ABSTRACT

Title of dissertation: NONLINEAR EVOLUTIONARY PDEs IN IMAGE PROCESSING AND COMPUTER VISION

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Evolutionary PDE-based methods are widely used in image processing and computer vision. For many of these evolutionary PDEs, there is little or no theory on the existence and regularity of solutions, thus there is little or no understanding on how to implement them effectively to produce the desired effects. In this thesis work, we study one class of evolutionary PDEs which appear in the literature and are highly degenerate.

The study of such second order parabolic PDEs has been carried out by using semi-group theory and maximum monotone operator in case that the initial value is in the space of functions of bounded variation. But the noisy initial image is usually not in this space, it is desirable to know the solution property under weaker assumption on initial image. Following the study of time dependent minimal surface problem, we study the existence and uniqueness of generalized solutions of a class of second order parabolic PDEs. Second order evolutionary

PDE-based methods preserve edges very well but sometimes they have undesirable staircase effect. In order to overcome this drawback, fourth order evolutionary PDEs were proposed in the literature. Following the same approach, we study the existence and regularity of generalized solutions of one class of fourth order evolutionary PDEs in space of functions of bounded Hessian and bounded Laplacian. Finally, we study some evolutionary PDEs which do not satisfy the parabolicity condition by adding a regularization term.

Through the rigorous study of evolutionary PDEs which appear in the literature of image processing and computer vision, we provide a solid theoretical foundation for them which helps us better understand the behaviors and properties of them. The existence and regularity theory is the first step toward effective numerical scheme. The regularity results also answer the questions to which function spaces the solutions of evolutionary PDEs belong and the questions if the processing results have the desired properties.

Nonlinear Evolutionary PDEs in Image Processing and Computer Vision

by

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Chapter 1

Introduction

Variational methods and PDE-based methods appear in a large variety of image processing and computer vision [1] areas ranging from optical flow computation to stereo vision and surface reconstruction.

1.1 Image smoothing

Images are unavoidably degraded during acquisition and transmission. Image smoothing is the process which is intended to reduce noise in the image in order to retrieve useful information.

[1]Mapping from images to abstract description (Computer vision) versus mapping from abstract description to images (Vision).

1.1.1 Linear evolutionary PDEs in image smoothing

From variational problem to evolutionary PDE

Assume that the original image of a real scene is denoted by $u \in L^2(\Omega)$, the observed and noisy image of the same scene is denoted by $u_0 \in L^2(\Omega)$. Assume that they satisfy the linear relationship $u_0 = Ru + n$, where R is a linear operator and n is the Gaussian noise. Given u_0, we want to recover u. According to maximum likelihood principle, we can find the approximation of u by solving the least square problem

$$\inf_u \int_\Omega |Ru - u_0|^2 \, dx \tag{1.1.1}$$

This problem is ill-posed [8]. The classic method to overcome ill-posed minimization problems is to add regularization term to the minimization functional [88]. Now let's consider

$$\inf_u \left\{ \int_\Omega |\nabla u|^2 \, dx + \lambda \int_\Omega |Ru - u_0|^2 \, dx \right\} \tag{1.1.2}$$

here λ is a positive weighting constant. The first term of minimization functional is a smoothing term, the second term measures the fidelity to the initial data. Under suitable assumptions on R, the minimization problem (1.1.2) admits a unique solution which is characterized by the Euler-Lagrange equation

$$-\Delta u + \lambda R^*(Ru - u_0) = 0 \tag{1.1.3}$$

with Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$, ν is the outward normal of $\partial\Omega$. We may introduce a scale-space variable t and use gradient decent method to solve the minimization problem which results in an evolutionary partial differential equation

$$\begin{cases} \dot{u} = \Delta u - \lambda R^*(Ru - u_0) \\ \frac{\partial u}{\partial \nu}|_\Gamma = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1.1.4}$$

here $\dot{u}$ is the derivative with respect to scale-space variable t. In case there is no confusion, we also call it time derivative.

Gaussian smoothing and linear evolutionary PDE

Gaussian filter is a classic method of smoothing noisy images and detecting edges. It was introduced by Marr and Hildreth [65], then further developed by Witkin [97], Koenderink [54], and Canny [15]. Let $u_0 \in L^2(\mathbb{R}^2)$ be the noisy image and $G_\sigma = \frac{1}{2\pi\sigma^2} e^{-|x|^2/2\sigma^2}$. Then the smoothed version of u_0 is

$$(G_\sigma * u_0)(x) = \int_{\mathbb{R}^2} G_\sigma(x-y) u_0(y)\, dy \tag{1.1.5}$$

On the other hand, consider the following linear parabolic PDE

$$\begin{cases} \dot{u} = \Delta u \\ u(x,0) = u_0 \end{cases} \tag{1.1.6}$$

Assume that $u_0 \in C(\mathbb{R}^2)$ and bounded, the solution of (1.1.6) is

$$u(x,t) = (G_{\sqrt{2t}} * u_0)(x) \tag{1.1.7}$$

It is unique if we impose that u does not grow too fast

$$|u(x,t)| \le Ce^{a|x|^2} \tag{1.1.8}$$

for some positive constants C and a. Therefore, smoothing a noisy image using a Gaussian filter with parameter σ is the same as the solution of a linear parabolic PDE at $t = \frac{\sigma^2}{2}$.

Figure 1.1: Gaussian smoothing, left: original image; middle: $\sigma = 2$; right: $\sigma = 4$.

1.1.2 Advantages of using evolutionary PDE to process images

We have seen some evolutionary PDEs in image processing. Are there advantages to cast image processing problems into this frame work? It is well known that images usually contain structures at a large variety of scales. The advantage of casting image processing problem into evolutionary PDE frame work is that it allows an image represented at multiple scales. By comparing the structure at different scales, we obtain a hierarchy of image structures which are very useful for image interpretation.

A scale-space is an image interpretation at continuum scales, embedding the image u_0 into a family $\{T_t u_0 : t \geq 0\}$ of gradually simplified versions of it, provided that it satisfies certain requirements which are very natural from the image processing point of view [96]. Alvarez, Guichard, Lions and Morel [2] showed that every scale space satisfies some axioms and invariance properties is governed by a PDE with the original image as initial condition. In addition, if we impose the linearity

$$T_t(au_1 + bu_2) = aT_t u_1 + bT_t u_2 \quad \forall\, t \geq 0, a, b \in \mathbb{R} \tag{1.1.9}$$

The only candidate of linear scale space is Gaussian scale space [97, 94].

1.1.3 Nonlinear second order evolutionary PDEs in image smoothing

The linear PDE quickly removes noise, but at the same time it blurs the edge (see Figure 1.1). Since Gaussian filter is the only candidate in the linear framework, people began to consider nonlinear filters. Perona and Malik [76] proposed nonlinear PDEs to smooth images and detect edges

$$\begin{cases} \dot{u} = \nabla \cdot (g(|\nabla u|^2)\nabla u) \\ \frac{\partial u}{\partial \nu}|_\Gamma = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1.1.10}$$

here $g(s^2) = \frac{1}{1+(s/k)^2}$ or $g(s^2) = e^{-(s/k)^2}$, k is some positive constant. The edges of images smoothed by (1.1.10) are well localized from finer to coarser level, but this PDE is not well-posed. It may suffer instability problems caused by very noisy initial image.

We may also consider to modify model (1.1.2). Because the over-smoothing is due to the L^2 norm of the gradient, one feasible solution is to decrease the regularity, which leads Rudin, Osher and Fatemi [79] to propose the Total Variation (TV) model

$$\inf_u \{F(u) = \int_\Omega |\nabla u| \, dx + \frac{\lambda}{2} \int_\Omega |u - u_0|^2 \, dx\} \tag{1.1.11}$$

TV model preserves edges much better than Gaussian smoothing, which is the direct result of L^1 norm instead of L^2 norm. Later, a class of such minimization functionals was proposed for image smoothing [7]

$$F(u) = \int_\Omega \Phi(|\nabla u|) \, dx + \frac{\lambda}{2} \int_\Omega |Ru - u_0|^2 \, dx \tag{1.1.12}$$

here R is a linear continuous operator, $\Phi(\cdot)$ is an even convex function from $\mathbb{R} \to \mathbb{R}^+$ and approximately linear increasing. Thus, TV minimization functional is a special case of (1.1.12). Let R^* is the adjoint of R, the Euler-Lagrange equation associated with the minimization problem can be formally written as

$$-\nabla\Big(\frac{\Phi'(|\nabla u|)}{|\nabla u|}\nabla u\Big) + \lambda R^*(Ru - u_0) = 0 \tag{1.1.13}$$

Let $g(s^2) = \frac{\Phi'(s)}{s}$ and use gradient decent method to solve the minimization problem, we obtain

$$\begin{cases} \dot{u} = \nabla \cdot (g(|\nabla u|^2)\nabla u) - \lambda R^*(Ru - u_0) \\ \frac{\partial u}{\partial \nu}\big|_\Gamma = 0 \\ u(x,0) = u_0 \end{cases} \tag{1.1.14}$$

1.1.4 Nonlinear fourth order evolutionary PDEs in image smoothing

Although total variation minimization method has a great success for denoising and texture decomposition, sometimes it produces undesirable staircase effect (Figure 1.2). In order to deal with this issue, minimization method with second

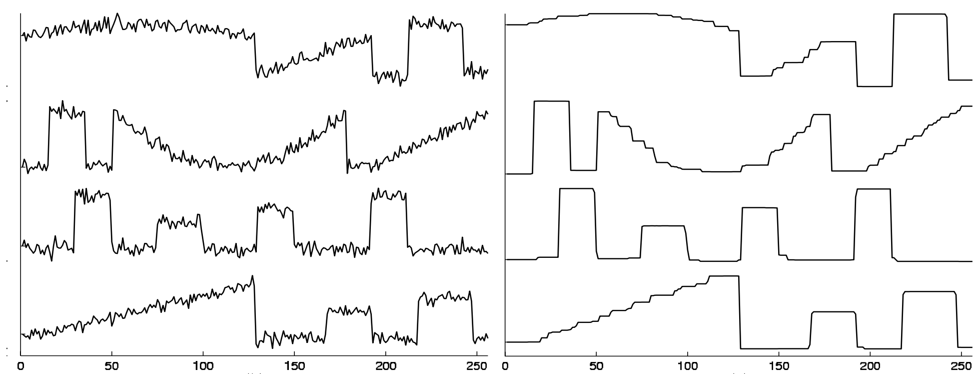

Figure 1.2: Staircase effect of second order model, left: noisy signal, right: denoised signal. Figure from Chan [21].

order derivatives in the functional and fourth order evolutionary PDEs were proposed in the hope of taking the image curvatures into account. Chambolle and

Lions [20] proposed the following minimization functional

$$J(u_1, u_2) = \int_\Omega |\nabla u_1| \, dx + \alpha \int_\Omega |\nabla^2 u_2| \, dx + \lambda \int_\Omega (u_1 + u_2 - u_0)^2 \, dx$$

to improve the staircase effects of total variation method. Here α, λ are weighting parameters. If we let $u = u_1 + u_2$ and $v = u_2$, we obtain

$$J(u, v) = \int_\Omega |\nabla (u - v)| \, dx + \alpha \int_\Omega |\nabla^2 v| \, dx + \lambda \int_\Omega (u - u_0)^2 \, dx$$

What is the idea behind the new functional? "In some sense, we first approximate locally the gradient of the function u_0 by ∇v, that has itself a very low total variation ($\alpha >> 1$). Then we find u as an approximation of u_0 such that $u - v$ has a low total variation"[20]. To the same purpose, Chan, Marquina and Muler [21] proposed minimization functional

$$J(u) = \int_\Omega \left[\alpha |\nabla u|_{\epsilon_1} + \beta \frac{\mathcal{L}(u)^2}{|\nabla u|^3_{\epsilon_2}} + \frac{1}{2}(u - u_0)^2 \right] dx \tag{1.1.15}$$

to smooth noisy images, here $|\nabla u|_{\epsilon_i} = \sqrt{|\nabla u|^2 + \epsilon_i}$ and $\mathcal{L}(u)$ is an elliptic operator and they restricted themselves to work with $\mathcal{L}(u) = \Delta u$. Lysaker, Lundervold and Tai [62] proposed the following minimization functionals in medical image processing

$$J_1(u) = \int_\Omega (|u_{xx}| + |u_{yy}|) \, dxdy + \frac{\lambda}{2} \Big[\int_\Omega (u - u_0)^2 \, dxdy - \sigma^2 \Big] \tag{1.1.16}$$

$$J_2(u) = \int_\Omega \sqrt{|\nabla^2 u|^2} \, dxdy + \frac{\lambda}{2} \Big[\int_\Omega (u - u_0)^2 \, dxdy - \sigma^2 \Big] \tag{1.1.17}$$

Tumblin and Turk [91] proposed an evolutionary PDE to preserve the details of high contrast scenes by building a coarse to fine order hierarchy of scene boundaries and shadings.

$$\begin{cases} \dot{u} = -\nabla \cdot (g(|\nabla^2 u|))\nabla \Delta u) - \lambda(u - u_0) \\ \frac{\partial u}{\partial \nu}|_\Gamma = 0 \\ \frac{\partial \Delta u}{\partial \nu}|_\Gamma = 0 \\ u(x,0) = u_0 \end{cases} \tag{1.1.18}$$

here $g(s) = \frac{k^2}{k^2+s^2}$. They call it "Lower Curvature Image Simplifiers". Later Tumblin pointed out that $|\nabla^2 u|$ is not rotational invariant. A better choice would be use Δu instead of $\nabla^2 u$ [11]. Thus, the new rotation invariant evolutionary PDE

$$\begin{cases} \dot{u} = -\nabla \cdot (g(|\Delta u|))\nabla \Delta u) - \lambda(u - u_0) \\ \frac{\partial u}{\partial \nu}|_\Gamma = 0 \\ \frac{\partial \Delta u}{\partial \nu}|_\Gamma = 0 \\ u(x,0) = u_0 \end{cases} \tag{1.1.19}$$

which is formally the gradient flow of the Euler-Lagrange equation of the following minimization problem

$$\inf_u \left\{ J(u) = \int_\Omega \left[k\Delta u \arctan \frac{\Delta u}{k} - \frac{k^2}{2} \log \left(\left(\frac{\Delta u}{k}\right)^2 + 1 \right) + \frac{\lambda}{2}(u - u_0)^2 \right] dx \right\} \tag{1.1.20}$$

You and Kaveh [99] propose the functional

$$J(u) = \int_\Omega f(|\Delta u|)\, dx \tag{1.1.21}$$

to eliminate the staircase effects of Perona-Malik PDE (1.1.10). Through gradient decent procedure, they formally derived the evolutionary PDE

$$\begin{cases} \dot{u} = -\Delta(g(|\Delta u|)\Delta u) \\ \frac{\partial u}{\partial \nu}|_{\Gamma} = 0 \\ \frac{\partial \Delta u}{\partial \nu}|_{\Gamma} = 0 \\ u(x,0) = u_0 \end{cases} \tag{1.1.22}$$

where $g(|\Delta u|) = \frac{f'(|\Delta u|)}{|\Delta u|}$. In numerical experiments, they chose $f(s) = \log(1 + (s/k)^2)$.

1.2 Image enhancement

Image enhancement is the process of improving the perceptual quality of a digitally stored image by manipulating the image with software. Osher and Rudin [72] used shock filters to improve image quality

$$\dot{u} = -|\nabla u|F(L(u)) \tag{1.2.1}$$

where F is a Lipschitz continuous function satisfying

$$\begin{cases} F(0) = 0 \\ \operatorname{sign}(s)F(s) > 0, \ s \neq 0 \end{cases} \tag{1.2.2}$$

L is a nonlinear elliptic operator such that zero crossing define the edges of the processed image. A typical example of (1.2.1) in 1D is

$$\begin{cases} \dot{u} + (u_{xx}\text{sign}(u_x))u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Gilboa, Sochen and Zeevi [41] proposed the following evolutionary PDE to enhance image features with middle gradients: neither low gradients nor very high gradients are enhanced.

$$\begin{cases} \dot{u} = \nabla \cdot \left(\frac{\nabla u}{\sqrt{1+(|\nabla u|/k_f)^2}} - \alpha \frac{\nabla u}{1+(|\nabla u|/k_b)^2} \right) - \lambda(u - u_0) - \epsilon \Delta^2 u \\ \frac{\partial u}{\partial \nu}|_\Gamma = 0 \\ \frac{\partial \Delta u}{\partial \nu}|_\Gamma = 0 \\ u(x,0) = u_0 \end{cases} \tag{1.2.3}$$

here $\alpha, \lambda, k_f, k_b$ are weighting parameters. Please see Figure 1.3 for the enhancement result of this PDE method.

1.3 Image texture decomposition

The minimization method and parabolic PDE-based method whose solutions are in the space of functions of bounded variation are very successful in image smoothing. Unfortunately, one drawback of these methods is that their inability to handle textures and small structures properly. In practice, smaller details, such as textures, are destroyed if the weighting parameter λ is too small. Gousseau and Morel [42] may be the first to challenge the idea that natural images are

Figure 1.3: Image enhancement by flows based on triple well potentials, top: original image; bottom: enhanced image. Figures are from `http://visl.technion.ac.il/~gilboa/ppt/huji02.pps`

in the space of functions of bounded variation (BV). Through an experimental study of the distribution of the bilevels [2] of natural images, they showed the total variation blows up to infinity with the increasing resolution. Meyer [68] took a study of this problem from mathematical point of view. He proved that the norm of error term $\|u-h\|$ of the Osher, Rudin and Fatemi model in Besov space $\dot{B}_{\infty}^{-1,\infty}$ is always small. Thus, it is more appropriate to represent textures or oscillatory

[2] Consider a digital image I whose gray levels are between 0 and N, the k-bilevels of I is defined by $I_l(i,j) = 1$ if $I(i,j) \in [(l-1)N/k, lN/k]$, 0 otherwise. $1 \leq l \leq k$.

by some weaker norms than L^2 norm. He proposed some alternative space G [3].

Definition 1.3.1. [68] Let G denote the Banach space consisting of all generalized functions $f(x)$ which can be written as

$$f(x) = \partial_1 g_1(x) + \partial_2 g_2(x) \quad g_1(x), g_2(x) \in L^\infty(\mathbb{R}^2) \tag{1.3.1}$$

The norm $\|f\|_*$ of f in G is defined as the lower bound of all L^∞ norms of the functions $|g|$ where $g = (g_1, g_2)$, $|g|(x) = \sqrt{|g_1|^2 + |g_2|^2}(x)$ where the infimum is computed over all decomposition (1.3.1) of f.

Vese, Osher [93], Aujol, Aubert [9], Osher, Solé, Vese [73] followed Meyer's idea to decompose image into Cartoon part and texture or noisy part in space G. The decomposition of Osher, Solé, Vese [73] (See Figure 1.4 for the decomposition result)

$$\inf_u \left\{ F(u) = \int_\Omega |\nabla u|\, dx + \frac{\lambda}{2} \int_\Omega |\nabla(\Delta^{-1}(u_0 - u)|^2\, dx \right\} \tag{1.3.2}$$

has almost the identical mathematical format as (1.1.12). From (3.4.93), they formally derived second order evolutionary PDE

$$\begin{cases} \dot{u} = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|} \right] - \lambda \Delta^{-1}(u - u_0) \\ \frac{\partial u}{\partial \nu}|_\Gamma = 0 \\ u(x, 0) = u_0 \end{cases}$$

[3]Together with this space G, two other functional spaces were also introduced. F is defined as G but the John and Nirenberg space $BMO(\mathbb{R}^2)$ is replacing the role of $L^\infty(\mathbb{R}^2)$. E is the Besov space $\dot{B}_\infty^{-1,\infty}$. We have $G \subset F \subset E$ [68].

Figure 1.4: Image texture decomposition of Osher, Solé, Vese model. Left: original image; middle: u (cartoon) part; right: v (texture) part. Figure from [73].

Tadmor, Nezzar [85] followed Rudin, Osher and Fatemi model (1.1.11) and took a step further, they represented an image using hierarchical (BV, L^2) decomposition. They argued that images could be realized as general L^2-objects and the more noticeable features of images are identified within a proper subclass of all L^2 objects. This subclass is known to be functions of bounded variation. Given initial image $f \in L^2(\Omega)$ and initial scale λ_0, their idea is to apply (1.1.11) to f recursively:

$$
\begin{array}{lll}
f = u_0 + v_0 & [u_0, v_0] & = \underset{u+v=f}{\operatorname{arginf}} \left\{ \int_\Omega |\nabla u| \, dx + \lambda_0 \int_\Omega |v|^2 \, dx \right\} \\
v_0 = u_1 + v_1 & [u_1, v_1] & = \underset{u_1+v_1=v_0}{\operatorname{arginf}} \left\{ \int_\Omega |\nabla u_1| \, dx + 2\lambda_0 \int_\Omega |v_1|^2 \, dx \right\} \\
\cdots & \cdots & \\
v_j = u_{j+1} + v_{j+1} & [u_{j+1}, v_{j+1}] & = \underset{u_{j+1}+v_{j+1}=v_j}{\operatorname{arginf}} \left\{ \int_\Omega |\nabla u_{j+1}| \, dx + 2^{j+1}\lambda_0 \int_\Omega |v_{j+1}|^2 \, dx \right\}
\end{array}
$$

After k such steps, it produces the following hierarchical decomposition of f:

$$u_0 + u_1 + \cdots + u_k + v_k$$

It was proved [85] that

$$\|f - \sum_{j=0}^{k} u_j\|_{W^{-1,\infty}(\Omega)} = \frac{1}{\lambda_0 2^{k+1}}$$

Here $\|f\|_{W^{-1,\infty}} = \sup\left\{ \int_\Omega f(x)g(x)\,dx : \|\nabla g\|_{L^1(\Omega)} \le 1 \right\}$.

1.4 Image segmentation

Image segmentation is the problem to distinguish objects from background. A segmentation is either a decomposition of the image domain into homogeneous regions with boundaries, or a set of boundary points (See Figure 1.5).

Figure 1.5: Image segmentation, left: original image; right: segmented image. Figure from [23].

1.4.1 Mumford and Shah functional

Mumford and Shah [70] proposed to obtain a segmented image u from u_0 by minimize the functional

$$F(u,K)=\int_{\Omega\setminus K}\left[(u-u_0)^2+\alpha|\nabla u|^2\right]dx+\beta\int_K d\sigma \tag{1.4.1}$$

where $K\subset\Omega\subset\mathbb{R}^d$ is the set of discontinuities, α,β are nonnegative constants and $\int_K d\sigma$ is the length of K. They conjectured that K is made of a finite set of $C^{1,1}$-curves. But this is too restrictive since one can't hope to obtain any compactness property. The difficulty is overcome by considering a wider class of sets of finite length rather than just a set of $C^{1,1}$-curves. The length of K is defined as its $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}(K)$. Therefore, the Mumford and Shah functional becomes

$$F(u,K)=\int_{\Omega\setminus K}\left[(u-u_0)^2+\alpha|\nabla u|^2\right]dx+\beta\mathcal{H}^{d-1}(K) \tag{1.4.2}$$

where $K\subset\bar{\Omega}$ is closed. If K is given then u is determined as the solution of the variational problem in the Sobolev space $W^{1,2}(\Omega\setminus K)$:

$$\min_u\left\{\int_{\Omega\setminus K}\left[|\nabla u|^2+\alpha(u-u_0)^2\right]dx\right\} \tag{1.4.3}$$

"The difficulty in studying F is that it involves two unknowns u and K of different nature: u is a function defined on an d-dimensional space, while K is an $(d-1)$-dimensional set"[8]. The existence of minimizer of Mumford-Shah functional is proved in the space of special functions of bounded variation (SBV), uniqueness is usually not true [8].

1.4.2 Deformable models

Deformable models are physics-based models that deform under the theory of elasticity. They are widely used techniques in image segmentation. These models (snakes, balloons) are active in the sense that they can adopt themselves to fit the given data. The active contour model algorithm, first introduced by Kass, Witkin and Terzopoulos [50], deforms a contour to lock onto features of interest within an image. Usually the features are lines, edges and object boundaries. The algorithm are named snakes because the deformable contours resemble snakes as they move. While 3-D active contour models are sometimes called active balloons.

Explicit models

Active contours can be thought as an energy-minimizing spline attracted by image features. The energy functional consists of two parts: an internal energy and external energy. Assume that the spline is represented by a curve $C(s) = (x_1(s), x_2(s))^T$ in an image f, the energies are defined as

$$E_{int}(C(s)) = \int_{C(s)} \frac{\alpha}{2}|C_s(s)|^2 + \frac{\beta}{2}|C_{ss}(s)|^2 \, ds \tag{1.4.4}$$

$$E_{ext}(C(s)) = -\int_{C(s))} \gamma|\nabla f(C(s))|^2 \, ds \tag{1.4.5}$$

Here α, β, γ are nonnegative parameters and serve as the weights of energies. Determining the weighting parameters is a difficult task for deformable models. In internal energy (1.4.4), the first term is an elasticity term causing the curve to shrink, the second one is a rigidity term encouraging straight contours. While external energy (1.4.5) pushes the contour to high gradients of the image f. Because this model makes direct use of the spline contour, it is also called an explicit model. Spline $C(s)$ should minimize the energy functional

$$E(C(s)) = E_{int}(C(s)) + E_{ext}(C(s)) \tag{1.4.6}$$

Solving (1.4.6) through gradient descent procedure gives

$$\dot{C} = \alpha C_{ss} + \beta C_{ssss} - \gamma \nabla(|\nabla f|^2) \tag{1.4.7}$$

The main shortcoming of the explicit model is that it can not split in order to segment several objects simultaneously [4].

Implicit models

Implicit model was proposed by Caselles, Catté, Coll and Dibos [17], and by Malladi, Sethian and Vemuri [64]. It overcomes the difficulty inherent in explicit model. The idea of implicit models is to embed the initial curve $C_0(s)$ as a zero level curve of a function $u_0 : \mathbb{R}^2 \to \mathbb{R}$, which is usually computed by using distance transformation. Then u_0 is evolved under a PDE which inherits knowledge from the original image f.

$$\dot{u} = g(|\nabla G_\sigma * f|^2)|\nabla u|\left(\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + \nu\right) \tag{1.4.8}$$

[4]Later, McInerney and Terzopoloulos [66, 67] proposed modified models to deal with several objects at the same time.

where $g(s)$ is a stopping function, it is small for large s; $\nu|\nabla u|$ is the motion in normal direction. When u does hardly change anymore at some time T, the final contour $C(s)$ is extracted as the zero level curve of $u(x, T)$. Alvarez, Lions and Morel [3] proposed the following model

$$\dot{u} = g(|G_\sigma * \nabla u|^2)|\nabla u|\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) \tag{1.4.9}$$

in the context of image smoothing. In (1.4.9), $u(t)$ is the denoised image at time t, while in (1.4.8), the zero level set of $u(t)$ is the evolved image feature. The well-posedness of (1.4.8) and (1.4.9) are studied in the viscosity sense [5]. Implicit model is very flexible in terms of topology. It allows contours splitting but it is difficult to interpret implicit model in terms of energy minimization.

Geometric models

Geometric models (Geodesic snakes) were proposed by Caselles, Kimmel and Sapiro [18] and by Kichenassamy [53]. They combine ideas of explicit and implicit models and are represented implicitly and evolve according to an Eulerian formulation. They are numerically implemented via level set algorithms [6]and can automatically handle changes in topology without resorting to dedicated contour tracking. In the evolving process, unknown numbers of multiple objects can be detected simultaneously. Geometric models are based on the minimization

[5]For the theory of viscosity solutions, please refer to [69].

[6]We refer to [71, 81] for the details of level set method.

functional

$$\int_{C(s)} g\big(|\nabla f_\sigma(C(s)|^2\big)|C_s(s)|\,ds \tag{1.4.10}$$

Embedding the initial curve as a level set of some image u_0, the gradient descent method leads to the following evolutionary PDE

$$\dot{u} = |\nabla u|\nabla\cdot\left(g\big(|\nabla f_\sigma|^2\big)\frac{\nabla u}{|\nabla u|}\right) \tag{1.4.11}$$

A new term $\nu g(|\nabla f_\sigma|^2)|\nabla u|$ is often added to achieve faster and more stable attraction to edges:

$$\dot{u} = |\nabla u|\left(\nabla\cdot\left(g\big(|\nabla f_\sigma|^2\big)\frac{\nabla u}{|\nabla u|}\right) + \nu g(|\nabla f_\sigma|^2)\right) \tag{1.4.12}$$

The theoretical analysis of (1.4.12) concerning existence, uniqueness and stability of a viscosity solution was studied in [18, 53].

Geodesic active contours have also been used for motion estimation and tracking [75, 74], for stereo vision [36, 37], for shape modeling and surface reconstruction [64, 47].

1.5 Optical flow problem

Optical flow field is defined as the velocity vector field of apparent motion of brightness patterns in a sequence of images [46] (see Figure 1.6 for an optical flow example). The computation of optical flow has proved to be an important tool for 3-D object reconstruction and 3-D scene analysis. Optical flow problem

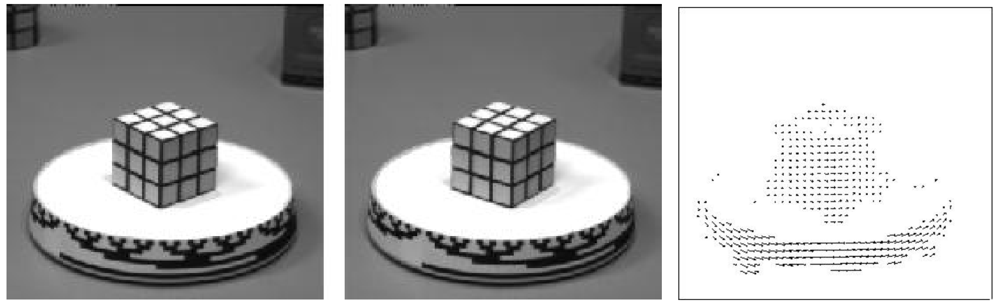

Figure 1.6: Optical flow, left two: a rubik's cube on a rotating turntable; right: optical flow. Figures taken from Russell and Norvig [80].

is ill posed. In order to get well-posedness, we have to impose suitable a priori knowledge. One constraint that has often been used in the literature is the "Optical Flow Constraint" (OFC). OFC is the result of the assumption of constant intensity $E(x, y, t)$ of the image points across all of the image frames. Based on this assumption, we have

$$\nabla E \cdot (u, v)^T + E_t = 0 \tag{1.5.1}$$

here $\nabla E = (E_x, E_y)^T$ and E_x, E_y, E_t are image intensity gradients in x, y and temporal directions. $u = \frac{\partial x}{\partial t}$, $v = \frac{\partial y}{\partial t}$, i.e. $(u, v)^T$ is the flow we are interested in. From (1.5.1), it is not difficult to see that the computation of optical flow (u, v) is not unique. It's uniqueness is only up to the computation of the flow along the intensity gradient ∇E at a point. This is called aperture problem. One way of treating the aperture problem is through the use of regularization in computation of optical flow. In their pioneering work, Horn and Schunk [48] used a L^2 smoothness constraint.

$$\int_{\Omega}\Big[\lambda(\nabla E\cdot(u,v)^T+E_t)^2+(|\nabla u|^2+|\nabla v|^2)\Big]\,dxdy \tag{1.5.2}$$

The first term measures the fidelity to OFC, and the second term imposes constraint on the smoothness of the flow field. The immediate difficulty with this constraint is that at the object boundaries, where it is natural to expect discontinuities in the flow, such a constraint will have difficulty to capturing the optical flow. Thus, Kumar, Tannenbaum and Balas [56] proposed the following minimization problem to compute optical flow.

$$\int_{\Omega}\Big[\frac{\lambda}{2}(\nabla E\cdot(u,v)^T+E_t)^2+(|\nabla u|+|\nabla v|)\Big]\,dxdy \tag{1.5.3}$$

This model reduces the regularity requirement of flow field from L^2 norm to L^1 norm. Then, they derived the Euler-Lagrange equation

$$\begin{cases} -\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)+\lambda E_x(\nabla E\cdot(u,v)^T+E_t)=0 \\ -\nabla\cdot\left(\frac{\nabla v}{|\nabla v|}\right)+\lambda E_y(\nabla E\cdot(u,v)^T+E_t)=0 \end{cases} \tag{1.5.4}$$

By introducing a new scale-space variable t' and use gradient decent method to solve (1.5.4), they obtained

$$\begin{cases} \dot{u}=\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)-\lambda E_x(\nabla E\cdot(u,v)^T+E_t) \\ \dot{v}=\nabla\cdot\left(\frac{\nabla v}{|\nabla v|}\right)-\lambda E_y(\nabla E\cdot(u,v)^T+E_t) \end{cases} \tag{1.5.5}$$

here $\dot{u}$, $\dot{v}$ are the partial derivatives with respect to scale-space variable t'. Aubert, Deriche and Kornprobst [6] proposed the following model

$$\inf_{u,v}\left\{\int_{\Omega}\Big[|\nabla E\cdot(u,v)^T+E_t|+\alpha(\phi(Du)+\phi(Dv))+\beta c(x)(u^2+v^2)\Big]\,dx\right\} \tag{1.5.6}$$

to compute optical flow. Here α and $c(x)$ are weighting parameters. A strict theoretical study of (1.5.6) is also provided [6]. Both of the authors reported that the L^1 norm approach preserves edges very well.

1.6 Shape from shading

Shape from shading is a method for determining the shape of a surface from the gradual variation of shading in its image (See Figure 1.7 for an example). Under

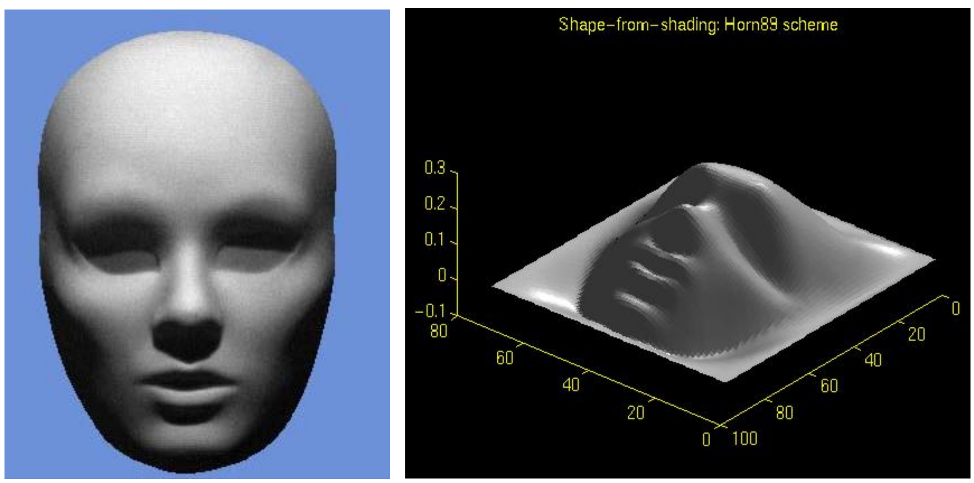

Figure 1.7: Shape from shading, left: face mask image; right: 3D shape from shading. Figures are from `http://www.cssip.edu.au/~danny/vision/shading.html`.

the assumption of Lambertian surface (each surface point appears equally bright from all viewing directions), the scene radiance is simply proportional to the dot product between the direction of the illuminant s and the surface normal n:

$$R_{\rho,s}(n) = \rho s \cdot n \tag{1.6.1}$$

where ρ is the effective albedo. This is a particular example of reflectance map. In general, the function $R_{\rho,s}$ is more complicated or known only numerically through experiments. If we make some approximations about the image brightness, we have the fundamental equation of shape from shading

$$E(x, y) = R_{\rho,s}(n) \tag{1.6.2}$$

here $E(x, y)$ is the image brightness. Thus from this equation, the surface normal (which is also called the needle map) can be recovered. Variational method is the classic approach of shape from shading. The pioneering work in this approach is due to Horn and his coworkers [49]. The Horn and Brooks functional uses a quadratic regularizer:

$$\int_\Omega \Big\{ \big[E(x, y) - s \cdot n\big]^2 + \lambda\Big[\big(\frac{\partial n}{\partial x}\big)^2 + \big(\frac{\partial n}{\partial y}\big)^2\Big] + \mu\big[\|n\|^2 - 1\big]\Big\}\, dxdy \tag{1.6.3}$$

The first term is brightness error which encourages data-closeness of the measured image intensity and the reflectance map. It directly exploits shading information. The second term is the regularizing term which imposes the smoothness constraint on recovered surface normals and penalizes large local changes in surface orientation. The third term forces n to close to a unit vector. Philip and Edwin [98] proposed the following minimization functional

$$\int_\Omega \Big\{ \big[E(x, y) - s \cdot n\big]^2 + \lambda\big[\rho_\sigma\big(\big|\frac{\partial n}{\partial x}\big|\big) + \rho_\sigma\big(\big|\frac{\partial n}{\partial y}\big|\big)\big] + \mu\big[\|n\|^2 - 1\big]\Big\}\, dxdy \tag{1.6.4}$$

to recover surface normal maps. If $\rho_\sigma(x) = \frac{\sigma}{\pi}\log\cosh\frac{\pi x}{\sigma}$, they reported good numerical results which offer reduced over-smoothing over discontinuities in real world image. It is not hard to verify that $\rho_\sigma(\cdot)$ is a convex function with linear increase at infinity in this case. Thus it is an L^1 norm version of Horn and Brooks functional.

1.7 Thesis outline

Although PDE techniques are widely used in image processing and computer vision, for many of these PDEs, there is little or no theory on the existence and regularity of solutions, thus there is little or no understanding on how to implement them effectively to produce the desired effects.

In this thesis work, we systematically study the regularity and existence of the generalized solution of one class of highly degenerate parabolic PDEs for given noisy initial data $u_0 \in L^2(\Omega)$, which is the case often met in image processing and computer vision. Through the rigorous study of these evolutionary PDEs, we provide a solid theoretical foundation for them which helps us better understand the behaviors and properties of them. The theory of existence and regularity is the first step toward effective numerical scheme. The regularity results also answer the questions to which function spaces the solutions of evolutionary PDEs belong and the questions if the processing results have the desired properties. The generalized solutions of these parabolic PDEs satisfy some variational inequalities and lie in the function spaces involving measures, similar function spaces involving measures have been used in the study of 2-D vertex by Liu and Xin [60, 61]. Following Lichnewsky and Temam [59], we explain why we introduce the weak formulation of parabolic equations and the concept of generalized solutions. Let's assume that Ω is bounded domain, $Q = [0, T] \times \Omega$, $\partial\Omega$, u_0 are sufficient regular (say $u_0 \in C^2(\bar{\Omega})$). Suppose that u is a classic solution of 1.1.14. Let v be a $C^2(\bar{Q})$ test function, from 1.1.14, we obtain:

$$\int_0^s \big\langle \dot{u}, v - u \big\rangle \, dt = \int_0^s \big\langle \nabla \cdot (g(|\nabla u|^2)\nabla u), v - u \big\rangle - \lambda \big\langle Ru - u_0, R(v - u) \big\rangle \, dt$$

On the other hand, by the convexity of $\Phi(\cdot)$ and L^2 norm, notice that $g(|\nabla u|^2) = \frac{\Phi'(|\nabla u|)}{|\nabla u|}$, we obtain

$$\int_\Omega \big[\Phi(|\nabla v|) - \Phi(|\nabla u|)\big] \, dx \geq \big\langle g(|\nabla u|^2)\nabla u, \nabla v - \nabla u \big\rangle$$
$$\frac{1}{2}\|Rv - u_0\|^2 - \frac{1}{2}\|Ru - u_0\|^2 \geq \big\langle Ru - u_0, R(v - u) \big\rangle$$

Using integration by parts, we obtain

$$\int_0^s \big\langle \dot{v} - \dot{u}, v - u \big\rangle \, dt = \frac{1}{2}\big[\|v(s) - u(s)\|^2 - \|v(0) - u_0\|^2\big]$$
$$\big\langle g(|\nabla u|^2)\nabla u, \nabla v - \nabla u \big\rangle = -\big\langle \nabla \cdot (g(|\nabla u|^2)\nabla u), v - u \big\rangle$$

If we define $\hat{J}_R(u) = \int_\Omega \Phi(|\nabla u|) \, dx + \frac{\lambda}{2} \int_\Omega (Ru - u_0)^2 \, dx$, we obtain

$$\int_0^s \big\langle \dot{v}, v - u \big\rangle \, dt + \int_0^s \big[\hat{J}_R(v) - \hat{J}_R(u)\big] \, dt \geq \frac{1}{2}\big[\|v(s) - u(s)\|^2 - \|v(0) - u_0\|^2\big] \quad (1.7.1)$$

Conversely, if $u \in C^2(\bar{Q})$, $u(0) = u_0$, u is satisfying homogeneous Neumann boundary condition and satisfies (1.7.1) for all $v \in C^2(\bar{Q})$, then

$$\int_0^s \langle \dot{u}, v - u \rangle \, dt + \int_0^s \left[\hat{J}_R(v) - \hat{J}_R(u) \right] dt \geq 0$$

Let $v = u + tw$ with $t > 0$, we obtain

$$\int_0^s \langle \dot{u}, tw \rangle \, dt + \int_0^s \left[\hat{J}_R(u + tw) - \hat{J}_R(u) \right] dt \geq 0$$

This inequality is divided by t and let $t \to 0$, we get

$$\int_0^s \langle \dot{u}, w \rangle \, dt + \int_0^s \left[\langle g(|\nabla u|^2) \nabla u, \nabla w \rangle + \lambda \langle Ru - u_0, Rw \rangle \, dt \geq 0 \right.$$

Integration by parts, we get

$$\int_0^s \langle \dot{u}, w \rangle \, dt - \int_0^s \left[\langle \nabla \cdot (g(|\nabla u|^2) \nabla u), w \rangle - \lambda \langle R^*(Ru - u_0), w \rangle \, dt \geq 0 \right.$$

Since $w \in C^2(\bar{Q})$ is arbitrary, we obtain

$$\dot{u} = \nabla \cdot (g(|\nabla u|^2) \nabla u) - \lambda R^*(Ru - u_0)$$

In Chapter 3, we study a class of second order parabolic PDEs

$$\begin{cases} \dot{u} = \nabla \cdot (g(|\nabla u|^2) \nabla u) - \lambda R^*(Ru - h) \\ \frac{\partial u}{\partial \nu}|_\Gamma = 0 \\ u(x, 0) = u_0(x) \end{cases} \tag{1.7.2}$$

here $\dot{u}$ denotes the partial derivative with respect to t, ν is the boundary normal pointing outward. λ is some positive constant. Under the following assumptions on $g(\cdot)$:

$$\begin{cases} g(s) : [0, +\infty) \to [0, +\infty) \text{ decreasing} \\ \alpha s - \beta \leq s^2 g(s^2) \leq \alpha s + \beta \\ c(s) = g(s) + 2sg'(s) \geq 0 \end{cases} \tag{1.7.3}$$

Equation (3.1.1) is a parabolic equation and highly degenerate [7]. Although it has been studied in [7] by using semi-group theory and maximum monotone operator in case that the initial value is in space of functions of bounded variation (BV) [100, 35, 5], unfortunately, the noisy initial image u_0 is usually not in this space, it is desirable to know the solution property under weaker assumption on u_0 [8]. Following the study of time dependent minimal surface problem [86, 39] and total variation flow problem [38], we prove the existence and regularity of generalized solution of (3.1.1) if $u_0 \in L^2(\Omega)$. If $u_0 \in BV(\Omega) \cap L^2(\Omega)$, the existence and uniqueness of generalized solution is proved, i.e. we have the following theorem,

Theorem 1.7.1 (Generalized Solution). *Let Ω be a bounded open domain with Lipschitz boundary. $\hat{J}_R(u) = \int_\Omega \Phi(|Du|)\,dx + \frac{1}{2}\int_\Omega |Ru - h|^2\,dx$.*

(a) Suppose that $u_0, h \in L^2(\Omega)$, then there exists a function u such that

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;BV(\Omega))$$

$$u \in L^\infty(s_0,T;BV(\Omega)) \cap C([s_0,T];L^2(\Omega)),\ s_0 \in (0,T]$$

$$\dot{u} \in L^2(0,T;H^{-1}(\Omega))$$

$u(t)$ is weakly continuous from $[0,T] \to L^2(\Omega)$.

