ABSTRACT

Title of Dissertation: BINORMAL MOTION OF CURVES AND SURFACES IN A MANIFOLD

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In the present work we consider a curve embedded in a three-dimensional Riemannian manifold moving in the binormal direction proportional to its curvature. We study how an appropriate orthonormal basis, the Frenet-Serret frame, along the curve evolves in order to deduce that a nonlinear Schrödinger-type equation rules the motion of the curve. Although there exist no conserved quantities, we establish, as one of our main results, local and global existence of solutions for the derived Cauchy problem on a unit circle (corresponding to closed curves) and on the real line (corresponding to open curves). We also study the motion by mean curvature of a surface embedded in the four-dimensional Euclidean space. Introducing the language of gauge fields as an appropriate framework for presenting the structural properties of the surface and the evolution equations of its geometric quantities, we derive that the complex mean curvature of the evolving surface satisfies a nonlinear Schrödiger-type equation.

BINORMAL MOTION OF CURVES AND SURFACES IN A MANIFOLD

by

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Preface

The nonlinear Schrödinger equation (NLSE) arises in various physical contexts. It plays a natural role in nonlinear fiber optics [17] describing nonlinear waves such as water waves at the free surface of an ideal fluid and plasma waves. In quantum field theory [10] it is used to study the statistical mechanics aspects of its measure preserving flow. It also appears in the study of geometric problems such as binormal motion of a curve in $\mathbb{R}^3$. This type of motion is particularly important because it preserves the length of the curve. For the particular case when the binormal velocity is proportional to the curvature, which is a crude approximation of the motion of a line vortex , Hasimoto [8] showed that this motion is governed by the celebrated nonlinear Schrödinger equation, namely

$$i\partial_\phi + \partial_{xx}\phi + \frac{1}{2}|\phi|^2\phi = 0. \tag{1}$$

This equation is completely integrable and it has propagating localized solution called solitons that have the remarkable property of retaining their shape after collisions. The one-soliton solution of equation (1) is given by the simple formula

$$\psi(t,x) = 2\nu \sec h[\nu(x - ct)],$$

where c is the propagation velocity and ν is the amplitude. Equation (1) and particular periodic solutions have been extensively studied in connection with the propagation of vortices, see [14].

It is also known, for instance, that for any $\phi_o \in H^1(\mathbb{R})$ there is a unique solution ϕ in $C(\mathbb{R}, H^1(\mathbb{R}))$ satisfying the initial $\phi(0,x) = \phi_o(x), x \in \mathbb{R}.$ Furthermore, it posses the following conserved quantities

$$N(t) \stackrel{\text{def}}{=} \int |\phi|^2(t,x)\, dx = N(0) \tag{2}$$

$$H(t) \stackrel{\text{def}}{=} \int \left(|\nabla\phi|^2(t,x) - \frac{1}{4}|\phi|^4(t,x) \right)\, dx = H(0) \tag{3}$$

and its flow is Hamiltonian. A similar result holds in the periodic case, i.e., the space variable x belongs to $\mathbb{T}$, where $\mathbb{T}$ denotes the unit circle S^1. For details, see [4].

This dissertation is divided into four chapters. Chapter 1 is concerned with the extension of Hasimoto's results to motion by binormal curvature of a curve embedded in a three-dimensional Riemannian manifold. We construct an appropriate orthonormal frame, the Frenet-Serret frame, at every point of the curve and we then study its evolution in time in order to deduce that a nonlinear Schrödinger-type equation governs the motion of the curve.

In chapter 2 we discuss global existence of a periodic solution for the Cauchy problem derived in chapter 1. The evolution equation is not Hamiltonian and in particular, there is no conservation of L^2-norm under the flow. As matter of fact, there are no obvious conserved quantities. The above observations mean that the standard global existence argument does not work in the present case. Therefore, we reformulate the initial problem as an equivalent integral equation and use Picard's fixed point theorem in order to construct a local solution with initial data in $L^2(\mathbb{T})$. Careful analysis of the growth of the size of L^2-norm of a local solution previously constructed allow us to extend it globally.

Chapter 3 considers existence of a global solution for the same Cauchy problem as in Chapter 2 but with the space variable x in $\mathbb{R}$. We start by constructing a local solution using an approximate sequence and then we extend it globally by analyzing carefully again the size of the L^2-norm of the local solution.

In chapter 4 we study the motion by mean curvature of a surface embedded in the four-dimensional Euclidean space $\mathbb{R}^4$. We introduce the language of gauge fields as an appropriate framework for presenting the structural properties of the surface and the evolution equations of its geometric quantities. We finally derive that a nonlinear Schrödiger-type equation is satisfied by the introduced complex mean curvature of the evolving surface.

DEDICATION

A mi amado hijo Diego y a mis preciosas nietas Ivanna y Arianna.

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All my love and my gratitude to my wife Margarita. She has been with me in every stage of my graduate education. She has supported me without hesitation or boundaries. One of the main reasons I have finished is her belief that I could do it. *Margaret, lo logramos.*

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Chapter 1

Binormal Motion of a Curve in a Three-dimensional Riemannian Manifold

1.1 Introduction

The present chapter studies the motion by binormal mean curvature of a curve embedded in a three-dimensional Riemannian manifold. Since the motion is related to the local geometry of the curve, it is advantageous to describe it in terms of intrinsic properties that do not depend on the choice of a coordinate system. The basic idea is to construct an appropriate frame, the Frenet-Serret frame, at every point of the curve and then examine its infinitesimal changes. The first section is dedicated to achieve this goal. The next step will be to deduce the evolution equations for the Frenet-Serret frame, the curvature, and the torsion. Finally, under some assumptions on the metric, we derive a nonlinear Schrödinger equation that governs the motion of the curve.

1.2 Structural Equations

Let us consider a curve ζ moving in a three-dimensional Riemannian manifold M. We describe the curve by $X(t, s)$, where t is the time variable and s, a parameter, is the space variable. We assume that $X : \Omega \mapsto M$ is a smooth map with $(t, s) \in \Omega$. Here Ω is either $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{T}$. $\mathbb{T}$ stands for the unit circle S^1.

We define

$$\mathbf{X}_s \stackrel{\text{def}}{=} dX(\boldsymbol{\partial}_s), \qquad \mathbf{X}_t \stackrel{\text{def}}{=} dX(\boldsymbol{\partial}_t). \tag{1.2.1}$$

$\mathbf{X}_s$, $\mathbf{X}_t$ are vector fields on M. If X lies in the domain of a chart (U, φ), $\varphi = (x^1, x^2, x^3)$, then its coordinates functions $X^i = x^i \circ X, i = 1, 2, 3$, are real-valued functions on U and we can write

$$\mathbf{X}_s \;=\; \partial_s X^i \;\boldsymbol{\partial}_i, \qquad \mathbf{X}_t \;=\; \partial_t X^i \;\boldsymbol{\partial}_i, \tag{1.2.2}$$

where summation over repeated indices is assumed and $\boldsymbol{\partial}_i$ is the ith coordinate vector field of φ.

The components of the metric h (first fundamental form) of M on U are given by

$$h_{ij} \;=\; \left\langle \boldsymbol{\partial}_i, \boldsymbol{\partial}_j \right\rangle \qquad (1 \leq i, j \leq 3). \tag{1.2.3}$$

At each point of U, the matrix h_{ij} is positive definite and invertible and its inverse matrix will be denoted by h^{ij}, i.e.,

$$(h_{ij})^{-1} = (h^{ij}) \quad ; \quad h_{ik}h^{kj} = \delta_i^{\ j}.$$

For a smooth vector field $\mathbf{Z}$ on X, its partial covariant derivatives $\nabla_{\mathbf{X}_s}\mathbf{Z}$ and $\nabla_{\mathbf{X}_t}\mathbf{Z}$ are, respectively, the covariant derivative of $\mathbf{Z}$ along s-parameter curves and the covariant derivative of $\mathbf{Z}$ along t-parameter curves. In terms of coordinates, $\mathbf{Z}$ is expressed as $\mathbf{Z} = Z^i \boldsymbol{\partial}_i$ and we therefore have

$$\nabla_{\mathbf{X}_\alpha} Z^k = \left[\partial_\alpha Z^k + \Gamma^k_{ij} Z^i \partial_\alpha X^j\right], \quad \alpha \in \{s, t\}, \tag{1.2.4}$$

where Γ_{ij}^k are Christoffel symbols. $\nabla_{\mathbf{X}_\alpha}\mathbf{Z}$ will be also denoted by either $\mathbf{Z}_\alpha$ or $\nabla_\alpha\mathbf{Z}$. The following well known facts will be very useful in the sequel.

Proposition 1.2.1. *(a)*

$$\nabla_{\mathbf{X}_t}\mathbf{X}_s = \nabla_{\mathbf{X}_s}\mathbf{X}_t. \tag{1.2.5}$$

(b) If $\mathbf{Z}$ is an arbitrary vector field on M, then

$$\nabla_{\mathbf{X}_t}\nabla_{\mathbf{X}_s}\mathbf{Z} - \nabla_{\mathbf{X}_s}\nabla_{\mathbf{X}_t}\mathbf{Z} \overset{def}{=} [\nabla_{\mathbf{X}_t}, \nabla_{\mathbf{X}_s}]\,\mathbf{Z} = R(\mathbf{X}_t, \mathbf{X}_s)\mathbf{Z}, \tag{1.2.6}$$

where $R^l{}_{ijk}$ is the Riemannian curvature tensor of M.

Proof. From (1.2.2) and (1.2.4) we have

$$\nabla_{\mathbf{X}_t}\nabla_{\mathbf{X}_s}\mathbf{Z} = \left[\partial_{st}X^k + \Gamma_{ij}^k\partial_s X^i\partial_t X^j\right]\boldsymbol{\partial}_k. \tag{1.2.7}$$

Since Γ_{ij}^k is symmetric in i and j, (1.2.7) is symmetric in s and t. This proves (1.2.5).

For the proof of identity (1.2.6), a coordinate computation gives

$$\begin{aligned}
[\nabla_{\mathbf{X}_t}, \nabla_{\mathbf{X}_s}]\,\mathbf{Z} &= \left[\partial_t\left(\partial_s Z^k + \Gamma_{ij}^k Z^i\partial_s X^j\right) + \Gamma_{ij}^k\left(\partial_s Z^i + \Gamma_{lm}^i Z^l\partial_s X^m\right)\partial_t X^j\right]\boldsymbol{\partial}_k \\
&= \left[\partial_t\left(\Gamma_{li}^i\right)Z^l\partial_s X^i - \partial_s\left(\Gamma_{lj}^k\right)Z^l\partial_t X^j\right]\boldsymbol{\partial}_k \\
&\quad + \left[\Gamma_{ij}^k\Gamma_{lm}^i Z^l\partial_s X^m\partial_t X^j - \Gamma_{ij}^k\Gamma_{lm}^i Z^l\partial_t X^m\partial_s X^j\right]\boldsymbol{\partial}_k.
\end{aligned}$$

The proof is completed by noticing that the last formula on the right-hand side is the corresponding expression of $R(\mathbf{X}_t, \mathbf{X}_s)\mathbf{Z}$ in coordinates. See [1, page 102]

$\square$

Remark. We also utilize $\nabla_{[t}, \nabla_{s]}$ to denote the commutator $[\nabla_{\mathbf{X}_t}, \nabla_{\mathbf{X}_s}]$.

Let us now assume that s is the arc length. This assumption implies that $\mathbf{X}_s$ is a unit tangent vector to the curve ζ at any point and consequently it will be called the *tangent* vector and denoted by $\mathbf{T}$. It is worthwhile to note that

$$|\mathbf{T}|^2 = h_{ij}\partial_s X^i \partial_s X^j = 1.$$

It follows that the vector field $\mathbf{T}_s = \nabla_{\mathbf{X}_s}\mathbf{T}$ given by

$$T_s^k = \partial_{ss}X^k + \Gamma_{ij}^k \partial_s X^i \partial_s X^j \tag{1.2.8}$$

must be orthogonal to $\mathbf{T}$. The *curvature* κ of the curve ζ is defined to be the length of $\mathbf{T}_s$ and the *normal* vector $\mathbf{N}$ as the unit vector in the direction of $\mathbf{T}_s$, i.e., we define

$$\kappa \overset{\text{def}}{=} |\mathbf{T}_s| \ , \quad N^k \overset{\text{def}}{=} \frac{1}{\kappa} T_s^k \ . \tag{1.2.9}$$

provided of course that the vector $\mathbf{T}_s$ does not vanish. In order to construct a vector field orthogonal to $\mathbf{T}$ and $\mathbf{N}$, we need to define a totally antisymmetric tensor on M. For this, let ϵ_{rjk} be the antisymmetric tensor

$$\epsilon_{rjk} = \begin{cases} 1 & \text{if } (r,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (r,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{otherwise.} \end{cases}$$

We define

$$\epsilon^i_{jk} = h^{ir}\epsilon_{rjk}. \tag{1.2.10}$$

Now, we can set the vector field $\mathbf{B}$ on ζ to be

$$B^i \stackrel{\text{def}}{=} \epsilon^i_{jk} T^j N^k. \tag{1.2.11}$$

$\mathbf{B}$ is called the *binormal vector field* and it is orthogonal to $\mathbf{T}$ and to $\mathbf{N}$. A straightforward but tedious computation shows $|\mathbf{B}| = 1$. The three orthogonal unit vectors $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$ define an orthonormal basis of the tangent space to M at any point of the curve and it is called the *Frenet trihedron*. This frame satisfies the *Frenet-Serret equations*:

$$\nabla_{\mathbf{X}_s}\mathbf{T} = \kappa\mathbf{N}, \quad \nabla_{\mathbf{X}_s}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}, \quad \nabla_{\mathbf{X}_s}\mathbf{B} = -\tau\mathbf{N}, \tag{1.2.12}$$

where τ, called the *torsion*, is the projection of $\mathbf{N}_s$ on the binormal $\mathbf{B}$. That is, we set

$$\tau \stackrel{\text{def}}{=} \big\langle \nabla_{\mathbf{X}_s}\mathbf{N}, \mathbf{B} \big\rangle. \tag{1.2.13}$$

Notice that the projections of the vector $\nabla_{\mathbf{X}_s}\mathbf{B}$ on $\mathbf{T}$ and $\mathbf{N}$ are fixed by the identities $\partial_s\left(\big\langle \mathbf{T}, \mathbf{B} \big\rangle\right) = 0$ and $\partial_s\left(\big\langle \mathbf{T}, \mathbf{B} \big\rangle\right) = 0$. Curvature and torsion define locally the rotation of the Frenet frame and they therefore determine the shape of a smooth curve. If the functions $\kappa(., t)$ and $\tau(., t)$ are known, the curve can be reconstructed, in principle, by integrating equations (1.2.12), which are a system of ODE's.

The complexified version of the Frenet-Serret equations (1.2.12) is obtained by expressing them through the complex null vector

$$\mathbf{m} \stackrel{\text{def}}{=} \mathbf{N} + i\mathbf{B}. \tag{1.2.14}$$

We adopt the convention for the inner product of two complex vectors $\mathbf{a}$ and $\mathbf{b}$

$$\big\langle \mathbf{a}, \mathbf{b} \big\rangle \stackrel{\text{def}}{=} h_{ij} a^i b^j = \big\langle \mathbf{b}, \mathbf{a} \big\rangle, \tag{1.2.15}$$

where a^k and b^k are the complex component of $\mathbf{a}$ and $\mathbf{b}$ respectively. The following orthogonality relations for $\mathbf{m}$ are immediate

$$\langle \mathbf{m}, \overline{\mathbf{m}} \rangle = 2 \quad , \quad \langle \mathbf{m}, \mathbf{m} \rangle = 0 \quad , \quad \langle \mathbf{T}, \mathbf{m} \rangle = 0.$$

Here $\overline{\mathbf{m}}$ denotes the complex conjugate of $\mathbf{m}$. In the present context equations (1.2.12) are therefore transformed into a compact form

$$\nabla_{\mathbf{X}_s} \mathbf{T} = \frac{1}{2} \kappa \left(\mathbf{m} + \overline{\mathbf{m}} \right), \tag{1.2.16a}$$

$$\nabla_{\mathbf{X}_s} \mathbf{m} = -\kappa \mathbf{T} - i\tau \mathbf{m}. \tag{1.2.16b}$$

Let us now introduce an arbitrary angle function $\theta(t, s)$, defined on the curve ζ, that rotates the vector $\mathbf{m}$ by $e^{i\theta}$. This new vector will be denoted by $\mathbf{M}$ and it is then defined to be

$$\mathbf{M} \overset{\text{def}}{=} e^{i\theta} \left(\mathbf{N} + i\mathbf{B} \right). \tag{1.2.17}$$

Remark. $\mathbf{M}$ satisfies the orthogonality relations

$$\langle \mathbf{M}, \overline{\mathbf{M}} \rangle = 2 \quad , \quad \langle \mathbf{M}, \mathbf{M} \rangle = 0 \quad , \quad \langle \mathbf{T}, \mathbf{M} \rangle = 0. \tag{1.2.18}$$

Our next step is to define a complex curvature ϕ and a gauge field A on ζ by

$$\phi \overset{\text{def}}{=} \kappa e^{i\theta}, \quad A \overset{\text{def}}{=} \partial_s \theta - \tau. \tag{1.2.19}$$

Equations (1.2.16) can be now rewritten in a gauge invariant form as

$$\nabla_{\mathbf{X}_s} \mathbf{T} = \frac{1}{2} \left(\overline{\phi} \mathbf{M} + \phi \overline{\mathbf{M}} \right), \tag{1.2.20a}$$

$$\nabla_{\mathbf{X}_s} \mathbf{M} = -\phi \mathbf{T} + iA\mathbf{M}. \tag{1.2.20b}$$

The equations above are manifestly gauge invariant under gauge transformations

$$\phi \mapsto e^{i\beta}\phi, \quad \mathbf{M} \mapsto \mathbf{M}e^{i\beta}, \quad A \mapsto A + \partial_s \beta. \tag{1.2.21}$$

Here β denotes an arbitrary function depending on (t, s).

1.3 Binormal Motion

Suppose we wish to set in motion the curve so that it satisfies the equation

$$\mathbf{X}_t = \kappa \mathbf{B}. \tag{1.3.1}$$

We are interested in obtaining the evolution equations of the frame $\{\mathbf{T}, \mathbf{M}, \overline{\mathbf{M}}\}$ and of the functions κ and τ.

In terms on the complex notation introduced in Section 1.2 we can reformulate the equation of motion as follows

$$\mathbf{X}_t = \frac{i}{2}\left(-\bar{\phi}\mathbf{M} + \phi\overline{\mathbf{M}}\right). \tag{1.3.2}$$

From Proposition 1.2.1 and (1.2.20) the evolution equation of the tangent vector $\mathbf{T}$ reads

$$\mathbf{T}_t = \frac{i}{2}\left[\left(-\overline{\partial_s^A \phi}\right)\mathbf{M} + \left(\partial_s^A \phi\right)\overline{\mathbf{M}}\right]. \tag{1.3.3}$$

Here ∂_s^A denotes the covariant derivative with respect to the gauge field A, i.e.,

$$\partial_s^A \stackrel{\text{def}}{=} \partial_s - iA. \tag{1.3.4}$$

Remark. Equation (1.3.3) can be also written classically as

$$\mathbf{T}_t = -\tau\kappa\mathbf{N} + (\partial_s\kappa)\,\mathbf{B} \tag{1.3.5}$$

The orthogonality relations (1.2.18) imply

$$\big\langle\mathbf{M}_t,\mathbf{T}\big\rangle\ = -\big\langle\mathbf{M},\mathbf{T}_t\big\rangle \quad ; \quad \big\langle\mathbf{M}_t,\mathbf{M}\big\rangle\ = 0 \quad ; \quad \Re\left(\big\langle\mathbf{M}_t,\overline{\mathbf{M}}\big\rangle\right) = 0.$$

Therefore, the evolution equation of the complex vector $\mathbf{M}$ must be of the form

$$\mathbf{M}_t = i\partial_s^A\phi\mathbf{T} + iB\mathbf{M}. \tag{1.3.6}$$

Since equations (1.3.3) and (1.3.6) are invariant under gauge transformations

$$\phi \mapsto e^{i\beta},\ \ \mathbf{M}\mapsto\mathbf{M}e^{i\beta},\ \ A\mapsto\partial_s\beta + A,\ \ B\mapsto B+\partial_t\beta, \tag{1.3.7}$$

the real scalar function B can be determined uniquely only when we fix the gauge.

We next proceed to find the evolution equations of the vectors $\mathbf{N}$ and $\mathbf{B}$. Invoking equations (1.2.6), (1.2.9), (1.2.12),and (1.3.5), we obtain the string of equalities

$$\begin{aligned}\mathbf{N}_t &= \nabla_t\left(\frac{1}{\kappa}\mathbf{T}_s\right)\\ &= -\frac{1}{\kappa}(\partial_t\kappa)\mathbf{N} + \frac{1}{\kappa}\left(\nabla_s\mathbf{T}_t + R(\mathbf{X}_t,\mathbf{X}_s)\mathbf{T}\right)\\ &= \ \tau\kappa\mathbf{T} - \frac{1}{\kappa^2}\left(\frac{1}{2}\partial_t\kappa^2 + \partial_s(\tau\kappa^2) + \kappa^2\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{N}\big\rangle\right)\mathbf{N}\\ &\qquad + \frac{1}{\kappa}\left(\partial_{ss}\kappa - \tau^2\kappa - \kappa\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{B}\big\rangle\right)\mathbf{B}.\end{aligned}$$

Requiring the projection $\big\langle\mathbf{N},\mathbf{N}_t\big\rangle$ to vanish gives us the evolution equation of the curvature κ

$$\frac{1}{2}\partial_t\kappa^2 + \partial_s(\tau\kappa^2) + \kappa^2\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{N}\big\rangle \;=\; 0. \tag{1.3.8}$$

Therefore, the evolution equation of $\mathbf{N}$ reduces to

$$\mathbf{N}_t = -\tau\kappa\mathbf{T} + \frac{1}{\kappa}\left(\partial_{ss}\kappa - \tau^2\kappa - \kappa\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{B}\big\rangle\right)\mathbf{B}. \tag{1.3.9}$$

Since $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is an orthonormal set, the evolution equation of the vector field $\mathbf{B}$ is of the form

$$\mathbf{B}_t = -\left(< \mathbf{B},\mathbf{T}_t > \mathbf{T} + \big\langle\mathbf{B},\mathbf{N}_t\big\rangle\mathbf{B}\right). \tag{1.3.10}$$

Hence, we can write the evolution equation of the binormal $\mathbf{B}$ as

$$\mathbf{B}_t = -(\partial_s\kappa)\mathbf{T} - \frac{1}{\kappa}\left(\partial_{ss}\kappa - \tau^2\kappa - \kappa\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{B}\big\rangle\right)\mathbf{N} \tag{1.3.11}$$

with the help of equations (1.3.5) and (1.3.9).

Now, a straightforward calculation shows

$$\nabla_t\nabla_s\mathbf{X}_t = \nabla_t\left(\nabla_s\left(\kappa\mathbf{B}\right)\right) = \nabla_t\left((\partial_s\kappa)\mathbf{B} - \tau\kappa\mathbf{N}\right) = a\mathbf{T} + b\mathbf{N} + c\mathbf{B},$$

where, for simplicity, we set

$$\begin{aligned}
a &= -\tau^2\kappa^2 - (\partial_s\kappa)^2, \\
b &= \frac{1}{\kappa}(\partial_s\kappa)\left[-\partial_{ss}\kappa - \tau^2\kappa + \kappa\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{B}\big\rangle\right] - \partial_t(\tau\kappa), \\
c &= \quad \partial_{ts}\kappa - \tau\partial_{ss}\kappa + \tau^3\kappa + \tau\kappa\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{B}\big\rangle.
\end{aligned}$$

On the other hand, we have

$$\nabla_s \nabla_t \mathbf{X}_t + R(\mathbf{X}_t, \mathbf{X}_s)\mathbf{X}_t = \nabla_s \left(\nabla_t(\kappa \mathbf{B})\right) + \kappa^2 R(\mathbf{B}, \mathbf{T})\mathbf{B} = a\mathbf{T} + (b_1 + b_2)\mathbf{N} + c\mathbf{B}$$

with the notation

$$\begin{aligned}
b_1 &= -\tau \partial_t \kappa - \kappa^2 \partial_s \kappa - \partial_{sss}\kappa + 2\kappa\tau\partial_s \tau + \tau^2 \partial_s \kappa + \kappa \partial_s \left(\left\langle R(\mathbf{T}, \mathbf{B})\mathbf{T}, \mathbf{B} \right\rangle \right), \\
b_2 &= -(\partial_s \kappa) \left\langle R(\mathbf{T}, \mathbf{B})\mathbf{T}, \mathbf{B} \right\rangle - \kappa^2 \left\langle R(\mathbf{T}, \mathbf{B})\mathbf{B}, \mathbf{N} \right\rangle.
\end{aligned}$$

Proposition 1.2.1 states the equality

$$\nabla_t \nabla_s \mathbf{X}_t = \nabla_s \nabla_t \mathbf{X}_t + R(\mathbf{X}_t, \mathbf{X}_s)\mathbf{X}_t. \tag{1.3.12}$$

This identity implies that the components of the vector fields in (1.3.10) with respect $\mathbf{N}$ are equal. It follows that $b = b_1 + b_2$. After some manipulation, this identity transforms into the evolution equation of the torsion τ given by the formula

$$\partial_t \tau = -\partial_s \left(\tau^2 - \frac{1}{2}\kappa^2 - \frac{1}{\kappa}\partial_{ss}\kappa + \left\langle R(\mathbf{T}, \mathbf{B})\mathbf{T}, \mathbf{B} \right\rangle \right) + \kappa \left\langle R(\mathbf{T}, \mathbf{B})\mathbf{B}, \mathbf{N} \right\rangle. \tag{1.3.13}$$

Remark. The evolution equation (1.3.8) of the curvature κ can be also deduced from the identity

$$\nabla_t \mathbf{T}_s = \nabla_s \mathbf{T}_t + R(\mathbf{X}_t, \mathbf{X}_s)\mathbf{T}. \tag{1.3.14}$$

1.4 Equation of Motion

Let us adopt the notation

$$\mathbf{F} \stackrel{\text{def}}{=} \left[\mathbf{T}, \mathbf{M}, \overline{\mathbf{M}}\right] \tag{1.4.1}$$

with $\mathbf{T}, \mathbf{M}$, and $\overline{\mathbf{M}}$ written as column vectors.

Equations (1.2.20) are therefore reformulated in matrix notation as

$$\begin{aligned} \nabla_s \mathbf{F} &= \frac{1}{2}\mathbf{F}S \\ &\equiv \frac{1}{2}\left[\mathbf{T}, \mathbf{M}, \overline{\mathbf{M}}\right] \begin{bmatrix} 0 & -2\phi & -2\bar{\phi} \\ \bar{\phi} & 2iA & 0 \\ \phi & 0 & -2iA \end{bmatrix}. \end{aligned} \tag{1.4.2}$$

Similarly, evolution equations of the frame $\{\mathbf{T}, \mathbf{M}, \overline{\mathbf{M}}\}$ take the form

$$\begin{aligned} \nabla_t \mathbf{F} &= \frac{1}{2}\mathbf{F}E \\ &\equiv \frac{1}{2}\left[\mathbf{T}, \mathbf{M}, \overline{\mathbf{M}}\right] \begin{bmatrix} 0 & -2i\partial_s^a \phi & 2i\overline{\partial_s^A \phi} \\ -i\overline{\partial_s^A \phi} & 2iB & 0 \\ i\partial_s^A \phi & 0 & -2iB \end{bmatrix}. \end{aligned} \tag{1.4.3}$$

Notice that S and E are complex 3×3 matrices.

