ABSTRACT

Title of Dissertation: A COGNITIVE FRAMEWORK FOR ANALYZING AND DESCRIBING INTRODUCTORY STUDENTS' USE AND UNDERSTANDING OF MATHEMATICS IN PHYSICS

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Many introductory, algebra-based physics students perform poorly on mathematical problem solving tasks in physics. There are at least two possible, distinct reasons for this poor performance: (1) students simply lack the mathematical skills needed to solve problems in physics, or (2) students do not know how to apply the mathematical skills they have to particular problem situations in physics. While many students do lack the requisite mathematical skills, a major finding from this work is that the majority of students possess the requisite mathematical skills, yet fail to use or interpret them in the context of physics.

In this thesis I propose a theoretical framework to analyze and describe students' mathematical thinking in physics. In particular, I attempt to answer two questions. What are the cognitive tools involved in formal mathematical thinking in physics? And, why do students make the kinds of mistakes they do when using mathematics in physics?

According to the proposed theoretical framework there are three major theoretical constructs: *mathematical resources*, which are the knowledge elements that are activated in mathematical thinking and problem solving; *epistemic games*, which are patterns of activities that use particular kinds of knowledge to create new knowledge or solve a problem; and *frames*, which are structures of expectations that determine how individuals interpret situations or events.

The empirical basis for this study comes from videotaped sessions of college students solving homework problems. The students are enrolled in an algebra-based introductory physics course. The videotapes were transcribed and analyzed using the aforementioned theoretical framework.

Two important results from this work are: (1) the construction of a theoretical framework that offers researchers a vocabulary (ontological classification of cognitive structures) and grammar (relationship between the cognitive structures) for understanding the nature and origin of mathematical use in the context physics, and (2) a detailed understanding, in terms of the proposed theoretical framework, of the errors that students make when using mathematics in the context of physics.

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A COGNITIVE FRAMEWORK FOR ANALYZING AND DESCRIBING INTRODUCTORY STUDENTS' USE AND UNDERSTANDING OF MATHEMATICS IN PHYSICS

by

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Introduction

Galileo wrote that “the book of nature is written in the language of mathematics.” Over the 400 years since Galileo wrote these words, the mathematical language needed to read the book of nature has become increasingly complex – so complex that it led Einstein to say, “Do not worry about your difficulties in Mathematics. I can assure you mine are still greater.”

Since mathematics is the language of physics, a complete understanding of the concepts in physics requires fluency in the mathematical language in which these concepts are couched. However, most instructors of physics would agree that mathematical problem solving tasks in physics are, in general, a struggle for students.

Among most physics faculty and instructors, there exist two common interpretations for students’ poor performance on mathematical problem solving in physics. One interpretation is that students lack the requisite mathematical knowledge to solve mathematical problems in physics. An alternative interpretation is that students do not know how to apply the mathematical knowledge they have learned in mathematics classes to the context of physics. Fleshing out exactly why students perform poorly on mathematical problem solving tasks in physics could have important implications for physics curriculum and instruction.

Implications for physics curriculum

Students' poor performance on mathematical problem solving tasks in physics has led many physics departments and instructors to adopt *conceptual physics courses*, which remove explicit use of equations from the curriculum. If students do not possess the requisite mathematical knowledge, these conceptual physics courses provide students with exposure to many important physics concepts to which they would otherwise not have access. However, if students do have the relevant mathematical knowledge, then the dilution or removal of mathematical problem solving tasks in physics does not help the students learn to appropriately apply their mathematical knowledge in the context of physics; rather, it deprives them of the opportunity to do so.

Implications for physics instruction

An instructor's (tacit or explicit) interpretation for why students perform poorly on mathematical problem solving tasks can have implications for physics instruction. To illustrate this point, consider the following example of a student (pseudonym Mary) working on a homework problem.

The particular problem that Mary is working on states:

> You are driving on the New Jersey Turnpike at 65 mi/hr. You pass a sign that says "Lane ends 500 feet." How much time do you have in order to change lanes?

Mary has difficulty, so she discusses her approach with an instructor:

> *…all right if I convert 65 mph to feet per second, which is the other thing that's given in feet… So then I got 95 feet per second is what you're moving, so in 500 feet like how long? So, I was trying to do a proportion, but that doesn't work. I was like 95 feet per second...oh wait...yeah in 500 feet, like, x would be like the time...that doesn't—I get like this huge number and that doesn't make any sense.*

Mary correctly identifies that using a proportion could help her solve this problem, but has trouble implementing this strategy.

One interpretation for Mary's difficulties is that she lacks the mathematical sophistication to solve this problem. That is, she doesn't know how to set up the proportion correctly, or, worse, she doesn't know how to perform division reliably! If this interpretation is correct, a legitimate pedagogical approach is to assign many mathematical exercises, in the hope that Mary's problem solving skills will improve through inculcation on proportion and/or division problems.

However, there is an alternative interpretation: it may be that Mary has the relevant mathematical knowledge, but has difficulty using it. Mary's difficulty in using her mathematics knowledge may stem from one of three reasons: (1) she doesn't know how to use her knowledge in the context of physics to arrive at an answer, (2) her strategy for solving this problem precludes her from using the appropriate mathematics knowledge, or (3) the mathematics knowledge that she is remembering and using precludes her from using the appropriate strategies to solve this problem.

If this alternative interpretation is correct (*i.e.* Mary has the knowledge, but doesn't use it) then Mary might not benefit from inculcation on proportion problems, and, indeed, that might make things worse! Rather, she needs guidance on how to activate and effectively apply the relevant mathematics knowledge she learned in her mathematics classes to the context of physics.

We have no compelling reason to favor one interpretation over the other without a theoretical framework and supporting empirical evidence for analyzing and interpreting how students use mathematics in physics.

Research Questions

In this dissertation I propose a theoretical framework to analyze and describe students' mathematical thinking in physics. In particular, this theoretical framework is my attempt to answer two research questions:

- *What are the cognitive tools involved in formal mathematical thinking in physics?*
- *Why do students make the kinds of mistakes they do when using mathematics in physics?*

Main Contributions of this Dissertation

Constructivism is the dominant paradigm in modern educational theory. Redish (2004) defines *constructivism* as follows:

> The belief, common among educational researchers today, that new knowledge must be constructed out of existing knowledge, by establishment of new associations, transformation, and processing.

The educator's role in the constructivist paradigm is to help students construct new knowledge. In order to assist the students the educator needs to be able to determine what the students are thinking and why they make the mistakes that they do. That is, educators and researchers need to be able to describe and understand *how* students construct new knowledge.

The major contribution of this dissertation is a theoretical, cognitive framework for analyzing and describing how students use and understand mathematics in the context of physics. The theoretical framework in this dissertation offers educators and researchers a technical language capable of describing students' (correct and incorrect) use of mathematics in physics. That is, this theoretical framework offers a vocabulary (definition of the relevant cognitive structures) and grammar (relationship between the cognitive structures) for analyzing and describing students' mathematical thinking and problem solving in the context physics. It is useful to researchers and educators in three important ways: it synthesizes previous research into one coherent framework, it can be used as a diagnostic tool during instruction, and it can be used as a guide for future instruction and curriculum development.

Theoretical framework as a synthesis of previous research

Cognitive scientists, sociolinguists, and education researchers have posited the existence of many different cognitive structures and frameworks to explain how students parse, interpret, and understand the myriad of stimuli that inundate them in all learning environments. The proposed cognitive structures vary in their grain-size, ranging from small cognitive building blocks (diSessa, 1993; Minsky, 1985; Minstrell, 1992; Sherin, 2001) to large cognitive structures that describe how students interpret the world (Rumelhart, 1975; Tannen, 1993). Despite the efforts of a few (Redish, 2004), these theoretical constructs exist as isolated ideas, without consensus about their range of applicability and relationship to each other.

The theoretical framework presented in this dissertation attempts to synthesize the isolated theoretical constructs into one coherent framework. This framework incorporates the ideas of *phenomenological primitives* (diSessa, 1993), *symbolic forms* and *interpretive devices* (Sherin, 1996), *epistemic games* (Collins and Ferguson, 1993), and *frames* (Fillmore, 1985; Goffman, 1974; Tannen, 1993) into one coherent theoretical framework for describing how students understand and use mathematics in physics.

The theoretical framework as a diagnostic tool

The technical language developed in this theoretical framework can help educators and researchers diagnose students' mathematical difficulties, on a case by case basis, and offer instructional interventions to help students utilize the mathematical knowledge that they already possess. Simply stated, this framework makes sense of introductory physics students' seemingly bizarre use of mathematics in the context of physics. That is, students sometimes use mathematics in physics in a manner that is in stark contrast to how an expert would use the same mathematics. This theoretical framework can help experts understand students' use of mathematics.

The theoretical framework as a guide for instruction

Many pedagogical attempts to improve mathematical problem solving focuses on teaching a systematic, step-by-step method that could be applied to all problem solving tasks (Pólya, 1945; Schoenfeld, 1978; Reif & Heller, 1982). While instruction based on these types of prescriptive methods can produce improvements in students' abilities to solve mathematical problems, exactly *why* these approaches work in not very well understood. That is, it is not clear how these instructional methods help students use the mathematical knowledge they already possess – in the constructivists' paradigm of learning.

The theoretical framework developed in this dissertation offers instructors and curriculum developers a more thorough understanding of the cognitive building blocks and processes involved in mathematical thinking and problem solving in the context of physics. With this improved understanding perhaps more effective and efficient instructional strategies can be developed.[1]

Brief introduction to the theoretical framework

My theoretical framework identifies three levels of cognitive structures relevant to mathematical thinking and problem solving in the context of physics:

- *Mathematical Resources* – the basic knowledge elements that are activated in mathematical thinking and problem solving;
- *Epistemic Games* – coherent activities that use particular kinds of knowledge and processes associated with that knowledge to create knowledge or solve a problem; and
- *Frames* – structures of expectations that determine how individuals interpret situations or events.

Each of these types of cognitive structures is described in more detail below.

Mathematical Resources

Mathematical resources are abstract knowledge elements – the cognitive tools involved in mathematical thinking and problem solving. Within the category of mathematical resources there are

- o *Intuitive Mathematics Knowledge* – knowledge of mathematics that is learned at a very early age; examples are *counting* and *subitizing*. Subitizing is the ability that humans have to immediately differentiate sets of one, two, and three objects from each other (Fuson, 1992).

[1] I will offer more speculations and anecdotal evidence for this in chapter 8.

- *Reasoning Primitives* – abstract knowledge elements which describe students' intuitive sense of physical mechanism. Reasoning primitives are a generalization of diSessa's p*henomenological primitives* (diSessa, 1993).
- *Symbolic Forms* – combination of the conceptual knowledge of reasoning primitives and syntactic knowledge of mathematical symbolism into one single knowledge element (Sherin, 1996, 2001). Symbolic forms consist of a symbol template and conceptual schema. The *symbol template* is an element of knowledge that gives structure to mathematical expressions; e.g. □ = □ or □ + □ + □... (where the boxes can contain any type of mathematical expression). The *conceptual schema* is a simple structure associated with the symbolic form that offers a conceptualization of the knowledge contained in the mathematical expression; this part of the symbolic form is similar to the reasoning primitives discussed in the previous section.
- *Interpretive Devices* – interpretive strategies used to extract information from a physics equation (Sherin, 1996).

Resources can exist in three states of activation: inactive, primed, and active. Inactive and primed resources are abstract knowledge elements that can potentially be used in different problem situations; as such they are neither right nor wrong. *Facets* are resources that are active and mapped into specific problem situations – in accordance with Redish's (2004) refinement of Minstrell's (1992) term. As such, facets can be right or wrong depending on how they are used. Whereas there are small numbers of mathematical resources, there are countless numbers of facets corresponding to the myriad different situations into which mathematical resources can be mapped.

Epistemic Games

During mathematical thinking and problem solving in the context of physics, students appear to engage in activities that are associated with each other. Epistemic games, which were first proposed by Collins and Ferguson (1993), are used to describe these associated activities. The epistemic games that Collins and Ferguson identify are normative; their games are used to describe expert scientific inquiry across all scientific disciplines. I extend their idea of epistemic games to include an observational categorization of what students actually do. My identification of epistemic games is descriptive, rather than normative, and specific to physics rather than common to all scientific disciplines. I identify six different epistemic games that can be used to describe how students actually use and understand mathematics in the context of physics. I follow Redish (2004) and define *epistemic games* as coherent activities that use particular kinds of knowledge and processes associated with that knowledge to create knowledge or solve a problem.

Epistemic games can be distinguished from each other by their ontology[2] and structure. There are two components that make up the ontology of an epistemic game: the knowledge base and the epistemic form. The *knowledge base* is the set of mathematical resources that are activated during the playing of the epistemic game. The *epistemic form* is a target structure that guides the inquiry. The structure of an epistemic game also consists of two components: the entry conditions and the moves. The *entry conditions* are determined by an individual's expectations about the particular situation or

[2] *Ontology* – the description of a system in terms of the kinds of objects relevant for its description and their characteristics (Redish, 2004).

problem. To describe students' expectations I introduce the concept of *frames*, which are discussed in the next section. The *moves* of an epistemic game are the collections of activities that occur during the course of the problem solving activity.

To understand the different components of an epistemic game, consider the epistemic game that Collins and Ferguson call *list making*. All lists are inherently the attempt to answer a question. Such as: "What do I need from the grocery store?"; "What were the causes of the American Civil War?": Or, "What are the constituents of matter?" The knowledge that one uses to answer anyone of these questions is the knowledge base. The epistemic form in *list making* is the list itself; the list is the target structure that guides the inquiry. The condition for entering the *list making* game is the expectation that a list can help answer the initial question. Legitimate moves in *list making* are adding a new item, combining two or more items into one, changing an item, splitting an item into two or more items, or deleting an item.

Frames and Framing

The concepts of *frames* and *framing* help us understand how or why students "choose" to play a particular epistemic game in a particular context. (I put the word "choose" in quotes, because I don't mean a conscious choice, but rather a tacit decision.) Frames and framing have a long history in the linguistics and cognitive science communities (Goffman, 1974; Fillmore, 1985; Tannen, 1993).

As a working definition of a frame, an individual's frame helps her answer the question "What kind of activity is this?" A *frame* is the definition of a situation that guides interpretation. One's expectations about a situation or event determine how the situation or event is interpreted. The moment-by-moment interpretation of the situation is the frame.

Overview of Dissertation

Chapter 2 offers a review of previous research on mathematical problem solving. In chapter 3, I discuss the data and the methodologies I employ to analyze the data. Chapters 4 and 5 are the major theoretical chapter. In chapter 4, I introduce the mathematical resources that describe the cognitive tools involved in mathematical thinking and problem solving. In particular, I discuss four different kinds of mathematical resources: intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices.

In chapter 5, I discuss epistemic games and frames. Students play six different epistemic games during mathematical thinking and problem solving in the context of physics: *Mapping Meaning to Mathematics*, *Mapping Mathematics to Meaning*, *Physical Mechanism Game*, *Pictorial Analysis*, *Recursive Plug-and-Chug*, and *Transliteration to Mathematics*. In addition, there are three different frames in which these games are couched: *quantitative sense-making*, *qualitative sense-making*, and *rote equation chasing*.

In chapter 6, I use this framework to analyze, in depth, a particular problem solving episode involving four students. In particular, I show how this framework allows educators and researchers to 'see' and examine all the knowledge and reasoning that is involved in mathematical thinking and problem solving. In chapter 7, I show how this framework can be used to interpret the kinds of mathematical errors that students make when using mathematics in the context of physics. In particular, I show how this framework helps make sense of students' semantic mathematics errors in physics. That is, this framework helps make sense of some of our students' seemingly bizarre use of mathematics. Lastly, in chapter 8, I summarize the theoretical framework and results, and discuss some instructional implications and possible future research that arise from this theoretical framework.

Chapter 2: Review of previous research on students' use of mathematics

Introduction

In my dissertation research I am interested in the cognitive tools involved in mathematical thinking and problem solving in the context of physics, and why students make the kinds of mathematical mistakes that they do. There is ample research in the literature that can shed light on these issues. In this chapter I summarize the work of a few key researchers and discuss how their work relates to my own research.

Previous research on mathematical thinking and problem solving can be divided into two major categories: empirical research on student mathematics use and theoretical approaches to student mathematics use. In the next section I discuss empirical research on student mathematics use. In particular, I discuss research of students' use of mathematics in the context of mathematics courses and physics courses. I focus more on the latter, since I am interested in how students' use mathematics in physics. In the third section, I discuss theoretical approaches to understanding students' use of mathematics. First, I describe some general theories of knowledge structures, and then I discuss some specific theories of the structure of students' mathematical knowledge.

The final section offers a brief discussion about how these different approaches fit together in a coherent whole, and how I use and build upon these approaches in my own research.

Empirical research on student mathematics use

The necessary first step in understanding students' use of mathematics in physics is observing what students do with math. It cannot be assumed that students necessarily use mathematics in the manner that they are instructed to by their teachers. There is empirical research of students' use of mathematics in the context of mathematics and the context of physics.

Empirical research in mathematics context

The mathematics education and mathematical psychology research communities have made significant progress on understanding students' use of mathematics in the context of mathematics courses. (Stephen Reed (1998) offers a review of research on problem solving in mathematics.) Research has focused on students' understanding of addition and subtraction (Carpenter and Moser, 1983; Riley, Greeno, and Heller, 1983; Kintsch and Greeno, 1985; Fuson, 1992), multiplication and division (Greer, 1992; Vergnaud, 1983, 1988; Schwartz, 1988), and understanding and describing student mathematical errors in mathematics courses (Ben-Zeev, 1996, 1998; Matz, 1982; VanLehn, 1986). In addition, researchers in the mathematics education research community have pushed to incorporate the results and methods of cognitive science in their pursuits of understanding students' use of mathematics (Schoenfeld, 1992; Silver, 1987). In this vein, Lakoff and

Núñez (2000) offer cognitive mechanisms by which humans can make sense of abstract mathematical concepts.

These results in the mathematics education community are very useful, however a coherent and meaningful description of a highly context dependent phenomenon – like mathematics achievement in the context of physics – can only be achieved if the phenomenon is studied in its' original setting. Therefore, although the results from the mathematics education community have colored my own interpretations of my data, the bulk of my attention has focused on previous research on mathematics use in the context of physics. Hence, the remainder of this chapter will focus on research on mathematics in the context of physics.

Empirical research in physics context

Previous empirical research on the role of mathematics in physics can be classified by the various methods employed by researchers to probe how students use mathematics in physics. Two approaches have emerged: the observational approach and modeling approach. In the observational approach, researchers observe students' use of mathematics in physics and attempt to explain these observations without explicit reference to the students' knowledge structure or cognitive state. The modeling approach generally starts by observing differences between experts and novices when using mathematics in physics, and then proceeds by constructing computer models that mimic the respective performances of the two groups.

Observational Approach

The *observational approach* is a relatively straightforward approach used to probe students' use of mathematics in physics. This approach tacitly assumes that students are rational thinkers who make mistakes when using mathematics in physics because of a small number of inappropriate interpretations. It is the presence of these inappropriate interpretations that explain student errors when using mathematics in physics.

Every algebraic equation has two main structural features: an equal symbol and variables. From the arrangement of these structures, the relationship between the variables can be deduced. So, in order to understand an algebraic equation one must successfully interpret at least three different things: the equal symbol, the variables, and the relationship between the variables. This section is broken up into three subsections that focus on students' misinterpretations of the equal symbol, the nature of a variable in an algebraic equation, and the relationship between the variables (in the context of thermodynamics).

The equal symbol. As a first attempt to understand students' use of mathematics in physics it is natural to assess their interpretation of what an equation really means. Herscovics and Kieran (1980), and later Kieran (1981), attempt to understand students' interpretations of the equals symbol. By examining previous research on a range of students from elementary school to early college students, Kieran concludes that students view the equal symbol as a "do something" symbol. These students' interpretation of the equal symbol is not necessarily harmful to their learning; it simply is illustrative of how students interpret one aspect of equations.

Elementary students when reading arithmetic equations like "3 + 5 = 8" would say "3 and 5 *make* 8." This reading of the arithmetic equation "3 + 5 = 8" was interpreted by Kieran to indicate that the students view the equal symbol as a symbolic prompt to add the first two numbers together. The following example supports this interpretation about how students view the equal symbol. First and second grade students when asked to read expressions like "□ = 3 + 4," would say, "blank equals 3 plus 4," but they would also include that "it's backwards! Am I supposed to read it backwards?" The students read the equations from left to right, like English sentences, in which case the result appears before the two numbers are added together. However, to these students three and four must be added together before a result can be computed.

The previous examples lend credence to the interpretation that elementary school students view the equal symbol as a "do something" symbol. Kieran argues, however, that this interpretation of the equal symbol is not specific to elementary school students. Kieran cites the following example, from a high school student's written solution, to argue that high school students also see the equal symbol as a do something symbol or an operator symbol:

Solve for *x*: (Byers and Herscovics, 1977)

$$\begin{aligned} x + 3 &= 7 \\ &= 7 - 3 \\ &= 4 \end{aligned}$$

Examining this example it is seen that both sides of the equations are not always equal. The equal symbol is traditionally used in algebraic equations to indicate a numerical equivalence between two mathematical expressions. That is, the equal symbol separates two mathematical expressions that represent the same numerical value. However, the student does not use the equal symbol in that way in the above example.

Kieran cites an example from Clement (1980), in which early college students enrolled in a calculus course use the equal symbol as a "do something" symbol. The student sees an equals symbol and spontaneously attempts to differentiate the function.

(Clement, 1980):

$$\begin{aligned} f(x) &= \sqrt{x^2+1} \\ &= (x^2+1)^{1/2} \\ &= \frac{1}{2}(x^2+1)^{-1/2} D_x(x^2+1) \\ &= \frac{1}{2}(x^2+1)^{-1/2}(2x) \\ &= x(x^2+1)^{-1/2} \\ &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

In this example it's as if the student sees the equal symbol as an arrow that leads to the next step in the problem solution. In the first line the student writes down what the function is. In the third line, which the student's solution implies is equal to the first line, the student is calculating the derivative of that function. The student connects these lines in the derivation by an equal symbol, which suggests that the student is using the equal symbol as an arrow or "do something" symbol and not as an equivalence symbol.

It's not clear from this research whether the interpretation of the equal symbol as a "do something" symbol is harmful to the students or not. That is, there are no direct instructional implications that can be drawn from this work. Rather, this research only gives insight about how students understand one aspect of equations: the equal symbol.

Variables in Algebraic Equations. Clement, Lochhead, and Monk (1981) videotaped college science students solving simple word problems. The students were instructed to talk aloud throughout the process of solving the problem. The observed students experienced great difficulty in translating the English words from the problem statement into algebraic expressions. Leery that the problem was "simply one of misunderstanding English," Clement *et al* developed a set of written questions to further probe this issue. One such question read:

> Write an equation for the following statement: "There are six times as many students as professors at this university." Use *S* for the number of students and *P* for the numbers of professors.

This question was given to 150 calculus-level and 47 non-science major students. The correct answer to this question is $S = 6P$; however, 37 percent of the calculus students and 57 percent of the non-science majors answered this question incorrectly, with the most common mistake being $6S = P$.

Clement *et al* offered two possible explanations for the students' mistakes. The first explanation, which they called *word order matching*, is direct mapping of the English words into algebraic symbols. So the sentence "there are six times as many students as professors" becomes $6S = P$, simply because that's the order in which the words "six," "student," and "professor" appear in the statement of the problem. However, they offer a second, more interesting explanation for the students' mistakes, which they call *static comparison*. According to this explanation students misinterpreted the very meaning of the variables. The variable *S*, to students using the static comparison interpretation, does not represent the number of students, but rather is a label or unit associated with the number six. Some students even drew figures like the one below (see Figure 1), which indicates that they recognized that there are more students than professors.

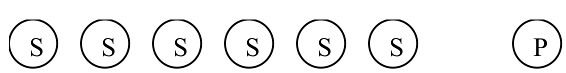

Figure 1. Figure that a student produced to assist in constructing an equation for the following statement: "There are six times as many students as professors at this university."

Relationship between variables. Research by Rozier and Viennot (1991) shows that, in the context of thermodynamics, some students have trouble parsing the relationship between variables in multivariable problems. Rozier and Viennot analyzed written responses to questions about thermodynamic processes on ideal gases, which could be understood using the equation of state for ideal gases $pV = nRT$. They found that students made two mistakes when interpreting multivariable processes.[3]

First, the students would chunk the variables by mentally reducing the number of variables they would consider in a given process. For example, Rozier and Viennot examined student responses to the following question:

> In an adiabatic compression of an ideal gas, pressure increases. Can you explain why in terms of particles?

The correct response involves the following string of reasoning:

> *volume goes down → number of particles per unit volume goes up* ***and*** *the average velocity of each particle goes up → number of collisions goes up* ***and*** *the average velocity of each particle goes up → the pressure goes up.*

However, a typical student response dropped any consideration about the velocity of the gas particles increasing and would only focus on the number of particles per unit volume increasing. The student response can be represented in the following way:

> *volume goes down → number of particles per unit volume goes up → number of collisions goes up → pressure goes up.*

---

[3] Loverude *et al* (2001) find results that are consist with Rozier and Viennot's results, but conclude that "general reasoning difficulties could not be completely separated from difficulties with specific concepts" (p. 141). That is, Loverude *et al* conclude that the difficulties are not only with the variables but also with the concepts the variables represent.