$\forall\, s \in (0,T]$, $\forall\, v \in L^1(0,T;BV(\Omega)) \cap L^2(0,T;L^2(\Omega)) \cap C([0,T];L^2(\Omega))$ such

[7] It has to be compared with the degenerate parabolic equations in [28]. There $p > 1$, here $p = 1$.

[8] In [38], the generalized solution of gradient flow of total variation is studied in case $u_0 \in L^2(\Omega)$.

that $\dot{v} \in L^2(0,T;L^2(\Omega))$, *we have*

$$\begin{aligned}
&\int_0^s \int_\Omega \dot{v}(v-u)\,dx\,dt + \int_0^s [\hat{J}_R(v) - \hat{J}_R(u)]dt \\
&\geq \frac{1}{2}\big[\|v(s)-u(s)\|^2 - \|v(0)-u_0\|^2\big]
\end{aligned} \tag{1.7.4}$$

(b) Suppose u_1 *and* u_2 *are two functions which satisfy* (1.7.4) *with initial data* u_{10}, h_1 *and* u_{20}, h_2 *respectively. If* $u_{10}, u_{20} \in L^2(\Omega) \cap BV(\Omega)$, $h_1, h_2 \in L^2(\Omega)$. *Then, there holds stability inequality*

$$\|u_1(s) - u_2(s)\|^2 \leq \|u_{10} - u_{20}\|^2 + s\|h_1 - h_2\|^2 \quad \forall\, s \in [0,T] \tag{1.7.5}$$

(c) If $u_0 \in BV(\Omega) \cap L^2(\Omega)$ *and* $h \in L^2(\Omega)$, *then* u *is unique,* $u(0) = u_0$ *and*

$$\begin{aligned}
&u \in L^\infty(0,T;BV(\Omega) \cap L^2(\Omega)) \cap C([0,T], L^2(\Omega)) \\
&\dot{u} \in L^2(0,T;L^2(\Omega))
\end{aligned}$$

$\forall\, s \in [0,T]$, $\forall\, v \in L^1(0,T;BV(\Omega)) \cap L^2(0,T;L^2(\Omega))$, *we have*

$$\int_0^s \int_\Omega \dot{u}(v-u)\,dx\,dt + \int_0^s \big[\hat{J}_R(v) - \hat{J}_R(u)\big]\,dt \geq 0$$

Remark 1.7.2. In case of $u_0 \in L^2(\Omega)$, the solution $u(t)$ is only weakly continuous from $[0,T] \to L^2(\Omega)$. The strong continuity is usually not true. The uniqueness of the solution is not proved either. In the literature, there are some mistakes regarding the proof of continuity and uniqueness of u when $u_0 \in L^2(\Omega)$. By looking at the proof of stability inequality in case $u_0 \in BV(\Omega) \cap L^2(\Omega)$, it is tempting to use a density argument to do it: suppose that $u_0^n \in BV(\Omega) \cap L^2(\Omega) \to$

$u_0 \in L^2(\Omega)$, u^n is the generalized solution corresponding to u_0^n, but it turns out that we don't know if $u_n \to u$ in any sense.

In Chapter 4, we turn to study some fourth order parabolic PDEs which are also highly degenerate. Among the PDEs in [20, 21, 99, 62], some are derived from the special cases of variational problem

$$\inf_u \left\{ J(u) = \int_\Omega \left[\Phi_1(|\nabla u|) + \Phi(|\nabla^2 u|) + \frac{\lambda}{2}\big(u-h\big)^2 \right] dx \right\} \tag{1.7.6}$$

here $\nabla^2 u$ is the Hessian matrix of u. $\Phi_1(\cdot), \Phi(\cdot)$ are even, convex functions from $\mathbb{R} \to \mathbb{R}^+$. They are nondecreasing in $\mathbb{R}^+$ and satisfy the following assumptions:

$$\begin{cases} \Phi(0) = 0 & \alpha|z| - \beta \le \Phi(|z|) \le \alpha|z| + \beta \\ \Phi_1(0) = 0 & \Phi_1(|z|) \le \alpha_1|z| + \beta_1 \end{cases} \tag{1.7.7}$$

where $\alpha, \alpha_1, \beta, \beta_1$ are positive constants. The rigorous study of fourth order evolutionary PDEs which appear in image processing is not common in literature. Greer and Bertozzi may be the first to study them. In [43], they study the traveling wave solutions of PDEs (1.1.10), (1.1.22), (1.1.19) in one space dimension by adding a Burger's convection term. In [44], they study the H^1 solution of mollifier regularized (1.1.19). Following the same approach as the second order parabolic PDEs, we prove the existence and regularity of generalized solution of one class of fourth order parabolic PDEs (1.7.8) in space of functions of bounded

Hessian (BH) [25] with initial condition $u_0 \in L^2(\Omega)$.

$$\begin{cases} \dot{u} = \nabla \cdot (\frac{\Phi_1'(|\nabla u|)}{|\nabla u|}\nabla u) - \nabla^2 \cdot (\frac{\Phi'(|\nabla^2 u|)}{|\nabla^2 u|}\nabla^2 u) - \lambda(u-h) \\ u(\cdot,t) \text{ is periodic} \\ u(x,0) = u_0(x) \end{cases} \tag{1.7.8}$$

here Φ, Φ_1 are smooth functions which satisfy the previous assumptions. We have the following theorem,

Theorem 1.7.3 (Generalized solution). *Suppose that* $\Omega = \prod_{i=1}^{d}(0, L_i)$*, a bounded open set in* $\mathbb{R}^d$*,* Φ_1, Φ *are smooth functions which satisfy previous assumptions.* $\hat{J}_h(u) := \int_\Omega \left[\Phi_1(|\nabla u|) + \Phi(|\nabla^2 u|) + \frac{\lambda}{2}(u-h)^2\right] dx.$

(a) If $u_0, h \in L^2(\Omega)$*, then there exists* u *such that*

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;BH_{per}(\Omega))$$

$$u \in L^\infty(s_0,T;BH_{per}(\Omega)) \cap C([s_0,T];L^2(\Omega)),\ s_0 \in (0,T]$$

$$\dot{u} \in L^2(0,T;V')$$

$u(t)$ *is weakly continuous from* $[0,T] \to L^2(\Omega)$.

$\forall v \in L^1(0,T;BH_{per}(\Omega)) \cap L^2(0,T;L^2(\Omega))$ *with* $\dot{v} \in L^2(0,T;L^2(\Omega))$

$$\int_0^s \int_\Omega \dot{v}(v-u)\,dxdt + \int_0^s (\hat{J}_h(v) - \hat{J}_h(u))\,dt \geq \frac{1}{2}\left[\|v(s)-u(s)\|^2 - \|v(0)-u_0\|^2\right] \quad \forall s \in (0,T] \tag{1.7.9}$$

(b) Suppose u_1*,* u_2 *satisfies* (1.7.9) *with initial data* u_{01}, h_1 *and* u_{02}, h_2 *respectively. Assume* $u_{01}, u_{02} \in BH_{per}(\Omega) \cap L^2(\Omega)$*,* $h_1, h_2 \in L^2(\Omega)$ *then*

$$\|u_1(s) - u_2(s)\|^2 \leq \|u_{01} - u_{02}\|^2 + \lambda s\|h_1 - h_2\|^2 \quad \forall\, s \in [0,T]$$

(c) Furthermore, if $u_0 \in L^2(\Omega) \cap BH_{per}(\Omega)$, $h \in L^2(\Omega)$, then u is unique and $u \in L^\infty(0,T; BH_{per}(\Omega)) \cap C([0,T]; L^2(\Omega))$, $\dot{u} \in L^2(0,T; L^2(\Omega))$, $u(0) = u_0$ such that

$$\int_0^s \int_\Omega \dot{u}(v-u)\,dxdt + \int_0^s (\hat{J}_h(v) - \hat{J}_h(u))\,dt \geq 0 \quad s \in [0,T] \tag{1.7.10}$$

$\forall v \in L^1(0,T; BH_{per}(\Omega)) \cap L^2(0,T; L^2(\Omega))$. Thus

$$\int_\Omega \dot{u}(v-u)\,dx + \hat{J}_h(v) - \hat{J}_h(u) \geq 0 \quad a.e.\ t \in [0,T] \tag{1.7.11}$$

$\forall v \in BH_{per}(\Omega) \cap L^2(\Omega)$.

The existence and uniqueness of the minimizer of the functional (1.7.6) in the space of functions of bounded Hessian is also proved. We then introduce a new function space — bounded Laplacian

$$BL^p(\Omega) = \left\{u \in W^{1,p}(\Omega) : \Delta u \in \mathcal{M}(\Omega)\right\}$$
$$BL^p_{per}(\Omega) = \left\{u \in W^{1,p}_{per}(\Omega) : \Delta u \in \mathcal{M}(\Omega)\right\}$$

to study fourth order evolutionary PDEs in [91, 99] which are Δu (Laplacian of u) instead of $\nabla^2 u$. If we let $\Phi_1 \equiv 0$, $\Phi(s) = ks \arctan(s/k) - \frac{k^2}{2}\log((s/k)^2+1)$, then $\Phi'(s) = k\arctan(s/k)$, we will recover PDE (1.1.19). Bertozzi and Greer [11] made a change of variables $w = \arctan(\Delta u)$ when $k = 1$ and $\lambda = 0$ and derived the equation satisfied by w

$$\dot{w} + \cos^2 w \Delta^2 w = 0 \tag{1.7.12}$$

They first proved the existence and uniqueness to the mollified equation with periodic boundary condition

$$\begin{cases} \dot{w}^\epsilon = -J_\epsilon \cos^2 w^\epsilon \Delta^2 J_\epsilon w^\epsilon \\ w^\epsilon(\cdot, 0) = w_0 \end{cases}$$

where J_ϵ is a standard mollifier. They then derived parameter ϵ independent energy estimates and proved the existence and uniqueness of the smooth solution of (1.1.19) when initial condition $w_0 \in H^6(\Omega)$. They also pointed out that an interesting point for further study is to better understand the theory for the LCIS equation for noisy initial data. Thanks to elliptic boundary value problem involving measures [16, 4] and the density result of [27], we can prove the existence and regularity of the generalized solution of fourth order parabolic PDEs with initial data $u_0 \in L^2(\Omega)$.

$$\begin{cases} \dot{u} = \nabla \cdot (\frac{\Phi_1'(|\nabla u|)}{|\nabla u|} \nabla u) - \Delta \cdot (\frac{\Phi'(|\Delta u|)}{|\Delta u|} \Delta u) - \lambda(u - h) \\ u(\cdot, t) \ \text{ is } \ \text{periodic} \\ u(x, 0) = u_0(x) \end{cases} \tag{1.7.13}$$

We have the following theorem,

Theorem 1.7.4 (Generalized solution). *Suppose that* $\Omega = \prod_{i=1}^{d} (0, L_i)$, Φ_1, Φ *are smooth functions which satisfy previous assumptions.* $\hat{J}_h(u) := \int_\Omega \left[\Phi_1(|\nabla u|) + \Phi(|\Delta u|) + \frac{\lambda}{2}(u - h)^2\right] dx.$

(a) If $u_0, h \in L^2(\Omega)$, then there exists u such that

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;BL^p_{per}(\Omega))$$

$$u \in L^\infty(s_0,T;BL^p_{per}(\Omega)) \cap C([s_0,T];L^2(\Omega)),\ s_0 \in (0,T]$$

$$\dot{u} \in L^2(0,T;V')$$

$u(t)$ is weakly continuous from $[0,T] \to L^2(\Omega)$.

$\forall v \in L^1(0,T;BL^p_{per}(\Omega)) \cap L^2(0,T;L^2(\Omega))$ with $\dot{v} \in L^2(0,T;L^2(\Omega))$

$$\begin{aligned}&\int_0^s \int_\Omega \dot{v}(v-u)\,dxdt + \int_0^s (\hat{J}_h(v) - \hat{J}_h(u))\,dt \\ &\geq \frac{1}{2}\left[\|v(s)-u(s)\|^2 - \|v(0)-u_0\|^2\right] \quad \forall s \in (0,T]\end{aligned} \tag{1.7.14}$$

(b) Suppose u_1, u_2 satisfies (1.7.14) with initial data u_{01}, h_1 and u_{02}, h_2 respectively. Assume $u_{01}, u_{02} \in BL^p_{per}(\Omega) \cap L^2(\Omega)$, $h_1, h_2 \in L^2(\Omega)$ then

$$\|u_1(s) - u_2(s)\|^2 \leq \|u_{01} - u_{02}\|^2 + \lambda s\|h_1 - h_2\|^2 \quad \forall\, s \in [0,T]$$

(c) Furthermore, if $u_0 \in L^2(\Omega) \cap BL^p_{per}(\Omega)$, $h \in L^2(\Omega)$, then u is unique and $u \in L^\infty(0,T;BH_{per}(\Omega)) \cap C([0,T];L^2(\Omega))$, $\dot{u} \in L^2(0,T;L^2(\Omega))$, $u(0) = u_0$ such that

$$\int_0^s \int_\Omega \dot{u}(v-u)\,dxdt + \int_0^s (\hat{J}_h(v) - \hat{J}_h(u))\,dt \geq 0 \quad s \in [0,T] \tag{1.7.15}$$

$\forall v \in L^1(0,T;BL^p_{per}(\Omega)) \cap L^2(0,T;L^2(\Omega))$. Thus

$$\int_\Omega \dot{u}(v-u)\,dx + \hat{J}_h(v) - \hat{J}_h(u) \geq 0 \quad a.e.\ t \in [0,T] \tag{1.7.16}$$

$\forall v \in BL^p_{per}(\Omega) \cap L^2(\Omega)$.

Remark 1.7.5. In Theorem 1.7.3 and 1.7.4, if $u_0 \in L^2(\Omega)$, $u(t)$ is only weakly continuous from $[0,T] \to L^2(\Omega)$. The uniqueness is usually not true. The reason is mentioned in Remark 1.7.2. By the trace theorems of BH functions and BL^p functions in Chapter 2, it makes sense to consider the Neumann boundary value problem. But we can't prove the convergence of boundary condition. The trace operator is continuous in the norm topology, or a weaker topology so called strict (tight) convergence, but not in the weak* topology. The convergence we can obtain is weak* topology, we can't find a way to prove that the sequence does not concentrate on the boundary of the domain. Thus, we failed to prove the uniqueness of the generalized solution even u_0 is sufficiently smooth in case of Neumann boundary condition.

Finally, we study some evolutionary PDEs which even do not satisfy parabolicity condition. In practice, nonconvex functional minimization methods and the corresponding evolutionary PDEs often perform better [8] in image smoothing. Some such evolutionary PDEs are used for image smoothing and enhancement [41].

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|^2)\nabla u) \\ \frac{\partial u}{\partial \nu}|_\Gamma = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1.7.17}$$

The study of (1.7.17) are much more challenging. By adding a high order regularization term, we prove the existence and regularities of regularized evolutionary

PDEs which appear in [76, 41], i.e. we prove the following theorem,

Theorem 1.7.6 (Existence, uniqueness, and energy identity). *Let* $u_0 \in L^2(\Omega)$. *Then, the initial-boundary-value problem* (1.7.17) *has a unique weak solution* $u : \Omega \times [0,T] \to \mathbb{R}$ *such that* $u \in L^2(0,T; H_n^2(\Omega))$ *and* $\dot{u} \in L^2(0,T; (H_n^2(\Omega))')$. *For any* $v \in H_n^2(\Omega))$, *for a.e.* $t \in [0,T]$,

$$\begin{cases} \big\langle \dot{u}, v \big\rangle + \big\langle g(|\nabla u|^2)\nabla u, \nabla v \big\rangle + \epsilon \big\langle \Delta u, \Delta v \big\rangle \, dt = 0 \\ u(x,0) = u_0 \end{cases}$$

Furthermore, if $u_0 \in H^2(\Omega)$, *then* $u \in L^\infty(0,T; H^2(\Omega))$, $\dot{u} \in L^2(0,T; L^2(\Omega))$, *for a.e.* $t \in [0,T]$, *u satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 \, dx + \int_\Omega g(|\nabla u|^2)|\nabla u|^2 \, dx + \epsilon \int_\Omega |\Delta u|^2 \, dx = 0$$
$$\frac{d}{dt}\int_\Omega \left(\Phi(|\nabla u|) + \frac{\epsilon}{2}|\Delta u|^2 \right) dx + \int_\Omega |\dot{u}|^2 \, dx = 0$$

here $H_n^2(\Omega) = \Big\{ v \in H^2(\Omega) : \int_\Omega v \, dx = 0 \, , \, \partial_\nu v|_{\partial\Omega} = 0 \Big\}$, $g(\cdot) : \mathbb{R} \to \mathbb{R}$ is a C^1 function and satisfies:

$$\begin{cases} |g(s)| \le C, \quad \forall \, s \in \mathbb{R} \\ |s g'(s)| \le C, \quad \forall \, s \in \mathbb{R}. \end{cases}$$

If $g(s^2) = \frac{1}{1+s^2}$ or $g(s) = 1 - s^2$, we will recover the PDEs in [57] which have been studied with periodic boundary condition. If $g(s^2) = \frac{1}{\sqrt{1+(s/k_f)^2}} - \alpha \frac{1}{1+(s/k_b)^2}$, we will recover (1.2.3).

Chapter 2

Mathematical preliminary

Before we go to the details of studying those evolutionary PDEs, let's recall some mathematical preliminaries first.

2.1 Mathematical notations

Ω	bounded open domain in $\mathbb{R}^d$, $d = 1, 2, 3$
1^*	$= \frac{d}{d-1}$
Γ	the boundary of the domain Ω
$\bar{\Omega}$	$= \Omega \cup \Gamma$
$\mathcal{L}^d$	d-dimensional Lebegue measure, sometimes it is denoted by dx
$C_0^\infty(\Omega)$	$= C_c^\infty(\Omega) = \mathscr{D}(\Omega),$ the space of C^∞functions with compact support in Ω
$C_c(\Omega)$	the space of continuous functions with compact support in Ω
$C(\bar{\Omega})$	the space of uniformly continuous functions on $\Omega,$ thus there is a unique continuous extension to $\bar{\Omega}$
$C_0(\Omega)$	the completion of $C_c(\Omega)$ under sup-norm
$\mathscr{D}'(\Omega)$	the space of distributions on Ω
$\mathcal{M}(\Omega)$	$= \left[C_0(\Omega)\right]',$ the space of bounded Radon measures on Ω
$\mathcal{M}(\bar{\Omega})$	$= \left[C(\bar{\Omega})\right]',$ the space of bounded Radon measures on $\bar{\Omega}$
$\Vert \cdot \Vert$	the L^2 norm

2.2 Generalized Sobolev spaces

$W^{k,p}(\Omega)$, $k \geq 0$ integer, $1 \leq p \leq \infty$ is the Sobolev space of all functions $u : \Omega \to \mathbb{R}$ having all distributional derivatives onto order k in $L^p(\Omega)$. The space $W^{k,p}(\Omega)$, equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left\{ \sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right\}^{1/p} \tag{2.2.1}$$

is a Banach space. For $s > 0$ non-integer, we denote by $[s]$ the integer part of s, then $W^{s,p}(\Omega)$ is a subspace of $W^{[s],p}(\Omega)$ consisting of functions $u \in W^{[s],p}(\Omega)$ for which

$$[D^\alpha u]^p_{s-[s],p} = \int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{d+p(s-[s])}}\, dxdy \tag{2.2.2}$$

is finite for all α, $|\alpha| = [s]$. $W^{s,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left\{ \|u\|^p_{W^{[s],p}(\Omega)} + [D^\alpha u]^p_{s-[s],p} \right\}^{1/p} \tag{2.2.3}$$

these spaces are called Sobolev-Slobodeckii spaces. They are very special cases of the scales of Besov and Triebel-Lizorkin spaces [89, 90].

Theorem 2.2.1 (Sobolev embeddings [63]). *Let Ω be a bounded open domain in $\mathbb{R}^d$ with Lipschitz boundary and let $0 \le s_2 < s_1$, $1 \le p, q < \infty$.*

(a) If $(s_1 - s_2)p < d$, then

$$\begin{aligned} s_1 - s_2 \ge d\Big(\frac{1}{p} - \frac{1}{q}\Big) &\implies W^{s_1,p}(\Omega) \subset W^{s_2,q}(\Omega) \\ s_1 - s_2 > d\Big(\frac{1}{p} - \frac{1}{q}\Big) &\implies W^{s_1,p}(\Omega) \subset\subset W^{s_2,q}(\Omega) \end{aligned} \tag{2.2.4}$$

(b) If $(s_1 - s_2)p > d$, then for $\alpha \in [0,1)$,

$$\begin{aligned} (s_1 - s_2 - \alpha)p \ge d &\implies W^{s_1,p}(\Omega) \subset C^{s_2,\alpha}(\bar{\Omega}) \\ (s_1 - s_2 - \alpha)p > d &\implies W^{s_1,p}(\Omega) \subset\subset C^{s_2,\alpha}(\bar{\Omega}) \end{aligned} \tag{2.2.5}$$

(c) If $(s_1 - s_2)p = d$, then $\forall\, q \in [1, +\infty)$,

$$W^{s_1,p}(\Omega) \subset\subset W^{s_2,q}(\Omega) \tag{2.2.6}$$

The proof of this theorem can be found in Kufner [55].

Remark 2.2.2. Let $\Omega = \prod_{i=1}^{d}(0, L_i)$, $C^{\infty}_{per}(\bar{\Omega})$ be the set of all restrictions onto $\bar{\Omega}$ of real-valued, $L = (L_1, \ldots, L_d)$ periodic, C^{∞} functions on $\mathbb{R}^d$. For any number $s > 0$, any $p \in [0, \infty]$, let $W^{s,p}_{per}(\Omega)$ be the closure of $C^{\infty}_{per}(\bar{\Omega})$ under the Sobolev norm of $W^{s,p}(\Omega)$. Note that $W^{0,p}_{per}(\Omega) = L^p(\Omega)$. We write $H^s_{per}(\Omega) = W^{s,2}_{per}(\Omega)$.

2.3 Spaces involving time

Spaces involving time comprising functions mapping time into Banach spaces. Let X denote a real Banach space, with norm $\|\cdot\|$.

Definition 2.3.1. The space $L^p(0, T; X)$ consists of all measurable functions $u : [0, T] \mapsto X$ with

$$\|u\|_{L^p(0,T;X)} := \Big(\int_0^T \|u(t)\|^p \, dt \Big)^{1/p} < \infty \tag{2.3.1}$$

for $1 \leq p < \infty$, and

$$\|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0 \leq t \leq T} \|u(t)\| < \infty \tag{2.3.2}$$

Definition 2.3.2. The space $C([0, T]; X)$ comprises all continuous functions $u : [0, T] \to X$ with $\|u\|_{C([0,T];X)} := \max_{0 \leq t \leq T} \|u(t)\| < \infty$

Theorem 2.3.3 (Time Continuity). *Let $u \in L^p(t_0, T; X)$ and $\dot{u} \in L^p(t_0, T; X)$ for some $1 \leq p \leq \infty$. Then*

(a) $u \in C([t_0, T]; X)$ (after possibly being redefined on a set of measure zero).

(b) $u(t) = u(s) + \int_s^t \dot{u}(\tau)\, d\tau \;\forall\, t_0 \le s \le t \le T$.

(c) Furthermore, we have the estimate

$$\max_{t_0 \le t \le T} \|u(t)\| \le C\big[\|u\|_{L^p(t_0,T;X)} + \|\dot{u}\|_{L^p(t_0,T;X)}\big] \tag{2.3.3}$$

Proof. See Evans [34] Chapter 5.9 Theorem 2.

Theorem 2.3.4 (Time Continuity). *Suppose that V is a Banach space, V' denotes it's dual space, $V \subset L^2(\Omega) \subset V'$, $u \in L^2(0,T;V)$, $\dot{u} \in L^2(0,T;V')$. Then*

(a) $u \in C([0,T];L^2(\Omega))$ *(after possibly being redefined on a set of measure zero)*

(b) The mapping $t \to \|u(t)\|^2$ *is absolutely continuous, with* $\frac{d}{dt}\|u(t)\|^2 = 2\big\langle \dot{u}(t), u(t)\big\rangle$ *for a.e.* $t \in [0,T]$.

(c) Furthermore, we have the estimate

$$\max_{0 \le t \le T} \|u(t)\| \le C\big[\|u\|_{L^2(0,T;V)} + \|\dot{u}\|_{L^2(0,T;V')}\big] \tag{2.3.4}$$

Proof. Follow the same approach as Evans [34] Chapter 5.9 Theorem 3. See also the Lemma 3.2 of Temam [87].

Lemma 2.3.5 (Lemma 3.3 of [87], see also [84]). *Let X and Y be two Banach spaces such that $X \subset Y$, if a function $u \in L^\infty(0,T;X)$ and is weakly continuous with values in Y, then u is weakly continuous from $[0,T]$ into X. i.e. $t \mapsto \big\langle u(t), v\big\rangle$ is continuous, $\forall\, v \in X$.*

2.4 Compactness result

Theorem 2.4.1 (Weak sequential compactness [8]).

(a) Let X be a reflexive Banach space, $K > 0$, and $x_n \in X$ a sequence such that $|x_n|_X \leq K$. Then there exists $x \in X$ and a subsequence x_{n_j} of x_n such that $x_{n_j} \rightharpoonup x(j \to \infty)$ weakly in X.

(b) Let X be a separable Banach space, $K > 0$, and $l_n \in X'$ such that $|l_n|_{X'} \leq K$. Then there exists $l \in X'$ and a subsequence l_{n_j} of l_n such that $l_{n_j} \rightharpoonup l(j \to \infty)$ weakly in X'.*

Theorem 2.4.2 (Simon's compactness result [83]). *Assume X, B, Y are Banach spaces, $X \subset B \subset Y$ with the compact embedding $X \subset\subset B$. Let $F = \{f : f \in F\}$ be bounded in $L^p(0, T; X)$ where $1 \leq p < \infty$, and $\frac{\partial F}{\partial t} = \{\frac{\partial f}{\partial t} : f \in F\}$ be bounded in $L^1(0, T; Y)$. Then F is relatively compact in $L^p(0, T; B)$.*

2.5 Lower semicontinuity

Definition 2.5.1 (Lower semicontinuity). F is called lower semicontinuous (l.s.c) for weak topology if for all sequence $x_n \rightharpoonup x_0$ we have

$$\lim_{x_n \rightharpoonup x_0} \inf F(x_n) \geq F(x_0) \tag{2.5.1}$$

The same definition can be given with a strong topology.

Theorem 2.5.2 (Convexity [8]). *Let $F : X \to \mathbb{R}$ be convex. Then F is weakly lower semicontinuous if and only if F is strongly lower semicontinuous.*

2.6 Measures and function spaces

We review some basic measure concepts first and then recall the definition of function spaces involving measures. Many of the definitions and lemmas are from [5]. We also refer to [78] for measure theory.

2.6.1 Measure, Radon measure, Hausdorff measure

Definition 2.6.1 (σ-algebras and measure spaces). Let X be a nonempty set and let $\mathcal{E}$ be a collection of subsets of X.

(a) We say that $\mathcal{E}$ is an algebra if $\emptyset \in \mathcal{E}$, $E_1 \cup E_2 \in \mathcal{E}$ and $X \setminus E_1 \in \mathcal{E}$ whenever $E_1, E_2 \in \mathcal{E}$.

(b) We say that an algebra $\mathcal{E}$ is a σ-algebra if for any sequence $\{E_n\} \subset \mathcal{E}$ its union $\bigcup_n E_n \in \mathcal{E}$.

(c) For any collection $\mathcal{C}$ of subsets of X, the σ-algebra generated by $\mathcal{C}$ is the smallest σ-algebra containing $\mathcal{C}$. If (X, τ) is a topological space, we denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X, i.e., the σ-algebra generated by the open subsets of X.

(d) If $\mathcal{E}$ is a σ-algebra in X, we call the pair $(X, \mathcal{E})$ a measure space.

Definition 2.6.2 (Measure). Let $(X, \mathcal{E})$ be a measure space and let $m \geq 1$ be an integer.

(a) We say that $\mu : \mathcal{E} \to \mathbb{R}^m$ is a measure if $\mu(\emptyset) = 0$ and for any sequence $\{E_n\}_n$ of pairwise disjoint elements of $\mathcal{E}$,

$$\mu(\bigcup_{n}^{\infty} E_n) = \sum_{n=0}^{\infty} \mu(E_n)$$

(b) If μ is a measure, the total variation $|\mu|(E)$ is defined as

$$|\mu|(E) = \sup \left\{ \sum_{n=0}^{\infty} |\mu(E_n)| : E_n \in \mathcal{E} \text{ pairwise disjoint, } E = \bigcup_{n=0}^{\infty} E_n \right\}$$

(c) If μ is a real measure, we define its positive and negative parts respectively as follows:

$$\mu^+ := \frac{|\mu| + \mu}{2} \quad \text{and} \quad \mu^- := \frac{|\mu| - \mu}{2}$$

Definition 2.6.3 (Radon Measure). Let X be an l.c.s (locally compact and separable) metric space, $\mathcal{B}(X)$ its Borel σ-algebra, and consider the measure space $(X, \mathcal{B}(X))$. A real or vector set function defined on the relatively compact Borel subsets of X that is a measure on $(K, \mathcal{B}(K))$ for every compact set $K \in X$ is called a real or vector Radon measure on X. If $\mu : \mathcal{B}(X) \to \mathbb{R}^m$ is a measure, we say that is a bounded Radon measure which is denoted by $\left[\mathcal{M}(\Omega)\right]^m$.

Remark 2.6.4. (a) Notice that if μ is a Radon measure and $\sup\{|\mu|(K) : K \in X \text{ compact } \} < \infty$ then it can be extended to the whole of $\mathcal{B}(X)$ and the resulting set function is a bounded Radon measure.

(b) Let $X = \Omega$, the bounded Radon measure on Ω is denoted by $\big[\mathcal{M}(\Omega)\big]^m$ which is the dual space of $\big[C_0(\bar{\Omega})\big]^m$ under the pairing

$$\langle \phi, \mu \rangle = \sum_{i=1}^{m} \int_{\Omega} \phi_i \, d\mu_i \tag{2.6.1}$$

(c) Let $X = \bar{\Omega}$, the bounded Radon measure on $\bar{\Omega}$ is denoted by $\big[\mathcal{M}(\bar{\Omega})\big]^m$ which is the dual space of $\big[C(\bar{\Omega})\big]^m$ under the pairing

$$\langle \phi, \mu \rangle = \sum_{i=1}^{m} \int_{\bar{\Omega}} \phi_i \, d\mu_i \tag{2.6.2}$$

(c) It is easy to see that $L^1(\Omega) \subset \mathcal{M}(\bar{\Omega})$. Since for any $f \in L^1(\Omega)$, by extending to any Lebesgue measurable functions $\bar{f}$ defined on $\bar{\Omega}$

$$\bar{f}(\phi) = \int_{\Omega} f(x)\phi(x)\, dx \quad \forall\, \phi \in C(\bar{\Omega}) \tag{2.6.3}$$

defines a continuous linear functional on $C(\bar{\Omega})$. Consequently, $\bar{f} \in \mathcal{M}(\bar{\Omega})$ with $\bar{f}(\Gamma) = 0$, moreover

$$|\bar{f}|(\bar{\Omega}) \leq \|f\|_{L^1(\Omega)} \tag{2.6.4}$$

Lemma 2.6.5. *Let X be an locally compact and separable metric space and $\mu : \mathcal{B}(X) \to \mathbb{R}^d$ is a bounded Radon measure on it. Then any open set $E \in X$*

$$|\mu|(E) = \sup\Big\{ \sum_{j=1}^{d} \int_X \phi_j \, d\mu_j : v \in \big[C_0(E)\big]^d, \|\phi\|_\infty \leq 1 \Big\} \tag{2.6.5}$$

Definition 2.6.6 (Hausdorff measures). Let $k \in [0, \infty)$ and $E \in \mathbb{R}^d$. The k-dimensional Hausdorff measure of E is defined by

$$\mathcal{H}^k(E) := \lim_{\delta \to 0} \mathcal{H}^k_\delta(E) \tag{2.6.6}$$

where $\forall\delta > 0$, $\mathcal{H}^k_\delta(E)$ is defined by

$$\mathcal{H}^k_\delta(E) := \frac{\omega_k}{2^k}\inf\left\{\sum_{i=1}^{\infty}[\mathrm{diam}(E_i)]^k : \mathrm{diam}(E_i) < \delta, E \subset \bigcup_{i=1}^{\infty} E_i\right\} \tag{2.6.7}$$

for finite for countable covers $\{E_i\}$, with the convention $\mathrm{diam}(\emptyset) = 0$ and $\omega_k = \frac{\pi^{k/2}}{\Gamma(k/2+1)}$ (here $\Gamma(\cdot)$ is Gamma function).

Definition 2.6.7 (Absolutely continuity and singularity). Let μ be a positive measure and ν a real or vector measure on the measure space $(X, \mathcal{E})$.

(a) We say that ν is absolutely continuous with respect to μ, and write $\nu \ll \mu$, if for every $E \in \mathcal{E}$, $\mu(E) = 0 \implies |\nu|(E) = 0$.

(b) We say they are mutually singular and write $\nu \perp \mu$, if there exists $E \in \mathcal{E}$ such that $\mu(E) = 0$ and $|\nu|(X \setminus E) = 0$.

Theorem 2.6.8 (Radon-Nikodým). *Let μ, ν be measures, assume that μ is a positive measure and σ-finite. Then there is a unique pair of $\mathbb{R}^m$ valued measures ν^a, ν^s such that $\nu^a \ll \mu$, $\nu^s \perp \mu$ and $\nu = \nu^a + \nu^s$. Moreover, there is a unique function $f \in [L^1(X, \mu)]^m$ such that $\nu^a = f\mu$. The function f is called the density of ν with respect to μ and is denoted by ν/μ.*

2.6.2 Convex functions of a measure

Assume that Φ is a continuous convex function from $\mathbb{R}^l$ to $\mathbb{R}$ which has at most a linear growth at infinity

$$|\Phi(\xi)| \leq C(1 + |\xi|) \tag{2.6.8}$$

for some constant $C > 0$.

Definition 2.6.9 (Recession function). The recession function $\Phi_\infty(\cdot)$ of $\Phi(\cdot)$ is defined by

$$\Phi_\infty(\xi) = \limsup_{s\to\infty} \frac{\Phi(s\xi)}{s} \quad \forall \xi \in \mathbb{R}^l \tag{2.6.9}$$

We assume furthermore that Φ possesses an recession function $\Phi_\infty(\cdot)$. It is easy to see that Φ_∞ is continuous and positive homogeneous on $\mathbb{R}^l$. Given a measure $\mu \in \mathcal{M}(\Omega)$, we consider its Lebesgue decomposition $\mu = f dx + \mu^s$, where μ^s is singular with respect to Lebesgue measure dx. We now define $\Phi(\mu)$ by setting

$$\Phi(\mu) = \Phi \circ f dx + \Phi_\infty(\mu^s) \tag{2.6.10}$$

The formula (2.6.10) makes sense: $\Phi \circ f$ makes sense as a function in $L^1(\Omega)$ because of (2.6.8); $\Phi_\infty(\mu^s)$ is defined as

$$\Phi_\infty(\mu^s) = \Phi_\infty \circ h|\mu^s| \tag{2.6.11}$$

where h is a $|\mu^s|$-measurable function such that $\mu^s = h|\mu^s|$.

Remark 2.6.10. In (1.1.11), $\Phi(z) = |z|$, then $\Phi_\infty(z) = |z|$. In (1.1.20), $\Phi(z) = k|z| \arctan \frac{|z|}{k} - \frac{k^2}{2} \log\left(\left(\frac{|z|}{k}\right)^2 + 1\right)$, then $\Phi_\infty(z) = \frac{k\pi}{2}|z|$. In (1.6.4), $\Phi(z) = \rho_\sigma(z) = \frac{\sigma}{\pi} \log \cosh \frac{\pi z}{\sigma}$, then, $\Phi_\infty(z) = |z|$.

We refer to Demengel and Temam [27], Ambrosio [5] on functions defined on the space of bounded Radon measure $\mathcal{M}(\Omega)$.

2.6.3 Functions of bounded variation

Definition 2.6.11. Let $u \in L^1(\Omega)$; we say that u is a function of bounded variation in Ω if the distributional derivative of u is representable by a bounded Radon measure in Ω, i.e. if

$$\int_\Omega u\nabla \cdot \phi \, dx = -\sum_{i=1}^{d} \int_\Omega \phi_i dD_i u \quad \forall \phi \in \left[C_c^1(\Omega)\right]^d \tag{2.6.12}$$

for some $\mathbb{R}^d$ valued measure $Du = (D_1 u, \cdots, D_d u)$ in Ω. The vector space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$. For functions $u \in \left[BV(\Omega)\right]^m$, Du is an $m \times d$ matrix of measures $D_i u^j$ in Ω satisfying

$$\sum_{j=1}^{m} \int_\Omega u^j \nabla \cdot u^j \, dx = -\sum_{j=1}^{m} \sum_{i=1}^{d} \int_\Omega \phi_i^j \, dD_i u^j \quad \forall \, \phi \in \left[C_c^1(\Omega)\right]^{md} \tag{2.6.13}$$

We represent by ∇u the absolutely continuous part of Du with respect to Lebesgue measure dx, $D^s u$ the singular part of Du with respect to dx. By Theorem 2.6.8,

$$Du = \nabla u dx + D^s u \tag{2.6.14}$$

Definition 2.6.12 (Variation). Let $u \in \left[L^1(\Omega)\right]^m$. The *Variation* $V(u, \Omega)$ of u in Ω is defined by

$$V(u, \Omega) = \sup\Big\{ \sum_{j=1}^{m} \int_\Omega u^j \nabla \cdot \phi^j \, dx : \phi \in \left[C_c^1(\Omega)\right]^{md}, \|\phi\|_\infty \leq 1 \Big\} \tag{2.6.15}$$

A simple integration by parts proves that $V(u, \Omega) = \int_\Omega |\nabla u| \, dx$ if u is continuously differentiable in Ω.

