mathcal{O}) = \sum_{\substack{x \in L_\mu^+(r,\mathcal{O}) \\ \bmod \Gamma^+}} \mathrm{pr}^+(z_x^+),$$

where $\mathrm{pr}^+ : D^+ \to \Gamma^+\backslash D^+$.

Proof. Proposition 6.18 and Corollary 6.20 imply

$$\Gamma\backslash L_\mu(r,\mathcal{O}) \simeq T(\mathbb{Q})\backslash \mathrm{Opt}(\phi_0,\mathcal{O}) \simeq T(\mathbb{Q})\backslash \left(D_{x_0}^+ \times T(\mathbb{A}_f)/\hat{\mathcal{O}}^\times \right). \tag{6.25}$$

Let $t \in T(\mathbb{A}_f), t = \gamma^{-1}k_0$. Looking at (6.25) in reverse order, $T(\mathbb{Q})(z_0^+, t\hat{\mathcal{O}}^\times)$ gets mapped to $[tR_0] = [\gamma^{-1}R_0]$ under the second isomorphism. Then this gets sent to $[\gamma \cdot x_0]_\Gamma \in \Gamma\backslash L_\mu(r,\mathcal{O})$. For the sequence of maps

$$T(\mathbb{Q})\backslash\left(D_{x_0}^+ \times T(\mathbb{A}_f)/\hat{\mathcal{O}}^\times\right) \to G(\mathbb{Q})\backslash\left(D \times G(\mathbb{A}_f)/K\right) \simeq \Gamma\backslash D, \tag{6.26}$$

the first map sends $T(\mathbb{Q})(z_0^+, t\hat{\mathcal{O}}^\times) \mapsto G(\mathbb{Q})(z_0^+, tK)$, and the isomorphism maps this to $\mathrm{pr}(\gamma \cdot z_0^+)$ for pr: $D \to \Gamma\backslash D$. Note that

$$\begin{aligned} \gamma^{-1}D_{\gamma x_0} &= \left\{\gamma^{-1}z \mid z \in D, (z, \gamma x_0) = 0\right\} \\ &= \left\{\gamma^{-1}z \mid z \in D, (\gamma^{-1}z, x_0) = 0\right\} \\ &= D_{x_0}. \end{aligned}$$

That is, $\gamma D_{x_0} = D_{\gamma x_0}$. So if $\gamma x_0 = x$, then

$$\gamma z_0^+ = z_x^{\mathrm{sgn}(x,x_0)}.$$

From (6.24), we can always choose x such that $\mathrm{sgn}(x, x_0) = +1$. So using (6.25) and (6.26), we see the image of $\Gamma^+\backslash L_\mu^+(r,\mathcal{O})$ in $\Gamma^+\backslash D^+$ is $Z_\mu(r,\mathcal{O})$. □

We conclude this section by giving an explicit description of $D_x = \{z_x^+, z_x^-\}$ as a pair of conjugate points in $\mathfrak{H}^+ \cup \mathfrak{H}^-$. To do this, we follow the appendix of [13]. Let $x = \begin{pmatrix} \frac{a}{1+\mu} & b \\ c & -\frac{a}{1+\mu} \end{pmatrix} \in L_\mu(r, \mathcal{O})$.

Lemma 6.24. $D_x = \left\{ \frac{a+(1+\mu)\sqrt{-r}}{(1+\mu)c}, \frac{a-(1+\mu)\sqrt{-r}}{(1+\mu)c} \right\}$.

Proof. In [13], (A.4) gives an identification of $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ with D by

$$z \mapsto w(z) = \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \bmod \mathbb{C}^\times.$$

Then (A.8) of [13] implies

$$D_x = \left\{ z \in \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) \mid (x, w(z)) = 0 \right\}.$$

We have

$$\begin{aligned}
(x, w(z)) &= \operatorname{tr} \left(\begin{pmatrix} \frac{a}{1+\mu} & b \\ c & -\frac{a}{1+\mu} \end{pmatrix} \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}^\iota \right) \\
&= \operatorname{tr} \left(\begin{pmatrix} \frac{a}{1+\mu} & b \\ c & -\frac{a}{1+\mu} \end{pmatrix} \begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix} \right) \\
&= cz^2 - \frac{2a}{1+\mu} z - b,
\end{aligned}$$

and the roots of this equation are

$$\frac{\frac{2a}{1+\mu} \pm \sqrt{\frac{4a^2}{(1+\mu)^2} + 4bc}}{2c} = \frac{a \pm (1+\mu)\sqrt{-r}}{(1+\mu)c}.$$

∎

Chapter 7

Recovering Gross-Zagier

In this chapter we reproduce a classic result of Gross and Zagier as a special case of our main theorem.

7.1 Gross-Zagier

Theorem 7.1 (Theorem 1.3 of [7]). *Let d_1 and d_2 be two negative fundamental discriminants which are relatively prime, and let $D = d_1 d_2$. Let w_1 and w_2 be the number of roots of unity in the quadratic orders of discriminants d_1 and d_2, respectively. Let*

$$J(d_1, d_2) = \left(\prod_{\substack{[\tau_1],[\tau_2] \\ \mathrm{disc}(\tau_i)=d_i}} (j(\tau_1) - j(\tau_2)) \right)^{\frac{4}{w_1 w_2}},$$

where $[\tau_i]$ denotes an equivalence class modulo $SL_2(\mathbb{Z})$. Then

$$J(d_1, d_2)^2 = \pm \prod_{\substack{x,n,n' \in \mathbb{Z} \\ n,n' > 0 \\ x^2 + 4nn' = D}} n^{\epsilon(n')}.$$

Here ϵ is defined as follows. If l is a prime such that $l \mid D$ or $l \nmid D$ and $(D,l)_l = 1$,

$$\epsilon(l) = \begin{cases} (d_1, l)_l & \text{if } (l, d_1) = 1, \\ (d_2, l)_l & \text{if } (l, d_2) = 1. \end{cases}$$

Then for $n = \prod_i l_i^{a_i}$, where for each i either $l_i \mid D$ or $(D, l_i)_{l_i} = 1$, define

$$\epsilon(n) = \prod_i \epsilon(l_i)^{a_i}.$$

In [7], two proofs of this theorem are given. The first is algebraic and the second is analytic. In the algebraic proof, they restrict to the case where $-d_1$ is a prime $q > 3, q \equiv 3 \pmod 4$. This algebraic proof is then generalized by Dorman, [6], to the case where $d_1 = -m$ is any odd (negative) fundamental discriminant. This gives the full result since $(d_1, d_2) = 1$. Here we recover Theorem 7.1, but our method of proof is completely different from that in [7] or [6]. For simplicity, we assume $d_1, d_2 < -4$.