By only considering the effect that the increase in the number of particles per unit volume had, the students reduced the number of variables that influence this process, and thereby resorted to what Rozier and Viennot refer to as *linear reasoning*. The student's response, in this example, is not necessarily wrong—that is, it doesn't lead to an incorrect conclusion—rather, it demonstrates that students may use a simplified reasoning track to reach the correct conclusion. This example serves only to give insight about the reasoning processes that students use when reasoning about the relationship between variables in multivariable causation.

The second mistake that Rozier and Viennot observed students making when interpreting multivariable causation was the unwarranted incorporation of a chronological interpretation to certain thermodynamic processes. An example of a student response helps bring this point out. When asked to explain why the volume would increase for an ideal gas that is being heated at constant pressure, a student responded:

> *The temperature of the gas increases. Knowing that in a perfect gas $pV = nRT$, therefore at constant volume, pressure increases: the piston is free to slide, therefore it moves and volume increases.*

In this example the student's response is wrong. It is clear that by allowing the pressure to increase in the solution the student has contradicted the statement of the problem; i.e. that the gas is heated at constant pressure. Rozier and Viennot argue that this contradiction disappears if the stipulation of constant pressure is only temporary, so that the interpretation by the student is understood to progress in time. That is, if the word "therefore" in the student's solution is interpreted to mean "later," the student's solution is no longer contradictory. However, the chronological interpretation present in the student's solution does not come from the equation of state for an ideal gas. The equation $pV = nRT$ represents simultaneous changes in the variables, whereas the student interprets the multivariable causation as being temporal.

The research presented here does not have any direct instructional implications; rather, it serves as a "jumping-off-point" to help us understand how students interpret the different features of an equation. This section focused on student interpretations and student reasoning about equations. The next section will focus on student (and expert) performance while using equations during problem solving.

Modeling Approach

There are two basic components to what I call the modeling approach. First, one observes the difference between the problem solving skills of the novice and the expert through talk aloud problem solving sessions, or written questionnaires, or both. The second component of the modeling approach is the reason for the name 'modeling approach.' Computer programs are developed with the intent of modeling the performance of either the novice or the expert on similar problem solving tasks.

Larkin, McDermott, Simon, and Simon (1980) articulated four novice/expert differences when solving problems; (i) speed of solution, (ii) backward vs. forward chaining, (iii) uncompiled vs. compiled knowledge, and (iv) syntax vs. semantic interpretations of English statements. The speed of the solution is an obvious difference between novice and expert problem solvers; experts solve problems faster than novices.

A difference that was articulated by Larkin *et. al* is that novices tend to "backward chain," whereas experts tend to "forward chain" when solving problems. This means that novices tend to attack the problem by determining what the end goal is and then working backwards from the end goal toward the initial conditions that are given in the problem statement. In contrast, the expert tends to starts with the initial conditions given in the problem statement and work toward the end goal. This is surprising because backward chaining is generally thought to be a sophisticated problem solving technique.

The third novice/expert difference mentioned above is not a result from direct observations; rather it is a theoretical conjecture about how knowledge is structured for the novice and the expert. Larkin *et. al.* argue that the novice's knowledge must be processed "on the spot" in order to arrive at the problem solution; that is, the novice's knowledge exists in what Redish (2004) calls an uncompiled form (much like a computer program that is uncompiled). However, the expert may have portions of the problem solution compiled from experience in solving similar problems. Because of these chunks of compiled knowledge, not all of the expert's knowledge must be processed "on the spot" to generate the problem solution; i.e. some of the expert's knowledge exists in compiled form. The difference in the speed of solution for the expert and novice may be accounted for by this difference in knowledge structure; procedures using compiled knowledge can be executed much faster than procedures relying on uncompiled knowledge.

The fourth novice/expert difference concerns the manner in which English statements are translated into algebraic notation. The novice tends to write algebraic expressions that correspond with the syntax of the English statements (this is similar to Clement's *word order matching* discussed above). The expert, on the other hand, tends to translate the English statements semantically—that is, in terms of the physics knowledge relevant to the problem—in order to construct algebraic expressions.

Larkin *et. al.* discuss the computer program developed in 1968, called STUDENT, which translates English problem statements into algebraic expressions using the same syntax mapping that is generally associated with students. Larkin *et. al.* use the following problem to discuss how STUDENT works:

> A board was sawed into two pieces. One piece was one-third as long as the whole board. It was exceeded in length by the second piece by 4 feet. How long was the board before it was cut?

To solve the problem STUDENT starts by assigning a variable name (x) to the "length of the board." The first piece mentioned then becomes $x/3$ and the next piece becomes ($x/3 + 4$); therefore, the algebraic expression to be solved is $x = x/3 + (x/3 + 4)$.

It was mentioned above that experts use their knowledge of physics to translate English statements into algebraic expressions. The program ISAAC was developed to model this type of expert performance. ISAAC uses schemata to understand ordinary language in terms of idealized levers, fulcrum, ropes, frictionless surfaces, etc.; i.e. it uses its physics knowledge to generate equations from the English statements. For example, ISAAC will recognize a ladder leaning up against a wall as a lever, and associate with that lever the specific properties mentioned in the problem.

What can be concluded from these computer programs? Research by Hinsley and Hayes (1977) suggests that students can quickly (within the first few words of the problem statement) categorize mathematics word problems. If these categorizations match with known problem solving strategies, then the students tend to employ these strategies – in much the same way that ISAAC attempts to solve algebraic word problems. However, if the problem statement does not match with a known problem strategy, the students tend to employ a line by line translation of the text into mathematical expressions – in much the same way as STUDENT does. Because of the correspondence of student performance and computer programs like STUDENT and ISAAC, Larkin *et al* argue that intuition and problem solving "need no longer to be considered mysterious and inexplicable"; with our increased understanding of the expert's knowledge will come new avenues by which to understand the learning processes involved in the acquisition of such knowledge.

Theoretical approaches to student mathematics use

Theoretical approaches to understand students' use of mathematics establish principles for understanding reasoning in general and in mathematics in particular. First, I discuss general theories of knowledge structures. In particular, I describe two different frameworks that have emerged for describing the general structure of knowledge. Second, I discuss specific theories of the structure of mathematics knowledge – in particular, the types of scientific knowledge and the ontological structure of mathematical entities.

General theories of knowledge structure

The *general theories of knowledge structure* method posits the existence of various kinds of cognitive constructs to understand the structure of concepts in general, not restricting the focus to simply concepts in mathematics. A cognitive mechanism that explains the use of concepts in learning can be constructed from the theoretical cognitive structures.

Two ostensibly distinct frameworks have emerged in the debate about the structure of student knowledge; (i) the *unitary*, *misconceptions*, or *alternative theories* framework

(Chi, 1992; Clement, 1983; Carey, 1986; McCloskey, 1983) and (ii) the *manifold* or *knowledge-in-pieces* framework (diSessa, 1993; Minsky, 1985; Minstrell, 1992). In short, the unitary story of knowledge is that students possess robust cognitive structures, or misconceptions, that need to be torn down, so the correct conception can be erected in its stead. The manifold framework claims that students possess small pieces of knowledge that have developed through everyday reasoning about the world. These small pieces of knowledge are activated by different contexts, and can be built upon to foster learning during formal instruction (diSessa, 1993; Smith *et al*, 1993; Hammer, 1996).

More recently, researchers have attempted to combine these two theoretical perspectives into one coherent framework. Scherr (2002a, 2002b, 2003) shows how some aspects of student reasoning within the context of special relativity fall within the misconceptions framework, whereas, other aspects of their reasoning are better understood in terms of a knowledge-in-pieces framework. Redish (2004) proposes a *theoretical superstructure* that subsumes the unitary and knowledge-in-pieces framework into one overarching framework and argues that both unitary and knowledge-in-pieces frameworks have explanatory power in different contexts.

The remainder of this section discusses two representative theories about concepts that emerge from the unitary and knowledge-in-pieces frameworks.

Unitary Knowledge Structure

Chi's (1992) central claim is that concepts exist within ontological categories, and the ontological categories admit an *intrinsic* and a *psychological* reality. The intrinsic reality is "a distinct set of constraints [that] govern the behavior and properties of entities in each ontological category." The psychological reality is "a distinct set of predicates [that] modify concepts in one ontological category versus another, based on sensibility judgment task." So, the intrinsic reality is an objective reality that is imposed by a "sensible" (scientific) community; whereas, the psychological reality is a subjective reality created by the individual. Chi argues that there should be an isomorphism between these two realities in order for learning to occur. Figure 2 shows what an idealized ontology might look like, where an idealized ontology is "based on certain scientific disciplinary standards."

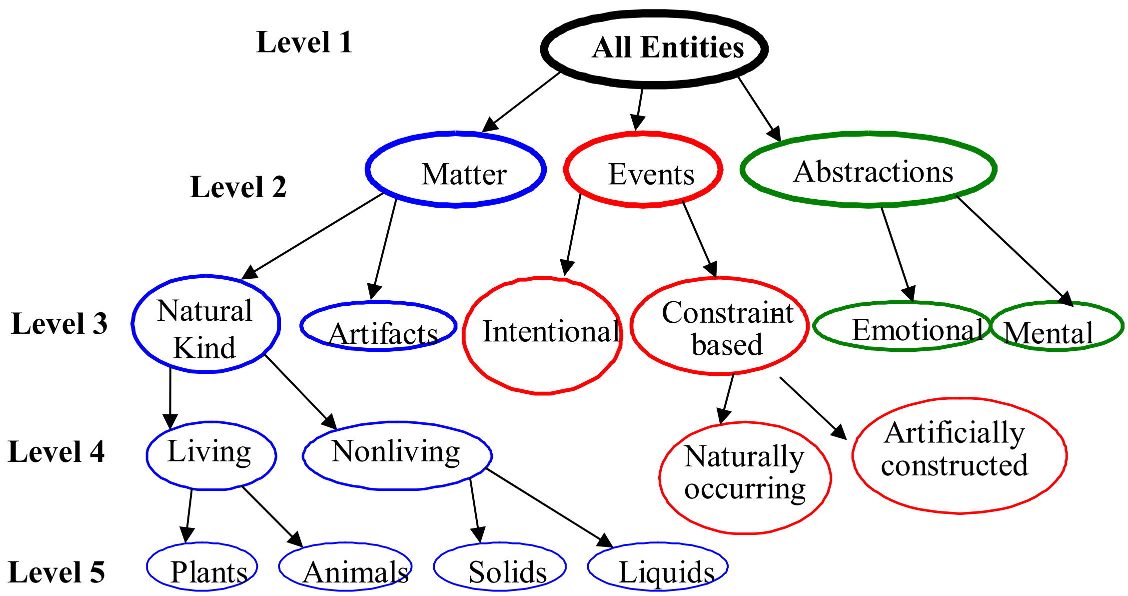

Figure 2. Idealized ontology (Chi, 1992).

Students do not start out knowing everything; they must change their mental state, *i.e.* undergo conceptual change, in order to learn. To understand conceptual change in Chi's ontological categories model, the details of Figure 2 must be discussed. The six entries along level 3—namely, the ovals entitled Natural Kind, Artifacts, Intentional, Constraint-based, Emotional, and Mental—are six different *branches* or ontological categories. The ontological *tree* refers to the collection of branches or ontological categories that are linked across different levels by arrows (in the figure the ontological tree associated with Matter is in blue). The ontological structure permits two kinds of conceptual change: conceptual change *within* an ontological category and conceptual change *across* ontological categories. Chi argues that the latter is more difficult and requires different cognitive processes to occur; therefore, it would better be classified as the acquisition of new conceptions rather than conceptual change.

The theory asserts that conceptual change across ontological categories—henceforth called *radical conceptual change*—requires two independent processes. First, the new category must be learned and understood. An example from physics would be the acquisition of the scientific notion of **Force** as a new ontological category. Secondly, radical conceptual change requires the realization that the original assignment of the concept to a particular category is inconsistent with the properties of that category; therefore, the concept must be reassigned to a different category. Staying with the same example from physics, one must realize that the concept of **Impetus**, as articulated by McCloskey (1983), does not belong in the ontological category of **Force**.

The first requirement for radical conceptual change—stated in the previous paragraph—is achieved by learning the new ontological category's properties and learning the meaning of the individual concepts contained within this ontological category. The second requirement for radical conceptual change—reassignment of a concept to a new ontological category—can be achieved in one of three ways.

Firstly, one can "actively abandon the concept's original meaning and replace it with a new meaning." For example, actively realizing that a thrown ball does not posses a quality like **Impetus**, rather the ball simply interacts with other objects via **Forces**.

The second method to reassign a concept to a new ontological category is to allow both meanings of the concept to coexist, in different ontological categories, with either meaning being accessible depending on context. Chi argues that this is probably the most common type of change since many professional "physicists will occasionally revert back and use naive notions to make predictions of everyday events." (It should be noted that some authors see this same example as evidence for knowledge fragments, like *p-prims*, instead of unitary knowledge structures like ontological categories.)

Third, the coherence and strength of the new meaning can be so robust that the replacement of the concept is automatic.

To summarize this subsection, Chi proposes a theoretical framework to understand conceptual change that occurs in learning science. In this framework, concepts exist within a rigid hierarchical structure. In the next subsection the very concept of a scientific concept is brought into question.

Manifold Knowledge Structure

diSessa and Sherin (1998) espouse a theory of one kind of concept[4] that is based on the linkage of fragmented knowledge structures, which they call a *coordination class*. The word "coordinate" is used in two different senses in the definition of a coordination class. The first is the *integration* of a particular situation into a whole, and the second is

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[4] They argue that the word 'concept' is used rather broadly in the research literature. The theory of coordination classes only refers to a narrowly defined type of concept.

the *invariance* of the interpretation across contexts. Along with the two uses of coordination, there are two structural components that make up a coordination class: the *read-out strategies* and the *causal net*. The information that one uses to construct a coordination class is gathered through various *read-out strategies*. Read-out strategies refer to the methods one employs to extract information in various contexts and situations. The *causal net* is the set of implications associated with the coordination class. For example, the existence of a force 'causes' an acceleration, which is essentially captured in Newton's Second Law: $\vec{F} = m\vec{a}$. The meaning of these abstract definitions will be extracted from an example found in the literature.

Wittmann (2002) applies diSessa and Sherin's theory to interpret students' understanding of wave pulses. This work will serve as a concrete example of how the theory of coordination classes may be used by researchers in education research. Wittmann's central claim is that students understand waves as object-like things instead of event-like things. One example that Wittmann discusses involves students' beliefs about pulses traveling on a string. Flicking a taut string with one's hand will generate a wave pulse that travels down the string. The students in Wittmann's study believe the pulse will travel faster if the string is flicked faster. If one is thinking of the wave as being like an object, for example a ball, this interpretation would be true. This is consistent with a common *phenomenological primitive* associated with objects, namely *faster means faster*. (See chapter 4 for more on *phenomenological primitives* and mathematical resources.) For example, throwing a ball is accurately described by the *faster means faster* p-prim, since moving one's hand faster when throwing a ball will cause the ball to move faster.

However, in the case of waves—which Wittmann describes as event-like—the *faster means faster* p-prim can be misleading. That is, the *faster means faster* p-prim does not apply to the transverse velocity of the wave, which is how the students are using it. So, in this example the p-prim is simply mapped incorrectly onto the physical situation. The speed of the pulse is only dependent on the properties of the media in which it travels, in this case the string. The relative speed at which the hand is moved to generate the pulse has no effect on the relative speed at which the pulse travels down the string.

Wittmann's conclusion is that students coordinate wave around the idea of objects; i.e. the students coordinate waves around the **Object** coordination class, whereas waves would be coordinated by an expert around the **Event** coordination class. This coordination, according to Wittmann, occurs along three dimensions. First, the students use their read-out strategies to associate *wave as solid* and *object as point*. Second, the students' motion resources, like *faster means faster* point to wave as object. Third, from examples that are not discussed in this review, the students' interaction resources, like *adding* and *bouncing*, point to wave as object. The motion resources, interaction resources, and read-out strategies all coordinate around *wave as object*. Figure 3 (Wittmann, 2002) summarizes this conclusion.

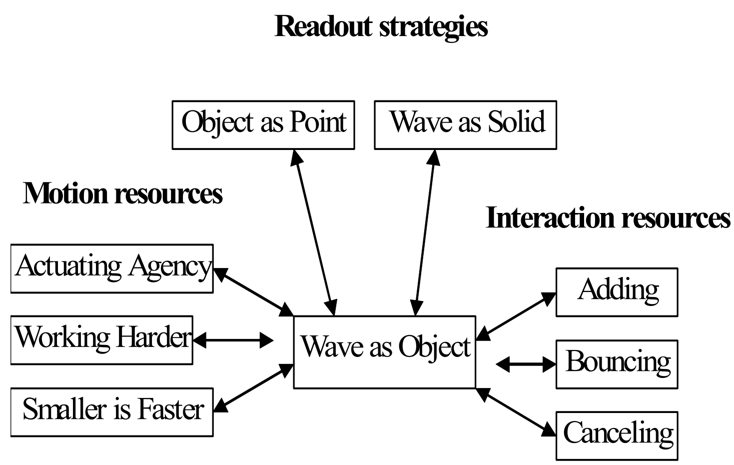

Figure 3. Possible schematic showing reasoning resources that describe an object-like model of waves.

Specific theories of the structure of students' mathematics knowledge

In the mathematics knowledge structure approach, researchers posit various theoretical cognitive structures. A cognitive mechanism, which explains the observed phenomena of the novice and/or the expert using mathematics, can then be constructed from the theoretical cognitive structures. This section will be divided into three subsections entitled *Types of Scientific Knowledge*, *Symbolic Forms*, and *Ontological Structure of Mathematical Entities*. These three subsections will focus on work by Reif and Allen (1992), Sherin (2001), and Sfard (1991), respectively.

Types of Scientific Knowledge

Reif and Allen (1992) developed a cognitive model of "ideally good scientific concept interpretation," which they used to understand the difference between 5 experts and 5 novices solving problems about acceleration. Reif and Allen's model starts by proposing knowledge that falls in three different categories (see Figure **4**):

(i) *main interpretation knowledge*,

(ii) *ancillary knowledge*, and

(iii) *form of knowledge*.

Main interpretation knowledge, as the name suggests, is the primary structure implicated in interpreting a scientific concept. Main interpretation knowledge has two major components:

1. **General knowledge**. General knowledge about a scientific concept is divided into three parts.
    a. A precise *definition* is important for any scientific concept and makes up the first part of general knowledge.
    b. *Entailed knowledge* is derivable from the definition, but is not explicitly articulated in the definition.
    c. Lastly, *supplementary knowledge* is related to, but not derivable from the definition.
2. **Case-specific knowledge**. This is knowledge that is applicable in a narrow domain of phenomena. As an example, consider an object moving with constant speed on an oval path. Many students say that the acceleration of the object is directed toward the center of the oval. Although this is true for a circular path, this result is not true for a generic oval path.

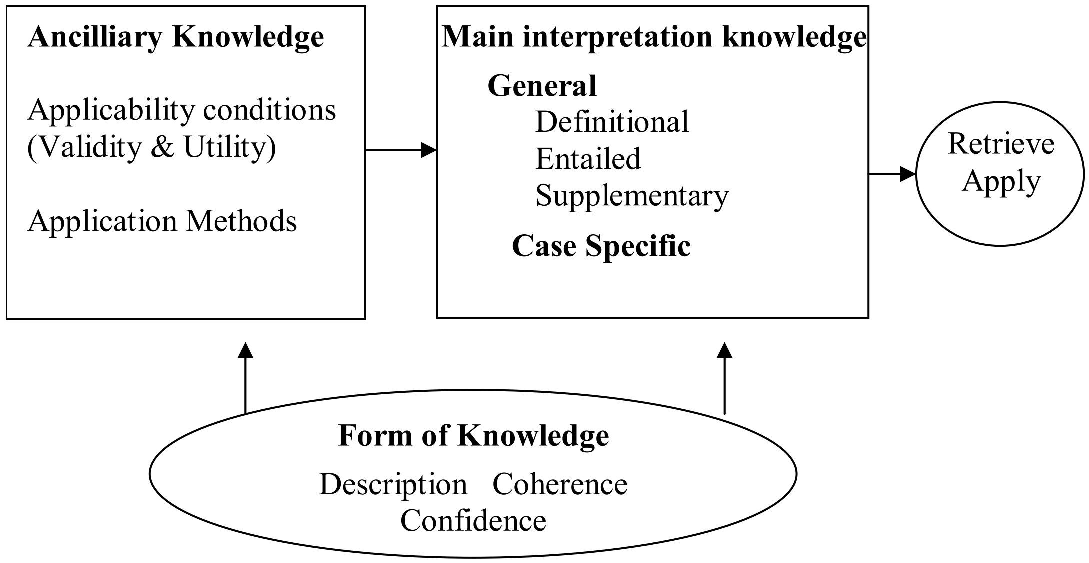

Figure **4**. Kinds of knowledge facilitating interpretation of a scientific concept.

(Reif and Allen, 1992, p. 10)

The second type of knowledge in Reif and Allen's framework is ancillary knowledge. Like main interpretation knowledge, there are two major components that make up ancillary knowledge. First, interpreting a scientific concept requires one to know when to use their knowledge; i.e. when is it *applicable* [validity] and when is it *useful* [utility]. Second, interpreting a scientific concept requires one to know *how* to use their knowledge; i.e. knowing the rules for applying one's knowledge.

The form of knowledge is the third type of knowledge that Reif and Allen proposed, which deals with the organization of the individual's knowledge. The following three components are contained in the form of knowledge:

1. **Description**. An individual's knowledge can have a very precise description or it could be described in vague terms. Either description will affect how the knowledge is applied.

2. **Coherence**. Individual knowledge elements may fit together into a coherent structure or they may be loosely connected fragments.
3. **Confidence**. Confidence in one's knowledge can affect how that knowledge is applied. Over-confidence in one's knowledge may lead to careless mistakes or "incorrect application of the knowledge," whereas under-confidence in one's knowledge may prevent the application of appropriate knowledge.

Reif and Allen attempt to categorize the different types of knowledge implicated in the understanding of a scientific concept – in this case, acceleration. However, Reif and Allen's model does not offer a constructivist's account of how students develop expertise based on their intuitive ideas. If a hypothetical group of students' were asked questions about acceleration, even before they were formally taught about the concept, they would have intuitive ideas about it. (See diSessa, 1993, for an account of students' intuitive reasoning in physics.) However, Reif and Allen's model has difficulty explaining students' intuitive ideas about scientific concepts.

Symbolic Forms

Sherin (2001) tries to gain insight as to "how students understand physics equations." He starts by collecting data on how students used equations. His data consists of videotaped sessions in which engineering students solved problems in pairs at a whiteboard. They are fairly advanced and do not make structural math errors. From this data Sherin developes a framework, called *symbolic forms*, to interpret how students understand physics equations.

Symbolic forms consist of two parts. The *symbol template* is an element of knowledge that gives structure to mathematical expressions; e.g. □ = □ or □ + □ + □...

(where the boxes can contain any type of mathematical expression). The *conceptual schema* is a simple structure associated with the symbolic form that offers a conceptualization of the knowledge contained in the mathematical expression; this part of the symbolic form is similar to diSessa's *p-prims* (diSessa, 1993).

Examples of the difference between the symbol template and conceptual schema may serve to clarify these definitions (Table 1). A student would use the symbol template, □ = □, when invoking the conceptual schema of *balancing*. For instance, the utterance, "the normal force of a table on a block is *balancing* the gravitational force of the earth on the block," corresponds with the algebraic expression $N_{T\ on\ B} = W_{E\ on\ B}$, a clear use of the symbol template □ = □. The student also utilizes the same symbol template, □ = □, in association with the conceptual schema *same amount*. For instance, the mathematical expression associated with the utterance, "the velocity of block A is the *same* as the velocity of block B," is $v_A = v_B$; this, again, is a clear use of the symbol template □ = □. To summarize, although the symbol templates were the same for both cases, the conceptual schemata associated with the symbol templates were different; therefore, different symbolic forms are implicated in the two cases.

Utterance	*Conceptual Schema*	*Symbol template*	*Mathematical expression*
"The normal force of a table on a block is balancing the gravitational force of the earth on the block"	Balancing	□ = □	$N_{T\ on\ B} = W_{E\ on\ B}$
"The velocity of block A is the same as the velocity of block B"	Same amount	□ = □	$v_A = v_B$

Table 1. Different conceptual schema associated with the same symbol template.

Sherin's framework was developed to accommodate algebraic equations for structureless quantities. That is, his framework does not discuss the nature of the structure of physics equations. For example, in physics there are algebraic equations, vector equations, operator equations, and matrix equations. It's unclear whether Sherin's framework would accommodate different types of equations—like vector equations and operator equations—or, if this framework needs to be extended in some way to handle equations that are not simply algebraic equations containing structureless quantities. It may be that different mathematical entities—like vector equations and operator equations—are conceptualized in different ways by the students. The next section discusses two different ways in which mathematical entities can be conceptualized.