Lemma 2.6.13 (Variation of BV functions). *Let $u \in \left[L^1(\Omega)\right]^m$. Then, $u \in \left[BV(\Omega)\right]^m$ iff $V(u,\Omega) < \infty$. In addition $V(u,\Omega) = |Du|(\Omega)$.*

Lemma 2.6.14 (BV embedding). *Assume Ω is bounded domain with Lipschitz boundary. Then*

$$BV(\Omega) \subset L^p(\Omega)$$

with continuous embedding if $1 \leq p \leq 1^*$. If $1 \leq p < 1^*$, the embedding is compact.

Usually, we can introduce three topologies in the space of functions of bounded variation. $\left[BV(\Omega)\right]^m$, endowed with the norm

$$\|u\|_{BV(\Omega)} = \int_\Omega |u|\, dx + |Du|(\Omega) \tag{2.6.16}$$

is a Banach space, but the norm topology is too strong for many applications. Even continuously differentiable functions are not dense in $\left[BV(\Omega)\right]^m$. For an example, in case $m = 1$, let's consider any $u \in BV(\Omega)$ such that $Du \neq 0$ and singular with respect to Lebesgue measure dx. Since $|\mu_1 - \mu_2| = |\mu_1| + |\mu_2|$ for mutually singular measures μ_1, μ_2, we obtain

$$|D(u-v)|(\Omega) = |Du|(\Omega) + |Dv|(\Omega) \geq |Du|(\Omega) > 0 \tag{2.6.17}$$

for any $v \in C^1(\Omega) \cap BV(\Omega)$. However, $\left[BV(\Omega)\right]^m$ functions can be approximated by smooth functions in an intermediate topology, which is weaker than norm topology and defined by the following distance:

$$d(u,v) = \int_\Omega |u-v|\, dx + \big||Du|(\Omega) - |Dv|(\Omega)\big| \tag{2.6.18}$$

The convergence under this distance is called strictly convergence. We have the following lemma:

Lemma 2.6.15. *For any $u \in \big[BV(\Omega)\big]^m$, there exists a sequence of functions $u_n \in \big[W^{1,1}(\Omega)\big]^m \cap \big[C^\infty(\Omega)\big]^m$ such that*

$$u_n \to u \quad \text{strictly in } \big[BV(\Omega)\big]^m \tag{2.6.19}$$

Moreover, if Ω is a bounded Lipschitz domain, we can choose $u_n \in \big[W^{1,1}(\Omega)\big]^m \cap \big[C^\infty(\bar{\Omega})\big]^m$ (c.f.[5] Remark 3.22).

In fact, a slightly stronger density result is valid [1].

Definition 2.6.16 (Weak* convergence). Let $u, u_n \in BV(\Omega)$. We say that $\big\{u_n\big\}$ weakly* converges in $BV(\Omega)$ to u if $\big\{u_n\big\}$ converges to u in $L^1(\Omega)$ and $\big\{Du_n\big\}$ weakly* converges to Du in Ω, i.e.

$$\lim_{n\to\infty}\int_\Omega \phi\, dDu_n = \int_\Omega \phi\, dDu \qquad \forall\, \phi \in C_0(\Omega) \tag{2.6.20}$$

Weak* convergence is weaker than strictly convergence. Under this convergence $BV(\Omega)$ has the compactness result.

Lemma 2.6.17 (Strict convergence [5]). *If $\big\{u_h\big\} \in [BV(\Omega)]^m$ strictly converges to u, and $f : \mathbb{R}^{md} \to \mathbb{R}$ is a continuous and positively 1-homogeneous function, we have*

$$\lim_{h\to\infty}\int_\Omega \phi f\Big(\frac{Du_h}{|Du_h|}\Big)\, dDu_h = \int_\Omega \phi f\Big(\frac{Du}{|Du|}\Big)\, dDu \tag{2.6.21}$$

[1] Please refer to Section 2.6.6 or Demengel and Temam [27].

for any bounded continuous function $\phi : \Omega \to \mathbb{R}$.

Theorem 2.6.18 (Boundary trace theorem [5]). *Let $\Omega \subset \mathbb{R}^d$ be an open set with bounded Lipschitz boundary and $u \in \left[BV(\Omega)\right]^m$. Then, for $\mathcal{H}^{d-1}$-almost every $x \in \partial\Omega$ there exists $Tu(x) \in \mathbb{R}^m$ such that*

$$\lim_{r\to 0} \fint_{\Omega\cap B_r(x)} |u(y) - Tu(x)|\, dy = 0 \tag{2.6.22}$$

Moreover, $\|Tu\|_{L^1(\partial\Omega)^m} \le C\|u\|_{BV}$ for some constant C depending only on Ω, the extension $\bar{u}$ of u to 0 out of Ω belongs to $\left[BV(\mathbb{R}^d)\right]^m$ and, viewing Du as a measure on the whole of $\mathbb{R}^d$ and concentrated on Ω, $D\bar{u}$ is given by

$$D\bar{u} = Du - (Tu \otimes \nu)\mathcal{H}^{d-1}(\partial\Omega) \tag{2.6.23}$$

where $a \otimes b$ is the $m \times d$ matrix with (i, j)-th entry $a_i b_j$ (for $a_i \in \mathbb{R}^m, b \in \mathbb{R}^d$). Furthermore, for any $i = 1, \cdots, d$, $j = 1, \cdots, m$ and $\phi \in C_c^1(\bar{\Omega})$ there holds

$$\int_\Omega u^j \frac{\partial\phi}{\partial x_i}\, dx = -\int_\Omega \phi\, dD_i u^i + \int_{\partial\Omega} (Tu)^j \nu_i \phi d\mathcal{H}^{d-1} \tag{2.6.24}$$

where $\nu = (\nu_1, \cdots, \nu_d)$ is the unit outer norm of the Ω.

In (2.6.14), ∇u is also called the approximate derivative of u. Now let's define the approximate upper limit $u^+(x)$ and the approximate lower limit $u^-(x)$ by

$$u^+(x) = \inf\left\{t \in [-\infty, +\infty] : \lim_{r\to 0} \frac{dx(\{u > t\} \cap B(x, r))}{r^d} = 0\right\}$$
$$u^-(x) = \sup\left\{t \in [-\infty, +\infty] : \lim_{r\to 0} \frac{dx(\{u > t\} \cap B(x, r))}{r^d} = 0\right\}$$

If $u \in L^1(\Omega)$, then

$$\lim_{r\to 0} \fint_{B(x,r)} |u(x) - u(y)| \, dy = 0 \quad \text{a.e. } x \tag{2.6.25}$$

A point x for which (2.6.25) holds is called a Lebesgue point of u, and we have

$$u(x) = \lim_{r\to 0} \fint_{B(x,r)} u(y) \, dy, \quad u(x) = u^+(x) = u^-(x)$$

We denote by S_u the jump set, that is, the complement, up to a set of $\mathcal{H}^{d-1}$ measure zero, of the set of Lebesgue points

$$S_u = \left\{ x \in \Omega : u^-(x) < u^+(x) \right\}$$

For $\mathcal{H}^{d-1}$ a.e. $x \in S_u$, we can define a normal $n_u(x)$. Then, Du can be decomposed as [5]:

$$Du = \nabla u dx + (u^+ - u^-) n_u \mathcal{H}^{d-1}|_{S_u} + C_u \tag{2.6.26}$$

We define $SBV(\Omega)$ as the space of special functions of bounded variation, which is the space of $BV(\Omega)$ functions such that $C_u = 0$.

2.6.4 Functions of bounded Hessian

We now introduce the space of functions of bounded Hessian.

Definition 2.6.19 (Bounded Hessian).

$$\begin{aligned} BH(\Omega) &= \left\{ u \in W^{1,1}(\Omega) : D^2 u \in \left[\mathcal{M}(\Omega)\right]^{d\times d} \right\} \\ &= \left\{ u \in L^1(\Omega) : D^2 u \in \left[\mathcal{M}(\Omega)\right]^{d\times d} \right\} \\ &= \left\{ u \in L^1(\Omega) : Du \in \left[BV(\Omega)\right]^{d} \right\} \end{aligned} \tag{2.6.27}$$

where D^2u denotes the distributional Hessian matrix of u.

If endowed norm $\|u\|_{BH(\Omega)} = \|u\|_{W^{1,1}(\Omega)} + |D^2u|(\Omega)$, $BH(\Omega)$ is a Banach space. If $\Omega = \prod_{i=1}^{d}(0, L_i)$, we also define

$$BH_{per}(\Omega) = W^{1,1}_{per}(\Omega) \cap BH(\Omega) \tag{2.6.28}$$

Definition 2.6.20 (BH^* convergence). $\{u_n\}_{n\geq 1}$ BH^*ly converges to u is defined as:

$$\begin{aligned} &u_n \to u \text{ strongly in } W^{1,1}(\Omega) \\ &D^2u_n \rightharpoonup D^2u \text{ weakly}^* \text{ in } \left[\mathcal{M}(\Omega)\right]^{d\times d} \end{aligned} \tag{2.6.29}$$

For various properties of $BH(\Omega)$, we refer to Demengel [25, 26].

Lemma 2.6.21 (BH embedding [25]). *Let $\Omega \in \mathbb{R}^d$ be bounded open set with Lipschitz boundary, then*

$$BH(\Omega) \subset W^{1,p}(\Omega) \tag{2.6.30}$$

with continuous embedding if $1 \leq p \leq 1^*$; the embedding is compact if $1 \leq p < 1^*$.

Lemma 2.6.22 (BH interpolation [25]). *Let $\Omega \in \mathbb{R}^d$ be Lipschitz, bounded open set, for every $\delta > 0$, there is a $C(\delta) > 0$, such that*

$$\|\nabla u\|_{L^1(\Omega)} \leq C(\delta)\|u\|_{L^1(\Omega)} + \delta|D^2u|(\Omega) \tag{2.6.31}$$

2.6.5 Functions of bounded Laplacian

In order to study evolutionary PDEs which appear in [91, 99], we introduce a new function space $BL^p(\Omega)$, which defined by

$$BL^p(\Omega) = \{u \in W^{1,p}(\Omega) : \Delta u \in \mathcal{M}(\Omega)\} \tag{2.6.32}$$

where $1 \leq p < 1^*$. If $\Omega = \prod_{i=1}^{d}(0, L_i)$, define

$$BL^p_{per}(\Omega) = \{u \in W^{1,p}_{per}(\Omega) : \Delta u \in \mathcal{M}(\Omega)\} \tag{2.6.33}$$

Lemma 2.6.23. *$BL^p(\Omega)$ is a Banach space if endowed norm topology:*

$$\|u\|_{W^{1,p}(\Omega)} + |\Delta u|(\Omega) \tag{2.6.34}$$

Proof. Let $\{u_n\}$ be a Cauchy sequence in $BL^p(\Omega)$. Then $\{u_n\}$ is a Cauchy sequence in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is Banach space, there exists $u \in W^{1,p}(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. On the other hand, there exists $\mu \in \mathcal{M}(\Omega)$ such that $\Delta u_n \to \mu$ in $\mathcal{M}(\Omega)$ since $\mathcal{M}(\Omega)$ is a Banach space. For any $\phi \in \mathscr{D}(\Omega)$, we have

$$\int_\Omega \Delta u\phi = -\int_\Omega \nabla u \cdot \nabla\phi\, dx = -\lim_{n\to\infty}\int_\Omega \nabla u_n \cdot \nabla\phi\, dx \tag{2.6.35}$$

$$\int_\Omega \mu\phi = \lim_{n\to\infty}\int_\Omega \Delta u_n\phi = -\lim_{n\to\infty}\int_\Omega \nabla u_n \cdot \nabla\phi\, dx \tag{2.6.36}$$

Therefore, $\mu = \Delta u$ in the distributional sense, i.e. the distributional derivative Δu is a Radon measure on Ω. Consequently, $BL^p(\Omega)$ is a Banach space.

Similarly, $BL^p_{per}(\Omega)$ is a Banach space if endowed with norm topology.

Theorem 2.6.24 (Trace theorem for $BL^p(\Omega)$ [16]). *Assume $1 < p < 1^*$, there exists a unique linear and continuous mapping γ_ν such that*

$$\gamma_\nu : BL^p(\Omega) \to W^{-1/p,p}(\Gamma) \tag{2.6.37}$$

$$\gamma_\nu(u) = \nabla u|_\Gamma \cdot \nu \ , \forall\, u \in C^1(\bar{\Omega}) \tag{2.6.38}$$

$$\big\langle \gamma_\nu(u), \gamma(z) \big\rangle = \int_\Omega \nabla u \cdot \nabla z \, dx + \int_\Omega z \, d\Delta u \ , \forall\, z \in W^{1,q}(\Omega) \tag{2.6.39}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. This proof is from [16]. Let us take $g \in W^{1/p,q}(\Gamma) = \gamma(W^{1,q}(\Omega))$ and $z \in W^{1,q}(\Omega)$ such that $\gamma(z) = g$. Then we define

$$\big\langle \gamma_\nu(u), g \big\rangle = \int_\Omega \nabla u \cdot \nabla z \, dx + \int_\Omega z \, d\Delta u \tag{2.6.40}$$

Let us prove that γ_ν is well defined. First, from the inequality $p < \frac{d}{d-1}$, it follows that $q > d$, and therefore $W^{1,q}(\Omega) \subset C(\bar{\Omega})$. On the other hand, if $z_1, z_2 \in W^{1,q}(\Omega)$ and $\gamma(z_1) = \gamma(z_2) = g$, then we must prove that

$$\int_\Omega \nabla u \cdot \nabla z_1 \, dx + \int_\Omega z_1 \, d\Delta u = \int_\Omega \nabla u \cdot \nabla z_2 \, dx + \int_\Omega z_2 \, d\Delta u \tag{2.6.41}$$

To do this, let us take $z = z_1 - z_2 \in W_0^{1,q}(\Omega)$ and $\{z_k\} \in \mathscr{D}(\Omega)$ a sequence converging to z in $W_0^{1,q}(\Omega)$. Since $q > d$, we have that $\nabla z_k \to \nabla z$ in $\left[L^q(\Omega)\right]^d$ and $z_k \to z$ in $C(\bar{\Omega})$, from where we obtain that

$$\int_\Omega \nabla u \cdot \nabla z \, dx + \int_\Omega z \, d\Delta u = \lim_{k\to\infty} \Big\{ \int_\Omega \nabla u \cdot \nabla z_k \, dx + \int_\Omega z_k \, d\Delta u \Big\} = 0 \tag{2.6.42}$$

the last inequality being a consequence of the definition of derivative in the distributional sense. So we have that γ_ν is well defined, and obviously, it is linear.

Let us prove the continuity.

$$\begin{aligned}\left|\left\langle \gamma_\nu(u), g\right\rangle\right| &\leq \|\nabla u\|_{L^p(\Omega)^d}\|\nabla z\|_{L^q(\Omega)^d} + \|\Delta u\|_{\mathcal{M}(\Omega)}\|z\|_{C(\bar{\Omega})} \\ &\leq C\|u\|_{BL^p(\Omega)}\|z\|_{W^{1,q}(\Omega)}\end{aligned} \tag{2.6.43}$$

Taking now the infimum we obtain that

$$\left|\left\langle \gamma_\nu(u), g\right\rangle\right| \leq C\|u\|_{BL^p(\Omega)} \inf_{\gamma(z)=g} \|z\|_{W^{1,q}(\Omega)} = C\|u\|_{BL^p(\Omega)}\|g\|_{W^{1/p,q}(\Omega)} \tag{2.6.44}$$

which implies the continuity of γ_ν. From the definition of γ_ν and using the Green's formula for regular functions, it is immediate to prove that (2.6.38) is satisfied. The uniqueness follows from (2.6.39) and the surjectivity of $\gamma : W^{1,q}(\Omega) \to W^{1/p,q}(\Omega)$.

2.6.6 Density result in space involving measures

X is the space defined by

$$X = \left\{u \in \left[L^1(\Omega)\right]^m : Su \in \left[\mathcal{M}(\Omega)\right]^l\right\} \tag{2.6.45}$$

where S is a linear differential operator with constant coefficients which operates from $\left[C_0^\infty(\Omega)\right]^m$ into $\left[C_0^\infty(\Omega)\right]^l$. Let $\Phi(\cdot)$ be a convex function such that $\Phi(0) = 0$ and with at most linear growth. We denote by X_Φ the space X equipped with the intermediate topology defined by the distance

$$d(u,v) = \|u-v\|_{L^1(\Omega)^m} + \left|\int_\Omega |Su| - \int_\Omega |Sv|\right| + \left|\int_\Omega |\Phi(Su)| - \int_\Omega |\Phi(Sv)|\right| \tag{2.6.46}$$

We set

$$Y = \left\{u \in \left[L^1(\Omega)\right]^m : Su \in \left[L^1(\Omega)\right]^l\right\} \tag{2.6.47}$$

Theorem 2.6.25 (Density result [27]). *Assume that Ω is an open bounded domain with Lipschitz boundary* [2] *and*

$$\forall\, u \in X, \forall\, \phi \in C^{\infty}(\bar{\Omega}), S(\phi u) - \phi S u \in \left[L^1(\Omega)\right]^l \tag{2.6.48}$$

Then for every u given in X, there exists a sequence $u_n \in \left[C^{\infty}(\Omega)\right]^m \cap Y$ such that $u_n \to u$ in X_Φ as $n \to \infty$.

Suppose that Φ is a convex function which satisfies (2.6.8) and $\Phi(0) = 0$. We denote $BL^p_\Phi(\Omega)$ is the space $BL^p(\Omega)$ equipped with the topology defined by

$$d(u,v) = \|u-v\|_{W^{1,p}(\Omega)} + \left|\int_\Omega |\Delta u| - \int_\Omega |\Delta v|\right| + \left|\int_\Omega \Phi(\Delta u) - \int_\Omega \Phi(\Delta v)\right| \tag{2.6.49}$$

This topology is stronger than the weak* topology on $BL^p(\Omega)$ corresponding the family of distance and semi-distances

$$\|u-v\|_{W^{1,p}(\Omega)}, \quad d(u,v) = \left|\int_\Omega \phi \Delta u - \int_\Omega \phi \Delta v\right| \quad \phi \in C_0(\Omega) \tag{2.6.50}$$

but it is weaker than the topology induced by norm. Following the approach of Demengel [27], it is not hard to prove the following theorem.

Theorem 2.6.26. *Assume that Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary, for any $u \in BL^p(\Omega)$, then there is a sequence $\{u_n\} \in C^{\infty}(\Omega) \cap Y$, such that*

$$u_n \to u \quad \text{in } BL^p_\Phi(\Omega) \tag{2.6.51}$$

Here $Y = \left\{u \in W^{1,p}(\Omega) : \Delta u \in L^1(\Omega)\right\}$.

[2] In Demengel and Temam's paper, the boundary of Ω is C^1.

2.6.7 Elliptic BVP involving measures

Elliptic boundary value problems (BVP) involving L^1 data or measures have been intensively studied in the literatures, please refer to [16, 10, 14, 58, 30, 77, 4], [82] Chapter two, [33] and the references therein. In our application, we are particularly interested in the existence, regularity and uniqueness of the following elliptic BVP problem

$$\begin{cases} -\Delta u &= \mu \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \Gamma \end{cases} \tag{2.6.52}$$

where μ is a Radon measure on $\bar{\Omega}$. Amann [4] studied a more general elliptic problems involving measures.

$$\begin{cases} -\Delta u &= h(x,u) + \mu & \text{in } \Omega \\ u &= \sigma_0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} &= \sigma_1 & \text{on } \Gamma_1 \end{cases} \tag{2.6.53}$$

Here $\Gamma = \Gamma_0 \cup \Gamma_1$. He casted the problem to functional analysis frame work:

$$\begin{cases} \mathcal{A}u &= \mu \text{ in } \Omega \\ \mathcal{B}u &= \sigma \text{ on } \Gamma \end{cases} \tag{2.6.54}$$

where $(\mathcal{A}, \mathcal{B})$ is a strongly uniformly elliptic BVP and is a linear isomorphism from $W^{2,p}(\Omega)$ to $L^p(\Omega) \times \partial W^{2,p}$ $(1 < p < \infty)$. Assume that $\Gamma = \partial\Omega$ is C^2, $\mu \in \mathcal{M}(\Omega \cup \Gamma_1)$, $\sigma = \mathcal{M}(\Gamma)$, if $\sigma|_{\Gamma_0} = 0$, then the elliptic boundary value problem has a unique weak solution such that $\forall\ 0 < s \leq 1$

$$\|u\|_{W^{2-s,1}(\Omega)} \leq C\big(|\mu|(\Omega \cup \Gamma_1) + |\sigma|(\Gamma_1)\big) \tag{2.6.55}$$

where C only depends on s, Ω and $(\mathcal{A}, \mathcal{B})$. Casas [16] has also studied the existence and uniqueness of the linear elliptic equation with Neumann boundary condition and measure data. He used a different approach with the assumption of Γ to be in $C^{1,1}$ and get a weaker estimate of the solution in terms of initial data

$$\|u\|_{W^{1,p}} \leq C\Big[|\mu|(\Omega) + |\sigma|(\Gamma) + \|u\|_{L^1(\Omega)}\Big] \tag{2.6.56}$$

where $1 \leq p < 1^*$. For (2.6.52), the problem has pure Neumann boundary condition. Following Amann's approach, we have the following theorem,

Theorem 2.6.27. *If $\mu \in \mathcal{M}(\bar{\Omega})$, $\mu(\bar{\Omega}) = 0$, Γ is C^2 or $\Omega = \prod_{i=1}^{d}(0, L_i)$, then* (2.6.52) *has a weak solution which satisfies:*

$$\left\| u - \fint_{\Omega} u \right\|_{W^{2-s,1}(\Omega)} \leq C|\mu|(\bar{\Omega}) \tag{2.6.57}$$

where $0 < s \leq 1$. Up to a constant, the solution is unique.

Remark 2.6.28. If $\Omega = \prod_{i=1}^{d}(0, L_i)$, the elliptic boundary value problem with periodic boundary condition has the similar result.

2.7 Monotone property of convex function

Lemma 2.7.1. *Assume that $\Phi(\cdot) : \mathbb{R} \to \mathbb{R}$ convex and smooth, $\Phi(\cdot)$ is nondecreasing on $\mathbb{R}^+$. $\xi, \eta \in \mathbb{R}^l$, then, we have*

$$\Big\langle \frac{\Phi'(|\xi|)}{|\xi|}\xi - \frac{\Phi'(|\eta|)}{|\eta|}\eta, \xi - \eta \Big\rangle \geq 0 \tag{2.7.1}$$

Proof. Let $f(\xi) = \frac{\Phi'(|\xi|)}{|\xi|}\xi$, then $f(\cdot) : \mathbb{R}^l \to \mathbb{R}^l$.

$$f'(\xi) = \frac{\Phi'(|\xi|)}{|\xi|} I_{l\times l} + \frac{\Phi''(|\xi|)|\xi| - \Phi'(|\xi|)}{|\xi|^3}\xi\xi^t$$

By Rolle's theorem, for some ζ which lies on the line between ξ and η,

$$\begin{aligned}
&\big\langle f(\xi) - f(\eta), \xi - \eta\big\rangle = \big\langle f'(\zeta)(\xi-\eta), \xi - \eta\big\rangle \\
&= (\xi-\eta)^T\left(\frac{\Phi'(|\zeta|)}{|\zeta|} I_{l\times l} + \frac{\Phi''(|\zeta|)|\zeta| - \Phi'(|\zeta|)}{|\zeta|^3}\zeta\zeta^T\right)(\xi-\eta) \\
&= \frac{\Phi''(|\zeta|)}{|\zeta|^2}(\xi-\eta)^T(\xi-\eta) + \frac{\Phi'(|\zeta|)}{|\zeta|^3}\big[|\zeta|^2|\xi-\eta|^2 - (\xi-\eta)^T\zeta\zeta^T(\xi-\eta)\big] \geq 0
\end{aligned}$$

The last step holds because Cauchy inequality and $\Phi''(s) \geq 0$, $\Phi'(s) \geq 0$ for $s \geq 0$.

Lemma 2.7.2 (Convexity inequality). *Assume that $\Phi(\cdot) : \mathbb{R} \to \mathbb{R}$ is a smooth convex function, $\xi, \eta \in \mathbb{R}^l$, then*

$$\Phi(|\xi|) - \Phi(|\eta|) \geq \big\langle \frac{\Phi'(|\eta|)}{|\eta|}\eta, \xi - \eta\big\rangle \tag{2.7.2}$$

Proof. This is a direct result of convex function.

Chapter 3

The study of second order parabolic PDEs

In order to avoid over-smoothing of linear filter, many nonlinear second order evolutionary equations were proposed [76, 3, 2, 92, 8]. In this chapter, we study a class of highly degenerated second order parabolic PDEs which appear in [92, 8] and have been derived in Section 1.1.3. For this class of parabolic PDE, the coefficients of the second order terms will vanish if $|\nabla u| \to \infty$. A classic method to study such PDEs is using the so-called vanish viscosity method (also called weak convergence method). First, we study the regularized PDE which is obtained by adding a regularization term $\epsilon \Delta u$ to the original equation. The existence of weak solutions for regularized PDE is proved by using Galerkin method and the property of monotone operator. For any $\epsilon > 0$, we obtain u^ϵ which is the weak solution of the regularized equation and satisfies some ϵ independent energy estimates. Next, we pass the limit $\epsilon \to 0$, by using the weak compactness result in $L^p(0, T; B)$, here B is a Banach space, $1 < p < \infty$ and the compactness result in $L^1(0, T; B)$, we will obtain u as the limit of u^ϵ. Then, by the lower semicontinuity property of L^2 norm and the lower semicontinuity property of variational functional involving measures, we will obtain that u satisfies a variational inequality.

3.1 Nonlinear second order parabolic equations

We consider

$$\begin{cases} \dot{u} = \nabla \cdot (g(|\nabla u|^2)\nabla u) - \lambda R^*(Ru - h) & \text{in} \quad \Omega \times (0, +\infty) \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Gamma \times [0, +\infty) \\ u(x, 0) = u_0(x) & \text{on} \quad \Omega \end{cases} \tag{3.1.1}$$

here $\dot{u}$ denotes the partial derivative with respect to t, ν is the boundary normal pointing outward, λ is some positive constant. $R : L^2(\Omega) \to L^2(\Omega)$, a linear continuous operator, and R^* is the adjoint. u_0 and h are initial data functions [1].

Under the following assumptions on $g(\cdot)$

$$\begin{cases} g(s) & : [0, +\infty) \to [0, +\infty) \\ g(s) & \approx (s^{-\frac{1}{2}}) \;\; \text{as } s \to +\infty\,, \\ c(s) & = g(s) + 2sg'(s) \geq 0. \end{cases} \tag{3.1.2}$$

we will see that (3.1.1) is a parabolic equation. Let's look at the principle terms of (3.1.1):

$$\begin{aligned} \nabla \cdot (g(|\nabla u|^2)\nabla u) &= g(|\nabla u|^2)\Delta u + 2g'(|\nabla u|^2) \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i u \partial_j u \partial_{ij} u \\ &= g(|\nabla u|^2)\Delta u + 2g'(|\nabla u|^2)(\nabla u)^T \nabla^2 u \nabla u \end{aligned} \tag{3.1.3}$$

[1] For the sake of generality, in (3.1.1), a function h is being introduced. In practice, $h = u_0$.

The coefficient matrix of the second order partial derivatives is

$$
\begin{aligned}
& g(|\nabla u|^2)I + 2g'(|\nabla u|^2) \begin{pmatrix} \partial_1 u \partial_1 u & \cdots & \partial_1 u \partial_d u \\ \vdots & \ddots & \vdots \\ \partial_d u \partial_1 u & \cdots & \partial_d u \partial_d u \end{pmatrix} \\
&= g(|\nabla u|^2)I + 2g'(|\nabla u|^2)\nabla u (\nabla u)^T
\end{aligned} \tag{3.1.4}
$$

where I is a $d \times d$ identity matrix. $\forall \eta \in \mathbb{R}^d$, notice that $g'(s) \leq 0$ and $\eta^T \nabla u (\nabla u)^T \eta \leq |\eta|^2 |\nabla u|^2$, we obtain

$$
\begin{aligned}
& \eta^T g(|\nabla u|^2) I \eta + 2g'(|\nabla u|^2)\eta^T \nabla u (\nabla u)^T \eta \\
& \geq \eta^T \left[g(|\nabla u|^2) + 2g'(|\nabla u|^2)|\nabla u|^2 \right] \eta \geq 0
\end{aligned} \tag{3.1.5}
$$

Therefore, (3.1.1) is a parabolic equation. It is worth to point out that each equation in the system of evolutionary PDEs (1.5.5) is a special case of (3.1.1). PDE (1.1.10) proposed by Perona and Malik [76] has a similar form as (3.1.1), but it does not have the reaction term. They restricted themselves to functions $g(s) = \frac{1}{1+s/k^2}$ or $g(s) = e^{-s/k^2}$ which do not satisfies (3.1.2) either. There are some general results for degenerate parabolic equations in the literature [28]:

$$
\dot{u} - \nabla \cdot a(t, x, u, \nabla u) = b(x, t, u, \nabla u)
$$

where the functions a, b satisfy the structural conditions

$$
a(t, x, u, \nabla u) \cdot \nabla u \geq c_0 |\nabla u|^p - \phi_0(x, t)
$$

$$
|a(t, x, u, \nabla u)| \leq c_1 |\nabla u|^{p-1} + \phi_1(x, t)
$$

$$
|b(x, t, u, \nabla u)| \leq c_2 |\nabla u|^p + \phi_2(x, t)
$$

a.e. (x,t) with $p>1$. c_0, c_1, c_2 are given constants and ϕ_0, ϕ_1, ϕ_2 are given non-negative functions satisfying some integrability conditions. But we can't apply them because we have $p=1$ here. The difficulty of studying (3.1.1) comes from the highly degenerate behavior of it due to vanishing condition $g(s) \approx (s^{-\frac{1}{2}})$ as $s \to +\infty$ and is closely related to the fact that L^1 is not a reflexive Banach space.

3.2 Notations

Now, let's introduce some notations. For any $z \in \mathbb{R}$, let $\Phi(z) = \int_0^{|z|} \tau g(\tau^2)\, d\tau$, then $\Phi''(z) = g(|z|^2) + 2|z|^2 g'(|z|^2)$, according to (3.1.2), $\Phi''(z) \geq 0$, thus $\Phi(\cdot)$ is a convex function. Since $g(s) \approx \frac{1}{\sqrt{s}}$ as $s \to +\infty$, without loss of generality, let $\alpha = \lim\limits_{s\to+\infty} g(s^2)s$. We further assume [2]

$$\alpha s - \beta \leq s^2 g(s^2) \leq \alpha s + \beta \ , \forall\, s \in [0, +\infty) \tag{3.2.1}$$

$$\alpha|z| - \beta \leq \Phi(z) \leq \alpha|z| + \beta \ , \forall\, z \in \mathbb{R} \tag{3.2.2}$$

here β is some positive constant. There are many functions which satisfy (3.1.2), such as: $\Phi(z) = |z|$ $(g(s) = \frac{1}{\sqrt{s}})$, the total variation function, was introduced by Rudin and Osher [79]; $\Phi(z) = \sqrt{1+z^2} - 1$ $(g(s) = \frac{1}{\sqrt{1+s}})$, the function of minimal surfaces. We refer to [22] for more such functions. Notice the assumptions on

[2] In fact, from (3.2.2) and the convexity of $\Phi(\cdot)$, we get $s^2 g(s^2) = s\Phi'(s) \geq \Phi(s) \geq \alpha s - \beta$. On the other hand, due to the assumptions on $g(s)$, $sg(s^2)$ is nondecreasing, we get $sg(s^2) \leq \alpha$. Thus (3.2.1) is redundant.

$g(\cdot)$, we obtain the recession function [3] of $\Phi(\cdot)$ is $\Phi_\infty(z) = \alpha|z|$. It is easy to verify that $\Phi_\infty(\cdot)$ is a positively 1-homogeneous function, i.e.

$$\Phi_\infty(tz) = t\Phi_\infty(z) \ \ \forall\, z \in \mathbb{R},\ \forall\, t \geq 0$$

Let $u \in BV(\Omega)$, recall that the distributional derivative of u can be decomposed as $Du = \nabla u dx + D^s u$, here dx is Lebesgue measure, $D^s u$ is singular with respect to dx. Define

$$\hat{J}(u) = \int_\Omega \Phi(|\nabla u|)\, dx + \int_\Omega \Phi_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u| \tag{3.2.3}$$

Since $\Phi(\cdot)$ satisfies (3.2.2), we have

$$\hat{J}(u) = \int_\Omega \Phi(|\nabla u|)\, dx + \alpha |D^s u|(\Omega) \tag{3.2.4}$$

$\hat{J}(u)$ is lower semicontinuous with respect to the convergence in $L^1(\Omega)$ (cf. [5] Theorem 5.47). If $u \in W^{1,1}(\Omega)$, then the second term vanishes. For any given $h \in L^2(\Omega)$ and linear continuous operator R, define

$$\hat{J}_R(u) = \int_\Omega \Phi(|\nabla u|)\, dx + \alpha |D^s u|(\Omega) + \frac{1}{2}\int_\Omega |Ru - h|^2\, dx \tag{3.2.5}$$

Definition 3.2.1 (Subdifferential). The subdifferential $\partial \hat{J}_R$ at u is defined as:

$$\partial \hat{J}_R(u) = \left\{ \xi \in L^2(\Omega) : \hat{J}_R(v) - \hat{J}_R(u) \geq \big\langle \xi, v - u \big\rangle,\ \forall\, v \in L^2(\Omega) \right\} \tag{3.2.6}$$

[3] Refer to definition 2.6.9 for recession function.

3.3 Semigroup approach

Chambolle and Lions [20] first studied PDE (3.1.1) in the case $g(s) = \sqrt{s}$ by using nonlinear semigroup theory and monotone operator. Following the same approach, later, Vese [92] studied the general case. She proved the following theorem [92, 8].

Theorem 3.3.1. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded, and connected subset of $\mathbb{R}^d$ ($d = 1, 2$) with Lipschitz boundary. Let $u_0 \in Dom(\partial \hat{J}_R)$ and $h = u_0$. Then there exists a unique function $u(t) : [0, +\infty) \to L^2(\Omega)$ such that*

$$u(t) \in Dom(\partial \hat{J}_R), \quad \forall\, t > 0, \quad \dot{u} \in L^\infty(0, +\infty, L^2(\Omega)) \tag{3.3.1}$$

$$-\dot{u} \in \partial \hat{J}_R(u(t)), \quad a.e.\, t \in (0, +\infty), \quad u(0) = u_0 \tag{3.3.2}$$

If u_1 and u_2 are two solutions with u_{01}, u_{02} as initial conditions respectively, then

$$\|u_1(t) - u_2(t)\| \le \|u_{01} - u_{02}\| \quad \forall\, t \ge 0 \tag{3.3.3}$$

3.4 Vanish viscosity approach

PDE (3.1.1) is a generalization of the classical time dependent minimal surface problem, whose study is carried out by using vanish viscosity method [59, 39]. Following this approach, Feng and Prohl [38] studied (3.1.1) in case $g(s) = \sqrt{s}$. We shall follow the same approach to study (3.1.1). The generalized solution of PDE (3.1.1) is studied in two cases: $u_0, h \in L^2(\Omega)$; $u_0 \in BV(\Omega) \cap L^2(\Omega), h \in$

$L^2(\Omega)$. Our approach is to consider first the regularized problem

$$\begin{cases} \dot{u} = \nabla \cdot (g(|\nabla u|^2)\nabla u) - \lambda R^*(Ru - h) + \epsilon \Delta u & \text{in} \quad \Omega \times (0, +\infty) \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times [0, +\infty) \\ u(x,0) = u_0(x) & \text{on} \quad \Omega \end{cases} \tag{3.4.1}$$

We then derive some ϵ independent estimates and pass the limit $\epsilon \to 0$. For simplicity, let's assume $\lambda = 1$ from now on.

Lemma 3.4.1. *Suppose that g satisfies* (3.1.2) *then,*

$$\langle g(|\xi|^2)\xi - g(|\eta|^2)\eta, \xi - \eta \rangle \geq 0 \tag{3.4.2}$$

Proof. Since $g(|z|^2) = \frac{\Phi'(z)}{|z|}$ and $\Phi(\cdot)$ is an increasing convex function, by Lemma 2.7.1, we conclude that (3.4.2) holds.

3.4.1 Weak solution of regularized PDE

Before moving on, we need to clarify what weak solution means for the regularized equation.

Definition 3.4.2 (Weak Solution). A function $u : \Omega \times [0, T] \to \mathbb{R}$ is called a weak solution of the initial-boundary-value problem (3.4.1), if

(a) $u \in L^2(0, T; H^1(\Omega))$ and $\dot{u} \in L^2(0, T; H^{-1}(\Omega))$;

(b) $\forall\, v \in H^1(\Omega)$, a.e. $t \in [0, T]$,

$$\langle \dot{u}, v \rangle + \langle g(|\nabla u|^2)\nabla u, \nabla v \rangle + \langle Ru - h, Rv \rangle + \epsilon \langle \nabla u, \nabla v \rangle = 0 \tag{3.4.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the action of a distribution on a test function or the inner product of $L^2(\Omega)$.

(c) $u(0) = u_0$

Remark 3.4.3. According to Theorem 2.3.4, $u \in C([0,T];L^2(\Omega))$.

Theorem 3.4.4 (Existence and uniqueness of weak solution). *Let $u_0\,, h \in L^2(\Omega)$. Then, the initial-boundary-value problem* (3.4.1) *has a unique weak solution $u : \Omega \times [0,T] \to \mathbb{R}$, $u \in C([0,T], L^2(\Omega))$ and satisfies the following inequalities:*

$$
\begin{aligned}
&\frac{1}{2}\|u\|^2 + \int_0^t \int_\Omega \Big[g(|\nabla u|^2)|\nabla u|^2 + \frac{1}{2}(Ru-h)^2 + \epsilon|\nabla u|^2\Big]\,dx \\
&\le \frac{1}{2}\Big[\|u_0\|^2 + T\|h\|^2\Big] && (3.4.4)\\
&\int_0^t t\|\dot{u}\|^2\,dt + t\int_\Omega \Phi(|\nabla u|)\,dx + \frac{1}{2}t\|Ru-h\|^2 + \frac{\epsilon}{2}t\|\nabla u\|^2 \\
&\le \frac{1}{2}\Big[\|u_0\|^2 + T\|h\|^2\Big] + 2\beta T \quad \forall\, t \in [0,T] && (3.4.5)\\
&\int_0^T \|\dot{u}\|^2_{H^{-1}(\Omega)}\,dt \le 6\Big[\|u_0\|^2 + T\|h\|^2\Big] + 3\alpha^2 m(\Omega)T && (3.4.6)
\end{aligned}
$$

here α is the constant in (3.2.1), *$m(\Omega)$ is the Lebesgue measure of Ω. Furthermore, if $u_0 \in H^1(\Omega)$, we have*

$$
\begin{aligned}
&\int_0^T \|\dot{u}\|^2 + \int_\Omega \Phi(|\nabla u|)\,dx + \frac{1}{2}\|Ru-h\|^2 + \frac{\epsilon}{2}\|\nabla u\|^2 \\
&\le 2\Big[\int_\Omega \Phi(|\nabla u_0|)\,dx + \frac{1}{2}\|Ru_0-h\|^2 + \frac{\epsilon}{2}\|\nabla u_0\|^2\Big]
\end{aligned}
\qquad (3.4.7)
$$

Galerkin method

We use Galerkin method to prove the existence of weak solution of (3.4.1). Assume that the functions $\omega_k = \omega_k(x)$ $(k = 1, \cdots)$ are the eigenfunctions of the following problem

$$\begin{cases} -\Delta u = 0 \\ \frac{\partial u}{\partial \nu}|_\Gamma = 0 \end{cases} \tag{3.4.8}$$

We have $\{\omega_k\}_{k=1}^{\infty} \in C^\infty(\Omega)$. If Γ, the boundary of Ω is of class $C^{k,1}$, $k >= 3 + [\frac{d}{2}]$, then $\{\omega_k\}_{k=1}^{\infty} \in C^2(\bar{\Omega})$ [4]. In image processing, we often have $\Omega = \prod_{i=1}^{d}(0, L_i)$. In this case, $\{\omega_k\}_{k=1}^{\infty} \in C^\infty(\bar{\Omega})$ [5]. Furthermore,

$$\{\omega_k\}_{k=1}^{\infty} \text{ is an orthogonal basis of } H^1(\Omega) \tag{3.4.9}$$

$$\{\omega_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } L^2(\Omega) \tag{3.4.10}$$

if we normalize $\{\omega_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$. Let $V_m = \text{span}\{\omega_k\}_{k=1}^{m}$ and $\mathcal{P}_m$ is the finite dimensional projection from $L^2(\Omega)$ to V_m.