7.2 Applying the $(n, 2)$-Theorem

Assume $d_1 = -m$ is odd. As in chapter 6, take $V = \{x \in M_2(\mathbb{Z}) \mid \mathrm{tr}(x) = 0\}$ and $L = M_2(\mathbb{Z}) \cap V$. We begin by choosing a primitive vector $x_0 \in L_1\left(\frac{m}{4}\right)$, where x_0 has the form

$$x_0 = \begin{pmatrix} \frac{a}{2} & b \\ c & -\frac{a}{2} \end{pmatrix}.$$

This vector determines a splitting of our space $V = \mathbb{Q}x_0 + x_0^{\perp}$. Next, we refer to section 6.2. Proposition 6.4 tells us $L_+ = 2\mathbb{Z}x_0$ and $L_+^{\vee} = \frac{1}{m}\mathbb{Z}x_0$, while the negative lattice is $L_- = l_1\mathbb{Z} + l_2\mathbb{Z}$, where

$$l_1 = ue_1 + ve_2 = \begin{pmatrix} (b,c) & -au \\ av & -(b,c) \end{pmatrix}, l_2 = \frac{b}{a(b,c)}e_1 + \frac{c}{a(b,c)}e_2 = \begin{pmatrix} & -\frac{b}{(b,c)} \\ \frac{c}{(b,c)} & \end{pmatrix}.$$

Lemmas 6.5 and 6.6 imply $L/(L_+ + L_-) \simeq \mathrm{pr}_+(L)/L_+$ and $\mathrm{pr}_+(L) = \frac{2}{m}\mathbb{Z}x_0$. Given $\lambda_+ = \frac{2}{m}x_0$, part (3) of Lemma 6.11 says the corresponding element λ_- such that $(\lambda_+, \lambda_-) < 0$ is $\lambda_- = Al_1 + Bl_2$, where

$$A = \frac{m\alpha - a}{m(b,c)},\ B = \frac{(b,c)(2b - m\beta) - au(m\alpha - a)}{mb}.$$

Letting $\xi = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}$ represent the nontrivial coset of L in $L^\vee$, we have

$$\mathrm{pr}_+(\xi) = -\frac{a}{m}x_0,\ \mathrm{pr}_-(\xi) = \frac{1}{m}\begin{pmatrix} -2bc & ab \\ ac & 2bc \end{pmatrix}.$$

To ease the notation, let $\xi_\pm = \mathrm{pr}_\pm(\xi)$. We view $x_0^\perp \simeq k = \mathbb{Q}(\sqrt{-m})$ and if $\mathfrak{A} \subseteq \mathcal{O}_k$ is the ideal corresponding to L_-, then Theorem 3.3 gives

$$\frac{1}{h_k}\sum_t \Phi(z_0^+, t; F) = \sum_{n\geq 0}\Bigg\{c_0(-n)\sum_{y\in\mathbb{Z}/m\mathbb{Z}}\sum_{w\in y\lambda_+ + L_+}\kappa(n - Q(w), y\lambda_-, \mathfrak{A}) +$$
$$c_1(-n)\sum_{y\in\mathbb{Z}/m\mathbb{Z}}\sum_{w\in\xi_+ + y\lambda_+ + L_+}\kappa(n - Q(w), \xi_- + y\lambda_-, \mathfrak{A})\Bigg\}.$$

We assume $c_0(0) = 0$ and use the relation

$$\Phi(z, t; F) = -2\log\left|\Psi(z, t; F)^2\right|,$$

where Ψ is a Borcherds form of weight 0 on $D \times T(\mathbb{A}_f)$. This, along with Corollary 6.20, gives

$$\frac{2}{h_k}\sum_{x\in\Gamma\backslash L_1\left(\frac{m}{4}\right)}\log\left|\Psi(z_0^+, z_x^+; F)^2\right| =$$

$$-\sum_{n>0}\Bigg\{c_0(-n)\sum_{y\in\mathbb{Z}/m\mathbb{Z}}\sum_{w\in y\lambda_+ + L_+}\kappa(n - Q(w), y\lambda_-, \mathfrak{A}) +$$
$$c_1(-n)\sum_{y\in\mathbb{Z}/m\mathbb{Z}}\sum_{w\in\xi_+ + y\lambda_+ + L_+}\kappa(n - Q(w), \xi_- + y\lambda_-, \mathfrak{A})\Bigg\}, \tag{7.1}$$

where $z_x^+ \in D_x$. In order to simplify this expression, we define

$$\kappa_1(t,\mu) = \sum_{q|m} \eta_q(t,\mu)\log(q)(\mathrm{ord}_q(t)+1)\rho(mt),$$

and

$$\kappa_2(t,\mu) = \sum_{p \text{ inert}} \eta_p(t,\mu)\log(p)(\mathrm{ord}_p(t)+1)\rho\left(\frac{mt}{p}\right).$$

Then Theorem 4.1 tells us

$$\kappa(t,\mu,\mathfrak{A}) = -\frac{2^{k(\mu)}}{h_k}\begin{cases}\kappa_1(t,0)+\kappa_2(t,0) & \text{if } \mu = 0,\\ \prod_{q|m}\mathrm{char}(Q(\mu_q)+\mathbb{Z}_q)(t)(\kappa_1(t,\mu)+\kappa_2(t,\mu)) & \text{if } \mu \neq 0.\end{cases}$$

Plugging this into (7.1), we see we can cancel off the term $-\frac{1}{h_k}$ on each side.