A direct calculation gives

$$\nabla_{[t}\nabla_{s]}\mathbf{F} = \frac{1}{2}\mathbf{F}\left[\partial_t S - \partial_s E + ES - SE\right] = \frac{1}{2}\mathbf{F}\left[\partial_t S - \partial_s E + [E, S]\right]. \tag{1.4.4}$$

After computing all terms in (1.4.4), we get

$$\nabla_{[t}\nabla_{s]}\mathbf{F} = \frac{1}{2}\mathbf{F} \begin{bmatrix} 0 & -2\Phi & -2\overline{\Phi} \\ \overline{\Phi} & 2i\Psi & 0 \\ \Phi & 0 & -2i\Psi \end{bmatrix}, \tag{1.4.5}$$

where

$$\Phi \stackrel{\text{def}}{=} \left(\partial_t - iB\right)\phi - i\partial_s^A\left(\partial_s^A\phi\right) \tag{1.4.6a}$$

$$\Psi \stackrel{\text{def}}{=} \partial_t A - \partial_s\left(B - \frac{|\phi|^2}{2}\right) \tag{1.4.6b}$$

On the other hand, the complexified version of (1.2.6) yields

$$\nabla_{[t}\nabla_{s]}\mathbf{F} = R(\mathbf{X}_t, \mathbf{T})\mathbf{F} = |\phi|\left[R(\mathbf{B},\mathbf{T})\mathbf{T}, R(\mathbf{B},\mathbf{T})\mathbf{M}, R(\mathbf{B},\mathbf{Y})\overline{\mathbf{M}}\right]. \tag{1.4.7}$$

Expanding each column vector in the right-hand side of (1.4.7), we obtain

$$R(\mathbf{B},\mathbf{T})\mathbf{T} = \bar{a}\mathbf{M} + a\overline{\mathbf{M}} \quad ; \quad a \stackrel{\text{def}}{=} \frac{1}{2}\big\langle R(\mathbf{B},\mathbf{T})\mathbf{T}, \mathbf{M}\big\rangle, \tag{1.4.8}$$

$$R(\mathbf{B},\mathbf{T})\mathbf{M} = -2a\mathbf{T} + ib\mathbf{M} \quad ; \quad b \stackrel{\text{def}}{=} -\frac{i}{2}\big\langle R(\mathbf{B},\mathbf{T})\mathbf{M}, \overline{\mathbf{M}}\big\rangle, \tag{1.4.9}$$

$$R(\mathbf{B},\mathbf{T})\overline{\mathbf{M}} = -2\bar{a}\mathbf{T} - ib\overline{\mathbf{M}}. \tag{1.4.10}$$

Remark. Notice that

$$\big\langle R(\mathbf{B},\mathbf{T})\mathbf{M}, \mathbf{M}\big\rangle = 0 \ \text{ and } \ \big\langle R(\mathbf{B},\mathbf{T})\mathbf{M}, \overline{\mathbf{M}}\big\rangle = 2i\big\langle R(\mathbf{B},\mathbf{T})\mathbf{B}, \mathbf{N}\big\rangle.$$

Therefore, b is a real number.

Consequently, equation (1.4.7) becomes

$$\nabla_{[t}\nabla_{s]}\mathbf{F} = |\phi|\left[\mathbf{T}, \mathbf{M}, \overline{\mathbf{M}}\right]\begin{bmatrix} 0 & -2a & -2\bar{a} \\ \bar{a} & ib & 0 \\ a & 0 & -ib \end{bmatrix} \equiv \mathbf{F}|\phi|Q. \tag{1.4.11}$$

From (1.4.5) and (1.4.11) we deduce $\Phi = 2a|\phi|$ and $\Omega = b|\phi|$, i.e.,

$$(\partial_t - iB)\,\phi - i\partial_s^A\left(\partial_s^A \phi\right) = a|\phi|, \tag{1.4.12a}$$

$$\partial_t A - \partial_s\left(B - \frac{|\phi|^2}{2}\right) = b|\phi|. \tag{1.4.12b}$$

Remark. Equation (1.4.12b) is equivalent to

$$\partial_t A = \partial_s\left(B - \frac{|\phi|^2}{2} + \int b|\phi|\,ds\right). \tag{1.4.13}$$

We therefore choose β such that

$$\partial_s\beta = A \quad \text{and} \quad \partial_t\beta = B - \frac{|\phi|^2}{2} + \int b|\phi|\,ds. \tag{1.4.14}$$

Let us define ψ as $\psi \stackrel{\text{def}}{=} \phi e^{-i\beta}$ and consider the equalities (1.4.14). Since equations (1.4.12) are clearly invariant under the gauge transformations in (1.2.21), equation (1.4.12a) transforms into

$$\partial_t\psi - i\partial_{ss}\psi - i\left(\frac{|\psi|^2}{2} - \int b|\psi|\,ds\right)\psi = 2a\psi, \tag{1.4.15}$$

which is equivalent to

$$i\partial_t\psi + \partial_{ss}\psi + \frac{1}{2}|\psi|^2\psi = W(\psi)\psi, \tag{1.4.16}$$

in view of (1.4.9). Here $W(\psi)$ stands for

$$\begin{aligned} W(\psi) \stackrel{\text{def}}{=} \ &\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{B}\big\rangle - \int |\psi|\big\langle R(\mathbf{T},\mathbf{B})\mathbf{B},\mathbf{N}\big\rangle\,ds \\ &- i\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{N}\big\rangle. \end{aligned} \tag{1.4.17}$$

Following [8], we next discuss another way to deduce (1.4.16). Let us first recall equations (1.3.8) and (1.3.13). They are

$$\frac{1}{2}\partial_t\kappa^2 + \partial_s(\tau\kappa^2) + \kappa^2\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{N}\big\rangle \;=\; 0, \tag{1.4.18a}$$

$$\partial_t\tau = \partial_s\left(-\tau^2 + \frac{1}{2}\kappa^2 + \frac{1}{\kappa}\partial_{ss}\kappa \;-\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{B}\big\rangle\right) + \kappa\big\langle R(\mathbf{T},\mathbf{B})\mathbf{B},\mathbf{N}\big\rangle. \tag{1.4.18b}$$

Let ψ be a function such that

(a) $\psi = \kappa\exp(i\theta)$ with $\theta(s,t) = \int\tau(t,s)\,ds$.

(b) κ and τ satisfy equations (1.4.18).

Using the mentioned equations a straightforward computation gives

$$i\partial_t\psi = (h + i\partial_t\kappa)\,e^{i\theta}, \tag{1.4.19a}$$

$$\partial_{ss}\psi = \big[\partial_{ss}\kappa - \tau^2\kappa - i\left(\partial_t\kappa + \;\kappa\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{N}\big\rangle\right)\big]\,e^{i\theta}, \tag{1.4.19b}$$

where

$$h = \kappa\tau^2 - \frac{1}{2}\kappa^3 - \partial_{ss}\kappa + \kappa\big\langle R(\mathbf{T},\mathbf{B})\mathbf{T},\mathbf{B}\big\rangle - \kappa\int\kappa\big\langle R(\mathbf{T},\mathbf{B})\mathbf{B},\mathbf{N}\big\rangle\,ds.$$

Combining (1.4.18) and (1.4.19) yields (1.4.16).

We conclude this chapter giving an example. Let g be a real-valued smooth function on M. We now assume that M has curvature tensor given by

$$R(\mathbf{V},\mathbf{Y})\mathbf{Z} = g[\big\langle\mathbf{Z},\mathbf{V}\big\rangle\mathbf{Y} - \big\langle\mathbf{Z},\mathbf{Y}\big\rangle\mathbf{V}].$$

This curvature formula has a simple geometric meaning. If $\mathbf{x}, \mathbf{y}$ is an orthonormal basis for a tangent plane Π at the point $p \in M$, then $R(\mathbf{x}, \mathbf{y})$ is zero on $\Pi^{\perp}$, and on Π is the rotation sending $\mathbf{x}$ to $\mathbf{y}$ and $\mathbf{y}$ to $-\mathbf{x}$, followed by a scalar multiplication by $g(p)$. It follows that the sectional curvature of any tangent plane Π to M at p is $g(p)$ and that the following relations are valid.

$$
\begin{aligned}
R(\mathbf{T}, \mathbf{B})\mathbf{T} &= g\mathbf{B}, \\
R(\mathbf{T}, \mathbf{B})\mathbf{B} &= -g\mathbf{T}, \\
R(\mathbf{T}, \mathbf{B})\mathbf{N} &= 0.
\end{aligned}
$$

Hence, equation (1.4.16) reduces to

$$
i\partial_t \psi + \partial_{ss}\psi + \left[\frac{1}{2}|\psi|^2 - g\right]\psi = 0. \tag{1.4.20}
$$

However, the Schur lemma [13, page 96] implies that g is a constant function on M, say $g(p) = c$ for every $p \in M$. Setting $\varphi = \psi e^{ict}$ φ satisfies the famous cubic nonlinear Schrödinger equation

$$
i\partial_t \varphi + \partial_{ss}\varphi + \frac{1}{2}|\varphi|^2\varphi = 0.
$$

Chapter 2

Periodic in space Solutions

2.1 Introduction

Our main concern in this chapter is existence of a global solution of the initial value problem for the nonlinear Schrödinger type equation (NLSE)

$$i\partial_t\psi + \partial_{xx}\psi = -\frac{1}{2}|\psi|^2\psi + W(\psi)\psi, \;\; t \in \mathbb{R}, \;\; x \in \mathbb{T}, \tag{2.1.1a}$$

with initial condition

$$\psi(0,x) = \psi_o(x), \;\; x \in \mathbb{T}. \tag{2.1.1b}$$

$\mathbb{T}$ denotes the unit circle S^1 and W is a complex-valued function.

In this setting, this Cauchy problem is different from the $\mathbb{R}$-case. The local theory for solution of the NLSE in $\mathbb{R}$ uses the dispersive effect of the free Schrödinger operator in the form of the Strichartz inequalities. When the domain is periodic such inequalities do not hold. However, it is still possible to prove existence of a solution of (2.1.1) for small enough time t (local existence) by solving the integral equation

$$\psi(t) = U(t)\psi_o - i\int_0^t U(t-s)F(\psi)(s)\,ds$$

using a Picard's fixed point technique where $U(t)$ is the linear Schrödinger group and F is an appropriate mapping. Although there is no conservation of $\|\phi(\cdot,t)\|_{L^2(\mathbb{T})}$ under the flow, existence for all time (global existence) holds by estimating carefully the growth in size of the L^2-norm of the local solution.

In the case of pure power nonlinearity, i.e., $W \equiv 0$, equation (2.1.1) has been studied by Bourgain [4]. Using refined properties of trigonometric series, he developed estimates similar to classical Strichartz inequalities and established that the solution for this particular case is in $C(\mathbb{R}, H^s(\mathbb{T}))$ for all $\phi_o \in H^s(\mathbb{T}),\ s \geq 0$.

2.2 An Integral Equation

Here and in the sequel, we will consider functions of two variables, $\psi(t,x)$, with $t \in \mathbb{R}$ the time variable and $x \in \mathbb{T}$ the space variable. We will denote by $\widetilde{\psi}$ the partial Fourier transform of the function ψ with respect to the space variable and by $\widehat{\psi}$ the Fourier transform of ψ with respect to both the space variable and the time variable, i.e.,

$$\widetilde{\psi}(t,\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-ix\xi}\psi(t,x)\,dx \ \text{ and } \ \widehat{\psi}(\tau,\xi) = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{T}} e^{-i(x\xi+t\tau)}\psi(t,x)\,dxdt.$$

We will also denote numerical constants by C and without loss of generality, we will assume that $C \geq 1$.

The free Schrödinger operator $U(t) := e^{it\Delta}$ plays an essential role in the study of the equation (2.1.1). Recall that in the case of periodic boundary conditions, the operator $U(t)$ is given by an exponential sum

$$U(t)\phi(x) = \sum_{\xi\in\mathbb{Z}} \widetilde{\phi}(\xi) e^{i(x\xi - t\xi^2)} \tag{2.2.1}$$

and that $\phi(t,x) = U(t)\phi_o(x)$ is the solution of the Schrödinger equation

$$\begin{cases} i\partial_t\phi + \partial_{xx}\phi = 0 \;\; t \in \mathbb{R}, \;\; x \in \mathbb{T}, \\ \\ \phi(0,x) = \phi_o(x), \;\; x \in \mathbb{T}. \end{cases} \tag{2.2.2}$$

Our first step is to reformulate equation (2.1.1) as an equivalent integral equation.

Proposition 2.2.1. *Consider the Cauchy problem*

$$i\psi_t + \psi_{xx} = F(\psi)\,, \;\; (t,x) \in \mathbb{R} \times \mathbb{T}, \tag{2.2.3a}$$

$$\psi(0,x) = \psi_o(x), \;\; x \in \mathbb{T}, \tag{2.2.3b}$$

where F is a complex-valued function. If ψ is a solution of (2.2.3)*, then ψ satisfies the integral equation*

$$\psi(t) = U(t)\psi_o - i\int_0^t U(t-s)F(\psi)(s)\,ds. \tag{2.2.4}$$

Proof. We use the partial Fourier transform to calculate

$$i\widetilde{\psi}_t - \xi^2\widetilde{\psi} = \widetilde{F(\psi)}, \tag{2.2.5a}$$

$$\widetilde{\psi}(0,\xi) = \widehat{\psi_o}(\xi). \tag{2.2.5b}$$

Solving (2.2.5), we obtain

$$\widetilde{\psi}(t,\xi) = e^{-it\xi^2}\widehat{\psi_o}(\xi) - i\int_0^t e^{-i(t-s)\xi^2}\widetilde{F(\psi)}(s,\xi)\,ds. \tag{2.2.6}$$

(2.2.4) follows from (2.2.6) by taking inverse Fourier transform. This completes the proof.

□

In view of Proposition 2.2.1, we replace the equations (2.2.3) by the equivalent equation (2.2.4) and try to solve for ψ using Picard's fixed point theorem. Let us denote by T the mapping defined by the right side of (2.2.4), i.e.,

$$T(\psi) = U(\cdot)\psi_o - i\int_0^{\cdot} U(\cdot - s)F(\psi)(s)\,ds. \tag{2.2.7}$$

Now, we want to use a cutoff function to decompose the integral in (2.2.4) into parts near and away from the level curves of $\tau + \xi^2$. For this reason, let us introduce a smooth function $\widehat{b}$ such that $0 \leq \widehat{b} \leq 1$, $\widehat{b} = 1$ on a neighborhood $[-r, r]$ of 0 and $\operatorname{supp}\widehat{b} \subseteq [-2r, 2r]$, where r is a positive number to be determined later. Let $\widehat{c}$ be the function given by $\widehat{c} \overset{\text{def}}{=} 1 - \widehat{b}$.

Proposition 2.2.2. $\widehat{T(\psi)}$ *can be written as*

$$\begin{aligned}\widehat{T(\psi)}(\tau,\xi) = \delta(\tau+\xi^2)\left(\widehat{\psi_o}(\xi) + \widehat{B_0}(\xi)\right) - \frac{\widehat{F(\psi)}_f(\tau,\xi)}{\tau+\xi^2} \\ - \sum_{k=1}^{\infty} \frac{(-2r)^{k-1}}{k!}\ \delta^{(k)}(\tau+\xi^2)\ \widehat{B_k}(\xi)\end{aligned} \tag{2.2.8}$$

where

$$\widehat{F_f(\psi)}(\tau,\xi) = \widehat{F(\psi)}(\tau,\xi)\ \hat{c}(\tau+\xi^2), \tag{2.2.9a}$$

$$\widehat{F_n(\psi)}(\tau,\xi) = \widehat{F(\psi)}(\tau,\xi)\ \hat{b}(\tau+\xi^2), \tag{2.2.9b}$$

$$\widehat{B_0}(\xi) = \int_{\mathbb{R}} \frac{\widehat{F_f(\psi)}(\sigma,\xi)}{\sigma+\xi^2}\,d\sigma, \tag{2.2.9c}$$

$$\widehat{B_k}(\xi) = \int_{\mathbb{R}} \left[\frac{\sigma+\xi^2}{2r}\right]^{k-1} \widehat{F_n(\psi)}(\sigma,\xi)\,d\sigma, \quad k = 1, 2, \ldots. \tag{2.2.9d}$$

Proof. Let be $\chi_{[0,t]}$ be the characteristic function defined by

$$\chi_{[0,t]}(s) = \begin{cases} 1 & 0 \leq s \leq t, \\ 0 & \text{otherwise}. \end{cases}$$

The second term on the right-hand side of (2.2.6) can be rewritten as

$$\int_0^t e^{-i(t-s)\xi^2}\widetilde{F(\psi)}(s,\xi)\,ds = e^{-it\xi^2}\int_{\mathbb{R}} e^{is\xi^2}\chi_{[0,t]}(t)\widetilde{F(\psi)}(s,\xi)\,ds. \tag{2.2.10}$$

We have by Parseval's formula

$$\int_0^t e^{is\xi^2}\widetilde{F(\psi)}(s,\xi) = \int_{\mathbb{R}} \overline{\widehat{\chi_{[0,t]}}}(\tau+\xi^2)\widehat{F(\psi)}(\tau,\xi)\,d\tau. \tag{2.2.11}$$

A straightforward computation allows us to obtain

$$\widehat{\chi_{[0,t]}}(\tau) = \int_0^t e^{-is\tau}\,ds = -i\frac{(1-e^{-it\tau})}{\tau}. \tag{2.2.12}$$

Substituting (2.2.12) back to (2.2.11) and then in (2.2.10), we get

$$\int_0^t e^{-i(t-s)\xi^2}\widetilde{F(\psi)}(s,\xi)\,ds = \int_{\mathbb{R}} \frac{e^{-it\xi^2}-e^{it\tau}}{\tau+\xi^2}\widehat{F(\psi)}(\tau,\xi)\,d\tau. \tag{2.2.13}$$

Therefore,

$$\widetilde{T(\psi)}(t,\xi) = e^{-it\xi^2}\widehat{\psi_o}(\xi) + \int_{\mathbb{R}} \frac{e^{-it\xi^2}-e^{it\tau}}{\tau+\xi^2}\widehat{F(\psi)}(\tau,\xi)\,d\tau. \tag{2.2.14}$$

Now, we write $\widehat{F(\psi)} = \widehat{F_n(\psi)} + \widehat{F_f(\psi)}$, see (2.2.9a) and (2.2.9b) , and substitute it in (2.2.13) to obtain

$$\begin{aligned}-i\int_0^t e^{-i(t-s)\xi^2}\widetilde{F(\psi)}(s,\xi)\,ds &= e^{-it\xi^2}\int_{\mathbb{R}} \frac{\widehat{F_f(\psi)}(\tau,\xi)}{\tau+\xi^2}\,d\tau - \int_{\mathbb{R}} e^{it\tau}\frac{\widehat{F_f(\psi)}(\tau,\xi)}{\tau+\xi^2}\,d\tau \\ &\quad - e^{-it\xi^2}\int \frac{e^{i(\tau+\xi^2)t}-1}{\tau+\xi^2}\widehat{F_n(\psi)}(\tau,\xi)\,d\tau. \end{aligned} \tag{2.2.15}$$

If we expand the expression $e^{i(\tau+\xi^2)t}-1$ in power series, the last term in (2.2.15) becomes

$$e^{it\xi^2}\int_{\mathbb{R}}\sum_{k=1}^{\infty}\frac{(it)^k}{k!}(\tau+\xi^2)^{k-1}\widehat{F_n(\psi)}(\tau,\xi)\,d\tau, \tag{2.2.16}$$

and hence

$$\begin{aligned}\widetilde{T(\psi)}(t,\xi) = e^{-it\xi^2}\widehat{\psi_o}(\xi) + \frac{1}{2\pi}\left(e^{-it\xi^2}\int_{\mathbb{R}}\frac{\widehat{F_f(\psi)}(\tau,\xi)}{\tau+\xi^2}\,d\tau - \int_{\mathbb{R}} e^{it\tau}\frac{\widehat{F_f(\psi)}(\tau,\xi)}{\tau+\xi^2}\,d\tau\right) \\ - \frac{e^{it\xi^2}}{2\pi}\int_{\mathbb{R}}\sum_{k=1}^{\infty}\frac{(it)^k}{k!}(\tau+\xi^2)^{k-1}\widehat{F_n(\psi)}(\tau,\xi)\,d\tau.\end{aligned} \tag{2.2.17}$$

Finally, we take Fourier transform in t variable of 2.2.17. Recall that

$$(it)^k e^{i\xi^2 t} \xrightarrow{\text{Fourier transform}} \delta^{(k)}(\tau+\xi^2),$$

with δ denoting the delta distribution. Therefore, we obtain

$$\begin{aligned}\widehat{T(\psi)}(\tau,\xi) = \delta(\tau+\xi^2)\left(\widehat{\psi_o}(\xi)+\widehat{B_0}(\xi)\right) - \frac{\widehat{F_f(\psi)}(\tau,\xi)}{\tau+\xi^2} \\ - \sum_{k=1}^{\infty}\ \frac{(-2r)^{k-1}}{k!}\ \delta^{(k)}(\tau+\xi^2)\ \ \widehat{B_k}(\xi).\end{aligned}$$

The proof is completed.

$\square$

Since $\widehat{T(\psi)}$ contains the delta distribution and its derivatives, we need to localize them in the variable t. For this reason, we consider a smooth cutoff function $a(t)$ which is 1 if $|t| \leq 1$, identically 0 if $|t| \geq 2$, and $0 \leq a(t) \leq 1$ for any real number t. Denote by $a_\beta(t) = a(\frac{t}{\beta})$, $\beta > 0$, its dilation. The next proposition is a very useful technical result.

Proposition 2.2.3. *Assume that $0 < \beta \leq 1$ and $0 \leq \gamma < 1$. If k is a nonnegative integer, then the following inequality holds.*

$$I \stackrel{def}{=} \left(\int_R (1+|\tau|)^{2\gamma} |\widehat{a_\beta}^{(k)}(\tau)|^2 \, d\tau \right)^{1/2} \le K(\gamma, a)(3\beta)^{k+\frac{1}{2}-\gamma}. \tag{2.2.18}$$

Proof. Observe that

$$\widehat{t^k a_\beta}(\tau) = \beta^{k+1} \widehat{t^k a}(\beta\tau), \quad k \in \mathbb{N}. \tag{2.2.19}$$

Identity (2.2.19) and the change of variable $u = \beta\tau$ yield

$$\begin{aligned} I^2 = \int_R (1+|\tau|)^{2\gamma} |\widehat{t^k a_\beta}(\tau|^2 \, d\tau &= \beta^{2(k+1)} \int_R (1+|\tau|)^{2\gamma} |\widehat{t^k a}(\beta\tau)|^2 \, d\tau \\ &= \beta^{2k+1-2\gamma} \int_R (\beta+|u|)^{2\gamma} |\widehat{t^k a}(u)|^2 \, du. \end{aligned}$$

Since $0 < \beta \le 1$, we have

$$I^2 \; \le \; \beta^{2k+1-2\gamma} \left(\int_{|u|\le 1} (1+|u|)^{2\gamma} |\widehat{t^k a}(u)|^2 \, du + \int_{|u|\ge 1} \frac{(1+|u|)^{2\gamma}}{u^2} \; |u\widehat{t^k a}(u)|^2 \, du \right).$$

Invoking the inequality

$$\frac{(1+|u|)^{2\gamma}}{u^2} \le 2^{2\gamma}, \;\text{ if }\; |u| \ge 1$$

and properties of the Fourier transform, we have the following string of inequalities

$$\begin{aligned} I^2 &\le 2^{3/4} \;\; \beta^{2k+1-2\gamma} \left(\int_{|u|\le 1} |\widehat{t^k a}(u)|^2 \, du + \int_{|u|\ge 1} |u\widehat{t^k a}(u)|^2 \, du \right), \\ &\le 2^{2\gamma} \;\; \beta^{2k+1-2\gamma} \left(\int_R |t^k a(t)|^2 \, dt + \int_R |\frac{d}{dt}(t^k a)(t)|^2 \, dt \right) \\ &= 2^{2\gamma} \;\; \beta^{2k+1-2\gamma} \left(\int_R |t^k a(t)|^2 \, dt + \int_R |kt^{k-1}a(t) + t^k a\prime(t)|^2 \, dt \right). \end{aligned} \tag{2.2.20}$$

The condition supp $a \subseteq [-2,2]$ and the inequalities

$$(a+b)^2 \le 2(a^2+b^2) \text{ for all } a,b \in \mathbb{R},$$
$$\left(1+\frac{k^2}{2}\right) \le \left(\frac{9}{4}\right)^k \text{ for any } k \in \mathbb{N}$$

imply that

$$\begin{aligned}
I^2 &\le 2^{2\gamma}\,\beta^{2k+1-2\gamma}\left(4^k\|a\|^2_{L^2(R)} + 2\int_R \left(|kt^{(k-1)}a(t)|^2 + |t^k a'(t)|^2\right)\,dt\right)\\
&\le 2^{2\gamma}\,\beta^{2k+1-2\gamma}\left(4^k\|a\|^2_{L^2(R)} + 2\left(k^2 4^{k-1}\|a\|^2_{L^2(R)} + 4^k\|a'\|^2_{L^2(R)}\right)\right)\\
&= 2^{2\gamma}\,\beta^{2k+1-2\gamma}\;\;4^k\left((1+\frac{k^2}{2})\|a\|^2_{L^2(R)} + 2\|a'\|^2_{L^2(R)}\right)\\
&\le 2^{2\gamma+1}\,\beta^{2k+1-2\gamma}\,9^k\|a\|^2_{H^1(R)}\\
&\le \frac{2}{3}\,6^{2\gamma}\,(3\beta)^{2k+1-2\gamma}\|a\|^2_{H^1(R)}\\
&\equiv K^2(\gamma,a)\,(3\beta)^{2k+1-2\gamma}.
\end{aligned} \tag{2.2.21}$$

Notice that $K(\gamma,a) \le 2\sqrt{3}\|a\|_{H^1(R)}$. This completes the proof. $\square$

The product of $T(\psi))$ and a_β will be denoted by $T_\beta(\psi)$. From (2.2.8), its Fourier transform $\widehat{T_\beta(\psi)}$ can be written in the following manner:

$$\begin{aligned}
\widehat{T_\beta(\psi)}(\tau,\xi) = \;& \widehat{a_\beta}(\tau+\xi^2)\left(\widehat{\psi_o}(\xi) + \widehat{B_0}(\xi)\right) - \widehat{\Phi}(\tau,\xi)\\
&- \sum_{k=1}^{\infty}\frac{(-2r)^{k-1}}{k!}\;\;\widehat{a_\beta}^{(k)}(\tau+\xi^2)\,\widehat{B_k}(\xi),
\end{aligned} \tag{2.2.22}$$

where

$$\widehat{\Phi}(\tau,\xi) = (\widehat{a_\beta} * \widehat{G})(\tau,\xi), \qquad \widehat{G}(\tau,\xi) = \frac{\widehat{F_f(\psi)}(\tau,\xi)}{\tau+\xi^2}. \tag{2.2.23}$$

2.3 A Priori Estimates

The estimates needed to prove that equation (2.2.3) has a solution locally in time are provided in this section. Let us define the multiplier

$$\hat{S}(\tau,\xi) = 1 + |\tau + \xi^2|. \tag{2.3.1}$$

Theorem 2.3.1. *Let f(t,x) be a function with $(t,x) \in \mathbb{R} \times \mathbb{T}$ and denote by $\hat{f}(\tau,\xi)$ its Fourier transform, with $(\tau,\xi) \in \mathbb{R} \times \mathbb{Z}$. The following estimates hold*

$$\|f\|_{L^4(\mathbb{R}\times\mathbb{T})} \le C\|\hat{S}^{\frac{3}{8}}\hat{f}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \tag{2.3.2}$$

and its dual

$$\left\| \ \frac{\hat{f}}{\hat{S}^{\frac{3}{8}}} \ \right\|_{L^2(\mathbb{R}\times\mathbb{Z})} \le C\|f\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}. \tag{2.3.3}$$

The proof of Theorem 2.3.1 will be given in the last section of this chapter.

Theorem 2.3.2. *Assume that $\psi_o \in L^2(\mathbb{T})$. Then there is a constant C such that*

$$\|T_\beta(\psi)\|_{L^4(\mathbb{R}\times\mathbb{T})} \le C \ \beta^{\frac{1}{8}} \left(\|\psi_o\|_{L^2(\mathbb{T})} + \beta^{\frac{1}{8}} \ \|F(\psi)\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}\right). \tag{2.3.4}$$

Proof. From Theorem 2.3.1 we have

$$\|T_\beta(\psi)\|_{L^4(\mathbb{R}\times\mathbb{T})} \le C\|\hat{S}^{\frac{3}{8}}\widehat{T_\beta(\psi)}\|_{L^2(\mathbb{R}\times\mathbb{Z})}. \tag{2.3.5}$$

In order to control the right member of (2.3.5), we have to consider the contributions of

$$\hat{A}(\tau,\xi) = \hat{S}^{\frac{3}{8}}(\tau,\xi)\widehat{a_\beta}(\tau+\xi^2)\widehat{\psi_o}(\xi), \tag{2.3.6a}$$

$$\hat{B}(\tau,\xi) = \hat{S}^{\frac{3}{8}}(\tau,\xi)\widehat{a_\beta}(\tau+\xi^2)\widehat{B_0}(\xi), \tag{2.3.6b}$$

$$\hat{D}(\tau,\xi) = -\hat{S}^{\frac{3}{8}}(\tau,\xi)(\widehat{a_\beta} * \widehat{G})(\tau,\xi), \qquad \widehat{G}(\tau,\xi) = \frac{\widehat{F_f(\psi)}(\tau,\xi)}{\tau+\xi^2}, \tag{2.3.6c}$$

$$\hat{E}(\tau,\xi) = -\hat{S}^{\frac{3}{8}}(\tau,\xi)\sum_{k=1}^{\infty}\frac{(-2r)^{k-1}}{k!}\ \widehat{a_\beta}^{(k)}(\tau+\xi^2)\,\widehat{B_k}(\xi). \tag{2.3.6d}$$

See equations (2.2.9) and (2.2.22).