Ontological Structure of Mathematical Entities

There is no explicit mention of any ontological structure in Sherin's symbolic forms, however Sfard (1991) argues there is an ontological structure to all abstract mathematical notions. According to Sfard, these abstract mathematical notions can be viewed "*structurally*—as objects, and *operationally*—as processes," and that these two views are complementary. For example, a circle can be viewed *structurally* as the locus of all points equidistance from a given point. Or, a circle can be viewed *operationally* as the figure obtained by rotating a compass about a fixed point. Sfard gives various examples of mathematical notions viewed structurally and operationally (these are summarized in Table 2).

	Operational	*Structural*
Function	Computational process or Well defined method of getting from one system to another (Skemp, 1971)	Set of ordered pairs (Bourbaki, 1934)
Symmetry	[Invariance under] transformation of a geometrical shape	Property of a geometrical shape
Natural number	0 or any number obtained from another natural number by adding one ([the result of] counting)	Property of a set or The class of all sets of the same finite cardinality
Rational number	[the result of] division of integers	Pair of integers (a member of a specially defined set of pairs)
Circle	[a curve obtained by] rotating a compass around a fixed point	The locus of all points equidistant from a given point

Table 2. Operational and structural descriptions of mathematical notions (Sfard, p5). Note: At some level these maybe formally the same, i.e. to identify a property of a shape one may have to transform the object in their mind—but may not be aware of this mental transformation. That is, the operational and structural interpretations are cognitive not formal differences.

Sfard argues that from a historical point of view a structural understanding of a mathematical notion is conceptually more difficult to achieve than an operational understanding. The transition from an operational to a structural understanding involves the following three-stage process:

1. *Interiorization*: At this stage, in order for the mathematical notion "to be considered, analyzed and compared it needs no longer to be actually performed" (p. 18).

2. *Condensation*: This phase involves a greater familiarity with the process as a whole, without the need of going through all the details of the process to understand it. That is, "it is like turning a recurrent part of a computer program into an autonomous procedure."
3. *Reification*: This stage is characterized by an ontological shift in how the mathematical notion is viewed, from process to object. This is a sudden and radical shift that offers the "ability to see something familiar in a totally new light."

Sfard summarizes the difference between an operational and structural conception of a mathematical notion along four dimensions (see Table 3): (1) the general characteristics, (2) the internal representation, (3) its place in concept development, and (4) its role in cognitive processes. Sfard concludes that the operational and structural conceptions of a mathematical entity are complementary and are both useful in problem solving.

	Operational Conception	*Structural Conception*
General Characteristics	Mathematical entity is conceived as a product of a certain process or is identified with the process itself	A mathematical entity is conceived as a static structure as if it was a real object
Internal Representation	Is supported by verbal representations	Is supported by visual imagery
Its place in concept development	Develops at the first stages of concept formation	Evolves from the operational conception
Its role in cognitive processes	Is necessary, but not sufficient, for effective problem-solving learning	Facilitates all the cognitive processes (learning, problem-solving)

Table 3. Differences between an operational and structural conception of a mathematical notion.

Although, the structural conception comes later than the operational conception of a mathematical notion in Sfard's story, she claims they are two "sides of the same coin." Both conceptions of a mathematical notion are important for understanding and for problem solving.

Discussion

The chapter looks fairly closely at empirical and theoretical attempts to understand how students use mathematics in physics. Can one distill a common thread between these approaches? There appears to be a logical flow that leads one approach into the next. The first step to understand how students use mathematics in physics is to systematically observe situations in which students use mathematics or simply document the problems students have when using mathematics in physics. This is the crux of the program in the **observational** approach. The second step in this logical flow—the **modeling** approach—attempts to model the performance or behavior of the students by creating runnable programs. The third step—the **general** and **mathematical knowledge structures** approaches—attempts to understand the internal cognitive structures that are responsible for the students' performance.

My own dissertation research is an attempt to construct a cognitive model for describing how students understand and use mathematics in the context of physics. The observational approach offers the necessary first step, lending insight into what students do with mathematics in physics, and where they have difficulties. The general knowledge structures approach has established general principles for describing the cognitive mechanisms involved in understanding scientific concepts – principles that can be applied to describing the cognitive mechanisms involved in mathematical thinking in the context of physics. I use Sherin's (1996, 2001) cognitive description of how students' *understand* physics equations (in terms of *symbolic forms* and *interpretive devices*), and extend his work to include a description of how students actually *use* mathematics in the course of solving problems in physics (in terms of *epistemic games*). Lastly, I show how one can use the cognitive framework that I have developed to understand student mathematical errors in physics.

I do not create a runnable, computational model of the mind in my own research. To date, I have only identified the cognitive structures and mechanisms to describe and analyze students' use of mathematics in physics. Future work could involve developing computer models, based on the cognitive model outlined in this current work, to model students' use of mathematics in physics. I have more to say about this in chapter 8.

Chapter 3: Data and research methodology

Introduction

Researchers have studied human problem solving in different contexts: problem solving associated with games such as chess (Newell and Simon, 1972), problem solving in mathematics (Kintsch & Greeno, 1985; Schoenfeld, 1992), and mathematical problem solving in the context of physics (Chi *et al*, 1981; Clement, 1987; Clement, 1988; Larkin, 1979; Larkin *et. al.*, 1980; Trowbridge & McDermott, 1980; Viennot & Rozier, 1991). My own research is not *per se* about mathematical problem solving; rather, it's about how students *use* mathematics in the context of physics. In particular, my research goal is to construct a theoretical framework for describing how students – correctly and incorrectly – understand and utilize mathematics in physics; *i.e.* what are the cognitive tools and processes they employ to understand mathematics in physics? From a detailed understanding of how students use and understand the mathematics I can then interpret the students' mathematical errors. Eventually, this work may lead to new instructional strategies and environments that improve students' use of mathematics in physics.

However, before I can outline a solution (a theoretical framework for analyzing and describing students use of mathematics in physics) I must clearly articulate the problem (what is it that students do with mathematics in physics). In this chapter I describe the empirical basis for this study. In the first section I discuss how math in math courses is different than math in physics courses. The second section gives some background into the existing cognitive theory of mathematical thinking and problem solving. In the third section I give a brief preview of the theoretical framework and how it incorporates existing ideas from cognitive theory. In the fourth and fifth sections I describe the reformed physics course in which the data was taken for this study and the actual data set. In the penultimate section I discuss the research methodologies employed in this study.

Math in physics courses is different than math in math courses

The first thing to note is that students use mathematics in physics courses differently than they do in mathematics course. My support for this claim rests on three non-orthogonal dimensions: (1) students have difficulty *mapping* concepts from mathematics courses to concepts in physics, (2) there are *ontological* differences between the mathematics taught in mathematics courses and that used in physics courses, and (3) students *think* there is a difference between the mathematics in math courses and the mathematics in physics courses.

Difficulties mapping concepts from mathematics courses

Mathematics is required in physics; algebra is a prerequisite for almost all physics courses (with the exception of some conceptual physics courses). In fact, many students have already taken two semesters of calculus by the time they enroll in a college level physics course. (In this study, greater than 95% of the students enrolled in the algebra-based physics course had taken two semesters of calculus.) However, research by Steinberg, Saul, Wittmann, and Redish (1996) indicates that introductory physics students don't apply what is learned in math classes to problems in physics – a reality that many physics professors have observed first hand. Students have had repeated exposure to mathematics in their previous course work, and yet they continue to perform poorly on mathematical problem solving tasks in physics.

We could simply require more mathematical preparation. However, our students are already very busy; requiring more course work does not seem like the answer. Even if we did require more course work, it is not likely to success if we do not understand *why* the students are not applying what they have learned in mathematics classes to problem solving in physics. With a detailed understanding of why students don't apply their mathematics knowledge to physics courses (or when they do) we can begin to develop instructional strategies and environments to help them apply their previous knowledge – *i.e.* we can help our students learn more efficiently. The only way this can be done is by investigating students' use of mathematics in the context of physics.

Ontological discord between math used in physics and math taught in mathematics

There is an ontological discord between the mathematics taught in introductory, college-level math courses and introductory, college-level physics courses.[5] By an ontological discord, I simply mean that the mathematical objects used in introductory, college-level physics courses are often more complex than the mathematical objects used in introductory, college-level math courses. Open a standard textbook used for an introductory college-level (or calculus) math course and you will see mostly single

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[5] The ontological discord between math courses and physics courses is not simply relegated to introductory courses. This discord has led many physics departments across the country (including the one at the University of Maryland) to adopt mathematical physics courses for advanced undergraduates (at UMd it is labeled PHYS 374 "Intermediate Theoretical Methods"). Future research could involve using the theoretical framework developed in this dissertation to study advanced physics undergraduate and graduate students' mathematical difficulties.

variable equations and relationships.[6] Redish *et al* (1996) notice that a standard introductory physics course contains many different mathematical entities that students must successfully interpret:

1. numbers: 2, e, 5/7
2. universal constants: c, G, h, k (Boltzmann)
3. experimental parameters: m, R, T, k (spring)
4. initial conditions: x_0, v_0
5. independent variables: x, y, z, t
6. dependent variables: x, y, v
7. quantities and net quantities: F_{net}, $F_{applied}$, $f_{friction}$

Students in a typical introductory mathematics course are not asked to discern a difference between a quantity and a net quantity (like F and F_{net}). This issue may seem relatively subtle or simply unimportant – one might think that students would have no trouble distinguishing force and net force. In fact, the opposite is true. Students have a great deal of trouble with the mathematical differences between force and net force, velocity and change in velocity, momentum and change in momentum, *etc*. Since these distinctions are not generally emphasized in mathematics – and since they are particularly important distinctions in physics – the obvious place to study students' difficulties with these distinctions is in the context of physics and not mathematics.

Students think there is a difference between math in physics and math in math

A third reason that indicates that the mathematics in math course is different from that in physics courses is that students think and act as if there is a difference. A conversation

[6] Multi-variable relationships are usually not taught until 3rd semester calculus.

between two students working on the Fuel Efficiency Problem (Appendix A, #8) illustrates this point. The students find the relationship between the European fuel efficiency e (measured in liters/100 km) and the American fuel efficiency f (measured in miles/gallon) to be $fe = 227$. In order to interpret what this equation means the students must translate it into "a regular math example":

> *S4: So, let's say, e is equal to x, e is the thing that you don't know, and f is equal to 2. That's, that's given in the equation. That's given in the example. So when you have a regular math example like this, a number is equal to 2 x, what do you do?*
> *S1: Divide.*
> *S4: You just divide by 2.*
> *S1: So, then that would give you e.*

The point of this quotation is to illustrate that the students do not immediately interpret an expression like $fe = 227$ as a "regular math example." It is not until e is mapped into x and f is mapped into 2 that the students are able to interpret the meaning of $fe = 227$. This is one example of a more general student belief that the mathematics in a physics course is different than the mathematics in a mathematics course. If the students perceive a difference between the mathematics in a physics course and the mathematics in a mathematics course, they may use different knowledge elements and reasoning strategies when using mathematics in these two different settings. To understand students' use of mathematics in physics courses, we should observe students using mathematics in physics courses – observations of students using mathematics in math courses is not enough.

The cognitive science of mathematical thinking and problem solving

As I mentioned earlier, I am interested in the cognitive tools involved in formal mathematical thinking in physics and understanding students' correct and incorrect use of mathematics in physics. However, before I explain the cognitive tools students bring to bear on problem solving in physics, I review some basic aspects from cognitive science.

Basics of cognitive theory

Research in cognitive science has reached a consensus on certain aspects of human memory. Most cognitive scientists divide memory into working memory and long term memory. Anderson (1983) further divides long term memory into declarative memory and production memory.

I will use the schematic diagram of the structure of memory contained in Figure 5 to discuss the different aspects of memory articulated by cognitive scientists. *Working memory*, or short-term memory, is where we encode and store input from the outside world. However, the memory space in working memory is limited and fleeting. For example, most of us cannot remember a ten digit phone number we just looked up in the phonebook between the time we read it and the time we dial unless we actively recite the digits in the phone number. Information that is elevated from working memory to permanent knowledge is stored in *declarative memory*. In contrast to working memory, *declarative memory* appears to have unlimited capacity; however, there is an issue with retrieving information stored in declarative memory. *Production memory* stores information about scripts and strategies for solving problems. If the encoding of a situation in working memory matches a strategy that exists in production memory, then the strategy is called and executed in working memory.

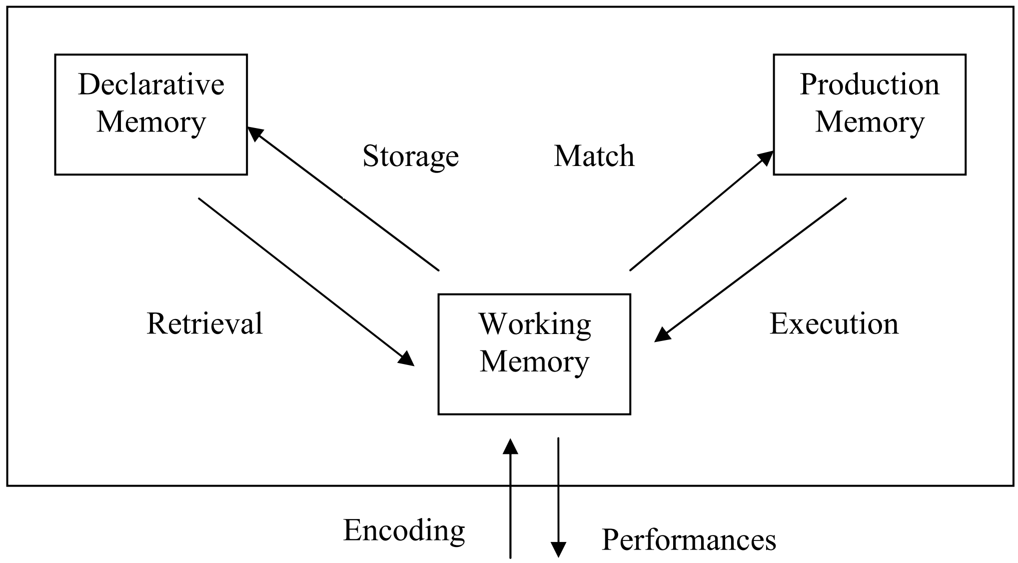

Figure 5. Schematic view of cognitive theorists' view of the ontological structure of human memory (Anderson, 1983).

Basics of cognitive theory applied to mathematical thinking and problem solving

In light of the structure of human memory, articulated by cognitive scientists, mathematics education researchers have established five generally accepted aspects that are important for any inquiry into mathematical thinking and problem solving (Schoenfeld, 1992):

1. *The knowledge base* – refers to the organization of and access to information stored in long term memory. What do students know? How is it organized? How do they access what they know?
2. *Problem solving strategies* – the strategies and heuristics that students employ during mathematical problem solving tasks. The modern discussion of problem solving strategies and heuristics starts with Pólya's book *How to Solve It* (1945).

3. *Monitoring and control* – human problem solvers awareness of and control over their progress during the problem solving process. It has been observed that human problem solvers must limit their attention to certain aspects of the problem situation – humans cannot parse and interpret the myriad of stimuli inundating them every second due to limitations of working memory. An umbrella term that encompasses monitoring and control is *metacognition*.
4. *Beliefs and affects* – an individual's beliefs about and feelings toward mathematics influences when and how she approaches and utilizes her knowledge about the subject.
5. *Practices* – the environments in which mathematics is taught that affect what and how the mathematical information is learned. It seems that some learning environments contribute to students developing a disconnect between "school mathematics" and "real mathematics." For these and other reasons, Schoenfeld (1992) espouses teaching mathematics as an "enculturation." In this view, students are immersed in the process and culture of thinking mathematically in the attempt to get the students to "[see] the world through the lens of the mathematician" (Schoenfeld, p. 341).

To address the first four aspects listed above, I distinguish three different cognitive structures implicated in mathematical thinking: *resources* (the cognitive building blocks of student thinking), *epistemic games* (the collections of reasoning strategies employed during problem solving), and *frames* and *framing* (individuals' interpretation of a situation or event based on their expectations of the situation or event). Each of these structures will be discussed in turn.

Although I believe the study of the practices of mathematical education is very important, I do not address practices in my study. However, the theoretical framework that I develop in this dissertation can be used as a guidepost for future work in developing learning environments and practices that enhance and improve students' use of mathematics in physics.

Theoretical Framework, in brief

Resources as knowledge base

The knowledge base refers to the organization of, and access to, information that is stored in long term memory. Students' mathematical knowledge consists of loosely organized bits of knowledge, or *resources*.[7] The cognitive mechanism governing access to these resources is *activation*. The following example helps illustrate the difference between resources and their activation.

A student (pseudonym Mary) working on the Conversion Problem (Appendix A, #4) explains her method to the TA:

> *Mary: I'm trying to—this one seems like it should be not too bad. This one you're driving on the New Jersey turnpike at 65 mph...so I was thinking—all right if I convert 65 mph to feet per second which is the other thing that's given in feet.*
>
> *TA: OK.*
>
> *Mary: So then I got 95 feet per second is what you're moving, so in 500 feet like how long? So, I was trying to do a proportion, but that doesn't work. I was like 95 feet per second...oh wait...yeah in 500 feet, like, x would be like the time...that doesn't, I get like this huge number and that doesn't make any sense.*

---

[7] My own view of resources in based on the work of many researchers (diSessa, 1993; Hammer, 2000; Hammer and Elby, 2002; Minstrell, 1992; Minsky, 1985; Redish, 2004; Sherin, 2001). I give a more thorough description of resources in chapter 4.

Mary realizes that a proportion could help her solve this problem, but has trouble implementing her strategy. (I believe what she writes is an expression like this: $\frac{x}{500} = \frac{95\text{ feet}}{1\text{ second}}$. When she cross-multiplies she gets a "huge number" that "doesn't make any sense.") When the TA asks the same question with slightly different numbers the student immediately answers the question:

TA: So what if I said something like… if I was traveling 4 feet per second and I moved 20 feet, how long did it take me?
Mary: Yeah, 5 seconds.

Changing the numbers makes this question immediately transparent to Mary, but why is that? The second quotation indicates that Mary has the appropriate mathematical resources to answer the original question, but she initially does not have access to those resources – that is, they are not activated. Changing the numbers in the problem activated, these resources, giving Mary access to the appropriate knowledge. (I have more to say about resources in chapter 4.)

Epistemic games as problem solving strategies

Students employ a variety of strategies during problem solving in physics. As an example consider the following group of students thinking about the equation for conservation of momentum in the Colliding Gliders Problem (Appendix A, #3):

Arielle: So then the Fnet for A, the Fnet for M. This is a big mass and this is a little mass and these are equal, so this has got to be a big, what is it, a big velocity and this has got to be a small velocity. So, p for A and p for m—the change in velocity here has got to be sort of bigger. Big velocity little mass, big mass little velocity. But these are equal.
Betty: Right.
Tommy: Right.
Arielle: So the momentums got to be the same, right?

Arielle seems to understand and draw valid conclusions from the expression for momentum. However, she later processes this same expression in a very different manner:

> *Arielle: How could they be the same? If the masses are different and the change in velocities are different the momentums can't be the same.*

This quotation indicates that the student processes the information contained in the expression for momentum in a seemingly, completely different manner than she did in first quotation – she draws the conclusion that the "momentums *can't* be the same." From the first set of quotations it's clear that the student possesses the requisite mathematical resources; however, the second set of quotations indicates that she uses a different strategy for processing and coordinating these resources to arrive at an answer. Any theoretical framework of mathematical thinking has to be able to explain how this can happen.

According to the theoretical framework I propose, the various different problem solving strategies that students employ can be understood in terms of *epistemic games*. Collins and Ferguson (1992) introduced the idea of epistemic games to categorize the different methods that experts employ during scientific inquiry. I extend the idea to novices creating new knowledge. I follow Redish (2004) in defining an epistemic game as:

> *A coherent activity that uses particular kinds of knowledge and processes associated with that knowledge to create knowledge or solve a problem.*

According to the idea of epistemic games, I interpret Arielle's two different approaches to the Colliding Gliders Problem as the activation of two different *interpretive devices* within the same epistemic game. (I introduce *interpretive devices* in chapter 4. I have more to say about epistemic games in chapters 5 and 6.)

Frames and Framing as a mechanism for monitoring and control

According to the theoretical framework I propose, students' expectations govern what they monitor and control. An example of Mary and Emma working on the Paper Towel Problem (Appendix A, #10) illustrates this point:

> *Mary: If you pull it with one hand, so all the force is concentrated in one area of the towel, so it causes it to rip. You know. But, if you pull it with both hands, it's going to be a more equal distribution, maybe. So, you could (?), that's what I was thinking. But, if your hands are wet it makes the towel soggy, which makes it weak, so it's more likely to rip.*
> *Emma: It might make it more likely to rip, but still that's better than pulling it with one hand.*
> *Mary: Yeah, if both your hands--like (?) yeah.*
> *Emma: Is that all we're supposed to do with that? I feel like (?). Like, I feel like it should have something to do with like not just force. I feel like it should have something to do with what we've learned like recently. Having to do with like water and pressure (?).*

Mary offers an explanation for this problem, but Emma is not completely satisfied with this explanation. Emma's expectations about what this problem involves are the driving factor behind her dissatisfaction: "I feel like it should have something to do with like not just force. I feel like it should have something to do with what we've learned like recently." This example typifies how students' expectations govern what they pay attention to and how they evaluate their own performance.

Frames and framing are the main theoretical constructs that I use for describing structures of expectations. The ideas of frames and framing have a long history in the sociolinguistics community. (See Tannen, 1993, for a review.) Tannen explains that an individual's framing (*i.e.* structure of expectations) helps her answer the question, "What kind of activity is this?" I discuss frames and framing in more detail in chapter 5.

The setting of this study

Student Population

The students for this study come from an introductory, algebra-based physics course at the University of Maryland, College Park. The students enrolled in this course are approximately 60% female; 70% are juniors and seniors, about 50% are biological science majors, and about 40% are pre-meds.

A particularly interesting statistic for this study is that greater than 95% of the students have had two semester of calculus, yet they are enrolled in an algebra-based introductory physics course. One possible reason that the students enroll in the algebra-based course although they have the requisite calculus background may simply be that the calculus based-course is not required for their majors. In general, these are ambitious and busy students, who are extremely concerned about getting "an A" in every course. Therefore, it is not in their best interest to take what they imagine to be more difficult courses that are not required for their majors. However, related to the students' desire to get an A, they may lack confidence in their mathematics skills, and therefore opt to take the algebra-based course because it requires less mathematical sophistication. Calculus is usually taken by these students in the first year of college, and since 70% of these students are juniors and seniors many of these students probably haven't taken any formal mathematics courses for two years (or more) by the time we see them in our physics class. This long hiatus from formal mathematics suggests that lack of confidence in their mathematics skills may be a secondary reason for enrolling in the algebra-based, introductory physics course.

Structure of the modified, introductory, algebra-based course

The students involved in this study were enrolled in the introductory, algebra-based physics course that was reformed by the Physics Education Research Group (PERG) at the University of Maryland (UMd). This course had four major structural components that were all non-traditional in some fashion: a lecture, a discussion section, and a laboratory.

The Lecture: The lecture was taught by the instructor of the course and was given in a large lecture hall consisting of about 100-160 students. The lecture met 3 times a week, with each meeting lasting 50 minutes. Two modifications to this lecture significantly increased student participation during these lectures: (1) Each student was issued a ***R****emote* ***A****nswering* ***D****evice* (RAD) that they use to answer multiple choice questions in real-time (Mazur, 1997). The instructor periodically asks a multiple-choice question during the lecture to which the students respond. The students' responses were collected electronically. A computer program would automatically display a histogram of the student responses. In this way the students and instructor could see the fractions of students choosing each answer. This immediate feedback about the students' thinking was valuable for both the instructor and the students. (2) Most weeks the students participated in an ***I****nteractive* ***L****ecture* ***D****emonstration* (ILD). During an ILD the students received a worksheet outlining specific questions that would be discussed. The instructor would lead the students through the worksheet and lead a class discussion about the issues raised in the worksheet. The students were not graded on their answers to the ILD, but they were given homework and test questions to assess their understanding of the material discussed during ILDs. (See Sokoloff and Thornton, 2004, for more on ILDs.)

The discussion and laboratory: The students also attended a discussion and laboratory section taught by a teaching assistant. These sections were limited to 20 students per section and met once a week for three hours. In the first hour the students worked in groups of four on worksheets, called *tutorials*, which lead the students through conceptual physics content. Some of these tutorials were modified versions of the University of Washington's *Tutorials in Introductory Physics* (McDermott *et al*, 2002). During the second and third hours the students worked on a physics laboratory, called *Scientific Community Labs* (Lippmann, 2003). These laboratories were modified in many ways. First, the students were not given a lab manual of lengthy instructions. Rather, the students were given a brief description of a particular setting (for example, the pendulum of a grandfather clock) and were asked a question (for example, what properties of a pendulum affect the period). The students were expected to design an experiment to answer the question. A second major modification is that these laboratories focused on the process of doing science, rather than focusing on physics content. Many of the questions focused on physics topics that the students hadn't seen yet in lecture. In this way, ideally, the students would focus on *how* to arrive at and evaluate an answer in a scientific manner, rather than focusing on the answer that is accepted by the scientific community.