We would like to simplify (7.1). For $y \in \mathbb{Z}/m\mathbb{Z}$, write $k(y)$ for $k(y\lambda_-)$ and $\kappa_j(t,y)$ for $\kappa_j(t,y\lambda_-)$, $j = 1,2$. Let us first focus on the double sum next to $c_0(-n)$. For $y = 0$, we have $w = 2sx_0$ for some $s \in \mathbb{Z}$ giving

$$(-h_k)\sum_{w\in L_+}\kappa(n-Q(w),0,\mathfrak{A}) = 2^{k(0)}\sum_{s\in\mathbb{Z}}(\kappa_1\left(n-s^2m,0\right)+\kappa_2\left(n-s^2m,0\right)). \tag{7.2}$$

For $y \neq 0, w \in y\lambda_+ + L_+$ is of the form $w = y\lambda_+ + 2sx_0$ and $Q(w) = \frac{y^2}{m} + 2ys + s^2m = m\left(\frac{y}{m}+s\right)^2$. Considering the characteristic function in the formula for $\kappa(t,y\lambda_-,\mathfrak{A})$, we note that $Q(w) \equiv Q(y\lambda_+) \pmod{\mathbb{Z}}$. So for $q \mid m$,

$$\mathrm{ord}_q(n-Q(w)-Q(y\lambda_-)) = \mathrm{ord}_q(n-y^2Q(\lambda)). \tag{7.3}$$

For our purposes, (7.3) ≥ 0 since $n \in \frac{1}{4}\mathbb{Z}$ and $\lambda \in L$. Putting this together with (7.2) we have

$$c_0(-n)\sum_{y\in\mathbb{Z}/m\mathbb{Z}}\sum_{s\in\mathbb{Z}}2^{k(y)}\left(\kappa_1\left(n-m\left(\frac{y}{m}+s\right)^2,y\right)+\kappa_2\left(n-m\left(\frac{y}{m}+s\right)^2,y\right)\right).$$

For the double sum next to $c_1(-n)$, we refer to section 6.2. Using Lemma 6.9 and Corollary 6.10 we have

$$\mathrm{pr}_-(L)/L_- = \mathrm{pr}_-(L^\vee)/L_-,$$

and

$$\mathrm{pr}_+(L)/L_+ \subsetneq \mathrm{pr}_+(L^\vee)/L_+ = L_+^\vee/L_+.$$

Then Lemma 6.13 implies

$$\xi_- + y\lambda_- + L_- = \Big(y - \frac{m+a}{2}\Big)\lambda_- + L_-,$$

where $\frac{m+a}{2}$ is reduced modulo m, while a full set of representatives of L_+ in $L_+^\vee$ is given by $\{y\lambda_-\} \cup \{\xi_- + y\lambda_-\}$. For $w \in \xi_+ + y\lambda_+ + L_+, w = \xi_+ + y\lambda_+ + 2sx_0$ for some $s \in \mathbb{Z}$. Then

$$Q(w) = \frac{(2y-a)^2}{4m} + s(2y-a) + s^2 m = m\left(\frac{2y-a}{2m} + s\right)^2,$$

and as above we have

$$\mathrm{ord}_q\left(n - Q(w) - Q\left(\left(y - \frac{m+a}{2}\right)\lambda_-\right)\right) = \mathrm{ord}_q(n - Q(\xi + y\lambda))$$

for any $q \mid m$. Since $n \in \frac{1}{4}\mathbb{Z}$ and $Q(\xi + y\lambda) \in \frac{1}{4}\mathbb{Z}$ as well, we do not need to worry about the characteristic functions here either. For the second part of the right hand side of (7.1), we have

$$c_1(-n)\Bigg[\sum_{y\in\mathbb{Z}/m\mathbb{Z}}\sum_{s\in\mathbb{Z}} 2^{k\left(y-\frac{m+a}{2}\right)}\left(\kappa_1\left(n - m\left(\frac{2y-a}{2m}+s\right)^2, y - \frac{m+a}{2}\right) + \kappa_2\left(n - m\left(\frac{2y-a}{2m}+s\right)^2, y - \frac{m+a}{2}\right)\right)\Bigg].$$

7.3 A Theorem of Dorman

To prove Theorem 7.1, we prove the following theorem of Dorman, which is equivalent to the result of Gross and Zagier.

Theorem 7.2 (Theorem 1.2 of [6]). *Let l be a rational prime and e its ramification index in $\mathbb{Q}(\sqrt{-m})$. Then*

$$\operatorname{ord}_l(J(-m,-d)) = \frac{1}{2e}\sum_{s\in\mathbb{Z}}\sum_{n\geq 1}\varrho_l(s)\rho\left(\frac{md-s^2}{4l^n}\right), \tag{7.4}$$

where

$$\varrho_l(s) = \begin{cases} 0 & \text{if there is } q \mid m; q\neq l \text{ such that } \chi_q(s^2-md) = -1, \\ 2^{a(s)} & \text{otherwise, where } a(s) = \#\{q \mid (s,m)\}. \end{cases}$$

Proof. The proof is broken up into two cases, based on whether $d_2 = -d \equiv 0, 1 \pmod 4$. We first need to translate some things from [7] into our language. In particular, we need to see how the set

$$\{[\tau] \in SL_2(\mathbb{Z})\backslash\mathfrak{H} \mid \operatorname{disc}(\tau) = -d\}$$

relates to the set $Z_\mu(r, \mathcal{O})$ defined in section 6.3.

Lemma 7.3. *For $d \in \mathbb{Z}_{>0}$,*

$$\left\{[\tau] \in \Gamma^+\backslash\mathfrak{H} \mid \operatorname{disc}(\tau) = -d\right\} = \begin{cases} Z_0\left(\frac{d}{4}, \mathcal{O}_d\right) & \text{if } -d \equiv 0 \pmod 4, \\ Z_1\left(\frac{d}{4}, \mathcal{O}_d\right) & \text{if } -d \equiv 1 \pmod 4, \end{cases}$$

where $\mathcal{O}_d$ is the maximal order in the field $k = \mathbb{Q}\left(\sqrt{-d}\right)$ and $\Gamma^+ = SL_2(\mathbb{Z})$.