Theorem 3.4.5 (Galerkin approximation). *Let $u_0, h \in L^2(\Omega)$, for each integer $m \geq 1$, there exists a unique $u_m : \Omega \times [0, T] \to \mathbb{R}$ such that*

(a) $u_m \in C^\infty(\Omega \times [0, T])$ and $u_m(t) \in V_m$ for any $t \in [0, T]$.

[4] Please refer to [45].

[5] In this case, ω_k is the cosine sequence.

(b) Assume $h_m = \mathcal{P}_m h$, $u_{0m} = \mathcal{P}_m u_0$. $\forall v \in V_m$ and any $t \in [0,T]$,

$$\big\langle \dot{u}_m, v\big\rangle + \big\langle g(|\nabla u_m|^2)\nabla u_m, \nabla v\big\rangle + \big\langle Ru_m - h_m, Rv\big\rangle + \epsilon\big\langle \nabla u_m, \nabla v\big\rangle = 0$$

$$u_m(0) = u_{0m} \tag{3.4.11}$$

and the energy estimates

$$\frac{1}{2}\|u_m\|^2 + \int_0^T \int_\Omega \Big[g(|\nabla u_m|^2)|\nabla u_m|^2 + \frac{1}{2}(Ru_m - h_m)^2 + \epsilon|\nabla u_m|^2\Big]dx$$

$$\leq \frac{1}{2}\Big[\|u_0\|^2 + T\|h\|^2\Big] \quad \forall\, t \in [0,T] \tag{3.4.12}$$

$$\int_0^t t\|\dot{u}_m\|^2\, dt + t\int_\Omega \Phi(|\nabla u_m|)\, dx + \frac{1}{2}t\|Ru_m - h_m\|^2 + \frac{\epsilon}{2}t\|\nabla u_m\|^2$$

$$\leq \frac{1}{2}(\|u_0\|^2 + T\|h\|^2) + 2\beta T \tag{3.4.13}$$

$$\int_0^T \|\dot{u}_m\|^2_{H^{-1}(\Omega)}\, dt \leq 6(\|u_0\|^2 + T\|h\|^2) + 3\alpha^2 m(\Omega)T \tag{3.4.14}$$

where $m(\Omega)$ is the measure of Ω. Furthermore, if $u_0 \in H^1(\Omega)$, we have

$$\begin{aligned}&\int_0^T \|\dot{u}_m\|^2\, dt + \Big[\int_\Omega \Phi(|\nabla u_m|)\, dx + \frac{1}{2}\|Ru_m - h_m\|^2 + \frac{\epsilon}{2}\|\nabla u_m\|^2\Big]\\ &\leq 2\Big[\int_\Omega \Phi(|\nabla u_{0m}|)\, dx + \frac{1}{2}\|Ru_0 - h\|^2 + \frac{\epsilon}{2}\|\nabla u_0\|^2\Big]\end{aligned} \tag{3.4.15}$$

Proof of Galerkin approximation. Fix now a positive integer m. We will look for a function $u_m : [0,T] \to H^1(\Omega)$ of the form

$$u_m(t) = \sum_{k=1}^{m} a_m^k(t)\omega_k \tag{3.4.16}$$

We hope to select the coefficients $a_m^k(t)$ ($0 \leq t \leq T$, $k = 1, \cdots, m$) so that

$$a_m^k(0) = \big\langle u_0, \omega_k\big\rangle \tag{3.4.17}$$

$$\big\langle \dot{u}_m, \omega_k\big\rangle + \big\langle g(|\nabla u_m|^2)\nabla u_m, \nabla\omega_k\big\rangle + \big\langle Ru_m - h_m, R\omega_k\big\rangle + \epsilon\big\langle \nabla u_m, \nabla\omega_k\big\rangle = 0$$

We first note from the above equation in finite dimensional space that

$$\big\langle \dot{u}_m, \omega_k \big\rangle = \dot{a}_m^k(t), \quad k = 1, \cdots, m, \tag{3.4.18}$$

$$\dot{a}_m^k(t) = f_k(a_m^1(t), \cdots, a_m^m(t)), \quad k = 1, \cdots, m, \tag{3.4.19}$$

where all $f_k : \mathbb{R}^m \to \mathbb{R}$ $(1 \le k \le m)$ are smooth and locally Lipschitz. It follows from the theory for initial-value problems of ordinary differential equations that there exists $T_m > 0$ such that the initial-value problem (3.4.19) and (3.4.17), has a unique smooth solution $a_m^1(t), \cdots, a_m^m(t)$ for $t \in [0, T_m]$. For each $t \in [0, T_m]$, set $v = u_m(t)$ in (3.4.11), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \int_\Omega \Big[g(|\nabla u_m|^2)|\nabla u_m|^2 + (Ru_m - h_m)^2 + \epsilon|\nabla u_m|^2\Big]dx \\
&= \int_\Omega \Big[(Ru_m - h_m)h_m\Big]\,dx \le \frac{1}{2}\Big[\int_\Omega (Ru_m - h_m)^2\,dx + \int_\Omega h_m^2\,dx\Big] \\
&\le \frac{1}{2}(\int_\Omega (Ru_m - h_m)^2\,dx + \int_\Omega h^2\,dx)
\end{aligned} \tag{3.4.20}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \int_\Omega \Big[g(|\nabla u_m|^2)|\nabla u_m|^2 + \frac{1}{2}(Ru_m - h_m)^2 + \epsilon|\nabla u_m|^2\Big]dx \\
&\le \frac{1}{2}\|h\|^2 \quad \forall\, t \in [0, T_m]
\end{aligned} \tag{3.4.21}$$

Integrate (3.4.21) against t, we get,

$$\|u_m(t)\|^2 \le \|u_0\|^2 + T\|h\|^2 \quad \forall\, t \in [0, T_m] \tag{3.4.22}$$

This, together with the orthogonality of $\big\{\omega_k\big\}_{k=1}^m$, implies that

$$\sum_{k=1}^m [a_m^k(t)]^2 = \|u_m(t)\|^2 \le \|u_0\|^2 + T\|h\|^2$$

The solution $(a_m^1(t), \cdots, a_m^m(t))$ of the initial-value problem (3.4.19) and (3.4.17) can be uniquely extended to a smooth solution over $[0, T]$. Thus, (3.4.12) follows and

$$\begin{aligned} \|u_m\|^2_{L^\infty(0,T;L^2(\Omega))} &\le \|u_0\|^2 + T\|h\|^2 \\ \|\nabla u_m\|^2_{L^2(0,T;L^2(\Omega))} &\le \frac{\|u_0\|^2 + T\|h\|^2}{\epsilon} \end{aligned} \tag{3.4.23}$$

Set now $v = t\dot{u}_m(t)$ in (3.4.11)

$$\begin{aligned} &t\|\dot{u}_m\|^2 + \frac{d}{dt}\Big[t\int_\Omega \Phi(|\nabla u_m|)\,dx + \frac{1}{2}t\|Ru_m - h_m\|^2 + \frac{1}{2}\epsilon t\|\nabla u_m\|^2\Big] \\ &= \Big[\int_\Omega \Phi(|\nabla u_m|)\,dx + \frac{1}{2}\|Ru_m - h_m\|^2 + \frac{1}{2}\epsilon\|\nabla u_m\|^2\Big] \end{aligned} \tag{3.4.24}$$

Integrate (3.4.24) against t from 0 to s, and recall (3.2.2)

$$\begin{aligned} &\int_0^s t\|\dot{u}_m\|^2\,dt + \Big[s\int_\Omega \Phi(|\nabla u_m|)\,dx + \frac{1}{2}s\|Ru_m - h_m\|^2 + \frac{1}{2}\epsilon s\|\nabla u_m\|^2\Big] \\ &= \int_0^s \Big[\int_\Omega \Phi(|\nabla u_m|)\,dx + \frac{1}{2}\|Ru_m - h_m\|^2 + \frac{1}{2}\epsilon\|\nabla u_m\|^2\Big]dt \\ &\le \int_0^s \Big[\alpha\int_\Omega |\nabla u_m|\,dx + \frac{1}{2}\|Ru_m - h_m\|^2 + \frac{1}{2}\epsilon\|\nabla u_m\|^2\Big]dt + \beta T \end{aligned} \tag{3.4.25}$$

On the other hand, from (3.2.2) and (3.4.12), we know that

$$\begin{aligned} &\alpha\int_0^s\int_\Omega |\nabla u_m|\,dx + \int_0^s \big[\frac{1}{2}\|Ru_m - h_m\|^2 + \epsilon\|\nabla u_m\|^2\big]dt \\ &\le \frac{1}{2}\Big[\|u_0\|^2 + T\|h\|^2\Big] + \beta T \end{aligned} \tag{3.4.26}$$

Hence,

$$\begin{aligned} &\int_0^s t\|\dot{u}_m\|^2\,dt + \Big[s\int_\Omega \Phi(|\nabla u_m|)\,dx + \frac{1}{2}s\|Ru_m - h_m\|^2 + \frac{1}{2}\epsilon s\|\nabla u_m\|^2\Big] \\ &\le \frac{1}{2}\Big[\|u_0\|^2 + T\|h\|^2\Big] + 2\beta T \quad \forall\, s \in [0, T] \end{aligned} \tag{3.4.27}$$

Recall that

$$\|\dot{u}_m\|_{H^{-1}(\Omega)} = \sup\Big\{ < \dot{u}_m, v >: \|v\|_{H_0^1(\Omega)} \le 1\Big\} \tag{3.4.28}$$

It is easy to see that (3.4.11) is valid for all $v \in H_0^1(\Omega)$, then

$$\|\dot{u}_m\|_{H^{-1}(\Omega)} \leq \|g(|\nabla u_m|^2)\nabla u_m\| + \|Ru_m - h_m\| + \epsilon\|\nabla u_m\| \tag{3.4.29}$$

Take square of the above inequality and integrate t from 0 to T, we obtain

$$\begin{aligned}
&\int_0^T \|\dot{u}_m\|^2_{H^{-1}(\Omega)}\, dt \\
&\leq 3\int_0^T \Big[\|g(|\nabla u_m|^2)\nabla u_m\|^2 + \|Ru_m - h_m\|^2 + \epsilon\|\nabla u_m\|^2\Big] dt \\
&\leq 3\alpha^2 m(\Omega)T + 6(\|u_0\|^2 + T\|h\|^2)
\end{aligned} \tag{3.4.30}$$

If $u_0 \in H^1(\Omega)$, set $v = \dot{u}_m(t)$ in (3.4.11) to get (3.4.15). Therefore,

$$\begin{aligned}
&\int_0^T \|\dot{u}_m\|^2 dt + \Big[\int_\Omega \Phi(|\nabla u_m|)\, dx + \frac{1}{2}\|Ru_m - h_m\|^2 + \frac{\epsilon}{2}\|\nabla u_m\|^2\Big] \\
&\leq 2\Big[\int_\Omega \Phi(|\nabla u_{0m}|)\, dx + \frac{1}{2}\|Ru_{0m} - h_m\|^2 + \frac{\epsilon}{2}\|\nabla u_{0m}\|^2\Big] \\
&\leq 2\Big[\int_\Omega \Phi(|\nabla u_{0m}|)\, dx + \frac{1}{2}\|Ru_0 - h\|^2 + \frac{\epsilon}{2}\|\nabla u_0\|^2\Big]
\end{aligned} \tag{3.4.31}$$

Proof of theorem 3.4.4. According to energy estimates (3.4.14), (3.4.13) and (3.4.23), for any fixed $\epsilon > 0$, the sequence $\{u_m\}_{m=1}^\infty$ is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, $\{\dot{u}_m\}_{m=1}^\infty$ is bounded in $L^2(0,T;H^{-1}(\Omega))$ and $\{\sqrt{t}\dot{u}_m\}_{m=1}^\infty$ is bounded in $L^2(0,T;L^2(\Omega))$. Consequently, there exists a subsequence $\{u_{m_l}\}_{l=1}^\infty \subset \{u_m\}_{m=1}^\infty$ and a function $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, $\dot{u} \in L^2(0,T;H^{-1}(\Omega))$

and $\sqrt{t}\dot{u} \in L^2(0,T;L^2(\Omega))$ such that

$$\begin{aligned}
u_{m_l} &\rightharpoonup u && \text{weakly in } L^2(0,T;H^1(\Omega)) \\
\dot{u}_{m_l} &\rightharpoonup \dot{u} && \text{weakly in } L^2(0,T;H^{-1}(\Omega)) \\
u_{m_l} &\to u && \text{strongly in } L^2(0,T;L^2(\Omega)) \\
\sqrt{t}\dot{u}_{m_l} &\rightharpoonup \sqrt{t}\dot{u} && \text{weakly in } L^2(0,T;L^2(\Omega))
\end{aligned} \tag{3.4.32}$$

the strong convergence is due to $H^1(\Omega) \subset\subset L^2(\Omega)$ and a compactness result [83] (see also Theorem 2.4.2). Let $v \in H^1(\Omega)$ and $\eta \in C[0,T]$. For each $m \geq 1$, set $v = v_m(= \mathcal{P}_m v)$ in (3.4.11), multiply both sides of the identity by $\eta(t)$, and integrate against t to yield

$$\begin{aligned}
&\int_0^T \langle \eta(t)v_m, \dot{u}_m(t)\rangle\, dt + \int_0^T \langle \eta(t)\nabla v_m, g(|\nabla u_m(t)|^2)\nabla u_m(t)\rangle\, dt \\
&+ \int_0^T \langle Ru_m(t) - h_m, Rv_m\rangle\, dt + \epsilon \int_0^T \langle \eta(t)\nabla v_m, \nabla u_m(t)\rangle\, dt = 0
\end{aligned} \tag{3.4.33}$$

We still denote the subscript m_l of convergence sequence as m,

$$\begin{aligned}
&\int_0^T \langle \eta(t)v_m, \dot{u}_m(t)\rangle\, dt - \int_0^T \langle \eta(t)v, \dot{u}(t)\rangle\, dt \\
&= \int_0^T \langle \eta(t)(v_m - v), \dot{u}_m(t)\rangle dt + \int_0^T \langle \eta(t)v, \dot{u}_m(t) - \dot{u}(t)\rangle\, dt
\end{aligned} \tag{3.4.34}$$

The first term

$$\begin{aligned}
&\left|\int_0^T \langle \eta(t)(v_m - v), \dot{u}_m(t)\rangle dt\right| \leq \|\eta\|\|v_m - v\|_{H^1(\Omega)} \int_0^T \|\dot{u}_m\|_{H^{-1}(\Omega)}\, dt \\
&\leq \|\eta\|\|v_m - v\|_{H^1(\Omega)} T \int_0^T \|\dot{u}_m\|^2_{H^{-1}(\Omega)}\, dt \to 0 \text{ as } m \to \infty
\end{aligned} \tag{3.4.35}$$

It follows from the weak convergence of $\dot{u}_m$ in $L^2(0,T;H^{-1}(\Omega))$, the second term $\to 0$ as $m \to \infty$. Therefore

$$\int_0^T \langle \eta(t)v_m, \dot{u}_m(t)\rangle\, dt \to \int_0^T \langle \eta(t)v, \dot{u}(t)\rangle\, dt \text{ as } m \to \infty \tag{3.4.36}$$

From the strong convergences $u_m \to u$, $\mathcal{P}_m v \to v$, $u_{0m} = \mathcal{P}_m u_0 \to u_0$, $h_m = \mathcal{P}_m h \to h$, we obtain

$$\int_0^T \big\langle \eta(t) R u_m - h_m, R v_m > \, dt \to \int_0^T \big\langle \eta(t) R u - h, R v > \, dt \tag{3.4.37}$$

as $m \to \infty$. It follows from $u_m \rightharpoonup u$ weakly in $H^1(\Omega)$ and $\mathcal{P}_m v \to v$ strongly in $H^1(\Omega)$,

$$\int_0^T \big\langle \eta(t) \nabla u_m, \nabla v_m \big\rangle \, dt \to \int_0^T \big\langle \eta(t) \nabla u, \nabla v \big\rangle \, dt \text{ as } m \to \infty \tag{3.4.38}$$

Finally, let's consider the nonlinear term. $g(|\nabla u_m|^2)\nabla u_m$ is bounded $L^2(0,T;L^2(\Omega)^d)$, there exists some $\xi \in L^2(0,T;L^2(\Omega)^d)$, such that $g(|\nabla u_m|^2)\nabla u_m \rightharpoonup \xi$. Therefore, as $m \to \infty$

$$\int_0^T \big\langle \eta(t) \nabla v_m, g(|\nabla u_m|^2 \nabla u_m \big\rangle \, dt \to \int_0^T \big\langle \eta(t) \nabla v, \xi \big\rangle \, dt \tag{3.4.39}$$

Let $m \to \infty$, we get from (3.4.34), (3.4.37), (3.4.38) and (3.4.39) that

$$\int_0^T \eta(t) \Big[\big\langle \dot{u}, v \big\rangle + \big\langle \xi, \nabla v \big\rangle + \big\langle Ru - h, Rv > + \epsilon \big\langle \nabla u, \nabla v > \Big] dt = 0 \tag{3.4.40}$$

Since $\eta(t) \in C[0,T]$ is arbitrary, this implies:

$$\big\langle \dot{u}, v \big\rangle + \big\langle \xi, \nabla v \big\rangle + \big\langle Ru - h, Rv > + \epsilon \big\langle \nabla u, \nabla v >= 0 \quad a.e.\ t \in [0,T] \tag{3.4.41}$$

Notice that, by Theorem 2.3.4, after possibly being redefined on a set of measure zero, we have $u \in C([0,T];L^2(\Omega))$. Moreover, $u(t) = u(s) + \int_s^t \dot{u}(\tau)\, d\tau$ for any $s,t \in [0,T]$. Replace $\eta(t)$ in (3.4.40) by $\eta_T(t) = 1 - \frac{t}{T}$ and integrate by parts against t for the first term to get

$$\begin{aligned}&\int_0^T \eta_T(t)\Big[\big\langle \xi, \nabla v\big\rangle + \big\langle Ru - h, Rv\big\rangle + \epsilon\big\langle \nabla u, \nabla v\big\rangle\Big]\,dt\\ &+\int_0^T \frac{1}{T}\big\langle u(t), v\big\rangle\,dt = \big\langle u(0), v\big\rangle\end{aligned} \tag{3.4.42}$$

Repeat the same argument using (3.4.11) with $v_m = \mathcal{P}_m v$ to get

$$\begin{aligned}&\int_0^T \eta_T(t)\Big[\big\langle g(|\nabla u_m|^2)\nabla u_m, \nabla v_m\big\rangle + \big\langle Ru_m - h_m, Rv_m\big\rangle + \epsilon\big\langle \nabla u, \nabla v_m\big\rangle\Big]\,dt\\ &+\int_0^T \frac{1}{T}\big\langle u_m(t), v_m\big\rangle\,dt = \big\langle u_{0m}, v_m\big\rangle\end{aligned}$$

Let $m \to \infty$, we deduce from (3.4.34), (3.4.37), (3.4.38) and (3.4.39),

$$\begin{aligned}&\int_0^T \eta_T(t)\Big[\big\langle \xi, \nabla v\big\rangle + \big\langle Ru - h, Rv\big\rangle + \epsilon\big\langle \nabla u, \nabla v\big\rangle\Big]\,dt\\ &+\int_0^T \frac{1}{T}\big\langle u(t), v\big\rangle\,dt = \big\langle u_0, v\big\rangle\end{aligned} \tag{3.4.43}$$

Now, a comparison of (3.4.42) and (3.4.43), together with the arbitrariness of $v \in H^1(\Omega)$, we get $u(0) = u_0$. Similarly, let $\eta(t) = -\frac{t}{T}$, we deduce

$$\lim_{m\to\infty} \big\langle u_m(T), v_m\big\rangle = \big\langle u(T), v\big\rangle \tag{3.4.44}$$

On the other hand,

$$\Big|\big\langle u_m(T), v_m - v\big\rangle\Big| \le \|u_m(T)\|\|v_m - v\| \tag{3.4.45}$$

From (3.4.23), we know that $\lim_{m\to\infty} \big\langle u_m(T), v_m - v\big\rangle = 0$. Consequently,

$$\lim_{m\to\infty} \big\langle u_m(T), v\big\rangle = \lim_{m\to\infty} \big\langle u_m(T), v_m\big\rangle = \big\langle u(T), v\big\rangle \quad \forall\, v \in H^1(\Omega) \tag{3.4.46}$$

Let $v = u$ in (3.4.41) and integrate against t from 0 to T, we get

$$\begin{aligned}&\int_0^T \big\langle \xi, \nabla u\big\rangle\,dt = \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|u(T)\|^2\\ &-\int_0^T \big\langle Ru - h, Ru\big\rangle\,dt - \int_0^T \epsilon\big\langle \nabla u, \nabla u\big\rangle\,dt\end{aligned} \tag{3.4.47}$$

Let $v_m = u_m$ in (3.4.11) and integrate against t from 0 to T, we get

$$\begin{aligned}\int_0^T \big\langle g(|\nabla u_m|^2)\nabla u_m, \nabla u_m\big\rangle\, dt &= \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|u_m(T)\|^2 \\ - \int_0^T \epsilon\big\langle \nabla u_m, \nabla u_m\big\rangle dt &- \int_0^T \big\langle Ru_m - h_m, Ru_m\big\rangle\, dt\end{aligned} \tag{3.4.48}$$

Therefore

$$\begin{aligned}&\lim_{m\to\infty}\sup \int_0^T \big\langle g(|\nabla u_m|^2)\nabla u_m, \nabla u_m\big\rangle\, dt \\ &= \frac{1}{2}\|u_0\|^2 - \lim_{m\to\infty}\inf \frac{1}{2}\|u_m(T)\|^2 \\ &- \lim_{m\to\infty}\inf \int_0^T \Big[\big\langle Ru_m - h_m, Ru_m\big\rangle + \epsilon\big\langle\nabla u_m, \nabla u_m\big\rangle\Big]\, dt\end{aligned} \tag{3.4.49}$$

Notice that L^2 norm is lower semicontinuous [6], from (3.4.47) and (3.4.49), we deduce

$$\lim_{m\to\infty}\sup \int_0^T \big\langle g(|\nabla u_m|^2)\nabla u_m, \nabla u_m\big\rangle\, dt \le \int_0^T \big\langle \xi, \nabla u\big\rangle\, dt \tag{3.4.50}$$

Now, from Lemma 3.4.1, we obtain, $\forall\, v \in L^2(0,T;H^1(\Omega))$

$$\begin{aligned}0 &\le \lim_{m\to\infty}\sup \int_0^T \big\langle g(|\nabla u_m|^2)\nabla u_m - g(|\nabla v|^2)\nabla v, \nabla u_m - \nabla v\big\rangle\, dt \\ &\le \int_0^T \big\langle \xi - g(|\nabla v|^2)\nabla v, \nabla u - \nabla v\big\rangle\, dt \quad \forall\, v \in L^2(0,T;H^1(\Omega))\end{aligned} \tag{3.4.51}$$

Let $v = u - \theta w$ for some constant $\theta > 0$,

$$\int_0^T \big\langle \xi - g(|\nabla u - \theta\nabla w|^2)\nabla u - \theta\nabla w, \nabla w\big\rangle\, dt \ge 0 \tag{3.4.52}$$

Let $\theta \to 0$, we deduce

$$\int_0^T \big\langle \xi - g(|\nabla u|^2)\nabla u, \nabla w\big\rangle\, dt \ge 0 \quad \forall\, w \in L^2(0,T;H^1(\Omega)) \tag{3.4.53}$$

[6] L^2 norm is lower semicontinuous with respect to strong convergence, from Theorem in Section 2.5, L^2 norm is lower semicontinuous with respect to weak convergence.

i.e. $\langle \xi, \nabla v \rangle = \langle g(|\nabla u|^2)\nabla u, \nabla v \rangle$ for any $v \in H^1(\Omega)$ and $a.e.$ $t \in [0,T]$. Therefore

$$\langle \dot{u}, v \rangle + \langle g(|\nabla u|^2)\nabla u, \nabla v \rangle + \langle Ru - h, Rv \rangle + \epsilon \langle \nabla u, \nabla v \rangle = 0 \tag{3.4.54}$$

$\forall\, v \in H^1(\Omega)$, $a.e.$ $t \in [0,T]$. All the inequalities follow directly from the corresponding ones in Theorem 3.4.5.

Uniqueness

Suppose that u_1 and u_2 are the weak solutions of PDE (3.4.1) with initial values u_{10}, h_1 and u_{20}, h_2 respectively. Then,

$$\langle \dot{u_1}, v \rangle + \langle g(|\nabla u_1|^2)\nabla u_1, \nabla v \rangle + \langle Ru_1 - h_1, Rv \rangle + \epsilon \langle \nabla u_1, \nabla v \rangle = 0 \tag{3.4.55}$$

$$\langle \dot{u_2}, v \rangle + \langle g(|\nabla u_2|^2)\nabla u_2, \nabla v \rangle + \langle Ru_2 - h_2, Rv \rangle + \epsilon \langle \nabla u_2, \nabla v \rangle = 0 \tag{3.4.56}$$

(3.4.55) - (3.4.56) and let $w = u_1 - u_2$, $v = u_1 - u_2$ and $w_0 = u_{10} - u_{20}$,

$$\begin{aligned} &\langle \dot{w}, w \rangle + \langle g(|\nabla u_1|^2)\nabla u_1 - g(|\nabla u_2|^2)\nabla u_2, \nabla u_1 - \nabla u_2 \rangle \\ &+ \langle Rw - (h_1 - h_2), Rw \rangle + \epsilon \langle \nabla w, \nabla w \rangle = 0 \end{aligned} \tag{3.4.57}$$

Since $\langle g(|\nabla u_1|^2)\nabla u_1 - g(|\nabla u_2|^2)\nabla u_2, \nabla u_1 - \nabla u_2 \rangle \geq 0$, we get

$$\|u_1(t) - u_2(t)\|^2 \leq \|u_{10} - u_{20}\|^2 + t\|h_1 - h_2\|^2 \ \text{ a.e. } [0,T] \tag{3.4.58}$$

On the other hand, after possibly being redefined on a set of measure zero, $u_1, u_2 \in C([0,T], L^2(\Omega))$. Thus

$$\|u_1(t) - u_2(t)\|^2 \leq \|u_{10} - u_{20}\|^2 + t\|h_1 - h_2\|^2 \ \forall\, t \in [0,T] \tag{3.4.59}$$

This inequality ensures the uniqueness of the solution.

3.4.2 Existence and uniqueness of generalized solution

We have proved that the existence and uniqueness of the weak solution of regularized PDE and derived some ϵ independent energy estimates. Now let's study the properties of original PDE. We have the following theorem.

Theorem 3.4.6 (Generalized Solution). *Let Ω be a bounded open domain with Lipschitz boundary.*

(a) Suppose that $u_0, h \in L^2(\Omega)$, then there exists a function u such that

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;BV(\Omega))$$

$$u \in L^\infty(s_0,T;BV(\Omega)) \cap C([s_0,T];L^2(\Omega)),\ s_0 \in (0,T]$$

$$\dot{u} \in L^2(0,T;H^{-1}(\Omega))$$

$u(t)$ is weakly continuous from $[0,T] \to L^2(\Omega)$.

$\forall\, s \in (0,T]$, $\forall\, v \in L^1(0,T;BV(\Omega)) \cap L^2(0,T;L^2(\Omega)) \cap C([0,T];L^2(\Omega))$ such that $\dot{v} \in L^2(0,T;L^2(\Omega))$, we have

$$\int_0^s \int_\Omega \dot{v}(v-u)\,dx\,dt + \int_0^s [\hat{J}_R(v) - \hat{J}_R(u)]dt \geq \frac{1}{2}\left[\|v(s)-u(s)\|^2 - \|v(0)-u_0\|^2\right] \tag{3.4.60}$$

(b) Suppose u_1 and u_2 are two functions which satisfy (3.4.60) *with initial data u_{10}, h_1 and u_{20}, h_2 respectively. If $u_{10}, u_{20} \in L^2(\Omega) \cap BV(\Omega)$, $h_1, h_2 \in L^2(\Omega)$. Then, there holds stability inequality*

$$\|u_1(s) - u_2(s)\|^2 \leq \|u_{10} - u_{20}\|^2 + s\|h_1 - h_2\|^2 \quad \forall\, s \in [0,T] \tag{3.4.61}$$

(c) If $u_0 \in BV(\Omega) \cap L^2(\Omega)$ *and* $h \in L^2(\Omega)$*, then* u *is unique,* $u(0) = u_0$ *and*

$$u \in L^\infty(0,T; BV(\Omega) \cap L^2(\Omega)) \cap C([0,T], L^2(\Omega))$$

$$\dot{u} \in L^2(0,T; L^2(\Omega))$$

$\forall\, s \in [0,T]$, $\forall\, v \in L^1(0,T; BV(\Omega)) \cap L^2(0,T; L^2(\Omega))$, *we have*

$$\int_0^s \int_\Omega \dot{u}(v-u)\, dx\, dt + \int_0^s \left[\hat{J}_R(v) - \hat{J}_R(u)\right] dt \geq 0 \tag{3.4.62}$$

Remark 3.4.7. (a) In case of $u_0 \in L^2(\Omega)$, the solution $u(t)$ is only weakly continuous from $[0,T] \to L^2(\Omega)$. The strong continuity is usually not true. The uniqueness of the solution is not proved either. In the literature, there are some mistakes regarding the proof of continuity and uniqueness of u when $u_0 \in L^2(\Omega)$. By looking at the proof of stability inequality in case $u_0 \in BV(\Omega) \cap L^2(\Omega)$, it is tempting to use a density argument to do it: suppose that $u_0^n \in BV(\Omega) \cap L^2(\Omega) \to u_0 \in L^2(\Omega)$, u^n is the generalized solution corresponding to u_0^n, but it turns out that we don't know if $u_n \to u$ in any sense.

(b) From (3.4.62), it is easy to see that, for a.e. $t \in [0,T]$

$$\int_\Omega \dot{u}(v-u)\, dx + \hat{J}_R(v) - \hat{J}_R(u) \geq 0 \quad \forall\, v \in BV(\Omega) \cap L^2(\Omega) \tag{3.4.63}$$

Proof. The proof is carried out by using the same approach as Lichnewski and Temam [59], Gerhardt [39], Feng [38].

Part (a)

For each $\epsilon > 0$, consider the regularized problem (3.4.1), from theorem 3.4.4, we know, there exists a weak solution u^ϵ which satisfies the following ϵ independent bounds:

$$
\begin{aligned}
&\|u^\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\epsilon}\|\nabla u^\epsilon\| + \|u^\epsilon\|_{L^1(0,T;W^{1,1}(\Omega))} \le C_0(T, \|u_0\|, \|h\|) \\
&\|\dot{u}^\epsilon\|_{L^2(0,T;H^{-1}(\Omega))} \le C_1(T, \|u_0\|, \|h\|) \\
&\|\sqrt{t}\dot{u}^\epsilon\|_{L^2(0,T;L^2(\Omega))} + \sqrt{\epsilon}\|\sqrt{t}\nabla u^\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|tu^\epsilon\|_{L^\infty(0,T;W^{1,1}(\Omega))} \\
&\le C_2(T, \|u_0\|, \|h\|)
\end{aligned}
\tag{3.4.64}
$$

here C_0, C_1, C_2 are constants. These bounds imply that there exists a function $u \in L^\infty(0,T;L^2(\Omega))$, $tu \in L^\infty(0,T;BV(\Omega))$ and a subsequence $\{u^\epsilon\}_{\epsilon>0}$ (which is denoted by the same notation) such that as $\epsilon \to 0$

$$u^\epsilon \rightharpoonup u \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega))$$

$$u^\epsilon \rightharpoonup u \quad \text{weakly in } L^2(0,T;L^2(\Omega)) \tag{3.4.65}$$

$$u^\epsilon \to u \quad \text{strongly in } L^1(0,T;L^p(\Omega))$$

$$\dot{u}^\epsilon \rightharpoonup \dot{u} \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)) \tag{3.4.66}$$

$$\dot{u}^\epsilon \rightharpoonup \dot{u} \quad \text{weakly in } L^2(t_0,T;L^2(\Omega))\ \forall\, t_0 \in (0,T]$$

$$\sqrt{t}\dot{u}^\epsilon \rightharpoonup \sqrt{t}\dot{u} \quad \text{weakly in } L^2(0,T;L^2(\Omega)) \tag{3.4.67}$$

$$tu^\epsilon \to tu \quad \text{strongly in } L^p(\Omega) \text{ for } a.e.\, t \in [0,T] \tag{3.4.68}$$

$$u^\epsilon \to u \quad \text{strongly in } L^p(\Omega) \text{ for } a.e.\, t \in [t_0,T] \quad \forall\, t_0 \in (0,T]$$

where $1 \leq p < 1^*$. The strong convergence is due to the fact that $BV(\Omega)$ compactly embedded in $L^p(\Omega)$(cf. Lemma 2.6.14) and the compactness result of Simon [83] (See also Theorem 2.4.2). Since $u \in L^\infty(0, T, L^2(\Omega))$, u is continuous from $[0, T]$ into $H^{-1}(\Omega)$, by Lemma 2.3.5, we know that $u(t)$ is weakly continuous from $[0, T] \to L^2(\Omega)$. Since $L^1(0, T; BV(\Omega))$ is neither reflexive nor the dual of some separable Banach space, we can't directly conclude that $u \in L^1(0, T, BV(\Omega))$. Thanks to Fatou's lemma, we have

$$\liminf_{\epsilon \to 0} \int_0^T \Big[\int_\Omega |\nabla u^\epsilon| \, dx \Big] dt \geq \int_0^T \Big[\liminf_{\epsilon \to 0} \int_\Omega |\nabla u^\epsilon| \, dx \Big] dt \tag{3.4.69}$$

From $u^\epsilon \to u$ strongly in $L^1(0, T; L^p(\Omega))$, we have $u^\epsilon \to u$ strongly in $L^p(\Omega)$ a.e. $t \in [0, T]$. On the other hand, the variation of a function is lower semicontinuous with respect to L^1 convergence. Thus, we conclude

$$\int_0^T \Big[\liminf_{\epsilon \to 0} \int_\Omega |\nabla u^\epsilon| \, dx \Big] dt \geq \int_0^T \Big[\int_\Omega |Du| \, dx \Big] dt \tag{3.4.70}$$

i.e. $u \in L^1(0, T; BV(\Omega))$. Since $\forall \, s_0 > 0$, $\dot{u} \in L^2(s_0, T; L^2(\Omega))$, by Theorem 2.3.3, after possibly being redefined on a set of measure zero, we have $u \in C([s_0, T], L^2(\Omega))$, i.e. $u \in C((0, T], L^2(\Omega))$. $\forall \, s_1, s_2 > 0$,

$$u(s_2) = u(s_1) + \int_{s_1}^{s_2} \dot{u} \, dt, \qquad u^\epsilon(s_2) = u^\epsilon(s_1) + \int_{s_1}^{s_2} \dot{u}^\epsilon \, dt$$

From (3.4.65) and (3.4.66), we obtain

$$u^\epsilon(s) \rightharpoonup u(s) \quad \text{weakly in } L^2(\Omega) \; \forall \, s \in (0, T] \tag{3.4.71}$$

Substitute v by $v - u^\epsilon$ in (3.4.3) and integrate against t from 0 to $s(\le T)$, we obtain

$$\begin{aligned}
&\int_0^s \int_\Omega \dot{u}^\epsilon (v - u^\epsilon)\, dxdt + \int_0^s \int_\Omega g(|\nabla u^\epsilon|^2)\nabla u^\epsilon \cdot (\nabla v - \nabla u^\epsilon)\, dxdt \\
&+ \int_0^s \int_\Omega (Ru^\epsilon - h)(Rv - Ru^\epsilon)\, dxdt \\
&+ \epsilon \int_0^s \int_\Omega \nabla u^\epsilon \cdot (\nabla v - \nabla u^\epsilon)\, dxdt = 0
\end{aligned} \tag{3.4.72}$$

$\Phi(s)$ is a convex function and recall that $\Phi(s) = \int_0^s g(\tau^2)\tau\, d\tau$, thus,

$$\Phi(|\nabla v|) - \Phi(|\nabla u^\epsilon|) \ge g(|\nabla u^\epsilon|^2)\nabla u^\epsilon \cdot (\nabla v - \nabla u^\epsilon) \tag{3.4.73}$$

and we also have the following

$$\begin{aligned}
&\int_0^s \int_\Omega \dot{u}^\epsilon (v - u^\epsilon)\, dx\, dt = \int_0^s \int_\Omega \dot{v}(v - u^\epsilon)\, dxdt \\
&- \frac{1}{2}\big[\|v(s) - u^\epsilon(s)\|^2 - \|v(0) - u_0\|^2\big]
\end{aligned} \tag{3.4.74}$$

$$\begin{aligned}
&\frac{1}{2}\Big[\int_0^s \int_\Omega (Rv - h)^2\, dx\, dt - \int_0^s \int_\Omega (Ru^\epsilon - h)^2\, dx\, dt\Big] \\
&\ge \int_0^s \int_\Omega (Ru^\epsilon - h)(Rv - Ru^\epsilon)\, dxdt
\end{aligned} \tag{3.4.75}$$

Hence

$$\begin{aligned}
&\int_0^s \int_\Omega \dot{v}(v - u^\epsilon)\, dx\, dt + \int_0^s (\hat{J}_R(v) - \hat{J}_R(u^\epsilon))\, dt \\
&+ \epsilon \int_0^s \int_\Omega \nabla u^\epsilon \cdot (\nabla v - \nabla u^\epsilon)\, dx\, dt \\
&\ge \frac{1}{2}\big[\|v(s) - u^\epsilon(s)\|^2 - \|v(0) - u_0\|^2\big]
\end{aligned} \tag{3.4.76}$$

which holds $\forall\, v \in L^1(0, T; H^1(\Omega)) \cap L^2(0, T; L^2(\Omega))$ such that $\dot{v} \in L^2(0, T; L^2(\Omega))$.

By Fatou's Lemma and the lower semicontinuity of $\hat{J}(\cdot)$ with respect to L^1 norm,

$$\liminf_{\epsilon \to 0} \int_0^s \hat{J}(u^\epsilon)\, dt \ge \int_0^s \liminf_{\epsilon \to 0} \hat{J}(u^\epsilon)\, dt \ge \int_0^s \hat{J}(u)\, dt \tag{3.4.77}$$

It is not hard to verify that L^2 norm is lower semicontinuous with respect to strong convergence. Notice that L^2 norm is convex, from [8] Theorem 2.1.2, we conclude that it is lower semicontinuous with respect to weak convergence. Thus,

$$\begin{aligned}
&\liminf_{\epsilon\to 0} \|v(s) - u^\epsilon(s)\|^2 \geq \|v(s) - u(s)\|^2 \quad \forall\, s \in (0, T] \\
&\liminf_{\epsilon\to 0} \int_0^s \|Ru^\epsilon - h\|^2 \geq \int_0^s \|Ru - h\|^2
\end{aligned} \tag{3.4.78}$$

Now let $\epsilon \to 0$ and notice that $\sqrt{\epsilon}\|\nabla u^\epsilon\|$ is bounded, we have, for any $s \in [0, T]$

$$\begin{aligned}
&\int_0^s \int_\Omega \dot{v}(v - u)\, dx\, dt + \int_0^s [\hat{J}_R(v) - \hat{J}_R(u)]dt \\
&\geq \frac{1}{2}\big[\|v(s) - u(s)\|^2 - \|v(0) - u_0\|^2\big]
\end{aligned} \tag{3.4.79}$$

$\forall\, v \in L^1(0, T; H^1(\Omega)) \cap L^2(0, T; L^2(\Omega))$ such that $\dot{v} \in L^2(0, T; L^2(\Omega))$. However, for each $v \in BV(\Omega)$, there exists (cf.[27], see also Section 2.6.6) a sequence $\{v_n\}_{n\geq 1} \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that $v_n \to v$ strongly in $L^2(\Omega)$ and $\hat{J}(v) = \lim_{n\to\infty} \hat{J}(v_n)$. Thus, (3.4.60) holds [7].

Stability inequality

To this purpose, let's prove the following lemma which is using the techniques in [39, 59].

[7] The proof of the density result of [27] is based on mollification, for the time dependent function, the space variable mollification will make sure the time derivative of the sequence converge strongly.