Step 1. We invoke equation (2.2.18), with $k = 0$ and $\gamma = 3/8$, to bound the L^2-norm of $\hat{A}$.

$$\begin{aligned}
\|\hat{A}\|^2_{L^2(\mathbb{R}\times\mathbb{Z})} &= \sum_{\xi\in\mathbb{Z}}\int_{\mathbb{R}}|\widehat{a_\beta}(\tau+\xi^2)|^2\hat{S}^{\frac{3}{4}}(\tau,\xi)|\widehat{\psi_o}(\xi)|^2\,d\tau \\
&= \left[\int|\widehat{a_\beta}(\sigma)|^2\,(1+|\sigma|)^{\frac{3}{4}}\ d\sigma\right]\sum_{\xi\in\mathbb{Z}}|\widehat{\psi_o}(\xi)|^2 \\
&\leq\ K^2(a)\beta^{\frac{1}{4}}\|\psi_o\|^2_{L^2(\mathbb{T})}.
\end{aligned}$$

Consequently,

$$\|\hat{A}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq C\beta^{\frac{1}{8}}\|\psi_o\|_{L^2(\mathbb{T})}. \tag{2.3.7}$$

Step 2. Following the lines in step 1., we can show that

$$\|\hat{B}\|^2_{L^2(\mathbb{R}\times\mathbb{Z})} \leq C\beta^{\frac{1}{4}}\|\widehat{B_0}\|^2_{L^2(\mathbb{Z})}. \tag{2.3.8}$$

We now have by Hölder's inequality

$$
\begin{aligned}
\|\widehat{B_0}\|^2_{L^2(\mathbb{Z})} &= \sum_\xi \left(\int \frac{\widehat{F(\psi)}(\tau,\xi)}{\tau+\xi^2} \hat{c}(\tau+\xi^2)\, d\tau \right)^2 \\
&\le \sum_\xi \left[\int \frac{|\widehat{F(\psi)}(\tau,\xi)|^2}{(1+|\tau+\xi^2|)^{3/4}}\, d\tau \right] \; \left[\int \frac{(1+|\tau+\xi^2|)^{3/4}}{|\tau+\xi^2|^2} |\hat{c}(\tau+\xi^2)|^2\, d\tau \right] \\
&= 2\|\hat{S}^{-\frac{3}{8}} \widehat{F(\psi)}\|^2_{L^2(\mathbb{R}\times\mathbb{Z})} \int_r^\infty \frac{(1+\tau)^{3/4}}{\tau^2} |\hat{c}(\tau)|^2\, d\tau.
\end{aligned} \tag{2.3.9}
$$

Assuming that $r \ge 1$ the last term of (2.3.9) is estimated in the form

$$
\begin{aligned}
\|\widehat{B}_0\|^2_{L^2(\mathbb{T})} &\le C\|\hat{S}^{-\frac{3}{8}} \widehat{F(\psi)}\|^2_{L^2(\mathbb{R}\times\mathbb{Z})} \int_r^\infty \frac{1}{\tau^{5/4}}\, d\tau \\
&\le \frac{C}{r^{1/4}} \left\| \frac{\widehat{F(\psi)}}{\hat{S}^{3/8}} \right\|^2_{L^2(\mathbb{R}\times\mathbb{Z})}.
\end{aligned} \tag{2.3.10}
$$

Thus, by (2.3.3), inequality (2.3.10) becomes

$$
\|\widehat{B}_0\|^2_{L^2(\mathbb{T})} \le \frac{C}{r^{1/4}} \|F(\psi)\|^2_{L^{4/3}(\mathbb{R}\times\mathbb{T})}. \tag{2.3.11}
$$

Combining (2.3.8) with (2.3.11) and setting $r\beta = 1$, we get

$$
\|\hat{B}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \le C\beta^{1/4} \|F(\psi)\|_{L^{4/3}(\mathbb{R}\times\mathbb{T})}. \tag{2.3.12}
$$

Step 3. Taking in account (2.2.23) and the triangle inequality, we can write

$$
\begin{aligned}
\|\hat{D}\|^2_{L^2(\mathbb{R}\times\mathbb{Z})} &= \sum_\xi \int \Big| \int \hat{S}^{\frac{3}{8}}(\tau,\xi)\, \widehat{a_\beta}(\tau-\sigma)\, \frac{\widehat{F(\psi)}(\sigma,\xi)}{\sigma+\xi^2}\, d\sigma \Big|^2\, d\tau \\
&\le \sum_\xi \int \left[\int \left(|\tau-\sigma|^{\frac{3}{8}} + \hat{S}^{\frac{3}{8}}(\sigma,\xi) \right) |\widehat{a_\beta}(\tau-\sigma)| |\widehat{G}(\sigma,\xi)|\, d\sigma \right]^2 d\tau \\
&\le 2\sum_\xi \int_{\mathbb{R}} \left[J^2_\beta(\tau,\xi) + H^2_\beta(\tau,\xi) \right]\, d\tau,
\end{aligned} \tag{2.3.13}
$$

where

$$J_\beta(\tau,\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}} |\tau-\sigma|^{3/8}\, |\widehat{a_\beta}(\tau-\sigma)||\widehat{G}(\sigma,\xi)|\, d\sigma,$$
$$H_\beta(\tau,\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \hat{S}^{\frac{3}{8}}(\sigma,\xi)\, |\widehat{a_\beta}(\tau-\sigma)||\widehat{G}(\sigma,\xi)|\, d\sigma.$$

Invoking Hölder's inequality, we derive

$$\begin{aligned} J_\beta^2(\tau,\xi) &\le \left[\int_{\mathbb{R}} \frac{|\widehat{F(\psi)}(\sigma,\xi)|^2}{(1+|\sigma+\xi^2|)^{\frac{3}{4}}}\, d\sigma\right] \\ &\quad \cdot \left[\int_{\mathbb{R}} \frac{|\tau-\sigma|^{\frac{3}{4}}}{|\sigma+\xi^2|^2}\ |\widehat{a_\beta}(\tau-\sigma)|^2\, \hat{S}^{\frac{3}{4}}(\sigma,\xi)\, |\hat{c}(\sigma+\xi^2)|^2\, d\sigma\right] \\ &\equiv \left(\int_{\mathbb{R}} \frac{|\widehat{F(\psi)}(\sigma,\xi)|^2}{(1+|\sigma+\xi^2|)^{\frac{3}{4}}}\, d\sigma\right) V_\beta^2(\tau,\xi). \end{aligned} \tag{2.3.14}$$

The changes of variables $u = \tau+\xi^2$ and $\rho = \sigma+\xi^2$ allow us to show that $\Upsilon_\beta^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}} V_\beta^2(\tau,\xi)\, d\tau$ does not depend on the variable ξ. Moreover, since $\rho \ge 1$ and $r\beta = 1$, we obtain by (2.2.18)

$$\begin{aligned} \Upsilon_\beta^2 &= \int\int |u-\rho|^{\frac{3}{4}}\, |\widehat{a_\beta}(u-\rho)|^2\, |\hat{c}(\rho)|^2\, \frac{(1+|\rho|)^{\frac{3}{4}}}{\rho^2}\, d\rho du \\ &\le \left(\int_{\mathbb{R}} |v|^{\frac{3}{4}} |\widehat{a_\beta}(v)|^2\, dv\right)\left(\int_{|\rho|\ge r} \frac{(1+|\rho|)^{\frac{3}{4}}}{\rho^2}\, d\rho\right) \\ &\le C\beta^{\frac{1}{4}} \int_r^\infty \frac{(1+\rho)^{\frac{3}{4}}}{\rho^2}\, d\rho \\ &\le C\beta^{\frac{1}{2}}. \end{aligned} \tag{2.3.15}$$

Therefore, equation (2.3.13) transforms into

$$\|\hat{D}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \le C\beta^{\frac{1}{2}} \Big\| \frac{\widehat{F(\psi)}}{\hat{S}^{\frac{3}{8}}} \Big\|^2_{L^2(\mathbb{R}\times\mathbb{Z})} + 2\sum_{\xi} \int_{\mathbb{R}} H_\beta^2(\tau,\xi)\, d\tau. \tag{2.3.16}$$

On the other hand, we have

$$
\begin{aligned}
H_\beta(\tau,\xi) &= \int_{\mathbb{R}} \frac{\hat{S}^{\frac{3}{4}}(\sigma,\xi)}{|\sigma+\xi^2|}\, |\widehat{a_\beta}(\tau-\sigma)|\, \frac{|\widehat{F(\psi)}(\sigma,\xi)|}{\hat{S}^{\frac{3}{8}}(\sigma,\xi)}\, \hat{c}(\sigma+\xi^2)\, d\sigma \\
&\le C\beta^{\frac{1}{4}} \left(|\widehat{a_\beta}| \star \left(\frac{|\widehat{F(\psi)}|}{\hat{S}^{\frac{3}{8}}}(\cdot,\xi) \right) \right)(\tau). \qquad (2.3.17)
\end{aligned}
$$

As a consequence of the last inequality, we obtain

$$
\|H_\beta(\cdot\,,\xi)\|^2_{L^2(\mathbb{R})} \le C\beta^{\frac{1}{2}} \|\widehat{a}\|^2_{L^1(\mathbb{R})} \Big\| \frac{\widehat{F(\psi)}}{\hat{S}^{\frac{3}{8}}}(\,\cdot\,,\xi) \Big\|^2_{L^2(\mathbb{R})}. \qquad (2.3.18)
$$

We plug (2.3.18) into (2.3.16) to get

$$
\|\widehat{D}\|^2_{L^2(\mathbb{R}\times\mathbb{Z})} \le C\,\beta^{\frac{1}{2}} \Big\| \frac{\widehat{F(\psi)}}{\hat{S}^{\frac{3}{8}}} \Big\|^2_{L^2(\mathbb{R}\times\mathbb{Z})}. \qquad (2.3.19)
$$

Employing (2.3.3), we can rewrite the inequality (2.3.19) as

$$
\|\hat{D}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \le C\beta^{\frac{1}{4}} \|F(\psi)\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}. \qquad (2.3.20)
$$

Step 4. Let us express (2.3.6d) as

$$
\hat{E}(\tau,\xi) = -\sum_{k=1}^{\infty} \frac{(-2r)^{k-1}}{k!} \widehat{E_k}(\tau,\xi)
$$

with

$$
\widehat{E_k}(\tau,\xi) = \hat{S}^{\frac{3}{8}}(\tau,\xi)\, \widehat{a_\beta}^{(k)}(\tau+\xi^2) \int_{\mathbb{R}} \left[\frac{\sigma+\xi^2}{2r} \right]^{k-1} \widehat{F_n(\psi)}(\sigma,\xi)\, d\sigma. \qquad (2.3.21)
$$

In order to estimate $\|\hat{E}\|_{L^2(\mathbb{R}\times\mathbb{Z})}$, we will estimate $\|\widehat{E_k}\|_{L^2(\mathbb{R}\times\mathbb{Z})}$ and then using the triangle inequality, we will have

$$\|\hat{E}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq \sum_{k=1}^{\infty} \frac{(2r)^{k-1}}{k!} \|\widehat{E_k}\|_{L^2(\mathbb{R}\times\mathbb{Z})}. \tag{2.3.22}$$

For each $k \geq 1$ we have by (2.2.9d) and by Proposition 2.2.3 that

$$\begin{aligned}
\|\widehat{E_k}\|^2_{L^2(\mathbb{R}\times\mathbb{Z})} &= \sum_{\xi\in\mathbb{Z}} \int_{\mathbb{R}} (1+|\tau+\xi^2|)^{\frac{3}{4}} |\widehat{a_\beta}^{(k)}(\tau+\xi^2))|^2 \, |\widehat{B_k}(\xi)|^2 \, d\tau \\
&\overset{u=\tau+\xi^2}{=} \left(\int_{\mathbb{R}} (1+|u|)^{\frac{3}{4}} |\widehat{a_\beta}^{(k)}(u)|^2 \, du\right) \left(\sum_{\xi\in\mathbb{Z}} |\widehat{B_k}(\xi)|^2\right) \\
&\leq K^2(a)(3\beta)^{2k+\frac{1}{4}} \, \|\widehat{B_k}\|^2_{L^2(\mathbb{Z})}.
\end{aligned} \tag{2.3.23}$$

We will denote $\|\widehat{B_k}\|^2_{L^2(\mathbb{Z})}$ by Λ_k^2.

We utilize (2.2.9d) and Hölder's inequality to compute

$$\begin{aligned}
\Lambda_k^2 &\leq \sum_{\xi\in\mathbb{Z}} \left[\int_{|\tau+\xi^2|\leq 2r} |\frac{\sigma+\xi^2}{2r}|^{k-1} \, |\widehat{F(\psi)}(\sigma,\xi)| \, \hat{b}(\sigma+\xi^2) \, d\sigma\right]^2 \\
&\leq \sum_{\xi\in\mathbb{Z}} \left(\int_{|\sigma+\xi^2|\leq 2r} |\widehat{F(\psi)}(\sigma,\xi)| \, d\sigma\right)^2 \\
&\leq \sum_{\xi\in\mathbb{Z}} \left(\int_{\mathbb{R}} \frac{|\widehat{F(\psi)}(\sigma,\xi)|^2}{\hat{S}^{\frac{3}{4}}(\sigma,\xi)} \, d\sigma\right) \left(\int_{|\sigma+\xi^2|\leq 2r} (1+|\sigma+\xi^2|)^{\frac{3}{4}} \, d\sigma\right) \\
&\leq C(2r)^{\frac{7}{4}} \Big\| \frac{\widehat{F(\psi)}}{\hat{S}^{\frac{3}{8}}} \Big\|^2_{L^2(\mathbb{R}\times\mathbb{Z})}.
\end{aligned} \tag{2.3.24}$$

Inserting (2.3.24) and (2.3.23) into (2.3.22) gives the estimate

$$\begin{aligned}
\|\hat{E}\|_{L^2(\mathbb{R}\times\mathbb{Z})} &\leq (2r)^{\frac{7}{8}} C \Big\| \frac{\widehat{F(\psi)}}{\hat{S}^{\frac{3}{8}}} \Big\|_{L^2(\mathbb{R}\times\mathbb{Z})} \left(\sum_{k=1}^{\infty} \frac{(2r)^{k-1}(3\beta)^{k+\frac{1}{8}}}{k!}\right) \\
&= C \left(\frac{\beta}{r}\right)^{\frac{1}{8}} \left[\frac{e^{6r\beta}-1}{6r\beta}\right] \Big\| \frac{\widehat{F(\psi)}}{\hat{S}^{\frac{3}{8}}} \Big\|_{L^2(\mathbb{R}\times\mathbb{Z})}.
\end{aligned} \tag{2.3.25}$$

Since $r\delta = 1$, the last inequality is equivalent to

$$\|\hat{E}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq C\beta^{\frac{1}{4}}\|F(\psi)\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \tag{2.3.26}$$

because of Theorem 2.3.1.

Collecting equations (2.3.5), (2.3.7), (2.3.12), (2.3.20), and (2.3.26), we have the desired estimate (2.3.4). This concludes the proof.

□

2.4 Local Solutions

We want to prove existence of a global solution of (2.2.3), but we will start by constructing a local in time solution of the equation

$$i\partial_t\psi + \partial_{xx}\psi = -\frac{1}{2}|\psi|^2\psi + W(\psi)\psi, \;\; t \in \mathbb{R}, \;\; x \in \mathbb{T},$$

with initial condition

$$\psi(0, x) = \psi_o(x), \;\; x \in \mathbb{T},$$

using a fixed point argument. The construction will be accomplished in an interval $[0, \beta]$ with β chosen appropriately small. After the local construction, a global existence will be achieved by an iteration scheme.

In what follows, we assume that W is a complex-valued function satisfying the following conditions:

$$|W(u) - W(v)| \leq C|u - v|, \;\; u, v \in \mathbb{C}, \tag{2.4.1a}$$

$$|W(\psi)(t, \cdot)| \leq C(1 + \|\psi(\cdot, t)\|_{L^2(\mathbb{T})}), \tag{2.4.1b}$$

$$\Im W(u) \leq K, \;\; u \in \mathbb{C}, \tag{2.4.1c}$$

where K is a constant.

Remark. It is clear that if $\psi \in L^4(\mathbb{R} \times \mathbb{T})$, then

$$\||\psi|^2\psi\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} = \|\psi\|^3_{L^4(\mathbb{R}\times\mathbb{T})}. \tag{2.4.2}$$

Remark. In Section 1.4 we proved that the equation governing the binormal motion of curve ζ embedded in a three-dimensional Riemannian manifold M is given by (see (1.4.16))

$$i\partial_t\psi + \partial_{ss}\psi + \frac{1}{2}|\psi|^2\psi = W(\psi)\psi,$$

where R is the Riemannian curvature tensor of the manifold M, $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is the Frenet trihedron on ζ, and $W(\psi)$ stands for

$$\begin{aligned} W(\psi) \overset{\text{def}}{=} \; &\big\langle R(\mathbf{T}, \mathbf{B})\mathbf{T}, \mathbf{B}\big\rangle - \int |\psi| \big\langle R(\mathbf{T}, \mathbf{B})\mathbf{B}, \mathbf{N}\big\rangle \, ds \\ &- i\big\langle R(\mathbf{T}, \mathbf{B})\mathbf{T}, \mathbf{N}\big\rangle. \end{aligned}$$

If we assume that R is smooth and bounded, then the conditions (2.4.1b) and (2.4.1c) follow immediately.

Proposition 2.4.1. *Let $\psi, \phi \in L^4(\mathbb{R} \times \mathbb{T})$. If W satisfies condition* (2.4.1b)*, then*

$$\|a_\beta W(\phi)\psi\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \leq C\beta^{\frac{1}{4}} \left(\beta^{\frac{1}{4}} + \|\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}\right) \|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})}. \tag{2.4.3}$$

Proof. Since $\phi \in L^4(\mathbb{R}\times\mathbb{T})$, it follows that $\|\phi(t,\cdot)\|_{L^4(\mathbb{T})} < \infty$ almost everywhere in time. Therefore,

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{T})} \le C\|\phi(t,\cdot)\|_{L^4(\mathbb{T})},\ a.e.\ t\in\mathbb{R}. \tag{2.4.4}$$

Combining (2.4.1b) and (2.4.4), we have

$$|W(\phi)(t,\cdot)| \le C\left(1+\|\phi(t,\cdot)\|_{L^4(\mathbb{T})}\right). \tag{2.4.5}$$

By Hölder's inequality, (2.4.5) and the inequality $(1+y)^4 \le 8(1+y^4)$, $y\in\mathbb{R}$, we obtain

$$\begin{aligned}
\|a_\beta W(\phi)\psi\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} &\le \|a_\beta\|^{\frac{1}{2}}_{L^2(\mathbb{R}\times\mathbb{T})}\|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})}\left(\int_{\mathbb{R}\times\mathbb{T}}|a_\beta(t)|^2|W(\phi)(t,x)|^4\,dxdt\right)^{\frac{1}{4}}\\
&\le C\beta^{\frac{1}{4}}\|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})}\left(\int_{\mathbb{R}\times\mathbb{T}}|a_\beta(t)|^2\left(1+\|\phi(t,\cdot)\|^4_{L^4(\mathbb{T})}\right)\,dt\right)^{\frac{1}{4}}\\
&\le C\beta^{\frac{1}{4}}\left(\beta^{\frac{1}{4}}+\|\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}\right)\|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})}
\end{aligned} \tag{2.4.6}$$

as asserted. ☐

Let us recall the definition of the map T_β

$$T_\beta(\psi)(t) = a_\beta(t)\left(U(t)\psi_o - i\int_0^t U(t-s)F(\psi)(s)\,ds\right). \tag{2.4.7}$$

Since we are looking for a local solution of (2.2.3), we set

$$F(\psi) = -\frac{1}{2}|\psi|^2\psi + a_\beta W(\psi)\psi. \tag{2.4.8}$$

Theorem 2.4.2. *Assume that $\psi_o \in L^2(\mathbb{T})$. The map T_β, with F given by* (2.4.8), *is a contraction of the unit ball in $L^4(\mathbb{R}\times\mathbb{T})$ into itself, provided β is small enough. Moreover, β depends on the L^2-norm of the initial data ψ_o.*

Proof. We combine Theorem 2.3.2, equation (2.4.2), and Proposition 2.4.1 to get

$$\begin{aligned}\|T_\beta(\psi)\|_{L^4(\mathbb{R}\times\mathbb{T})} \leq C\beta^{\frac{1}{4}} \left[\|\psi\|^3_{L^4(\mathbb{R}\times\mathbb{T})} + \beta^{\frac{1}{4}} \left(\beta^{\frac{1}{4}} + \|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})}\right) \|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})}\right] \\ + C\beta^{\frac{1}{8}}\|\psi_o\|_{L^2(\mathbb{T})}. \qquad (2.4.9)\end{aligned}$$

Hence,

$$\|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})} \leq 1 \Rightarrow \|T_\beta(\psi)\|_{L^4(\mathbb{R}\times\mathbb{T})} \leq 1, \qquad (2.4.10)$$

provided we make the choice

$$\beta^{\frac{1}{4}} = \frac{1}{4C^2(\rho + \|\psi_o\|^2_{L^2(\mathbb{T})})}, \quad \rho \geq 3. \qquad (2.4.11)$$

We consider next the difference $T_\beta(\psi) - T_\beta(\phi)$. The first term in (2.4.7) disappears and $F(\psi)$ in the integral term has to be replaced by $F(\psi) - F(\phi)$. Repeating previous estimates in the proof of Theorem 2.3.2, we get

$$\|T_\beta(\psi) - T_\beta(\phi)\|_{L^4(\mathbb{R}\times\mathbb{T})} \leq C\beta^{\frac{1}{4}}\|F(\psi) - F(\phi)\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}. \qquad (2.4.12)$$

We have by Hölder's inequality

$$\begin{aligned}\|F(\psi) - F(\phi)\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \leq \frac{1}{2}\|\psi|\psi|^2 - \phi|\phi|^2\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \\ + \|a_\beta \left[W(\psi)\psi - W(\phi)\phi\right]\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}. \qquad (2.4.13)\end{aligned}$$

The following algebraic manipulations

$$\psi|\psi|^2 - \phi|\phi|^2 = (\psi - \phi)|\psi|^2 + \phi\overline{\psi}(\psi - \phi) + \phi^2(\overline{\psi} - \overline{\phi}), \qquad (2.4.14a)$$

$$W(\psi)\psi - W(\phi)\phi = W(\psi)(\psi - \phi) + (W(\psi) - W(\phi))\phi, \qquad (2.4.14b)$$

Hölder's inequality once more, Proposition 2.4.1, and condition (2.4.1a) give the estimates

$$\|\psi|\psi|^2-\phi|\phi|^2\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}\leq\left(\|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})}+\|\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}\right)^2\|\psi-\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}, \quad \text{(2.4.15a)}$$

$$\|a_\beta W(\psi)(\psi-\phi)\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}\leq C\beta^{\frac{1}{4}}\left(\beta^{\frac{1}{4}}+\|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})}\right)\|\psi-\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}, \quad \text{(2.4.15b)}$$

$$\|a_\beta\left(W(\psi)-W(\phi)\right)\phi\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}\leq C\beta^{\frac{1}{4}}\|\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}\|\psi-\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}. \quad \text{(2.4.15c)}$$

We substitute equations (2.4.15) back into (2.4.13) and then the resulting inequality back into (2.4.12) to obtain

$$\begin{aligned}\|T_\beta(\psi)-T_\beta(\phi)\|_{L^4(\mathbb{R}\times\mathbb{T})}\leq{}& C\beta^{\frac{1}{4}}\|\psi-\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}\left(\|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})}+\|\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}\right)^2\\ &+C\beta^{\frac{1}{2}}\left(\beta^{\frac{1}{4}}+\|\psi\|_{L^4(\mathbb{R}\times\mathbb{T})}+\|\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}\right)\\ &\cdot\|\psi-\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}.\end{aligned} \quad \text{(2.4.16)}$$

Hence, for β given by (2.4.11), we get the inequality

$$\|T_\beta(\psi)-T_\beta(\phi)\|_{L^4(\mathbb{R}\times\mathbb{T})}\leq\frac{1}{2}\|\psi-\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}. \quad \text{(2.4.17)}$$

This establishes the theorem. ☐

Picard's theorem yields a function $\psi\in L^4(\mathbb{R}\times\mathbb{T})$ satisfying $T_\beta(\psi)=\psi$. We therefore have proved the theorem

Theorem 2.4.3. *Consider the problem*

$$\begin{cases}i\partial_t\psi+\partial_{xx}\psi=-\frac{1}{2}|\psi|^2\psi+W(\psi)\psi, & t\in\mathbb{R},\ \ x\in\mathbb{T},\\ \psi(0,x)=\psi_o(x) & x\in\mathbb{T}.\end{cases} \quad \text{(2.4.18)}$$

Assume that the initial data $\psi_o \in L^2(\mathbb{T})$. Then the Cauchy problem (2.4.18) *has a unique weak solution for $t \in [-\beta, \beta]$ where β depends on the initial data.*

Remark. The same argument that we used in the construction of a local solution also proves well-posedness, i.e., continuous dependence on the initial data,

$$\|T_\beta(\psi) - T_\beta(\phi)\|_{L^4(\mathbb{R}\times\mathbb{T})} \leq C\|\psi_o - \phi_o\|_{L^2(\mathbb{T})}. \tag{2.4.19}$$

2.5 Global Solutions

Our next purpose is to show that problem (2.4.18) has a global solution. We constructed a solution of that problem in a slice

$$\mathbb{T} \times [0, \beta],$$

where β is given by

$$\beta = \frac{1}{\left(4C^2\left[\rho + \|\psi_o\|^2_{L^2(\mathbb{T})}\right]\right)^4}, \tag{2.5.1}$$

with ρ being any real number greater than 3 (recall (2.4.11)) and C is a given constant. The drawback of the previous argument in section 2.4 which establishes local well-posedness is that the size of β depends on L^2-norm of ψ on which we do not have a priori bound.

To simplify matters, we assume that ψ_o is smooth so that ψ is also smooth and we adopt the notation $\mathrm{X}(t) = \|\psi(t,\cdot)\|^2_{L^2(\mathbb{T})}$. Although $\mathrm{X}(t)$ is not conserved by the flow in (2.4.18), we have an inequality of the form

$$\frac{d\mathrm{X}(t)}{dt} \leq K\mathrm{X}(t), \tag{2.5.2}$$

by invoking the condition (2.4.1c), where K is an a priori constant. Therefore,

$$X(\tau) \leq e^{K(\tau-\sigma)} X(\sigma). \tag{2.5.3}$$

In order to show global existence, let us start setting $t_0 = \beta_0 = 0$. We next define

$$\beta_1 \stackrel{\text{def}}{=} \beta = \frac{1}{\left(4C^2\left[\rho + X(t_0)\right]\right)^4},$$
$$t_1 \stackrel{\text{def}}{=} t_0 + \beta_1,$$

and by induction

$$\beta_{n+1} \stackrel{\text{def}}{=} \frac{1}{\left(4C^2\left[\rho + X(t_n)\right]\right)^4}, \tag{2.5.4a}$$
$$t_{n+1} \stackrel{\text{def}}{=} t_n + \beta_{n+1} \quad . \tag{2.5.4b}$$

Setting

$$X_n = X(t_n), \tag{2.5.5}$$
$$\lambda = \frac{K}{4^4 C^8}, \tag{2.5.6}$$

the estimate in (2.5.3) together with (2.5.4) implies that

$$X_{n+1} \leq \exp\left(\frac{\lambda}{\left(\rho + X_n\right)^4}\right) X_n. \tag{2.5.7}$$

Denoting the constant $|\lambda| e^{4\lambda}$ by ν, we now claim

Proposition 2.5.1. *There exists a number $\rho \geq 3$ such that if n is any nonnegative integer, then the following inequality holds:*

$$\mathrm{X}_n \leq \sqrt[4]{(\rho + \mathrm{X}_0)^4 + 8\nu n}. \tag{2.5.8}$$

Proof. If $\lambda \leq 0$, then equation (2.5.7) implies that

$$\mathrm{X}_{n+1} \leq \mathrm{X}_n, \quad n \in \mathbb{Z}, \ n \geq 0, \tag{2.5.9}$$

from where (2.5.8) is deduced for any $\rho \geq 3$.