Proof. If $\tau \in \mathfrak{H}$ has $A\tau^2 + B\tau + C = 0$ for $A, B, C \in \mathbb{Z}$ with $\gcd(A, B, C) = 1$, then $\operatorname{disc}(\tau) = B^2 - 4AC$. This is, of course, the same as the discriminant of the quadratic form $(A, B, C) = AX^2 + BXY + CY^2$. The matrix for (A, B, C) is $\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}$ and the action of Γ^+ on (A, B, C) is given by

$$y = \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \mapsto {}^t\gamma y \gamma.$$

If $\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma^+$, then $\gamma \cdot (A, B, C)$ equals

$$\left(Aa_1^2 + Ba_1c_1 + Cc_1^2\right) X^2 + \left(2Aa_1b_1 + B(b_1c_1 + a_1d_1) + 2Cc_1d_1\right) XY +$$

$$\left(Ab_1^2 + Bb_1d_1 + Cd_1^2\right) Y^2,$$

while $A(\gamma\tau)^2 + B(\gamma\tau) + C$ equals

$$\tau^2 \left(Aa_1^2 + Ba_1c_1 + Cc_1^2\right) + \tau \left(2Aa_1b_1 + B(b_1c_1 + a_1d_1) + 2Cc_1d_1\right) +$$

$$\left(Ab_1^2 + Bb_1d_1 + Cd_1^2\right).$$

This means

$$\left\{[\tau] \in \Gamma^+ \backslash \mathfrak{H} \mid \operatorname{disc}(\tau) = -d\right\} \rightleftharpoons \left\{\Gamma^+(A, B, C) \mid \operatorname{disc}(A, B, C) = -d\right\}.$$

We have

$$\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} = \frac{1}{2} J^{-1} x = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B & 2C \\ -2A & -B \end{pmatrix},$$

and the above action on (A, B, C) corresponds to $x \mapsto \gamma^{-1} x \gamma$. If we have a primitive vector $x = \begin{pmatrix} \frac{a}{1+\mu} & b \\ c & -\frac{a}{1+\mu} \end{pmatrix} \in L_\mu(r, \mathcal{O}_{4r})$ for some $r \in \mathbb{Q}$, then Lemma 6.24 implies

$$z_x^+ = \frac{a + (1+\mu)\sqrt{-r}}{(1+\mu)c},$$

which is a root of $c\tau^2 - \frac{2a}{1+\mu}\tau - b = 0$. We see $\operatorname{disc}(z_x^+) = -4r$ and we want $\operatorname{disc}(z_x^+) = -d$, so we choose $r = \frac{d}{4}$. Primitivity tells us if $-d \equiv 0 \pmod 4$, then $\mu = 0$, while if $-d \equiv 1 \pmod 4$ we have $\mu = 1$. $\square$

Lemma 7.4. *If $-d$ is a negative fundamental discriminant, then*

(i) $-d \equiv 0 \pmod 4$ *implies* $L_0\left(\frac{d}{4}, \mathcal{O}_d\right) = L_0\left(\frac{d}{4}\right)$,

(ii) $-d \equiv 1 \pmod 4$ *implies* $L_1\left(\frac{d}{4}, \mathcal{O}_d\right) = L_1\left(\frac{d}{4}\right)$.

Proof. For $\mu = 0, 1$, let $x \in L_\mu\left(\frac{d}{4}\right)$. Then $x \in L_\mu\left(\frac{d}{4}, \mathcal{O}\right)$ for some order $\mathcal{O}$. If $\mu = 0$, we have $\mathbb{Z}[\sqrt{-d/4}] \subseteq \mathcal{O} \subseteq \mathcal{O}_d$, while if $\mu = 1$, then $\mathbb{Z}[\sqrt{-d}] \subseteq \mathcal{O} \subseteq \mathcal{O}_d$. In case (i), $d = 4d'$ where d' is square-free and $-d' \equiv 2, 3 \pmod 4$. Then $\mathcal{O}_d = \mathbb{Z}[\sqrt{-d'}]$ so $\mathcal{O} = \mathcal{O}_d$. For (ii), we have

$$\phi_x\left(\frac{1+\sqrt{-d}}{2}\right) = \frac{1}{2}I + x = \frac{1}{2}I + \begin{pmatrix} \frac{a}{2} & b \\ c & -\frac{a}{2} \end{pmatrix} = \begin{pmatrix} \frac{a+1}{2} & b \\ c & -\frac{a+1}{2} \end{pmatrix},$$

which is in $M_2(\mathbb{Z})$ since a is odd. Thus, $\mathcal{O} = \mathcal{O}_d$. □

<u>Case 1</u>: $-d \equiv 0 \pmod 4$.

Let

$$J_d(\tau) = \prod_{\tau_2 \in Z_0\left(\frac{d}{4}, \mathcal{O}_d\right)} \Big(j(\tau) - j(\tau_2)\Big).$$

Then the zero set of $J_d(\tau)$ is $Z_0\left(\frac{d}{4}, \mathcal{O}_d\right)$. Theorem 1.3 of [12], which is a restatement of Theorem 13.3 of [2], says

$$\operatorname{div}(\Psi(F)^2) = \sum_{\mu \in L^\vee/L} \sum_{n>0} c_\mu(-n) Z(n, \mu, K).$$

The $c_\mu(-n)$'s are the negative Fourier coefficients of F while, in our situation, (A.16) of [13] implies

$$Z(n, \mu, K) = \sum_{\substack{x \in L_\mu(n) \\ \bmod \Gamma^+}} \operatorname{pr}^+(z_x^+). \tag{7.5}$$

This sum is taken over $\Gamma^+ \backslash L_\mu(n)$. If we take $n = \frac{d}{4}$ for d as in Lemma 7.4, then $L_\mu\left(\frac{d}{4}\right) = L_\mu\left(\frac{d}{4}, \mathcal{O}_d\right)$. Since $L_\mu\left(\frac{d}{4}, \mathcal{O}_d\right) = L_\mu^+\left(\frac{d}{4}, \mathcal{O}_d\right) \sqcup L_\mu^-\left(\frac{d}{4}, \mathcal{O}_d\right)$ and Γ^+

preserves each piece, this implies

$$Z\left(\frac{d}{4}, \mu, K\right) = 2Z_\mu\left(\frac{d}{4}, \mathcal{O}_d\right).$$