Lemma 3.4.8. *Let $\eta > 0$ and u_η be the solution of the following ODE:*

$$\begin{cases} \eta \dot{u}_\eta + u_\eta &= u \quad \text{for } 0 < t < T; \\ u_\eta(0) &= u_0 \end{cases} \tag{3.4.80}$$

If u satisfies (3.4.60) *and $u_0 \in BV(\Omega) \cap L^2(\Omega)$, then as $\eta \to 0$*

$$\begin{aligned} &u_\eta \to u \quad \text{strongly in } L^2(0,T; L^2(\Omega)) \\ &u_\eta \to u \quad \text{strongly in } L^1(0,T; BV(\Omega)) \\ &u_\eta(s) \to u(s) \quad \text{strongly in } L^2(\Omega)\ \forall\, s \in [0,T] \end{aligned} \tag{3.4.81}$$

Furthermore,

$$\|\dot{u}\|^2_{L^2(0,T;L^2(\Omega))} \leq \hat{J}_R(u_0) \tag{3.4.82}$$

Proof of lemma. It is easy to see that for any $t > 0$

$$u_\eta(t) = e^{-t/\eta} u_0 + 1/\eta \int_0^t e^{(\tau - t)/\eta} u(\tau)\, d\tau = e^{-t/\eta} u_0 + (u * \rho_\eta)(t) \tag{3.4.83}$$

where the definition of u has been extended by setting $u = 0$ for $t < 0$ and $\rho_\eta(t) = (1/\eta)\rho(t/\eta)$, $\rho(t) = e^{-t}$. It is checked in a standard way that if $u \in L^q(0,T;X)$ (where $1 \leq q \leq +\infty$ and X is a Banach space), then $u * \rho_\eta \to u$ in $L^q(0,T;X)$ as $\eta \to 0$. On the other hand, if $u_0 \in X$, $u_0 e^{-t/\eta} \to 0$ in $L^q(0,T;X)$ as $\eta \to 0$. Therefore, (3.4.81) hold. Now, let's take $v = u_\eta$ in (3.4.60), we get

$$\frac{1}{2}\|u_\eta(s) - u(s)\|^2 + \eta \int_0^s \int_\Omega |\dot{u}_\eta|^2\, dx\, dt \leq \int_0^s (\hat{J}_R(u_\eta) - \hat{J}_R(u))\, dt \tag{3.4.84}$$

We write u_η as a convex combination

$$u_\eta = e^{-t/\eta} u_0 + (1 - e^{-t/\eta}) \frac{1}{\eta(1 - e^{-t/\eta})} \int_0^t e^{(\tau - t)/\eta} u(\tau)\, d\tau$$

From the convexity of $\hat{J}_R(u)$ and Jensen's inequality, we obtain

$$\hat{J}_R(u_\eta) \le e^{-t/\eta}\hat{J}_R(u_0) + 1/\eta \int_0^t e^{(\tau - t)/\eta}\hat{J}_R(u(\tau))\,d\tau$$

Thus, we get from (3.4.84)

$$\begin{aligned}
&\eta \int_0^T \int_\Omega |\dot{u}_\eta|^2\,dx\,dt + \int_0^T \hat{J}_R(u)\,dt \\
&\le \hat{J}_R(u_0)\int_0^T e^{-t/\eta}dt + 1/\eta \int_0^T \int_0^t e^{(\tau - t)/\eta}\hat{J}_R(u(\tau))\,d\tau\,dt \\
&\le \eta(1 - e^{-T/\eta})\hat{J}_R(u_0) + \int_0^T \hat{J}_R(u)\,dt
\end{aligned}$$

Thus $\|\dot{u}_\eta\|^2_{L^2(0,T;L^2(\Omega))} \le \hat{J}_R(u_0)$. $\|\dot{u}_\eta\|_{L^2(0,T;L^2(\Omega))}$ is bounded and $u_\eta \to u$ in $L^2(0,T;L^2(\Omega))$, upon take a subsequence, $\dot{u}_\eta \rightharpoonup \dot{u}$. Consequently, (3.4.82) holds.

Now let's prove the stability inequality (3.4.61). Let u_1 and u_2 be two functions which satisfy (3.4.60) with initial data u_{10}, h_1 and u_{20}, h_2 respectively. Notice that $u_{10}, u_{20} \in L^2(\Omega) \cap BV(\Omega)$. Set

$$u := \frac{u_1 + u_2}{2}, \quad u_0 := \frac{u_{10} + u_{20}}{2}$$

For any $\eta > 0$, define u_η as in Lemma 3.4.8, now take $v = u_\eta$ in each inequality (3.4.60) with u_1, u_2 in place of u, u_{10} and u_{20} in place of u_0, h_1 and h_2 in place of h, add the two resulting inequalities

$$\begin{aligned}
&-2\eta \int_0^s \|\dot{u}_\eta\|^2\,dt + \int_0^s \Big[2\hat{J}(u_\eta) - \hat{J}(u_1) - \hat{J}(u_2) \\
&+ \frac{1}{2}\big(\|Ru_\eta - h_1\|^2 + \|Ru_\eta - h_2\|^2 - \|Ru_1 - h_1\|^2 - \|Ru_2 - h_2\|^2\big)\Big]\,dt \\
&\ge \frac{1}{2}\Big[\|u_\eta(s) - u_1(s)\|^2 + \|u_\eta(s) - u_2(s)\|^2 - \frac{1}{2}\|u_{10} - u_{20}\|^2\Big] \qquad (3.4.85)
\end{aligned}$$

Notice that $\hat{J}(\cdot)$ is a convex functional, we have $2\hat{J}(u) \leq \hat{J}(u_1) + \hat{J}(u_2)$. Thus, we get

$$
\begin{aligned}
&-2\eta \int_0^s \|\dot{u}_\eta\|^2 \, dt + \int_0^s \Big[2\hat{J}(u_\eta) - 2\hat{J}(u) + \frac{1}{2}\big(\|Ru_\eta - h_1\|^2 \\
&+ \|Ru_\eta - h_2\|^2 - \|Ru_1 - h_1\|^2 - \|Ru_2 - h_2\|^2\big)\Big] \, dt \\
&\geq \frac{1}{2}\Big[\|u_\eta(s) - u_1(s)\|^2 + \|u_\eta(s) - u_2(s)\|^2 - \frac{1}{2}\|u_{10} - u_{20}\|^2\Big]
\end{aligned}
\tag{3.4.86}
$$

Let $\eta \to 0$ and by Lemma 3.4.8, we get (3.4.61).

Part (c)

Since $u_0 \in BV(\Omega) \cap L^2(\Omega)$, by stability inequality, we know that u is unique. From (3.4.82), we have $\dot{u} \in L^2(0,T;L^2(\Omega))$. Combine this and $u \in L^\infty(0,T;L^2(\Omega))$, by Theorem 2.3.3, we know that $u \in C([0,T];L^2(\Omega))$ after possibly being redefined on a set of measure zero and $u(s_2) = u(s_1) + \int_{s_1}^{s_2} \dot{u} \, dt$. This unique u is the limit of a subsequence $\{u^\epsilon\}_{\epsilon>0}$ which satisfies $u^\epsilon(s_2) = u^\epsilon(s_1) + \int_{s_1}^{s_2} \dot{u}^\epsilon \, dt$. Combine with (3.4.65) and (3.4.66), we obtain

$$
u^\epsilon(s) \rightharpoonup u(s) \quad \text{weakly in } L^2(\Omega) \; \forall \, s \in [0,T] \tag{3.4.87}
$$

Since $u^\epsilon(0) = u_0$ for all $\epsilon > 0$, thus $u(0) = u_0$ and

$$
\begin{aligned}
&\int_0^s \int_\Omega \dot{v}(v-u) \, dx \, dt = \int_0^s \int_\Omega \dot{u}(v-u) \, dx \, dt \\
&+ \frac{1}{2}\big[\|v(s) - u(s)\|^2 - \|v(0) - u_0\|^2\big]
\end{aligned}
\tag{3.4.88}
$$

Notice that the time derivative of v has been transferred to u, (3.4.62) holds $\forall \, v \in L^1(0,T;BV(\Omega) \cap L^2(0,T;L^2(\Omega))$. Now, let's prove $u \in L^\infty(0,T;BV(\Omega))$.

Assume first $u_0 \in H^1(\Omega)$, by energy estimate (3.4.7), we know that this unique u can be regarded as the limit of sequence $\{u^\epsilon\}_{\epsilon>0}$ which satisfies

$$\hat{J}_R(u^\epsilon) \leq C\Big[\int_\Omega \Phi(|\nabla u_0|)\,dx + \frac{\epsilon}{2}\|\nabla u_0\|^2 + \frac{1}{2}\|Ru_0 - h\|^2\Big]$$

Thus, we get $\hat{J}_R(u) \leq C(\int_\Omega \Phi(|\nabla u_0|)\,dx + \frac{1}{2}\|Ru_0 - h\|^2)$ for a.e. $t \in [0,T]$. For $u_0 \in L^2(\Omega) \cap BV(\Omega)$, there exists a sequence of $\{u_0^n\}_{n=1}^\infty \subset H^1(\Omega)$ such that

$$u_0^n \to u_0 \text{ strongly in } L^2(\Omega)$$

$$u_0^n \to u_0 \text{ strictly in } BV(\Omega)$$

Using the lower semicontinuity of $\hat{J}_R$, we obtain that $\hat{J}_R(u) \leq C(\int_\Omega \Phi(|Du_0|) + \frac{1}{2}\|Ru_0 - h\|^2)$ for a.e. $t \in [0,T]$ still holds. Thus $u \in L^\infty(0,T;BV(\Omega))$.

3.4.3 Evolutionary PDE and variational problem

Theorem 3.4.9. *Suppose $u_0 \in BV(\Omega) \cap L^2(\Omega)$ and $g \in L^2(\Omega)$. Let u satisfies* (3.4.63) *and $\bar{u}$ be the minimizer of $\hat{J}_R(u)$. Then,*

$$\lim_{t\to\infty} \|u(t) - \bar{u}\|_{L^p(\Omega)} = 0 \quad \forall\, p \in [1, 1^*) \tag{3.4.89}$$

Proof. We follow the approach of Feng [38]. The existence and uniqueness of the minimizer $\bar{u}$ of $\hat{J}_R(u)$ was proved in Vese [7]. Take $v(t) = u(t-\tau)$ for $\tau > 0$ in (3.4.62) with $s = T$, dividing the resulted inequality by $-\tau$ and then let $\tau \to 0$ yields

$$\int_0^T \|\dot{u}\|^2\,dt + \hat{J}_R(u(T)) \leq \hat{J}_R(u_0) < \infty \tag{3.4.90}$$

Hence, there exists a sequence $\{t_j\}$ with $t_j \to \infty$ as $j \to \infty$ such that

$$\begin{aligned} &\lim_{j\to\infty} \|\dot{u}(t_j)\| = 0 \\ &\|u(t_j)\|_{BV(\Omega)\cap L^2(\Omega)} \leq C \text{ for any } j \geq 1 \end{aligned} \tag{3.4.91}$$

By compactness of $BV(\Omega)$, there exists a subsequence of $\{u(t_j)\}$ (still denoted by the same notation) and $\hat{u} \in BV(\Omega) \cap L^2(\Omega)$ such that $u(t_j)$ converges to $\hat{u}$ weakly* in $BV(\Omega)$, strongly in $L^p(\Omega)$ for $1 \leq p < 1^*$, and weakly in $L^2(\Omega)$ as $j \to \infty$. Finally, let $j \to \infty$ in (3.4.63) after choosing $t = t_j$ and using the fact that $\hat{J}_R$ is lower semicontinuous with respect to L^1 convergence, we get

$$\hat{J}_R(v) \geq \hat{J}_R(\hat{u}) \quad \forall\, v \in BV(\Omega) \cap L^2(\Omega) \tag{3.4.92}$$

which implies that $\hat{u}$ is a minimizer of $\hat{J}_R$. By the uniqueness of minimizer, we conclude that $\hat{u} = \bar{u}$ and that the whole sequence $\{u(t)\}$ converges to $\bar{u}$ as $t \to \infty$.

It is worth to point out that the solution of the minimization problem is in $W^{1,1}(\Omega)$ (cf. [31]) provided that the operator R is coercive, i.e. $\|Ru\| \geq \theta\|u\|$, the initial data $h \in H^1(\Omega)$ and Ω satisfies some regularity condition.

3.4.4 Relationship with texture decomposition PDE

The texture decomposition model of Osher, Solé, Vese [73]

$$\inf_u \left\{ F(u) = \int_\Omega |\nabla u|\, dx + \frac{\lambda}{2} \int_\Omega |\nabla(\Delta^{-1}(h-u)|^2\, dx \right\} \tag{3.4.93}$$

has almost the identical mathematical format as

$$\inf_u \left\{ F(u) = \int_\Omega |\nabla u| \, dx + \frac{\lambda}{2} \int_\Omega |h - Ru|^2 \, dx \right\} \tag{3.4.94}$$

The difference is that the linear operator $R_o = \nabla \Delta^{-1}$ ($R_o^* R_o = \Delta^{-1} \nabla \cdot \nabla \Delta^{-1} = \Delta^{-1}$) acts on original image h too in OSV model. The study of the formally derived second order evolutionary PDE from OSV model

$$\begin{cases} \dot{u} = -\nabla \cdot \left[\frac{\nabla u}{|\nabla u|} \right] + \lambda \Delta^{-1}(h - u) \\ u(0) = u_0 \\ \frac{\partial u}{\partial \nu}|_\Gamma = 0 \end{cases} \tag{3.4.95}$$

is essentially the same as (3.1.1).

Chapter 4

The study of fourth order parabolic PDEs

In chapter 3, we studied the solution existence and regularity of generalized solutions of one class of second order parabolic PDEs. Although they have a great success for denoising, edge detection and texture decomposition, sometimes they produce undesirable staircase effect, namely, the transformation of smooth regions (ramps) into piecewise constant regions (stairs) [29, 20, 21, 13]. Thus, minimization functionals with second order derivatives of u and the fourth order PDEs are proposed in the literatures [20, 91, 21, 99, 62] to eliminate the staircase effects suffered by first order derivative models. It is not a surprise that fourth order parabolic PDEs appear in image processing literatures since many such PDEs have been appeared widely in material science and fluid dynamics [12, 24, 40]. For this class of fourth parabolic PDE, the coefficients of the fourth order terms will vanish if $|Su| \to \infty$, here S is a differential operator, $S = \nabla^2$ or Δ. We use a classic method — vanish viscosity method to study them. First, by using Galerkin method and the property of monotone operator, we prove the existence of weak solutions for regularized PDEs which are obtained by adding a regularization term $-\epsilon\Delta^2 u$ to the original equations. Thus, For any $\epsilon > 0$, we obtain u^ϵ which is the weak solution of the regularized equation and satisfies some ϵ independent energy estimates. Next, we pass the limits $\epsilon \to 0$, by using the weak compactness result in $L^p(0,T;B)$, here B is a Banach space, $1 < p < \infty$ and the compactness result in $L^1(0,T;B)$, we will obtain u as the limit of u^ϵ. Finally, by the lower semicontinuity property of L^2 norm and the lower semicontinuity property of variational functional involving measures, we will obtain that u satisfies a variational inequality.

4.1 Minimization functional

We consider the following minimization functional

$$J(u) = \int_\Omega \left[\Phi_1(|\nabla u|) + \Phi(|\nabla^2 u|) + \frac{\lambda}{2}(u-h)^2\right] dx \tag{4.1.1}$$

where ∇u, $\nabla^2 u$ are the gradient and Hessian matrix of u respectively. Minimization functionals in [20, 99, 62] are the special cases of (4.1.1). We shall study the existence and uniqueness of the solution of (4.1.1) in $BH(\Omega)$. Assume

H.1 $\Phi(\cdot)$ and $\Phi_1(\cdot)$ are even, convex functions from $\mathbb{R}$ to $\mathbb{R}^+$. They are nondecreasing in $\mathbb{R}^+$.

H.2 $\Phi(0) = 0$, $\Phi_1(0) = 0$ (without loss of generality).

H.3 $\Phi(\cdot)$ has linear growth and satisfies

$$\alpha|z| - \beta \leq \Phi(|z|) \leq \alpha|z| + \beta \tag{4.1.2}$$

where α, β are positive constants.

H.4 $\Phi_1(\cdot)$ satisfies

$$\Phi_1(|z|) \leq \alpha_1|z| + \beta_1 \tag{4.1.3}$$

α_1, β_1 are some nonnegative constants.

Remark 4.1.1. (a) For smooth convex function $\Phi(\cdot)$ defined on $\mathbb{R}$, we have

$$\Phi(s_0) - \Phi(s) \geq (s_0 - s)\Phi'(s) \quad \forall\, s_0, s \in \mathbb{R} \tag{4.1.4}$$

Set $s_0 = 0$ and $s_0 = 2s$ respectively, we obtain:

$$\Phi'(s)s \geq \Phi(s), \qquad\qquad \Phi'(s)s \leq \Phi(2s) - \Phi(s) \tag{4.1.5}$$

Thus, $\Phi'(s) \leq \frac{\Phi(2s)}{s} \leq \frac{2\alpha s+\beta}{s}$, $\lim\limits_{s\to+\infty} \Phi'(s) \leq 2\alpha$. Notice $\forall\, s \geq 0$, $\Phi'(s)$ is nondecreasing, thus it is bounded, i.e. $\Phi'(s) \leq C$. Similarly, $\Phi_1'(s) \leq C$.

(b) Since Φ is a convex and linear growth function, the recession function [1]of Φ, $\Phi_\infty(z) = \alpha|z|$. For example, in (1.1.20), $\Phi(z) = kz \arctan\frac{z}{k} - \frac{k^2}{2}\log(\frac{z^2}{k^2}+1)$, $\Phi_\infty(z) = \frac{k\pi}{2}|z|$. Although functional $J(u) = \int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi(|\nabla^2 u|)\,dx + \frac{\lambda}{2}\int_\Omega (u-h)^2\,dx$ is well defined and finite on $W^{2,1}$, unfortunately $W^{2,1}$ is not a reflexive Banach space and the minimization problem

[1]Please refer to definition 2.6.9

may not have solution in this space. Following the ideas of Chambolle and Lions [20], Vese [92], we study this minimization problem in $BH(\Omega)$ [2].

Theorem 4.1.2. *Let Ω be a domain in $\mathbb{R}^d$ with Lipschitz boundary, $h \in L^2(\Omega)$, $\lambda > 0$. Under the above assumption about $\Phi(\cdot)$ and $\Phi_1(\cdot)$, the minimization problem*

$$\inf_u \left\{ \hat{J}(u) = \int_\Omega \left[\Phi(|\nabla^2 u|) + \Phi_1(|\nabla u|) + \lambda(u-h)^2\right] dx + \alpha |D_s^2 u|(\Omega) \right\} \qquad (4.1.6)$$

for $u \in BH(\Omega)$, $D^2 u = \nabla^2 u dx + D_s^2 u$ the Lebesgue decomposition of $D^2 u$, has a unique solution $u \in BH(\Omega)$.

The functional $\hat{J} : BH(\Omega) \to [0, +\infty)$ is lower semicontinuous with respect to BH^* topology and less than or equal to J, where J is defined by

$$J(u) = \begin{cases} \int_\Omega \left[\Phi(|\nabla^2 u|) + \Phi_1(|\nabla u|) + \frac{\lambda}{2}(u-h)^2\right] dx & u \in W^{2,1}(\Omega) \\ +\infty \quad u \in BH(\Omega) \setminus W^{2,1}(\Omega) \end{cases} \qquad (4.1.7)$$

$J(\cdot)$ is not lower semicontinuous on $BH(\Omega)$. The so called relaxed functional $\bar{J}$ is defined by

$$\bar{J}(u) = \inf \left\{ \lim_{n\to\infty} \inf J(\cdot) : u_n \in W^{2,1}(\Omega), u_n \to u \in W^{1,1}(\Omega) \right\} \qquad (4.1.8)$$

for any $u \in W^{2,1}(\Omega)$. $\bar{J}(u)$ is the largest lower semicontinuous functional which is less than or equal to $J(u)$. Obviously, $\hat{J}(u) \le \bar{J}(u)$. However, From theorem 2.3 in Demengel and Temam [27], for any $u \in BH(\Omega)$, there exists a sequence

[2] Please refer to Section 2.6.4 for definition of BH and various properties of it.

$\{u_n\}_{n\geq 1} \in C^\infty(\Omega) \cap W^{2,1}(\Omega)$ such that

$$\begin{aligned} &u_n \to u \quad \text{strongly in } W^{1,1}(\Omega) \\ &(|D^2 u_n|)(\Omega) \to (|D^2 u|)(\Omega) \\ &\Phi(|D^2 u_n|)(\Omega) \to \Phi(|D^2 u|)(\Omega) \end{aligned} \tag{4.1.9}$$

Hence $\bar{J}(u) \leq \hat{J}(u)$. Therefore, $\hat{J}(\cdot)$ is the relaxation of $J(\cdot)$ on $BH(\Omega)$ with respect to weak* topology.

Remark 4.1.3. The proof of the above theorem is based on mollification. In [27], Demengel and Temam assumed the regularity of the boundary Γ of Ω to be C^1. In fact, by using a slightly modified technique (see [35]), it is not hard to see that the theorem is valid when Γ is Lipschitz.

Existence. Let C will be some constant which may differ from line to line. Assume that $\{u_n\}_{n\geq 1}$ be a minimizing sequence for (4.1.6), due to the linear assumption on $\Phi(\cdot)$, We have

$$|D^2 u_n|(\Omega) \leq C, \quad \|u_n - h\| \leq C \tag{4.1.10}$$

From (2.6.31), we obtain that u_n is bounded in $BH(\Omega)$. Therefore, there exists $u \in BH(\Omega)$, such that

$$u_n \to u \quad \text{strongly in } W^{1,p}(\Omega), \quad D^2 u_n \rightharpoonup D^2 u \quad \text{weakly}^* \text{ in } \mathcal{M}(\Omega) \tag{4.1.11}$$

where $1 \leq p < 1^*$. We have used the fact (2.6.30). From the lower semicontinuity of $\hat{J}(u)$, we have

$$\hat{J}(u) \leq \liminf_{n\to\infty} \hat{J}(u_n) \tag{4.1.12}$$

Thus, u is a minimizer of $\hat{J}$.

Uniqueness. Let u, $v \in BH(\Omega)$ be two different solutions of the minimization problem (4.1.6), from the strict convexity of $\hat{J}$, we have

$$\hat{J}(\frac{1}{2}u + \frac{1}{2}v) < \frac{1}{2}[\hat{J}(u) + \hat{J}(v)] = \inf \hat{J} \tag{4.1.13}$$

It is a contradiction! Thus, the minimizer is unique.

It is not hard to see that the relaxation functional of (1.1.20) in one space dimension and $J_2(u)$ of (1.1.16) have unique solutions in $BH(\Omega)$.

4.2 Fourth order parabolic equations

In section 4.1, we mentioned that Lysaker, Lundervold, and Tai [62] proposed the minimization functional to denoising medical images. For minimization functional (1.1.17), by deriving Euler-Lagrange equation and employing gradient decent method to solve minimization problem, they obtained the evolutionary partial differential equation:

$$\dot{u} + \left(\frac{u_{xx}}{|\nabla^2 u|}\right)_{xx} + \left(\frac{u_{xy}}{|\nabla^2 u|}\right)_{xy} + \left(\frac{u_{yx}}{|\nabla^2 u|}\right)_{yx} + \left(\frac{u_{yy}}{|\nabla^2 u|}\right)_{yy} + \lambda(u - u_0) \quad = \quad 0 \tag{4.2.1}$$

with homogeneous Neumann boundary conditions. This evolutionary PDE, together with the one dimensional case of PDE proposed in [91] are the special cases of the following PDEs:

$$\begin{cases} \dot{u} = \nabla \cdot (\frac{\Phi_1'(|\nabla u|)}{|\nabla u|}\nabla u) - \nabla^2 \cdot (\frac{\Phi'(|\nabla^2 u|)}{|\nabla^2 u|}\nabla^2 u) - \lambda(u-h) \\ u(0) = u_0 \end{cases} \tag{4.2.2}$$

here Φ, Φ_1 are smooth functions which satisfy H.1 to H.4. From now on, we restrict ourself to only consider $\Omega = \prod_{i=1}^{d}(0, L_i)$, $L = (L_1, \cdots, L_d)$. In this case, the homogeneous Neumann boundary condition problem can be mapped to a periodic boundary condition problem by reflection symmetry (See figure 4.1). Following the same approach as the second order evolutionary equations, we shall

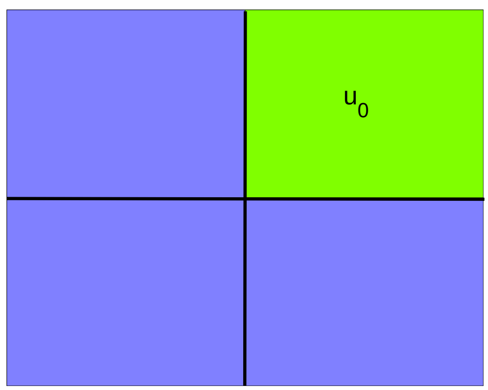

Figure 4.1: Extension of u_0 to periodic boundary

prove the existence and regularity of the generalized solution if $u_0, h \in L^2(\Omega)$ and $u_0 \in BH_{per}(\Omega) \cap L^2(\Omega), h \in L^2(\Omega)$.

4.2.1 Solution of regularized equation and energy estimates

For this purpose, first, we will prove the existence and uniqueness of the weak solution of the regularized equation:

$$\begin{cases} \dot{u} = \nabla \cdot \left(\frac{\Phi_1'(|\nabla u|)}{|\nabla u|}\nabla u\right) - \nabla^2 \cdot \left(\frac{\Phi'(|\nabla^2 u|)}{|\nabla^2 u|}\nabla^2 u\right) - \lambda(u-h) - \epsilon\Delta^2 u \\ u(t) \text{ is } L-\text{periodic } \forall\, t \in (0,T] \\ u(0) = u_0 \end{cases} \tag{4.2.3}$$

here $L = (L_1, \cdots, L_d)$. Then we derive some ϵ independent bounds and pass the limit to $\epsilon \to 0$. Let $V = H^2_{per}(\Omega)$, V' is the dual space. We define

$$B^\epsilon[u,v;t] = \big\langle g_1(\nabla u), \nabla v\big\rangle + \big\langle g(\nabla^2 u), \nabla^2 v\big\rangle + \epsilon\big\langle \Delta u, \Delta v\big\rangle \tag{4.2.4}$$

$$J^\epsilon[u,h;t] = \int_\Omega \Big[\Phi_1(|\nabla u|) + \Phi(|\nabla^2 u|) + \frac{\lambda}{2}(u-h)^2 + \frac{\epsilon}{2}|\Delta u|^2\Big]dx \tag{4.2.5}$$

where $g_1(\nabla u) = \frac{\Phi_1'(|\nabla u|)}{|\nabla u|}\nabla u$, $g(\nabla^2 u) = \frac{\Phi'(|\nabla^2 u|)}{|\nabla^2 u|}\nabla^2 u$, $\big\langle \nabla^2 u, \nabla^2 v\big\rangle = \sum_{i,j=1}^{d} \partial_{ij}u\partial_{ij}v$.

Definition 4.2.1 (Weak Solution). A weak solution of (4.2.3) is defined as $u \in L^2(0,T,V) \cap C([0,T], L^2(\Omega))$ such that $\dot{u} \in L^2(0,T,V')$ and

$$\begin{cases} \big\langle \dot{u}, v\big\rangle + B^\epsilon[u,v;t] + \lambda\big\langle u-h, v\big\rangle = 0 \quad \text{a.e. } t \in [0,T]\ \forall\, v \in V \\ u(0) = u_0 \end{cases} \tag{4.2.6}$$

Existence and uniqueness of weak solution

Theorem 4.2.2 (Existence and Uniqueness of Weak Solution). *Assume that $u_0 \in L^2(\Omega), h \in L^2(\Omega)$, Φ_1, Φ are smooth functions which satisfy H.1-H.4* *of section 4.1, Then there is a unique weak solution of* (4.2.3) *which satisfies the following energy estimates:*

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + B^\epsilon[u,u;t] + \frac{\lambda}{2}\int_\Omega (u-h)^2\,dx \le \frac{\lambda}{2}\|h\|^2 \tag{4.2.7}$$

$$\int_0^T t\|\dot{u}\|^2\,dt + tJ^\epsilon[u,u_0;t] \le C\big(\|u_0\|^2 + \|h\|^2\big) \tag{4.2.8}$$

$$\int_0^T \|\dot{u}\|^2_{V'}\,dt \le C\big(\|u_0\|^2 + \|h\|^2\big) \tag{4.2.9}$$

Moreover, if $u_0 \in V$, we have

$$\begin{aligned}&\int_0^t \|\dot{u}\|^2 + \alpha\int_\Omega |\nabla^2 u|\,dx + \frac{\lambda}{2}\int_\Omega (u-h)^2\,dx + \frac{\epsilon}{2}\|\Delta u\|^2\\ &\le \alpha_1\int_\Omega |\nabla u_0|\,dx + \beta_1 + \alpha\int_\Omega |\nabla^2 u_0|\,dx + 2\beta\\ &+ \frac{\epsilon}{2}\|\Delta u_0\|^2 + \frac{\lambda}{2}\|u_0 - h\|^2\end{aligned} \tag{4.2.10}$$

Proof. Assume the functions $\{\omega_k\}_{k\ge1}$ are smooth and

$$\begin{aligned}&\{\omega_k\}_{k=1}^\infty \text{ is an orthogonal basis of } V\\ &\{\omega_k\}_{k=1}^\infty \text{ is an orthonormal basis of } L^2(\Omega)\end{aligned} \tag{4.2.11}$$

We could take $\{\omega_k\}_{k=1}^\infty$ be the appropriately normalized eigenfunctions of the following periodic boundary value problem [3]:

$$\begin{cases} -\Delta u = 0 \\ u \text{ is } L-\text{periodic} \end{cases}$$

[3] In fact, the eigenfunctions are cosine and sine functions.

Solution in finite dimensional space Fix a positive integer m, we will look for the weak solutions of (4.2.3) in a finite dimensional space in the form of

$$u_m = \sum_{k=1}^{m} a_k(t)\omega_k \tag{4.2.12}$$

$u_m : [0, T] \mapsto V$ which satisfies:

$$\begin{cases} \langle \dot{u}_m, \omega_k \rangle + B^\epsilon[u_m, \omega_k; t] + \lambda \langle u_m - h_m, \omega_k \rangle = 0 \\ \langle u_m(0), \omega_k \rangle = \langle u_0, \omega_k \rangle \end{cases} \tag{4.2.13}$$

for $0 \le t \le T, k = 1, \cdots, m$, here h_m is the finite dimensional projection of h onto linear space generated by $\{\omega_k\}_{k=1}^m$. u_{0m} is the finite dimensional projection of u_0 onto the same space.

Theorem 4.2.3 (Galerkin approximation). *For each integer* $m = 1, 2, \cdots$, *there exists a unique function* u_m *of the form* (4.2.12) *satisfying* (4.2.13).

Proof. Assuming u_m has the structure (4.2.12), from (4.2.11), we first notice $\langle \dot{u}_m(t), \omega_k \rangle = a_k'(t)$. Therefore,

$$\begin{cases} a_k'(t) = f_k(a_1(t), \cdots, a_m(t)) \quad k = 1, \cdots, m \\ a_k(0) = \langle u_0, \omega_k \rangle \quad k = 1, \cdots, m \end{cases} \tag{4.2.14}$$

where $f_k : \mathbb{R}^m \mapsto \mathbb{R}(1 \le k \le m)$ are locally Lipschitz. It follows from the Picard theorem on a Banach Space that there exists a $T_m > 0$ such that (4.2.14) has a unique absolutely continuous solution $(a_1(t), \cdots, a_m(t))$ for $t \in [0, T_m]$. For each $t \in [0, T_m]$, multiply (4.2.13) by $a_k(t)$ and sum for $k = 1, \cdots, m$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + B^\epsilon[u_m, u_m; t] + \frac{\lambda}{2}\|u_m - h_m\|^2 \le \frac{\lambda}{2}\|h_m\|^2 \le \frac{\lambda}{2}\|h\|^2 \tag{4.2.15}$$

The orthogonality of $\{\omega_k\}_{k=1}^{\infty}$ implies that

$$\sum_{k=1}^{m} |a_k(t)|^2 = \|u_m\|^2 \leq \lambda \|h\|^2 + T\|u_0\|^2 \tag{4.2.16}$$

The solution of (4.2.13) is bounded on $[0, T_m]$, hence can be uniquely extended to $[0, \infty)$.

Energy estimates in finite dimension

Theorem 4.2.4 (Energy estimates). *There exists a constant C, depending only on Ω, T, λ and Φ_1, Φ, such that*

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + B^{\epsilon}[u_m, u_m; t] + \frac{\lambda}{2}\int_{\Omega}(u_m - h_m)^2\, dx \leq \frac{\lambda}{2}\|h\|^2 \tag{4.2.17}$$

$$\int_0^t t\|\dot{u}_m\|^2\, dt + tJ^{\epsilon}[u_m, h_m; t] = \int_0^t J^{\epsilon}[u_m, h_m; t]\, dt \tag{4.2.18}$$

$$\int_0^T \|\dot{u}_m\|_{V'}^2 \leq C\big(\|u_0\|^2 + \|h|^2\big) \tag{4.2.19}$$

If $u_0 \in V$, then

$$\begin{aligned}
&\int_0^t \|\dot{u_m}\|^2 + \alpha \int_{\Omega} |\nabla^2 u_m|\, dx + \frac{\lambda}{2}\int_{\Omega}(u_m - h_m)^2\, dx + \frac{\epsilon}{2}\|\Delta u_m\|^2 \\
&\leq \alpha_1 \int_{\Omega} |\nabla u_0|\, dx + \beta_1 + \alpha \int_{\Omega} |\nabla^2 u_0|\, dx \\
&+ 2\beta + \frac{\epsilon}{2}\|\Delta u_0\|^2 + \frac{\lambda}{2}\|u_0 - h\|^2
\end{aligned} \tag{4.2.20}$$

Proof. Multiply equation (4.2.13) by $a_k(t)$, sum for $k = 1, \cdots, m$, and then recall (4.2.12) to find

$$\big\langle \dot{u}_m, u_m \big\rangle + B^{\epsilon}[u_m, u_m; t] + \lambda \big\langle u_m - h_m, u_m \big\rangle = 0 \tag{4.2.21}$$

Thus, we obtain (4.2.17). Multiply equation (4.2.13) by $ta_k'(t)$, sum for $k = 1, \cdots, m$,

$$t\big\langle \dot{u}_m, \dot{u}_m \big\rangle + \frac{d}{dt} t J^\epsilon[u_m, h_m; t] = J^\epsilon[u_m, h_m; t] \tag{4.2.22}$$

Thus, we obtain (4.2.18). Notice (4.1.5) and assumptions on Φ, Φ_1, from (4.2.17) we obtain

$$\begin{aligned} &\int_0^T \Big[\int_\Omega (\alpha |\nabla^2 u_m| - \beta)\, dx + \frac{\lambda}{2} \|u_m - h_m\|^2 \Big] dt \\ &\leq \int_0^T \Big[B^\epsilon[u_m, u_m; t] + \frac{\lambda}{2} \|u_m - h_m\|^2 \Big] dt \\ &\leq \frac{\lambda}{2} \|h\|^2 + \frac{1}{2} \|u_0\|^2 \end{aligned} \tag{4.2.23}$$

On the other hand,

$$\begin{aligned} &\int_0^T J^\epsilon[u_m, h_m; t]\, dt \leq \int_0^T \Big[\int_\Omega \alpha_1 |\nabla u_m| + \beta_1 \Big] dxdt \\ &+ \int_0^T \Big[\alpha |\nabla^2 u_m| + \beta + \frac{\lambda}{2} (u_m - h_m)^2 + \frac{\epsilon}{2} |\Delta u_m|^2 \Big] dxdt \end{aligned} \tag{4.2.24}$$

Notice $u_m \in H^2(\Omega) \subset BH(\Omega)$, combine Lemma 2.6.22 (see also Adams [1], the interpolation inequality), (4.2.23), (4.2.24) we obtain, for some C **does not** depend on ϵ, but could depend on $\Omega, T, \alpha_1, \alpha, \beta, \beta_1, \lambda$,

$$\int_0^T t \|\dot{u}_m\|^2\, dt + t J^\epsilon[u_m, h_m; t] \leq C \big(\|u_0\|^2 + \|h\|^2 \big) \tag{4.2.25}$$

Recall that

$$\|\dot{u}_m\|_{V'} = \sup \big\{ \big\langle \dot{u}_m, v \big\rangle : \|v\|_V \leq 1 \big\} \tag{4.2.26}$$

$\forall\, v \in V$ with $\|v\|_{H^2(\Omega)} \leq 1$, we have

$$v = v_1 + v_2; \quad v_1 = \sum_{k=1}^m b_k \omega_k \tag{4.2.27}$$

and $\langle \omega_k, v_2 \rangle = 0$ $(k = 1, \cdots, m)$. Since the functions $\{\omega_k\}_{k=1}^{\infty}$ are orthogonal in V, $\|v_1\|_{H^2(\Omega)} \le \|v\|_{H^2(\Omega)} \le 1$. From (4.2.13),

$$\langle \dot{u}_m, v \rangle + B^{\epsilon}[u_m, v_1; t] + \lambda \langle u_m - h_m, v_1 \rangle = 0 \tag{4.2.28}$$

Consequently,

$$\begin{aligned}
\langle \dot{u}_m, v \rangle \le & \int_{\Omega} \Big[\Phi_1'(|\nabla u_m|)|\nabla v_1| + \Phi'(|\nabla^2 u_m|)|\nabla^2 v_1| \\
& + \epsilon |\Delta u_m||\Delta v_1| + |u_m - h_m||v_1| \Big] \, dx \\
& \le \Big\{ \Big[\int_{\Omega} \Phi_1'(|\nabla u_m|)^2 \, dx \Big]^{\frac{1}{2}} + \Big[\int_{\Omega} \Phi'(|\nabla^2 u_m|)^2 \, dx \Big]^{\frac{1}{2}} \\
& + \epsilon \Big[\int_{\Omega} |\Delta u_m|^2 \, dx \Big]^{\frac{1}{2}} + \Big[\int_{\Omega} |u_m - h_m|^2 \, dx \Big]^{\frac{1}{2}} \Big\} \|v_1\|_{H^2(\Omega)}
\end{aligned} \tag{4.2.29}$$

By Cauchy inequality,

$$\begin{aligned}
\langle \dot{u}_m, v \rangle^2 \le 4 \Big\{ \int_{\Omega} \Big[& \Phi_1'(|\nabla u_m|)^2 + \Phi'(|\nabla^2 u_m|)^2 \\
& + \epsilon |\Delta u_m|^2 + |u_m - h_m|^2 \Big] \, dx \Big\} \|v_1\|^2_{H^2(\Omega)}
\end{aligned} \tag{4.2.30}$$

Therefore

$$\begin{aligned}
\|\dot{u}_m\|^2_{V'} \le 4 & \int_{\Omega} \Big[\Phi_1'(|\nabla u_m|)^2 + \Phi'(|\nabla^2 u_m|)^2 \\
& + \epsilon |\Delta u_m|^2 + (u_m - h_m)^2 \Big] \, dx
\end{aligned} \tag{4.2.31}$$

By Remark 4.1.1, $\Phi_1'(|z|) \le C$ and $\Phi'(|z|) \le C$ for some constant C. Consequently,

$$\|\dot{u}_m\|^2_{V'} \le 4 \int_{\Omega} \Big[\epsilon |\Delta u_m|^2 + (u_m - h_m)^2 \Big] \, dx + 4C^2 m(\Omega) \tag{4.2.32}$$

here $m(\Omega)$ is the Lebesgue measure of Ω. From (4.2.17), we obtain:

$$\int_0^T \int_{\Omega} (u_m - h_m)^2 \, dx dt + \epsilon \int_0^T \int_{\Omega} |\Delta u_m|^2 \, dx dt \le C\big(\|u_0\|^2 + \|h\|^2\big) \tag{4.2.33}$$

Thus (4.2.31) - (4.2.33) implies (4.2.19). Now assume $u_0 \in V$, multiply (4.2.13) by $a_k'(t)$, sum for $k = 1, \cdots, m$, we find

$$\big\langle \dot{u}_m, \dot{u}_m \big\rangle + \frac{d}{dt} J^\epsilon[u_m, h_m; t] = 0 \tag{4.2.34}$$

Integrate against t, we obtain

$$\begin{aligned}
&\int_0^t \|\dot{u}_m\|^2 \, dt + J^\epsilon[u_m, h_m; t] \\
&\le \Big[\int_\Omega \Phi_1(|\nabla u_{0m}|) + \int_\Omega \Phi(|\nabla^2 u_{0m}|) + \frac{\lambda}{2}(u_{0m} - h_m)^2 + \frac{\epsilon}{2}|\Delta u_{0m}|^2 \Big] \, dx \qquad (4.2.35) \\
&\le \Big[\int_\Omega \Phi_1(|\nabla u_0|) + \int_\Omega \Phi(|\nabla^2 u_0|) + \frac{\lambda}{2}(u_0 - h)^2 + \frac{\epsilon}{2}|\Delta u_0|^2 \Big] \, dx
\end{aligned}$$

Notice the assumptions on $\Phi(\cdot)$, $\Phi_1(\cdot)$, we deduce (4.2.20).