On the other hand, inequality (2.5.8) is obviously true for $n = 0$. Assume that $\mathrm{X}_n \leq \sqrt[4]{(\rho + \mathrm{X}_0)^4 + 8\nu n}$. We consider two cases:

$$\mathrm{X}_n \leq e^{-\lambda} \sqrt[4]{(\rho + \mathrm{X}_0)^4 + 8\nu n}, \tag{A}$$

$$\mathrm{X}_n \geq e^{-\lambda} \sqrt[4]{(\rho + \mathrm{X}_0)^4 + 8\nu n}. \tag{B}$$

For the case (A), we have by (2.5.7) and by induction hypothesis

$$\begin{aligned}
\mathrm{X}_{n+1} \leq &\ \exp\left(\frac{\lambda}{(\rho + \mathrm{X}_n)^4}\right) \mathrm{X}_n \\
\leq &\ e^{\lambda} \mathrm{X}_n \\
\leq &\ e^{\lambda} e^{-\lambda} \left[(\rho + \mathrm{X}_0)^4 + 8\nu n\right]^{\frac{1}{4}} \\
\leq &\ \left[(\rho + \mathrm{X}_0)^4 + 8\nu(n+1)\right]^{\frac{1}{4}}.
\end{aligned}$$

We next consider case (B). Using again (2.5.7) and induction hypothesis we obtain

$$\begin{aligned}
\mathrm{X}_{n+1} \leq &\ \exp\left(\frac{\lambda}{(\rho + \mathrm{X}_n)^4}\right) \mathrm{X}_n \\
\leq &\ \exp\left(\frac{\lambda}{(\rho + \mathrm{X}_n)^4}\right) \left[(\rho + \mathrm{X}_0)^4 + 8\nu n\right]^{\frac{1}{4}} \\
\leq &\ \exp\left[\frac{\nu}{(\rho + \mathrm{X}_0)^4 + 8\nu n}\right] \left[(\rho + \mathrm{X}_0)^4 + 8\nu n\right]^{\frac{1}{4}}.
\end{aligned} \tag{2.5.10}$$

It is therefore enough to check that

$$
\begin{aligned}
&\left[(\rho+\mathrm{X}_0)^4+8\nu n\right]^{\frac{1}{4}} \exp\left[\frac{\nu}{(\rho+\mathrm{X}_0)^4+8\nu n}\right] \\
&\leq \left[(\rho+\mathrm{X}_0)^4+8\nu(n+1)\right]^{\frac{1}{4}}, \qquad (2.5.11)
\end{aligned}
$$

or equivalently,

$$
\frac{\nu}{(\rho+\mathrm{X}_0)^4+8\nu n} \leq \frac{1}{4}\ln\left(1+\frac{8\nu}{(\rho+\mathrm{X}_0)^4+8\nu n}\right). \qquad (2.5.12)
$$

Let

$$
x_n = \frac{\nu}{(\rho+\mathrm{X}_0)^4+8\nu n}.
$$

Then inequality (2.5.12) can be written as

$$
4x_n \leq \ln(1+8x_n). \qquad (2.5.13)
$$

Since $\ln(1+8x)-4x \geq 0$ holds for $x \in [0,0.3]$, we choose $\rho \geq 3$ such that

$$
\frac{\nu}{(\rho+\|\psi_o\|^2_{L^2(\mathbb{T})})^4} \leq 0.3. \qquad (2.5.14)
$$

Hence, inequality (2.5.13) holds by (2.5.14) proving the claim made above.

□

Corollary 2.5.2. *The Cauchy problem* (2.4.18) *is globally well-posed.*

Proof. Inequality (2.5.8) and the inequality $(a+b)^4 \leq 8\left(a^4+b^4\right)$, $a, b \in \mathbb{R}$ allow us to compute

$$
\begin{aligned}
t_{n+1} = \sum_{k=0}^{n} \beta_{k+1} &= \sum_{k=0}^{n} \frac{C}{(\rho + \mathrm{X}_k)^4} \\
&\geq \frac{C}{8} \sum_{k=0}^{n} \frac{1}{\rho^4 + \mathrm{X}_k^4} \\
&\geq \frac{C}{8} \sum_{k=0}^{n} \frac{1}{\rho^4 + (\rho + \mathrm{X}_0)^4 + 8\nu k} \\
&\geq \frac{C}{8} \sum_{k=0}^{n} \frac{1}{9\rho^4 + 8\mathrm{X}_0 + 8\nu k}. \qquad (2.5.15)
\end{aligned}
$$

Therefore, $t_{n+1} \sim \sum_{k=1}^{n} \frac{1}{k}$ implying that $t_n \to \infty$. This gives the desired conclusion.

$\square$

2.6 A Schrödinger Multiplier Estimate

We begin by restating Theorem 2.3.1.

Theorem 2.6.1. *Let f(t,x) be a function with $(t,x) \in \mathbb{R} \times \mathbb{T}$ and denote by $\hat{f}(\tau,\xi)$ its Fourier transform, with $(\tau,\xi) \in \mathbb{R} \times \mathbb{Z}$. The following estimates hold*

$$
\|f\|_{L^4(\mathbb{R}\times\mathbb{T})} \leq C\|\hat{S}^{\frac{3}{8}}\hat{f}\|_{L^2(\mathbb{R}\times\mathbb{Z})}. \qquad (2.6.1)
$$

and its dual

$$
\left\| \frac{\hat{f}}{\hat{S}^{\frac{3}{8}}} \right\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq C\|f\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}. \qquad (2.6.2)
$$

Proof. With some modifications, we will proceed as in the proof of Theorem 2.1 in [7]. We start making a dyadic decomposition of $\hat{f}$ as $\hat{f} = \sum_j \widehat{f_j}$ with $\widehat{f_j}$ supported in the region $2^{j-1} \leq \tau + \xi^2 \leq 2^{j+1}$. Without loss of generality, we can assume that $j \geq 1$. Next, we write

$$\hat{f} = \sum_j \widehat{f}_j \leftrightarrow f = \sum_j f_j, \tag{2.6.3}$$

where $f_j = \mathcal{F}^*[\widehat{f}_j]$ with $\mathcal{F}^*$ denoting inverse Fourier transform. Let us consider the quadratic expression

$$f_j f_k(t,x) = C \int \sum_{\xi_1.,\xi_2} e^{i[t(\tau_1+\tau_2)+x(\xi_1+\xi_2)]} \widehat{f}_j(\tau_1,\xi_1)\ \widehat{f}_k(\tau_2,\xi_2)\, d\tau_1\, d\tau_2. \tag{2.6.4}$$

Again, without loss of generality, we can suppose $j \geq k$. Our next step is to change variables as follows:

$$\xi_1 + \xi_2 = \xi, \qquad\qquad \xi_1^2 + \xi_2^2 = \nu, \tag{2.6.5a}$$

$$\tau_1 + \tau_2 = \tau, \qquad\qquad \tau_2 + \xi_2^2 = \rho, \tag{2.6.5b}$$

i.e., $(\tau_1, \tau_2, \xi_1, \xi_2) \longmapsto (\tau, \rho, \xi, \nu)$. Observe that the inverse formulae are

$$\begin{aligned} \xi_{1,2} &= \frac{\xi \pm \sqrt{2\nu - \xi^2}}{2}, \\ \tau_2 &= \rho - \xi_2^2 \quad \text{with} \quad 2^{k-1} \leq \rho \leq 2^{k+1}, \\ \tau_1 &= \tau - \tau_2. \end{aligned} \tag{2.6.6}$$

Let us call $\mu \stackrel{\text{def}}{=} \tau_1 + \xi_1^2$ so that $2^{j+1} \leq \mu \leq 2^{j+1}$ Since we have $2^{k-1} + 2^{j-1} \leq \rho + \mu \leq 2^{k+1} + 2^{j+1}$, we conclude that

$$2^{k-1} + 2^{j-1} - \tau \leq \gamma \leq 2^{k+1} + 2^{j+1} - \tau, \tag{2.6.7a}$$

$$\gamma \stackrel{\text{def}}{=} \xi_1^2 + \xi_2^2. \tag{2.6.7b}$$

Equation (2.6.4) can be expressed in the new variables as

$$f_j f_k(t,x) = C \int \sum_{\xi} e^{i(t\tau+x)} \int_{\rho\sim 2^k} \sum_{\gamma} \widehat{f}_j(\tau_1,\xi_1)\ \widehat{f}_k(\tau_2.\xi_2)\, d\rho\, d\tau. \tag{2.6.8}$$

We apply Plancherel's theorem to the equation above to obtain

$$\|f_j f_k\|^2_{L^2(\mathbb{R}\times\mathbb{T})} = \|\int \sum_{\gamma} \widehat{f}_j \widehat{f}_k\, d\rho\|^2_{L^2(\mathbb{R}\times\mathbb{T})} = \int \sum_{\xi} |\int \sum_{\gamma} \widehat{f}_j \widehat{f}_k|^2\, d\tau, \tag{2.6.9}$$

which is bounded by

$$B \equiv 2^k |\{\#\gamma\}| \int \sum_{\xi,\gamma} |\widehat{f}_j|^2 |\widehat{f}_k|^2\, d\tau\, d\rho.$$

This bound can be estimated in an equivalent way as

$$\begin{aligned} B &= 2^k |\{\#\gamma\}| \int \sum_{\xi_1,\xi_2} |\widehat{f}_j\,(\tau_1,\xi_1)|^2 |\widehat{f}_k\,(\tau_2,\xi_2)|^2\, d\tau_1\, d\tau_2 \\ &= 2^k |\{\#\gamma\}| \|\widehat{f}_j\|^2_{L^2(\mathbb{R}\times\mathbb{T})} \|\widehat{f}_k\|^2_{L^2(\mathbb{R}\times\mathbb{T})}, \end{aligned} \tag{2.6.10}$$

where $|\{\#\gamma\}|$is the number of γ that satisfies restriction (2.6.7) for a given $\xi \in \mathbb{Z}$, i.e.,

$$\{\#\gamma\} = \{\gamma = \xi_1^2 + \xi_2^2 \in \mathbb{Z}^+\ :\ \xi = \xi_1 + \xi_2\, ;\ 2^{k-1} + 2^{j-1} - \tau \le \gamma \le 2^{k+1} + 2^{j+1} - \tau\}. \tag{2.6.11}$$

As in [7], there is a constant C such that the size of the set $\{\#\gamma\}$ satisfies

$$|\{\#\gamma\}| \le C 2^{\frac{j}{2}}. \tag{2.6.12}$$

Therefore, equation (2.6.9) reads

$$\|f_j f_k\|^2_{L^2(\mathbb{R}\times\mathbb{Z})} \le 2^k 2^{\frac{j}{2}} \|f_j\|^2_{L^2(\mathbb{R}\times\mathbb{Z})} \cdot \|f_k\|^2_{L^2(\mathbb{R}\times\mathbb{Z})}. \tag{2.6.13}$$

The triangle inequality and estimate (2.6.13) allow us to compute

$$\begin{aligned}\|f\|_{L^2(\mathbb{R}\times\mathbb{Z})} &\le \sum_{j,k} \frac{1}{2^{\frac{j-k}{4}}} \left[2^{\frac{3k}{8}} \|f_k\|_{L^2(\mathbb{R}\times\mathbb{Z})}\right] \left[2^{\frac{3j}{8}} \|f_j\|_{L^2(\mathbb{R}\times\mathbb{Z})}\right] \\ &\le C \left[2^{\frac{3j}{8}} \|f_j\|_{L^2(\mathbb{R}\times\mathbb{Z})}\right]^2. \end{aligned} \tag{2.6.14}$$

The preceding equation can be rewritten as

$$\begin{aligned}\|f\|^2_{L^4(\mathbb{R}\times\mathbb{T})} &\le C \left[2^{\frac{3j}{8}} \|f_j\|_{L^2(\mathbb{R}\times\mathbb{Z})}\right]^2 \\ &\sim C\, \|\hat{S}^{\frac{3}{8}} \hat{f}\|_{L^2(\mathbb{R}\times\mathbb{Z})}. \end{aligned} \tag{2.6.15}$$

This proves (2.6.1).

In order to prove (2.6.2), we just write for a test function h

$$\begin{aligned}|<\ \hat{f}\,\hat{S}^{-\frac{3}{8}}, \hat{h}\ >| &= |<\ f, \mathcal{F}^*\left(\hat{h}\,\hat{S}^{-\frac{3}{8}}\right)\ >| \\ &\le \|f\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \|\mathcal{F}^*\left(\hat{h}\,\hat{S}^{-\frac{3}{8}}\right)\|_{L^4(\mathbb{R}\times\mathbb{T})}, \end{aligned} \tag{2.6.16}$$

with the help of Hölder inequality. Finally, estimate (2.6.1) implies that the right-hand side of (2.6.16) is bounded by

$$\|f\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \|\hat{h}\|_{L^2(\mathbb{R}\times\mathbb{T})}.$$

This finishes the proof. $\square$

Chapter 3

Non-periodic Solutions

3.1 Introduction

This present chapter concentrates on global existence of solutions of the initial value problem for the nonlinear Schrödinger-type equation (NLSE)

$$i\partial_t\phi + \partial_{xx}\phi = -\frac{1}{2}|\phi|^2\phi + W(\phi)\phi, \;\; (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{3.1.1a}$$

with initial condition

$$\phi(0,x) = \phi_o(x), \;\; x \in \mathbb{R}. \tag{3.1.1b}$$

Here W denotes a complex-valued function.

As in the periodic case, the problem (3.1.1) will be rewritten conveniently in the the integral equation

$$\phi(t) = \mathcal{U}(t)\phi_o - i\int_0^t \mathcal{U}(t-s)(-\frac{1}{2}|\phi|^2\phi + W(\phi)\phi)(s)\,ds, \tag{3.1.2}$$

where the Schrödinger operator $\mathcal{U}(t) := e^{it\Delta}$ defines an one-parameter unitary group.

The existence of a solution for small enough t (local existence) will be obtained by constructing a family of approximate solutions. Although there is no conservation of

$\|\phi(\cdot,t)\|_{L^2(\mathbb{R})}$ under flow, existence for all time (global existence) holds by extending the local solution in the large in time by means of priori estimates for the L^2-norm of the local solution.

3.2 A Basic Framework

In the following, $\phi(t,x)$ will denote a complex-valued function of two variables (t,s) with $t \in \mathbb{R}$ the time variable and $x \in \mathbb{R}$ the space variable. Numerical constants will be also denoted by C as in Chapter 2.

Let us recall that the equation

$$\begin{cases} i\partial_t\phi + \partial_{xx}\phi = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ \phi(0,x) = \phi_o(x), & x \in \mathbb{R}, \end{cases} \tag{3.2.1}$$

is solved as $\phi(t,x) = \mathcal{U}(t)\phi_o(x)$, where $\mathcal{U}(t)$ given by

$$\mathcal{U}(t)\phi_o(x) = \int_{\mathbb{R}} \widetilde{\phi}_o(\xi)e^{i(x\xi - t\xi^2)}\,d\xi \tag{3.2.2}$$

defines a unitary transformation group in L^2.

The observation that (3.1.1) can be reformulated equivalently as the integral equation (3.1.2) is a direct consequence of the next proposition.

Proposition 3.2.1. *Consider the Cauchy problem*

$$i\phi_t + \phi_{xx} = F\,, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{3.2.3a}$$

$$\phi(0,x) = \phi_o(x), \quad x \in \mathbb{R}, \tag{3.2.3b}$$

where F is a complex-valued function of the variable (t,x). If ϕ is a solution of (3.2.3), *then ϕ satisfies the integral equation*

$$\phi(t) = \mathcal{U}(t)\phi_o - i\int_0^t \mathcal{U}(t-s)F(s)\,ds. \tag{3.2.4}$$

Proof. Proof is quite similar to that for Proposition 2.2.4. Calculating the partial Fourier transform of the solution ϕ to equation (3.2.3), we get

$$i\widetilde{\phi}_t - \xi^2\widetilde{\phi} = \widetilde{F(t,\cdot)}, \tag{3.2.5a}$$

$$\widetilde{\phi}(0,\xi) \;\; = \widehat{\phi}_o(\xi). \tag{3.2.5b}$$

We next solve (3.2.5). Its solution can be expressed as

$$\widetilde{\phi}(t,\xi) = e^{-it\xi^2}\widehat{\phi}_o(\xi) - i\int_0^t e^{-i(t-s)\xi^2}\widetilde{F(s,\cdot)}(\xi)\,ds. \tag{3.2.6}$$

Equation (3.2.4) follows from (3.2.6) by taking inverse Fourier transform. This completes the proof.

$\square$

Exactly as in Chapter 2, we introduce a smooth function $\widehat{b}$ such that $0 \le \widehat{b} \le 1$, $\widehat{b} = 1$ on a neighborhood $[-r, r]$ of 0 and $\operatorname{supp}\widehat{b} \subseteq [-2r, 2r]$, where r is a positive number to be determined later. Let $\widehat{c}$ be the function defined by $\widehat{c} \overset{\text{def}}{=} 1 - \widehat{b}$.

The same argument used to prove Proposition 2.2.8 can be utilized to prove the next result. Its proof therefore will be omitted.

Proposition 3.2.2. *If ϕ is a solution of* (3.2.3)*, then $\widehat{\phi}$ can be written as follows*

$$\begin{aligned}\widehat{\phi}(\tau,\xi) &= \delta(\tau+\xi^2)\left(\widehat{\phi}_o(\xi) + \widehat{B_0}(\xi)\right) - \frac{\widehat{F}_f(\tau,\xi)}{\tau+\xi^2} \\ &\quad - \sum_{k=1}^{\infty} \frac{(-2r)^{k-1}}{k!}\;\delta^{(k)}(\tau+\xi^2)\;\widehat{B_k}(\xi),\end{aligned} \tag{3.2.7}$$

where

$$\widehat{F}_f(\tau,\xi) = \widehat{F}(\tau,\xi)\;\;\hat{c}(\tau+\xi^2), \tag{3.2.8a}$$

$$\widehat{F}_n(\tau,\xi) = \widehat{F}(\tau,\xi)\;\;\hat{b}(\tau+\xi^2), \tag{3.2.8b}$$

$$\widehat{B_0}(\xi) = \int_{\mathbb{R}} \frac{\widehat{F}_f(\sigma,\xi)}{\sigma+\xi^2}\,d\sigma, \tag{3.2.8c}$$

$$\widehat{B_k}(\xi) = \int_{\mathbb{R}} \left[\frac{\sigma+\xi^2}{2r}\right]^{k-1} \widehat{F}_n(\sigma,\xi)\,d\sigma,\;\; k=1,2,\ldots \tag{3.2.8d}$$

Since $\widehat{\phi}$ contains the delta function and its derivatives, we need to localize them in the variable t. For this reason. we consider a smooth cutoff function $a(t)$ which is 1 if $|t| \leq 1$, identically 0 if $|t| \geq 2$, and $0 \leq a(t) \leq 1$ for any real number t. Denote by $a_\beta(t) = a(\frac{t}{\beta}),\ \beta > 0$, its dilation. From (3.2.7) $\widehat{a_\beta\phi}$ is then given by the formulae

$$\begin{aligned}\widehat{a_\beta\phi}(\tau,\xi) = {} & \widehat{a_\beta}(\tau+\xi^2)\left(\widehat{\phi_o}(\xi)+\widehat{B_0}(\xi)\right) - \widehat{\Phi}(\tau,\xi)\\ & - \sum_{k=1}^{\infty} \frac{(-2r)^{k-1}}{k!}\;\; \widehat{a_\beta}^{(k)}(\tau+\xi^2)\,\widehat{B_k}(\xi)\end{aligned} \tag{3.2.9}$$

with

$$\widehat{\Phi}(\tau,\xi) = (\widehat{a_\beta} * \widehat{G})(\tau,\xi), \qquad \widehat{G}(\tau,\xi) = \frac{\widehat{F}_f(\tau,\xi)}{\tau+\xi^2}. \tag{3.2.10}$$

3.3 A Basic Estimate

Let us define the multiplier

$$\hat{S}(\tau,\xi) = 1 + |\tau+\xi^2|. \tag{3.3.1}$$

Theorem 3.3.1. *Assume that ϕ is a solution of* (3.2.3)*, with $\phi_o \in L^2(\mathbb{R})$, and that $\frac{1}{2} < \gamma < 1$. Then there is a constant C(γ, a) such that*

$$\|\hat{S}^\gamma \widehat{a_\beta \phi}\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq C(a,\gamma) \left(\frac{1}{\beta^{\gamma-\frac{1}{2}}} \|\phi_o\|_{L^2(\mathbb{R})} + \left\| \frac{\widehat{F}}{\hat{S}^{1-\gamma}} \right\|_{L^2(\mathbb{R}\times\mathbb{R})} \right). \tag{3.3.2}$$

Proof. We will proceed in a similar way as in the proof of Theorem (2.3.2). In order to bound the left member of (3.3.2), we need to consider the contributions

$$\hat{A}(\tau,\xi) = \hat{S}^\gamma(\tau,\xi)\widehat{a_\beta}(\tau+\xi^2)\widehat{\phi_o}(\xi), \tag{3.3.3a}$$

$$\hat{B}(\tau,\xi) = \hat{S}^\gamma(\tau,\xi)\widehat{B_0}(\xi), \tag{3.3.3b}$$

$$\hat{D}(\tau,\xi) = -\hat{S}^\gamma(\tau,\xi)(\widehat{a_\beta} * \widehat{G})(\tau,\xi), \qquad \widehat{G}(\tau,\xi) = \frac{\widehat{F}_f(\tau,\xi)}{\tau+\xi^2}, \tag{3.3.3c}$$

$$\hat{E}(\tau,\xi) = -\hat{S}^{\frac{3}{8}}(\tau,\xi) \sum_{k=1}^{\infty} \frac{(-2r)^{k-1}}{k!} \; \widehat{a_\beta}^{(k)}(\tau+\xi^2)\, \widehat{B_k}(\xi), \tag{3.3.3d}$$

according to equation (3.2.9). We then proceed in four steps.

Step 1. We employ (2.2.18) with $k=0$ to bound the L^2-norm of $\hat{A}$. In fact, we have

$$\begin{aligned} \|\hat{A}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} &= \int_{\mathbb{R}^2} |\widehat{a_\beta}(\tau+\xi^2)|^2 \hat{S}^{2\gamma}(\tau,\xi)|\widehat{\phi_o}(\xi)|^2\, d\tau\, d\xi \\ &= \left[\int_{\mathbb{R}} |\widehat{a_\beta}(\sigma)|^2 \left(1+|\sigma|\right)^{2\gamma} \, d\sigma \right] \int_{\mathbb{R}} |\widehat{\phi_o}(\xi)|^2\, d\xi \\ &\leq \; K^2(\gamma,a)\beta^{1-2\gamma}\|\phi_o\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{3.3.4}$$

Therefore,

$$\|\hat{A}\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq C(\gamma,a)\beta^{\frac{1}{2}-\gamma}\|\phi_o\|_{L^2(\mathbb{R})}, \tag{3.3.5}$$

as desired.

Step 2. By calculations similar to those leading to (3.3.4), we can obtain

$$\|\hat{B}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \leq C(\gamma,a)\beta^{1-2\gamma}\|\widehat{B_0}\|^2_{L^2(\mathbb{R})}. \tag{3.3.6}$$

Now, we have by Hölder's inequality

$$\begin{aligned}
\|\widehat{B_0}\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\widehat{F}(\tau,\xi)}{\tau+\xi^2}\, \hat{c}(\tau+\xi^2)\, d\tau \right)^2 d\xi \\
&\leq \int \left[\int \frac{|\widehat{F}(\tau,\xi)|^2}{(1+|\tau+\xi^2|)^{2-2\gamma}}\, d\tau \right] \\
&\quad \cdot \left[\int \frac{(1+|\tau+\xi^2|)^{2-2\gamma}}{|\tau+\xi^2|^2} |\hat{c}(\tau+\xi^2)|^2\, d\tau \right] d\xi \\
&= \left\| \frac{\widehat{F}}{\hat{S}^{1-\gamma}} \right\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \int_{|u|\geq r} \frac{(1+|u|)^{2-2\gamma}}{u^2}\, |\hat{c}(u)|^2\, du.
\end{aligned} \tag{3.3.7}$$

Assuming that $r \geq 1$ the last term of (3.3.7) is estimated in the form

$$\begin{aligned}
\|\widehat{B}_0\|^2_{L^2(\mathbb{R})} &\leq 2^{3-2\gamma} \left\| \frac{\widehat{F}}{\hat{S}^{1-\gamma}} \right\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \int_r^\infty \frac{1}{u^{2\gamma}}\, du \\
&\leq \frac{2^{3-2\gamma}}{2\gamma-1}\, r^{1-2\gamma} \left\| \frac{\widehat{F}}{\hat{S}^{1-\gamma}} \right\|^2_{L^2(\mathbb{R}\times\mathbb{R})}.
\end{aligned} \tag{3.3.8}$$

At this stage, we have proved that

$$\|\widehat{B}_0\|^2_{L^2(\mathbb{R})} \leq C(\gamma,a) r^{1-2\gamma} \left\| \frac{\widehat{F}}{\hat{S}^{1-\gamma}} \right\|^2_{L^2(\mathbb{R}\times\mathbb{R})}. \tag{3.3.9}$$

Substituting (3.3.9) into (3.3.6) and setting $r\beta = 1$, we get

$$\|\hat{B}\|_{L^2(\mathbb{R}\times\mathbb{Z})} \leq C(\gamma,a) \left\| \frac{\widehat{F}}{\hat{S}^{1-\gamma}} \right\|_{L^2(\mathbb{R}\times\mathbb{R})}. \tag{3.3.10}$$

Step 3. We utilize (3.2.10) and the triangle inequality to write

$$
\begin{aligned}
\|\hat{D}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} &= \int_{\mathbb{R}^2} |\int_{\mathbb{R}} \hat{S}^\gamma(\tau,\xi)\,\widehat{a_\beta}(\tau-\sigma)\,\frac{\widehat{F}(\sigma,\xi)}{\sigma+\xi^2}\,d\sigma|^2\,d\tau\,d\xi \\
&\leq \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} \left(|\tau-\sigma|^\gamma + \hat{S}^\gamma(\sigma,\xi)\right)\,|\widehat{a_\beta}(\tau-\sigma)||\widehat{G}(\sigma,\xi)|\,d\sigma\right]^2\,d\tau\,d\xi \\
&\leq 2\int_{\mathbb{R}^2} \left[J^2_{\beta,\gamma}(\tau,\xi) + H^2_{\beta,\gamma}(\tau,\xi)\right]\,d\tau\,d\xi, \qquad (3.3.11)
\end{aligned}
$$

where

$$
\begin{aligned}
J_{\beta,\gamma}(\tau,\xi) &\stackrel{\text{def}}{=} \int_{\mathbb{R}} |\tau-\sigma|^\gamma\,|\widehat{a_\beta}(\tau-\sigma)||\widehat{G}(\sigma,\xi)|\,d\sigma, \\
H_{\beta,\gamma}(\tau,\xi) &\stackrel{\text{def}}{=} \int_{\mathbb{R}} \hat{S}^\gamma(\sigma,\xi)\,|\widehat{a_\beta}(\tau-\sigma)||\widehat{G}(\sigma,\xi)|\,d\sigma.
\end{aligned}
$$

Applying Hölder's inequality, we can see that

$$
\begin{aligned}
J^2_{\beta,\gamma}(\tau,\xi) &\leq \left[\int_{\mathbb{R}} \frac{|\widehat{F}(\sigma,\xi)|^2}{(1+|\sigma+\xi^2|)^{2-2\gamma}}\,d\sigma\right] \\
&\quad \cdot \left[\int_{\mathbb{R}} \frac{|\tau-\sigma|^{2-2\gamma}}{|\sigma+\xi^2|^2}\;|\widehat{a_\beta}(\tau-\sigma)|^2\,\hat{S}^{2-2\gamma}(\sigma,\xi)\,|\hat{c}(\sigma+\xi^2)|^2\,d\sigma\right] \\
&\equiv \left(\int_{\mathbb{R}} \frac{|\widehat{F}(\sigma,\xi)|^2}{(1+\sigma+\xi^2)^{2-2\gamma}}\,d\sigma\right) V^2_{\beta,\gamma}(\tau,\xi). \qquad (3.3.12)
\end{aligned}
$$