This means if we want to have $\mathrm{div}(\Psi(F)^2) = \mathrm{div}(J_d(\tau)^2)$, we need to choose F such that $c_0\left(-\frac{d}{4}\right) = 1$ and all other negative Fourier coefficients equal zero. By Lemma 6.3, we know that such a function exists. Then Proposition 6.23 implies we sum over $x \in \Gamma^+\backslash L_1^+\left(\frac{m}{4}, \mathcal{O}_k\right)$ and evaluate Ψ at z_x^+. This gives

$$4\log|J(-m,-d)| =$$

$$\sum_{y\in\mathbb{Z}/m\mathbb{Z}}\sum_{s\in\mathbb{Z}} 2^{k(y)}\left(\kappa_1\left(\frac{d}{4} - m\left(\frac{y}{m}+s\right)^2, y\right) + \kappa_2\left(\frac{d}{4} - m\left(\frac{y}{m}+s\right)^2, y\right)\right)$$

$$\begin{aligned}
= \sum_{y\in\mathbb{Z}/m\mathbb{Z}}\sum_{s\in\mathbb{Z}} 2^{k(y)}\Bigg[&\sum_{q|m}\eta_q\left(\frac{d}{4} - m\left(\frac{y}{m}+s\right)^2, y\right)\log(q)\times \\
&\left(\mathrm{ord}_q\left(\frac{d}{4} - m\left(\frac{y}{m}+s\right)^2\right)+1\right)\rho\left(\frac{md-4m^2\left(\frac{y}{m}+s\right)^2}{4}\right) + \\
&\sum_{p\text{ inert}}\eta_p\left(\frac{d}{4} - m\left(\frac{y}{m}+s\right)^2, y\right)\log(p)\times \\
&\left(\mathrm{ord}_p\left(\frac{d}{4} - m\left(\frac{y}{m}+s\right)^2\right)+1\right)\rho\left(\frac{md-4m^2\left(\frac{y}{m}+s\right)^2}{4p}\right)\Bigg],
\end{aligned}$$

where we write $\eta_q(t, y)$ for $\eta_q(t, y\lambda_-)$ and $\eta_p(t, y)$ for $\eta_p(t, y\lambda_-)$. From the definitions of η_q and η_p, we see that $\eta_q(mt, \mu) = \eta_q(t, \mu)$ and similarly for η_p, so the above is

$$= \sum_{y\in\mathbb{Z}/m\mathbb{Z}} \sum_{s\in\mathbb{Z}} 2^{k(y)} \left[\sum_{q|m} \eta_q \left(\frac{md - 4m^2\left(\frac{y}{m}+s\right)^2}{4}, y \right) \log(q) \times \right.$$

$$\operatorname{ord}_q \left(\frac{md - 4m^2\left(\frac{y}{m}+s\right)^2}{4} \right) \rho \left(\frac{md - 4m^2\left(\frac{y}{m}+s\right)^2}{4} \right) +$$

$$\sum_{p \text{ inert}} \eta_p \left(\frac{md - 4m^2\left(\frac{y}{m}+s\right)^2}{4}, y \right) \log(p) \times$$

$$\left.\left(\operatorname{ord}_p \left(\frac{md - 4m^2\left(\frac{y}{m}+s\right)^2}{4} \right) + 1 \right) \rho \left(\frac{md - 4m^2\left(\frac{y}{m}+s\right)^2}{4p} \right) \right]$$

$$= \sum_{y\in\mathbb{Z}/m\mathbb{Z}} \sum_{s\in m\mathbb{Z}} 2^{k(y)} \left[\sum_{q|m} \eta_q \left(\frac{md - 4\left(y+s\right)^2}{4}, y \right) \log(q) \times \right.$$

$$\operatorname{ord}_q \left(\frac{md - 4\left(y+s\right)^2}{4} \right) \rho \left(\frac{md - 4\left(y+s\right)^2}{4} \right) +$$

$$\sum_{p \text{ inert}} \eta_p \left(\frac{md - 4\left(y+s\right)^2}{4}, y \right) \log(p) \times$$

$$\left.\left(\operatorname{ord}_p \left(\frac{md - 4\left(y+s\right)^2}{4} \right) + 1 \right) \rho \left(\frac{md - 4\left(y+s\right)^2}{4p} \right) \right]. \tag{7.6}$$

For any $s \in m\mathbb{Z}$, we have $k(y+s) = \#\{q \text{ ramified} \mid ((y+s)\lambda_-)_q = 0\}$, and $((y+s)\lambda_-)_q = 0$ if and only if $(y\lambda_-)_q = 0$ since $s \in m\mathbb{Z}$. So $k(y+s) = k(y)$ and it follows that $\eta_q(t, y+s) = \eta_q(t,y)$ and $\eta_p(t, y+s) = \eta_p(t,y)$. Now we can write (7.6) as

$$\sum_{s\in\mathbb{Z}} 2^{k(s)} \left[\sum_{q|m} \eta_q \left(\frac{md - s^2}{4}, s \right) \log(q) \operatorname{ord}_q \left(\frac{md - s^2}{4} \right) \rho \left(\frac{md - s^2}{4} \right) + \right.$$

$$\left. \sum_{p \text{ inert}} \eta_p \left(\frac{md - s^2}{4}, s \right) \log(p) \left(\operatorname{ord}_p \left(\frac{md - s^2}{4} \right) + 1 \right) \rho \left(\frac{md - s^2}{4p} \right) \right]. \tag{7.7}$$

Note that, in (7.7), if $s \in \mathbb{Z}$ is odd, then md is not congruent to $s^2 \pmod 4$ and so $\rho\left(\frac{md-s^2}{4p^i}\right) = 0$ for $i = 0, 1$, and p inert.

Recall that (7.7)$=$ $\log|J(-m,-d)^4|$. We now compare formula (7.7) with formula (7.4) in Theorem 7.2 and show that they agree. The proof is done with several lemmas, and we work separately with the cases where the prime is inert or ramified. Let p be an inert prime. Then inside the logarithm we have

$$\mathrm{ord}_p(7.7) = \sum_{s\in\mathbb{Z}} 2^{k(s)}\eta_p\left(\frac{md-s^2}{4}, s\right)\left(\mathrm{ord}_p\left(\frac{md-s^2}{4}\right)+1\right)\rho\left(\frac{md-s^2}{4p}\right). \tag{7.8}$$

Both [7] and [6] state, without proof, that $\epsilon\left(\frac{D-s^2}{4}\right) = -1$ holds for general relatively prime d_1 and d_2. This fact is useful so we state it as a lemma and give a proof of it.