Description of the Course Center

Since the discussion sections were modified, the students did not have time to discuss the problems on the homework set with a TA. To mitigate this deficiency a room was set up, called the *course center*, where students could gather to work on the homework problems together. Most of the data for this study comes from video-taped sessions of students working on homework problems in the course center. (The data will be discussed in more detail below.)

The course center was staffed during specified hours of the week by a teaching assistant or instructor. The TA or instructor was present to offer assistance but not to explicitly solve the problems for the students (as is often done in many traditional recitation sessions). The special features of this room were its architecture, the white boards, and the audio-video set up.

Architecture: Many students expect recitation sessions in which a teaching assistant stands at the front of the room and solves problems, while the students frantically copy down the solutions. The architecture of the course center was altered in the attempt to modify this expectation, by removing the 'front' of the room. All the tables were removed from the room and replaced with five long work benches. (See Figure 6 for a schematic lay out of the room.) This seating arrangement did not direct the attention of the students to any one location in the room – as is the case in all lecture halls in which the seating is arranged to face the 'front,' directing attention to the lecturer.

White boards: As a second alteration to the course center, white boards were mounted on all the walls and the students were provided with dry erase markers. The reason for this was twofold. First, the white boards facilitate group problem solving. Research on expert and novice problem solving has show that external representations are a helpful – and sometimes necessary – tool in the problem solving process (Kintsch and Greeno, 1985; Larkin, 1979). The white boards offered the students a medium to share their external representations with each other. Second, the white boards help me with my research agenda. A video-taped record of the students' writing during problem solving assisted me in my goal to understand how students use mathematics in the context of physics.

Audio-video set-up: The course center was equipped with a digital video camera and microphones. The microphones were mounted in the middle of the tables to ensure quality audio reception.

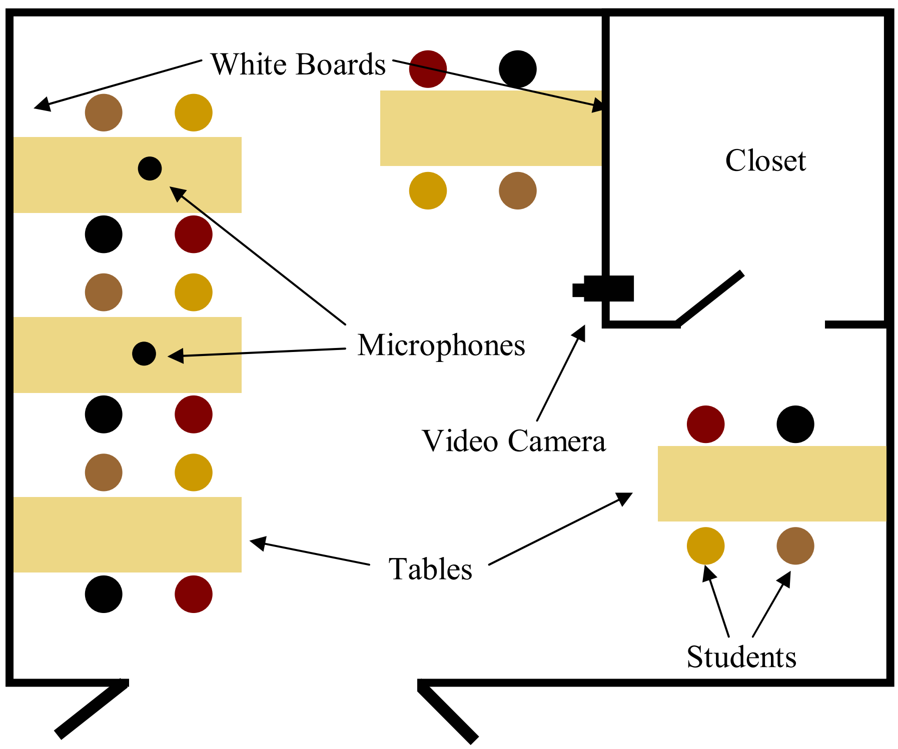

Figure 6. Top-view of the lay-out of the course center.

The video camera was mounted about seven feet above the floor on the wall of the closet across from the tables that were equipped with microphones. The elevation of the camera was ideal for three reasons. First, students and staff members walking by the closet would not be in the camera's field of vision. Second, from this vantage point, the students sitting closer to the camera did not block the students who sat closer to the wall. Third, the location of the camera allowed a clear view of what the students wrote on the whiteboards, offering me a videotaped record of the students' written work.

Types of problems

The students worked on problems assigned from their introductory, algebra-based physics course. Because these problems were assigned as homework, this gave us an authentic look at how students attempt to solve their actual homework problems – as opposed to problems artificially posed to them in an interview environment.

The problems that the students worked in the course center are also an important aspect of the modified, introductory course, because they were not standard end-of-the-chapter problems or exercises. The problems asked both quantitative and qualitative questions. The instructor expected that each problem would take the students about an hour to complete. In accordance with his expectation, the instructor only assigned about five problems each week. (Some of the problems that the students worked on appear in Appendix A. For more on these types of problems see Redish, 2003.)

Data and analysis

The data set

The majority of the data for this study comes from about 60 hours of video-taped sessions of groups of students solving homework problems in the course center. Additional video-taped data comes from a tutorial session involving a discussion about conservation of momentum. In addition to the video data, all the homework and exams that the students turned in were electronically scanned and stored on compact disc. The scanned homework data served to corroborate video data collected in the course center.

Selecting episodes

The complete data set consists of 60 hours of video-tapes of students working on homework problems. All the video-tapes were watched, from which I selected 11 to analyze in more detail. (Table 4 lists the problems and students that appear in the 11 different episodes.) In addition, I selected 18 clips of video, each about one minute long. A video-tape was selected for further detail based on two heuristic criteria:

1. <u>Rich in student thinking</u>. Since this is an ecological study, the students dictate how they choose to work on the problems. Even students solving problems in group may go several minutes without speaking to each other. So, if a video contained discussions rich in student thinking it was immediately flagged for further investigation.
2. <u>Explicit use of mathematics</u>. Since this is a study about the use of mathematics in physics, videos that contain students using and discussing mathematics were selected for transcription.

Problems	Students
Impulse discussion	Adib, Marco
Pulling Two Boxes	Alisa, Emma, Mary, Joe, Carrie
Elevator Problem	Mary, Lynn, Tony
Ladder and friction on the wall problem	Lynn, Mary, Kristy, Sabrina
Rotational kinetic energy	Lynn, Mary, Kristy, Sabrina
Units and melting ice problem	Mary, Emma, Tony, Carrie, Liz
PV=nRT	Valerie and Sarah
Conversion Problem	Mary, Emma, Kristy, Carrie
First course center hours	Mary
Joe's hours	Monica, Mike
Three-Charge Problem	Alisa, Bonnie, Darlene, Edgar
Conservation of Momentum	Arielle, Tommy, Betty, Allen

Table 4: A list of the 11 episodes and the pseudonyms of the students that appear in the video.

Transcribing the episodes

The first step in analyzing any of the video episodes was to transcribe the episode into a written form. This type of transcription is valuable for any type of fine-grained analysis of student thinking for two important reasons. First, transcribing the video episodes requires that the video be watched several times. This allows me to see subtle details in the video episode that may otherwise be missed after only one viewing. Second, the written transcribe facilitates comparison of key moments across episodes. This sort of comparison would be much more difficult with only the video.

Communication is more than a collection of words. The reader has probably heard the following statement: "It's not *what* you said. It's *how* you said it." A statement like this illustrates that there are cues other than the spoken words – like gestures, tone, and volume – that can contribute to the interpretation of verbal discourse. In an attempt to capture the richness of the communication in the course center, these additional cues (*e.g.* gestures, tone, and volume) were also recorded in the transcript, set off in brackets.

All the one-hour problem solving episodes contained in Table 4 were transcribed. In addition, 18 shorter clips (each about 1 minute long) were transcribed. In total over 11 hours of video data were transcribed in detail.

Parsing the video data

Following the transcription of the video data, the next stage of the analysis is to parse the data according to various time scales. I used the following list of time scales, adapted from Sherin (1996), to parse the video data:

1. *The thought time scale* (~ 1 second). This time scale is associated with the time it takes a student to look at an equation (or graph) and then say something about it.
2. *The problem heuristics time scale* (~ 10 minutes). As shown by Schoenfeld (1985), students (and experts) engage in different problem solving strategies, or heuristics, during the course of solving a single problem.

3. *The problem solving time scale* (~ 1 hour). Students were expected to spend about an hour on the homework problems in this study. In fact, it was often the case that students spent *at least* an hour on these problems.
4. *The learning time scale* (~ 1 year). During the course of a two semester course we would hope that some of our students would learn some physics.

Roughly speaking, the thought time scale corresponds with the activation of mathematical resources, the problem heuristic time scale corresponds with epistemic games, and the problem solving time scale corresponds with frames.

Identifying and coding Mathematical Resources

I did not develop a strict set of rules for identifying and coding mathematical resources. Instead, following diSessa (1993) and Sherin (1996), I used a list of heuristic principles.

1. *Verbal cues*. What the students say is one of the strongest pieces of evidence for identifying and coding mathematical resources. In some cases the use of specific words can be indications of particular mathematical resources. For example, phrases like "in the way" or "in the middle" can be an indication of the reasoning primitive of *blocking*.
2. *Non-verbal cues*. As mentioned above, there is more to communication than a collection of words. Non-verbal cues can contribute in coding interpretive utterances – *e.g.* gestures, volume, and the pace of the speech.
3. *External representations*. The students' use of external representations or lack thereof, can be used to identify mathematical resources. For example, reasoning primitives do not involve explicit reference to physics equations, whereas

symbolic forms do. So, one clue for distinguishing between reasoning primitives and symbolic forms is explicit reference to physics equations.

4. *Global as well as local evidence*. In addition to local evidence, global evidence can also be used to identify mathematical resources. Isolated interpretive utterances are difficult to code. Couching an individual student's isolated utterances into a larger context can facilitate coding. For example, sometimes a student will repeat a line of reasoning in a more articulate manner in a later episode. This more articulate interpretive utterance can be used to help code the earlier utterance.

Identifying and coding Epistemic Games

Similar to identifying mathematical resources, I used a list of heuristic principles to identify and code epistemic games.

1. *Types of problem solving activities.* How the students use the mathematics in the context of solving a physics problem is the main source of evidence for identifying epistemic games. In some cases, the order in which the problem solving activities occur is an indication of the game being played. For example, *Mapping Mathematics to Meaning* starts with the identification of a mathematical relationship between entities in a particular problem and then progresses to a conceptual story, whereas *Mapping Meaning to Mathematics* starts with a conceptual story that is translated into a mathematical relationship.
2. *Coherence of problem solving activities*. Students' problem solving behavior appears to consist of coherent units of activities. The coherence of the students' problem solving activities can serve to distinguish between different epistemic

games. If a particular problem solving activity always follows another, then those two activities are probably part of the same epistemic game.

3. *Types of knowledge being used*. The type of knowledge that the students use during problem solving activities can serve to distinguish between different epistemic games. In the previous section I discussed how mathematical resources are coded. The mathematical resources that are active during the different problem solving activities help in the identification and coding of epistemic games. For example, two problem solving activities may make reference to the same physics equation, but involve the activation of a different set of mathematical resources. Since the two activities involve different mathematical resources, they are coded as different epistemic games.
4. *Epistemic form.* The target structure that guides the students' inquiry (*i.e.* the epistemic form) is a major piece of evidence for identifying and coding epistemic games. In many cases, since the epistemic form is associated with a particular type of external representation, it can be used to identify an epistemic game. For example, if a student draws a free-body diagram, then this is an indication that she is playing *Pictorial Analysis*.

Identifying and coding Frames

Frames are theoretical structures that can be used to describe longer time scales than epistemic games or mathematical resources. To identify frames I use both local and global cues.

1. *Linguistic cues*. Tannen (1993) list sixteen linguistic cues that indicate expectations: omission, repetition, false starts, back tracks, hedges and other

qualifying words or expressions, negatives, contrastive connectives, modals, inexact statements, generalizations, inferences, evaluative language, interpretation, moral judgment, incorrect statements, and addition.

2. *Global as well as local evidence.* Global evidence must be used to compare interpretive utterances that occur at different times during a given episode. In addition, episodes in a particular situation need to be compared to other situations.

Checking the reliability of the coding

During weekly researcher meetings, a group of researchers (Tuminaro, Redish, and Scherr) scrutinized the transcription and coding of the episodes. During this process the transcript and coding were refined and polished. In addition, two different coders (Tuminaro and Scherr) independently analyzed a sample episode in terms of epistemic games, with an inter-rater reliability of 80%. After discussion, the two codings agreed at the 100% level.

Summary

In this chapter I describe the empirical basis for this study. First, I outline some reasons why the mathematics in math courses is different from the mathematics in physics courses. Second, since I am interested in developing a cognitive framework for analyzing and describing students' use of mathematics in physics, I review some basic ideas from cognitive science and showed how they apply to inquiries into mathematical thinking and problem solving. Third, I briefly mention how the theoretical framework I developed incorporates what has been learned from cognitive science and mathematics education. Fourth, I described the setting from which the data is taken. Finally, I describe the data and analysis used in this dissertation, including how I identify and code Mathematical Resources, Epistemic Games, and Frames.

In the next chapter I develop of the idea of resources and give examples of mathematical resources that introductory physics students' employ while using mathematics in physics.

Chapter 4: The cognitive building blocks students use to understand mathematics in physics: An introduction to *Resources*

Introduction: Describing the knowledge base

As I discussed in chapter 3, previous research established five generally accepted aspects that are important for any inquiry into mathematical thinking and problem solving: (i) knowledge base, (ii) problem solving strategies, (iii) monitoring and control, (iv) beliefs and affect, and (v) practices. The first aspect dictates that any theoretical description of students' use of mathematics in physics must begin with a model that describes the students' existing knowledge.

In this chapter I lay the groundwork for a theoretical framework to describe and analyze students' use of mathematics in physics. I begin by modeling the students' mathematical knowledge base in terms of mathematical resources. In particular, I identify four different kinds of mathematical resources: intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices (see Table 5).

- *Intuitive mathematics knowledge* – knowledge of mathematics that is learned at a very early age; examples are counting and subitizing. Subitizing is the ability that humans have to immediately differentiate sets of one, two, and three objects from each other (Fuson, 1992).

- *Reasoning primitives* – abstract knowledge elements which describe students' intuitive sense of physical mechanism. Reasoning primitives are abstractions of *phenomenological primitives* (diSessa, 1993).
- *Symbolic forms* – combination of the conceptual knowledge of reasoning primitives and syntactic knowledge of mathematical symbolism into one single knowledge element (Sherin, 1996, 2001). Symbolic forms consist of a symbol template and conceptual schema. The *symbol template* is an element of knowledge that gives structure to mathematical expressions; e.g. □ = □ or □ + □ + □... (where the boxes can contain any type of mathematical expression). The *conceptual schema* is a simple structure associated with the symbolic form that offers a conceptualization of the knowledge contained in the mathematical expression; this part of the symbolic form is similar to the reasoning primitives discussed in the previous section.
- *Interpretive devices* – interpretive strategies used to extract information from a physics equation (Sherin, 1996).

Before I describe the students' knowledge base in terms of my theoretical framework, I discuss two different paradigms for modeling student thinking: the unitary (or misconception) framework and the manifold (or resource) framework. In section 3, I describe some general characteristics of resources. In section 4, I identify and discuss *intuitive mathematics knowledge*. In section 5, I introduce p-prims and reasoning primitives. In addition, I show how the idea of reasoning primitives reduces the huge number of p-prims and how it creates knowledge elements that exist at the same level of abstraction. In section 6, I discuss symbolic forms and give some examples. In addition, I contrast the theory of symbolic forms with other theories of students' conceptualization of physics equations. In section 7, I discuss *interpretive devices*. Sherin (1996) identifies three different classes of interpretive devices – Narrative, Static, and Specific Case – from his data corpus. I identify a fourth class of interpretive devices, which I call the *intuitive class*. Interpretive devices in the intuitive class are reasoning strategies abstracted from everyday experiences that are applied to physics equations.

Mathematical Resources	
Intuitive Mathematics Knowledge	A collection of primitive cognitive capacities that are required for and involved in advanced and abstract mathematical thought.
Reasoning Primitives[8]	Abstract cognitive elements that describe students' intuitive sense of physical mechanism.
Symbolic Forms[9]	Abstract cognitive elements that describe students' intuitive understanding of physics equations.
Interpretive Devices[10]	Reasoning strategies that when activated determine how students interpret meaning in physics equations.

Table 5. List of students' knowledge base in terms of Mathematical Resources.

Unitary versus manifold models of student thinking

The notion of the students' knowledge base grew out of the idea of *constructivism*, the dominant paradigm in modern theories about student thinking and learning. The major tenet of constructivism is that students construct new knowledge from their

[8] Abstracted from diSessa's (1993) *phenomenological primitives*.

[9] From Sherin's (1996, 2001) work.

[10] Generalized from Sherin's (1996) work.

existing knowledge. They are not empty containers to be filled with vast quantities of school knowledge; they enter formal instruction with a wealth of existing knowledge and previous experiences from which they build their interpretations and understanding of concepts taught in a school setting.

Two different models of student thinking have emerged in this constructivist paradigm, which I described in chapter 2 as the *unitary* and *manifold* frameworks of student thinking. In the unitary framework – which includes *misconceptions* and *alternative theories* (Chi, 1992; Clement, 1983; McCloskey, Caramazza, and Green, 1980; McCloskey, 1983; Whitaker, 1982) – researchers view students' existing knowledge as robust, coherent cognitive structures that are resistant to formal instruction. From the unitary perspective students are not blank slates. Rather, in this view, students enter formal instruction with a wealth of knowledge about physical phenomena that is often in opposition to the generally accepted scientific explanations. Therefore, the students' existing knowledge acts as an obstacle that the physics instructor must overcome, avoid, or eliminate in order for the student to achieve expert understanding.

DiSessa (1983, 1993), Smith *et al* (1993), and later Hammer (1996) take issue with the unitary view that students' existing knowledge obstructs formal education in science. "If students construct new understanding out of their current knowledge, then there must be aspects of their current knowledge that are useful for that construction" (Hammer, p. 1319). These researchers espouse a manifold, or knowledge-in-pieces, view of student thinking based on resources that if appropriately organized could contribute to expert reasoning. Students' existing resources are not seen as an obstruction that physics instructors need to overcome during formal instruction; rather, students' existing resources can be utilized by instructors during formal physics teaching.

More recently, researchers have attempted to combine these two theoretical perspectives into one coherent framework. Scherr (2002a, 2002b, 2003) shows how some aspects of student reasoning within the context of special relativity fall within the misconceptions framework, whereas other aspects of their reasoning are better understood in terms of a knowledge-in-pieces framework. Redish (2004) proposes a *theoretical superstructure* that subsumes the unitary and knowledge-in-pieces framework in one overarching framework and argues that both unitary and knowledge-in-pieces frameworks have explanatory power in different contexts.

I follow Redish (2004) and start from the manifold perspective of student knowledge in my goal to construct a theoretical framework for student mathematical thinking in physics. The issue of whether student thinking is better modeled according to the unitary or fragmented view becomes an empirical question. Where appropriate I contrast my own framework with the unitary perspective.

According to the manifold or knowledge-in-pieces view of thinking and learning, students possess a wealth of previous knowledge and experiences that are stored in long term memory, which can be called upon, or activated, during the problem solving process (diSessa, 1993; diSessa and Sherin, 1998; Minsky, 1985; Minstrell, 1992; Redish, 2004). I use the generic term *resources* to describe all the previous knowledge and experiences that can potentially be used in understanding physical and mathematical phenomena. With this view of the mind, understanding any physical phenomenon or mathematical expression involves the activation of resources.

General discussion about resources

Resources are *cognitive structures* – units of thought or reasoning with which cognitive scientists (and education researchers) describe and understand human thinking and learning. Resources are not a physical structure within the brain; they are not neurons. A probe of a student's brain would not yield the location of any resources. However, neurons and their interactions can be used as a metaphor for understanding resources and their interactions (Redish, 2004). In this section I discuss the different activation states and associational structure of resources. In addition, I discuss whether some resources are inherently "correct" while others are inherently "incorrect."

Activation states of resources: inactive, primed, active

Resources can exist at three different levels of activation: inactive, primed, and active. Inactive resources exist in long term memory but are not cued for sense making in a given situation. For example, a situation involving energy conservation (a block sliding down a frictionless incline) may, for a physicist, immediately cue the notion of *balancing* (the potential energy of the block at the top of the incline must equal or *balance* the kinetic energy of the block at the bottom). However, the same situation may not cue *balancing* for a student. The student has this resource and uses it frequently – but it is not turned on here.

Resources can also exist in a *primed* state of activation – a sort of limbo state that is not active, but is more easily activated than a completely inactive resource. For example, if you are asked to list four vegetables and then asked to list the names of four objects beginning with the letter 'b,' you are likely to include the names of some vegetables (like 'Brussels sprouts' and 'broccoli') in your list.

When resources are active they are mapped into particular situations. Such mapped and active resources I call *facets*, in accordance with Minstrell's (1992) use of the term. When watching students, what we directly observe are facets, from which resources can be inferred.

Associational structure of resources and spreading activation

Resources exist within a loosely organized associational pattern (Sabella, 1999). Activating one leads automatically (depending on context) to activations of other associated resources. All resources are connected to other resources – the strength of the connection between resources determines the *cuing probability*, which is highly context dependent. For instance, you may have had the following experience: everyday at work, you say hi to a colleague of yours. One day you see that individual in a place other than work – perhaps the shopping mall. You have the feeling that you know him, but you can't place his name. You easily remember his name when seeing him in a familiar context (at work), but when you see him out of context (at the shopping mall) it's not so easy to remember. The situation illustrates that the associational pattern of resources is highly context dependent: in one context, "Bob" is strongly associated with his face and in another it is not.

In this chapter I focus on individual resources that are involved in mathematical thinking in physics. In the next chapter I will focus on the organizational structure of resources and how they are used in concert during activities for constructing new knowledge.

Abstract and specific resources

Resources are abstracted from everyday phenomena and exposure to mathematical formalism. They are classes of different experiences and events. For example, *more is more* may be abstracted from an array of experiences: from the experience that 'lifting more boxes requires more effort' to the experience that 'adding more logs on a campfire results in a larger flame.'

The last two examples lead to an important question about resources: are some resources 'correct' and others 'incorrect'? The answer to this question is that resources exist at a level of abstraction in which they are neither correct nor incorrect. It is not until a resource is mapped into a particular situation that the correctness of its *usage* can be determined. Asking if *more is more* is a correct resource is a meaningless question; however, asking whether *more is more* is used correctly in a particular situation is a meaningful question. Adding more logs to a campfire sometimes results in larger flames; in these cases, *more is more* results in a correct conclusion. However, if the logs are too big (or, wet) adding more logs may smother the fire, making the flames smaller; mapping *more is more* in these situations yields an incorrect statement.

Precursors to formal mathematical reasoning: *Intuitive Mathematics Knowledge*

The mathematics utilized in physics is a formal, rigorous subject matter that takes years of schooling and practice to learn; however, many of the cognitive building blocks necessary to understand this subject are present in very young children – even infants. I call these cognitive building blocks *intuitive mathematics knowledge*. Research involving human infants demonstrates their ability to differentiate sets of one, two, and three objects from each other (Fuson, 1992). This ability has been dubbed *subitizing* in the research literature, and has also been observed in various species of primates and birds. Another, more familiar, cognitive building block that is necessary to understand mathematics in physics is *counting* – a cognitive ability that should be familiar to all readers.

The concepts of subitizing and counting are particularly important for understanding students' use and understanding of mathematics in physics at the introductory college level. I examine the episode of Mary discussing her approach to the Conversion Problem, which states (Appendix A, #4):

> You are driving on the New Jersey Turnpike at 65 mi/hr. You pass a sign that says "Lane ends 500 feet." How much time do you have in order to change lanes?

Mary has difficulty, so she calls the TA over to explain her problem:

> *...all right if I convert 65 mph to feet per second, which is the other thing that's given in feet... So then I got 95 feet per second is what you're moving, so in 500 feet like how long? So, I was trying to do a proportion, but that doesn't work. I was like 95 feet per second...oh wait...yeah in 500 feet, like, x would be like the time...that doesn't—I get like this huge number and that doesn't make any sense.*

Mary correctly identifies that using a proportion could help her solve this problem, but has trouble implementing this strategy. The TA attempts to redirect Mary:

> *So what if I said something like...if you're traveling 8 feet per second and you go 16 feet, how long would that take you?*

The TA changes how Mary approaches this problem by replacing 95 feet per second and 500 feet with 8 feet per second and 16 feet, respectively. With this replacement, Mary immediately responds "2 seconds." Her immediate response is an indication that the knowledge she uses to arrive at this answer is readily available to her – suggesting she is using intuitive mathematics knowledge. In particular, she could be *counting* or *subitizing*. That is, she could be counting up the number of seconds needed to make up

16 feet. Alternatively, she could be visualizing the number of ‘8 feet per second’ blocks in ‘16 feet,’ then using her subitizing ability she arrives at the answer of 2 seconds.