The changes of variables $u=\tau+\xi^2$ and $\rho=\sigma+\xi^2$ show that $\Upsilon^2_{\beta,\gamma} \stackrel{\text{def}}{=} \int_{\mathbb{R}} V^2_\beta(\tau,\xi)\,d\tau$ does not depend on the variable ξ. Moreover, a straightforward computation gives

$$
\Upsilon^2_{\beta,\gamma} \leq \left(\int_{\mathbb{R}} (1+|v|)^{2\gamma}\,|\widehat{a_\beta}(v)|^2\,dv\right)\left(\int_{|\rho|\geq r} \frac{(1+|\rho|)^{2-2\gamma}}{\rho^2}\,d\rho\right). \qquad (3.3.13)
$$

Whereas Proposition (2.2.3) implies

$$
\int_{\mathbb{R}} (1+|v|)^{2\gamma}\,|\widehat{a_\beta}(v)|^2\,dv \leq K^2(\gamma,a)\beta^{1-2\gamma}, \qquad (3.3.14)
$$

conditions $r \geq 1$ and $2\gamma > 1$ yield

$$\int_{|\rho|\geq r} \frac{(1+|\rho|)^{2-2\gamma}}{\rho^2}\, d\rho \leq C(\gamma) r^{1-2\gamma}. \tag{3.3.15}$$

Since $r\beta = 1$, the two last inequalities allow us to deduce

$$\Upsilon^2_{\gamma,\beta} \leq C(\gamma, a). \tag{3.3.16}$$

Therefore, equation (3.3.11) transforms into

$$\|\hat{D}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \leq C^2(\gamma, a)\Big\|\frac{\widehat{F}}{\hat{S}^{1-\gamma}}\Big\|^2_{L^2(\mathbb{R}\times\mathbb{R})} + 2\int_{\mathbb{R}^2} H^2_\beta(\tau,\xi)\, d\tau d\xi. \tag{3.3.17}$$

On the other hand, we have

$$\begin{aligned} H_{\beta,\gamma}(\tau,\xi) &= \int_{\mathbb{R}} \frac{\hat{S}(\sigma,\xi)}{|\sigma+\xi^2|}\, |\widehat{a_\beta}(\tau-\sigma)|\, \frac{|\widehat{F}(\sigma,\xi)|}{\hat{S}^{1-\gamma}(\sigma,\xi)}\, \hat{c}(\sigma+\xi^2)\, d\sigma \\ &\leq 2\left(|\widehat{a_\beta}| * \left(\frac{|\widehat{F}|}{\hat{S}^{1-\gamma}}(\cdot,\xi)\right)\right)(\tau). \end{aligned} \tag{3.3.18}$$

We apply Young's inequality, Theorem 4.2 in [11], to obtain

$$\|H_{\beta,\gamma}(\cdot\,,\xi)\|^2_{L^2(\mathbb{R})} \leq\, C\|\widehat{a}\|^2_{L^1(\mathbb{R})}\, \Big\|\frac{\widehat{F}}{\hat{S}^{1-\gamma}}(\,\cdot\,,\xi)\Big\|^2_{L^2(\mathbb{R})}. \tag{3.3.19}$$

As consequence of the inequalities (3.3.17) and (3.3.19), we have

$$\|\widehat{D}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \leq C(\gamma, a)\Big\|\frac{\widehat{F}}{\hat{S}^{1-\gamma}}\Big\|^2_{L^2(\mathbb{R}\times\mathbb{R})}. \tag{3.3.20}$$

Step 4. Let us express (3.3.3d) as

$$\hat{E}(\tau,\xi) = -\sum_{k=1}^{\infty} \frac{(-2r)^{k-1}}{k!}\widehat{E_k}(\tau,\xi) \tag{3.3.21a}$$

with

$$\widehat{E_k}(\tau,\xi)=\hat{S}^{2\gamma}(\tau,\xi)\,\widehat{a_\beta}^{(k)}(\tau+\xi^2)\int_{\mathbb{R}}\left[\frac{\sigma+\xi^2}{2r}\right]^{k-1}\widehat{F}_n(\sigma,\xi)\,d\sigma. \tag{3.3.21b}$$

According to (3.3.21), we will estimate $\|\widehat{E_k}\|_{L^2(\mathbb{R}\times\mathbb{R})}$ and then using the triangle inequality, we will have

$$\|\hat{E}\|_{L^2(\mathbb{R}\times\mathbb{Z})}\leq\sum_{k=1}^{\infty}\frac{(2r)^{k-1}}{k!}\|\widehat{E_k}\|_{L^2(\mathbb{R}\times\mathbb{Z})}. \tag{3.3.22}$$

For each $k\geq 1$ we have by (3.2.8d) and by Proposition 2.2.3

$$\begin{aligned}\|\widehat{E_k}\|^2_{L^2(\mathbb{R}\times\mathbb{R})}&=\int_{\mathbb{R}^{\not{2}}}(1+|\tau+\xi^2|)^{2\gamma}|\widehat{a_\beta}^{(k)}(\tau+\xi^2))|^2\,|\widehat{B_k}(\xi)|^2\,d\tau\,d\xi\\&=\left(\int_{\mathbb{R}}(1+|u|)^{2\gamma}|\widehat{a_\beta}^{(k)}(u)|^2\,du\right)\left(\int_{\mathbb{R}}|\widehat{B_k}(\xi)|^2\,d\xi\right)\\&\leq K^2(a)(3\beta)^{2k+1-2\gamma}\left(\int_{\mathbb{R}}|\widehat{B_k}(\xi)|^2\,d\xi\right)\end{aligned} \tag{3.3.23}$$

We will denote $\int_{\mathbb{R}}|\widehat{B_k}(\xi)|^2\,d\xi$ by Λ_k^2.

By (3.2.8d) and by Hölder's inequality, we obtain

$$\begin{aligned}\Lambda_k^2&\leq\int_{\mathbb{R}}\left[\int_{|\tau+\xi^2|\leq 2r}|\frac{\sigma+\xi^2}{2r}|^{k-1}\,|\widehat{F}(\sigma,\xi)|\,\hat{b}(\sigma+\xi^2)\,d\sigma\right]^2d\xi\\&\leq\int_{\mathbb{R}}\left(\int_{|\sigma+\xi^2|\leq 2r}|\widehat{F}(\sigma,\xi)|\,d\sigma\right)^2d\xi\\&\leq\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{|\widehat{F}(\sigma,\xi)|^2}{\hat{S}^{2-2\gamma}(\sigma,\xi)}\,d\sigma\right)\left(\int_{|\sigma+\xi^2|\leq 2r}(1+|\sigma+\xi^2|)^{2-2\gamma}\,d\sigma\right)d\xi.\end{aligned} \tag{3.3.24}$$

A direct computation verifies the following estimate

$$\int_{|\rho|\geq 2r}(1+|\rho|)^{2-2\gamma}\,d\rho\leq C(\gamma)r^{3-2\gamma}. \tag{3.3.25}$$

Thus

$$\Lambda_k^2 \le C(\gamma)\, r^{3-2\gamma} \Big\| \frac{\widehat{F}}{\hat{S}^{1-\gamma}} \Big\|^2_{L^2(\mathbb{R}\times\mathbb{R})}. \tag{3.3.26}$$

Inserting (3.3.26) and (3.3.23) into (3.3.22) gives the estimate

$$\begin{aligned}
\|\hat{E}\|_{L^2(\mathbb{R}\times\mathbb{R})} &\le (2r)^{\frac{3-2\gamma}{2}} C(\gamma,a) \Big\| \frac{\widehat{F}}{\hat{S}^{1-\gamma}} \Big\|_{L^2(\mathbb{R}\times\mathbb{R})} \left(\sum_{k=1}^{\infty} \frac{(2r)^{k-1}(3\beta)^{k+\frac{1}{2}-\gamma}}{k!} \right) \\
&= C(\gamma,a)(r\beta)^{\frac{1}{2}-\gamma} \left[\frac{e^{6r\beta}-1}{6r\beta} \right] \Big\| \frac{\widehat{F}}{\hat{S}^{1-\gamma}} \Big\|_{L^2(\mathbb{R}\times\mathbb{R})} \\
&= C(\gamma,a) \Big\| \frac{\widehat{F}}{\hat{S}^{1-\gamma}} \Big\|_{L^2(\mathbb{R}\times\mathbb{R})}.
\end{aligned} \tag{3.3.27}$$

Recall that $r\beta = 1$.

Collecting equations (3.3.5), (3.3.10), (3.3.20), and (3.3.27), we have (3.3.2) as desired. This concludes the proof.

□

3.4 Existence Results

In this section our goal is to obtain a global solution to the problem

$$i\partial_t \phi + \partial_{xx}\phi = -\frac{1}{2}|\phi|^2\phi + W(\phi)\phi, \qquad (t,x) \in \mathbb{R}\times\mathbb{R},$$

subject to the boundary condition

$$\phi(0,x) = \phi_o(x), \quad x \in \mathbb{R}.$$

We will construct a family of approximate solutions for which we apply Strichartz's estimates in order to get a solution local in time.We will prove that such solution can be extended to a global one by estimating the time interval of existence.

Strichartz's estimates are space-time estimates that are essential for solving non-linear Schrödinger equations. We begin by introducing the notion of an admissible pair.

Definition. A pair (q, r) is said to be admissible if

$$\frac{2}{q} = \left(\frac{1}{2} - \frac{1}{r}\right) \tag{3.4.1}$$

and $2 \leq r \leq \infty$.

Remark. The pairs $(\infty, 2), (8, 4), (6, 6)$, and $(4, \infty)$ are some examples.

The two following theorems state Strichartz's estimates that we will use in this section. Their proofs can be found in [17, page 43–48].

Theorem 3.4.1. *(Strichartz's estimate) For every $\varphi \in L^2(\mathbb{R})$ and for every admissible par (q,r), the function t $\mapsto \mathcal{U}(t)\varphi$ belongs to*

$$L^q(\mathbb{R}, L^2(\mathbb{R})) \cap C(\mathbb{R}, L^2(\mathbb{R})).$$

Furthermore, there is a constant C depending only on q such that

$$\|\mathcal{U}(\cdot)\varphi\|_{L^q(\mathbb{R},L^2(\mathbb{R}))} \leq C\|\varphi\|_{L^2(\mathbb{R})} \quad \textit{for every} \quad \varphi \in L^2(\mathbb{R}^d). \tag{3.4.2}$$

The next theorem establishes an extension of the previous result to solutions of the non-homogeneous linear Schrödinger equation.

Theorem 3.4.2. *(Strichartz's estimate) Let I be an interval of $\mathbb{R}$, J = $\overline{I}$, and t_o ∈J. Let (μ, ν) an admissible and $f \in L^{\mu'}(I, L^{\nu'}(\mathbb{R}))$, where μ' and ν' denote the conjugates of μ and ν, respectively. Then, for every admissible pair (q,r), the function*

$$t \in I \longmapsto \Phi_f(t) = \int_{t_o}^{t} \mathcal{U}(t-s) f(s)\, ds \tag{3.4.3}$$

belongs to $L^q(I, L^r(\mathbb{R})) \cap C(J, L^2(\mathbb{R}))$. *Moreover, there exists a constant* C *depending only on* q *and* μ *such that*

$$\|\Phi_f\|_{L^q(I,L^r(\mathbb{R}))} \leq C \|f\|_{L^{\mu'}(I,L^{\nu'}(\mathbb{R}))}. \tag{3.4.4}$$

Remark. When there is no risk of confusion, we denote $\| \ \|_{L^p(I,X)}$ by $\| \ \|_{L^p(I)}$.

In what follows, we assume that W is a complex-valued function satisfying the following conditions:

(i) There exists a constant K such that

$$\Im m \ W(z) \leq K \quad \text{for all } z \in \mathbb{C}, \tag{3.4.5a}$$

(ii) There is a constant C such that if $u, v \in L^2(\mathbb{R})$, then

$$\|h(u) - h(v)\|_{L^2(\mathbb{R})} \leq C \ \|u - v\|_{L^2(\mathbb{R})}, \tag{3.4.5b}$$

where $h(z) = zW(z)$.

We next construct a sequence of approximate solutions. In order to do this, we first set $\varphi_0 = 0$ and

$$f(z) \ \stackrel{\text{def}}{=} \ -\frac{1}{2}|z|^2 z + zW(z) \equiv g(z) + h(z).$$

Given $\phi_o \in L^2(\mathbb{R})$, let φ_1 be the function defined by

$$\varphi_1(t) = \mathcal{U}(t)\phi_o. \tag{3.4.6}$$

It is well known that $\|\varphi_1(t)\|_{L^2(\mathbb{R})} = \|\phi_o\|_{L^2(\mathbb{R})}$ for each $t \in \mathbb{R}$. Moreover, for every admissible pair (q,r), $\varphi_1 \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^q(\mathbb{R}, L^r(\mathbb{R}))$ and

$$\|\varphi_1\|_{L_t^q L_x^r} \leq C(q)\|\phi_o\|_{L^2(\mathbb{R})}. \tag{3.4.7}$$

Here $\| \, \|_{L_t^q L_x^r}$ also denotes the norm in $L^q(I, L^r(\mathbb{R}))$.

We claim that the mapping $t \mapsto f(a_\beta(t)\varphi_1(t))$ belongs to $L^1(\mathbb{R}, L^2(\mathbb{R}))$ and satisfies the inequality

$$\|f(a_\beta\varphi_1)\|_{L_t^1 L_x^2} \leq 2\beta^{\frac{1}{2}}\|\varphi_1\|_{L_t^\infty L_x^2}\left(2C\beta^{\frac{1}{2}} + \|\varphi_1\|^2_{L_t^4 L_x^\infty}\right). \tag{3.4.8}$$

Indeed, since (4,∞) is an admissible pair, Theorem 3.4.1 implies that $\varphi_1(t) \in L^\infty(\mathbb{R})$ for a.e. $t \in \mathbb{R}$. Invoking Hölder's inequality, we get

$$\|g(a_\beta(t)\varphi_1(t))\|_{L^2(\mathbb{R})} \leq a_\beta^3(t)\|\varphi_1(t)\|^2_{L^\infty(\mathbb{R})}\|\varphi_1\|_{L_t^\infty L_x^2}. \tag{3.4.9}$$

Consequently, we obtain

$$\|g(a_\beta\varphi_1)\|_{L_t^1 L_x^2} \leq 2\beta^{\frac{1}{2}}\|\varphi_1\|^2_{L_t^4 L_x^\infty}\ \|\varphi_1\|_{L_t^\infty L_x^2}, \tag{3.4.10}$$

by using Hólder's inequality in time and (3.4.9).

On the other hand, we have from (3.4.5b)

$$\begin{aligned}\int_{\mathbb{R}}\|h(a_\beta(t)\varphi_1(t))\|_{L^2(\mathbb{R})}\,dt &\leq C\int_{I_{2\beta}}\|\varphi_1(t)\|_{L^2(\mathbb{R})}\,dt\\ &= 4\beta C\|\varphi_1\|_{L_t^\infty L_x^2},\end{aligned} \tag{3.4.11}$$

where $I_\beta \overset{\text{def}}{=} (-\beta,\beta)$. Next we can combine (3.4.10) and (3.4.11) to obtain (3.4.8) as claimed.

Assume that φ_k is given satisfying $f(a_\beta\varphi_k) \in L^1(\mathbb{R}, L^2(\mathbb{R}))$. We define φ_{k+1} inductively to be

$$\varphi_{k+1}(t) = \mathcal{U}(t)\phi_o - i\int_0^t \mathcal{U}(t-s)f(a_\beta(s)\varphi_k(s))\,ds, \qquad t \in \mathbb{R}. \tag{3.4.12}$$

Thus φ_{k+1} is in $L^q(\mathbb{R}, L^r(\mathbb{R})) \cap C(\mathbb{R}, L^2(\mathbb{R}))$ for any admissible pair (q, r). Moreover, calculations similar to those leading to (3.4.8) give

$$\|f(a_\beta\varphi_{k+1})\|_{L_t^1 L_x^2} \leq 2\beta^{\frac{1}{2}}\|\varphi_{k+1}\|_{L_t^\infty L_x^2}\left(2C\beta^{\frac{1}{2}} + \|\varphi_{k+1}\|^2_{L_t^4 L_x^\infty}\right). \tag{3.4.13}$$

Hence, we have proved the following theorem.

Theorem 3.4.3. *If ϕ_o is in $L^2(\mathbb{R})$ and β is an arbitrary positive number then there is a sequence $\{\varphi_k\}_{k=0}^\infty$, with $\varphi_0 = 0$ and φ_{k+1} given inductively by* (3.4.12)*, such that*

(i) For any admissible pair (q, r)

$$\varphi_k \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^q(\mathbb{R}, L^r(\mathbb{R})).$$

(ii) For each $k \geq 1$ and for any bounded open interval I, with $0 \in I$,

$$\varphi_k \in W^{1,1}(I, H^{-2}(\mathbb{R})),$$

and satisfies

$$\begin{cases} i\partial_t\varphi_k + \Delta\varphi_k = f(a_\beta\varphi_{k-1}), & t \in I, \\ \varphi_k(0) = \phi_o. \end{cases} \tag{3.4.14}$$

Remark. The property (ii) above is a well known result in theory of semi-groups of linear operator. See [5, page 90].

We define

$$\phi_k = a_\beta \varphi_k, \quad k \geq 0. \tag{3.4.15}$$

It is easy to check directly that

(i) For every admissible pair (q, r), $\phi_k \in L^q(\mathbb{R}L^r(\mathbb{R}))$.

(ii)

$$\|\phi_k\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq 2\sqrt{\beta}\|\phi_k\|_{L_t^\infty L_x^2}. \tag{3.4.16}$$

(iii) $\phi_k \in L^6(\mathbb{R}\times\mathbb{R})$.

Thus $\phi_k \in L^p(\mathbb{R}\times\mathbb{R}),\ p \in [2, 6]$, and

$$\|\phi_k\|_{L^p(\mathbb{R}\times\mathbb{R})} \leq (4\beta)^{\frac{3}{2p}-\frac{1}{4}} \|\phi_k\|_{L_t^\infty L_x^2}^{\frac{3}{p}-\frac{1}{2}} \|\phi_k\|_{L^6(\mathbb{R}\times\mathbb{R})}^{3\left(\frac{1}{2}-\frac{1}{p}\right)}. \tag{3.4.17}$$

For the proof of Theorem 3.4.5, we will use the following result.

Proposition 3.4.4. *Let p be an increasing function on $I = (a, b)$. Suppose p has a fixed point $x_o \in I$. If $\{x_k\}_{k=1}^\infty$ is a sequence of real number in I such that*

(a) $x_1 \leq x_o$,

(b) $x_{k+1} \leq p(x_k), \ \ k \geq 1$,

then $x_k \leq x_o$ *for every* k.

Proof. By (a), our conclusion is true for $k = 0$. Assume $x_n \leq x_o$. Since p is increasing, $p(x_n) \leq p(x_o)$. Plugging this inequality into (b), we obtain the desired conclusion for $k = n + 1$. ☐

Theorem 3.4.5. *Suppose $\phi_o \in L^2(\mathbb{R})$. Then there is a positive number β_o such that if $\beta \in (0, \beta_o)$ is given, then there exists ϕ_β in $L^2(\mathbb{R} \times \mathbb{R})$ so that*

(i) $a_\beta \varphi_k \to \phi_\beta$ *in* $L^2(\mathbb{R} \times \mathbb{R})$,

(ii) $\hat{S}^{\frac{5}{8}} \widehat{a_\beta \varphi_k} \to \hat{S}^{\frac{5}{8}} \widehat{\phi_\beta}$ *in* $L^2(\mathbb{R} \times \mathbb{R})$,

(iii) $\|\hat{S}^{\frac{5}{8}} \widehat{\phi_\beta}\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq C(\beta) \|\phi_o\|_{L^2(\mathbb{R})}$.

Furthermore, β_o depends on L^2-norm of ϕ_o.

Proof. We proceed in six steps. It is very important to track carefully all of constants involved in this proof.

Step 1. We begin by applying the estimate in Theorem 3.3.1 to ϕ_{k+1} with the choice $\gamma = \frac{5}{8}$ to get

$$\|\hat{S}^{\frac{5}{8}} \widehat{a_\beta \varphi_{k+1}}\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq C(a, \gamma) \left(\frac{1}{\beta^{\frac{1}{8}}} \|\phi_o\|_{L^2(\mathbb{R})} + \left\| \frac{\widehat{f_k}}{\hat{S}^{\frac{3}{8}}} \right\|_{L^2(\mathbb{R}\times\mathbb{R})} \right), \tag{3.4.18}$$

where we set up $f_k = f(\phi_k) \equiv g_k + h_k$.

From Corollary 3.5.4, (3.4.17), (3.5.6), and (3.5.7), we deduce with $\alpha = 3/8$ and $\epsilon = 1/4$

$$\begin{aligned} \left\| \frac{\widehat{g_k}}{\hat{S}^{\frac{3}{8}}} \right\|_{L^2(\mathbb{R}\times\mathbb{R})} &\leq C(\alpha) \|\phi_k\|^3_{L^4(\mathbb{R}\times\mathbb{R})} \\ &\leq \sqrt[4]{2} C(\alpha) K^{\frac{9}{4}}(\epsilon) M^{\frac{3}{4}}(\epsilon) \|\hat{S}^{\frac{5}{8}} \widehat{\phi_k}\|^3_{L^2(\mathbb{R}\times\mathbb{R})}. \end{aligned} \tag{3.4.19}$$

Similarly, we employ (3.5.7) and (3.4.16) to obtain

$$
\begin{aligned}
\left\| \frac{\widehat{h_k}}{\hat{S}^{\frac{3}{8}}} \right\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq \|\widehat{h_k}\|_{L^2(\mathbb{R}\times\mathbb{R})} &= \|h_k\|_{L^2(\mathbb{R}\times\mathbb{R})} \\
&\leq C\|\phi_k\|_{L^2(\mathbb{R}\times\mathbb{R})} \\
&\leq 2CM(\epsilon)\beta^{\frac{1}{2}}\|\hat{S}^{\frac{5}{8}}\widehat{\phi_k}\|_{L^2(\mathbb{R}\times\mathbb{R})}. \qquad (3.4.20)
\end{aligned}
$$

We used assumption (3.4.5b) on h in the second line.

Setting

$$x_k \overset{\text{def}}{=} \|\hat{S}^{\frac{5}{8}}\widehat{\phi_k}\|_{L^2(\mathbb{R}\times\mathbb{R})}, \qquad (3.4.21a)$$

$$A \equiv A(a) \overset{\text{def}}{=} \sqrt[4]{2}C(a,\gamma)C(\alpha)K^{\frac{9}{4}}(\epsilon)M^{\frac{3}{4}}(\epsilon), \qquad (3.4.21b)$$

$$B \equiv B(a) \overset{\text{def}}{=} 2CC(a,\gamma)M(\epsilon), \qquad (3.4.21c)$$

$$D \equiv D(a) \overset{\text{def}}{=} C(a,\gamma), \qquad (3.4.21d)$$

$$E \equiv E(a) \overset{\text{def}}{=} \frac{27AD^2}{4}, \qquad (3.4.21e)$$

we can state the following recursive formula

$$x_{k+1} \leq A\beta^{\frac{3}{8}}x_k^3 + B\sqrt{\beta}x_k + \frac{D}{\beta^{\frac{1}{8}}}\|\phi_o\|_{L^2(\mathbb{R})} \qquad (3.4.22)$$

by substituting (3.4.19) and (3.4.20) into (3.4.18)

Step 2. Let

$$p_\beta(x) = \beta^{\frac{3}{8}}Ax^3 + \sqrt{\beta}Bx + \frac{D}{\beta^{\frac{1}{8}}}\|\phi_o\|_{L^2(\mathbb{R})}.$$

We first observe that (3.4.22) reads

$$x_{k+1} \leq p_\beta(x_k), \quad k \geq 0. \qquad (3.4.23)$$

Defining

$$\ell_\beta(x) \stackrel{\text{def}}{=} p_\beta(x) - x,$$

we claim that there exists $\beta_o > 0$ such that if $0 < \beta < \beta_o$, then $(\hat{S}^{\frac{5}{8}}\widehat{a_\beta \varphi_k})_{k\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}\times\mathbb{R})$.

To see this, let

$$\zeta(\beta) \stackrel{\text{def}}{=} \frac{2A}{D}\left[\frac{1-\delta^4 B}{3\sqrt[3]{\delta}A}\right]^{\frac{3}{2}}, \qquad \delta = \beta^{\frac{1}{8}}. \tag{3.4.24}$$

It is not difficult to verify that ζ is a decreasing function and that

$$\lim_{\beta\to 0^+} \zeta(\beta) = \infty.$$

Hence, there is β_o such that

$$\|\phi_o\|_{L^2(\mathbb{R})} = \zeta(\beta_o), \tag{3.4.25a}$$

$$\|\phi_o\|_{L^2(\mathbb{R})} < \zeta(\beta) \quad \text{for any } \beta \in (0, \beta_o). \tag{3.4.25b}$$

Notice that the inequality $\|\phi_o\|_{L^2(\mathbb{R})} < \zeta(\beta)$ is equivalent to

$$\delta E \|\phi_o\|^2_{L^2(\mathbb{R})} < (1-\delta^4 B)^3, \quad \delta = \beta^{\frac{1}{8}}. \tag{3.4.26}$$

Fix $0 < \beta < \beta_o$. A straightforward computation shows that if

$$x_\beta = \sqrt{\frac{1-\delta^4 B}{3\delta^3 A}}, \qquad \delta = \beta^{\frac{1}{8}}, \tag{3.4.27}$$

then $\ell'_\beta(x_\beta) = 0$ and

$$\ell_\beta(x_\beta) = \frac{D\|\phi_o\|_{L^2(\mathbb{R})}}{\beta} - \frac{2}{3}\left(1 - \delta^4 B\right) x_\beta \tag{3.4.28a}$$

$$= \frac{D}{\delta}\left(\|\phi_o\|_{L^2(\mathbb{R})} - \zeta(\beta)\right) \tag{3.4.28b}$$

$$< 0. \tag{3.4.28c}$$

Consequently, there is $z_\beta \in (0, x_\beta)$ such that $l(z_\beta) = 0$.

By (3.4.28a), we can deduce

$$\frac{D\|\phi_o\|_{L^2(\mathbb{R})}}{\delta} < \frac{2}{3}\left(1 - \delta^4 B\right) x_\beta, \qquad \delta = \beta^{\frac{1}{8}}. \tag{3.4.29}$$

from where the next inequality follows immediately

$$\frac{D\|\phi_o\|_{L^2(\mathbb{R})}}{\delta} < x_\beta, \qquad \delta = \beta^{\frac{1}{8}}. \tag{3.4.30}$$

Since ℓ is decreasing on $(0,\ x_\beta)$ and $\ell(p_\beta(0)) > 0$, we have

$$p_\beta(0) < z_\beta. \tag{3.4.31}$$

Proposition 3.4.4 allows us to conclude

$$x_k \leq z_\beta, \text{ for all } k \geq 0. \tag{3.4.32}$$

We remark that the next inequality also holds for every $k \geq 0$

$$x_k < x_\beta. \tag{3.4.33}$$

Step 3. We now want to estimate

$$m_k \stackrel{\text{def}}{=} \|\hat{S}^{\frac{5}{8}}\left(\widehat{\phi_k} - \widehat{\phi_{k-1}}\right)\|_{L^2(\mathbb{R}\times\mathbb{R})}, \quad k \geq 1. \tag{3.4.34}$$

Since $u \stackrel{\text{def}}{=} \varphi_{k+1} - \varphi_k$ satisfies

$$\begin{cases} i\partial_t u + \Delta u = f_k - f_{k-1}, \quad t \in \mathbb{R}, \\ u(0) = 0, \end{cases}$$

Theorem 3.3.2 implies with $\gamma = \frac{5}{8}$

$$m_{k+1} \leq C(a, \gamma) \Big\| \frac{\widehat{f_k - f_{k-1}}}{\hat{S}^{\frac{3}{8}}} \Big\|_{L^2(R)}. \tag{3.4.35}$$

Using the inequality

$$\|u|u|^2 - v|v|^2\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{R})} \leq \Big(\|u\|^2_{L^4(\mathbb{R}\times\mathbb{R})} + \|u\|_{L^4(\mathbb{R}\times\mathbb{R})} \|v\|_{L^4(\mathbb{R}\times\mathbb{R})} + \|v\|^2_{L^4(\mathbb{R}\times\mathbb{R})} \Big) \|u - v\|_{L^4(\mathbb{R}\times\mathbb{R})}.$$

and proceeding as in step 1., we can show that the sequence $(m_k)_{k\geq 1}$ satisfies the recursive formula

$$m_{k+1} \leq \Big(\beta^{\frac{3}{8}} A \left[x_k^2 + x_k\, x_{k-1} + x_{k-1}^2 \right] + \beta^{\frac{1}{2}} B \Big) m_k. \tag{3.4.36}$$

Let us point out that A and B are the same constants as in (3.4.22).