Lemma 7.5. *If d_1 and d_2 are two negative fundamental discriminants which are relatively prime, and $D = d_1d_2$, then $\epsilon\left(\frac{D-s^2}{4}\right) = -1$ for any $s \in \mathbb{Z}$ with $s^2 < D$ and $s^2 \equiv D \pmod 4$.*

Proof. Assume, without loss of generality, that $d_1 \equiv 1 \pmod 4$. Write $d_1 = -p_1\cdots p_u$ and $d_2 = -q_1^a q_2\cdots q_v$, where $p_1,\ldots,p_u,q_2,\ldots,q_v$ are all odd primes and either q_1 is an odd prime and $a = 1$ or $q_1 = 2$ and $a = 2, 3$. Assume we have

$$\frac{D-s^2}{4} = \prod_{i=1}^{u} p_i^{a_i}\prod_{j=1}^{v} q_j^{b_j}\prod_{k=1}^{w} l_k^{c_k},$$

where $l_k \nmid D, a_i, b_j \geq 0, c_k > 0$. Then

$$\epsilon\left(\frac{D-s^2}{4}\right) = \prod_{i=1}^{u}(d_2,p_i)_{p_i}^{a_i}\prod_{j=1}^{v}(d_1,q_j)_{q_j}^{b_j}\prod_{k=1}^{w}(d_1,l_k)_{l_k}^{c_k}. \tag{7.9}$$

Now, $(d_2,p_i)_{p_i}^{a_i} = (d_2, p_1^{a_1}\cdots p_u^{a_u})_{p_i} = \left(d_2, \frac{D-s^2}{4}\right)_{p_i}$ since for all $l \neq p_i, (d_2,l)_{p_i} = 1$. This also works if we replace d_2 with d_1 and p_i with l_k or q_j. Note for $l \neq 2$, $(d_1,l)_2 = (-1)^{\frac{d_1-1}{2}\frac{l-1}{2}} = 1$ because $d_1 \equiv 1 \pmod 4$. So (7.9) becomes

$$\begin{aligned}\epsilon\left(\frac{D-s^2}{4}\right) &= \prod_{i=1}^{u}\left(d_2,\frac{D-s^2}{4}\right)_{p_i}\prod_{j=1}^{v}\left(d_1,\frac{D-s^2}{4}\right)_{q_j}\prod_{k=1}^{w}\left(d_1,\frac{D-s^2}{4}\right)_{l_k}\\ &= \prod_{p\mid d_1}\left(d_2,\frac{D-s^2}{4}\right)_{p}\prod_{p'\nmid d_1}\left(d_1,\frac{D-s^2}{4}\right)_{p'},\end{aligned}$$

since if $p' \nmid D$ and $p' \nmid \frac{D-s^2}{4}$, then $\left(d_1, \frac{D-s^2}{4}\right)_{p'} = 1$. $\frac{D-s^2}{4}$ is positive which means the product formula for the Hilbert symbol implies

$$\epsilon\left(\frac{D-s^2}{4}\right) = \prod_{p|d_1}\left(d_2, \frac{D-s^2}{4}\right)_p \prod_{p|d_1}\left(d_1, \frac{D-s^2}{4}\right)_p = \prod_{p|d_1}\left(D, D-s^2\right)_p.$$

Fix a prime $p \mid d_1$. If $p \nmid s$, then $(D, D-s^2)_p = \left(\frac{D-s^2}{p}\right) = \left(\frac{-1}{p}\right)$ since $D \equiv 0 \pmod{p}$. If $p \mid s$, then $\frac{D-s^2}{p}$ is a unit in $\mathbb{Z}_p$ and $\frac{D-s^2}{p} = d_1' d_2 - p s_1^2 \equiv d_1' d_2 \pmod{p}$, where $d_1 = p d_1', s = p s_1$. We see

$$\begin{aligned}\left(D, D-s^2\right)_p &= (D,p)_p\left(D, \frac{D-s^2}{p}\right)_p = (D,p)_p(D, d_1' d_2)_p \\ &= (D,D)_p = (D,-1)_p = \left(\frac{-1}{p}\right).\end{aligned}$$

So

$$\epsilon\left(\frac{D-s^2}{4}\right) = \prod_{p|d_1}\left(\frac{-1}{p}\right) = -1,$$

since $-d_1 \equiv 3 \pmod{4}$ implies $-d_1$ is divisible by an odd number of primes $p \equiv 3 \pmod{4}$. ☐

Lemma 7.6. *$\sum_{n\geq 1} \rho\left(\frac{md-s^2}{4p^n}\right) = \frac{1}{2}\left(\mathrm{ord}_p\left(\frac{md-s^2}{4}\right)+1\right)\rho\left(\frac{md-s^2}{4p}\right)$ for an inert prime p.*

Proof. Inert primes satisfy $\epsilon(p) = -1$, so Lemma 7.5 implies there must be an odd number of inert primes which are raised to an odd power in the factorization of $\frac{md-s^2}{4}$. If there are more than one or none of them are p, then $\rho\left(\frac{md-s^2}{4p^n}\right) = 0$ for all $n \geq 1$. Otherwise, we can write

$$\frac{md-s^2}{4} = p^{2a+1} l_1^{2a_1} \cdots l_u^{2a_u} q_1^{b_1} \cdots q_v^{b_v},$$

where $\epsilon(p) = \epsilon(l_i) = -1$ and $\epsilon(q_i) = +1$. Then

$$\begin{aligned}
\sum_{n\geq 1} \rho\left(\frac{md-s^2}{4p^n}\right) &= \sum_{n=1}^{2a+1} \rho\left(\frac{md-s^2}{4p^n}\right) \\
&= \Big(\#\{n \mid 1 \leq n \leq 2a+1, n \text{ odd}\}\Big)(b_1+1)\cdots(b_v+1) \\
&= \frac{1}{2}(2a+2)(b_1+1)\cdots(b_v+1) \\
&= \frac{1}{2}\left(\operatorname{ord}_p\left(\frac{md-s^2}{4}\right)+1\right)\rho\left(\frac{md-s^2}{4p}\right).
\end{aligned}$$

□

Lemma 7.6 tells us that (7.8) can be written

$$\operatorname{ord}_p(7.7) = 2\sum_{s\in\mathbb{Z}}\sum_{n\geq 1} 2^{k(s)}\eta_p\left(\frac{md-s^2}{4}, s\right)\rho\left(\frac{md-s^2}{4p^n}\right),$$

while Theorem 7.2 gives

$$\operatorname{ord}_p\left(J(-m,-d)^4\right) = 2\sum_{s\in\mathbb{Z}}\sum_{n\geq 1} \varrho_p(s)\rho\left(\frac{md-s^2}{4p^n}\right).$$

We now prove a very useful lemma.