The evidence in this case does not distinguish between these interpretations. However, the evidence does indicate that changing the numbers in the problem cues Mary to use a new set of resources: intuitive mathematics knowledge. In Mary’s initial approach she is attempting to use a formal, symbolic approach involving proportions. By using “easier numbers,” Mary is able to tap into intuitive knowledge that she already has to eventually construct a general relationship between distance, speed, and time – a relationship she uses to get the answer to the problem as it was originally stated.

This example episode illustrates that the use of intuitive mathematics knowledge can serve as a vehicle for students to the more sophisticated and formal mathematics used in college level physics. I do not offer an exhaustive list of intuitive mathematics knowledge. I am simply drawing attention to the fact that this aspect of students’ previous knowledge can be used by instructors during formal instruction that involves more advanced mathematics. Lakoff and Núñez (2000) offer a more extensive list of primitive cognitive capacities – like *counting*, *ordering*, and *pairing* – that are required for and involved in advanced and abstract mathematical thought. Table 6 lists some different forms of intuitive mathematics knowledge.

Intuitive Mathematics Knowledge	
<u>*Subitizing*</u>	The ability to distinguish between sets of one, two, and three objects.
<u>*Counting*</u>	The ability to enumerate a series of objects.
<u>*Pairing*</u>	The ability to group two objects for collective consideration.
<u>*Ordering*</u>	The ability to rank relative magnitudes of mathematical objects.

Table 6. List of Intuitive Mathematics Knowledge.

Students' sense of physical of mechanism: *Abstract Reasoning Primitives*

In addition to intuitive mathematics knowledge, students use a form of intuitive knowledge about physical phenomena and processes, which they have learned in their everyday life experiences, to make sense of the physical world. DiSessa (1993) proposes that students develop an intuitive sense of physical mechanism from abstractions of everyday experience. This intuitive sense of physical mechanism arises from the interaction and activation of myriad of cognitive resources that he calls *phenomenological primitives* (p-prims).

The name, *phenomenological primitives*, is used to convey several key aspects of these cognitive structures. The word "phenomenological" is used to reflect the idea that these resources are abstracted from everyday phenomena. (*Closer is stronger* could be abstracted from the phenomena that the closer one is to a fire the warmer it is.) These resources are "primitive" in the sense that they are "irreducible and undetectable" to the user – they are often used as if they were self-explanatory. (Asked why is it warmer closer to a fire, a student using *closer is stronger* may respond, "it just is.")

Because of his focus on the irreducibility of p-prims with respect to the user diSessa identifies p-prims at differing levels of abstraction: for example, *force as mover* and *abstract balancing*. *Force as mover* involves the very specific concept of force; whereas, *abstract balancing* involves the very general notion that two abstract influences can be in a state of equilibrium. Because of the specific nature of p-prims like *force as mover*, diSessa proposes that there are thousands of p-prims corresponding to the myriad of physical experiences one may have in this complex world.

To reduce the extremely large number of p-prims and propose cognitive structures that exist at the same level of abstraction, I follow Redish (2004) and abstract from p-prims the notion of intuitive pieces of knowledge called *reasoning primitives*. Reasoning primitives are abstractions of everyday experiences that involve generalizations of classes of objects and influences. In this view a p-prim like *force as mover* results from mapping an abstract reasoning primitive like *agent causes effect* into a specific situation that involves forces and motion. The specific agent, in this case, is a force and the effect it causes is movement. *Agent causes effect* could also be mapped into *force as spinner*, another p-prim identified by diSessa. This makes it clear how the notion of reasoning primitives compared to p-prims reduces the total number of resources necessary to describe students' previous knowledge about physical phenomena. In addition, *agent causes effect* and *abstract balancing* both reflect relationships between abstract influences, and therefore exist at the same level of abstraction.

Examples of Abstract Reasoning Primitives from the data

To illustrate the usefulness of reasoning primitives I discuss some of the reasoning primitives that are prevalent in my data. I do not offer an exhaustive list of reasoning primitives that students may use to describe and understand all the complex physical interactions they may encounter during formal physics instruction. Rather, I offer a few examples of reasoning primitives that commonly occur.

Abstract Reasoning Primitives	
Blocking[*]	The abstract notion that inanimate objects are not active agents in any physical scenario.
Overcoming[*]	The abstract notion that two opposing influences attempt to achieve mutually exclusive results, with one of these influences beating out the other.
Balancing[*]	The abstract notion that two opposing influences exactly cancel each other out to produce no apparent result.
More is more	The abstract notion that more of one quantity implies more of a related quantity.

Table 7. List of Abstract Reasoning Primitives identified.

Blocking

Many introductory physics students view inanimate objects (such as tables or walls) as hindrances or obstacles for more active agents (such as people or cars). The inanimate objects do not play active roles in determining the outcome of any physical situation; they are simply in the way. For example, many introductory physics students do not think that a table can exert a normal force on a book placed on top of it. Rather, these students think the book does not fall to the floor simply because the table is "in the way."

The following discussion between Alisa and Darlene illustrates the use of *blocking*. Alisa and Darlene are working on the Three Charge Problem (Appendix A, #15):

[*] Discussed below.

> *Alisa: Like—q_2 is—q_2 is pushing this way, or attracting--whichever. There's a certain force between two Q, or q_2 that's attracting.*
> *Darlene: q_3.*
> *Alisa: But at the same time you have q_1 repelling q_3.*
> *Darlene: How is it repelling when it's got this charge in the middle?*

The presence of q_2, is seen by Darlene, to hinder the affect of q_1 on q_3, since q_2 is "in the middle."

Overcoming

Many physical situations may be perceived by students as involving two opposing influences attempting to achieve mutually exclusive results. The reasoning primitive of *overcoming* may be activated if one of those influences is seen as overcoming the other. A student discussing her ideas about the Pulling Two Boxes Problem (Appendix A, #13) illustrate this:

> *Alisa: Well, if you pull with a small force it's not going to overcome the friction coefficient, necessarily. So, they won't move, so nothing will happen. And, you keep pulling then as soon as you overcome that that friction force it moves. I don't know how else to answer.*

Alisa conceptualizes friction as an influence that her pull must "overcome" in order for the book to move. Her concession that she doesn't "know how else to answer" is another indication that the knowledge she uses seems self-explanatory to her – a sign that she is using a reasoning primitive.

Balancing[11]

Balancing is often activated when it appears that two opposing influences exactly cancel each other out to produce no apparent result. It appears that Alisa, Darlene, and Betty all utilize the reasoning primitive of *balancing* in their explanation of why q_3 remains in equilibrium in the Three Charge Problem (Appendix A, #15):

Alisa: Because this is in equilibrium, there's some force...
Darlene: Pulling it that way and some force pulling ex--equally back on it.
Bonnie: Yeah.
Alisa: And, they're equal.
Bonnie: Yes.

These students state that the physical mechanism keeping the charge in equilibrium is the action of two forces pulling in opposite directions with equal magnitudes. The two influences, in this case forces, are attempting to achieve mutually exclusive goals (*i.e.* pull the charge in opposite directions), but it happens to be the case that these two influences exactly balance to yield no net result.

Resources involved in understanding physics equations: *Symbolic Forms*

In the last section we saw how students can use an intuitive sense of physical mechanism to understand various physical situations. Sherin (1996, 2001) was interested in the cognitive mechanisms and processes involved when students look at an equation and understand and interpret its meaning. He argues that students use an intuitive sense of physical mechanism in concert with knowledge of mathematical symbolism and protocols to make sense of equations in physics. In order to understand and describe how

[11] diSessa makes a distinction between *abstract balancing* and *dynamic balancing* in the following way: Abstract balancing is the tendency to believe that two influences must or should be equal; whereas, dynamic balancing occurs through the result of some accident or conspiracy. I abstract both of these ideas into one single reasoning primitive: *balancing*.

students use and understand physics equations we need two cognitive constructs: a symbol template and a conceptual schema.

The *symbol template* is an element of knowledge that gives structure to mathematical expressions; e.g. □ = □ or □ + □ + □... (where the boxes can contain any type of mathematical expression). That is, the symbol template is a general symbol pattern in which specific quantities can be mapped. The *conceptual schema* is a knowledge structure that offers a conceptualization of the knowledge contained in the mathematical expression; the conceptual schema is similar to diSessa's p-prims. A *symbolic form* is the combination of a symbol template and conceptual schema.

An example of a student deriving an equation for air drag in the Air Drag Problem (Appendix A, #1) will facilitate this discussion about symbolic forms.

Amy: So basically what you have to do-
Monica: So like when you think about it, you can think that if you increase density, the air can - that - it would have to be directly proportional, cause you increase density, the resistance with the air has to also increase.
Amy: Yeah. So...
Monica: And as you increase the radius, that also increases. So they're all directly proportional-
Amy: Right
Monica: So you multiply them-
Amy: Right, so it's all multiplied-
Monica: Instead of dividing them.

Monica has *more is more* activated when she states that "if you increase density…the resistance with the air has to also increase"; *i.e.* more density is more resistance. This conceptual idea is associated with the symbol template $\square = [...x...]$. The left side of the equation is associated with the drag force. The density appears on the right side of the equation; since it is directly proportional to the drag force it. Therefore, the drag force and density are mapped into the symbol template, $\square = [...x...]$, resulting in the specific expression, $D = [...\rho...]$. Monica goes on to identify that an increase in radius also results in an increase in air drag, which is also associated with the symbol template $\square = [...x...]$, *i.e.* $D = ...r...$. Since an increase in density and radius both result in an increase in resistance, Monica realizes that they both must appear in the numerator: "So you multiply them." The association of the conceptual schema of *more is more* with the symbol template $\square = [...x...]$ occurs often in students' interpretive utterances, and is given the name *proportionality plus* (*prop+*, for short).

Sherin identifies collections of symbolic forms, which he organizes into *clusters*. The symbolic forms within a given cluster tend to involve "entities of the same or similar ontological type. For example, [symbolic] forms in the Competing Terms Cluster are primarily concerned with influences" (Sherin, 1996, p. 75). That is, symbolic forms in the Competing Terms Cluster do not involve specific physics concepts (like force or velocity), rather they involve everyday concepts (like push or motion). Table 8 lists the different clusters and symbolic forms that Sherin identifies. I draw out examples of *balancing* and *canceling* from my data set and discuss them below.

Competing Terms Cluster		**Terms are Amounts Cluster**	
Competing Terms	□ ± □ ± □ …	*Parts-of-a-Whole*	[□ + □ + □ …]
Opposition	□ - □	*Base ± Change*	[□ ± Δ]
Balancing[*]	□ = □	*Whole – Part*	[□ - □]
Canceling[*]	□ - □ = 0	*Same Amount*	□ = □
Dependence Cluster		**Coefficient Cluster**	
Dependence	[…x…]	*Coefficient*	[x □]
No Dependence	[…]	*Scaling*	[n □]
Sole Dependence	[…x…]	**Other**	
Multiplication Cluster		*Identity*	$x = \ldots$
Intensive•Extensive	$x \times y$	*Dying Away*	$\left[e^{-x\ldots}\right]$
Extensive•Extensive	$x \times y$		
Proportionality Cluster			
Prop+[◊]	$\left[\frac{\ldots x \ldots}{\ldots}\right]$	*Ratio*	$\left[\frac{x}{y}\right]$
Prop-	$\left[\frac{\ldots}{\ldots x \ldots}\right]$	*Canceling(B)*	$\left[\frac{\ldots x \ldots}{\ldots x \ldots}\right]$

Table 8. List of symbolic forms identified by Sherin (1996, p. 75).

Examples of symbolic forms in the data

<u>Balancing □=□</u>

The symbolic form of *balancing* results from the association of the reasoning primitive of *balancing* along with the symbol template of □=□. Alisa's explanation of her solution to the Three Charge Problem (Appendix A, #15) seems to involve the

[*] Discussed below.

[◊] Discussed above.

symbolic form of *balancing*. Alisa writes the following two equations on the white board: $F_{q2 \to q3} = \frac{kQq_3}{d^2}$ and $F_{q1 \to q3} = \frac{kxQq_3}{4d^2}$. Then she proceeds to explain how she uses these two equations:

> *Alisa:* *Then, I set [the forces] equal to each other, and I crossed out like the q_2 and the k and the d squared and that gave me q equals x q over four. And, then x q equals four q, so x would have to be equal to four. That's how you know it's four q.*
> *TA:* *How did—why did you set it equal?*
> *Alisa:* *Because, they're equal [forces]. Like these two have to cancel*[12] *each other out for this to be in that equilibrium.*

Alisa mentions that she set the two forces equal, which involves the symbol template □=□. When asked why she set them equal, she replied they "have to" be "for the [system] to be in equilibrium."

The conceptual content contained in Alisa's explanation above is similar to the conceptual contain of the example for the reasoning primitive of *balancing* (discussed on p. 82). However, the above example is coded as the symbolic form of *balancing*, because Alisa makes explicit reference to an equation. The symbolic form of *balancing* and the reasoning primitive of *balancing* are different in one fundamental aspect: The symbolic form of *balancing* incorporates the symbol template, □=□. Symbolic forms have two components: a symbol template and a conceptual schema. As mentioned earlier, the conceptual schema is similar to a reasoning primitive, so it is natural that the symbolic form of *balancing* is conceptually similar to the reasoning primitive of *balancing*. But Alisa's explanation above involves explicit reference to an equation. That is, Alisa associates one side of an equation with one force and the other side with another force,

[12] Although the student uses the phrase 'cancel each other out,' it is associated with the symbol template □=□. Also, terms like 'equilibrium' and 'equal' are explicit clues that the *balancing* symbolic form is activated, rather than the *canceling* symbolic form.

which she claims should be equal because the system is in "equilibrium." The conceptual idea of balancing is associated with the symbol template, $\square = \square$. This suggests that Alisa is using the symbolic form of *balancing* in her explanation, rather than the reasoning primitive of *balancing*.

Canceling $\square - \square = 0$

The *canceling* symbolic form is viewed as a process of two influences acting toward mutually exclusive goals, yielding no resultant effect. It is different from *balancing* in that *canceling* is view as the active process of one influence negated the effect of another, and is associated with the symbol template $\square - \square = 0$. Monica uses *canceling* when explaining to Amy what it means to have the '*ma* term be negligible' in the Paramecium Problem (Appendix A, #12):

> *Amy: How can you have any – I'm just curious – how can you have any force at all if you don't have, if you don't have any ma?*
> *Monica: Well, they're just saying it's so small that when, if you bring one to the other, if you bring one of the forces to the other side it'll cancel [the other force] out.*

Monica's explanation of what it means for *ma* to be 'so small,' *i.e.* nearly zero, seems to involve a process. In this case, the process is to bring one of the forces to the other side of the equation; *i.e.* this process involves the symbol template $\square - \square$. The result of this process is that the two forces will 'cancel out,' yielding a very small *ma* term.

Comparing Symbolic Forms and the Principle-based Representation Model

A different theory describing students' understanding of physics equations was developed by Larkin (1983). According to Larkin, students' ability to understand and write physics equations involves the generation and interpretation of two different representations: the *naïve representation* and the *physical representation*. Below I contrast Larkin's approach, which I call the *principle-based representation* model, with that of symbolic forms. Although I ultimately argue in favor of symbolic forms, there are aspects of the *principle-based representation* model that are useful. In fact, I believe a complete description of expert use and understanding of physics equations involves aspects of both symbolic forms and the principle-based representation model.

Succinctly, the naïve representation is the student's mental representation of the situation in terms of objects that are familiar from everyday life. This representation involves the student's "envisionment" (Larkin, 1983) of the process in question – the ability to visualize what will happen. The processes governing this visualization are not based on any physical principles.

In contrast, the physical representation involves physical principles (like Newton's 2nd Law and conservation of energy) and entities (like forces and energies). Qualitative relationships between physical entities are developed based on physical principles. From these qualitative relationships quantitative relations can be written. So, according to Larkin, students' generation and understanding of equations are strictly guided by physics principles and stems from their physical representation of the particular situation. For these reasons I call Larkin's approach a *principle-based representation* model of student understanding of physics equations.

Sherin's description of student understanding of physics equations is fundamentally different from Larkin's: students' generation of physics equations, in Sherin's description, does not necessarily involve formal physics principles. For example, the symbolic form of *balancing* involves reasoning primitive of balancing – two mutually exclusive influences in equilibrium – and the symbol template of □=□. It can be the case that a student may identify forces as influences that balance to write an equation like $F_1 = F_2$, but the theory of symbolic forms does not require that the influences be forces. In contrast, according to Larkin, if a student writes an equation like $F_1 = F_2$, this action is necessarily guided by the student thinking about Newton's second law: $\sum_i \vec{F}_i = m\vec{a}$.

Reexamining Alisa's explanation for why she set the two forces equal in the Three Charge Problem (Appendix A, #15), she says:

> *Because they're equal [forces]. Like these two have to cancel each other out for this to be in that equilibrium.*

The interpretation of Alisa's equation, according to the principle-based representation model involves four steps. First, Alisa identifies Newton's 2nd Law as the relevant physics principle in this problem. Second, she sums up all the forces acting on the charge, q_3, and places that on the left side of the equation. Third, the acceleration of q_3 is set equal to zero, and therefore the right side of the equation is zero. Fourth, she brings one of the forces that was on the left side of the equation over to the right side, to conclude that the forces are equal.

According to the theory of symbolic forms, Alisa's explanation can be understood as the activation and use of the symbolic form of *balancing*. That is, Alisa associates two mutually exclusive influences that are in equilibrium with two sides of an equation. In this particular case, the influences that Alisa identifies are forces.

Looking back at Alisa's explanation, she does not make explicit reference to Newton's second law. The principle-based representation model requires that the generation of a physics equation be guided by physics principles (like Newton's 2nd

Law). In contrast, according to the theory of symbolic forms, the generation of a physics equation is guided by the students' intuitive sense of physical mechanism. In addition, Alisa's reason seems to be self-explanatory to her: "these two *have* to cancel each other out for this to be in that equilibrium" (emphasis added). As I mentioned earlier, reasoning primitives are used in a self-explanatory fashion (p. 78). Since Alisa's explanation does not involve explicit reference to physics principles and her reasoning seems to be self-explanatory to her, it seems that the generation of the equation is better understood in terms of symbolic forms, rather than Larkin's principle-based representation model.

Although I argue in favor of symbolic forms, there are two important aspects in the principle-based representation model that cannot be overlooked. First, students (and experts) do use formal physics principles in their discussions about and interpretations of physics equations. Aspects of students' use and understanding of physics equations must be associated with physics principles. So, symbolic forms cannot be the entire story for describing expert symbol use. Second, constructivism teaches that students have a wealth of previous experience that they bring into the physics classroom. The connection between everyday experience and physics principles is often not emphasized in models of student thinking. However, the principle-based representation model attempts to understand the mapping between everyday experience and physics principles. In the next section I will describe how students apply "everyday" reasoning strategies to extract information from physics equations.

Reasoning strategies for interpreting physics equations: *Interpretive Devices*

Symbolic forms cannot be the entire story for how students understand and interpret equations. Students (and experts) appear to have compiled strategies for extracting information from physics equations. I follow Sherin (1996, 2001) and call these compiled interpretive strategies *interpretive devices*.[13] Sherin identifies three different classes of interpretive devices – Narrative, Static, and Specific Moment – that students in his data corpus use to interpret physics equations. In addition to these three, I propose a fourth class of interpretive devices: *intuitive interpretive devices*. (Table 9 lists the different interpretive devices according to class.) The interpretive devices in the Narrative, Static, and Specific Moment classes all derive from and rely on the formal properties of equations. Therefore, I will lump all of these classes into one class, which I call *formal interpretive devices*. In contrast, intuitive interpretive devices are reasoning strategies that are abstracted from everyday reasoning and applied to physics equations.

[13] In his dissertation, Sherin uses the term *representational devices* instead of *interpretive devices*.

Narrative	**Static**
Changing Parameters[*]	*Specific Moment*
Physical Change	*Generic Moment*
Changing Situation	*Steady State*
Special Case	*Static Forces*
Restricted Value	*Conservation*
Specific Value	*Accounting*
Limiting Case	**Intuitive**[14]
Relative Values	*Feature Analysis*[*]
	Ignoring[*]

Table 9. List of interpretive devices by class

Formal versus intuitive interpretive devices

Feature Analysis

I use an example episode of four students working on the Colliding Gliders Problem (Appendix A, #3) to illustrate the difference between formal and intuitive interpretive devices. In particular, in Arielle's first attempt to solve this problem it appears that she actives the formal interpretive device of *changing parameters* to conclude the change in momenta must be the same. However, she later uses the intuitive interpretive device of *feature analysis* to conclude that the momenta are different.

The students' first attempt seems correct:

> *Arielle: So then the Fnet for A, the Fnet for M. This is a big mass and this is a little mass and [the Δt] are equal, so this has got to be a big, what is it, a big velocity and this has got to be a small velocity. So, p*

[*] Discussed below.

[14] Class of interpretive devices not identified by Sherin.

for A and p for m – the change in velocity here has got to be sort of bigger. Big velocity little mass. Big mass little velocity. But [the net forces] are equal.
Tommy: Right.
Betty: Right.
Arielle: So the momentums got to be the same, right?

It seems that Arielle is using *prop+*: the mass and the velocity are directly proportional to the net force. In addition, it appears that she is using a particular strategy for extracting meaning from this equation – in this case, the formal interpretive device called *changing parameters*. *Changing parameters* is an interpretive device in which "a quantity, usually corresponding to an individual symbol in the expression, is imagined to vary while other quantities are held fixed" (Sherin, 1996, p. 467). Arielle imagines how changing a parameter on the right side of the equation (*i.e.* mass and change in velocity) will affect quantities on the left (*i.e.* the net forces). Since glider A has a smaller mass than glider M she imagines changing the values of the change in the velocities to maintain the equality between the forces. (Figure 7 shows this reasoning schematically.)

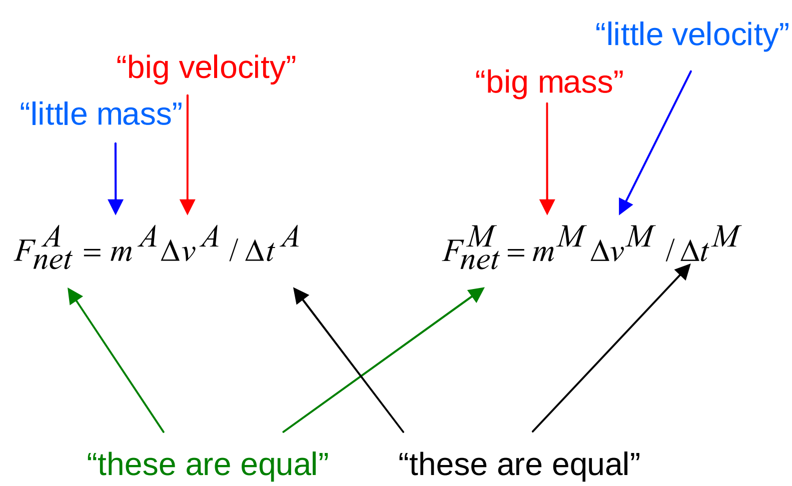

Figure 7. Schematic view of interpretation of equation using the formal interpretive device of *changing parameters*.

At first glance Arielle's reasoning appears to be very good. However, she is not satisfied with the conclusion that the momenta should be the same, so she continues the discussion:

Arielle: I don't know… No, this is not right.
Betty: It's right. But—I think it's right, but it's like--
Tommy: No, I think that's correct.
Betty: ...but see you have the subset so you have the change—the change in momentum...
Arielle: But the change in velocities are not the same though.
Betty: The change—
Tommy: Yeah, the change in velocities aren't the same. And also—
Arielle: Yeah, that's the problem, I was thinking they were the same.

The first line in this set of quotations indicates that Arielle is uncertain about the conclusion that the momenta would be the same. However, at first glance it appears that the last line in this set of quotations is in direct contradiction with what Arielle had said in the first set of quotations. In the first set of quotations she had said that the change in velocity for glider A had to be large, while the change in velocity for glider M had to be small; now, however she's stating that she was thinking the change in velocities were the same. This seems like a contradiction; however, what she says later helps clear up this apparent contradiction.

Tommy: Momentum might—could be the same. It could be.
Arielle: ...All right...they're in opposite directions.
Tommy: Wait, wait, wait. They're in opposite directions but they could be the same.
Arielle: Opposite directions—how could they be the same? If the masses are different and the change in velocities are different the momentums can't be the same.

It appears that Arielle is using a different interpretive device than she was before to conclude that the momenta cannot be the same. I suggest that she's using the intuitive interpretive device of *feature analysis* – a form of pattern recognition in which the features of a stimulus are evaluated individually. That is, she is comparing the features of the individual momenta (the features of the momenta are the masses and change in velocities). The more features that are different between the two momenta the easier it is to tell that the two momenta are different. (See Figure 8 for a schematic of her reasoning.) *Feature analysis* is an intuitive interpretive device that can be abstracted from such situations as determining if two faces are different (Figure 9).