We combine the above formula and (3.4.32) to obtain

$$m_{k+1} \leq (3\beta^{\frac{3}{8}} A z_\beta^2 + \beta^{\frac{1}{2}} B) m_k \equiv \lambda(\beta) m_k \tag{3.4.37}$$

for every $\beta \in (0, \beta_o)$.

Step 4. This step is the crux of the proof. We assert

$$\lambda(\beta) < 1, \quad \beta \in (0, \beta_o). \tag{3.4.38}$$

In fact, since $z_\beta \in (0, x_\beta)$, we have by (3.4.27)

$$\lambda(\beta) < 3\beta^{\frac{3}{8}}Ax_\beta^2 + \beta^{\frac{1}{2}}B = 3\beta^{\frac{3}{8}}A\frac{1-\beta^{\frac{1}{2}}B}{3A\beta^{\frac{3}{8}}} + \beta^{\frac{1}{2}}B = 1,$$

as asserted.

Step 5. Fix $0 < \beta < \beta_o$. Step 4 tells us that $(\hat{S}^{\frac{5}{8}}\widehat{\phi_k})_{k\geq 0}$ is a Cauchy sequence in $L^2(\mathbb{R}\times\mathbb{R})$. Therefore, (i) and (ii) follow immediately.

Step 6. According to (3.4.33), the right hand side of (3.4.22) is bounded by

$$A\beta^{\frac{3}{8}}x_k x_\beta^2 + B\sqrt{\beta}x_k + \frac{D}{\beta^{\frac{1}{8}}}\|\phi_o\|_{L^2(\mathbb{R})},$$

which is equal to

$$\frac{1}{3}\left(1+2B\sqrt{\beta}\right)x_k + \frac{D}{\beta^{\frac{1}{8}}}\|\phi_o\|_{L^2(\mathbb{R})} \tag{3.4.39}$$

by (3.4.27).

Consequently, inequality (3.4.22) transforms into

$$x_{k+1} < \frac{1}{3}\left(1+2B\sqrt{\beta}\right)x_k + \frac{D}{\beta^{\frac{1}{8}}}\|\phi_o\|_{L^2(\mathbb{R})}. \tag{3.4.40}$$

Taking limit when $k\to\infty$ on both side of the previous inequality, we obtain

$$\|\hat{S}^{\frac{5}{8}}\widehat{\phi_\beta}\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq \frac{1}{3}\left(1+2B\sqrt{\beta}\right)\|\hat{S}^{\frac{5}{8}}\widehat{\phi_\beta}\|_{L^2(\mathbb{R}\times\mathbb{R})} + \frac{D}{\beta^{\frac{1}{8}}}\|\phi_o\|_{L^2(\mathbb{R})}. \tag{3.4.41}$$

Hence

$$\|\hat{S}^{\frac{5}{8}}\widehat{\phi_\beta}\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq \frac{3D}{2\delta\left(1-\delta^4 B\right)}\|\phi_o\|_{L^2(\mathbb{R})}, \quad \delta = \beta^{\frac{1}{8}}. \tag{3.4.42}$$

The proof is complete. $\square$

Remark. The same arguments used in the proof of Theorem 3.4.5 can be utilized to show well-posedness, namely

$$\|\hat{S}^{\frac{5}{8}}(\widehat{\phi_\beta} - \widehat{\psi_\beta})\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq C(\beta)\|\phi_o - \psi_o\|_{L^2(\mathbb{R})}. \tag{3.4.43}$$

In the rest of this section, we assume $\beta \in (0, \beta_o)$. Also, when there is no risk of confusion, we write ϕ instead of ϕ_β. As a direct consequence of the previous theorem and Theorem 3.5.3, we have

Theorem 3.4.6. *Let I be a bounded interval of $\mathbb{R}$ and $r \in [2, 6]$. Then $(\phi_k)_{k\in\mathbb{N}}$ also converges to ϕ in $L^r(\mathbb{R}\times\mathbb{R})$ and in $L^\infty(I, L^2(\mathbb{R}))$. In particular,*

$$\varphi_k \;\; \rightarrow \;\; \phi \;\; in \;\; L^\infty(I_\beta, L^2(\mathbb{R})). \tag{3.4.44}$$

Theorem 3.4.7. *Let I be a bounded interval of $\mathbb{R}$. Then sequences $(g(\phi_k))_{k\geq 0}$ and $(h(\phi_k))_{k\geq 0}$ converge to $g(\phi)$ and $h(\phi)$, respectively, in $L^1(I, L^2(\mathbb{R}))$.*

Proof. An elementary calculation based on the identity

$$u|u|^2 = |u|^2(u-v) + \overline{u}v(u-v) + v^2(\overline{u} - \overline{v})$$

and on the inequality

$$\|uvw\|_{L^2(\mathbb{R})} \leq \|u\|_{L^6(\mathbb{R})}\,\|v\|_{L^6(\mathbb{R})}\,\|w\|_{L^6(\mathbb{R})},$$

shows that

$$\|g(\phi_k(t)) - g(\phi(t))\|_{L^2(\mathbb{R})} \leq \|\phi_k(t) - \phi(t)\|_{L^6(\mathbb{R})}\,\Theta, \tag{3.4.45}$$

where

$$\Theta \stackrel{\text{def}}{=} \|\phi_k(t)\|^2_{L^6(\mathbb{R})} + \|\phi_k(t)\|_{L^6(\mathbb{R})} \, \|\phi(t)\|_{L^6(\mathbb{R})} + \|\phi(t)\|^2_{L^6(\mathbb{R})}.$$

Applying Hölder's inequality in time, we obtain

$$\|g(\phi_k) - g(\phi)\|_{L^1(I)} \leq \sqrt{|I|} \, \|\phi_k - \phi\|_{L^6(\mathbb{R}\times\mathbb{R})} \, \Upsilon, \tag{3.4.46}$$

$$\Upsilon \stackrel{\text{def}}{=} \|\phi_k\|^2_{L^6(\mathbb{R}\times\mathbb{R})} + \|\phi_k\|_{L^6(\mathbb{R}\times\mathbb{R})} \, \|\phi\|_{L^6(\mathbb{R}\times\mathbb{R})} + \|\phi\|^2_{L^6(\mathbb{R}\times\mathbb{R})},$$

from where we can conclude that $(g(\phi_k))_{k\geq 0}$ converges to $g(\phi)$ in $L^1(I, L^2(\mathbb{R}))$.

On the other hand, we have from (3.4.5b)

$$\begin{aligned}\int_I \|h(\phi_k(t)) - h(\phi(t))\|_{L^2(\mathbb{R})} \, dt &\leq C \int_I \|\phi_k(t) - \phi(t)\|_{L^2(\mathbb{R})} \, dt \\ &\leq C \, |I| \, \|\phi_k - \phi\|_{L^\infty_t L^2_x}.\end{aligned} \tag{3.4.47}$$

Theorem 3.4.6 implies

$$\|h(\phi_k(t)) - h(\phi(t))\|_{L^1(I, L^2(\mathbb{R}))} \;\to\; 0 \quad \text{as} \;\; k \;\to\; \infty.$$

□

The main result of this section is now the following theorem.

Theorem 3.4.8. *Let I be a bounded, open interval of $\mathbb{R}$, with $0 \in I$, and $\phi_o \in L^2(\mathbb{R})$. Then there exists*

$$\varphi \in L^\infty(I, L^2(\mathbb{R})) \cap C(\overline{I}, L^2(\mathbb{R}))$$

such that

(i) $\varphi_k \;\to\; \varphi$ *in* $L^\infty(I, L^2(\mathbb{R}))$.

(ii)

$$\varphi(t) = \mathcal{U}(t)\phi_o - i\int_o^t \mathcal{U}(t-s) f(\phi(s))\,dt, \quad t \in I. \tag{3.4.48}$$

(iii) $\varphi \in W^{1,1}(I, H^{-2}(\mathbb{R}))$ *and*

$$\begin{cases} i\partial_t\varphi + \Delta\varphi = f(\phi) \quad \text{for } t \in I, \\ \varphi(0) = \phi_o. \end{cases} \tag{3.4.49}$$

Proof. We start the proof by recalling that φ_{k+1} is defined by the recursive formula

$$\varphi_{k+1}(t) = \mathcal{U}(t)\phi_o - i\int_0^t \mathcal{U}(t-s) f(a_\beta(s)\varphi_k(s))\,ds, \qquad t \in \mathbb{R}. \tag{3.4.50}$$

Since $f(\phi_k) \;\to\; f(\phi)$ in $L^1(I, L^2(\mathbb{R}))$, Theorems 3.4.1 and 3.4.2 imply that there exists $\varphi \in L^\infty(I, L^2(\mathbb{R}))$ such that $\varphi_k \;\to\; \varphi$ in $L^\infty(I, L^2(\mathbb{R}))$ and φ satisfies (3.4.48). Moreover,

$$\varphi \in L^q(I, L^r(\mathbb{R})) \cap C(\overline{I}, L^2(\mathbb{R}))$$

for all admissible pair (q, r).

Finally, (3.4.49) is well-known result in theory of semi-groups of linear operator.

$\square$

The following corollary is an immediate consequence of the previous result and (3.4.44)

Corollary 3.4.9. *Suppose* $\phi_o \in L^2(\mathbb{R})$. *If* $\beta \in (0, \beta_o)$ *is given, then* ϕ_β *satisfies the following properties:*

(i) For every admissible pair (q,r)

$$\phi_\beta \in L^q(I_\beta, L^r(\mathbb{R})) \cap C(\overline{I}_\beta, L^2(\mathbb{R})) \cap W^{1,1}(I_\beta, H^{-2}(\mathbb{R}))$$

Recall that I_β denotes the open interval $(-\beta, \beta)$.

(ii)

$$\begin{cases} i\partial_t \phi_\beta + \Delta\phi_\beta = -\frac{1}{2}\phi_\beta|\phi_\beta|^2 + \phi_\beta W(\phi_\beta) \quad \textit{for } \ t \ \in \ I_\beta, \\ \phi_\beta(0) = \phi_o. \end{cases} \tag{3.4.51}$$

Now that we have established local existence, we are ready to study global existence. First of all, a remark is needed at this point.

Remark. Assume that μ is a nonnegative real number. If $3B + \mu \geq 1$, then

$$\delta = \frac{1}{3B + \mu + E\|\phi_o\|^2_{L^2(\mathbb{R})}}, \qquad \delta = \beta^{\frac{1}{8}}, \tag{3.4.52}$$

satisfies (3.4.26).

According to this remark, we set

$$\beta = \frac{1}{E^8\left(\rho + \|\phi_o\|^2_{L^2(\mathbb{R})}\right)^8} \quad E\rho \equiv 3B + \mu \geq 1 \tag{3.4.53}$$

with ρ to be determined. All of results in Section 2.5 can be now easily adapted to prove

Theorem 3.4.10. *Let $\phi_o \in L^2(\mathbb{R})$. Then the problem*

$$i\partial_t \phi + \Delta\phi = -\frac{1}{2}|\phi|^2\phi + W(\phi)\phi, \qquad (t, x) \in \mathbb{R} \times \mathbb{R},$$

subject to the initial condition

$$\psi(0,x)=\phi_o(x),\;\; x\in\mathbb{R},$$

has a global unique solution. Moreover,

$$\phi\in L^q_{loc}(\mathbb{R},L^r(\mathbb{R}))\cap C(\mathbb{R},L^2(\mathbb{R}))\cap W^{1,1}_{loc}(\mathbb{R},H^{-2}(\mathbb{R}))$$

for any admissible pair (q,r).

3.5 A Priori Estimates

This last section is devoted to establish the estimates needed to prove that the equation (3.2.3) has a solution local in time. We start by reviewing briefly a few properties of linear Schrödinger operator. The estimates discussed here play an important role in the proof of Theorem 3.5.3. Although results quoted below hold in more general context, we restrict ourselves to the one-dimensional case.

The conservation of the L^2-norm $\|\mathcal{U}(t)\varphi\|_{L^2(\mathbb{R})}=\|\varphi\|_{L^2(\mathbb{R})}$, together with the classical estimate $|\mathcal{U}(t)\varphi(x)|\leq(4\pi|t|)^{-\frac{1}{2}}\|\varphi\|_{L^1(\mathbb{R})}$ leads to the following result

Theorem 3.5.1. *(Decay estimates) If* $p\in[2,\infty],\;\; t\neq 0$*, then* $\mathcal{U}(t)$ *maps* $L^{p'}(\mathbb{R})$ *continuously to* $L^p(\mathbb{R}),\;\; \frac{1}{p}+\frac{1}{p'}=1$*, and*

$$\|\mathcal{U}(t)\varphi\|_{L^p(\mathbb{R})}\leq\frac{1}{(4\pi|t|)^{\frac{1}{2}-\frac{1}{p}}}\|\varphi\|_{L^{p'}(\mathbb{R})}\quad \textit{for all}\;\; \varphi\in L^{p'}(\mathbb{R}). \tag{3.5.1}$$

The proof of Theorem 3.5.1 relies on Riesz-Thorin interpolation theorem and on the following proposition

Proposition 3.5.2. *Given* $t\neq 0$*, define the function* $\mathcal{K}(t)$ *by*

$$\mathcal{K}(t)(x) = \frac{1}{\sqrt{4i\pi t}} e^{\frac{ix^2}{4t}} \quad for \;\; x \in \mathbb{R}. \tag{3.5.2}$$

It follows that $\mathcal{U}(t)\varphi = \mathcal{K}(t) * \varphi$*, i.e.,*

$$\mathcal{U}(t)\varphi(x) = \frac{1}{\sqrt{4i\pi t}} \int_{\mathbb{R}} e^{\frac{i(x-y)^2}{4t}} \varphi(y)\, dy \tag{3.5.3}$$

for all $t \neq 0$ *and all* φ *in the Schwartz space* $\mathcal{S}(\mathbb{R})$*.*

A remark is in order.

Remark. By duality, $\mathcal{U}(t)$ can be extended to $\mathcal{S}'(\mathbb{R})$ and

$$\mathcal{U}(t)\varphi \in C(\mathbb{R}, \mathcal{S}'(\mathbb{R}))$$

for every $\varphi \in S'(\mathbb{R})$. For example, the generalized solution of the initial value problem

$$\begin{cases} i\partial_t u + \Delta u = 0, \\ u(0) = \delta \end{cases}$$

is given by (3.5.2) and it satisfies

$$\widehat{\mathcal{K}(t)}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-it\xi^2}. \tag{3.5.4}$$

We now proceed to establish the main theorem in this section.

Theorem 3.5.3. *Let f(t,x) be a function with* $(t, x) \in \mathbb{R} \times \mathbb{R}$ *and denote by* $\hat{f}(\tau, \xi)$ *its Fourier transform, with* $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}$*. If* $0 \leq \alpha < 1$ *and* $\epsilon > 0$*, the following estimates hold:*

$$\|f\|_{L^p(\mathbb{R}\times\mathbb{R})} \leq C(\alpha)\|\hat{S}^{\frac{\alpha}{2}}\hat{f}\|_{L^2(\mathbb{R}\times\mathbb{R})}, \qquad p(\alpha) = \frac{6}{3-2\alpha}, \tag{3.5.5}$$

$$\|f\|_{L^6(\mathbb{R}\times\mathbb{R})} \leq K(\epsilon)\|\hat{S}^{\frac{1+\epsilon}{2}}\hat{f}\|_{L^2(\mathbb{R}\times\mathbb{R})}, \tag{3.5.6}$$

$$\|f\|_{L^2_x L^\infty_t} \leq M(\epsilon)\|\hat{S}^{\frac{1+\epsilon}{2}}\hat{f}\|_{L^2(\mathbb{R}\times\mathbb{R})}. \tag{3.5.7}$$

Proof. • We will prove (3.5.5) and (3.5.6) by a TT* argument.

We first make the substitution $\sigma = \tau + \xi^2$ to find

$$\mathcal{F}^*(\hat{S}^{-\alpha})(t,x) = \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\int \frac{e^{it\sigma}}{(|\sigma|+1)^\alpha}\,d\sigma\right)\left(\int e^{i(x\xi - t\xi^2)}\frac{1}{\sqrt{2\pi}}\,d\xi\right) \tag{3.5.8a}$$

$$= \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\int \frac{e^{it\sigma}}{(|\sigma|+1)^\alpha}\,d\sigma\right)\frac{1}{\sqrt{4i\pi t}}\,e^{\frac{ix^2}{4t}} \tag{3.5.8b}$$

$$\equiv \sqrt{2\pi}\,\mathcal{P}_\alpha(t)\,\frac{1}{\sqrt{4i\pi t}}\,e^{\frac{ix^2}{4t}}. \tag{3.5.8c}$$

Here $\mathcal{F}^*$ denotes inverse Fourier transform.

Let us call

$$\mathcal{E}_\alpha(t,x) \stackrel{\text{def}}{=} \sqrt{2\pi}\,\mathcal{P}_\alpha(t)\;\mathcal{K}(t)(x). \tag{3.5.9}$$

and consider the extension operator defined by

$$\mathcal{O}_\alpha(t)[\varphi] \stackrel{\text{def}}{=} \int_{\mathbb{R}} \mathcal{E}_\alpha(t,\cdot - y)\varphi(y)\,dy. \tag{3.5.10}$$

It is important to remark that

$$\mathcal{O}_\alpha(t)[\varphi] = \sqrt{2\pi}\,\mathcal{P}_\alpha(t)\,\mathcal{U}(t)\varphi. \tag{3.5.11}$$

Therefore, Theorem 3.5.1 gives the following estimate

$$\|\mathcal{O}_\alpha(t)[\varphi]\|_{L^p(\mathbb{R})} \leq \frac{\sqrt{2\pi}\,|\mathcal{P}_\alpha(t)|}{(4\pi|t|)^{\frac{1}{2}-\frac{1}{p}}}\,\|\varphi\|_{L^{p'}(\mathbb{R})} \quad \text{for all } \varphi \in L^{p'}(\mathbb{R}), \tag{3.5.12}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Denote by $< \cdot, \cdot >$ the pairing $f, g \mapsto \int fg dx$. For $f \in L^2(\mathbb{R}\times\mathbb{R})$ and $g \in C_0^\infty(\mathbb{R}\times\mathbb{R})$, Parseval identity and Cauchy-Schwarz inequality allow us to obtain

$$\begin{aligned} |< f, g >| &= |< \hat{f}, \mathcal{F}^* g >| = |< \hat{S}^{\frac{\alpha}{2}} \hat{f}, \hat{S}^{-\frac{\alpha}{2}} \widehat{g} >| \\ &\leq \|\hat{S}^{\frac{\alpha}{2}} \hat{f}\|_{L^2(\mathbb{R}\times\mathbb{R})}\, \|\hat{S}^{-\frac{\alpha}{2}} \widehat{g}\|_{L^2(\mathbb{R}\times\mathbb{R})}. \end{aligned} \tag{3.5.13}$$

Invoking Parseval identity again and (3.5.8), we can express $\|\hat{S}^{-\frac{\alpha}{2}}\widehat{g}\|^2_{L^2(\mathbb{R}\times\mathbb{R})}$ as

$$\begin{aligned} \|\hat{S}^{-\frac{\alpha}{2}}\widehat{g}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} &= < \hat{S}^{-\alpha}\widehat{g}, \mathcal{F}^* g > \\ &= \frac{1}{\sqrt{2\pi}} \; < \widehat{(\mathcal{E}_\alpha \star \bar{g})}, \mathcal{F}^* g > = \frac{1}{\sqrt{2\pi}} \; < \widehat{\mathcal{E}_\alpha \star \bar{g}}, g > . \end{aligned} \tag{3.5.14}$$

Fubini's Theorem justifies an interchange of order of integration giving

$$\begin{aligned} \|\hat{S}^{-\frac{\alpha}{2}}\widehat{g}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} &= \frac{1}{\sqrt{2\pi}} \int \left(\int \mathcal{E}_\alpha(t-s, x-y)\bar{g}(s,y)\, dsdy \right) g(t,x)\, dtdx \\ &= \frac{1}{\sqrt{2\pi}} \int < \mathcal{O}_\alpha(t-s)[\bar{g}(s)], g(t) > \; dsdt, \end{aligned}$$

which is bounded by

$$\frac{1}{\sqrt{2\pi}} \int \|\mathcal{O}_\alpha(t-s)[\bar{g}(s)]\|_{L^p(\mathbb{R})} \|g(t)\|_{L^{p'}(\mathbb{R})}\, dsdt, \qquad \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.5.15}$$

Reintroducing (3.5.12) in (3.5.15), we deduce easily that

$$\|\hat{S}^{-\frac{\alpha}{2}}\widehat{g}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \leq \int |\mathcal{P}_\alpha(t-s)| \frac{1}{(4\pi|t-s|)^{\frac{1}{2}-\frac{1}{p}}} \|g(s)\|_{L^{p'}(\mathbb{R})} \|g(t)\|_{L^{p'}(\mathbb{R})}\, dsdt. \tag{3.5.16}$$

(a) If we set $\alpha = 1+\epsilon$, $\epsilon > 0$, then $\mathcal{P}_\alpha \in L^\infty(\mathbb{R})$ and for every $t \in \mathbb{R}$

$$|\mathcal{P}_\alpha(t)| \leq \mathcal{P}_\alpha(0).$$

Hence, equation (3.5.16) becomes

$$\|\hat{S}^{-\frac{\alpha}{2}}\widehat{g}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \leq \mathcal{P}_\alpha(0) \int \frac{1}{(4\pi|t-s|)^{\frac{1}{2}-\frac{1}{p}}} \|g(s)\|_{L^{p'}(\mathbb{R})} \|g(t)\|_{L^{p'}(\mathbb{R})}\, dsdt.$$

Now, Hardy-Littlewood-Sobolev inequality, [16, page 119], implies

$$\|\hat{S}^{-\frac{\alpha}{2}}\widehat{g}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \leq K(\epsilon)\|g\|^2_{L^{p'}(\mathbb{R}\times\mathbb{R})} \tag{3.5.17}$$

provided $\frac{2}{p'} + \frac{1}{2} - \frac{1}{p} = 2$, i.e., $p = 6$. By density and duality from (3.5.13), we obtain (3.5.6).

(b) Assume that $0 \leq \alpha < 1$. Notice that if $\alpha = 0$, then the estimate (3.5.5) is trivial.

Theorem (3.5.5) states for every $0 < \alpha < 1$ there is a constant $D(\alpha)$ such that

$$|\mathcal{P}_\alpha(t)| \leq \frac{D(\alpha)}{|t|^{1-\alpha}}, \qquad t \neq 0. \tag{3.5.18}$$

In this case, (3.5.16) transforms into

$$\|\hat{S}^{-\frac{\alpha}{2}}\widehat{g}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} \leq D(\alpha) \int \frac{1}{(4\pi|t-s|)^{\frac{3}{2}-\frac{1}{p}-\alpha}} \|g(s)\|_{L^{p'}(\mathbb{R})} \|g(t)\|_{L^{p'}(\mathbb{R})}\, dsdt. \tag{3.5.19}$$

According to Hardy-Littlewood-Sobolev inequality, the right hand side of the last inequality is bounded by $C(\alpha)\|g\|^2_{L^{p'}(\mathbb{R}\times\mathbb{R})}$ if p', p and α satisfy

$$\frac{2}{p'} + \frac{3}{2} - \frac{1}{p} - \alpha = 2.$$

It follows that p' must be equal to $\frac{6}{3+2\alpha}$. Consequently, we have

$$|< f, g >| \leq C(\alpha)\|\hat{S}^{\frac{\alpha}{2}}\widehat{f}\|_{L^2(\mathbb{R}\times\mathbb{R})}\|g\|_{L^{\frac{6}{3+2\alpha}}(\mathbb{R}\times\mathbb{R})}, \qquad g \in C_0^\infty(\mathbb{R}\times\mathbb{R}). \tag{3.5.20}$$

Our desired estimate (3.5.5) follows immediately.

- Finally, we want to prove the inequality

$$\|f\|_{L^2_x L^\infty_t} \leq M(\epsilon)\|\hat{S}^{\frac{1+\epsilon}{2}}\hat{f}\|_{L^2(\mathbb{R}\times\mathbb{R})} \tag{3.5.21}$$

The above inequality is local in time. We then consider a smooth cut-off function $a(t)$ depending only on variable t and write

$$\widehat{af} = \frac{1}{\sqrt{2\pi}}\hat{S}^{-\frac{\alpha}{2}}\hat{h}, \qquad \alpha = 1+\epsilon,$$

with $\hat{h}$ defined by

$$\hat{h}(\tau,\xi) = \hat{S}^{\frac{\alpha}{2}}(\tau,\xi)\left[\hat{a} * \left(\hat{f}(\cdot,\xi)\right)\right](t)$$

We now claim that

$$\|\hat{h}\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq C(a,\alpha)\left(\|\widehat{f}\|_{L^2(\mathbb{R}\times\mathbb{R})} + \|\hat{S}^{\frac{\alpha}{2}}\widehat{f}\|_{L^2(\mathbb{R}\times\mathbb{R})}\right). \tag{3.5.22}$$

Indeed, utilizing inequality $|\tau+\xi^2| \leq |\tau-\sigma| + |\sigma+\xi^2|,\ (a+b)^2 \leq 2(a^2+b^2)$, and defining $J(\tau) \overset{\text{def}}{=} |\tau|^{\frac{\alpha}{2}}|\widehat{a}|(\tau)$, we have

$$|\hat{h}(\tau,\xi)|^2 \le C(\alpha)\left(\left[J * |\hat{f}|(\cdot,\xi)\right]^2 + \left[|\hat{a}| * \left(\hat{S}^{\frac{\alpha}{2}}(\cdot,\xi)|\hat{f}|(\cdot,\xi)\right)\right]^2(\tau)\right). \quad (3.5.23)$$

We next obtain by applying Young's inequality (Theorem 4.2 in [11])

$$\|\hat{h}\|^2_{L^2(\mathbb{R})} \le C(\alpha)\left(\|J\|^2_{L^1(\mathbb{R})}\,\|\widehat{f}\|^2_{L^2(\mathbb{R}\times\mathbb{R})} + \|\hat{a}\|_{L^1(\mathbb{R})}\,\|\hat{S}^{\frac{\alpha}{2}}\widehat{f}\|^2_{L^2(\mathbb{R}\times\mathbb{R})}\right), \quad (3.5.24)$$

from where (3.5.22) can be deduced. This establishes our claim.

Denoting by $\tilde{f}$ the partial Fourier transform (only in the x-variable) of f, we have

$$a(t)\tilde{f}(t,\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{P}_{\frac{\alpha}{2}}(t-s)e^{-i(t-s)\xi^2}\tilde{h}(s,\xi)\,d\xi. \quad (3.5.25)$$

This follows from (3.5.8) by noticing that

$$\widetilde{\mathcal{E}}(t,\xi) = \mathcal{P}_{\alpha/2}(t)e^{-it\xi^2}.$$

Applying Minkowski's integral inequality, we obtain for each t

$$\begin{aligned}|a(t)|\|f(t,\cdot)\|_{L^2(\mathbb{R})} &\le \frac{1}{2\pi}\int_{\mathbb{R}}\left[\int_{\mathbb{R}}|\mathcal{P}_{\alpha/2}|^2(t-s)\,|\tilde{h}|^2(s,\xi)\,d\xi\right]^{\frac{1}{2}}\,ds \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}|\mathcal{P}_{\alpha/2}|^2(t-s)\|\tilde{h}(s,\cdot)\|_{L^2(\mathbb{R})}\,ds. \quad (3.5.26)\end{aligned}$$

Theorem (3.5.5), Hölder's inequality, and (3.5.22) give (3.5.7)

□

Corollary 3.5.4. *Under the same hypothesis in Theorem 3.5.3, we assert that*

$$\left\|\ \frac{\hat{f}}{\hat{S}^{\frac{\alpha}{2}}}\ \right\|_{L^2(\mathbb{R}\times\mathbb{R})} \le C(\alpha)\|f\|_{L^q(\mathbb{R}\times\mathbb{R})}, \qquad q = \frac{6}{2\alpha+3}. \quad (3.5.27)$$

Proof. Inequality (3.5.27) can be proved by a standard duality argument.