Existence and uniqueness of weak solution From (4.2.17) and (4.2.19), it is not hard to see

$$\begin{aligned}
&\|u_m\|_{L^\infty(0,T;L^2(\Omega))} + \|u_m\|_{L^2(0,T;V)} \\
&+ \|\dot{u}_m\|_{L^2(0,T;V')} \le C(\epsilon)\big(\|u_0\|_{L^2(\Omega)} + \|h\|^2\big)
\end{aligned} \tag{4.2.36}$$

where $C(\epsilon)$ is a constant which could be depending on Ω, T, ϵ. According to this energy estimate, we see that the sequence $\big\{u_m\big\}_{m=1}^\infty$ is bounded in $L^2(0, T; V)$, and $\big\{\dot{u}_m\big\}_{m=1}^\infty$ is bounded in $L^2(0, T; V')$. Consequently, there exists a subsequence $\big\{u_{m_l}\big\}_{l=1}^\infty \subset \big\{u_m\big\}_{m=1}^\infty$ and a function $u \in L^2(0, T; V) \cap L^\infty(0, T; L^2(\Omega))$, with $\dot{u} \in L^2(0, T; V')$, such that

$$\begin{aligned}
u_{m_l} &\rightharpoonup u \qquad \text{weakly in } L^2(0, T; V) \\
u_{m_l} &\rightharpoonup u \qquad \text{weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)) \\
\dot{u}_{m_l} &\rightharpoonup \dot{u} \qquad \text{weakly in } L^2(0, T; V')
\end{aligned} \tag{4.2.37}$$

Since $V \subset\subset H^1_{per}(\Omega)$, By the compactness result in Simon [83] (see Theorem 2.4.2) or Temam [87], we obtain

$$u_{m_l} \to u \quad \text{strongly in } L^2(0,T;H^1_{per}(\Omega)) \tag{4.2.38}$$

On the other hand, $g(\nabla^2 u_m)$ is bounded in $L^2(0,T;L^2(\Omega)^{d\times d})$, upon picking up a subsequence from $\{u_{m_l}\}_{l=1}^{\infty}$, we still denote this subsequence as $\{u_{m_l}\}_{l=1}^{\infty}$,

$$g(\nabla^2 u_{m_l}) \rightharpoonup \xi \quad \text{weakly in } L^2(0,T;L^2(\Omega)^{d\times d}) \tag{4.2.39}$$

$g_1(\nabla u_{m_l})$ is bounded in $L^2(0,T;L^2(\Omega)^d)$, upon picking up another subsequence from $\{u_{m_l}\}_{l=1}^{\infty}$, we still denote this subsequence as $\{u_{m_l}\}_{l=1}^{\infty}$,

$$g_1(\nabla u_{m_l}) \rightharpoonup \xi_1 \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d) \tag{4.2.40}$$

Next fix an integer N and choose a function $v \in C^1([0,T];V)$ having the form

$$v(t) = \sum_{k=1}^{N} d_k(t)\omega_k \tag{4.2.41}$$

where $\{d_k\}_{k=1}^{N}$ are given smooth functions. We choose $m \geq N$, multiply (4.2.13) by $d_k(t)$, sum for $k = 1, \cdots, N$, and then integrate with respect to t to obtain

$$\int_0^T \Big[\big\langle \dot{u}_m, v\big\rangle + B^{\epsilon}[u_m, v; t]\Big]\, dt + \lambda \int_0^T \big\langle u_m - h_m, v\big\rangle\, dt = 0 \tag{4.2.42}$$

Set $m = m_l$ and recall (4.2.37) - (4.2.40), let $l \to \infty$, we find

$$\int_0^T \Big[\big\langle \dot{u}, v\big\rangle + \big\langle \xi_1, \nabla v\big\rangle + \big\langle \xi, \nabla^2 v\big\rangle + \epsilon\big\langle \Delta u, \Delta v\big\rangle\Big]\, dt + \lambda \int_0^T \big\langle u - h, v\big\rangle\, dt = 0 \tag{4.2.43}$$

Since the functions v of form (4.2.41) is dense in $L^2(0,T;V)$, we conclude that (4.2.42) holds for all function $v \in L^2(0,T;V)$. From theorem 2.3.4, we have

$u \in C([0,T], L^2(\Omega))$. In order to prove $u(x,0) = u_0(x)$, we first note from (4.2.43) that

$$\begin{aligned}&\int_0^T \Big[-\big\langle u, \dot{v}\big\rangle + \big\langle \xi_1, \nabla v\big\rangle + \big\langle \xi, \nabla^2 v\big\rangle + \epsilon\big\langle \Delta u, \Delta v\big\rangle\Big]\, dt + \\ &\lambda \int_0^T \big\langle u - h, v\big\rangle\, dt = -\big\langle u(0), v(0)\big\rangle\end{aligned} \tag{4.2.44}$$

for each $v \in C^1([0,T]; V)$ with $v(T) = 0$. Similarly, from (4.2.42), we deduce

$$\begin{aligned}&\int_0^T \Big[-\big\langle u_m, \dot{v}\big\rangle + B^\epsilon[u_m, v; t]\Big]\, dt + \lambda \int_0^T \big\langle u_m - h_m, v\big\rangle\, dt \\ &= -\big\langle u_m(0), v(0)\big\rangle\end{aligned} \tag{4.2.45}$$

Set $m = m_l$ and let $l \to \infty$, once again employ (4.2.37) - (4.2.40) to find

$$\begin{aligned}&\int_0^T \Big[-\big\langle u, \dot{v}\big\rangle + \big\langle \xi_1, \nabla v\big\rangle + \big\langle \xi, \nabla^2 v\big\rangle + \epsilon\big\langle \Delta u, \Delta v\big\rangle\Big]\, dt + \\ &\lambda \int_0^T \big\langle u - h, v\big\rangle\, dt = -\big\langle u_0, v(0)\big\rangle\end{aligned} \tag{4.2.46}$$

since $u_{m_l}(0) \to u_0$ in $L^2(\Omega)$. As $v(0)$ is arbitrary, from (4.2.44) and (4.2.46), we conclude $u(0) = u_0$. Pick up $v \in C^1([0,T], V)$ such that $v(0) = 0$, we can deduce $u_{m_l}(T) \rightharpoonup u(T)$ weakly in $L^2(\Omega)$. Let $v = u$ in (4.2.43), we obtain

$$\begin{aligned}&\int_0^T \Big[\big\langle \xi_1, \nabla u\big\rangle + \big\langle \xi, \nabla^2 u\big\rangle\Big]\, dt \\ &= \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|u(T)\|^2 - \int_0^T \Big[\epsilon\big\langle \Delta u, \Delta u\big\rangle + \lambda\big\langle u - h, u\big\rangle\Big]\, dt\end{aligned} \tag{4.2.47}$$

From (4.2.42), we deduce

$$\begin{aligned}&\int_0^T \Big[\big\langle g_1(\nabla u_m), \nabla u_m\big\rangle + \big\langle g(\nabla^2 u_m), \nabla^2 u_m\big\rangle\Big]\, dt \\ &= \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|u_m(T)\|^2 - \int_0^T \Big[\epsilon\big\langle \Delta u_m, \Delta u_m\big\rangle + \lambda\big\langle u_m - h_m, u_m\big\rangle\Big]\, dt\end{aligned} \tag{4.2.48}$$

It can be easily verified that L^2 norm is lower semicontinuous with respect to strong convergence. Since L^2 norm is convex, by theorem 2.5.2, we conclude that L^2 norm is lower semicontinuous with respect to weak convergence. As consequences,

$$\begin{aligned}
&\lim_{l\to\infty}\inf \|u_{m_l}(T)\|^2 \geq \|u(T)\|^2 \\
&\lim_{l\to\infty}\inf \int_0^T \|\Delta u_{m_l}\|^2\,dt \geq \int_0^T \|\Delta u\|^2\,dt
\end{aligned} \tag{4.2.49}$$

From (4.2.38), $\lim_{l\to\infty}\int_0^T \big\langle u_{m_l}-h_{m_l},u_{m_l}\big\rangle\,dt = \int_0^T \big\langle u-h,u\big\rangle\,dt$. Thus, we have

$$\begin{aligned}
&\lim_{l\to\infty}\sup \int_0^T \Big[\big\langle g_1(\nabla u_{m_l}),\nabla u_{m_l}\big\rangle + \big\langle g(\nabla^2 u_{m_l}),\nabla^2 u_{m_l}\big\rangle\Big]\,dt \\
&= \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\lim_{l\to\infty}\inf\ \|u_{m_l}(T)\|^2 - \lim_{l\to\infty}\inf\ \int_0^T \epsilon\|\Delta u_{m_l}\|^2\,dt \\
&- \lim_{l\to\infty}\inf \int_0^T \lambda\big\langle u_{m_l}-h_{m_l},u_{m_l}\big\rangle\,dt \\
&\leq \frac{1}{2}\|u_0\|^2 - \|u(T)\|^2 - \int_0^T \Big[\epsilon\|\Delta u\|^2 + \lambda\big\langle u-h,u\big\rangle\Big]\,dt \\
&\leq \int_0^T \Big[\big\langle \xi_1,\nabla u\big\rangle + \big\langle \xi,\nabla^2 u\big\rangle\Big]\,dt
\end{aligned} \tag{4.2.50}$$

$\Phi(\cdot)$, $\Phi_1(\cdot)$ are convex and smooth, by Lemma 2.7.1, $\forall\, w\in L^2(0,T;V)$,

$$\begin{aligned}
&\big\langle g_1(\nabla u_m)-g_1(\nabla w),\nabla u_m-\nabla w\big\rangle \geq 0 \\
&\big\langle g(\nabla^2 u_m)-g(\nabla^2 w),\nabla^2 u_m-\nabla^2 w\big\rangle \geq 0
\end{aligned} \tag{4.2.51}$$

Set $m=m_l$ and let $l\to\infty$, we find

$$\begin{aligned}
&0 \leq \lim_{l\to\infty}\sup \int_0^T \big\langle g_1(\nabla u_{m_l})-g_1(\nabla w),\nabla u_{m_l}-\nabla w\big\rangle\,dt \\
&+ \lim_{l\to\infty}\sup \int_0^T \big\langle g(\nabla^2 u_{m_l})-g(\nabla^2 w),\nabla^2 u_{m_l}-\nabla^2 w\big\rangle\,dt \\
&= \lim_{l\to\infty}\sup \int_0^T \Big[\big\langle g_1(\nabla u_{m_l}),\nabla u_{m_l}\big\rangle + \big\langle g(\nabla^2 u_{m_l}),\nabla^2 u_{m_l}\big\rangle\Big]\,dt \\
&- \lim_{l\to\infty}\int_0^T \Big[\big\langle g_1(\nabla u_{m_l}),\nabla w\big\rangle + \big\langle g(\nabla^2 u_{m_l}),\nabla^2 w\big\rangle\Big]\,dt
\end{aligned}$$

$$
\begin{aligned}
&- \lim_{l \to \infty} \int_0^T \Big[\big\langle g_1(\nabla w), \nabla u_{m_l} \big\rangle + \big\langle g(\nabla^2 w), \nabla^2 u_{m_l} \big\rangle \Big]\, dt \\
&+ \int_0^T \Big[\big\langle g_1(\nabla w), \nabla w \big\rangle + \big\langle g(\nabla^2 w), \nabla^2 w \big\rangle \Big]\, dt
\end{aligned} \tag{4.2.52}
$$

By (4.2.50), (4.2.40), (4.2.39), (4.2.38) and (4.2.37), we obtain

$$
\begin{aligned}
0 &\le \int_0^T \Big[\big\langle \xi_1, \nabla u \big\rangle + \big\langle \xi, \nabla^2 u \big\rangle \Big]\, dt - \int_0^T \Big[\big\langle \xi_1, \nabla w \big\rangle + \big\langle \xi, \nabla^2 w \big\rangle \Big]\, dt \\
&- \lim_{l \to \infty} \int_0^T \Big[\big\langle g_1(\nabla w), \nabla u \big\rangle + \big\langle g(\nabla^2 w), \nabla^2 u \big\rangle \Big]\, dt \\
&+ \int_0^T \Big[\big\langle g_1(\nabla w), \nabla w \big\rangle + \big\langle g(\nabla^2 w), \nabla^2 w \big\rangle \Big]\, dt \\
&= \int_0^T \big\langle \xi_1 - g_1(\nabla w), \nabla u - \nabla w \big\rangle\, dt \\
&+ \int_0^T \big\langle \xi - g(\nabla^2 w), \nabla^2 u - \nabla^2 w \big\rangle\, dt
\end{aligned} \tag{4.2.53}
$$

Fix any $v \in L^2(0,T;V)$ and set $w := u - \theta v$ ($\theta > 0$) in (4.2.52). We obtain then

$$
\int_0^T \big\langle \xi_1 - g_1(\nabla(u - \theta v)), \nabla v \big\rangle\, dt + \int_0^T \big\langle \xi - g(\nabla^2(u - \theta v)), \nabla^2 v \big\rangle\, dt \ge 0
$$

Let $\theta \to 0$

$$
\int_0^T \Big[\big\langle \xi_1 - g_1(\nabla u), \nabla v \big\rangle + \big\langle \xi - g(\nabla^2 u), \nabla^2 v \big\rangle \Big]\, dt \ge 0 \tag{4.2.54}
$$

Replace v by $-v$, we deduce that

$$
\int_0^T \Big[\big\langle \xi_1 - g_1(\nabla u), \nabla v \big\rangle + \big\langle \xi - g(\nabla^2 u), \nabla^2 v \big\rangle \Big]\, dt \le 0 \tag{4.2.55}
$$

Therefore,

$$
\begin{aligned}
&\int_0^T \Big[\big\langle \xi_1, \nabla v \big\rangle + \big\langle \xi, \nabla^2 v \big\rangle \Big]\, dt \\
&= \int_0^T \Big[\big\langle g_1(\nabla u), \nabla v \big\rangle + \big\langle g(\nabla^2 u), \nabla^2 v \big\rangle \Big]\, dt
\end{aligned} \tag{4.2.56}
$$

Substitute (4.2.56) into (4.2.43), to find

$$\int_0^T \Big[\big\langle \dot{u}, v\big\rangle + B^\epsilon[u,v;t]\Big]\,dt + \lambda\int_0^T \big\langle u-h, v\big\rangle\,dt = 0 \tag{4.2.57}$$

This inequality holds for all functions $v \in L^2(0,T;V)$, Hence in particular

$$\big\langle \dot{u}, v\big\rangle + B^\epsilon[u,v;t] + \lambda\big\langle u-h, v\big\rangle = 0 \tag{4.2.58}$$

for each $v \in V$ and a.e. $0 \le t \le T$. Let $v = u$, in (4.2.58), we deduce (4.2.7). The other energy estimates (4.2.8), (4.2.9) and (4.2.10) are direct consequences of (4.2.25), (4.2.19) and (4.2.20) when $m \to \infty$.

Stability of weak solution

Theorem 4.2.5 (Stability). *If u_1, u_2 are two solutions of* (4.2.3) *with initial datum u_{01}, h_1 and u_{02}, h_2 respectively, then*

$$\|u_1(t) - u_2(t)\|^2 \le \|u_{01} - u_{02}\|^2 + \lambda t\|h_1 - h_2\|^2 \tag{4.2.59}$$

Proof. Since u_1, u_2 are weak solutions of (4.2.3), we have

$$\big\langle \dot{u}_1, v\big\rangle + B^\epsilon[u_1,v;t] + \lambda\big\langle u_1-h_1, v\big\rangle = 0$$

$$\big\langle \dot{u}_2, v\big\rangle + B^\epsilon[u_2,v;t] + \lambda\big\langle u_2-h_2, v\big\rangle = 0$$

Therefore,

$$\begin{aligned}
&\big\langle \dot{u}_1 - \dot{u}_2, v\big\rangle + \big\langle g_1(\nabla u_1) - g_1(\nabla u_2), \nabla v\big\rangle \\
&+ \big\langle g(\nabla^2 u_1) - g(\nabla^2 u_2), \nabla^2 v\big\rangle + \epsilon\big\langle \Delta u_1 - \Delta u_2, \Delta v\big\rangle \\
&+ \lambda\big\langle u_1 - u_2 - (h_1 - h_2), v\big\rangle = 0
\end{aligned}$$

Let $v = u_1 - u_2$, recall Lemma 2.7.1, we obtain

$$\left\langle g_1(\nabla u_1) - g_1(\nabla u_2), \nabla u_1 - \nabla u_2 \right\rangle \geq 0$$

$$\left\langle g(\nabla^2 u_1) - g(\nabla^2 u_2), \nabla^2 u_1 - \nabla^2 u_2 \right\rangle \geq 0$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|\dot{u}_1 - \dot{u}_2\|^2 + \lambda\|u_1 - u_2\|^2 \leq \lambda\left\langle h_1 - h_2, u_1 - u_2 \right\rangle$$
$$\leq \frac{\lambda}{2}\|u_1 - u_2\|^2 + \frac{\lambda}{2}\|h_1 - h_2\|^2$$

which implies

$$\frac{1}{2}\frac{d}{dt}\|\dot{u}_1 - \dot{u}_2\|^2 \leq \frac{\lambda}{2}\|h_1 - h_2\|^2$$

Integrate against t, we obtain (4.2.59).

4.2.2 Existence and uniqueness of generalized solution

Recall (4.1.6),

$$\hat{J}(u) := \int_\Omega \left[\Phi_1(|\nabla u|) + \Phi(|\nabla^2 u|)\right] dx + \alpha|D_s^2 u|(\Omega) \tag{4.2.60}$$

which is lower semicontinuous with respect to $W^{1,1}$ convergence. Let

$$\hat{J}_h(u) := \int_\Omega \left[\Phi_1(|\nabla u|) + \Phi(|\nabla^2 u|) + \frac{\lambda}{2}(u-h)^2\right] dx + \alpha|D_s^2 u|(\Omega) \tag{4.2.61}$$

Theorem 4.2.6 (Generalized solution). *Suppose that* $\Omega = \prod_{i=1}^{d}(0, L_i)$, *a bounded open set in* $\mathbb{R}^d$, Φ_1, Φ *are smooth functions which satisfy H.1-H.4.*

(a) *If $u_0, h \in L^2(\Omega)$, then there exists u such that*

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;BH_{per}(\Omega))$$

$$u \in L^\infty(s_0,T;BH_{per}(\Omega)) \cap C([s_0,T];L^2(\Omega)),\ s_0 \in (0,T]$$

$$\dot{u} \in L^2(0,T;V')$$

$u(t)$ is weakly continuous from $[0,T] \to L^2(\Omega)$.

$\forall\, v \in L^1(0,T;BH_{per}(\Omega)) \cap L^2(0,T;L^2(\Omega))$ *with* $\dot{v} \in L^2(0,T;L^2(\Omega))$

$$\begin{aligned}&\int_0^s \int_\Omega \dot{v}(v-u)\,dxdt + \int_0^s (\hat{J}_h(v) - \hat{J}_h(u))\,dt \\ &\geq \frac{1}{2}\big[\|v(s)-u(s)\|^2 - \|v(0)-u_0\|^2\big] \quad \forall\, s \in (0,T]\end{aligned} \tag{4.2.62}$$

Such u is called a generalized solution of (4.2.2).

(b) *Suppose u_1, u_2 satisfies* (4.2.62) *with initial data u_{01}, h_1 and u_{02}, h_2 respectively. Assume $u_{01}, u_{02} \in BH_{per}(\Omega) \cap L^2(\Omega)$, $h_1, h_2 \in L^2(\Omega)$ then*

$$\|u_1(s) - u_2(s)\|^2 \leq \|u_{01} - u_{02}\|^2 + \lambda s\|h_1 - h_2\|^2 \quad \forall\, s \in [0,T] \tag{4.2.63}$$

(c) *Furthermore, if $u_0 \in L^2(\Omega) \cap BH_{per}(\Omega)$, $h \in L^2(\Omega)$, then u is unique and $u \in L^\infty(0,T;BH_{per}(\Omega)) \cap C([0,T];L^2(\Omega))$, $\dot{u} \in L^2(0,T;L^2(\Omega))$, $u(0) = u_0$ such that*

$$\int_0^s \int_\Omega \dot{u}(v-u)\,dxdt + \int_0^s (\hat{J}_h(v) - \hat{J}_h(u))\,dt \geq 0 \quad s \in [0,T] \tag{4.2.64}$$

$\forall\, v \in L^1(0,T;BH_{per}(\Omega)) \cap L^2(0,T;L^2(\Omega))$. *Thus*

$$\int_\Omega \dot{u}(v-u)\,dx + \hat{J}_h(v) - \hat{J}_h(u) \geq 0 \quad a.e.\ t \in [0,T] \tag{4.2.65}$$

$\forall\, v \in BH_{per}(\Omega) \cap L^2(\Omega)$.

The case $u_0, h \in L^2$

Proof. The first part of the proof is devoted to the existence of the generalized solution if $u_0, h \in L^2(\Omega)$. Fix any $\epsilon > 0$, for $u_0 \in L^2(\Omega)$, according to Theorem 4.2.2, there exists a unique u^ϵ which satisfies the energy estimates:

$$\|u^\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|u^\epsilon\|_{L^1(0,T;BH(\Omega))} + \epsilon^{1/2}\left[\int_0^T \|\Delta u^\epsilon\|^2\, dt\right]^{1/2}$$

$$\leq C\big(\|u_0\| + \|h\|\big) \tag{4.2.66}$$

$$\|\sqrt{t}\dot{u}^\epsilon\|_{L^2(0,T;L^2(\Omega)} + \|tu^\epsilon\|_{L^\infty(0,T;BH(\Omega))} \leq C\big(\|u_0\| + \|h\|\big) \tag{4.2.67}$$

$$\|\dot{u}^\epsilon\|_{L^2(0,T;V')} \leq C\big(\|u_0\| + \|h\|\big) \tag{4.2.68}$$

here C is a ϵ independent constant, it may depend on T, Ω. Therefore, there exists $u \in L^\infty(0,T,L^2(\Omega))$, $\dot{u} \in L^2(0,T;V')$, $\sqrt{t}\dot{u} \in L^2(0,T;L^2(\Omega))$, $tu \in L^\infty(0,T,BH(\Omega))$ such that a subsequence of $\big\{u^\epsilon\big\}$ (we still denote it u^ϵ)

$$\begin{aligned}
&u^\epsilon \rightharpoonup u && \text{weakly in } L^2(\Omega) \ \text{ a.e } t \in [0,T]\\
&\dot{u}^\epsilon \rightharpoonup \dot{u} && \text{weakly in } L^2(0,T;V')\\
&u^\epsilon \to u && \text{strongly in } L^1(0,T;W^{1,p}_{per}(\Omega)) && (4.2.69)\\
&\sqrt{t}\dot{u}^\epsilon \rightharpoonup \sqrt{t}\dot{u} && \text{weakly in } L^2(0,T;L^2(\Omega))\\
&tD^2u^\epsilon \rightharpoonup tD^2u && \text{weakly* in } \mathcal{M}(\Omega) \text{ a.e. } t \in [0,T]\\
&tu^\epsilon \to tu && \text{strongly in } W^{1,p}_{per}(\Omega) \text{ a.e. } t \in [0,T] && (4.2.70)
\end{aligned}$$

where $1 \leq p < 1^*$. Notice that $W^{1,p} \subset\subset BH(\Omega)$, the strong convergence (4.2.69) and (4.2.70) are due to the compactness result of Simon [83] which is stated in Theorem 2.4.2. Since $u \in L^\infty(0,T,L^2(\Omega))$, u is continuous from $[0,T]$ into V', by Lemma 2.3.5, we know that $u(t)$ is weakly continuous from $[0,T] \to L^2(\Omega)$. Thanks to Fatou's lemma, we have

$$\liminf_{\epsilon\to 0} \int_0^T \int_\Omega |\nabla^2 u^\epsilon|\, dxdt \geq \int_0^T \liminf_{\epsilon\to 0} \int_\Omega |\nabla^2 u^\epsilon|\, dxdt \tag{4.2.71}$$

Notice that $\int_\Omega |D^2 u|$ is a special case of $\hat{J}(u)$, by the lower semicontinuity of $\hat{J}(u)$ with respect to $W^{1,1}$ convergence, we obtain

$$\int_0^T \liminf_{\epsilon\to 0} \int_\Omega |\nabla^2 u^\epsilon|\, dxdt \geq \int_0^T \int_\Omega |D^2 u| dt \tag{4.2.72}$$

Thus, $u \in L^1(0,T;BH_{per}(\Omega))$. Similarly, from (4.2.70), we know that, for any $0 < s_0 \leq T$, $u \in L^\infty(s_0,T;BH(\Omega))$, $\dot{u} \in L^2(s_0,T;L^2(\Omega))$. By Theorem 2.3.3, $u \in C([s_0,T],L^2(\Omega))$. Thus, $u \in C((0,T],L^2(\Omega))$. Since $\forall\, s_1, s_2 > 0$,

$$u^\epsilon(s_2) = u^\epsilon(s_1) + \int_{s_1}^{s_2} \dot{u}^\epsilon(t)\, dt, \qquad u(s_2) = u(s_1) + \int_{s_1}^{s_2} \dot{u}(t)\, dt$$

we obtain,

$$\lim_{\epsilon\to 0} \big\langle u(s_2) - u^\epsilon(s_2), v\big\rangle = \lim_{\epsilon\to 0} \big\langle u(s_1) - u^\epsilon(s_1), v\big\rangle \quad \forall\, v \in L^2(\Omega) \tag{4.2.73}$$

Consequently, $u^\epsilon(s) \rightharpoonup u(s)\ \forall\, s \in (0,T]$. For each u^ϵ, we have

$$\begin{cases} \big\langle \dot{u}^\epsilon, v\big\rangle + B^\epsilon[u^\epsilon, v; t] + \lambda\big\langle u^\epsilon - h, v\big\rangle = 0 \ \text{ a.e. } t \in [0,T]\ \forall\, v \in V \\ u^\epsilon(0) = u_0 \end{cases} \tag{4.2.74}$$

Substitute v by $v - u^\epsilon$ and integrate with respect to t from 0 to s, we obtain

$$\int_0^s \Big[\big\langle \dot{u}^\epsilon, v - u^\epsilon \big\rangle + B^\epsilon[u^\epsilon, v - u^\epsilon; t] + \lambda \big\langle u^\epsilon - h, v - u^\epsilon \big\rangle \Big]\, dt = 0 \tag{4.2.75}$$

Nevertheless,

$$\int_0^s \big\langle \dot{v} - \dot{u}^\epsilon, v - u^\epsilon \big\rangle\, dt = \frac{1}{2}\big[\|v(s) - u^\epsilon(s)\|^2 - \|v(0) - u_0\|^2\big] \tag{4.2.76}$$

Consequently, $\forall\, v \in L^1(0,T;V) \cap L^2(0,T;L^2(\Omega))$ such that $\dot{v} \in L^2(0,T;L^2(\Omega))$, we have

$$\begin{aligned} &\int_0^s \Big[\big\langle \dot{v}, v - u^\epsilon \big\rangle + B^\epsilon[u^\epsilon, v - u^\epsilon; t] + \lambda \big\langle u^\epsilon - h, v - u^\epsilon \big\rangle \Big]\, dt \\ &= \frac{1}{2}\big[\|v(s) - u^\epsilon(s)\|^2 - \|v(0) - u_0\|^2\big] \end{aligned} \tag{4.2.77}$$

It is easy to verify that $\frac{\lambda}{2}\|v - h\|^2 - \frac{\lambda}{2}\|u^\epsilon - h\|^2 \geq \lambda \big\langle u^\epsilon, v - u^\epsilon \big\rangle$. Since $\Phi_1(\cdot), \Phi(\cdot)$ are convex, from Lemma 2.7.2, we deduce:

$$\begin{aligned} \Phi_1(|\nabla v|) - \Phi_1(|\nabla u^\epsilon|) &\geq \big\langle g_1(\nabla u^\epsilon), \nabla v - \nabla u^\epsilon \big\rangle \\ \Phi(|\nabla^2 v|) - \Phi(|\nabla^2 u^\epsilon|) &\geq \big\langle g(\nabla^2 u^\epsilon), \nabla^2 v - \nabla^2 u^\epsilon \big\rangle \end{aligned} \tag{4.2.78}$$

Thus,

$$\begin{aligned} &\hat{J}_h(v) - \hat{J}_h(u^\epsilon) + \epsilon \big\langle \Delta u^\epsilon, \Delta v \big\rangle \geq \hat{J}_h(v) - \hat{J}_h(u^\epsilon) + \epsilon \big\langle \Delta u^\epsilon, \Delta v - \Delta u^\epsilon \big\rangle \\ &\geq B^\epsilon[u^\epsilon, v - u^\epsilon; t] + \lambda \big\langle u^\epsilon - h, v - u^\epsilon \big\rangle \end{aligned} \tag{4.2.79}$$

Since $\Phi_1'(\cdot)$ is bounded, we have

$$\int_0^s \int_\Omega \Big|\Phi_1(|\nabla u^\epsilon|) - \Phi_1(|\nabla u|)\Big|\, dxdt \leq C \int_\Omega \big||\nabla u^\epsilon| - |\nabla u|\big|\, dxdt \tag{4.2.80}$$

From the strong convergence (4.2.69), we obtain

$$\lim_{\epsilon \to 0} \int_0^s \int_\Omega \Phi_1(|\nabla u^\epsilon|)\, dxdt = \int_0^s \int_\Omega |\Phi_1(|\nabla u|)\, dxdt \tag{4.2.81}$$

Notice the lower semicontinuity, by Fatou's lemma and the strong convergence (4.2.69),

$$
\begin{aligned}
&\liminf_{\epsilon\to 0}\int_0^s\int_\Omega \Phi(|\nabla^2 u^\epsilon|)\,dxdt \ge \int_0^s \liminf_{\epsilon\to 0}\int_\Omega \Phi(|\nabla^2 u^\epsilon|)\,dxdt \\
&\ge \int_0^s\int_\Omega \Phi(|D^2 u|)dt = \int_0^s\int_\Omega \Phi(|\nabla^2 u|)\,dxdt + \alpha\int_0^s |D_s^2 u|(\Omega)dt
\end{aligned}
\tag{4.2.82}
$$

From (4.2.66), we know that $\epsilon\|\Delta u^\epsilon\|^2$ is bounded. Hence $\epsilon\big\langle \Delta u^\epsilon, \Delta v\big\rangle \to 0$ as $\epsilon \to 0$. Combine (4.2.77), (4.2.79) and the lower semicontinuity of L^2 norm with respect to weak convergence, let $\epsilon \to 0$, we obtain

$$
\begin{aligned}
&\int_0^s\int_\Omega \dot v(v-u)\,dxdt + \int_0^s \hat J_h(v)\,dt - \int_0^s \hat J_h(u)\,dt \\
&\ge \frac{1}{2}\big[\|v(s)-u(s)\|^2 - \|v(0)-u_0\|^2\big]
\end{aligned}
\tag{4.2.83}
$$

$\forall\, v \in L^1(0,T;V)\cap L^2(0,T;L^2(\Omega))$ and $\dot v \in L^2(0,T;L^2(\Omega))$. By a density argument, we deduce (4.2.62) holds for $\forall\, v \in L^1(0,T;BH_{per}(\Omega))\cap L^2(0,T;L^2(\Omega))$ and $\dot v \in L^2(0,T;L^2(\Omega))$.

Stability inequality

The following lemma is useful in the proof of the stability inequality. It's proof is same as the proof of Lemma 3.4.8.

Lemma 4.2.7. *Let $\eta > 0$ and u_η be the solution of the following ODE:*

$$
\begin{cases}
\eta \dot u_\eta + u_\eta &= u \quad \text{for } 0 < t < T; \\
u_\eta(0) &= u_0
\end{cases}
\tag{4.2.84}
$$

If u satisfies (4.2.62) *and* $u_0 \in BH_{per}(\Omega) \cap L^2(\Omega)$*, then as* $\eta \to 0$

$$
\begin{aligned}
&u_\eta \to u \quad \textit{strongly in } L^2(0,T;L^2(\Omega)) \\
&u_\eta \to u \quad \textit{strongly in } L^1(0,T;BH_{per}(\Omega)) \\
&u_\eta(s) \to u(s) \quad \textit{strongly in } L^2(\Omega)\ \forall\, s \in [0,T]
\end{aligned}
\tag{4.2.85}
$$

Furthermore,

$$
\|\dot{u}\|^2_{L^2(0,T;L^2(\Omega))} \le \hat{J}_h(u_0) \tag{4.2.86}
$$

Now let's prove the stability inequality (4.2.63). Set

$$
u := \frac{u_1 + u_2}{2}, \quad u_0 := \frac{u_{01} + u_{02}}{2}
$$

For any $\eta > 0$, define u_η as in Lemma 4.2.7, now take $v = u_\eta$ in each inequality (4.2.62) with u_1, u_2 in place of u, u_{01}, u_{02} in place of u_0, h_1, h_2 in place of h, add them together

$$
\begin{aligned}
&-2\eta \int_0^s \|\dot{u}_\eta\|^2\, dt + \int_0^s \Big[\hat{J}_{h_1}(u_\eta) + \hat{J}_{h_2}(u_\eta) - \hat{J}_{h_1}(u_1) - \hat{J}_{h_2}(u_2)\Big]\, dt \\
&\ge \frac{1}{2}\Big[\|u_\eta(s) - u_1(s)\|^2 + \|u_\eta(s) - u_2(s)\|^2 - \frac{1}{2}\|u_{01} - u_{02}\|^2\Big]
\end{aligned}
\tag{4.2.87}
$$

Notice that $\hat{J}(\cdot)$ is a convex functional, we have

$$
\begin{aligned}
&-2\eta \int_0^s \|\dot{u}_\eta\|^2\, dt + 2\int_0^s \Big[\hat{J}(u_\eta) - \hat{J}(u)\Big]\, dt \\
&+ \frac{\lambda}{2}\int_0^s \int_\Omega \big[(u_\eta - h_1)^2 + (u_\eta - h_2)^2 - (u_1 - h_1)^2 - (u_2 - h_2)^2\big] dxdt \\
&\ge \frac{1}{2}\Big[\|u_\eta(s) - u_1(s)\|^2 + \|u_\eta(s) - u_2(s)\|^2 - \frac{1}{2}\|u_{01} - u_{02}\|^2\Big]
\end{aligned}
\tag{4.2.88}
$$

Let $\eta \to 0$ and by Lemma 4.2.7, we have

$$\begin{aligned}
&\frac{\lambda}{4}\int_0^s \int_\Omega (h_1 - h_2)^2 \, dxdt \\
&\geq \frac{\lambda}{2}\int_0^s \int_\Omega \big[(u_2 - u_1)(h_2 - h_1) - (u_1 - u_2)^2\big] dxdt \\
&= \int_0^s \int_\Omega \big[(u - h_1)^2 + (u - h_2)^2 - (u_1 - h_1)^2 - (u_2 - h_2)^2\big] dxdt \qquad (4.2.89) \\
&\geq \frac{1}{2}\Big[\|u(s) - u_1(s)\|^2 + \|u_(s) - u_2(s)\|^2 - \frac{1}{2}\|u_{01} - u_{02}\|^2\Big] \\
&= \frac{1}{4}\Big[\|u_1(s) - u_2(s)\|^2 - \|u_{01} - u_{02}\|^2\Big]
\end{aligned}$$

Thus, (4.2.63) holds.

The case $u_0 \in BH_{per}(\Omega) \cap L^2(\Omega)$

Proof. Assume first $u_0 \in V$, for any $\epsilon > 0$, from Theorem 4.2.2, there exists u^ϵ such that (4.2.6) holds and satisfies the following energy estimates

$$\|u^\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C\big[\|u_0\| + \|h\|\big] \qquad (4.2.90)$$

$$\begin{aligned}
&\|\dot{u}^\epsilon\|_{L^2(0,T;L^2(\Omega))} + \|u^\epsilon\|_{L^\infty(0,T;BH(\Omega))} + \sqrt{\epsilon}\|\Delta u^\epsilon\| \\
&\leq C\big[\|u_0\|_{BH(\Omega)} + \|h\|\big] + \sqrt{\epsilon}\|\Delta u_0\| \qquad (4.2.91)
\end{aligned}$$

Consequently, there exists a subsequence of u^ϵ and $\dot{u}^\epsilon$ (we still use the same notation to denote the subsequence), $u \in L^\infty(0,T;BH(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$

and $\dot{u} \in L^2(0,T;L^2(\Omega))$, such that

$$\begin{aligned} u^\epsilon &\rightharpoonup u && \text{weakly in } L^2(0,T;L^2(\Omega)) \\ \dot{u}^\epsilon &\rightharpoonup \dot{u} && \text{weakly in } L^2(0,T;L^2(\Omega)) \\ u^\epsilon &\rightharpoonup u && \text{weakly in } L^2(\Omega) \text{ a.e. } t \in [0,T] \\ u^\epsilon &\to u && \text{strongly in } W^{1,p}_{per}(\Omega) \text{ a.e. } t \in [0,T] \end{aligned} \tag{4.2.92}$$

where $1 \leq p < 1^*$. The strong convergence is due to $W^{1,p}(\Omega) \subset\subset BH(\Omega)$ and the compactness result of Simon [83] (see Theorem 2.4.2). By Theorem 2.3.3, we know that, after possibly being redefined on a set of measure zero, $u \in C([0,T], L^2(\Omega))$ and $u(s_2) = u(s_1) + \int_{s_1}^{s_2} \dot{u}(t)\,dt$. On the other hand, $u^\epsilon \in C([0,T], L^2(\Omega))$ and $u^\epsilon(s_2) = u^\epsilon(s_1) + \int_{s_1}^{s_2} \dot{u}^\epsilon(t)\,dt$. Therefore

$$\big\langle u^\epsilon(s_2) - u(s_2), v \big\rangle = \big\langle u^\epsilon(s_1) - u(s_1), v \big\rangle + \int_{s_1}^{s_2} \big\langle \dot{u}^\epsilon(t) - \dot{u}(t), v \big\rangle \, dt \tag{4.2.93}$$

Let $\epsilon \to 0$, by the weak convergence $\dot{u}^\epsilon \rightharpoonup \dot{u}$ in $L^2(0,T;L^2(\Omega))$, we have

$$\lim_{\epsilon \to 0} \big\langle u^\epsilon(s_2) - u(s_2), v \big\rangle = \lim_{\epsilon \to 0} \big\langle u^\epsilon(s_1) - u(s_1), v \big\rangle \tag{4.2.94}$$

Since $u^\epsilon(s) \rightharpoonup u(s)$ in $L^2(\Omega)$ for a.e. $s \in [0,T]$, we conclude

$$u^\epsilon(s) \rightharpoonup u(s) \quad \forall\, s \in [0,T] \tag{4.2.95}$$

Replace u, v with u^ϵ and $v - u^\epsilon$ in (4.2.6) respectively, integrate against t from 0 to $s \in [0,T]$, we deduce

$$\int_0^s \big\langle \dot{u^\epsilon}, v - u^\epsilon \big\rangle \, dt + \int_0^s \big[B^\epsilon[u^\epsilon, v - u^\epsilon; t] + \lambda \big\langle u^\epsilon - h, v - u^\epsilon \big\rangle \big] \, dt = 0 \tag{4.2.96}$$

for a.e. $t \in [0,T]$. Notice the convexity of $\Phi_1(\cdot)$, $\Phi(\cdot)$, $(\cdot - h)^2$, by Lemma 2.7.1, we find

$$\begin{aligned}
&\Phi_1(|\nabla v|) - \Phi_1(|\nabla u^\epsilon|) \geq \big\langle g_1(\nabla u^\epsilon), \nabla v - \nabla u^\epsilon \big\rangle \\
&\Phi(|\nabla^2 v|) - \Phi(|\nabla^2 u^\epsilon|) \geq \big\langle g(\nabla^2 u^\epsilon), \nabla^2 v - \nabla^2 u^\epsilon \big\rangle \\
&\frac{\lambda}{2}(v-h)^2 - \frac{\lambda}{2}(u^\epsilon - h)^2 \geq \lambda(u^\epsilon - h)(v - u^\epsilon)
\end{aligned} \tag{4.2.97}$$

Therefore,

$$\begin{aligned}
&\int_0^s \big[\big\langle \dot{u^\epsilon}, v - u^\epsilon \big\rangle + \hat{J}_h(v) - \hat{J}_h(u^\epsilon) + \epsilon \big\langle \Delta u^\epsilon, \Delta v \big\rangle\big]\, dt \\
&\geq \int_0^s \big[\big\langle \dot{u^\epsilon}, v - u^\epsilon \big\rangle + \hat{J}_h(v) - \hat{J}_h(u^\epsilon) + \epsilon \big\langle \Delta u^\epsilon, \Delta v - \Delta u^\epsilon \big\rangle\big]\, dt \\
&\geq \int_0^s \big[\big\langle \dot{u^\epsilon}, v - u^\epsilon \big\rangle + B^\epsilon[u^\epsilon, v - u^\epsilon; t] + \lambda \big\langle u^\epsilon - h, v - u^\epsilon \big\rangle\big]\, dt = 0
\end{aligned} \tag{4.2.98}$$

Recall (4.2.92), from the lower semicontinuity of $\hat{J}$ in $BH(\Omega)$ with respect to convergence in $W^{1,1}(\Omega)$, we have

$$-\int_0^s \hat{J}(u)\, dt \geq -\int_0^s \liminf_{\epsilon \to 0}\ \hat{J}(u^\epsilon)\, dt \tag{4.2.99}$$

From the lower semicontinuity of L^2 norm with respect to weak convergence, we have

$$-\int_0^s \|u - h\|^2\, dt \geq -\int_0^s \liminf_{\epsilon \to 0}\ \|u^\epsilon - h\|^2\, dt \tag{4.2.100}$$

The weak convergence $\dot{u}^\epsilon \rightharpoonup \dot{u}$ implies

$$\lim_{\epsilon \to 0} \int_0^s \big\langle \dot{u}^\epsilon, v \big\rangle\, dt = \int_0^s \big\langle \dot{u}, v \big\rangle\, dt \tag{4.2.101}$$

The weak convergence $u^\epsilon(s) \rightharpoonup u(s)$ implies

$$-\|u(s)\|^2 \geq -\liminf_{\epsilon \to 0} \|u^\epsilon(s)\|^2 \tag{4.2.102}$$

From (4.2.98) and (4.2.99) - (4.2.102), notice that $\epsilon\big\langle \Delta u^\epsilon, \Delta v\big\rangle \to 0$ as $\epsilon \to 0$ we obtain

$$\int_0^s \big\langle \dot{u}, v\big\rangle\, dt + \frac{1}{2}\big[\|u_0\|^2 - \|u(s)\|^2\big] + \int_0^s \big[\hat{J}_h(v) - \hat{J}_h(u)\big]\, dt \geq 0 \qquad (4.2.103)$$

$\forall\, v \in L^2(0, T; V)$. $u_0 = u^\epsilon(0) \rightharpoonup u(0)$ implies $u(0) = u_0$. Thus

$$\int_0^s \big\langle \dot{u}, v - u\big\rangle\, dt + \int_0^s \big[\hat{J}_h(v) - \hat{J}_h(u)\big]\, dt \geq 0 \qquad (4.2.104)$$

By a standard density argument, (4.2.103) holds $\forall\, v \in L^1(0, T; BH_{per}(\Omega))$ and $\dot{v} \in L^2(0, T; L^2(\Omega))$. We just prove that (4.2.103) holds for $u_0 \in V$. For any function $u_0 \in L^2(\Omega) \cap BH_{per}(\Omega)$, notice the stability inequality, another density argument suffices.