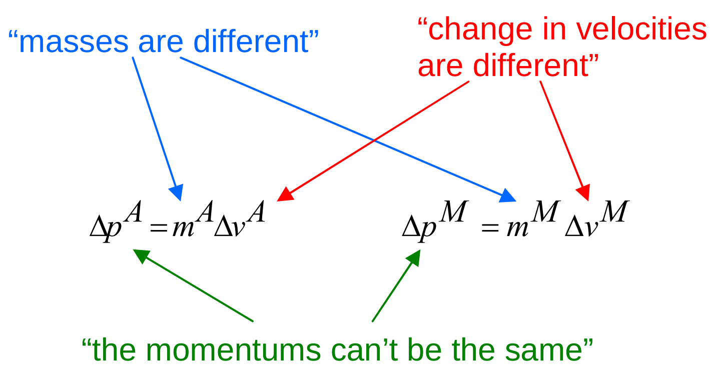

Figure 8. Schematic view of interpretation of equation using the intuitive interpretive device of *feature analysis*.

This interpretation of Arielle’s reasoning makes sense of her seemingly contradictory statement from the second set of quotations: “Yeah, that’s the problem, I was thinking they were the same.” In the first line of that set of quotations she indicates that she is uncertain about the conclusion that the change in momenta would be the same. I propose that at this time she started to search through her mind for different reasoning strategies that she could employ to corroborate the conclusion that the change in the momenta would be the same. *Feature analysis* could be a possible reasoning strategy that was tacitly cued. If one reasons with *feature analysis* the only way the momenta could be the same is if the change in velocities were also the same. This may be why she claims "I was thinking they were the same," even though in the first set of quotes she says "the change in velocity [for A] has got to be sort of bigger."

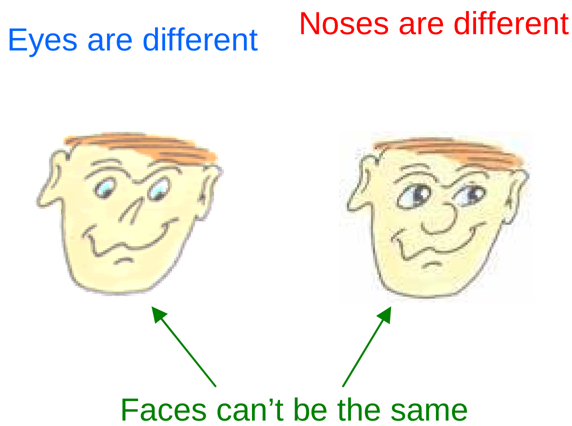

Figure 9. *Feature analysis* is possibly abstracted from situations like determining if two faces are the same. The more features that are different on the two faces the easier it is to determine that the faces are different.

Ignoring

In addition to *feature analysis*, I have identified another intuitive interpretive device that students use to extract information from physics equations: an intuitive interpretive device I call *ignoring*. When using the intuitive interpretive device of *ignoring*, students simply neglect certain terms or symbols in an equation.

It appears that Mary uses *ignoring* when working on part (b) of the Speed versus Pace Problem (Appendix A, #14). Mary is attempting to find the speed, in miles per hour, of a person walking on a treadmill at a pace of 17 minutes per mile. She explains her approach to me (I happened to be the teaching assistant in the course center at the time):

Mary: 'Cause you see if you have to end up with miles per hour, it has to be that way. That's the only way you're going to get those units on top and those on the bottom. Is by reversing [17 minutes per mile] at the beginning.

So, Mary thought she had to "flip" 17 minutes per mile, *i.e.* write $\frac{1\,\text{mile}}{17\text{minutes}}$. From Mary's "flipping" procedure she had written the following expression:

$\frac{60\,\text{minutes}}{1\,\text{hour}} \times \frac{1\,\text{mile}}{17\,\text{minutes}}$. Mary then described how she operationally interpreted this expression:

TA: How did you calculate this number 3.5? What did you put into your calculator?
Mary: 60 divided by that, ***'cause the 1s just like aren't there****. 60 divided by 17. And, then you're left over with miles per hour* (emphasis added).

From Mary's words it appears that she simply ignores the presence of the ones. She doesn't say, "60 times 1 is just 60, so *it's like* we can ignore the 1." She openly states, "the 1s just like aren't there."

Summary

In this chapter I discussed how the theoretical framework I propose describes students' knowledge base. According to my theoretical framework, students' knowledge base is described in terms of mathematical resources. I identify four different kinds of mathematical resources that contribute to students' knowledge base: intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices.

Intuitive mathematics knowledge is mathematics knowledge that is innate or learned at a very early age. This aspect of students' previous knowledge can be used by instructors to bridge the gap toward the more sophisticated and formal mathematics used in college level physics. *Reasoning primitives* are knowledge elements about physical phenomena that are abstracted from everyday experience. These reasoning primitives if correctly coordinated and organized could help lead to expert understanding. *Symbolic forms* offer a cognitive description of students' conceptual understanding of equations in physics. Lastly, there are both formal and intuitive *interpretive devices*, which are reasoning strategies that students employ to extract meaning from physics equations.

This chapter focused only on the mathematical resources to describe how students *understand* mathematics in physics. In the next chapter I focus on collections of mathematical resources to describe how students actually *use* mathematics in physics.

Chapter 5: Understanding the process of students' mathematics use in physics: An introduction to *Epistemic Games* and *Frames*

Introduction and Motivation

In the previous chapter I discussed the *ontological component* of mathematical problem solving in physics. That is, I introduced the cognitive "stuff" that can be used to describe students' mathematical thinking, which – in the theoretical framework that I propose – is made of *resources* (*e.g.* intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices). In this chapter I discuss the *process component* of mathematical problem solving in physics – *i.e.* how the students actually activate, combine, and use these resources to solve problems in physics.

Previous research tends not to focus on students' different problem solving approaches

The actual path that students follow during problem solving in physics varies from problem to problem and student to student, yet this fact is rarely addressed in two key areas of the research literature: (1) observational studies of students' problem solving, and (2) cognitive models of mathematical problem solving. Many of the observational studies of problem solving compare students to experts, not students to students – for the purpose of understanding students' problem solving (Larkin *et al*, 1980; Reif & Allen, 1992). For example, by observing students solving mathematics problems, Schoenfeld (1985) develops a representation depicting the amount of time students spend on different aspects of the problem solving process. However, he uses this representation to compare the differences between students' and experts' problem solving approaches; he does not emphasize the various different approaches that students employ.

Many of the cognitive models of problem solving and problem comprehension rely on idealizations of the problem solving process, not on the different approaches that students actually use during problem solving. As discussed in chapter 4, Larkin develops a cognitive model of physics problem solving based on the coordination of different mental representations (the *naïve representation* and the *physical representation*). In a similar vein, Nathan *et al* (1992) develops a model of algebra word-problem comprehension that is based on three components: an understanding of the problem statement, a qualitative understanding of the particular situation, and a quantitative understanding of the particular situation that "captures the algebraic problem structure." I argue later in this chapter that these models are normative, not descriptive. That is, they adequately model *ideal* student problem solving approaches, but they do not describe all the different approaches that students *actually use* during problem solving in physics.

My attempt to describe the students' different problem solving approaches

Through an observational categorization of students solving problems in physics, I identify two important aspects in their activities: (1) there seem to be collections of student activities that are associated, and (2) students' expectations about physics problems and problem solving factor into how they use mathematics in physics. To describe the associations of activities I introduce *epistemic games*.[15] To describe structures of student expectations I introduce *frames*.[16] Epistemic games and frames, taken together, help us understand the *process component* of students' mathematical thinking and problem solving in the context of physics.

In the next section I give an introduction to epistemic games. In the third section I discuss the epistemic games that account for the different problem solving approaches that appear in the data. In section four I discuss how the epistemic games I identify are different from previous attempts at understanding mathematical thinking and problem solving. The fifth section offers an introduction to frames and how they can be used to understand why students (usually tacitly) choose to play particular epistemic games. I conclude with a summary and some closing remarks.

Introduction to *Epistemic Games*

Epistemic games (or, *e-games*, for short) were introduced by Collins and Ferguson (1993) to describe expert scientists' approaches to scientific inquiry – expert scientists across all scientific disciplines. According to Collins and Ferguson, each epistemic game has an accompanying *epistemic form*. The epistemic game is the complex "set of rules and strategies the guide inquiry," whereas the epistemic form is the "target structure that guides scientific inquiry." The difference between these two concepts is best articulated by Collins and Ferguson:

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[15] Adapted from Collins and Ferguson (1993).

[16] Adapted from a term proposed by psychoanalyst Gregory Bateson (1972) and anthropologist Irving Goffman (1997), and used by socio-linguist Deborah Tannen (1993).

> The difference between forms and games is like the difference between the squares that are filled out in tic-tac-toe and the game itself. The game consists of rules, strategies, and different moves that players master over a period of time. The squares form a target structure that is filled out as any particular game is played (Collins and Ferguson, 1993, p. 25).

Epistemic games were introduced by Collins and Ferguson to describe expert scientific inquiry across all scientific disciplines. The students in introductory physics courses are far from experts. Using scientists' approaches to inquiry as a norm by which to describe students' inquiry would therefore be problematic. For this reason, I generalize epistemic games to be descriptive rather than normative. I use the main characteristics that Collins and Ferguson attribute to epistemic games to identify a set of games that introductory, algebra-based physics students play while solving problems in physics. The epistemic games that I identify can be used to describe and analyze introductory students' use of mathematics in physics.

The definition I use for an *epistemic game* comes from Redish (2004):

> *A coherent activity that uses particular kinds of knowledge and processes associated with that knowledge to create knowledge or solve a problem.*

The name 'epistemic game' is used to capture the most important aspects of the pattern of activities that it describes. The activities are 'epistemic' in the sense that students engage in these activities as a means to construct new knowledge. I use the word 'games' in a very real sense; a particular game (like checkers or chess) is a coherent activity that has an ontology (players, pieces, and a playing board) and structure (a beginning and an end, rules), which makes it distinguishable from other activities or games. In the same way, a particular epistemic game has an ontology and structure that makes it distinguishable from other activities or epistemic games.

In the next two subsections, in order to describe the ontology and structure of epistemic games, I use the simplest epistemic game identified by Collins and Ferguson: *list making*. (Table 10 summarizes the ontological and structural components of all epistemic games.) Every list is implicitly an answer to a question. Some examples are: "What do I need from the grocery store?"; "What are the fundamental forces of nature?"; and, "What are the constituents of all matter?"

Ontology of Epistemic Games

Epistemic games have two ontological components: the knowledge base and the epistemic form. An epistemic game is *not* simply a cognitive structure; it's a pattern of activities that can be *associated* with a collection of resources. The collection of resources that an individual draws on while playing a particular epistemic game constitutes the *knowledge base*. To answer a question like, "What are the fundamental forces of nature?" one needs to have some requisite knowledge to list the forces. The knowledge base for the epistemic games I identify below consists of all the resources that I introduced in chapter 4: intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices.

The *epistemic form* is a target structure that helps guide the inquiry during an epistemic game. For example, the epistemic form in the list making game is the list itself. The list is an external representation that cues particular resources and guides the progression of the inquiry.

Structure of Epistemic Games

Epistemic games have two structural components: the entry and ending condition, and the moves. The *entry* and *ending conditions* specify the beginning and the ending of the game. As I mentioned above, one may enter into the list making game as a means to answer a question. When solving physics problems, students' expectations about physics problems determine the entry and ending conditions. These expectations can depend on real-time categorizations of physics problems and/or on preconceived notions about the nature of problem solving in physics. Research by Hinsley and Hayes (1977) indicates that students can quickly categorize large classes of physics problems very shortly after reading the statement of the problem – often these categorizations can be made after reading the first sentence! The students' ability to very quickly categorize physics problems may stem from their expectations about physics problem solving, and vice versa. These expectations, and categorizations, of physics problems affect which epistemic game the students (tacitly) choose to play. In contrast, students' preconceived epistemological beliefs about problem solving in physics can affect their expectations. If students believe that problem solving in physics involves rote memorization of physics equations, then that can affect what strategy they employ (*i.e.* epistemic game they play) and what they believe an answer in physics is (*i.e.* how they know they are done playing a particular game). I say more about students' expectations and epistemic games in the discussion about the interplay between epistemic games and frames later in this chapter.

The second structural component of epistemic games is the moves. The moves are the steps that occur in an epistemic game. In the list making game the moves may be to add a new item, combine two (or more) items, substitute an item, split an item, and remove an item.

Ontological Components		**Structural Components**	
Knowledge Base	Cognitive resources associated with the game.	*Entry and ending conditions*	Conditions for when to begin and end playing a particular game.
Epistemic Form	Target structure that guides inquiry.	*Moves*	Activities that occur during the course of an e-game.

Table 10. The ontological and structural components of all epistemic games.

Epistemic games students play in introductory, algebra-based physics

In this section I discuss all the epistemic games that are necessary to account for the different problem solving strategies seen in my data. From an observational categorization, I identify six different epistemic games that students play while using mathematics in the context of problem solving in physics (see Table 11). These six games span the different problem solving approaches seen within the data. I do not claim that this list spans all the possible problem solving approaches that could be employed during problem solving in physics. If I had examined a different population of students or a different domain, the list of epistemic games would most certainly be different. However, the list contained below is sufficient for describing the problem solving approaches that introductory, algebra-based physics students employ in my data set. Each of these games is described in more detail below; however, I do not discuss the entry conditions for each game in the next subsections. This subject is discussed in the section on *frames* and *framing*. For each epistemic game I give a brief introduction, discuss its ontology and structure, and then I give an example of students playing that game.

List of epistemic games
Mapping Meaning to Mathematics
Mapping Mathematics to Meaning
Physical Mechanism Game
Pictorial Analysis
Recursive Plug-and-Chug
Transliteration to Mathematics

Table 11. List of epistemic games identified in my data set.

Mapping Meaning to Mathematics

The most intellectually complex epistemic game that I identify is *Mapping Meaning to Mathematics*. The name is derived from the structural nature of this game. Students begin from a conceptual understanding of the physical situation described in the problem statement, and then progress to a quantitative solution. There are five basic moves in *Mapping Meaning to Mathematics* (see Figure 10): (1) develop a story about the physical situation, (2) translate quantities in the physical story to mathematical entities, (3) relate the mathematical entities in accordance with the physical story, (4) manipulate symbols, and (5) evaluate solution.

The knowledge base for this game (as with all the games I identify) comes from the set of physics and mathematics resources; however, in general, different resources are activated during the different moves of the game. During the development of the conceptual story (move 1), reasoning primitives are most often activated. That is, students often rely on their own conceptual understanding to generate this story – not on fundamental physics principles. Translating the conceptual story into mathematical entities (move 2) is one of the most difficult moves in the entire game for most students. Intuitive mathematics knowledge, symbolic forms, and interpretive devices are usually activated during this move. Relating the mathematical entities to the physical story (move 3), again is difficult for students, and relies on intuitive mathematics knowledge, symbolic forms, and interpretive devices. Once the physics equations are written, the symbolic manipulations (move 4) usually goes by without a hitch; most introductory physics students have had ample practice manipulating symbols. The evaluation of the story (move 5) can occur in many different ways: checking the solution with a worked example (or solution in the back of the book), checking their quantitative answer with their conceptual story, or checking their solution against an iconic example.

The epistemic form for *Mapping Meaning to Mathematics* is the collection of mathematical expressions that the students generate during moves (2) and (3). These expressions lead the direction of the inquiry.

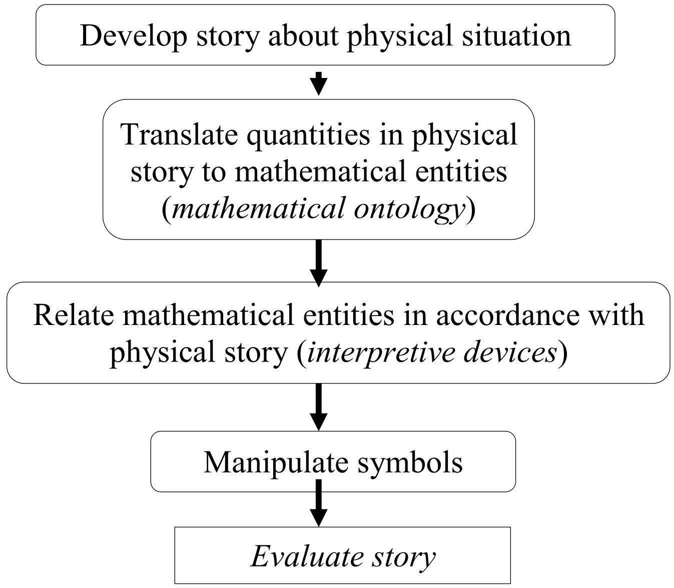

Figure 10. Schematic diagram of students' moves with *Mapping Meaning to Mathematics*.

An example of Mapping Meaning to Mathematics: Three Charge Problem

An example of a student playing *Mapping Meaning to Mathematics* comes from the Three Charge Problem (Appendix A, #15). Alisa summarizes her solution to this problem as Bonnie and Darlene listen. In move (1), Alisa develops a conceptual story:

> *Alisa: All right, so because [q_3] isn't moving the two forces that are acting on it are equal. The push and the pull.*

Alisa's story for why q_3 isn't moving seems to rely on the reasoning primitive of *balancing*. She identifies two influences ("the push and the pull"), which she correctly classifies as forces that are exactly "equal."

In move (2), Alisa translates the influences in the conceptual story into mathematical entities:

> *So, the F--I don't know if this is the right F symbol—but, the F q2 on q3 is equal to this. And, then the F q1 on q3 is equal to this, because the*

distance is twice as much, so it would be four d squared instead of d squared.

That is, Alisa uses the identity form, □ = …, along with Coulomb's Law to write the equations $F_{q_1 \to q_3} = \frac{kxQq_3}{4d^2}$ and $F_{q_2 \to q_3} = \frac{kQq_3}{d^2}$. That is, she *identifies* the forces on the left side of the equation with the appropriate arrangement of charges and distance according to Coulomb's Law. She continues to explain why she wrote the equations the way she did. She appears to use the symbolic form of *scaling*, *x*□,

> *Alisa: And, then I used x q like or you can even do—yeah—x q for the charge on q_1, because we know in some way it's going to be related to q like the big q we just got to find the factor that relates to that.*

In move (3), Alisa relates the mathematical entities she derived in (2) with the conceptual story she developed in (1):

> *Then, I set them equal to each other…*

In move (4), she manipulates her equation to arrive at a solution.

> *…and I crossed out like the q_2 and the k and the d squared and that gave me q equals x q over four. And, then x q equals four q, so x would have to be equal to four. That's how you it's four q.*

In move (5), Bonnie and Darlene critique Alisa's approach; however, Alisa's final comment makes it fairly clear that Alisa is confident in her conclusion.

> *Bonnie: Well, shouldn't it be--well equal and opposite, but...*
> *Alisa: Yeah, you could stick the negative.*
> *Bonnie: Yeah.*
> *Darlene: I didn't use Coulomb's equation, I just--but it was similar to that.*
> *Bonnie: That's a good way of proving it.*
> *Darlene: Uh-huh.*
> *Bonnie: Good explanation.*
> *Alisa: Can I have my A now?*

Figure 11 is a schematic diagram that displays how Alisa's activities match with the moves in *Mapping Meaning to Mathematics*.

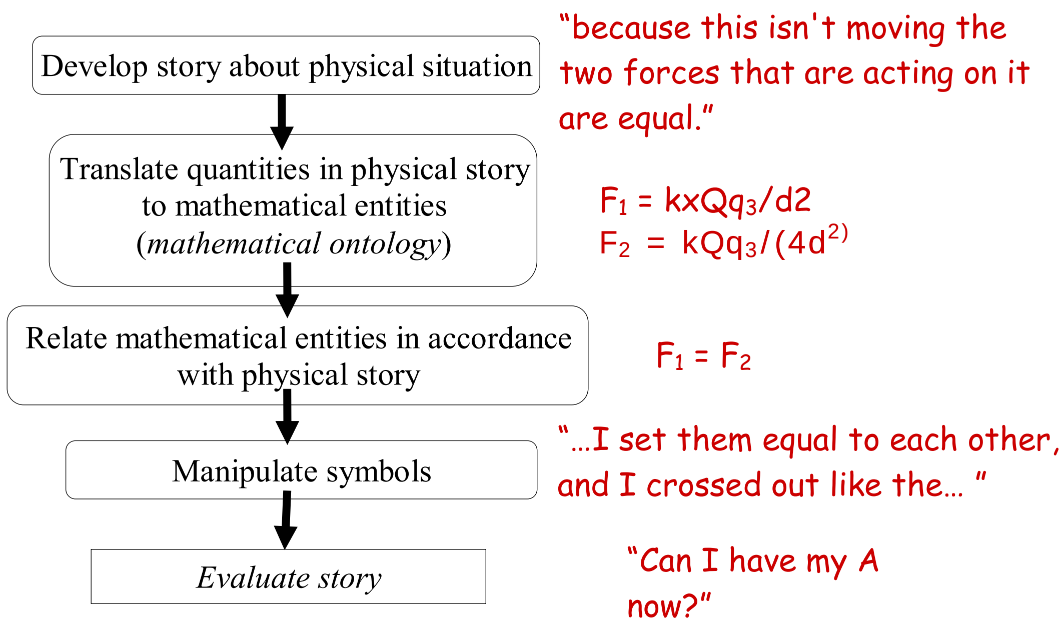

Figure 11. Schematic map of Alisa's moves within the *Mapping Meaning to Mathematics* epistemic game.

An example of Mapping Meaning to Mathematics: Melting Ice Problem

The previous example illustrates Alisa playing *Mapping Meaning to Mathematics* nearly flawlessly. However, as I mentioned above, move (2) in this game presents the greatest difficulty for most introductory physics students. Melissa's approach to the Melting Ice Problem (Appendix A, #10) illustrates this point. Melissa entered the course center because she was having trouble with this problem. The explanation of her approach is consistent with the *Mapping Meaning to Mathematics* epistemic game (see Figure 12). First, she develops a conceptual story.

> *Melissa: I kind of look at it differently than the way [the professor] did it [in class]. I calculate the—I separated the components. I put--I was thinking of it as where you put the ice in the cup and then you just pour the hot water in, and then finding the temperature of that. And*

then take that water after the ice melted and combine it with the 100 grams of water and then find what the temperature is from that.
Tuminaro: I am not sure I follow everything there.
Melissa: So basically, instead of putting the—cause at that time he said the ice and the water are together and their both at zero degrees. But, I separated that I put just the ice by itself. And, add the hot water poured into it to melt it first. And, then find out the temperature that it was after equiltherm—of the thermal equilibrium and then pour that water into the other water. But then it was a totally different answer from what he did in class.

Melissa's conceptual story involves two steps:

1. She pours the hot water (100^{o} C) onto the ice cube "to melt it first," and finds the temperature of that mixture.
2. Then, she combines that mixture with the 100 grams of water that is at 0^{o} C.

Melissa combines the second and third move of *Mapping Meaning to Mathematics* in one statement. She translates influences in her conceptual story into mathematical entities by writing the equation $mL_F = mc\Delta T$ on the whiteboard. Her explanation for why she wrote this equation follows:

Melissa: It's M L F, that heat of fusion to melt the ice—the heat gained by the ice. And then I took M C delta T was the—which is the energy that is lost by the hot water.

In move (4), she then plugs the numbers given in the problem into her equation: using 25 grams for the mass of ice, 50 grams for the mass of hot water, and "…the initial temperature is 100 degrees." Finally, she calculates the final temperature of the mixture (or, at least that's what she thinks she is calculating):

And then I found out what T F was, the final temperature. Knowing that, um, T, the initial temperature is 100 degrees. And I got nine point nine.

The evaluation in move (5) occurs by her checking her answer against the solution, and realizing her answer is different.

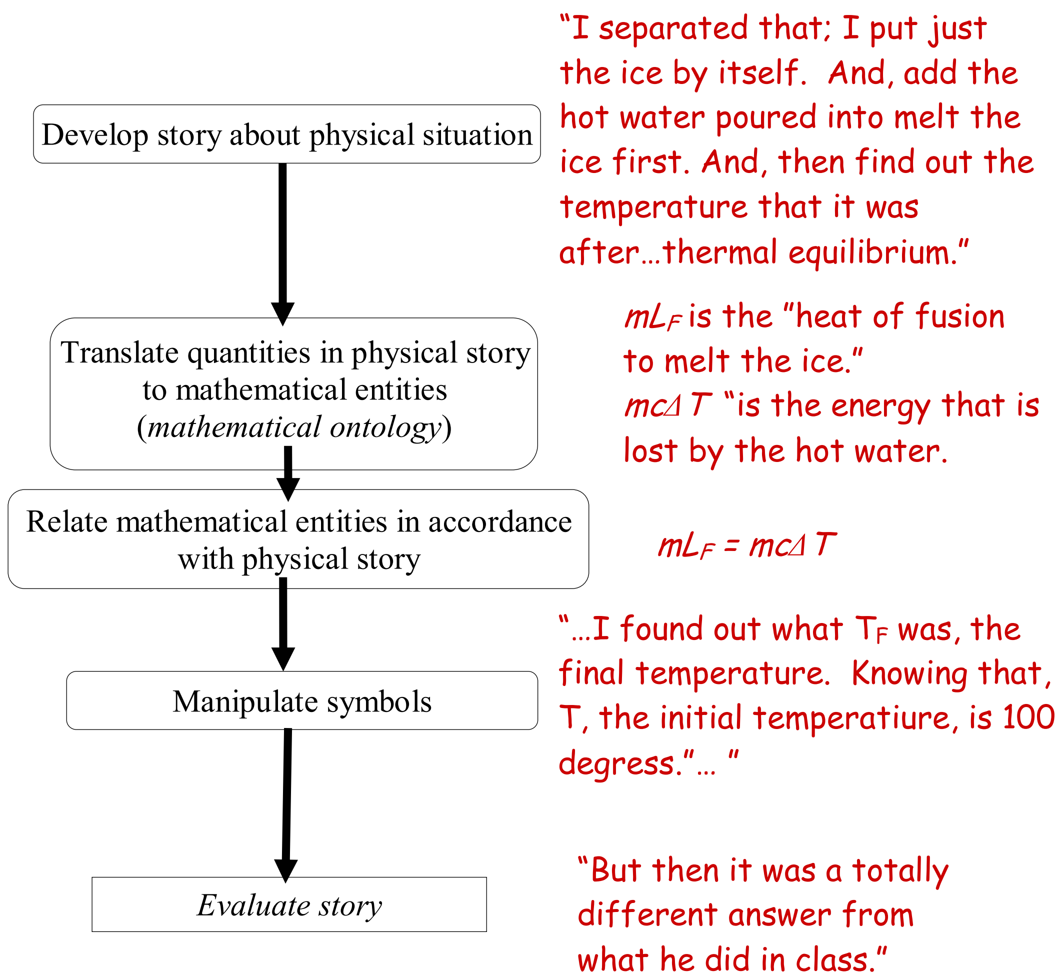

Figure 12. Schematic map of Melissa's moves within the *Mapping Meaning to Mathematics* epistemic game.