Lemma 7.7. *If q is a ramified prime, then $\operatorname{ord}_q(s^2-md) \leq 1$ and*

$$\chi_q(s^2-md) = \begin{cases} 1 & \text{if } \operatorname{ord}_q(s^2-md) = 0, \\ \epsilon(q) & \text{if } \operatorname{ord}_q(s^2-md) = 1. \end{cases}$$

Proof. Since $(d,m) = 1$, $\operatorname{ord}_q(s^2-md) \leq 1$. Then $\chi_q(s^2-md) = (s^2-md,-m)_q$ while $\epsilon(q) = (-d,q)_q$. Say $q \nmid (s^2-md)$. Then $\chi_q(s^2-md)$ only depends on $s^2-md \pmod q$ and $s^2-md \equiv s^2 \pmod q$, so $\chi_q(s^2-md) = 1$. If $q \mid (s^2-md)$, then we have $q \mid s$. Assume $s^2-md = q(qs_1^2-m_1d)$, where $m = qm_1, s = qs_1$. Then $\frac{s^2-md}{q} = qs_1^2-m_1d$ and we have

$$\chi_q(s^2-md) = (s^2-md,-m)_q = (q,-m)_q(qs_1^2-m_1d,-m)_q.$$

Then $(q,-m)_q = (q,-q)_q(q,m_1)_q = (q,m_1)_q$ and $qs_1^2 - m_1d \equiv -m_1d \pmod{q}$. Since $qs_1^2 - m_1d$ is a unit in $\mathbb{Z}_q$,

$$\chi_q(s^2 - md) = (q,-m)_q(-m_1d,-m)_q = (q,m_1)_q(-m_1d,q)_q(-m_1d,-m_1)_q.$$

m_1d and m_1 are both units in $\mathbb{Z}_q$ leaving

$$\chi_q(s^2 - md) = (-d,q)_q = \epsilon(q).$$

□

To finish the case for p inert, we just need to prove

Lemma 7.8. $\varrho_p(s) = 2^{k(s)}\eta_p\left(\frac{md-s^2}{4}, s\right)$.

Proof. $k(s) = \#\{q \text{ ramified} \mid (s\lambda_-)_q = 0\}$ and $(s\lambda_-)_q = 0$ if and only if $q \mid s$, which implies $k(s) = \#\{q \mid (s,m)\} = a(s)$. Since 4 is a square, we can ignore it in $\eta_p\left(\frac{md-s^2}{4}, s\right)$. We have

$$\eta_p(md - s^2, s) = \begin{cases} 0 & \text{if } \chi_q(s^2 - md) = -1 \text{ and } (s\lambda_-)_q = 0 \text{ for some } q \mid m, \\ 1 & \text{otherwise.} \end{cases}$$

If $(s\lambda_-)_q \neq 0$, then $\mathrm{ord}_q(s^2 - md) = 0$ and Lemma 7.7 implies $\chi_q(s^2 - md) = 1$. So

$$2^{k(s)}\eta_p(md - s^2, s) = \begin{cases} 0 & \text{if } \chi_q(s^2 - md) = -1 \text{ for some } q \mid m, \\ 2^{a(s)} & \text{otherwise,} \end{cases}$$

which equals $\varrho_p(s)$. □

Now let q be any ramified prime. Then looking at equation (7.7), in the argument of the logarithm we have

$$\mathrm{ord}_q(7.7) = \sum_{s\in\mathbb{Z}} 2^{k(s)}\eta_q\left(\frac{md-s^2}{4}, s\right)\mathrm{ord}_q\left(\frac{md-s^2}{4}\right)\rho\left(\frac{md-s^2}{4}\right).$$

Lemma 7.7 implies $\text{ord}_q\left(\frac{md-s^2}{4}\right) \le 1$ so $\sum_{n\ge1}\rho\left(\frac{md-s^2}{4q^n}\right) = \rho\left(\frac{md-s^2}{4q}\right)$ and Theorem 7.2 says

$$\text{ord}_q\left(J(-m,-d)^4\right) = \sum_{s\in\mathbb{Z}} \varrho_q(s)\rho\left(\frac{md-s^2}{4q}\right).$$

Lemma 7.9. $2^{k(s)}\eta_q\left(\frac{md-s^2}{4}, s\right)\text{ord}_q\left(\frac{md-s^2}{4}\right)\rho\left(\frac{md-s^2}{4}\right) = \varrho_q(s)\rho\left(\frac{md-s^2}{4q}\right).$

Proof. If $\text{ord}_q\left(\frac{md-s^2}{4}\right) = 0$, then $\rho\left(\frac{md-s^2}{4q}\right) = 0$ and both sides are zero. Assume $\text{ord}_q\left(\frac{md-s^2}{4}\right) = 1$. Ramified primes can have $\epsilon(q) = +1$ or -1. As in the proof of Lemma 7.6, assume we have a factorization

$$\frac{md-s^2}{4} = p_1^{2a+1} l_1^{2a_1} \cdots l_u^{2a_u} q_1^{b_1} \cdots q_v^{b_v},$$

where $\epsilon(p_1) = \epsilon(l_i) = -1$ and $\epsilon(q_i) = +1$. If p_1 is not ramified, then p_1 is inert and $\rho\left(\frac{md-s^2}{4q}\right) = 0 = \rho\left(\frac{md-s^2}{4}\right)$. If p_1 is ramified and $p_1 \ne q$, then we must have $a = 0$ and $\epsilon(q) = +1$, while $\chi_q(s^2-md) = +1$ and $\chi_{p_1}(s^2-md) = -1$ by Lemma 7.7. Then

$$\eta_q(md-s^2, s) = \begin{cases} 0 & \text{if } (s\lambda_-)_q \ne 0, \text{ or } (s\lambda_-)_q = 0 \text{ and } \chi_q(s^2-md) = 1, \\ & \text{or } \chi_q(s^2-md) = -1 = \chi_{q'}(s^2-md) \text{ for some ramified} \\ & \text{prime } q' \ne q \text{ with } (s\lambda_-)_{q'} = 0, \\ 1 & \text{if } (s\lambda_-)_q = 0, \chi_q(s^2-md) = -1, \text{ and } \chi_{q'}(s^2-md) = 1 \\ & \text{for all ramified primes } q' \ne q \text{ with } (s\lambda_-)_{q'} = 0. \end{cases}$$