4.3 Evolutionary PDE with $\nabla^2 u$ replaced by Δu

In higher dimensional space, the computation of $\nabla^2 u$ is quite time consuming. In order to reduce the computation cost, we consider PDEs in which $\nabla^2 u$ is replaced by Δu, here Δ denotes either the distributional derivative or weak derivative $\sum_{i=1}^{d} \partial_{x_i}^2$. Again, we restrict ourselves to consider only $\Omega = \prod_{i=1}^{d}(0, L_i)$. Consider the following evolutionary equation

$$\begin{cases} \dot{u} = \nabla\big(\frac{\Phi_1'(|\nabla u|)}{|\nabla u|}\nabla u\big) - \Delta\big(\frac{\Phi'(|\Delta u|)}{|\Delta u|}\Delta u\big) - \lambda(u - h) \\ u(t), L - \text{periodic} \\ u(0) = u_0 \end{cases} \qquad (4.3.1)$$

where Φ_1 and Φ satisfies the assumptions H.1-H.4 of section 4.1. If we let $\Phi_1 \equiv 0$, $\Phi(s) = ks \arctan(s/k) - \frac{k^2}{2}\log((s/k)^2+1)$, then $\Phi'(s) = k\arctan(s/k)$, we will recover PDE (1.1.19). Bertozzi and Greer [11] made a change of variables $w = \arctan(\Delta u)$ when $k=1$ and $\lambda = 0$ and derived the equation satisfied by w

$$\dot{w} + \cos^2 w \Delta^2 w = 0 \tag{4.3.2}$$

They first proved the existence and uniqueness to the mollified equation with periodic boundary condition

$$\begin{cases} \dot{w}^\epsilon = -J_\epsilon \cos^2 w^\epsilon \Delta^2 J_\epsilon w^\epsilon \\ w^\epsilon(0) = w_0 \end{cases} \tag{4.3.3}$$

where J_ϵ is a standard mollifier. They then derived parameter ϵ independent energy estimates and proved the existence and uniqueness of the smooth solution of (1.1.19) when initial condition $w_0 \in H^6(\Omega)$. They also pointed out that an interesting point for further study is to better understand the theory for the LCIS equation for noisy initial data. Through the vanish viscosity study of (4.3.1), we will get a clear idea on the generalized solution of (1.1.19).

4.3.1 Regularized equation and energy estimates

Now, let's consider the regularized equation:

$$\begin{cases} \dot{u} = \nabla(\frac{\Phi_1'(|\nabla u|)}{|\nabla u|}\nabla u) - \Delta(\frac{\Phi'(|\Delta u|^2)}{|\Delta u|}\Delta u) - \lambda(u-u_0) + \epsilon\Delta^2 u \\ u(t) \text{ is } L-\text{periodic} \\ u(0) = u_0 \end{cases} \tag{4.3.4}$$

Adopt the same approach as Section 4.2.1, we can prove the existence and uniqueness of weak solution and derive some ϵ independent energy estimates. V and V' are defined as Section 4.2. Define:

$$B^\epsilon[u,v;t] = \Big\langle \frac{\Phi_1'(|\nabla u|)}{|\nabla u|}\nabla u, \nabla v\Big\rangle + \Big\langle \frac{\Phi(|\Delta u|)}{|\Delta u|}\Delta u, \Delta v\Big\rangle + \epsilon\Big\langle \Delta u, \Delta v\Big\rangle \tag{4.3.5}$$

$$J^\epsilon[u,h;t] = \int_\Omega \Big[\Phi_1(|\nabla u|) + \Phi(|\Delta u|) + \frac{\lambda}{2}(u-h)^2 + \frac{\epsilon}{2}|\Delta u|^2\Big]\,dx \tag{4.3.6}$$

A weak solution of (4.3.4) is defined as $u \in L^2(0,T,V) \cap C([0,T], L^2(\Omega))$ such that $\dot{u} \in L^2(0,T,L^2(\Omega))$ and

$$\begin{cases} \Big\langle \dot{u}, v\Big\rangle + B^\epsilon[u,v;t] + \lambda\Big\langle u-h, v\Big\rangle = 0 \quad \text{a.e. } t \in [0,T]\ \forall\, v \in V \\ u(0) = u_0 \end{cases} \tag{4.3.7}$$

Theorem 4.3.1 (Existence and uniqueness). *Assume that* $u_0 \in L^2(\Omega), h \in L^2(\Omega)$, Φ_1, Φ *are smooth functions which satisfy H.1-H.4 of section 4.1, Then there is a unique weak solution of* (4.3.4) *which satisfies the following energy estimates:*

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + B^\epsilon[u,u;t] + \frac{\lambda}{2}\int_\Omega (u-h)^2\,dx \le \frac{\lambda}{2}\|h\|^2 \tag{4.3.8}$$

$$\int_0^T t\|\dot{u}\|^2\,dt + tJ^\epsilon[u,u_0;t] \le C\big(\|u_0\|^2 + \|h\|^2\big) \tag{4.3.9}$$

$$\int_0^T \|\dot{u}\|_{V'}^2\,dt \le C\big(\|u_0\|^2 + \|h\|^2\big) \tag{4.3.10}$$

Moreover, if $u_0 \in H^2(\Omega)$, we have

$$\begin{aligned}
&\int_0^t \|\dot{u}\|^2 + \alpha \int_\Omega |\Delta u|\,dx + \frac{\lambda}{2}\int_\Omega (u-h)^2\,dx + \frac{\epsilon}{2}\|\Delta u\|^2 \\
&\leq \alpha_1 \int_\Omega |\nabla u_0|\,dx + \beta_1 + \alpha \int_\Omega |\Delta u_0|\,dx \qquad (4.3.11)\\
&+ 2\beta + \frac{\epsilon}{2}\|\Delta u_0\|^2 + \frac{\lambda}{2}\int_\Omega (u_0-h)^2\,dx
\end{aligned}$$

4.3.2 Existence and uniqueness of generalized solution

Define

$$\hat{J}(u) := \int_\Omega \Big[\Phi_1(|\nabla u|) + \Phi(|\Delta u|)\Big]\,dx + \alpha|\Delta^s u|(\Omega) \tag{4.3.12}$$

$\hat{J}(u)$ is lower semicontinuous with respect to $W^{1,1}(\Omega)$ convergence.

$$\hat{J}_h(u) := \int_\Omega \Big[\Phi_1(|\nabla u|) + \Phi(|\Delta u|) + \frac{\lambda}{2}(u-h)^2\Big]\,dx + \alpha|\Delta^s u|(\Omega) \tag{4.3.13}$$

Theorem 4.3.2 (Generalized solution). *Suppose that $\Omega = \prod_{i=1}^d (0, L_i)$, Φ_1, Φ are smooth functions which satisfy H.1-H.4.*

(a) If $u_0, h \in L^2(\Omega)$, then there exists u such that

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;BL^p_{per}(\Omega))$$

$$u \in L^\infty(s_0,T;BL^p_{per}(\Omega)) \cap C([s_0,T];L^2(\Omega)),\ s_0 \in (0,T]$$

$$\dot{u} \in L^2(0,T;V')$$

$u(t)$ is weakly continuous from $[0,T] \to L^2(\Omega)$.

$\forall\, v \in L^1(0,T; BL^p_{per}(\Omega)) \cap L^2(0,T; L^2(\Omega))$ with $\dot v \in L^2(0,T; L^2(\Omega))$

$$\begin{aligned}&\int_0^s \int_\Omega \dot v(v-u)\,dxdt + \int_0^s (\hat J_h(v) - \hat J_h(u))\,dt \\ &\geq \frac{1}{2}\big[\|v(s)-u(s)\|^2 - \|v(0)-u_0\|^2\big] \quad \forall\, s \in (0,T]\end{aligned} \tag{4.3.14}$$

Such u is called a generalized solution of (4.3.1).

(b) Suppose u_1, u_2 satisfies (4.3.14) with initial data u_{01}, h_1 and u_{02}, h_2 respectively. Assume $u_{01}, u_{02} \in BL^p_{per}(\Omega) \cap L^2(\Omega)$, $h_1, h_2 \in L^2(\Omega)$ then

$$\|u_1(s) - u_2(s)\|^2 \leq \|u_{01} - u_{02}\|^2 + \lambda s \|h_1 - h_2\|^2 \quad \forall\, s \in [0,T] \tag{4.3.15}$$

(c) Furthermore, if $u_0 \in L^2(\Omega) \cap BL^p_{per}(\Omega)$, $h \in L^2(\Omega)$, then u is unique and $u \in L^\infty(0,T; BH_{per}(\Omega)) \cap C([0,T]; L^2(\Omega))$, $\dot u \in L^2(0,T; L^2(\Omega))$, $u(0) = u_0$ such that

$$\int_0^s \int_\Omega \dot u(v-u)\,dxdt + \int_0^s (\hat J_h(v) - \hat J_h(u))\,dt \geq 0 \quad s \in [0,T] \tag{4.3.16}$$

$\forall\, v \in L^1(0,T; BL^p_{per}(\Omega)) \cap L^2(0,T; L^2(\Omega))$. Thus

$$\int_\Omega \dot u(v-u)\,dx + \hat J_h(v) - \hat J_h(u) \geq 0 \quad a.e.\ t \in [0,T] \tag{4.3.17}$$

$\forall\, v \in BL^p_{per}(\Omega) \cap L^2(\Omega)$.

Remark 4.3.3. In Theorem 4.2.6 and 4.3.2, if $u_0 \in L^2(\Omega)$, u is only weakly continuous from $[0,T] \to L^2(\Omega)$. The uniqueness is usually not true. The reason is mentioned in Remark 3.4.7. By the trace theorems of BH functions and BL^p

functions in Chapter 2, it makes sense to consider the Neumann boundary value problem. But we can't prove the convergence of boundary condition. The trace operators are continuous in the norm topology, or a weaker topology so called strict (tight) convergence, but not in the weak* topology. The convergence we can obtain is weak* topology, we can't find a way to prove that the sequence does not concentrate on the boundary of the domain. Thus, we failed to prove the uniqueness of the generalized solution even u_0 is sufficiently smooth in case of Neumann boundary condition.

The proof of this theorem is essentially the same as the proof of Theorem 4.2.6. The main difference is to replace the compact embedding $W^{1,p} \subset\subset BH(\Omega)$ with $W^{1,p}_{per}(\Omega) \subset\subset BL^p_{per}(\Omega)$ which is a direct result of elliptic periodic boundary value problem.

The case $u_0, h \in L^2(\Omega)$

Proof. Fix any $\epsilon > 0$, for $u_0, h \in L^2(\Omega)$, according to Theorem 4.3.1, there exists a unique u^ϵ which satisfies the energy estimates:

$$\|u^\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|u^\epsilon\|_{L^1(0,T;BL^p_{per}(\Omega))} + \epsilon^{1/2}\Big[\int_0^T \|\Delta u^\epsilon\|^2\, dt\Big]^{1/2}$$

$$\leq C\big(\|u_0\| + \|h\|\big) \tag{4.3.18}$$

$$\|\sqrt{t}\dot{u}^\epsilon\|_{L^2(0,T;L^2(\Omega)} + \|tu^\epsilon\|_{L^\infty(0,T;BL^p_{per}(\Omega))} \leq C\big(\|u_0\| + \|h\|\big) \tag{4.3.19}$$

$$\|\dot{u}^\epsilon\|_{L^2(0,T;V')} \leq C\big(\|u_0\| + \|h\|\big) \tag{4.3.20}$$

here C is a ϵ independent constant, it may depend on T, Ω. Therefore, there exists $u \in L^\infty(0,T,L^2(\Omega))$, $\dot{u} \in L^2(0,T;V')$, $\sqrt{t}\dot{u} \in L^2(0,T;L^2(\Omega))$, $tu \in L^\infty(0,T,BL^p_{per}(\Omega))$ such that a subsequence of $\{u^\epsilon\}$ (we still denote it u^ϵ)

$$u^\epsilon \rightharpoonup u \qquad \text{weakly in } L^2(\Omega) \text{ a.e. } t \in [0,T] \tag{4.3.21}$$

$$\dot{u}^\epsilon \rightharpoonup \dot{u} \qquad \text{weakly in } L^2(0,T;V')$$

$$u^\epsilon \to u \qquad \text{strongly in } L^1(0,T;W^{1,p}_{per}(\Omega)) \tag{4.3.22}$$

$$\sqrt{t}\dot{u}^\epsilon \rightharpoonup \sqrt{t}\dot{u} \qquad \text{weakly in } L^2(0,T;L^2(\Omega))$$

$$tD^2u^\epsilon \rightharpoonup tD^2u \qquad \text{weakly* in } \mathcal{M}(\Omega) \text{ a.e. } t \in [0,T]$$

$$tu^\epsilon \to tu \qquad \text{strongly in } W^{1,p}_{per}(\Omega) \text{ a.e. } t \in [0,T] \tag{4.3.23}$$

where $1 \leq p < 1^*$. Notice that $W^{1,p}_{per} \subset\subset BL^p_{per}(\Omega)$, the strong convergence (4.3.22) and (4.3.23) are due to the compactness result of Simon [83] which is stated in Theorem 2.4.2. Since $u \in L^\infty(0,T,L^2(\Omega))$, u is continuous from $[0,T]$ into V', by Lemma 2.3.5, we know that $u(t)$ is weakly continuous from $[0,T] \to L^2(\Omega)$. Thanks to Fatou's lemma, we have

$$\liminf_{\epsilon\to 0} \int_0^T \int_\Omega |\Delta u^\epsilon|\, dxdt \geq \int_0^T \liminf_{\epsilon\to 0} \int_\Omega |\Delta u^\epsilon|\, dxdt$$

Notice that $\int_\Omega |\Delta u|$ is a special case of $\hat{J}(u)$. It is lower semicontinuous with respect to $W^{1,1}$ convergence, we obtain

$$\int_0^T \liminf_{\epsilon\to 0} \int_\Omega |\Delta u^\epsilon|\, dxdt \geq \int_0^T \int_\Omega |\Delta u| dt$$

Thus, $u \in L^1(0,T;BL^p_{per}(\Omega))$. Similarly, from (4.3.23), we know that, for any $0 < s_0 \le T$, $u \in L^\infty(s_0,T;BL^p_{per}(\Omega))$, $\dot u \in L^2(s_0,T;L^2(\Omega))$. By Theorem 2.3.3, $u \in C([s_0,T],L^2(\Omega))$. Thus, $u \in C((0,T],L^2(\Omega))$ and $u(s_2) = u(s_1) + \int_{s_1}^{s_2} \dot u(t)\,dt$, $\forall\, s_1, s_2 > 0$. On the other hand, $u^\epsilon \in C([0,T];L^2(\Omega))$ and $u^\epsilon(s_2) = u^\epsilon(s_1) + \int_{s_1}^{s_2} \dot u^\epsilon(t)\,dt$. Thus, we obtain

$$\lim_{\epsilon\to 0} \big\langle u(s_2) - u^\epsilon(s_2), v\big\rangle = \lim_{\epsilon\to 0} \big\langle u(s_1) - u^\epsilon(s_1), v\big\rangle \quad \forall\, v \in L^2(\Omega) \tag{4.3.24}$$

Consequently, $u^\epsilon(s) \rightharpoonup u(s)\;\forall\, s \in (0,T]$. For each u^ϵ, we have

$$\begin{cases} \big\langle \dot u^\epsilon, v\big\rangle + B^\epsilon[u^\epsilon, v; t] + \lambda\big\langle u^\epsilon - h, v\big\rangle = 0 \;\text{ a.e. } t \in [0,T]\;\forall\, v \in V \\ u^\epsilon(x,0) = u_0(x) \end{cases} \tag{4.3.25}$$

Substitute v by $v - u^\epsilon$ and integrate with respect to t from 0 to s, we obtain

$$\int_0^s \Big[\big\langle \dot u^\epsilon, v - u^\epsilon\big\rangle + B^\epsilon[u^\epsilon, v - u^\epsilon; t] + \lambda\big\langle u^\epsilon - h, v - u^\epsilon\big\rangle\Big]\,dt = 0 \tag{4.3.26}$$

Nevertheless,

$$\int_0^s \big\langle \dot v - \dot u^\epsilon, v - u^\epsilon\big\rangle\,dt = \frac{1}{2}\big[\|v(s) - u^\epsilon(s)\|^2 - \|v(0) - u_0\|^2\big] \tag{4.3.27}$$

Consequently, $\forall\, v \in L^1(0,T;V) \cap L^2(0,T;L^2(\Omega))$ such that $\dot v \in L^2(0,T;L^2(\Omega))$, we have

$$\begin{aligned} &\int_0^s \Big[\big\langle \dot v, v - u^\epsilon\big\rangle + B^\epsilon[u^\epsilon, v - u^\epsilon; t] + \lambda\big\langle u^\epsilon - h, v - u^\epsilon\big\rangle\Big]\,dt \\ &= \frac{1}{2}\big[\|v(s) - u^\epsilon(s)\|^2 - \|v(0) - u_0\|^2\big] \end{aligned} \tag{4.3.28}$$

It is easy to verify that $\frac{\lambda}{2}\|v-h\|^2 - \frac{\lambda}{2}\|u^\epsilon - h\|^2 \geq \lambda\big\langle u^\epsilon, v-u^\epsilon\big\rangle$. Since $\Phi_1(\cdot), \Phi(\cdot)$ are convex, from Lemma 2.7.2, we deduce:

$$\begin{aligned}\Phi_1(|\nabla v|) - \Phi_1(|\nabla u^\epsilon|) &\geq \big\langle g_1(\nabla u^\epsilon), \nabla v - \nabla u^\epsilon\big\rangle \\ \Phi(|\nabla^2 v|) - \Phi(|\nabla^2 u^\epsilon|) &\geq \big\langle g(\nabla^2 u^\epsilon), \nabla^2 v - \nabla^2 u^\epsilon\big\rangle\end{aligned} \tag{4.3.29}$$

Thus,

$$\begin{aligned}&\hat{J}_h(v) - \hat{J}_h(u^\epsilon) + \epsilon\big\langle \Delta u^\epsilon, \Delta v\big\rangle \geq \hat{J}_h(v) - \hat{J}_h(u^\epsilon) + \epsilon\big\langle \Delta u^\epsilon, \Delta v - \Delta u^\epsilon\big\rangle \\ &\geq B^\epsilon[u^\epsilon, v-u^\epsilon; t] + \lambda\big\langle u^\epsilon - h, v - u^\epsilon\big\rangle\end{aligned} \tag{4.3.30}$$

Since $\Phi_1'(\cdot)$ is bounded, we have

$$\begin{aligned}&\int_0^s \int_\Omega \Big|\Phi_1(|\nabla u^\epsilon|) - \Phi_1(|\nabla u|)\Big|\, dxdt \\ &\leq C \int_\Omega \big||\nabla u^\epsilon| - |\nabla u|\big|\, dxdt\end{aligned} \tag{4.3.31}$$

From the strong convergence (4.3.22), we obtain

$$\lim_{\epsilon\to 0}\int_0^s \int_\Omega \Phi_1(|\nabla u^\epsilon|)\, dxdt = \int_0^s \int_\Omega |\Phi_1(|\nabla u|)\, dxdt \tag{4.3.32}$$

Notice the lower semicontinuity, by Fatou's lemma and the strong convergence (4.3.22),

$$\begin{aligned}&\liminf_{\epsilon\to 0}\int_0^s \int_\Omega \Phi(|\Delta u^\epsilon|)\, dxdt \geq \int_0^s \liminf_{\epsilon\to 0}\int_\Omega \Phi(|\Delta u^\epsilon|)\, dxdt \\ &\geq \int_0^s \int_\Omega \Phi(|\Delta u|) dt\end{aligned} \tag{4.3.33}$$

From (4.3.18), we know that $\epsilon\|\Delta u^\epsilon\|^2$ is bounded. Hence $\epsilon\big\langle \Delta u^\epsilon, \Delta v\big\rangle \to 0$ as $\epsilon \to 0$. Combine (4.3.28), (4.3.30) and the lower semicontinuity of L^2 norm with respect to weak convergence, let $\epsilon \to 0$, we obtain

$$\begin{aligned}&\int_0^s \int_\Omega \dot{v}(v-u)\,dxdt + \int_0^s \hat{J}_h(v)\,dt - \int_0^s \hat{J}_h(u)\,dt \\ &\geq \frac{1}{2}\big[\|v(s)-u(s)\|^2 - \|v(0)-u_0\|^2\big]\end{aligned} \tag{4.3.34}$$

$\forall\, v \in L^1(0,T;V) \cap L^2(0,T;L^2(\Omega))$ and $\dot{v} \in L^2(0,T;L^2(\Omega))$. By a density argument, we deduce (4.3.14) holds for $\forall\, v \in L^1(0,T;BL^p_{per}(\Omega)) \cap L^2(0,T;L^2(\Omega))$ and $\dot{v} \in L^2(0,T;L^2(\Omega))$.

Stability inequality

Now let's prove the stability inequality (4.3.15).

Lemma 4.3.4. *Let $\eta > 0$ and u_η be the solution of the following ODE:*

$$\begin{cases} \eta \dot{u}_\eta + u_\eta &= u \quad \text{for } 0 < t < T; \\ u_\eta(0) &= u_0 \end{cases} \tag{4.3.35}$$

If u satisfies (4.3.14) and $u_0 \in BL^p_{per}(\Omega) \cap L^2(\Omega)$, then as $\eta \to 0$

$$\begin{aligned} &u_\eta \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \\ &u_\eta \to u \quad \text{strongly in } L^1(0,T;BL^p_{per}(\Omega)) \\ &u_\eta(s) \to u(s) \quad \text{strongly in } L^2(\Omega)\ \forall\, s \in [0,T] \end{aligned} \tag{4.3.36}$$

Furthermore,

$$\|\dot{u}\|^2_{L^2(0,T;L^2(\Omega))} \leq \hat{J}_h(u_0) \tag{4.3.37}$$

The proof of this lemma is almost identical to the proof of 3.4.8. We will omit it here. Set

$$u := \frac{u_1 + u_2}{2}, \quad u_0 := \frac{u_{01} + u_{02}}{2}$$

For any $\eta > 0$, define u_η as in Lemma 4.3.4, now take $v = u_\eta$ in each inequality (4.3.14) with u_1, u_2 in place of u, u_{01}, u_{02} in place of u_0, h_1, h_2 in place of h, add them together

$$\begin{aligned}&-2\eta\int_0^s \|\dot{u}_\eta\|^2\,dt + \int_0^s \Big[\hat{J}_{h_1}(u_\eta) + \hat{J}_{h_2}(u_\eta) - \hat{J}_{h_1}(u_1) - \hat{J}_{h_2}(u_2)\Big]\,dt \\ &\geq \frac{1}{2}\Big[\|u_\eta(s) - u_1(s)\|^2 + \|u_\eta(s) - u_2(s)\|^2 - \frac{1}{2}\|u_{01} - u_{02}\|^2\Big]\end{aligned} \tag{4.3.38}$$

Notice that $\hat{J}(\cdot)$ is a convex functional, we have

$$\begin{aligned}&-2\eta\int_0^s \|\dot{u}_\eta\|^2\,dt + 2\int_0^s \Big[\hat{J}(u_\eta) - \hat{J}(u)\Big]\,dt \\ &+\frac{\lambda}{2}\int_0^s\int_\Omega \big[(u_\eta - h_1)^2 + (u_\eta - h_2)^2 - (u_1 - h_1)^2 - (u_2 - h_2)^2\big]dxdt \\ &\geq \frac{1}{2}\Big[\|u_\eta(s) - u_1(s)\|^2 + \|u_\eta(s) - u_2(s)\|^2 - \frac{1}{2}\|u_{01} - u_{02}\|^2\Big]\end{aligned} \tag{4.3.39}$$

Let $\eta \to 0$ and by Lemma 4.3.4, we have

$$\begin{aligned}&\frac{\lambda}{4}\int_0^s\int_\Omega (h_1 - h_2)^2\,dxdt \\ &\geq \frac{\lambda}{2}\int_0^s\int_\Omega \big[(u_2 - u_1)(h_2 - h_1) - (u_1 - u_2)^2\big]dxdt \\ &= \int_0^s\int_\Omega \big[(u - h_1)^2 + (u - h_2)^2 - (u_1 - h_1)^2 - (u_2 - h_2)^2\big]dxdt \\ &\geq \frac{1}{2}\Big[\|u(s) - u_1(s)\|^2 + \|u_(s) - u_2(s)\|^2 - \frac{1}{2}\|u_{01} - u_{02}\|^2\Big] \\ &= \frac{1}{4}\Big[\|u_1(s) - u_2(s)\|^2 - \|u_{01} - u_{02}\|^2\Big]\end{aligned} \tag{4.3.40}$$

Thus, (4.3.15) holds.

The case $u_0 \in BL^p_{per}(\Omega) \cap L^2(\Omega)$

Proof. Since $u_0 \in BL^p_{per}(\Omega) \cap L^2(\Omega)$, by stability inequality, we know that there is a unique u which satisfies (4.3.14). From Lemma 4.3.4, we obtain $\dot{u} \in L^2(0, T; L^2(\Omega))$.

Since $u \in L^\infty(0,T;L^2(\Omega))$, by Theorem 2.3.3, we know that $u \in C([0,T];L^2(\Omega))$ after possibly being redefined on a set of measure zero and $u(s_2) = u(s_1) + \int_{s_1}^{s_2} \dot{u}(t)\,dt \;\forall\, s_1, s_2 \in [0,T]$. From the proof of part (a), u is the limit of a subsequence $\{u^\epsilon\}_{\epsilon>0}$ which satisfies $u^\epsilon(s_2) = u^\epsilon(s_1) + \int_{s_1}^{s_2} \dot{u}^\epsilon(t)\,dt\;, \forall\, s_1, s_2 \in [0,T]$ and $u^\epsilon(0) = u_0$. Therefore

$$\langle u(s_2) - u^\epsilon(s_2), v\rangle = \langle u(s_1) - u^\epsilon(s_1), v\rangle + \int_{s_1}^{s_2} \langle \dot{u}(t) - \dot{u}^\epsilon(t), v\rangle\,dt \;\forall\, v \in L^2(\Omega)$$

By (4.3.21), we obtain $u(t) \rightharpoonup u(t)$ in $L^2(\Omega)$ $\forall\, t \in [0,T]$. Thus $u(0) = u_0$ and

$$\int_0^s \int_\Omega \dot{v}(v-u)\,dx\,dt = \int_0^s \int_\Omega \dot{u}(v-u)\,dx\,dt \\ + \frac{1}{2}\big[\|v(s) - u(s)\|^2 - \|v(0) - u_0\|^2\big]$$

Notice that the time derivative of v has been transferred to u, (4.3.16) holds $\forall\, v \in L^1(0,T;BL^p_{per}(\Omega)) \cap L^2(0,T;L^2(\Omega))$. Now, let's prove $u \in L^\infty(0,T;BL^p_{per}(\Omega))$. First, assume that $u_0 \in H^2_{per}(\Omega)$, by energy estimate (4.3.11) and the embeddings in $BL^p_{per}(\Omega)$, we know that the unique generalized solution u can be regarded as the limit of a sequence $\{u^\epsilon\}_{\epsilon>0}$ which satisfies

$$\hat{J}_h(u^\epsilon) \le C\Big[\int_\Omega |\Delta u_0|\,dx + \frac{\lambda}{2}\|u_0 - h\|^2 + \frac{\epsilon}{2}\|\Delta u_0\|^2\Big]$$

Thus, we get $\hat{J}_h(u) \le C\big[\int_\Omega |\Delta u_0|\,dx + \frac{\lambda}{2}\|u_0 - h\|^2\big]$ for a.e. $t \in [0,T]$. For $u_0 \in L^2(\Omega) \cap BL^p_{per}(\Omega)$, there exists a sequence of $\{u_0^n\}_{n=1}^\infty \subset H^2_{per}(\Omega)$ such that

$$u_0^n \to u_0 \text{ strongly in } L^2(\Omega)$$

$$u_0^n \to u_0 \text{ strictly in } BL^p_{per}(\Omega)$$

Using the lower semicontinuity of $\hat{J}_h$, we obtain that $\hat{J}_h(u) \leq C\big[\int_\Omega |\Delta u_0|\,dx + \frac{\lambda}{2}\|u_0 - h\|^2\big]$ for a.e. $t \in [0,T]$ still holds. Thus $u \in L^\infty(0,T; BL^p_{per}(\Omega))$.

Chapter 5

The study of PDEs derived from nonconvex functional

In Chapter 3 and Chapter 4, we have mainly studied the properties of convex functional and corresponding PDEs (PDEs derived from the Euler-Lagrange equation of minimization problems). In practice, some nonconvex functional minimization often perform better than convex functional minimization in image smoothing [8]. Unfortunately, the study of the corresponding evolutionary PDEs is much more challenging because they even do not satisfy the parabolicity condition. A well known example is (1.1.10) proposed by Perona and Malik [76]. In this chapter, we will use regularization method to study a class of evolutionary PDEs which do not satisfy parabolicity condition. Following Galerkin method, we prove the existence of the weak solution of the regularized equation and obtain energy estimates. These energy estimates are usually are ϵ dependent which are different from the energy estimates in Chapter 3 and 4. Thus, we couldn't vanish the regularization term as before.

5.1 Smoothing-enhancing PDEs

In one space dimension, if $|\partial_x u| \le k$, the Perona-Malik PDE (1.1.10) is of forward parabolic type, and backward parabolic type for $|\partial_x u| > k$. In the backward region, Perona-Malik PDE resembles the backward diffusion equation $\dot{u} = -\partial_{xx} u$, a classical example for an ill-posed equation. In the same way as forward diffusion smoothes contrasts, backward diffusion enhances them. Thus, the Perona-Malik PDE may sharpen edges, if their gradient is larger than the contrast parameter k. Kichenassamy [52] limited himself to one space dimension and proved that (1.1.10) doesn't have a global weak solution. "The restriction to one space dimension is not a significant one: if the equation has no solution in this case, the only alternative would be to imagine that there is a solution which depends explicitly on y when it's initial condition does not — the equation therefore introduce new features. Such behavior, however, not observed numerically". Later, he proposed a notation of generalized solutions, which are piecewise linear and contain jumps. Kawohl and Kutev [51] proved that the Perona-Malik PDE does have a unique weak solution which is continuously differentiable, satisfies a maximum-minimum principle, and which is exists for some finite time, but not for the entire interval $[0, \infty)$. It is an open question whether the smooth Kawohl-Kutev solution, which exists for some finite time, turns into such a discontinuous one afterwards. Interestingly, practical implementation of the Perona-Malik model work often better than one would expected from theory. In the following, we add a fourth order term to a more general equation and study the property of the regularized equation. From now on, let's make some general assumptions on $g(\cdot)$:

$$\begin{cases} g(s) : \mathbb{R} \to \mathbb{R} \ \text{ is a } C^1 \text{ function} \\ |g(s)| \le C, \quad \forall\, s \in \mathbb{R} \\ |sg'(s)| \le C, \quad \forall\, s \in \mathbb{R}. \end{cases} \tag{5.1.1}$$

Obviously, $g(s^2) = \frac{1}{1+(s/k)^2}$ or $g(s^2) = e^{-(s/k)^2}$ satisfy these conditions. Let $\Phi(s) = \int_0^s \tau g(\tau^2) d\tau$. For simplicity, in this chapter, we assume that $\Omega = \prod_{i=1}^{d}(0, L_i)$. C is a constant which could depend on Ω, T and ϵ and may differ from line to line.

5.1.1 Existence and uniqueness of weak solution of regularized equation

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (g(|\nabla u|^2)\nabla u) - \epsilon \Delta^2 u & \text{on} \quad \Omega \times (0, +\infty) \\ u(x, 0) = u_0(x) & \text{on} \quad \Omega \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega \times (0, +\infty) \\ \partial_\nu \Delta u = 0 & \text{on} \quad \partial\Omega \times (0, +\infty) \end{cases} \tag{5.1.2}$$

where ν is the unit normal of the boundary of the domain Ω pointing outward. Suppose that $⨍_\Omega u_0(x)\, dx = \mu_0$ and $u(x,t)$ is a solution of (5.1.2), let $v(x,t) = u(x,t) - \mu_0$, it is easy to verify that v satisfies (5.1.2) and $\int_\Omega v(x, 0)\, dx = 0$ with initial condition $v(x, 0) = u_0(x) - \mu_0$. Therefore, without loss of generality, we assume $\int_\Omega u_0(x)\, dx = 0$.

Lemma 5.1.1. *If $u(x,t)$ is a solution of problem* (5.1.2), *then it satisfies*

$$\frac{d}{dt}\int_{\Omega} u(x,t)\,dx = 0$$

Proof. From (5.1.2), we have

$$\int_{\Omega} \dot{u}(x,t)\,dx = \int_{\Omega} \nabla\cdot(g(|\nabla u|^2)\nabla u)\,dx - \int_{\Omega} \epsilon\Delta^2 u\,dx$$

By Green's formula,

$$\int_{\Omega} \nabla\cdot(g(|\nabla u|^2)\nabla u)\,dx = \int_{\partial\Omega} g(|\nabla u|^2)\nabla u\cdot\nu\,ds = 0$$

$$\int_{\Omega} \Delta^2 u\,dx = \int_{\partial\Omega} \nabla(\Delta u)\cdot\nu\,ds = 0$$

Thus $\frac{d}{dt}\int_{\Omega} u(x,t)\,dx = 0$. i.e. $\int_{\Omega} u(x,t)\,dx = \int_{\Omega} u_0(x)\,dx$.

Define:

$$H_n^2(\Omega) = \left\{ v\in H^2(\Omega) : \int_{\Omega} v\,dx = 0\,,\ \partial_\nu v|_{\partial\Omega} = 0 \right\} \tag{5.1.3}$$

It is easy to see that $H_n^2(\Omega)$ is a Hilbert space. Denote $(H_n^2(\Omega))'$ the dual space of $H_n^2(\Omega)$. Assume that

$$\omega_k(x) = \prod_{i=1}^{d} \sqrt{\frac{2}{L_i}} cos(\pi k_i \frac{x_i}{L_i}) \tag{5.1.4}$$

with $x = (x_1,\cdots,x_d)^t$, $k = (k_1,\cdots,k_d)$. Thus $\{\omega_k\}_{|k|=1}^{\infty}$ is orthogonal basis of $H_n^2(\Omega)$ and is normalized under $L^2(\Omega)$ norm.

$$V_m \equiv \text{span}\,\{\omega_k : 1\le |k| \le m\} \tag{5.1.5}$$

Definition 5.1.2 (Weak solution). A function $u : \Omega\times[0,T]\to\mathbb{R}$ is called a weak solution of the initial boundary value problem (5.1.2), if

(a) $u \in L^2(0,T;H_n^2(\Omega)) \cap C([0,T];L^2(\Omega))$ and $\dot{u} \in L^2(0,T;(H_n^2(\Omega))')$;

(b) For any $v \in H_n^2(\Omega)$, a.e. $t \in [0,T]$,

$$\langle \dot{u}, v \rangle + \langle g(|\nabla u|^2)\nabla u, \nabla v \rangle + \epsilon \langle \Delta u, \Delta v \rangle \, dt = 0 \tag{5.1.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(\Omega)$ or the action of a distribution on a test function.

(c) $u(x,0) = u_0(x)$

Lemma 5.1.3 (Generalized Poincaré Inequality[87]). *If Ω is bounded and Lipschitz set in $\mathbb{R}^n$, and let p be a continuous seminorm on $H^1(\Omega)$ which is a norm on the constants ($p(a) = 0, a \in \mathbb{R}$). Then there exists a constant $c(\Omega)$ depending only on Ω such that*

$$\|u\| \leq c(\Omega)(\|\nabla u\| + p(u)), \quad \forall \, u \in H^1(\Omega). \tag{5.1.7}$$

Specifically, $p(u) = \left| \int_\Omega u(x) \, dx \right|$. Thus, $\forall \, u \in H_n^2(\Omega)$, we have $\|u\| \leq c(\Omega)\|\nabla u\|$.

Lemma 5.1.4. *For any $v \in H_n^2(\Omega)$ that*

$$\|\nabla v\|^2 \leq \|v\| \|\Delta v\| \tag{5.1.8}$$

$$C_2 \|\Delta v\|^2 \leq \|v\|_{H^2(\Omega)}^2 \leq C_1 \|\Delta v\|^2 \tag{5.1.9}$$

Proof. For (5.1.8), by Green's Formula,

$$\begin{aligned} \|\nabla v\|^2 &= \int_\Omega \nabla v \cdot \nabla v \, dx = -\int_\Omega v \Delta v \, dx + \int_{\partial\Omega} v \nabla v \cdot \nu \, ds \\ &= -\int_\Omega v \Delta v \, dx \leq \int_\Omega |v||\Delta v| \, dx \leq \|v\| \|\Delta v\|. \end{aligned}$$

The left hand side of (5.1.9) is trivial. From Temam [87, p. 154], we obtain, for any $\delta > 0$

$$\|v\|^2_{H^2(\Omega)} \le C_1(\|\Delta v\|^2 + \delta\|v\|^2)$$

Let $\delta = \frac{1}{2C_1}$,

$$\frac{1}{2}\|v\|^2_{H^2(\Omega)} \le \|v\|^2_{H^2(\Omega)} - \frac{1}{2}\|v\|^2 \le C_1\|\Delta v\|^2$$

Thus, (5.1.9) holds.

Lemma 5.1.5. *Let $\mathcal{P}_m$ be the $L^2(\Omega)$ projection operator onto V_m, $\forall\, u \in H^2_n(\Omega)$, we have*

$$\|\mathcal{P}_m u\|_{H^2(\Omega)} \le C\|u\|_{H^2(\Omega)}, \qquad \lim_{m\to\infty} \|\mathcal{P}_m u - u\|_{H^2(\Omega)} = 0$$

Proof. Let $\{\omega_k(x)\}_{|k|=1}^{\infty}$ be the orthogonal basis of $H_n(\Omega)$ which is defined in (5.1.4), then

$$u = \sum_{|k|=1}^{\infty} a_k w_k(x), \quad \mathcal{P}_m u = \sum_{|k|=1}^{m} a_k w_k(x)$$

Therefore,

$$\|\mathcal{P}_m u\|^2 \le \|u\|^2, \qquad \lim_{m\to\infty} \|\mathcal{P}_m u - u\| = 0$$

It is easy to verify that $\Delta w_k \in V_m$ for $1 \le |k| \le m$ and $\Delta w_k \notin V_m$ if $|k| > m$ (recall that $\omega_k(x)$ is the cosine sequence). Thus $\Delta\mathcal{P}_m u = \mathcal{P}_m \Delta u$. Consequently

$$\|\Delta\mathcal{P}_m u\|^2 \le \|\Delta u\|^2, \qquad \lim_{m\to\infty} \|\Delta\mathcal{P}_m u - \Delta u\| = 0$$

By (5.1.9), we have

$$\|\mathcal{P}_m u\|_{H^2(\Omega)} \le C\|u\|_{H^2(\Omega)}, \qquad \lim_{m\to\infty} \|\mathcal{P}_m u - u\|_{H^2(\Omega)} = 0$$

Theorem 5.1.6 (Galerkin approximation). *Let $u_0 \in L^2(\Omega)$, For each integer $m \geq 1$, there exists a unique $u_m : \Omega \times [0, T] \to \mathbb{R}$ such that*

(a) $u_m \in C^\infty(\bar{\Omega} \times [0, T])$ *and* $u_m(t) \in V_m$ *for any* $t \in [0, T]$.