What went wrong with her approach? As I mentioned earlier, many students have difficulties with the second and third moves in the *Mapping Meaning to Mathematics* e-game; Melissa also has difficulties with moves (2) and (3).

Melissa makes a few minor oversights in move (2) – she does not interpret the mathematical expressions in her equation with the appropriate measure of precision. The term mL_F, which appears on the left side of Melissa's equation, is identified as the "heat gained by the ice"; but, more exactly, it is the amount of heat needed to melt the entire 25 grams of ice. Mathematically, the amount of heat needed is

$$mL_F = (25 \text{ grams})\left(80 \text{ calories}/_{\text{gram}}\right) = 2000 \text{ calories}.$$

Her interpretation of the right side of the equation also lacks the appropriate amount of clarity. She states it is the "energy lost by the hot water." However, more exactly, each gram of water that contributes to the melting of the ice will *necessarily* lose 100 calories. That is, each drop of hot water that contributes to melting the ice will go from a temperature of 100 ^{o}C to 0 ^{o}C, which is written symbolically as

$mc\Delta T = (1 \text{ gram})\left(1 \text{ cal}/_{\text{g}\,^{o}\text{C}}\right)\left(100\,^{o}\text{C}\right) = 100 \text{ calories}$. Therefore, the maximum amount of heat that the hot water can provide to the ice is 50 times that (5000 calories), since there are 50 grams of hot water.

Melissa's inappropriate interpretations from move (2) get her into trouble in move (3) of *Mapping Meaning to Mathematics*. Melissa simply equates mL_F to $mc\Delta T$. However, as I showed in the previous two paragraphs the maximum amount of heat needed to melt the ice is 2000 calories, whereas the maximum amount of heat that all the hot water can provide is 5000 calories. Therefore, all the hot water is not needed to melt the ice – only 20 of the 50 grams are needed. Melissa's equation and subsequent interpretations do not capture that fact.

This example is one indication that students' mathematical difficulties may not be with the mathematics; rather, it lies in translation of their conceptual understanding into physics equations and expressions. I discuss this point in more detail in chapter 7.

Mapping Mathematics to Meaning

The ontological components of *Mapping Mathematics to Meaning* are exactly the same as those in *Mapping Meaning to Mathematics*. Both games involve the same kind of knowledge base (resources) and epistemic form (physics equation). However, the particular resources and physics equation that are used in each game will vary from problem to problem.

In addition, the structural components of the two games are different. In *Mapping Mathematics to Meaning* students begin with a physics equation, and then develop a conceptual story; whereas, in the *Mapping Meaning to Mathematics* students begin with a conceptual story, which is then translated into mathematical expressions. The structural differences between these two games make them distinguishable from each other.

There are four moves in this game (see Figure 13): (1) identify target "concept(s)," (2) find an equation relating target to other "concepts," (3) tell a story using this relationship between "concepts," and (4) evaluate story.

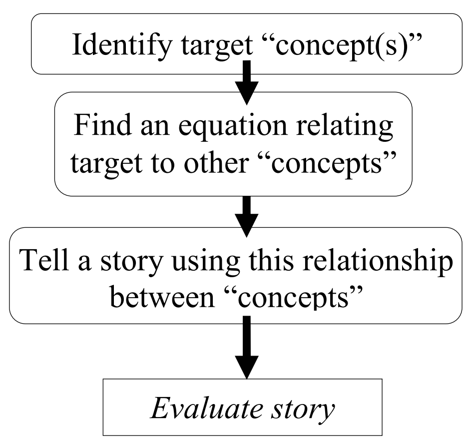

Figure 13. Schematic diagram of students' moves with *Mapping Mathematics to Meaning*.

In the remainder of this subsection on *Mapping Mathematics to Meaning* I give two examples of students playing this epistemic game. In the first example I discuss how Monica's solution to the Jogger Problem (Appendix A, #9) fits the moves of *Mapping Mathematics to Meaning*. In the second example I discuss how the resources that Arielle has active while playing *Mapping Mathematics to Meaning* leads her to two different solutions to the Colliding Gliders Problem (Appendix A, #3).

The Jogger Problem

Monica's approach to the Jogger Problem (Appendix A, #9) follows the moves in *Mapping Mathematics to Meaning* (see Figure 14). She discusses this problem with one of her classmates, named Mike. In move (1), Monica identifies the target "concept":

> *Monica: So her average velocity going from A to C is...*

Next, consistent with move (2), she finds and equation (in this case she finds two equations) relating the target concept to other concepts:

> *...both of these equations are going to figure out average velocity. Change in distance over change in time, or velocity final plus velocity initial divided by two, right?*

In move (3), she tells a story using the relationship between the "concepts":

> *Monica: They're both - so, here... you could do it either way, but, I think if you do it this way, like, if you look at her final velocity at C, we said was down four point seven.*
> *Mike: Oh, so that's negative?*
> *Monica: And, yeah, so it doesn't really matter. So we can say that's negative. And this one's up four point seven, divided by-*
> *Mike: It's going to be z-*
> *Monica: Two.*
> *Mike: It's going to be zero.*
> *Monica: It's going to be zero. So, average velocity, I think, is zero. Because the directions cancel each other out.*

Using the relationship in the equation $\langle v \rangle = \frac{v_f + v_i}{2}$, Monica concludes that the average velocity will be zero. (In this case, choosing the equation $\langle v \rangle = \frac{v_f + v_i}{2}$ is incorrect because the acceleration is not constant; however, Monica's problem solving approach is still consistent with *Mapping Mathematics to Meaning*. It just happens to be the case that this particular instantiation of the epistemic game leads to an erroneous solution. I discuss the association between students' mathematical errors and epistemic games in more detail in chapter 7.)

Lastly, in move (4), she evaluates[17] her story:

> *Monica: Velocity has to take into account direction. So speed, of course, is never changing.*
> *Mike: Ohhhh.*
> *Monica: Speed is immutable by direction.*

Her evaluation in this case is an acknowledgment that this conclusion seems to contradict her understanding of the notion of *speed*. She knows the jogger never stops (*i.e.* "Speed is immutable by direction."), so she simply justifies this apparent contradiction as a consequence of the physics concept of *velocity*.

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[17] I mean "evaluate" to be an umbrella term. This evaluation can be carried out using several different methods. See discussion of evaluation in *Mapping Meaning to Mathematics* on page 107 for more.

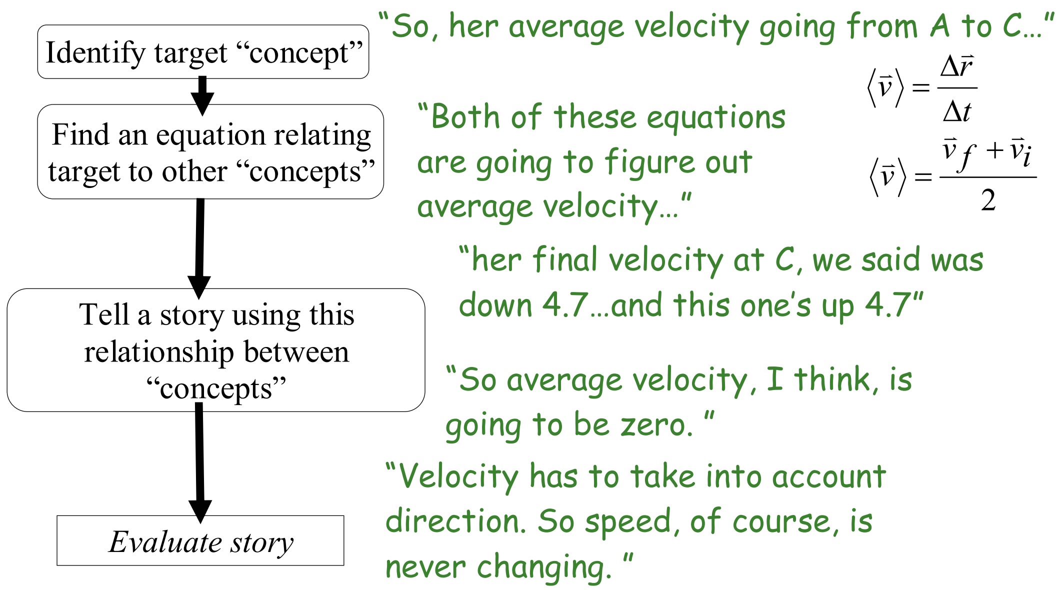

Figure 14. Schematic diagram of Monica's moves in the *Mapping Mathematics to Meaning* epistemic game.

The moves in *Mapping Mathematics to Meaning* make it distinguishable from *Mapping Meaning to Mathematics*. However, the moves only specify the general progression of activities; the moves do not specify exactly what the students do. The particular resources that are activated during the game will dramatically affect the outcome of that game. Arielle's work on the Colliding Gliders Problem (Appendix A, #3) is an extreme example of this fact. Arielle plays *Mapping Mathematics to Meaning* in two different ways for one single problem. In her first attempt she arrives at the correct answer, whereas in her second attempt she does not.

The Colliding Gliders Problem

In the statement of the Colliding Gliders Problem the target "concepts" (force and momentum) and equations ($F_{net} = m\Delta v/\Delta t$ and $\Delta p = m\Delta v$) are given. That is, the first two moves in *Mapping Mathematics to Meaning* are already completed for these students. Arielle jumps into this game at move (3), and develops a story using the relationship between the "concepts":

> *Arielle: So then the Fnet for A, the Fnet for M. This is a big mass and this is a little mass and these are equal, so this has got to be a big, what is it, a big velocity and this has got to be a small velocity. So, p for A and p for m – the change in velocity here has got to be sort of bigger. Big velocity little mass. Big mass little velocity. But these are equal.*
> *Tommy: Right.*
> *Betty: Right.*
> *Arielle: So the momentums got to be the same, right?*
> *Betty: Yeah, but the change in momentum from glider A—*
> *Arielle: I don't know. No, this is not right.*

Move (4): She doesn't articulate her evaluation of her story; however her comments indicate that she at least internally evaluates her story: "I don't know. No, this is not right."

Arielle "executes" all the moves in the *Mapping Mathematics to Meaning* game. What resources does Arielle draw on to generate this story? In chapter 4 I introduced the idea of *interpretive devices* – reasoning strategies for extracting information from equations. In the above example, Arielle draws on the interpretive device of *changing parameters*[18] to develop her story from the equation. That is, she images what will happened to the left side of the equation (the force), if a parameter on the right is changed (the mass). (She also images what will happen to the force if the change in velocity is varied.)

[18] See the end of chapter 4 (p. 93) for more on *changing parameters*.

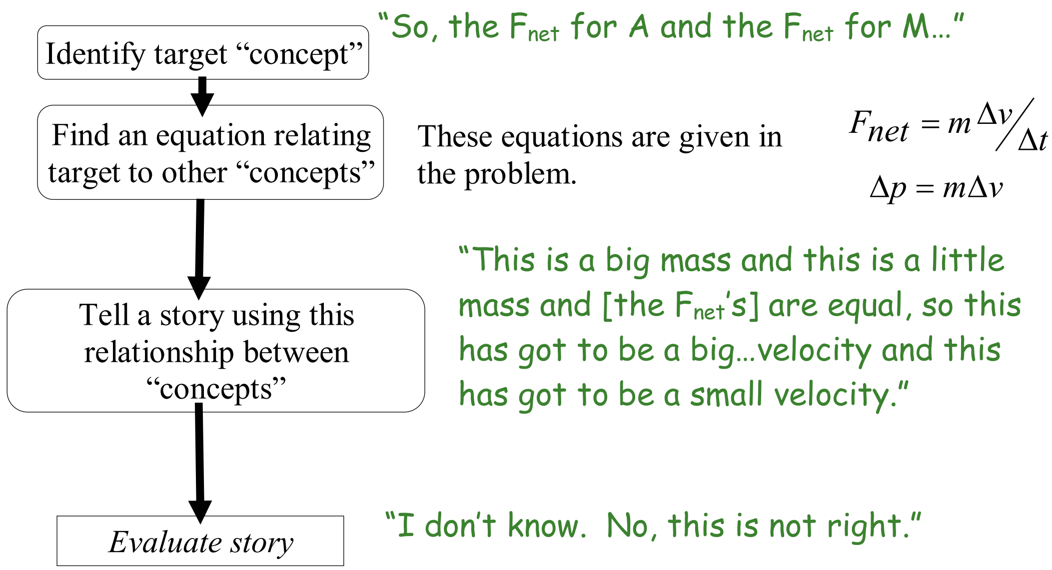

Figure 15. Schematic diagram of Arielle's moves in the *Mapping Mathematics to Meaning* epistemic game when using the interpretive device of *changing parameters*.

Later on in the same discussion, Arielle again plays *Mapping Mathematics to Meaning* (beginning at move (3)), but she uses a different interpretive device (*feature analysis*[19]) to develop a different story:

> *Arielle: Opposite directions—how could [the momenta] be the same? If the masses are different and the change in velocities are different the momentums can't be the same.*

In this instance, because *feature analysis* is activated, she develops a different story, even though she is still playing *Mapping Meaning to Mathematics*. That is, she realizes that the two features of the momenta, namely the mass and the change in velocity, are both different; therefore, she concludes the momenta must be different, as well.

To sum up, in both cases Arielle is making the same moves (she identifies the target concept, finds an equation, develops a story, and evaluates the story) – *i.e.* she is playing

[19] See the end of chapter 4 (p. 92) for a more complete discussion of *feature analysis*.

the same epistemic game. However, in the first case, it appears that *changing parameters* is the resource that is activated during the development of the story; whereas *feature analysis* appears to be the resource activated in the development of the second story. So, the moves of the epistemic game describe the general progression of Arielle's problem solving strategy, but the particular resources that are active during the epistemic game dictate how she actually plays the game. Said another way, the structure of *Mapping Mathematics to Meaning* is always the same (it always involves the same moves); however, the ontology (the resources that are active) may vary from problem to problem.

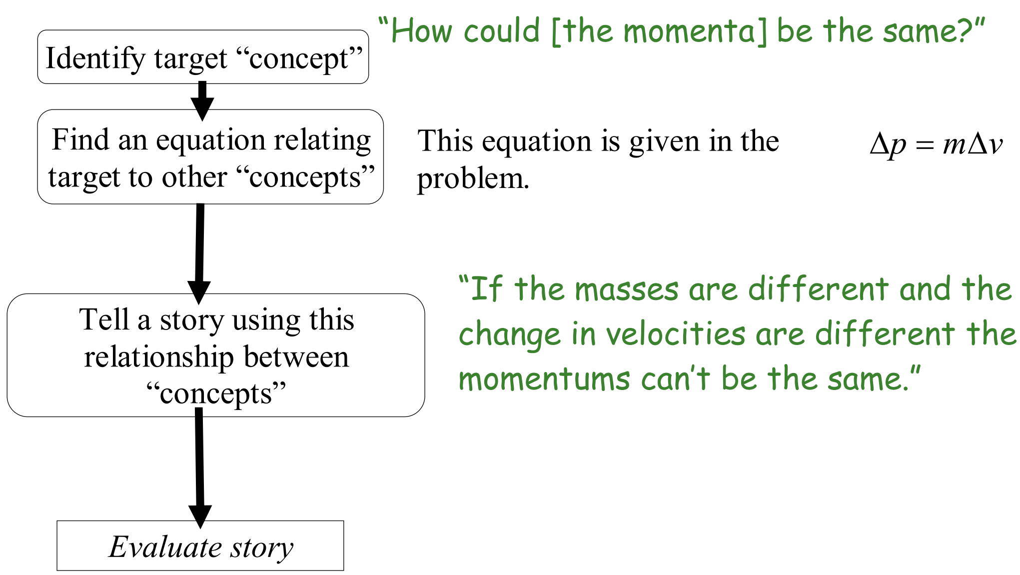

Figure 16. Schematic diagram of Arielle's moves in the *Mapping Mathematics to Meaning* epistemic game when using the interpretive device of *feature analysis*.

Physical Mechanism Game

In the *Physical Mechanism Game* students attempt to construct a physically coherent and descriptive story based on their intuitive sense of physical mechanism. The knowledge base for this game consists of reasoning primitives. In this game students do not make explicit reference to physics principles or equations.

The ontology of the *Physical Mechanism Game* is different than in *Mapping Meaning to Mathematics* and *Mapping Mathematics to Meaning*. The epistemic form in the latter two games explicitly involves physics equations; however the epistemic form in the *Physical Mechanism Game* does not.   Although the epistemic form is necessarily different, the same set of resources (intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices) may be active in this game as in the previous games.

The structure of the *Physical Mechanism Game* is similar to the first move in *Mapping Meaning to Mathematics – i.e.* both involve the development of a conceptual story. However, I set these two apart because the *Physical Mechanism Game* represents a separate, coherent unit of student activities; whereas, in *Mapping Meaning to Mathematics*, after move (1) students go on to move (2), then move (3), etc. The conceptual story developed in the *Physical Mechanism Game* stands alone. The activities that follow this game do not cohere with the conceptual story – in direct contrast with the activities that follow move (1) in *Mapping Meaning to Mathematics*. There are only two moves in the *Physical Mechanism Game*: (1) develop conceptual story and (2) evaluate story (see Figure 17).

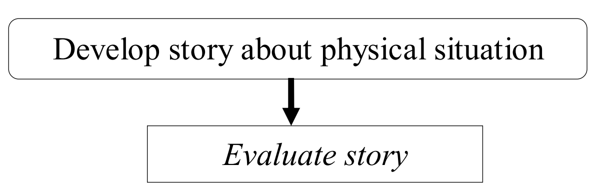

Figure 17. Schematic diagram of students' moves within the *Physical Mechanism Game*.

Now that I have given some background about the ontology and structure of this game I discuss an example. In this example, Lynn and Mary discuss their approach to the Elevator Problem (Appendix A, #6), while Tony listens. Lynn and Mary have already drawn the appropriate forces for the passenger and the scale when the elevator is at rest on the 33rd floor (see Figure 18).

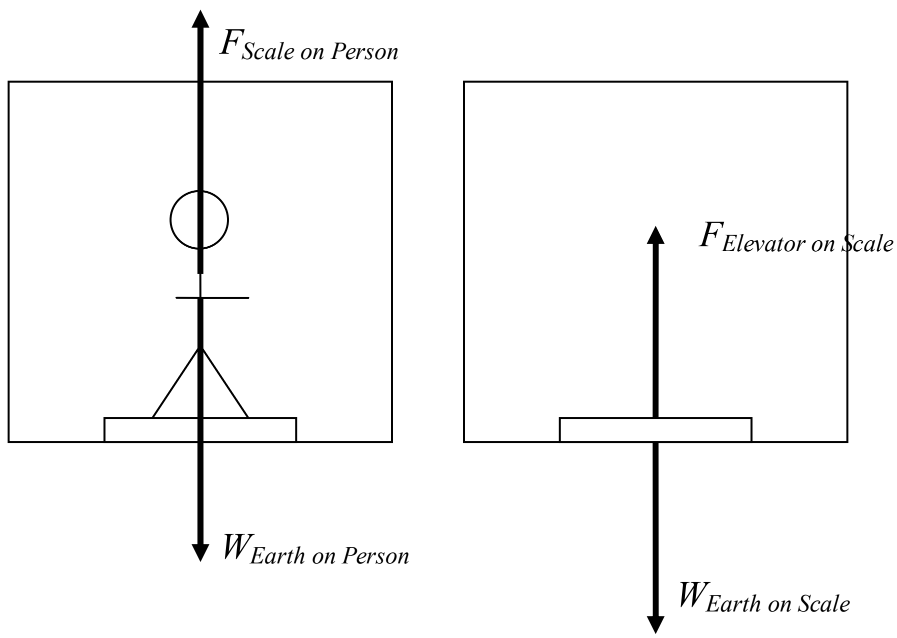

Figure 18. Lynn and Mary's free-body diagram for the person and the scale in the Elevator Problem.

Lynn and Mary are trying to determine which forces, if any, would change if the elevator begins to accelerate downward. To do this they calculate the numeric value for the acceleration from the numbers given in the problem, and begin identifying all the numeric values of the other quantities given in the problem statement.

> *Lynn: Oh, no. OK, so we know...they gave us the weights, so we know that the person is 80 kilograms and the scale is 7. And, we determined the acceleration.*

At this point Tony joins the discussion:

Tony: Do we even need to do all that calculation?
Lynn: I don't know.
Tony: I don't know if they're asking for it.
Lynn: They don't want numbers, but we couldn't really figure it out so we thought maybe numbers would help.

Tony's comments indicate that he does not expect that explicit calculations are necessary for this problem – conditions that are appropriate for the *Physical Mechanism Game*. Tony continues along this line of reasoning:

Tony: Yeah. Well, does um, let's see the the [normal force of the person on the scale] *would—don't you think that'd decrease? At--initially.*
Mary: When we're accelerating downward. The force of the...
Tony: Right. You know, it's almost likes you can look at it and like exaggerate it--like the elevator pulls away from the person. And the person has to catch up to it.
Lynn: Oh. That makes sense. And that's why the person would weigh less.
Tony: Right.
Lynn: Which is what I remember from high school physics.

An interesting feature about Tony's explanation is the type of reasoning he uses. Formal physical laws and principles are conspicuously absent from his explanation. He does not rely on arguments based on authoritative citations of abstruse physical laws. Rather, the support for his assertion rests on the other students "seeing" what he means: "Oh. That makes sense." This is evidence that he is relying on his intuitive sense of physical mechanism to generate this explanation. In particular, it appears that he images the elevator at a specific moment: when the *initially* starts to accelerate.

A second interesting feature about this exchange is that after Tony's explanation this activity basically stops. Lynn seems to think Tony's explanation "makes sense," and it confirms what she "remembers from high school physics." Therefore, there is nothing left to do. Tony's intuitive explanation answered their question. That is, the *Physical*

Mechanism Game ends with Tony's explanation. There is no need to translate this conceptual story into mathematical entities. The explanation in terms of Tony's intuitive sense of physical mechanism represents a coherent unit of activity.

Pictorial Analysis Game

In the *Pictorial Analysis Game* students generate an external spatial, representation that specifies the relationship between influences in the problem statement. Examples of students playing the *Pictorial Analysis Game* are familiar to most readers, even if the name is not. For instance, students that make a cartoon drawing of a physical situation, a free-body diagram, or a circuit diagram are all playing the *Pictorial Analysis Game*.

In this game, as with all the games previously discussed, the knowledge base consists of all the resources listed in chapter 4. The epistemic form in this game is a distinguishing characteristic. The epistemic form is the cartoon or diagram that the students generate. For example, if the students draw a circuit diagram during their inquiry, then that diagram serves as an epistemic form which guides their inquiry; in the same way, a cartoon drawing or free-body diagram could both serve as target structures that guide inquiry.

The moves in this game are largely determined by the particular external representation that the students choose to make. For example, if the students choose to draw a free-body diagram, then one move is to determine the forces that act upon the object in question; whereas, if the students choose to draw a circuit diagram, then one move is to identify the active elements (*e.g.* resistors, capacitors, batteries, etc.). So, the specific moves in this game vary depending on the external representation that the students choose. There are three moves that are common to all instantiations of the

Pictorial Analysis Game (see Figure 19): (1) determine the target concept, (2) choose an external representation, (3) tell a conceptual story about the physical situation based on the spatial relation between the objects, and (4) fill in the slots in this representation. Below is an example of students that choose to draw a free-body diagram while playing the *Pictorial Analysis Game*.