□

Remark. Notice that we obtain from (3.5.5) with $\alpha = 3/4$

$$\|f\|_{L^4(\mathbb{R}\times\mathbb{R})} \le C(\alpha)\|\hat{S}^{\frac{3}{8}}\hat{f}\|_{L^2(\mathbb{R}\times\mathbb{R})}. \tag{3.5.28}$$

Finally, we define the function p_α to be

$$p_\alpha(\sigma) = \frac{1}{(1+|\sigma|)^\alpha}, \qquad 0 < \alpha. \tag{3.5.29}$$

Therefore,

$$\mathcal{P}_\alpha(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\sigma t}\, p_\alpha(\sigma)\, d\sigma. \tag{3.5.30}$$

The next result was used in the proof of the Theorem 3.5.3.

Theorem 3.5.5. *The following properties hold:*

(a) For every $0 < \alpha < 1$ *there exists a constant D(α) such that*

$$\mathcal{P}_\alpha(t) \le \frac{D(\alpha)}{|t|^{1-\alpha}}. \tag{3.5.31}$$

(b) $\mathcal{P}_\alpha \in L^2(\mathbb{R}), \quad 1 < 2\alpha.$

Proof. It is easy to check

$$\mathcal{P}_\alpha(t) = \frac{2}{|t|^{1-\alpha}}\Re\left(\int_{|t|}^{\infty}\frac{e^{iu}}{u^\alpha}\,du\right) \quad t \ne 0, \tag{3.5.32}$$

by making the substitution $|t|(1+\sigma) = u$.

A long but straightforward computation using a contour integral shows that

$$|\mathcal{P}_\alpha(t)| \leq \frac{2}{|t|^{1-\alpha}} \left(\int_{|t|}^{\infty} \frac{e^{-x}}{x^\alpha}\, dx + \frac{|t|^{1-\alpha}}{e^{|t|}} C(\alpha) + A_\alpha(t) \right), \tag{3.5.33}$$

where we set

$$C(\alpha) = \int_0^1 \frac{1}{(1+x^2)^\alpha}\, dx, \tag{3.5.34a}$$

$$A_\alpha(t) = \int_0^{|t|} \frac{e^{-x}}{(1+x^2)^\alpha} \sin(\alpha\theta(x))\, dx, \tag{3.5.34b}$$

$$\theta(x) = \arctan\left(\frac{x}{|t|}\right), \quad 0 \leq x \leq |t|.$$

From (3.5.33) we obtain (3.5.31) by taking

$$D(\alpha) = 2\left(\Gamma(1-\alpha) + C(\alpha) \sup_{0<z} \frac{z^{1-\alpha}}{e^z}\right). \tag{3.5.35}$$

Here Γ denotes the gamma function.

It remains to prove (b). We first observe that $0p_\alpha \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ for $1 < \alpha$. In this case, Plancherel's theorem implies that $\mathcal{P}_\alpha \in L^2(\mathbb{R})$. Hence it is enough to consider the case $0.5 < \alpha \leq 1$. Substitution $x = |t|u$ allows us to rewrite (3.5.31) as

$$|\mathcal{P}_\alpha(t)| \leq 2\left(\int_1^{\infty} \frac{e^{-|t|u}}{u^\alpha}\, du + \frac{1}{e^{|t|}} C(\alpha) + \int_0^1 \frac{ue^{-|t|u}}{(u^2+1)^{(\alpha+1)/2}}\right). \tag{3.5.36}$$

Using inequalities

$$(a+b+c)^2 \leq 2(a^2+b^2+c^2),$$

$$\sqrt{a^2+b^2+c^2} \leq |a| + |b| + |c|,$$

and Minkowski's integral inequality, we can deduce that the L^2-norm of $\mathcal{P}_\alpha$ is bounded by

$$2\sqrt{2}\left(\int_1^\infty \frac{1}{u^{\alpha+\frac{1}{2}}}\,du + \frac{\sqrt{2}C(\alpha)}{2} + \int_0^1 \sqrt{\frac{u}{(u^2+1)^{\frac{\alpha+1}{2}}}}\,du\right),$$

which is finite under our assumption on α. This completes the proof.

$\square$

Chapter 4

Motion by Mean Curvature of a Surface in $\mathbb{R}^4$

The purpose of the present chapter is to examine the motion by mean curvature of a two dimensional surface embedded in a four dimensional space. We will start with the examination of the basic structural properties of the embedded surface before we proceed to the examination of the rule of motion in the section 4.2. The idea is to examine the infinitesimal changes of an appropriate frame i.e. orthogonal tetrad of vectors, constructed on every point on the surface. Section 4.1 will be devoted to this aim. We will see that a natural gauge structure arises connecting the curvature tensor with the torsion of the surface. Overall the theme will be the language of gauge fields as an appropriate framework for presenting the structural relations among various geometric quantities.

4.1 A Surface embedded in $\mathbb{R}^4$

Let us consider a surface embedded in four dimensional space, i. e., we will consider $\Sigma \subset \mathbb{R}^4$. The surface can be described in terms of some internal, but otherwise arbitrary, coordinates (u^1, u^2) by an expression of the type

$$\Sigma \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^4 \quad : \quad x^j(u^\alpha)\} \quad \text{where} \quad j = 1,2,3,4 \quad \text{and} \quad \alpha = 1,2. \tag{4.1.1}$$

Let us also assume that $\mathbb{R}^4$ is equipped with the flat Euclidean metric. Throughout the rest of this presentation we will use the summation over repeated indices convention. Let us use the notation

$$\delta_{jk} x^j x^k \stackrel{\text{def}}{=} <\mathbf{x}\,,\mathbf{x}> \tag{4.1.2}$$

to denote the usual inner product in $\mathbb{R}^4$. We will use bold letters to denote vectors in $\mathbb{R}^4$.

Our aim is to derive the structure equations for this surface. These equations will give us information on how the surface Σ is embedded in $\mathbb{R}^4$. Let us start by constructing an appropriate frame on the surface Σ. We can construct two tangent vectors on every point of the surface as follows

$$\mathbf{t}_\alpha \stackrel{\text{def}}{=} \frac{\partial \mathbf{x}}{\partial u^\alpha}, \qquad \alpha = 1,2. \tag{4.1.3}$$

For simplicity we will denote in what follows the partial derivative with respect to the u^α-coordinate by

$$\partial_\alpha \stackrel{\text{def}}{=} \frac{\partial}{\partial u^\alpha}, \qquad \alpha = 1,2, \tag{4.1.4}$$

The metric (first fundamental form) on the surface is given by the tensor

$$g_{\alpha\beta} \stackrel{\text{def}}{=} <\mathbf{t}_\alpha, \mathbf{t}_\beta> . \tag{4.1.5}$$

The square root of the determinant is an important quantity, which we will denote by g,

$$g \stackrel{\text{def}}{=} \sqrt{\det\left(g_{\alpha\beta}\right)}. \tag{4.1.6}$$

The importance of g lies in the fact that $d\sigma \stackrel{\text{def}}{=} g du^1 du^2$ is the infinitesimal area on the surface. We will denote the inverse of the matrix $g_{\alpha\beta}$ by $g^{\alpha\beta}$, i. e.,

$$g^{\alpha\beta} \stackrel{\text{def}}{=} \left(g_{\alpha\beta}\right)^{-1} \qquad ; \qquad g_{\alpha\gamma} g^{\gamma\beta} = \delta_{\alpha}{}^{\beta}. \tag{4.1.7}$$

The Christoffel symbols (first kind) are defined by

$$\Gamma_{\alpha\beta;\gamma} \stackrel{\text{def}}{=} \left\langle \partial_\alpha \partial_\beta \mathbf{x}, \partial_\gamma \mathbf{x} \right\rangle. \tag{4.1.8}$$

We will raise or lower Greek indices using the metric $g^{\alpha\beta}$ on the surface Σ. For example, we have

$$\Gamma_{\alpha\beta}{}^{\gamma} = g^{\gamma\delta} \Gamma_{\alpha\beta;\delta}. \tag{4.1.9}$$

A tedious calculation using the symmetries of $\Gamma_{\alpha\beta;\gamma}$ gives the equation

$$\Gamma_{\alpha\beta;\gamma} = \frac{1}{2}\left[\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}\right], \tag{4.1.10}$$

i.e., the Christoffel symbols are uniquely determined by the metric. From (4.1.10) above we can obtain the useful relation

$$\Gamma_{\alpha\beta}{}^{\alpha} = \frac{\partial_\beta g}{g}. \tag{4.1.11}$$

There is an intrinsic way to pick a vector normal to the surface by calculating the Laplacian with respect to the metric $g_{\alpha\beta}$ of the position vector $\mathbf{x}(u^\alpha)$, i.e., let us compute

$$\Delta_g \mathbf{x} \stackrel{\text{def}}{=} \frac{1}{g}\, \partial_\alpha \left(g\; g^{\alpha\beta} \partial_\beta \mathbf{x} \right). \tag{4.1.12}$$

Using (4.1.11) it is easy to check the orthogonality relations $< \Delta_g \mathbf{x}, \mathbf{t}_\alpha >= 0$, $\alpha = 1, 2$, which means that the vector $\Delta_g \mathbf{x}$ is indeed orthogonal to the surface Σ. Now, we can define the unit normal vector $\mathbf{n}$ by

$$\mathbf{n} \stackrel{\text{def}}{=} \frac{\Delta_g \mathbf{x}}{|\Delta_g \mathbf{x}|}, \tag{4.1.13}$$

provided of course that the vector $\Delta_g \mathbf{x}$ does not vanish. Once we have a choice for the normal vector we can pick a binormal vector which we will call $\mathbf{b}$ so that the tetrad of vectors $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}, \mathbf{b}\}$ form a frame at each point on the surface. If we differentiate the frame, we obtain a set of structure equations of the following type

$$\partial_\alpha \mathbf{t}_\beta = \Gamma_{\alpha\beta}{}^{\gamma} \mathbf{t}_\gamma + \kappa_{\alpha\beta} \mathbf{n} + \tau_{\alpha\beta} \mathbf{b}, \tag{4.1.14a}$$

$$\partial_\alpha \mathbf{n} = -\kappa_\alpha{}^{\gamma} \mathbf{t}_\gamma + Q_\alpha \mathbf{b}, \tag{4.1.14b}$$

$$\partial_\alpha \mathbf{b} = -\tau_\alpha{}^{\gamma} \mathbf{t}_\gamma - Q_\alpha, \mathbf{n}. \tag{4.1.14c}$$

In the above equations the tensors $\kappa_{\alpha\beta}$, $\tau_{\alpha\beta}$ and Q_α are defined by these relations, i.e., we define

$$\kappa_{\alpha\beta} \stackrel{\text{def}}{=} \left\langle \partial_\alpha \partial_\beta \mathbf{x}, \mathbf{n} \right\rangle, \tag{4.1.15a}$$

$$\tau_{\alpha\beta} \stackrel{\text{def}}{=} \left\langle \partial_\alpha \partial_\beta \mathbf{x}, \mathbf{b} \right\rangle, \tag{4.1.15b}$$

$$Q_\alpha \stackrel{\text{def}}{=} \left\langle \partial_\alpha \mathbf{n}, \mathbf{b} \right\rangle. \tag{4.1.15c}$$

The mean curvature, say H, is defined as the trace of the tensor $\kappa_{\alpha\beta}$, namely the scalar quantity

$$H \stackrel{\text{def}}{=} g^{\alpha\beta}\kappa_{\alpha\beta}. \tag{4.1.16}$$

Notice that $\tau_{\alpha\beta}$ is traceless by construction. There is an inherent ambiguity in this construction, if $\Delta_g \mathbf{x}$ vanishes, then the choice of the normal and binormal vectors is ambiguous.

We can complexify the equations (4.1.14) as follows. Let us define the complex vector $\mathbf{m}$ and a complex tensor $\lambda_{\alpha\beta}$ to be

$$\mathbf{m} \stackrel{\text{def}}{=} \mathbf{n} + i\mathbf{b} \qquad ; \qquad \lambda_{\alpha\beta} \stackrel{\text{def}}{=} \kappa_{\alpha\beta} + i\tau_{\alpha\beta}, \tag{4.1.17}$$

and use the convention for the inner product of two complex vectors, say $\mathbf{a}$ and $\mathbf{b}$,

$$\big\langle \mathbf{a}, \mathbf{b} \big\rangle \stackrel{\text{def}}{=} \delta_{jk} a^j b^k, \tag{4.1.18}$$

where a^j and b^k are the complex components of $\mathbf{a}$ and $\mathbf{b}$ respectively. The following orthogonality relations for the complex vector $\mathbf{m}$ are immediate

$$\big\langle \mathbf{m}, \overline{\mathbf{m}} \big\rangle = \big\langle \overline{\mathbf{m}}, \mathbf{m} \big\rangle = 2 \qquad ; \qquad \big\langle \mathbf{m}, \mathbf{m} \big\rangle = \big\langle \overline{\mathbf{m}}, \overline{\mathbf{m}} \big\rangle = 0. \tag{4.1.19}$$

In the present context the structure equations (4.1.14) read

$$\partial_\alpha \mathbf{t}_\beta = \Gamma_{\alpha\beta}{}^{\gamma} \mathbf{t}_\gamma + \frac{1}{2}\left[\lambda_{\alpha\beta}\overline{\mathbf{m}} + \overline{\lambda}_{\alpha\beta}\mathbf{m}\right], \tag{4.1.20a}$$

$$\partial_\alpha \mathbf{m} = \lambda_\alpha^{\ \gamma} \mathbf{t}_\gamma - iQ_\alpha \mathbf{m}. \tag{4.1.20b}$$

Let us now introduce an arbitrary angle function $\theta(u^\alpha)$, defined on the surface Σ and rotate the vector $\mathbf{m}$ and the complex tensor $\lambda_{\alpha\beta}$ by $e^{i\theta}$. The new quantities will be denoted by the same name

$$\lambda_{\alpha\beta} \stackrel{\text{def}}{=} e^{i\theta}\left(\kappa_{\alpha\beta} + i\tau_{\alpha\beta}\right), \tag{4.1.21a}$$

$$\mathbf{m} \stackrel{\text{def}}{=} e^{i\theta}\left(\mathbf{n} + i\mathbf{b}\right). \tag{4.1.21b}$$

Next, we can define a complex scalar mean curvature Ψ and a gauge field A_α on the surface via the relations

$$Q_\alpha \stackrel{\text{def}}{=} -A_\alpha + \partial_\alpha \theta \qquad ; \qquad \Psi \stackrel{\text{def}}{=} g^{\alpha\beta}\lambda_{\alpha\beta}. \tag{4.1.22}$$

Finally, let us adopt the notation

$$\partial^A_\alpha \stackrel{\text{def}}{=} \partial_\alpha - iA_\alpha \quad ; \quad \alpha = 1, 2 \tag{4.1.23}$$

for the covariant derivative with respect to the gauge field A_α. Equations (4.1.20) can be written in a gauge invariant form as

$$\partial_\alpha \mathbf{t}_\beta = \Gamma_{\alpha\beta}{}^{\gamma}\mathbf{t}_\gamma + \frac{1}{2}\left[\lambda_{\alpha\beta}\overline{\mathbf{m}} + \overline{\lambda}_{\alpha\beta}\mathbf{m}\right], \tag{4.1.24a}$$

$$\partial^A_\alpha \mathbf{m} = -\lambda_\alpha{}^{\gamma}\mathbf{t}_\gamma. \tag{4.1.24b}$$

The equations above are manifestly gauge invariant under a gauge transformation

$$\Psi \mapsto e^{i\theta}\Psi \quad ; \quad \lambda_{\alpha\beta} \mapsto e^{i\theta}\lambda_{\alpha\beta}, \tag{4.1.25}$$

where $\theta(u^\alpha)$ is an arbitrary function. We can impose an extra restriction on the gauge field in order to fix the gauge. This question will be addressed later. Notice that the angle function, θ, may not be trivial. For a closed path, say C, on the surface we will require that $\int_C d\theta = 2\pi n$, where n is an integer. This restriction assures us that all complex quantities are well defined.

The Gauss and Codazzi-Mainardi equations are derived from the equality of second derivatives, namely the fact that $\partial_\gamma \partial_\beta \mathbf{t}_\alpha = \partial_\beta \partial_\gamma \mathbf{t}_\alpha$ for the tangent vectors on the surface and that $\partial_\alpha \partial_\beta \mathbf{m} = \partial_\beta \partial_\alpha \mathbf{m}$ for the complex normal vector. The first restriction gives, after a tedious calculation, the following set of equations

$$\begin{aligned} & \partial_\gamma \Gamma_{\beta\alpha}{}^{\delta} - \partial_\beta \Gamma_{\gamma\alpha}{}^{\delta} + \Gamma_{\beta\alpha}^{\ \sigma} \Gamma_{\sigma\gamma}{}^{\delta} - \Gamma_{\gamma\alpha}^{\ \sigma} \Gamma_{\sigma\beta}{}^{\delta} \\ = \ & \frac{1}{2} \left[\lambda_{\beta\alpha} \overline{\lambda}_\gamma{}^{\delta} + \overline{\lambda}_{\beta\alpha} \lambda_\gamma{}^{\delta} - \lambda_{\gamma\alpha} \overline{\lambda}_\beta{}^{\delta} - \overline{\lambda}_{\gamma\alpha} \lambda_\beta{}^{\delta} \right]. \end{aligned} \tag{4.1.26}$$

by equating the coefficients of the tangent vectors, $\mathbf{t}_\alpha, \alpha = 1, 2$. After equating the coefficients of the vector $\mathbf{m}$, we obtain one more set of equations

$$\partial_\gamma^A \lambda_{\beta\alpha} + \Gamma_{\beta\alpha}^{\ \delta} \lambda_{\delta\gamma} = \partial_\beta^A \lambda_{\gamma\alpha} + \Gamma_{\gamma\alpha}^{\ \delta} \lambda_{\delta\beta}. \tag{4.1.27}$$

Remark. The computations above are made more transparent after the observation

$$\partial_\gamma \left(\lambda_{\beta\alpha} \overline{\mathbf{m}} \right) = \left(\partial_\gamma^A \lambda_{\beta\alpha} \right) \overline{\mathbf{m}} + \lambda_{\beta\alpha} \overline{\partial}_\gamma^A \overline{\mathbf{m}}. \tag{4.1.28}$$

At this point, it is appropriate to introduce covariant differentiation of a tensor with respect to the metric, specifically we define

$$\nabla_\gamma^A \lambda_{\beta\alpha} \stackrel{\text{def}}{=} \partial_\gamma^A \lambda_{\beta\alpha} - \Gamma_{\gamma\beta}^{\ \delta} \lambda_{\delta\alpha} - \Gamma_{\gamma\alpha}^{\ \delta} \lambda_{\beta\delta}. \tag{4.1.29}$$

The left hand side of (4.1.26) is, by definition, the Riemann curvature tensor, i.e.,

$$R_{\gamma\beta\alpha}{}^{\delta} \stackrel{\text{def}}{=} \partial_\gamma \Gamma_{\beta\alpha}{}^{\delta} - \partial_\beta \Gamma_{\gamma\alpha}{}^{\delta} + \Gamma_{\beta\alpha}^{\ \sigma} \Gamma_{\sigma\gamma}{}^{\delta} - \Gamma_{\gamma\alpha}^{\ \sigma} \Gamma_{\sigma\beta}{}^{\delta}, \tag{4.1.30}$$

while equation (4.1.27) can be written, using covariant derivatives, see (4.1.29),

$$\nabla_\gamma^A \lambda_{\beta\alpha} = \nabla_\beta^A \lambda_{\gamma\alpha}. \tag{4.1.31}$$

A crucial property of the covariant derivative is the fact that $\nabla_\gamma g_{\alpha\beta} = 0$, which means that we can raise or lower indices in an equation with covariant derivatives without introducing extra terms. The Ricci tensor is defined via a contraction of the Riemann tensor, namely

$$\widehat{R}_{\gamma\alpha} \stackrel{\text{def}}{=} R_{\gamma\delta\alpha}{}^{\delta}. \tag{4.1.32}$$

Contracting equations (4.1.30) and (4.1.31), we obtain the relations

$$\widehat{R}_{\gamma\alpha} = \frac{1}{2}\left[\lambda_\gamma{}^\delta\overline{\lambda}_{\delta\alpha} + \overline{\lambda}_\gamma{}^\delta\lambda_{\delta\alpha} - \lambda_{\gamma\alpha}\overline{\Psi} - \overline{\lambda}_{\gamma\alpha}\Psi\right], \tag{4.1.33a}$$

$$\nabla^A_\gamma\Psi = \nabla^A_\alpha\lambda_\gamma{}^\alpha. \tag{4.1.33b}$$

Let us notice here, first that since $\lambda_{\alpha\beta}$ and $g_{\alpha\beta}$ are symmetric, we have

$$\lambda_\alpha{}^\beta = \lambda^\beta{}_\alpha, \tag{4.1.34}$$

second that the covariant derivative of a scalar quantity coincides with the standard partial derivative, i.e. for scalars we have $\nabla^A_\gamma\Psi = \partial_\gamma\Psi - iA_\gamma\Psi$. It is a well known fact, easily checked, that for a surface the Ricci tensor is in fact $R_{\gamma\alpha} = \frac{1}{2}Rg_{\gamma\alpha}$ where R is the Gauss or scalar curvature. Contracting the Ricci curvature gives the Gauss equation

$$R = \left\langle\lambda;\overline{\lambda}\right\rangle - \left|\Psi\right|^2, \tag{4.1.35}$$

where we adopted the notation

$$\left\langle\nu\,;\overline{\nu}\right\rangle \stackrel{\text{def}}{=} \nu^{\alpha\beta}\bar{\nu}_{\alpha\beta} \tag{4.1.36}$$

for the total contraction of a tensor $\nu_{\alpha\beta}$. Let us introduce now the traceless symmetric complex tensor

$$\mu_{\alpha\beta} \stackrel{\text{def}}{=} \lambda_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \Psi. \tag{4.1.37}$$

Using the tensor $\mu_{\alpha\beta}$ the Gauss and Codazzi-Mainardi equations reduce to the equations

$$R + \frac{1}{2}\left|\Psi\right|^2 = \langle \mu \,; \overline{\mu} \rangle, \tag{4.1.38a}$$

$$\nabla^A_\beta {\mu_\alpha}^\beta = \frac{1}{2} \nabla^A_\alpha \Psi. \tag{4.1.38b}$$

It is a fundamental fact that the Gaussian curvature depends only on the first fundamental form, i.e., the metric $g_{\alpha\beta}$. This is immediately apparent from the observation that the Christoffel symbols can be derived from the metric, see (4.1.10). One can view (4.1.38) as a set of restrictions on Ψ and $\mu_{\alpha\beta}$, for example, given the metric $g_{\alpha\beta}$ and the gauge field A_α, the complex mean curvature and $\mu_{\alpha\beta}$ are not arbitrary but satisfy the restrictions (4.1.38). We will see later that given $g_{\alpha\beta}$, Ψ, A_α, one can compute μ by solving an elliptic equation. The fact that $\partial_\beta \partial_\alpha \mathbf{m} = \partial_\alpha \partial_\beta \mathbf{m}$ gives one more equation. First, notice the commutation relation

$$\left[\partial^A_\alpha \partial^A_\beta - \partial^A_\beta \partial^A_\alpha\right] \mathbf{m} = -i \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \mathbf{m}, \tag{4.1.39}$$

and the calculation

$\partial^A_\beta \left({\lambda_\alpha}^\gamma \mathbf{t}_\gamma\right) = (\partial^A_\beta {\lambda_\alpha}^\gamma) \mathbf{t}_\gamma + {\lambda_\alpha}^\gamma \partial_\beta \mathbf{t}_\gamma$. Another tedious calculation gives two equations more. The first is is obtained by equating the coefficients of the tangent vectors in the resulting equation

$$\nabla^A_\alpha \lambda_\beta{}^\gamma - \nabla^A_\beta \lambda_\alpha{}^\gamma = 0, \tag{4.1.40}$$

which is the same as (4.1.31).