(b) For any $v \in V_m$ *and any* $t \in [0, T]$

$$\langle v, \dot{u}_m \rangle + \langle \nabla v, g(|\nabla u_m|^2) \nabla u_m \rangle + \epsilon \langle \Delta v, \Delta u_m \rangle = 0 \tag{5.1.10}$$

(c) $u_m(0) = \mathcal{P}_m u_0$*; where* $\mathcal{P}_m$ *is the projection to finite subspace* V_m.

(d) u_m *satisfies energy estimate*

$$\|u_m\|_{L^\infty(0,T;L^2(\Omega))} + \|u_m\|_{L^2(0,T;H^2(\Omega))} + \|\dot{u}_m\|_{L^2(0,T;(H^2_n(\Omega))')} \leq C \tag{5.1.11}$$

Furthermore, if $u_0 \in H^2(\Omega)$*, we have*

$$\|u_m\|_{L^\infty(0,T;H^2(\Omega))} + \|u_m\|_{L^2(0,T;H^2(\Omega))} + \|\dot{u}_m\|_{L^2(0,T;L^2(\Omega))} \leq C \tag{5.1.12}$$

Galerkin Approximation. The proof is following [57]. Fix now a positive integer m, let $s(m) = \dim(V_m)$. We will look for a function $u_m : [0, T] \to H^2_n(\Omega)$ of the form

$$u_m(t) := \sum_{k=1}^{s(m)} a^k_m(t) \omega_k$$

where we hope to select the coefficient $a^k_m(t)$ $(0 \leq t \leq T, k = 1, \cdots, s(m))$, such that

$$\begin{cases} \langle \dot{u}_m, \omega_k \rangle + \langle g(|\nabla u_m|^2) \nabla u_m, \nabla \omega_k \rangle + \epsilon \langle \Delta u_m, \Delta \omega_k \rangle = 0 \\ a^k_m(0) = \langle u_0, \omega_k \rangle \end{cases} \tag{5.1.13}$$

From the orthonormality of $\{\omega_k : k = 1, \cdots, s(m)\}$, we obtain

$$\begin{cases} \frac{d}{dt} a_m^k(t) = f_m^k(a_m^1(t), \cdots, a_m^{s(m)}(t)), \quad k = 1, \cdots, s(m) \\ a_m^k(0) = \langle u_0, \omega_k \rangle, \qquad k = 1, \cdots, s(m) \end{cases} \tag{5.1.14}$$

where all $f_m^k : \mathbb{R}^{s(m)} \to \mathbb{R}(1 \le k \le s(m))$ are smooth and locally Lipschitz. It follows from the theory for initial-value problems of ordinary differential equations that there exists $T_m > 0$ such that the initial-value problem (5.1.14) has a unique smooth solution $(a_m^1(t), \cdots, a_m^{s(m)})$ for $t \in [0, T_m]$. For each $t \in [0, T_m]$, set $v = u_m(t) \in V_m$ in (5.1.10), we have

$$\frac{1}{2} \frac{d}{dt} \|u_m(t)\|^2 + \int_\Omega g(|\nabla u_m|^2) |\nabla u_m|^2 \, dx + \epsilon \|\Delta u_m\|^2 = 0 \tag{5.1.15}$$

Integrate against t, we obtain, for all $t \in [0, T_m]$,

$$\begin{aligned} &\|u_m(t)\|^2 + 2\epsilon \int_0^t \|\Delta u_m\|^2 \, dt \\ &\le \|u_0\|^2 + 2 \int_0^t \left| g(|\nabla u_m(t)|^2) \right| \|\nabla u_m(t)\|^2 \, dt \\ &\le \|u_0\|^2 + 2C \int_0^t \|\nabla u_m(t)\|^2 \, dt \\ &\le \|u_0\|^2 + \epsilon \int_0^t \|\Delta u_m\|^2 \, dt + \frac{C^2}{\epsilon} \int_0^t \|u_m\|^2 \, dt \end{aligned} \tag{5.1.16}$$

The last inequality is due to Cauchy inequality and Lemma 5.1.4. By Gronwall's inequality, we conclude $\|u_m(t)\|^2 \le C$. This, with the orthogonality of $\{\omega_k\}_{k=1}^{s(m)}$, implies that

$$\sum_{k=1}^{s(m)} \left[a_m^k(t) \right]^2 = \|u_m(t)\|^2 \le C \tag{5.1.17}$$

The solution $(a_m^1(t), \cdots, a_m^{s(m)}(t))$ of the initial-value problem (5.1.14) is thus bounded on $[0, T_m]$, hence can be uniquely extended to a smooth solution over

$[0,\infty)$. $\forall\, v \in H_n^2(\Omega)$, we can write $v = v_1 + v_2$ with $v_1 \in V_m$ and $v_2 \perp V_m$. Hence, (5.1.10) holds for any $v \in H_n^2(\Omega)$. Consequently,

$$\big\langle \dot{u}_m, v \big\rangle \leq \|g(|\nabla u_m|^2)\nabla u_m\| \|\nabla v\| + \epsilon \|\Delta u_m\| \|\Delta v\| \; a.e.\; t \in (0,T) \tag{5.1.18}$$

Notice the definition of $(H_n^2(\Omega))'$ norm

$$\|\dot{u}_m\|_{(H_n^2(\Omega))'} = \sup \big\{ \big\langle \dot{u}_m, v \big\rangle : \|v\|_{H_n^2(\Omega)} \leq 1 \big\} \tag{5.1.19}$$

We obtain

$$\|\dot{u}_m\|_{(H_n^2(\Omega))'} \leq \|g(|\nabla u_m|^2)\nabla u_m\| + \epsilon \|\Delta u_m\| \quad a.e.\; t \in (0,T) \tag{5.1.20}$$

Take square on both sides of (5.1.20) and employ Cauchy inequality, we obtain

$$\|\dot{u}_m\|^2_{(H_n^2(\Omega))'} \leq 2\big(\|g(|\nabla u_m|^2)\nabla u_m\|^2 + \epsilon \|\Delta u_m\|^2 \big) \quad a.e.\; t \in (0,T) \tag{5.1.21}$$

Integrate against t from 0 to T, we have

$$\begin{aligned} \|\dot{u}_m\|^2_{L^2(0,T;(H_n^2(\Omega))')} &\leq 2 \int_0^T \Big[\int_\Omega g(|\nabla u_m|^2)^2 |\nabla u_m|^2 \, dx\, dt + \epsilon \int_\Omega |\Delta u_m|^2 \, dx \Big] dt \\ \leq 2 \int_0^T \Big[C\|\nabla u_m\|^2 \, dt + \epsilon \|\Delta u_m\|^2 \Big] dt \leq C \end{aligned} \tag{5.1.22}$$

Notice (5.1.9), combine (5.1.22), (5.1.16) and (5.1.17), we obtain (5.1.11). If $u_0 \in H^2(\Omega)$, set $v = \dot{u}_m(t)$ in (5.1.10) to get for any $t \in [0,T]$ that

$$\|\dot{u}_m\|^2 + \frac{d}{dt} \int_\Omega \big[\Phi(|\nabla u_m|) + \frac{\epsilon}{2} |\Delta u_m|^2 \big] dx = 0 \tag{5.1.23}$$

Integrate against t, we obtain

$$\begin{aligned} &\|\dot{u}_m\|^2_{L^2(0,T;L^2(\Omega))} + \int_\Omega \Phi(|\nabla u_m|)\, dx + \frac{\epsilon}{2} \int_\Omega |\Delta u_m|^2 \, dx \\ &= \int_\Omega \Phi(|\nabla u_m(0)|)\, dx + \frac{\epsilon}{2} \int_\Omega |\Delta u_m(0)|^2 \, dx \end{aligned} \tag{5.1.24}$$

Therefore,

$$\|\dot{u}_m\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\epsilon}{2}\|\Delta u_m\|^2 \le C \tag{5.1.25}$$

From (5.1.9) of lemma 5.1.4 and(5.1.25), we obtain (5.1.12).

Theorem 5.1.7 (Existence, uniqueness, and energy identity). *Let $u_0 \in L^2(\Omega)$. Then, the initial-boundary-value problem* (5.1.2) *has a unique weak solution $u : \Omega \times [0,T] \to \mathbb{R}$. Furthermore, if $u_0 \in H^2(\Omega)$, then $u \in L^\infty(0,T;H^2(\Omega))$, $\dot{u} \in L^2(0,T,L^2(\Omega))$, for a.e. $t \in [0,T]$, u satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \int_\Omega g(|\nabla u|^2)|\nabla u|^2\,dx + \epsilon\int_\Omega |\Delta u|^2\,dx = 0 \tag{5.1.26}$$

$$\frac{d}{dt}\int_\Omega \left(\Phi(|\nabla u|) + \frac{\epsilon}{2}|\Delta u|^2\right)dx + \int_\Omega |\dot{u}|^2\,dx = 0 \tag{5.1.27}$$

Proof. It follows from Theorem 5.1.6 that there exists a sequence of functions $\{u_m\} \in L^2(0,T;H^2_n(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ with $\{\dot{u}_m\} \subset L^2(0,T;(H^2_n(\Omega))')$ such that for each $m \ge 1$, any $v_m \in V_m$,

$$\begin{cases} \langle \dot{u}_m, v_m\rangle + \langle g(|\nabla u_m|^2)\nabla u_m, \nabla v_m\rangle + \epsilon\langle \Delta u_m, \Delta v_m\rangle = 0 \\ u_m(0) = \mathcal{P}_m u_0 \end{cases} \tag{5.1.28}$$

Consequently, there exists $u \in L^2(0,T;H^2_n(\Omega))$ with $\dot{u} \in L^2(0,T;(H^2_n(\Omega))')$ such that

$$u_m \rightharpoonup u \quad \text{in } L^\infty(0,T;L^2(\Omega)) \tag{5.1.29}$$

$$\dot{u}_m \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;(H^2_n(\Omega))') \tag{5.1.30}$$

$$u_m \rightharpoonup u \quad \text{in } L^2(0,T;H^2(\Omega)) \tag{5.1.31}$$

$$u_m \to u \quad \text{in } L^2(0,T;H^1(\Omega)) \tag{5.1.32}$$

where the strong convergence (5.1.32) follows from (5.1.31) and the compactness result of Simon [83]. Therefore, part (a) of Definition 5.1.2 is satisfied. Let$v \in H_n^2(\Omega)$ and $\eta(t) \in C[0,T]$. For each $m \geq 1$, set $v_m = \mathcal{P}_m v$ in (5.1.28), multiply both sides of the resulting identity by $\eta(t)$, and integrate against t to yield,

$$\begin{aligned}&\int_0^T \big\langle \eta(t)\nabla\mathcal{P}_m v, g(|\nabla u_m(t)|^2)\nabla u_m(t)\big\rangle dt \\ &+\int_0^T \big\langle \eta(t)\mathcal{P}_m v, \dot{u}_m(t)\big\rangle dt + \epsilon\int_0^T \big\langle \eta(t)\Delta\mathcal{P}_m v, \Delta u_m(t)\big\rangle dt = 0\end{aligned} \tag{5.1.33}$$

From Lemma 5.1.5, (5.1.30), (5.1.31), we obtain

$$\begin{aligned}&\int_0^T \langle \eta(t)\mathcal{P}_m v, \dot{u}_m(t\rangle dt \to \int_0^T \big\langle \eta(t)v, \dot{u}(t)\big\rangle dt \text{ as } m\to\infty \\ &\int_0^T \big\langle \eta(t)\Delta\mathcal{P}_m v, \Delta u_m(t)\big\rangle dt \to \int_0^T \big\langle \eta(t)\Delta v, \Delta u(t)\big\rangle dt \text{ as } m\to\infty\end{aligned}$$

While

$$\begin{aligned}&\Big|\int_0^T \big\langle \eta(t)\nabla\mathcal{P}_m v, g(|\nabla u_m(t)|^2)\nabla u_m(t)\big\rangle dt \\ &-\int_0^T \big\langle \eta(t)v, g(|\nabla u(t)|^2)\nabla u(t)\big\rangle dt\Big| \\ &\leq \|\eta\|_{L^\infty(0,T)}\Big[\|\nabla\mathcal{P}_m v - \nabla v\|\int_0^T \big\|g(|\nabla u_m(t)|^2)\nabla u_m(t)\big\|\,dt \\ &+\|\nabla v\|\int_0^T \big\|g(|\nabla u_m(t)|^2)\nabla u_m(t) - g(|\nabla u(t)|^2)\nabla u(t)\big\|\,dt\Big]\end{aligned} \tag{5.1.34}$$

Notice (5.1.1) and (5.1.11), we obtain

$$\int_0^T \big\|g(|\nabla u_m(t)|^2)|\nabla u_m(t)|\big\|\,dt \leq C \tag{5.1.35}$$

$$\begin{aligned}&\big\|g(|\nabla u_m(t)|^2)\nabla u_m(t) - g(|\nabla u(t)|^2)\nabla u(t)\big\| \\ &= \big\|\big(g(|\xi|^2)I + 2g'(|\xi|^2)\xi^t\xi\big)\big(\nabla u_m(t) - \nabla u(t)\big)\big\| \\ &\leq C\|\nabla u_m(t) - \nabla u(t)\|\end{aligned} \tag{5.1.36}$$

From (5.1.34), (5.1.35) and (5.1.36), we have

$$
\begin{aligned}
&\Big| \int_0^T \big\langle \eta(t)\nabla \mathcal{P}_m v, g(|\nabla u_m(t)|^2)\nabla u_m(t)\big\rangle dt \\
&- \int_0^T \big\langle \eta(t) v, g(|\nabla u(t)|^2)\nabla u(t)\big\rangle dt \Big| \\
&\le C\|\mathcal{P}_m v - v\|_{H^2(\Omega)} + C\|u_m - u\|_{L^2(0,T;H^2(\Omega))} \\
&\to 0 \text{ as } m \to \infty
\end{aligned}
\tag{5.1.37}
$$

Therefore,

$$
\begin{aligned}
&\int_0^T \eta(t)\Big\{\big\langle v, \dot{u}(t)\big\rangle + \big\langle \nabla v, g(|\nabla u(t)|^2)\nabla u(t)\big\rangle \\
&+ \epsilon\big\langle \Delta v, \Delta u(t)\big\rangle\Big\} dt = 0
\end{aligned}
\tag{5.1.38}
$$

Since $\eta(t)$ is arbitrary, this implies (5.1.6). Notice that, after a possible modification of u on a set of measure zero, we have $u \in C([0,T]; L^2(\Omega))$ (cf. Theorem 2.3.3). Moreover, $u(t) = u(s) + \int_s^t u'(\tau)\, d\tau$ for any $s, t \in [0,T]$, where $u(t) = u(t) \in L^2(\Omega)$ and $u'(t) = \dot{u}(t)$. In (5.1.38), let $\eta(t) = -t/T + 1$ and integrate by parts against t for the first term to get

$$
\begin{aligned}
&\int_0^T \eta(t)\Big\{\big\langle \nabla v, g(|\nabla u(t)|^2)\nabla u(t)\big\rangle + \epsilon\big\langle \Delta v, \Delta u(t)\big\rangle\Big\} dt \\
&+ \int_0^T \frac{1}{T}\big\langle v, u(t)\big\rangle dt = \big\langle v, u(0)\big\rangle
\end{aligned}
\tag{5.1.39}
$$

In (5.1.28), let$v_m = \mathcal{P}_m v$ and use the same argument, we get

$$
\begin{aligned}
&\int_0^T \frac{1}{T}\big\langle \mathcal{P}_m v, u_m(t)\big\rangle dt + \int_0^T \eta(t)\big\langle \nabla \mathcal{P}_m v, g(|\nabla u_m(t)|^2)\nabla u_m(t)\big\rangle dt \\
&+ \int_0^T \eta(t)\epsilon\big\langle \Delta \mathcal{P}_m v, \Delta u_m(t)\big\rangle dt = \big\langle \mathcal{P}_m v, u_m(0)\big\rangle = \big\langle \mathcal{P}_m v, \mathcal{P}_m u_0\big\rangle
\end{aligned}
\tag{5.1.40}
$$

Let $m \to \infty$, we have

$$\begin{aligned} &\int_0^T \eta(t)\Big\{\big\langle \nabla v, g(|\nabla u(t)|^2)\nabla u(t)\big\rangle + \epsilon\big\langle \Delta v, \Delta u(t)\big\rangle\Big\}dt \\ &+ \int_0^T \frac{1}{T}\big\langle v, u(t)\big\rangle dt = \big\langle v, u_0\big\rangle \end{aligned} \tag{5.1.41}$$

Compare (5.1.38) and (5.1.41), we get $\big\langle v, u(0)\big\rangle = \big\langle v, u_0\big\rangle$. Since $v \in H_n^2(\Omega)$ is arbitrary, we have $u(0) = u_0$. Therefore, u is a weak solution. The uniqueness follows from the stability established in Theorem 5.1.8. Now if $u_0 \in H^2(\Omega)$, from energy estimate (5.1.12), we obtain $u \in L^\infty(0, T, H_n^2(\Omega))$ and $\dot{u} \in L^2(0, T; L^2(\Omega))$. The first energy identity can be obtained by setting $v = u(t)$ in (5.1.6). Notice (5.1.29) and (5.1.30), the second energy identity is obtained by letting $m \to \infty$ in (5.1.24).

Theorem 5.1.8 (Stability). *Let u_{01}, $u_{02} \in L^2(\Omega)$. Let u_1, u_2 be the given weak solutions of* (5.1.2) *with $u_1(x, 0) = u_{01}$ and $u_2(x, 0) = u_{02}$ a.e., respectively, Then,*

$$\|u_1 - u_2\|_{L^\infty(0,T;L^2(\Omega))} + \|u_1 - u_2\|_{L^2(0,T;H^2(\Omega))} \le C\|u_{01} - u_{02}\| \tag{5.1.42}$$

Proof. Let $w = u_1 - u_2$. Since u_1 and u_2 are two weak solutions, we have for any $v \in H_n^2(\Omega)$ and a.e $t \in (0, T)$ that

$$\big\langle v, \dot{w}\big\rangle + \big\langle \nabla v, g(|\nabla u_1|^2)\nabla u_1 - g(|\nabla u_2|^2)\nabla u_2\big\rangle + \epsilon\big\langle \Delta v, \Delta w\big\rangle = 0 \tag{5.1.43}$$

Since $w \in L^2(0, T; H_n^2(\Omega))$ and $\dot{w} \in L^2(0, T; L^2(\Omega))$, $\frac{d}{dt}\big\langle w, w\big\rangle = 2\big\langle w, \dot{w}\big\rangle$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \epsilon\|\Delta w\|^2 \le \|\nabla w\|^2 \le C\|w\|\|\Delta w\| \le \frac{\epsilon}{2}\|\Delta w\|^2 + \frac{C}{2\epsilon}\|w\|^2$$

By Gronwall's inequality,

$$\|w(t)\|^2 \leq e^{c(\epsilon)t}\big(\|u_{01}-u_{02}\|^2 - \int_0^t (\epsilon\|\Delta w\|^2)\,dt)\big)$$

Hence, (5.1.42) holds.

5.1.2 Relationship with other PDEs

Liu and Li [57] studied the following PDEs in the context of modeling epitaxial growth of thin films

$$\dot{u} = -\nabla\cdot\big(\frac{\nabla u}{1+|\nabla u|^2} + \epsilon\nabla u\big) \tag{5.1.44}$$

$$\dot{u} = -\nabla\cdot\big((1-|\nabla u|^2)\nabla u + \epsilon\nabla u\big) \tag{5.1.45}$$

These two PDEs are special cases of (5.1.2) if they are imposed homogeneous Neumann boundary condition. We also notice that (1.2.3) is a special case of (5.1.2). Thus, we proved the well-posedness of (1.2.3).

5.2 You-Kaveh PDE

In section Chapter 4 section 4.1, we mentioned that You and Kaveh [99] proposed a minimization functional of the form

$$\int_\Omega f(|\Delta u|)\,dx \tag{5.2.1}$$

to smoothing images. They intentionally used nonconvex function f because convex function will lead to globally planar images. Indeed, as we studied in

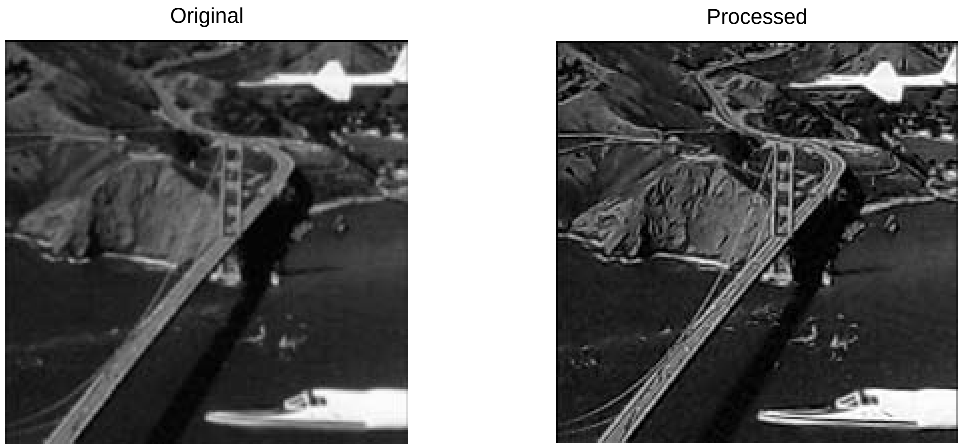

Figure 5.1: Image enhancement by flows based on triple well potentials. Figures are from `http://visl.technion.ac.il/~gilboa/ppt/huji02.pps`

Chapter 4, if f is convex and satisfies the conditions of Chapter 4, the solution of the corresponding evolutionary is in $W^{1,p}$ with $1 \leq p < 1^*$. The question is how it behaves if we use a nonconvex f? In their numerical experiment, You and Kaveh chose $g(s) = \frac{1}{1+(s/k)^2}$, here $g(s) = \frac{f'(s)}{s}$. Greer and Bertozzi [43] studied the traveling wave solution of the one dimensional You-Kaveh PDE (1.1.22) by adding a Burgers' convection term:

$$\dot{u} + \frac{1}{2}(u^2)_x = -(g(u_{xx})u_{xx})_{xx} \tag{5.2.2}$$

They proved that smooth traveling wave solution of (5.2.2) does not exist for sufficient large jump height. Following the ideas of Catte, Lions, Morel and Coll

[19] to study

$$\begin{cases} \dot{u} = \nabla \cdot \big(g(|\nabla G_\sigma * u|\nabla u\big) \quad \text{ on } (0,T) \times \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{ on } (0,T) \times \Gamma \\ u(x,0) = u_0(x) \end{cases} \tag{5.2.3}$$

it is possible to prove the well posedness of regularized You-Kaveh PDE

$$\dot{u} = \Delta\big(g(|\Delta(G_\sigma * u)|)\Delta u\big) \tag{5.2.4}$$

with L^2 initial condition and homogeneous Neumann boundary. Here G_σ is Gaussian filter.

Chapter 6

Numerical experiments

In this chapter, we use finite difference method to solve evolutionary PDEs and compare the processing results of different PDEs.

6.1 Second order method

We first describe the explicit finite difference method of second order evolutionary PDEs.

6.1.1 Explicit finite difference method in 1D

Let $u_i^k = u(i\Delta x, k\Delta t)$, $\Delta_f u_i^k = (u_{i+1}^k - u_i^k)/\Delta x$, $\Delta_b u_i^k = (u_{i-1}^k - u_i^k)/\Delta x$, $C_{f_i}^k = g(|\Delta_f u_i^k|^2)$, $C_{b_i}^k = g(|\Delta_b u_i^k|^2)$. Then the explicit finite difference discretization of (3.1.1) with $R = I$ is

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} = \frac{1}{\Delta x}(C_{f_i}^k \Delta_f u_i^k - C_{b_i}^k \Delta_b u_i^k) + \lambda(u_i^k - h_i^k) \tag{6.1.1}$$

In order to make sure the stability of the scheme, we need that $\Delta t/(\Delta x)^2 \leq \frac{1}{2}$.

6.1.2 Explicit finite difference method in 2D

Let $\Delta x = \Delta y$, $u_{i,j}^k = u(ih, jh, k\Delta t)$,

$$\begin{aligned}
\Delta_E u_{i,j}^k &= (u_{i+1,j}^k - u_{i,j}^k)/\Delta x, \quad \Delta_W u_{i,j}^k = (u_{i-1,j}^k - u_{i,j}^k)/\Delta x \\
\Delta_N u_{i,j}^k &= (u_{i,j+1}^k - u_{i,j}^k)/\Delta x, \quad \Delta_S u_{i,j}^k = (u_{i,j-1}^k - u_{i,j}^k)/\Delta x
\end{aligned} \tag{6.1.2}$$

and

$$\begin{aligned}
C_{E_{i,j}}^k &= g(|\Delta_E u_{i,j}^k|^2), \quad C_{W_{i,j}}^k = g(|\Delta_W u_{i,j}^k|^2) \\
C_{N_{i,j}}^k &= g(|\Delta_N u_{i,j}^k|^2), \quad C_{S_{i,j}}^k = g(|\Delta_S u_{i,j}^k|^2)
\end{aligned} \tag{6.1.3}$$

Then the explicit finite difference discretization of (3.1.1) with $R = I$ is [32]

$$\begin{aligned}
\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} &= \frac{1}{\Delta x}(C_{E_{i,j}}^k \Delta_E u_{i,j}^k - C_{W_{i,j}}^k \Delta_W u_{i,j}^k) \\
&+ \frac{1}{\Delta x}(C_{N_{i,j}}^k \Delta_N u_{i,j}^k - C_{S_{i,j}}^k \Delta_S u_{i,j}^k) + \lambda(u_{i,j}^k - h_{i,j}^k)
\end{aligned} \tag{6.1.4}$$

The stability of the scheme requires that

$$\Delta t/2(\Delta x)^2 \leq \frac{1}{2} \tag{6.1.5}$$

A semi-implicit scheme was proposed by Weickert [95]. It is stable even the time step and space step do not satisfy (6.1.5). Numerical experiments are carried out in one space dimension and two space dimension. The denoising results of three diffusion function are compared. They are $g(s^2) = \frac{1}{\sqrt{1+(s/k)^2}}$, the minimal surface diffusion function; $g(s^2) = \frac{\arctan(s/k)}{s}$, the Tumblin-Turk diffusion function; and $g(s^2) = \frac{1}{1+(s/k)^2}$, the Perona-Malik (You-Kaveh) diffusion function. The first two satisfies the conditions (3.1.2), while the third one does not.

6.1.3 Smoothing one dimensional signal

The original signal is a trapezoidal (Fig.6.1).

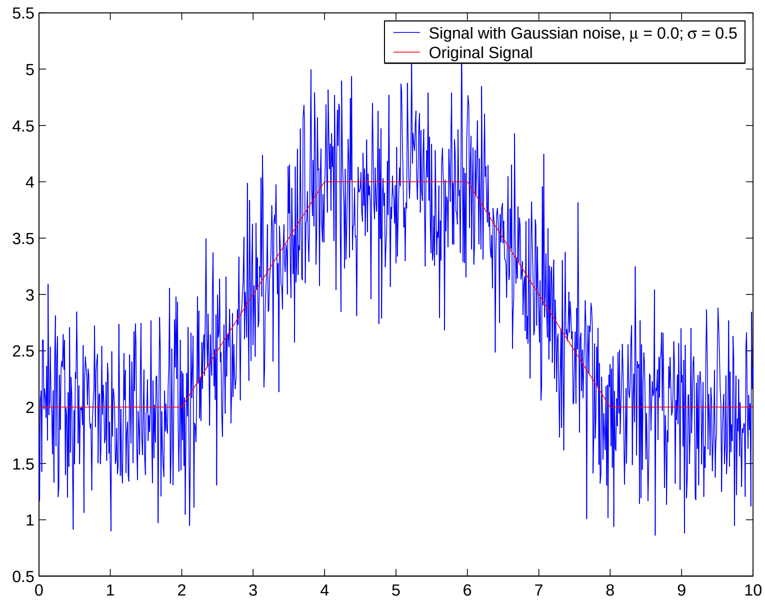

Figure 6.1: Original and noisy trapezoidal Signal

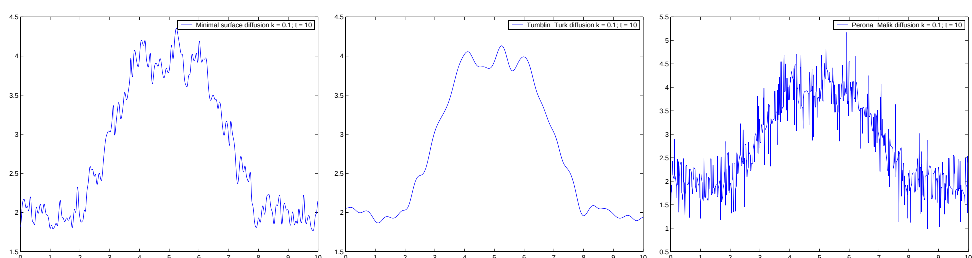

Figure 6.2: Denoising results at $t = 10$, from left to right Minimal surface, Tumblin-Turk, Perona-Malik diffusion function

6.1.4 Smoothing Lena image

Now, let's take a look at the denoising results on Lena image.

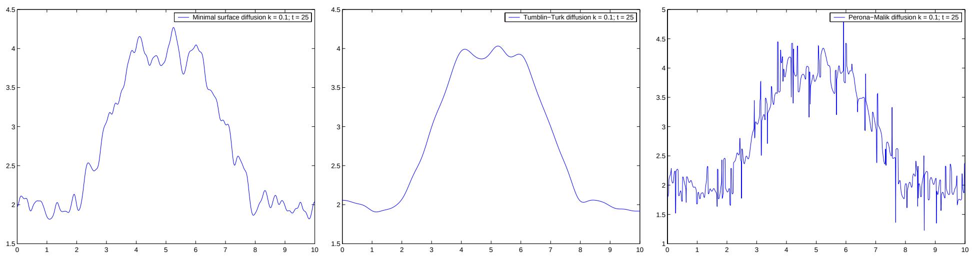

Figure 6.3: Denoising results at $t = 25$, from left to right Minimal surface, Tumblin-Turk, Perona-Malik diffusion function

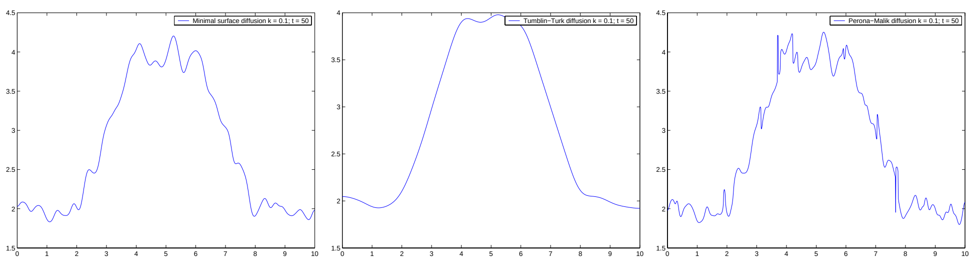

Figure 6.4: Denoising results at $t = 50$, from left to right Minimal surface, Tumblin-Turk, Perona-Malik diffusion function

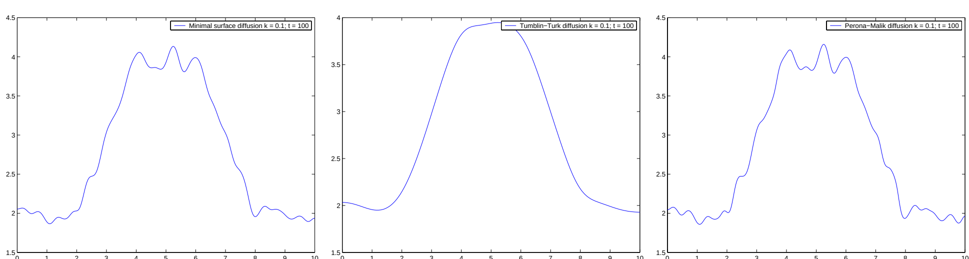

Figure 6.5: Denoising results at $t = 100$, from left to right Minimal surface, Tumblin-Turk, Perona-Malik diffusion function

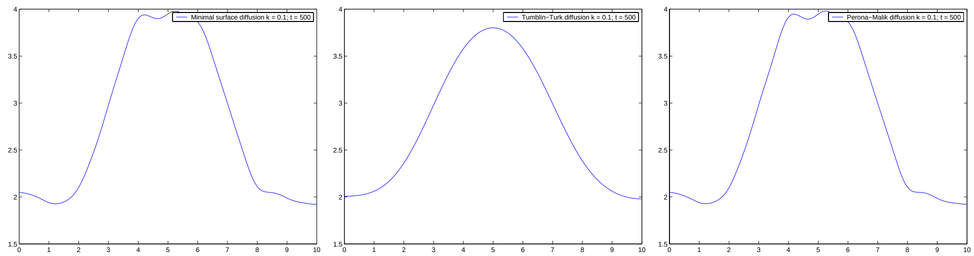

Figure 6.6: Denoising results at $t = 500$, from left to right Minimal surface, Tumblin-Turk, Perona-Malik diffusion function

Figure 6.7: Left: original Lena image; right: Lena image degraded by Gaussian noise $\sigma = 30$

Figure 6.8: Denoised image with Minimal surface ($k = 1.5$) and Tumblin-Turk ($k = 1$) diffusion function at $t = 15$

Figure 6.9: Denoised image with Perona Malik ($k = 8$) diffusion at $t = 25$ and linear diffusion at $t = 4$, i.e. Gaussian smoothing with $\sigma = 2\sqrt{2}$

Figure 6.10: Denoised image by Minimal surface ($k = 1.5$) and Tumblin-Turk ($k = 1$) diffusion function at $t = 25$

6.2 Fourth order method

Now let's describe the finite difference scheme of fourth order evolutionary PDEs. We only consider the case in Section 4.3 with $\Phi_1(\cdot) = 0$ and $g(s^2) = \frac{\Phi'(s)}{s}$.

6.2.1 Explicit finite difference method in 1D

Let $u_i^k = u(i\Delta x, k\Delta t)$, $\Delta u_i^k = (u_{i+1}^k + u_{i-1}^k - 2u_i^k)/(\Delta x)^2$. Then the explicit finite difference discretization of (4.3.1) in 1D is

$$\begin{aligned} \frac{u_i^{k+1} - u_i^k}{\Delta t} = \ & \frac{1}{(\Delta x)^2}(g(|\Delta u_{i+1}^k|^2)\Delta u_{i+1}^k + g(|\Delta u_{i-1}^k|^2)\Delta u_{i-1}^k \\ & -2g(|\Delta u_i^k|^2)\Delta u_i^k) + \lambda(u_i^k - h_i^k) \end{aligned} \tag{6.2.1}$$

6.2.2 Explicit finite difference method in 2D

Assume $\Delta x = \Delta y$, let $u_{i,j}^k = u(ih, jh, k\Delta t)$,

$$\Delta u_{i,j}^k = (u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^k)/(\Delta x)^2, \tag{6.2.2}$$

and

$$C_{i,j}^k = g(|\Delta u_{i,j}^k|^2 \Delta u_{i,j}^k \tag{6.2.3}$$

Then the explicit finite difference scheme of (4.3.1) with $\Phi_1(\cdot) = 0$ is

$$\begin{aligned} \frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} &= \frac{1}{(\Delta x)^2}(C_{i+1,j}^k + C_{i-1,j}^k + C_{i,j+1}^k + C_{i,j-1}^k - 4C_{i,j}^k)) \\ &+ \lambda(u_{i,j}^k - h_{i,j}^k) \end{aligned} \tag{6.2.4}$$

6.2.3 Smoothing one dimensional signal

The same 1D signal in 6.1.3 is being used.

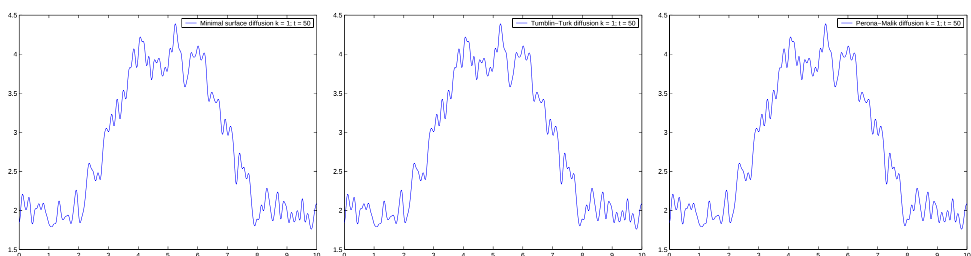

Figure 6.11: Fourth order PDE smoothing results for $k = 1$ at $t = 50$, from left to right, Minimal surface, Tumblin-Turk, You-Kaveh diffusion function.

6.2.4 Smoothing Lena image

Again, we choose Lena image as our test image.

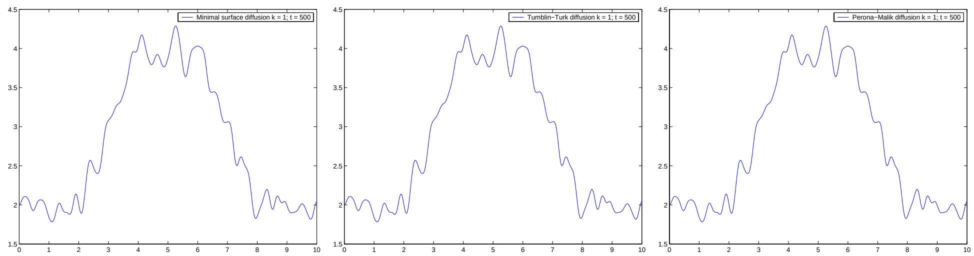

Figure 6.12: Fourth order PDE smoothing results for $k = 1$ at $t = 500$, from left to right, Minimal surface, Tumblin-Turk, You-Kaveh diffusion function.

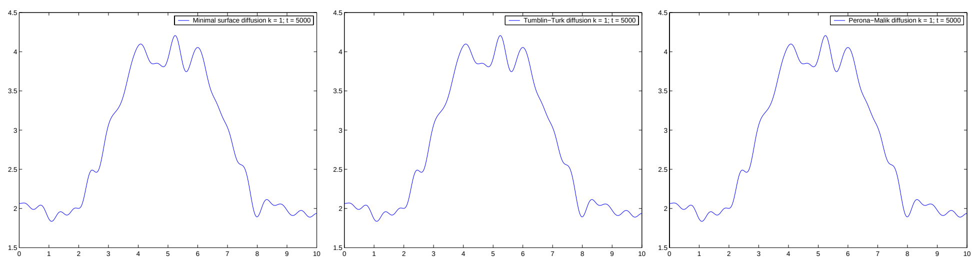

Figure 6.13: Fourth order PDE smoothing results for $k = 1$ at $t = 5000$, from left to right, Minimal surface, Tumblin-Turk, You-Kaveh diffusion function.

Figure 6.14: Fourth order PDE smoothing results for $k = 1$ at $t = 100$, from left to right, Minimal surface, Tumblin-Turk, You-Kaveh diffusion function.

Figure 6.15: Fourth order PDE smoothing results for $k = 1$ at $t = 500$, from left to right, Minimal surface, Tumblin-Turk, You-Kaveh diffusion function.

Figure 6.16: Fourth order PDE smoothing results for $k = 1$ at $t = 1000$, from left to right, Minimal surface, Tumblin-Turk, You-Kaveh diffusion function.

6.3 Conclusion

From the figures presented before, we know that the longer the diffusion time, the smoother the denoised signal. Perona-Malik diffusion function does not satisfy (3.1.2), hence we don't expect that it has a global solution in time [96, 8], but in practice, the only noticeable drawback of it is the staircase effects. The reason is that a standard discretization serves as a regularizer [96, 8]. The second order nonlinear diffusion equations do perform better than linear diffusion, they preserves edges much better than Gaussian smoothing. But we also notice that the second order method has staircase effects. For different functions $g(\cdot)$, the numerical results are very different. Especially in 1D, the result of Tumblin-Turk function is smoother than minimal surface function and Perona-Malik function. From theorem 3.4.6, we know that the solutions of the first two are in the space of functions of bounded variation, the different smoothing behaviors are due to the different nonlinear properties of them.

The fourth order PDEs take a much longer diffusion time to smooth signals and images and the computation cost of solving fourth order PDEs is much higher than solving second order PDEs. In case that the diffusion functions derived from convex functions which satisfy assumptions in Section 4.1, the smoothing results will be in $W^{1,p}$ for any diffusion time $t > 0$ as we studied in Chapter 4. Hence they do not keep edges as sharply as the second order PDEs, but they do not produce staircase effects either. For diffusion functions derived from non-convex functions such as You-Kaveh diffusion functional 1.1.21, there are speckles in the smoothing results for smaller k and relatively short diffusion time. The edges do not preserve as well as second order methods if we increase diffusion time and parameter k. You and Kaveh proposed [99] average method to post-process it to eliminate the speckles.

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