So $\chi_q(s^2-md) = +1$ implies $\eta_q(md-s^2, s) = 0$ and $\chi_{p_1}(s^2-md) = -1$ implies $\varrho_q(s) = 0$. We are left with the case of $p_1 = q$. In this case, $\rho\left(\frac{md-s^2}{4q}\right) = \rho\left(\frac{md-s^2}{4}\right)$ so we need to show

$$\varrho_q(s) = 2^{k(s)}\eta_q\left(\frac{md-s^2}{4}, s\right). \tag{7.10}$$

As in Lemma 7.8, $k(s) = a(s)$ and the 4 can be ignored. Since $\text{ord}_q(md - s^2) = 1$, Lemma 7.7 implies $\chi_q(s^2 - md) = -1$, so $\eta_q(md - s^2, s)$ simplifies to

$$\eta_q(md - s^2, s) = \begin{cases} 0 & \text{if } \chi_{q'}(s^2 - md) = -1 \text{ for some } q' \mid m, q' \neq q, \\ 1 & \text{otherwise.} \end{cases}$$

This implies (7.10) and so Lemma 7.9 is proved. □

This finishes the proof of Theorem 7.2 for the case where $-d \equiv 0 \pmod 4$.

For the second case, the proof begins in a similar fashion as above. Then we make one simple substitution and reduce the proof to that of case 1.

Case 2: $-d \equiv 1 \pmod 4$.

Here we let

$$J_d(\tau) = \prod_{\tau_2 \in Z_1\left(\frac{d}{4}, \mathcal{O}_d\right)} \Big(j(\tau) - j(\tau_2)\Big).$$

The zero set of $J_d(\tau)$ is $Z_1\left(\frac{d}{4}, \mathcal{O}_d\right)$. Proceeding as in case 1, in order to have $\text{div}(\Psi(F)^2) = \text{div}(J_d(\tau)^2)$ we choose our input function F with $c_1\left(-\frac{d}{4}\right) = 1$ and all other negative Fourier coefficients equal to zero. Again, Lemma 6.3 tells us such an input function exists, and summing over $x \in \Gamma^+ \backslash L_1^+\left(\frac{m}{4}, \mathcal{O}_k\right)$ and evaluating Ψ at z_x^+ gives

$$4 \log |J(-m, -d)| = \Bigg[\sum_{y \in \mathbb{Z}/m\mathbb{Z}} \sum_{s \in \mathbb{Z}} 2^{k\left(y - \frac{m+a}{2}\right)} \times$$

$$\left(\kappa_1 \left(\frac{d}{4} - m \left(\frac{2y - a}{2m} + s \right)^2, y - \frac{m+a}{2} \right) + \right.$$

$$\left. \kappa_2 \left(\frac{d}{4} - m \left(\frac{2y - a}{2m} + s \right)^2, y - \frac{m+a}{2} \right) \right) \Bigg].$$

Proceeding as in case 1, this is

$$= \sum_{y \in \mathbb{Z}/m\mathbb{Z}} \sum_{s \in m\mathbb{Z}} 2^{k\left(y - \frac{m+a}{2}\right)} \Bigg[\sum_{q \mid m} \eta_q \left(\frac{md - (2y - a + 2s)^2}{4}, y - \frac{m+a}{2} \right) \times$$

$$\log(q)\mathrm{ord}_q\left(\frac{md-(2y-a+2s)^2}{4}\right)\rho\left(\frac{md-(2y-a+2s)^2}{4}\right) +$$
$$\sum_{p \text{ inert}} \eta_p\left(\frac{md-(2y-a+2s)^2}{4}, y-\frac{m+a}{2}\right)\log(p)\times$$
$$\left(\mathrm{ord}_p\left(\frac{md-(2y-a+2s)^2}{4}\right)+1\right)\rho\left(\frac{md-(2y-a+2s)^2}{4p}\right)\Bigg].$$

Now we make the substitution $u = y - \frac{m+a}{2}$. Then $2y - a = 2u + m$ and we get

$$= \sum_{u=\frac{m+a}{2}}^{m-1-\frac{m+a}{2}} \sum_{s\in m\mathbb{Z}} 2^{k(u)}\left[\sum_{q|m}\eta_q\left(\frac{md-(2u+m+2s)^2}{4}, u\right)\log(q)\times\right.$$
$$\mathrm{ord}_q\left(\frac{md-(2u+m+2s)^2}{4}\right)\rho\left(\frac{md-(2u+m+2s)^2}{4}\right) +$$
$$\sum_{p \text{ inert}} \eta_p\left(\frac{md-(2u+m+2s)^2}{4}, u\right)\log(p)\times$$
$$\left(\mathrm{ord}_p\left(\frac{md-(2u+m+2s)^2}{4}\right)+1\right)\rho\left(\frac{md-(2u+m+2s)^2}{4p}\right)\Bigg].$$

We can replace $m + 2s$ with $s \in m\mathbb{Z}$ and again, by vanishing properties of the function ρ, we do not need to specify the parity of s. Also, for any $s \in m\mathbb{Z}$, $((u+s)\lambda_-)_q = 0$ if and only if $(u\lambda_-)_q = 0$, and this is equivalent to $q \mid u$. Since m is odd, this is also equivalent to $q \mid 2u$ and we see $k(u) = k(u+s) = k(2u+s)$. Similarly, we can write $2u+s$ in the second arguments of η_p and η_q. As y ranges from 0 to $m-1$, $2u$ ranges over all odd integers from 1 to $m-1$. Summing over $s \in m\mathbb{Z}$ implies we can replace the double sum by a single sum over all odd integers $s \in \mathbb{Z}$. Then if s is even, ρ vanishes so we can actually sum over all $s \in \mathbb{Z}$. We get

$$\sum_{s\in\mathbb{Z}} 2^{k(s)}\left[\sum_{q|m}\eta_q\left(\frac{md-s^2}{4}, s\right)\log(q)\mathrm{ord}_q\left(\frac{md-s^2}{4}\right)\rho\left(\frac{md-s^2}{4}\right) +\right.$$
$$\sum_{p \text{ inert}}\eta_p\left(\frac{md-s^2}{4}, s\right)\log(p)\left(\mathrm{ord}_p\left(\frac{md-s^2}{4}\right)+1\right)\rho\left(\frac{md-s^2}{4p}\right)\Bigg].$$

This is exactly (7.7) from case 1. Theorem 7.2 and the rest of the proof of case 1 do not depend on $d \pmod 4$, so we are done by case 1. $\square$

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