Remark. The covariant derivative of the tensor λ_β^γ is defined as follows

$$\nabla^A_\alpha \lambda_\beta{}^\gamma = \partial^A_\alpha \lambda_\beta{}^\gamma + \lambda_\beta{}^\delta \Gamma_{\alpha\delta}{}^\gamma - \lambda_\delta{}^\gamma \Gamma_{\alpha\beta}{}^\delta. \tag{4.1.41}$$

The new equation is obtained after equating the coefficients of the complex normal vector $\mathbf{m}$, i.e., we have the relation

$$\nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \frac{1}{2i} \left[\mu_\alpha{}^\gamma \overline{\mu}_{\gamma\beta} - \overline{\mu}_\alpha{}^\gamma \mu_{\gamma\beta} \right]. \tag{4.1.42}$$

The equation above describes the torsion on the surface Σ. Let us introduce at this point the totally antisymmetric invariant tensor on the surface $\epsilon_{\alpha\beta}$. It is well known that there is a unique antisymmetric tensor on the surface such that $\nabla_\gamma \epsilon_{\alpha\beta} = 0$, as a matter of fact, we can write

$$\epsilon_{\alpha\beta} = g\widehat{\epsilon}_{\alpha\beta} \qquad \text{where} \qquad \widehat{\epsilon}_{\alpha\beta}, \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.1.43}$$

We will define the torsion to be the antisymmetric real tensor

$$T_{\alpha\beta} \stackrel{\text{def}}{=} \nabla_\alpha A_\beta - \nabla_\beta A_\alpha. \tag{4.1.44}$$

Let us use the notation $[\cdot\,;\cdot]$ for the commutation of two matrices, i.e., we define

$$\left[\mu\,;\overline{\mu}\right]_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{2i} \left[\mu_\alpha{}^\gamma \overline{\mu}_{\gamma\beta} - \overline{\mu}_\alpha{}^\gamma \mu_{\gamma\beta} \right], \tag{4.1.45}$$

so that the equation of torsion reads

$$T_{\alpha\beta} = [\mu\,;\overline{\mu}]_{\alpha\beta}. \tag{4.1.46}$$

The scalar torsion will be defined as the quantity

$$T \stackrel{\text{def}}{=} \frac{1}{2}\epsilon^{\alpha\beta}T_{\alpha\beta}. \tag{4.1.47}$$

The torsion is obviously gauge invariant as well as the Codazzi-Mainardi equations. We can fix the gauge by requiring, for example, the Coulomb restriction $\nabla_\alpha A^\alpha = 0$. In conclusion. we have discovered the equations

$$R + \frac{1}{2}\left|\Psi\right|^2 = \langle \mu\,;\overline{\mu}\rangle, \tag{4.1.48a}$$

$$\nabla^A_\beta {\mu^\beta}_\alpha = \frac{1}{2}\nabla^A_\alpha \Psi, \tag{4.1.48b}$$

$$\nabla_\alpha A_\beta - \nabla_\beta A_\alpha = [\mu\,;\overline{\mu}]_{\alpha\beta}. \tag{4.1.48c}$$

A few remarks are in order. Notice that by differentiating the equation $\nabla^A_\alpha {\lambda^\alpha}_\beta = \nabla^A_\beta \Psi$ we obtain

$$\begin{aligned}
\nabla^A_\gamma \nabla^A_\alpha {\lambda^\alpha}_\beta &= \nabla^A_\alpha \nabla^A_\gamma {\lambda^\alpha}_\beta + \nabla^A_{[\gamma}\nabla^A_{\alpha]} {\lambda^\alpha}_\beta \\
&= \nabla^A_\alpha \nabla^{A,\alpha} \lambda\gamma\beta + \nabla^A_{[\gamma}\nabla^A_{\alpha]} {\lambda^\alpha}_\beta = \nabla^A_\gamma \nabla^A_\beta \Psi.
\end{aligned} \tag{4.1.49}$$

In order to compute the commutation of derivatives, let us observe that the fundamental property of the Riemann tensor is that for any vector field, say X^γ, we have

$$\nabla_{[\alpha}\nabla_{\beta]} X^\gamma = {R_{\alpha\beta\sigma}}^\gamma X^\sigma.$$

From the above equation we can obtain the commutation relation for a tensor, namely

$$\nabla_{[\alpha}\nabla_{\beta]}\lambda^{\gamma}_{\ \delta} = R_{\alpha\beta\sigma}^{\quad\gamma}\lambda^{\sigma}_{\ \delta} - R_{\alpha\beta\delta}^{\quad\sigma}\lambda^{\gamma}_{\ \sigma} \tag{4.1.50}$$

and contracting the β, γ indices, renaming δ as γ we derive

$$\nabla_{[\alpha}\nabla_{\beta]}\lambda^{\beta}_{\ \gamma} = \widehat{R}_{\alpha\sigma}\lambda^{\sigma}_{\ \gamma} - R_{\alpha\beta\gamma}^{\quad\sigma}\lambda^{\beta}_{\ \sigma}. \tag{4.1.51}$$

It is a well known that the Riemann curvature tensor for surfaces is in fact very simple. There is only one new quantity, the Gaussian (or scalar) curvature R so that

$$R_{\alpha\beta\gamma\sigma} = \frac{1}{2}R\left[g_{\alpha\gamma}g_{\beta\sigma} - g_{\alpha\sigma}g_{\beta\gamma}\right] \tag{4.1.52}$$

Thus equation (4.1.51) reads

$$\nabla_{[\alpha}\nabla_{\beta]}\lambda^{\beta}_{\ \gamma} = R\left[\lambda_{\alpha\gamma} - \frac{1}{2}\Psi g_{\alpha\gamma}\right] = R\mu_{\alpha\gamma}. \tag{4.1.53}$$

Let us use the convention

$$\nabla^{A}_{\alpha}\nabla^{A,\alpha} \stackrel{\text{def}}{=} \Delta^{A}_{g} \tag{4.1.54}$$

for the Laplacian. The commutation of derivatives reads, see (4.1.53),

$$\nabla^{A}_{[\gamma}\nabla^{A}_{\alpha]}\lambda^{\alpha}_{\ \beta} = -iT_{\gamma\alpha}\lambda^{\alpha}_{\ \beta} + R\left(\lambda^{\gamma\beta} - \frac{1}{2}\Psi g_{\gamma\beta}\right). \tag{4.1.55}$$

Hence, combining (4.1.54), (4.1.55) in (4.1.49) we can obtain an equation for $\mu_{\alpha\beta}$

$$\Delta^{A}_{g}\mu_{\alpha\beta} + R\mu_{\alpha\beta} - iT_{\alpha\gamma}\mu^{\gamma}_{\ \beta} = \widehat{\Omega}_{\beta\alpha} + \frac{i}{2}T_{\alpha\beta}\Psi, \tag{4.1.56}$$

where

$$\Omega_{\beta\alpha} \stackrel{\text{def}}{=} \nabla^{A}_{\alpha}\nabla^{A}_{\beta}\Psi \qquad ; \qquad \widehat{\Omega}_{\beta\alpha} \stackrel{\text{def}}{=} \Omega_{\beta\alpha} - \frac{1}{2}g_{\alpha\beta}\Delta^{A}\Psi. \tag{4.1.57}$$

The important point in equation (4.1.57) is that given the mean curvature Ψ and the gauge field A_α we can determine the tensor $\mu_{\alpha\beta}$. Finally, let us observe the useful formula

$$\mu_\alpha^{\ \gamma}\overline{\mu}_\gamma^{\ \beta} = \frac{1}{2}\left(R + \frac{1}{2}\left|\psi\right|^2\right)\delta_\alpha^{\ \beta} + iT_\alpha^{\ \beta}. \tag{4.1.58}$$

4.2 Binormal Motion by Mean Curvature

Let us consider again a surface Σ embedded in the four-dimensional space $\mathbb{R}^4$, i.e., $\Sigma \subset \mathbb{R}^4$. We describe Σ in terms of its position vector $x^j(u^\alpha)$, where $j = 1, 2, 3, 4$ and u^α are some coordinates with $\alpha = 1, 2$. Recall that the tangent vectors to the surface are $\mathbf{t}_\alpha = \partial_\alpha \mathbf{x}$ and the metric on the surface is given by

$$g_{\alpha\beta} \overset{\text{def}}{=} \left\langle \mathbf{t}_\alpha, \mathbf{t}_\beta \right\rangle \qquad ; \qquad g \overset{\text{def}}{=} \sqrt{\det\left(g_{\alpha\beta}\right)}. \tag{4.2.1}$$

The surface Laplacian with respect to the metric $g_{\alpha\beta}$ is defined to be

$$\Delta_g \mathbf{x} \overset{\text{def}}{=} \frac{1}{g}\partial_\alpha\left(g\, g^{\alpha\beta}\partial_\beta \mathbf{x}\right). \tag{4.2.2}$$

It is easy to check that $\left\langle \Delta_g \mathbf{x}, \mathbf{t}_\alpha \right\rangle = 0$ for $\alpha = 1, 2$, hence it is a natural choice for the normal direction to the surface provided of course that it does not vanish. Once the choice of the unit normal $\mathbf{n}$ is made so that it is parallel to $\Delta_g \mathbf{x}$ there is an essentially unique choice of the unit binormal vector $\mathbf{b}$ and the tetrad $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}, \mathbf{b}\}$ forms a frame on the surface. Let us define an antisymmetric surface form σ^{lm} on the surface using the two tangent vectors $\mathbf{t}_\alpha = \partial_\alpha \mathbf{x}$ and the totally antisymmetric forms $\epsilon_{\alpha\beta}$ on the surface by

$$\sigma^{lm} \stackrel{\text{def}}{=} \epsilon^{\alpha\beta}\left(t^l{}_\alpha t^m{}_\beta\right) \tag{4.2.3a}$$

and its dual normal form ω_{jk} using the antisymmetric tensor ϵ_{jklm} on the ambient space $\mathbb{R}^4$ by

$$\omega_{jk} \stackrel{\text{def}}{=} \epsilon_{jklm}\left(t^l{}_\alpha t^m{}_\beta\right)\epsilon^{\alpha\beta}. \tag{4.2.3b}$$

It is easy to see that σ^{lm} vanishes for vectors normal to the surface, while ω^{jk} vanishes for tangent vectors. Suppose we wish to move the surface using the mean curvature, which is the length of the vector $\Delta_g \mathbf{x}$. We have two possible directions to move, namely the normal $\mathbf{n}$ and the binormal $\mathbf{b}$. A general equation of motion will be

$$D_t x^j = \left(a\delta^j{}_k + b\omega^j{}_k\right)\Delta_g x^k, \tag{4.2.4}$$

where a, b are two real numbers normalized so that $a^2 + b^2 = 1$. The variable t denotes the time parameter and $D_t\mathbf{x}$ stands for

$$D_t\mathbf{x} \stackrel{\text{def}}{=} \partial_t \mathbf{x} - X^\gamma \mathbf{t}_\gamma, \tag{4.2.5}$$

with X^γ an arbitrary vector field defined on the surface. It is obvious that the term $X^\gamma \mathbf{t}_\gamma$ simply slides the points on the surface without changing the surface itself. Thus the vector field X^γ introduces an extra gauge freedom in our motion. We can choose X^γ in order to fix the evolution of the metric. In terms of the complex notation introduced in the first section we can write the equation of motion as

$$\partial_t \mathbf{x} = \frac{1}{2}\left[\Psi\bar{c}\,\overline{\mathbf{m}} + \overline{\Psi}c\,\mathbf{m}\right] + X^\gamma \mathbf{t}_\gamma, \tag{4.2.6}$$

where $c = e^{ia}$ is an arbitrary unit complex number. Different choices of a give different motions. We will be interested in particular in $a = \frac{\pi}{2}$ which describes motion in the binormal direction. The equation of motion for the tangent vector $\mathbf{t}_\alpha$ can be derived directly from (4.2.6). Differentiating with respect to u^α, we obtain

$$\partial_t \mathbf{t}_\alpha = \frac{1}{2}\left[\left(\partial^A_\alpha \Psi + c\lambda_{\alpha\gamma}X^\gamma\right)\bar{c}\,\overline{\mathbf{m}} + \left(\overline{\partial}^A_\alpha \overline{\Psi} + \bar{c}\overline{\lambda}_{\alpha\gamma}X^\gamma\right)c\,\mathbf{m}\right] + \left[-\frac{1}{2}\left(\Psi\bar{c}\overline{\lambda}_\alpha^{\ \gamma} + \overline{\Psi}c\lambda_\alpha^{\ \gamma}\right) + \nabla_\alpha X^\gamma\right]\mathbf{t}_\gamma. \tag{4.2.7}$$

In deriving the equation above the following elementary calculation helps

$$\partial_\alpha\left(\Psi\overline{\mathbf{m}}\right) = \left(\partial^A_\alpha \Psi\right)\overline{\mathbf{m}} + \Psi\left(\overline{\partial}^A_\alpha \overline{\mathbf{m}}.\right)$$

Equation (4.2.7) can be presented in a more compact form if we adopt the notation

$$L_\alpha \stackrel{\text{def}}{=} \partial^A_\alpha \Psi + c\lambda_{\alpha\gamma}X^\gamma \qquad ; \qquad N_\alpha^{\ \gamma} \stackrel{\text{def}}{=} -\frac{1}{2}\left[\Psi\bar{c}\overline{\lambda}_\alpha^{\ \gamma} + \overline{\Psi}c\lambda_\alpha^{\ \gamma}\right], \tag{4.2.8}$$

so that we can write equation (4.2.7) as

$$\partial_t \mathbf{t}_\alpha = \frac{1}{2}\left[L_\alpha \bar{c}\,\overline{\mathbf{m}} + \overline{L}_\alpha c\,\mathbf{m}\right] + \left(N_\alpha^{\ \gamma} + \nabla_\alpha X^\gamma\right)\mathbf{t}_\gamma. \tag{4.2.9}$$

From (4.2.9) we can derive the evolution of the metric. In fact, since $g_{\alpha\beta} = \langle \mathbf{t}_\alpha, \mathbf{t}_\beta\rangle$, we have

$$\partial_t g_{\alpha\beta} = 2N_{\alpha\beta} + \nabla_\alpha X_\beta + \nabla_\beta X_\alpha. \tag{4.2.10}$$

Moreover, the identity $g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha_\beta$ allows us to obtain the evolution equation of the inverse matrix $g^{\alpha\beta}$

$$\partial_t g^{\alpha\beta} = -2N^{\alpha\beta} - \nabla^\alpha X^\beta - \nabla^\beta X^\alpha. \tag{4.2.11}$$

A simple observation concerning the determinant, namely the fact that

$$\partial_t g^2 = \partial_t \left(g_{11}g_{22} - g_{12}g_{21}\right) = g^2 g^{\alpha\beta}\partial_t g_{\alpha\beta}$$

gives the evolution of g

$$\partial_t g = g \left(tr(N) + divX\right). \tag{4.2.12}$$

Let us set (compare with (4.1.10))

$$Y_\alpha{}^\beta \stackrel{\text{def}}{=} 2N_\alpha{}^\beta - tr(N)\delta_\alpha{}^\beta + \nabla_\alpha X^\beta + \nabla^\beta X_\alpha - div(X)\delta_\alpha{}^\beta \tag{4.2.13}$$

and denote by $\widehat{N}$ the traceless part of the tensor N, i.e.,

$$\widehat{N}_{\alpha\beta} \stackrel{\text{def}}{=} N_{\alpha\beta} - \frac{1}{2}tr(N)g_{\alpha\beta}. \tag{4.2.14}$$

The evolution equation for $g_{\alpha\beta}$ can be written as

$$\partial_t g_{\alpha\beta} = Y_\alpha{}^\gamma g_{\gamma\beta} + \big(tr(N) + divX\big)g_{\alpha\beta}. \tag{4.2.15}$$

We can choose the vector field X so that $Y_\alpha{}^\beta = 0$, namely, we set

$$\nabla_\alpha X^\beta + \nabla^\beta X_\alpha - \big(divX\big)\delta_\alpha^\beta = -2\widehat{N}_\alpha^\beta. \tag{4.2.16}$$

From the last equation we can obtain an elliptic equation for the vector field X_α. Taking the covariant derivative ∇_β, we obtain the equation

$$\left(\Delta_g + \frac{1}{2}R\right) X_\alpha = -2\nabla_\beta \widehat{N}_\alpha^\beta. \tag{4.2.17}$$

With this choice of X, the evolution equation of the metric reads

$$\partial_t g_{\alpha\beta} = \big(tr(N) + divX\big)g_{\alpha\beta}. \tag{4.2.18}$$

Now, the time evolution of the complex vector $\mathbf{m}$ must be of the form

$$\partial_t \mathbf{m} = -\bar{c}\, L^\alpha \mathbf{t}_\alpha + iB\mathbf{m}, \tag{4.2.19}$$

where the projection of $\partial_t \mathbf{m}$ onto $\mathbf{t}_\alpha$ is easily found from the fact that the vectors $\mathbf{m}$ and $\mathbf{t}_\alpha$ are orthogonal, i.e., $< \partial_t \mathbf{m}, \mathbf{t}_\alpha >= - < \mathbf{m}, \partial_t \mathbf{t}_\alpha >$. The relations $< \mathbf{m}, \overline{\mathbf{m}} >= 2$ and $< \mathbf{m}, \mathbf{m} >= 0$ imply that the quantity B must be a real scalar potential to be computed. Indeed, the gauge invariance of the equations means that B can be determined uniquely only after we fix the gauge. We will write

$$\partial_t^B \mathbf{m} = -\bar{c} L^\alpha \mathbf{t}_\alpha \quad \text{where} \quad \partial_t^B \stackrel{\text{def}}{=} \partial_t - iB, \tag{4.2.20}$$

and think of B as a temporal gauge field. We can derive evolution equations of λ_α^β and A_α as follows. First, observe that

$$\left(\partial_t^B \partial_\alpha^A - \partial_\alpha^A \partial_t^B\right)\mathbf{m} = -i\left(\partial_t A_\alpha - \partial_\alpha B\right)\mathbf{m}. \tag{4.2.21}$$

Next, we can compute the two sides and set them equal. On the left hand side, we have

$$\begin{aligned} \partial_t^B\left[\partial_\alpha^A \mathbf{m}\right) &= \left(-\partial_t^B {\lambda_\alpha}^\beta - {\lambda_\alpha}^\gamma\left({N_\gamma}^\beta + \nabla_\gamma X^\beta\right)\right]\mathbf{t}_\beta \\ &- \frac{1}{2}\left[{\lambda_\alpha}^\gamma L_\gamma \bar{c}\,\overline{\mathbf{m}} + {\lambda_\alpha}^\gamma \overline{L}_\gamma c\,\mathbf{m}\right]. \end{aligned} \tag{4.2.22}$$

On the right hand side, we have

$$\partial_\alpha^A\left(\partial_t^B \mathbf{m}\right) = \left(-\bar{c}\,\nabla_a^A L^\beta\right)\mathbf{t}_\beta - \frac{\bar{c}}{2}\left(L^\gamma \lambda_{\alpha\gamma}\overline{\mathbf{m}} + L^\gamma \overline{\lambda}_{\alpha\gamma}\mathbf{m}\right). \tag{4.2.23}$$

Substituting (4.2.22), (4.2.23) in (4.2.21) and equating the coefficients for the tangent vectors $\mathbf{t}_\alpha$ and the normal $\mathbf{m}$, we deduce the following two equations

$$\partial_t^B \lambda_\alpha{}^\beta + \lambda_\alpha^\gamma \left(N_\gamma{}^\beta + \nabla_\gamma X^\beta \right) = \bar{c}\, \nabla_\alpha^A L^\beta, \tag{4.2.24a}$$

$$\partial_t A_\alpha - \partial_\alpha B = \frac{1}{2i} \left[c\, \lambda_\alpha^\gamma \overline{L}_\gamma - \bar{c}\, \overline{\lambda}_\alpha^{\,\gamma} L_\gamma \right]. \tag{4.2.24b}$$

At this point, we can substitute L^β, see (4.2.8), to obtain

$$\begin{aligned} \nabla_\alpha^A L^\beta &= \nabla_\alpha^A \left(\nabla^{A\beta} \Psi + c \lambda^\beta{}_\gamma X^\gamma \right) \\ &= \nabla_\alpha^A \nabla^{A\beta} \Psi + c X^\gamma \nabla_\gamma^A \lambda^\beta{}_\alpha + c \lambda^\beta{}_\gamma \nabla_\alpha X^\gamma, \end{aligned} \tag{4.2.25}$$

where in the equation above we used (4.1.31) to commute the derivatives on the tensor $\lambda_{\alpha\beta}$. Remember that $\lambda_\alpha{}^\beta = \lambda^\beta{}_\alpha$ because $\lambda_{\alpha\beta}$ is symmetric. Let us also define the tensor

$$\Omega_{\beta\alpha} \stackrel{\text{def}}{=} \nabla_\alpha^A \nabla_\beta^A \Psi \tag{4.2.26}$$

for the second derivatives of Ψ. Notice that Ω is not symmetric, but it satisfies

$$\Omega_{\alpha\beta} - \Omega_{\beta\alpha} = -i T_{\alpha\beta} \Psi, \tag{4.2.27}$$

where $T_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. Equation (4.2.24a) can be written as

$$D_t^B \lambda_\alpha{}^\beta + \lambda_\alpha{}^\gamma N_\gamma{}^\beta + \left(\lambda_\alpha{}^\gamma \nabla_\gamma X^\beta - \nabla_\alpha X^\gamma \lambda_\gamma{}^\beta \right) = \bar{c} \Omega^\beta{}_\alpha, \tag{4.2.28}$$

where

$$D_t^B \stackrel{\text{def}}{=} \partial_t^B - X^\gamma \nabla_\gamma^A. \tag{4.2.29}$$

Contracting (4.2.28) gives an evolution equation of Ψ, namely

$$D_t^B\Psi + \langle\lambda; N\rangle = \bar{c}\Delta_g^A\Psi. \tag{4.2.30}$$

Equation (4.2.24b) can be written, after a straightforward calculation,

$$\partial_t A_\alpha - \partial_\alpha B = \frac{1}{2i}\left[c\,\lambda_\alpha{}^\beta\overline{\nabla}_\beta^A\overline{\Psi} - \bar{c}\,\overline{\lambda}_\alpha{}^\beta\nabla_\beta^A\Psi\right] + T_{\alpha\beta}X^\beta. \tag{4.2.31}$$

We can derive one more equation for the evolution of the Christoffel symbols of the second kind. Let us now compute

$$\begin{aligned}\partial_\beta\left(\partial_t\mathbf{t}_\alpha\right) = \frac{1}{2}&\left[\left(\partial_\beta^A L_\alpha + cS_\alpha{}^\gamma\lambda_{\gamma\beta}\right)\ \bar{c}\overline{\mathbf{m}} + \left(\overline{\partial}_\beta^A\overline{L}_\alpha +\ \bar{c}S_\alpha{}^\gamma\overline{\lambda}_{\gamma\beta}\right)\ c\mathbf{m}\right]\\ &+\left[\partial_\beta S_\alpha{}^\gamma + S_\alpha{}^\sigma\Gamma_{\beta\sigma}{}^\gamma - \frac{1}{2}\left(L_\alpha\bar{c}\overline{\lambda}_\beta{}^\gamma + \overline{L}_\alpha c\lambda_\beta{}^\gamma\right)\right]\mathbf{t}_\gamma.\end{aligned} \tag{4.2.32}$$

Differentiating with respect to time the structure equation (4.1.24a), we obtain

$$\begin{aligned}\partial_t\left(\partial_\beta\mathbf{t}_\alpha\right) = &\left(\partial_t\Gamma_{\beta\alpha}{}^\gamma - \frac{1}{2}\left(c\lambda_{\beta\alpha}\overline{L}^\gamma + \bar{c}\overline{\lambda}_{\beta\alpha}L^\gamma\right) + \Gamma_{\beta\alpha}{}^\sigma S_\sigma{}^\gamma\right)\mathbf{t}_\gamma\\ &+\frac{1}{2}\left[\left(\partial_t^B\lambda_{\beta\alpha} + \bar{c}\Gamma_{\beta\alpha}{}^\gamma L_\gamma\right)\overline{\mathbf{m}} + \left(\overline{\partial}_t^B\overline{\lambda}_{\beta\alpha} + c\Gamma_{\beta\alpha}{}^\gamma L_\gamma\right)\mathbf{m}\right].\end{aligned} \tag{4.2.33}$$

Equating the coefficients of the tangent vectors we obtain an evolution equation for $\Gamma_{\beta\alpha}{}^\gamma$, namely

$$\partial_t\Gamma_{\beta\alpha}{}^\gamma = \nabla_\beta S_\alpha{}^\gamma + \frac{1}{2}\left(c\lambda_{\beta\alpha}\overline{L}^\gamma + \bar{c}\overline{\lambda}_{\beta\alpha}L^\gamma\right) - \frac{1}{2}\left(c\lambda_\beta{}^\gamma\overline{L}_\alpha + \bar{c}\overline{\lambda}_\beta{}^\gamma L_\alpha\right). \tag{4.2.34}$$

Recall our definition of $L_\alpha = \nabla_\alpha^A\Psi + c\lambda_{\alpha\sigma}X^\sigma$ in (4.2.8). Substituting back into (4.2.34) and using the Gauss equation (4.1.35) and the symmetries of the Riemann curvature tensor, we obtain

$$\partial_t\Gamma_{\beta\alpha}{}^\gamma = \nabla_\alpha D_\beta{}^\gamma + \nabla_\beta D_\alpha{}^\gamma - \nabla^\gamma D_{\beta\alpha}, \tag{4.2.35a}$$

where the tensor $D_{\alpha\beta}$ is defined by

$$D_{\alpha\beta} \stackrel{\text{def}}{=} N_{\alpha\beta} + \frac{1}{2}\left(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha\right). \tag{4.2.35b}$$

Thus the time derivative of the Christoffel symbols is a tensor although they are not tensors.

Let us define two tensors that will appear often in subsequent calculations

$$M_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{2}\left[\mu_{\alpha\beta}\overline{\Psi} + \overline{\mu}_{\alpha\beta}\Psi\right] \qquad ; \qquad P_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{2i}\left[\mu_{\alpha\beta}\overline{\Psi} - \overline{\mu}_{\alpha\beta}\Psi\right]. \tag{4.2.36}$$

Let us now make the choice the choice $c = i$. This means that we evolve the surface in the binormal direction. It is easy to see that from (4.2.8) we have $N_\alpha^{\ \beta} = P_\alpha^{\ \beta}$, hence $tr(P) = 0$ and, see (4.2.31),

$$\frac{1}{2}\left[\lambda_\alpha^{\ \beta}\overline{\nabla}^A_\beta\overline{\Psi} + \overline{\lambda}_\alpha^{\ \beta}\nabla^A_\beta\Psi\right] = \nabla_\beta M_\alpha^{\ \beta}. \tag{4.2.37}$$

The evolution equation of $\lambda_\alpha^{\ \beta}$ becomes

$$D_t^B\lambda_\alpha^{\ \beta} + \lambda_\alpha^{\ \gamma}P_\gamma^{\ \beta} + \left(\lambda_\alpha^{\ \gamma}\nabla_\gamma X^\beta - \nabla_\alpha X^\gamma\lambda_\gamma^{\ \beta}\right) = -i\Omega^\beta_{\ \alpha}. \tag{4.2.38}$$

After contraction in (4.2.33), we have the evolution equation of Ψ

$$D_t^B\Psi + \left\langle\lambda\,;P\right\rangle = -i\Delta_g^A\Psi, \tag{4.2.39}$$

while the equation (4.2.31) becomes

$$\partial_t A_\alpha - \partial_\alpha B = \nabla_\beta M_\alpha^{\ \beta} + T_{\alpha\beta}X^\beta. \tag{4.2.40}$$

Let us compute

$$\langle \lambda; P \rangle = \langle \mu; P \rangle = -\frac{i}{2} \langle \mu; \mu \rangle \overline{\Psi} + \frac{i}{2} \langle \mu; \overline{\mu} \rangle \Psi. \tag{4.2.41}$$

Making use of (4.1.48a), we obtain the equation of Ψ

$$D_t^B \Psi - \frac{i}{2} \langle \mu \, ; \mu \rangle \overline{\Psi} = -i \left(\Delta_g^A + \frac{1}{2} R + \frac{1}{4} |\Psi|^2 \right) \Psi. \tag{4.2.42}$$

Moreover, we can compute, using (4.2.36) and (4.1.58),

$$\begin{aligned} \lambda_\alpha^{\ \gamma} P_\gamma^{\ \beta} = & -\frac{i}{2} |\Psi|^2 \mu_\alpha^{\ \beta} + \frac{i}{2} \Psi^2 \overline{\mu}_\alpha^{\ \beta} - \frac{i}{2} \mu_\alpha^{\ \gamma} \mu_\gamma^{\ \beta} \overline{\Psi} \\ & + \frac{i}{2} \left[\frac{1}{2} \big(R + \frac{1}{2} \big| \Psi \big|^2 \big) \delta_\alpha^{\ \beta} \Psi + i T_\alpha^{\ \beta} \Psi \right]. \end{aligned} \tag{4.2.43}$$

Using equation (4.2.38) above, the fact

$$\mu_\alpha^{\ \gamma} \mu_\gamma^{\ \beta} - \frac{1}{2} \langle \mu; \mu \rangle \delta_\alpha^{\ \beta} = 0,$$

and (4.1.56), we obtain an equation for the tensor $\mu_\alpha^{\ \beta}$

$$\begin{aligned} & i D_t^B \mu_\alpha^{\ \beta} - \frac{1}{2} \Psi^2 \overline{\mu}_\alpha^{\ \beta} + i \left(\mu_\alpha^{\ \gamma} \nabla_\gamma X^\beta - \nabla_\alpha X^\gamma \mu_\gamma^{\ \beta} \right) \\ = & \left[\left(\Delta_g^A + R - \frac{1}{2} |\Psi|^2 \right) \mu_\alpha^{\ \beta} - i T_{\alpha\gamma} \mu^{\gamma\beta} \right]. \end{aligned} \tag{4.2.44}$$

Let us rewrite the evolution equation of Ψ in a way that resembles a Schrödinger-type equation

$$i D_t^B \Psi + \frac{1}{2} \langle \mu; \mu \rangle \overline{\Psi} = \left(\Delta_g^A + \frac{1}{2} R + \frac{1}{4} |\Psi|^2 \right) \Psi. \tag{4.2.45}$$

The presence of the term $\langle \mu; \mu \rangle \overline{\Psi}$ means that the density $\big|\Psi\big|^2$ is not conserved in general. However, the Gaussian curvature must satisfy a conservation law. Starting from the relation $R = \langle \lambda \, ; \overline{\lambda} \rangle - \big|\psi\big|^2$, see (4.1.35), we can obtain a conservation law for the Gaussian curvature. Contract (4.2.38) with $\overline{\lambda}$ and (4.2.42) with $\overline{\Psi}$, and subtract them to obtain

$$\partial_t R - X^\gamma \nabla_\gamma R + 2\nabla_\alpha \nabla_\beta P^{\alpha\beta} = 0 \tag{4.2.46}$$

and to get after integration

$$\frac{d}{dt}\int_\Sigma R\, d\sigma = \int_\Sigma \nabla_\alpha \left\{-2\nabla_\beta P^{\alpha\beta} + RX^\alpha\right\}\, d\sigma. \tag{4.2.47}$$

Moreover, the square mean curvature satisfies an evolution equation

$$\partial_t\big(\frac{1}{2}\big|\Psi\big|^2\big) - X^\gamma \nabla_\gamma\big(\frac{1}{2}\big|\Psi\big|^2\big) + \big\langle P\,;M\big\rangle = \nabla_\alpha J^\alpha, \tag{4.2.48}$$

where the covector J_α is defined by

$$J_\alpha \overset{\text{def}}{=} \frac{1}{2i}\left(\overline{\Psi}\nabla^A_\alpha \Psi - \Psi\overline{\nabla}^A_\alpha \overline{\Psi}\right). \tag{4.2.49}$$

Thus we have after integrating over the surface

$$\frac{d}{dt}\int_\Sigma \left(\frac{1}{2}\left|\Psi\right|^2\right)\, d\sigma = -\int_\Sigma \big\langle P\,;M\big\rangle\, d\sigma. \tag{4.2.50}$$

Some final words. The equation (4.2.45) is very complicated. A possible first step in order to establish local existence could be to study the cited equation in the case of radially symmetric surfaces.

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