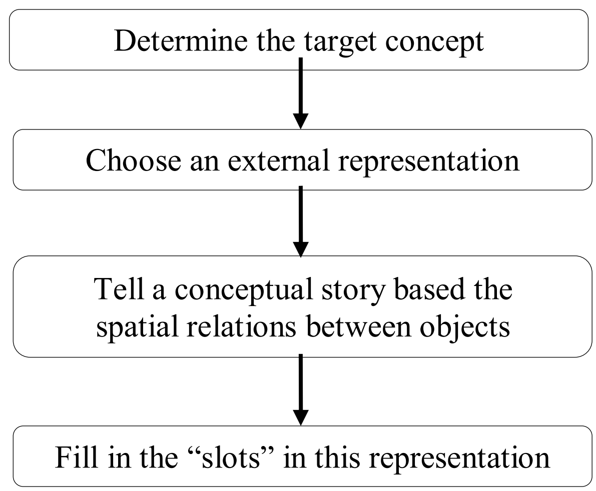

Figure 19. Moves in the *Pictorial Analysis Game*.

Alisa, Patty, Mary, and Emma play *Pictorial Analysis* while working on the Pulling Two Boxes Problem (Appendix A, #13). They are working on part A, which explicitly talks about forces. So, (1) Emma tacitly identifies force as the important concept, and then (2) decides that a free-body diagram is the appropriate external representation.

> *Emma: Like I think it would be a good idea to draw some free-body diagrams, but I don't know what—which ones we should draw.*
> *Alisa: Well, they want to know the friction between the crate and the floor.*

Alisa confirms Emma's tacit assumption that force is the important concept; in particular, Alisa notes that friction is what they ultimately need to determine.

Before the students can go to move (4) and begin to fill in the slots in the free-body diagram, they must decide what free-body diagram to draw. In move (3), they decide they can treat the two crates on top of each other as one big crate.

Emma: Because, I would assume that you could look at these two crates as being one unit, and look at it as one thing, like we did in class -- and, like we used for an example today.
Alisa: OK.
Emma: I mean for this, for this particular question – because, it's like one big crate.
Mary: Um-huh, just think of it as like one big thing.
Emma: But, maybe not, since they're [pushing up with her hands] I mean like with the boxes that you're pushing they're next to each other. But, when they're kind of of like, you know like this [pushing down on each other], it doesn't matter if you're trying to pull it.
Alisa: I think it does, because it makes it heavier. As long as that top box isn't stationary.
Emma: That's true.
Alisa: Or, is stationary, excuse me.
Emma: So, maybe it does matter when you're doing them liked stacked like that.
Alisa: Well, we can try it with both and then we could always ask, I guess.

Now that the students have decided for what object to draw a free-body diagram, they begin filling in the slots of this diagram – *i.e.* they begin move (4) of *Pictorial Analysis*.

Emma: So. OK, so like for the crates...they have...
Alisa: Well, they have weight.
Emma: They have weight from the earth on the crates.
Alisa: And, then they have that...
Emma: They have ground...acting on the—the normal force of the ground up against the, um, crates.
Patty: And, then we have the rope. Does the rope count?
Emma: Yeah, the ropes going to be something, I think.
Alisa: So then you have friction going [to the left].

These students specify four different forces that act on the crates (the weight, the normal force of the ground, the pull of the rope, and friction), and after a lengthy discussion they decide on the directions of all these forces (see Figure 20). Ultimately, the students do not correctly identify the direction of the tension force from rope on the crate. The students' activities follow the moves in *Pictorial Analysis*.

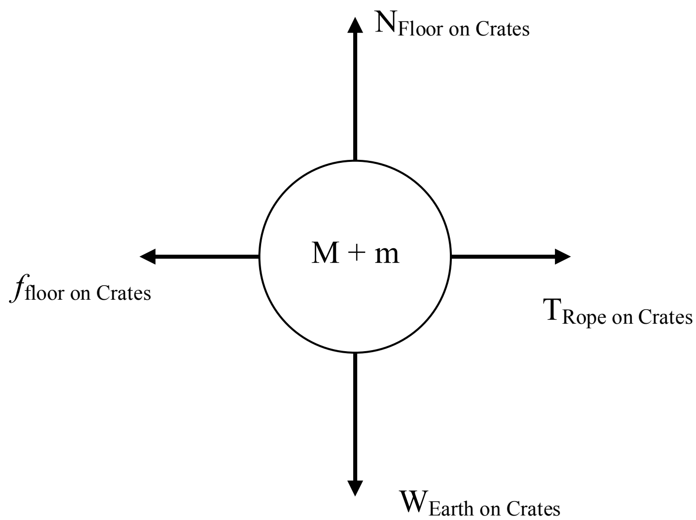

Figure 20. Recreation of the free-body diagram that the students created while playing *Pictorial Analysis*.

Recursive Plug-and-Chug

In the *Recursive Plug-and-Chug* e-game students plug quantities into physics equations and churn out numeric answers, without conceptual understanding the physical implications of their calculations.

Students do not generally draw on their intuitive knowledge base while playing this game; they simply identify quantities and plug them into an equation. Therefore, students usually just rely on their syntactic understanding of physics symbols, without attempting to understand these symbols conceptually. That is, their other cognitive resources (intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices) are usually not active during this game.

The epistemic form in *Recursive Plug-and-Chug* is similar to that in *Mapping Meaning to Mathematics* and *Mapping Mathematics to Meaning*: each game has physics equations as part of the epistemic form. As I stated in the previous paragraph, the resources that are active (*i.e.* knowledge base) in *Recursive Plug-and-Chug* are different than in these other games. Therefore, since the activated resources in *Recursive Plug-and-Chug* are different, the rules and strategies that are employed during this game differ from those in *Mapping Meaning to Mathematics* and *Mapping Mathematics to Meaning* – even though the epistemic form (target structure that guides inquiry) is the same in all these games. So, one of the distinguishing feature of *Recursive Plug-and-Chug* is the resources that are activated during this game.

Because the epistemic forms are similar, the structure of *Recursive Plug-and-Chug* is similar to *Mapping Mathematics to Meaning*. First, the students identify the target quantity. This is similar to the first move in *Mapping Mathematics to Meaning*, but it differs in this game in that the students only identify the quantity and its corresponding symbol – they do not attempt to understand conceptually what this quantity is. Second, the students identify an equation that relates the target quantity to other quantities. Third, the students identify which quantities are known and which quantities are unknown. If the target quantity is the only unknown, then they can proceed to calculate the answer. However, if there are additional unknowns, then they must choose a sub-goal and start this process over – herein lies the 'recursive' nature inherent in this game. Figure 21 shows a schematic depiction of the moves in this game.

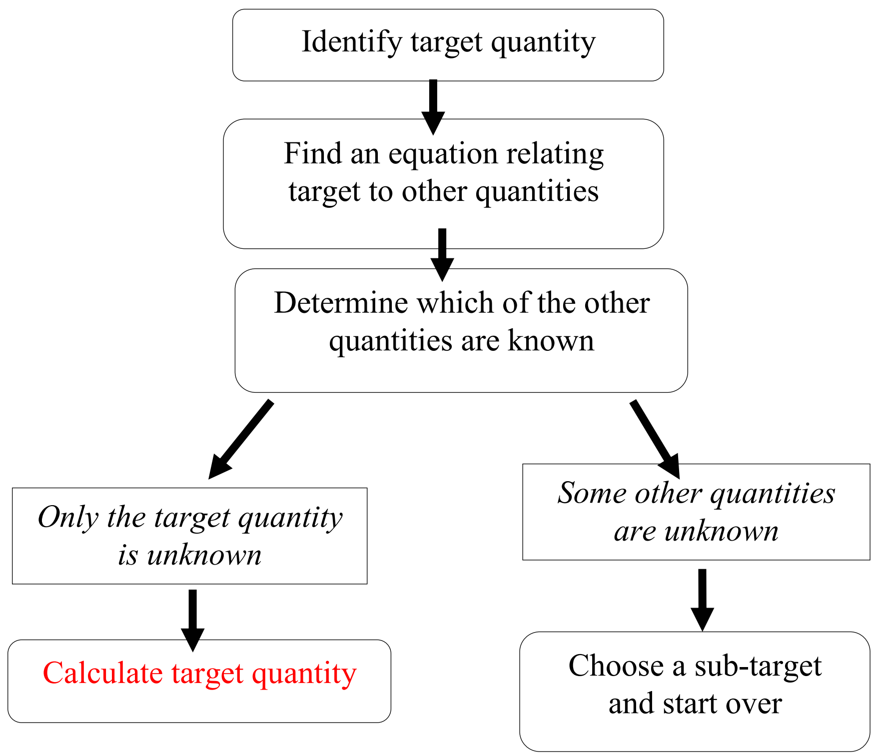

Figure 21. Schematic diagram of students' moves within *Recursive Plug-and-Chug*.

An example of students playing *Recursive Plug-and-Chug* occurs while Valerie and Sarah attempt to solve the Dorm Room Pressure Problem (Appendix A, #5). Valerie identifies 'pressure' as the target quantity, and then finds an equation relating pressure to other quantities:

Valerie: Pressure is equal to the radius, times the moles of the gas, times the temperature, divided by the volume. So what we need to do – we know the pressure…density is equal to…
Sarah: Are you using PV=nRT?
Valerie: Huh?
Sarah: Are you using PV=nRT?
Valerie: Yeah. Or…yeah.
Sarah: Or.
Valerie: Or P equals R times N T...
Sarah: Over V.
Valerie: Over V.

Two points can be interpreted from this exchange: the equation Valerie has chosen will not help them calculate the difference in pressure between the floor and the ceiling, and Valerie is not concerned with the conceptual meaning of the symbols in this equation – she incorrectly identifies *R* as the radius! The first point is an accidental feature of this particular instantiation of *Recursive Plug-and-Chug*. Students can play this game and get the correct answer. It just happens to be the case that Valerie chose an equation that won't lead her to the correct answer when playing this game. The second point, however, is an inherent feature of *Recursive Plug-and-Chug*. Since the cognitive resources for understanding the equations (*i.e.* symbolic forms and interpretive devices) are not activated during this game, conceptual understanding of the equation is not a part of this game. That is, the students need to be able to identify the symbols, but in this game the students do not need to understand the concepts that the symbols represent. The fact that Valerie identified *R* as the radius is an indication that she is playing *Recursive Plug-and-Chug*.

Consistent with the third move in *Recursive Plug-and-Chug*, Valerie and Sarah identify the 'knowns' and 'unknowns':

> *Sarah: We know the pressure.*
> *Valerie: We know the pressure. But we need to take the density to volume. Density is equal to...*
> *Sarah: Oh, we have the density.*
> *Valerie: Yeah, yeah, but that doesn't matter. We need the volume.*
> *Sarah: Oh, what did I just say?*
> *Valerie: Density is equal to volume over what mass, or something?*
> *Sarah: Density equals mass over volume.*

If the target quantity (pressure) was the only unknown they could proceed to calculate the target quantity. Since the target quantity is not the only unknown, they must choose a sub-target ("we need the volume") and return to the second move in this game ("Density equals mass over volume.")

To sum up, although there are some structural similarities (some of the moves are similar) between *Recursive Plug-and-Chug* and *Mapping Mathematics to Meaning*, the ontological components (the set of resources that are active) are different in the two games. Therefore, *Recursive Plug-and-Chug* represents a distinct set of activities that are distinguishable from *Mapping Mathematics to Meaning*. One of the distinguishing features is that in *Recursive Plug-and-Chug*, students use symbols without activating conceptual understanding.

Transliteration to Mathematics

Research on problem solving indicates that students often use worked examples to develop solutions to novel problems (Ben-Zeev, 1998). *Transliteration to Mathematics* is an epistemic game in which students use worked examples to generate a solution, yet they do so without developing a conceptual understanding of the worked example. The word 'transliterate' means "to represent (letters or words) in the corresponding characters of another alphabet."[20] In the *Transliteration to Mathematics* game students simply map the quantities from a target problem into the solution pattern of an example problem.

Because students use the symbolism in this game without conceptual meaning, usually only resources associated with the syntactic structure of equations are active during this game. The solution pattern of the target example serves as the epistemic form for the *Transliteration to Mathematics* game.

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[20] This definition comes from the The American Heritage® Dictionary of the English Language.

The moves in this game are simple: (1) identify target quantity, (2) find a solution pattern that relates to the current problem situation, (3) map quantities in current problem situation into that solution pattern, and (4) evaluate the mapping (see Figure 22). Moves (2) and (3) are very tricky for many students. Many times students may find a solution pattern that they think relates to the current problem, when in fact it does not.

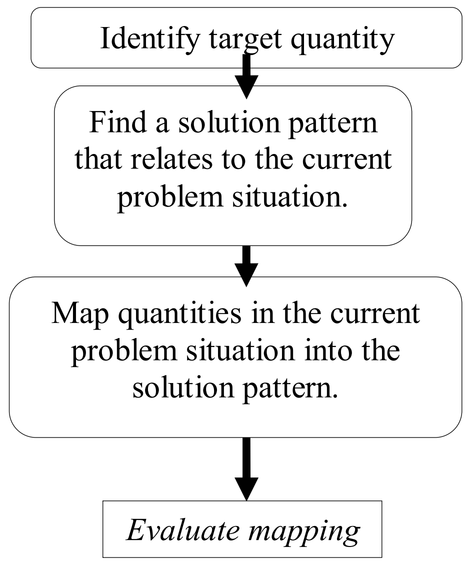


Figure 22. Schematic diagram of the moves in *Transliteration to Mathematics*.

Darlene, Bonnie, and Alisa play *Transliteration to Mathematics* while working on the Three Charge Problem (Appendix A, #15). (Figure 23 shows a schematic diagram of the students' moves within *Transliteration to Mathematics*.) First, Bonnie identifies the target quantities.

> *Bonnie: Yeah. So, if you double the distance how does that affect the charge, like does it--do you have to have the charge twice as big or four times big?*

Then, Darlene attempts to map the quantities in the Three Charge Problem into the solution from the Force-Distance Two-Charge Problem (Appendix A, #**7**).

Darlene: Where is that other problem [Force-Distance Two-Charge Problem]*? Three times as far apart as they were now what is the magnitude of the force?*
Bonnie: I think it should be four times.
Darlene: If it's three times as far apart it's...you divide...uh! I think it's q over two.
Bonnie: Q over two? So, if you think of it as half the force of q2.
Darlene: Look at this one [the Force-Distance Two-Charge Problem].
Bonnie: Is this one you're talking about?
Darlene: Uh-huh. If you increase the distance that they are from each other it's decreasing by the same amount. I thought it was four (?), but they said it was (?). I don't know why. Just three times...does it matter? I'm looking this one. Number three, isn't that like the same thing?
Alisa: Three was an estimation problem.
Darlene: No, no with the q and four q and all that, you know how there was this question that asked when you move the charges three times further apart than they originally were, what the resulting force is.
Alisa: OK.
Darlene: And, you said it was—we said it was four (?)--the charge would be like q, or nine, but it would got three times as far apart. Why it's not three I don't understand, but that's all right. So—
Alisa: Well, 'cause in the equation you square this—the distance between them. Like if you're multiplying by three...
Darlene: Oh! So, I would think this one would be q over four—negative q over four. Cause it's twice as far away, opposite charge. Does that make sense?
Alisa: But, then it's a smaller charge than this.
Bonnie: Yeah.
Alisa: So, I don't understand how it would be pushing three or pulling three whatever it's doing.

Darlene identifies the Force-Distance Two-Charge Problem (FDTCP) as being similar to the Three Charge Problem (TCP): “isn't that like the same thing?” The solution in the former problem has the epistemic form $\frac{\text{target quantity}}{\text{distance squared}}$, which becomes $\frac{F}{9}$ because of the situation in the FDTCP. Darlene attempts to use the same epistemic form as a target structure to guide her inquiry in the TCP. She simply maps the ‘charge’ in as the ‘target quantity’ and uses the distance specified in the Three Charge Problem: “Oh!

So, I would think this one would be q over four—negative q over four. Cause it's twice as far away, opposite charge."

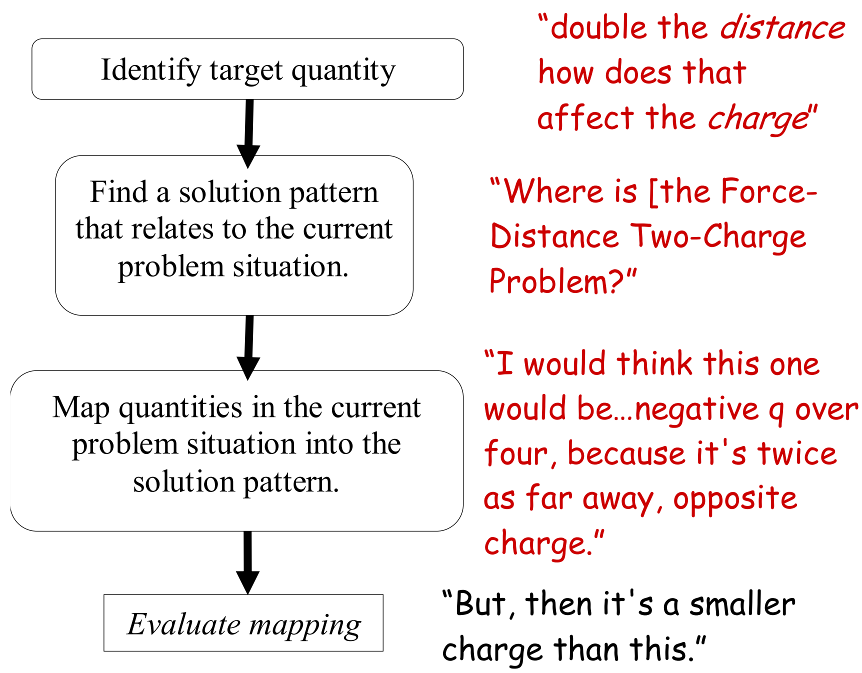


Figure 23. Darlene, Bonnie, and Alisa' moves in *Transliteration to Mathematics*.

An additional piece of evidence that indicates that Darlene is playing *Transliteration to Mathematics* comes from her admission that she doesn't understand the solution in the FDTCP: "I don't understand how it would be pushing three or pulling three whatever it's doing." One of the distinguishing features of *Transliteration to Mathematics* is that students can play this game without conceptual understanding of the solutions patterns. Darlene admits she doesn't understand the FDTCP, but according to her "that's all right."

Why students (tacitly) choose to play a particular e-game: Introduction to *Frames*

The introductory students in my study played six different epistemic games while using mathematics in physics. But why would a student choose to play any one particular e-game? My answer to this question is that the (tacit) decision to play a particular epistemic game is determined by a student's real-time and/or preconceived expectations about problem solving in physics. To describe students' expectations I introduce the concept of *frames*.

Background and history of Frames

The concept of frames has a long history across many different disciplines. Frames were proposed by the psychoanalyst Gregory Bateson (1972) and the anthropologist Irving Goffman (1997), and used by socio-linguist Deborah Tannen (1993). A *frame* is an individual's interpretation of a situation or event based on her expectations of the situation or event. The gerund, *framing*, is used to describe an individual's moment to moment parsing of a particular situation. That is to say, an individual's framing helps her answer the question, "what kind of activity is this?"

An example of a frame (*the restaurant frame*) comes from Tannen's (1993) discussion of a story by Schank and Abelson (1977), which reads:

> John went into the restaurant. He ordered a hamburger and a coke. He asked the waitress for the check and left.

Tannen discusses how Schank and Abelson's story illustrates the existence of frames in knowledge structures. Schank and Abelson's use of the term "script" in the following passage is synonymous with Tannen's use of the term "frame":

> One might ask how the story can refer to "the" waitress and "the" check "just as if these objects had been previously mentioned." The fact that they can is evidence of the existence of a script [or frame] which "has implicitly introduced them by virtue of its own introduction" (p.18).

That is, the waitress and the check don't need to be formally introduced in the story, because the reader has the expectation that waitresses and checks are present in restaurants; *i.e.* waitresses and checks are part of the *restaurant frame*.

So, if frames exist, then how do we know one when we 'see' it? Tannen identifies16 different linguistic cues that indicate an individual's structures of expectations: (1) omission, (2) repetition, (3) false starts, (4) back tracks, (5) hedges and other qualifying words or expressions, (6) negatives, (7) contrastive connectives, (8) modals, (9) inexact statements, (10) generalizations, (11) inferences, (12) evaluative language, (13) interpretation, (14) moral judgment, (15) incorrect statements, and (16) addition. I describe and use these cues in my categorization of the different frames involved in mathematical problem solving in introductory physics.

Frames and students' use of mathematics in physics

As articulated by Redish (2004), an individual's framing has many components: a social component ("Who will I interact with and how?"), a physical component ("What material will I be using?"), a skills component ("What will I actually be doing?"), an affect component ("How will I feel about what I'm going to be doing?"), and an epistemological component ("How will I learn / build new knowledge here?"). I draw on the skills and epistemological components to categorize epistemic games into three different frames: *rote equation chasing*, *qualitative sense-making*, *and quantitative sense-making*.

The *rote equation chasing* frame is students' expectations that problem solving in physics involves appropriately identifying a physics equation from a large memorized list, and then "plugging in" the corresponding quantities. In contrast, the *sense-making* frame is students' expectations that problem solving in physics should progress through the systematic application of common sense or physical principles – *i.e.* problem solving in physics should "make sense." In the *qualitative sense-making* frame students expect that the solution does not require formal mathematics; whereas, in the *quantitative sense-making* frame students expect that the solution does require formal mathematics. Table 12 shows the different epistemic games organized by frame. In the next three subsections, I discuss each frame in more detail.

Rote equation chasing frame	**Qualitative sense-making frame**
Recursive Plug-and-Chug *Transliteration to Mathematics*	*Physical Mechanism Game* *Pictorial Analysis*
	Quantitative sense-making frame
	Mapping Mathematics to Meaning *Mapping Meaning to Mathematics*

Table 12. Epistemic games organized by frame.

Rote equation chasing frame

I identify two epistemic games that fall into the rote equation chasing frame: *recursive plug-and-chug* and *transliteration to mathematics*. As I alluded to earlier, students' 'decisions' to enter into these games is based on their expectations. These expectations can be based on real-time assessments of the problem statement (*i.e.* moment-to-moment activation of epistemological resources) and/or on preconceived epistemological beliefs about problem solving in physics (*i.e.* a particular epistemological frame).

Research by Hinsley and Hayes (1989) indicates that students tend to “use a line-by-line procedure, especially in solving nonstandard problems” (p. 476). This result suggests that if students’ real-time assessment of the problem statement does not cue the appropriate knowledge that would allow the students to make sense of the problem, then they may be nudged into the rote equation chasing frame. That is, if the problem statement does not cue the appropriate mathematical resources for making sense of the problem, then students may resort to an equation hunting technique. So, the students enter the *rote equation chasing* frame based on a moment-to-moment framing of the problem statement.

Alternatively, if the students believe that problem solving in physics is simply picking the correct equation out of the book or a worked example, then they will likely be in the rote equation chasing frame. That is, their preconceived notion about problem solving in physics puts them into the *rote equation chasing* frame.

There is evidence that indicates that *Recursive Plug-and-Chug* occurs in a rote equation chasing frame. In particular, there are three pieces of evidence that indicate while Sarah and Valerie are playing *Recursive Plug-and-Chug* to solve the Dorm Room Pressure Problem (Appendix A, # 5) they are in the *rote equation chasing* frame. *First*, the question asks for the *difference* in pressure between the floor and the ceiling in a dorm room. They simply identify the pressure as the target quantity and the equation that they find (“$P = \frac{nRT}{V}$”) cannot help them find the difference in pressure between the floor and the ceiling – a fact that they don’t seem to give a second thought.

Second, Valerie identifies R in the equation as the “radius.” When it is brought to her attention that R does not represent the ‘radius,’ she is not fazed at all, in fact this pleases her: “Is it a constant?...Awesome, one less thing for us to find!” Valerie is not concerned with the semantic content contained within the equation; she’s simply using the equation as a calculating tool, without thinking about what the equation or the symbols mean.

The recursive nature of this game is the *third* indication that it occurs in a rote equation chasing frame. Valerie realizes that in order to solve for the target quantity using the equation $P = \frac{nRT}{V}$, she needs to determine the volume. The equation she finds to relate the volume to other quantities is $D = \frac{m}{V}$. From this she realizes that the volume is unknown and the mass is unknown. Therefore, she identifies the ‘mass’ as the new sub-target. The equation she finds to relate the ‘mass’ to other quantities is $D = \frac{m}{V}$. However, the ‘volume’ is unknown and the ‘mass’ is unknown. This leaves her in a recursive loop, because in order to find the ‘volume’ she needs the ‘mass,’ but in order to find the ‘mass’ she needs the ‘volume.’ The recursive nature of this game is inconsistent with sense making – another indication that *Recursive Plug-and-Chug* occurs in a rote equation chasing frame.

In addition to *Recursive Plug-and-Chug*, *Transliteration to Mathematics* occurs in the *rote equation chasing* frame. As I discussed above, Darlene plays *Transliteration to Mathematics* in an attempt to solve the Three Charge Problem – she attempts to map the solution pattern from the Force-Distance Two-Charge Problem into the Three Charge Problem. Darlene’s comments show that she does not have conceptual understanding of how the solution was obtained in the FDTCP: “Why it's not three I don't understand, **but** that’s all right” (emphasis added). Her use of the contrastive connective ‘but’ is particular telling. Tannen (1993) argues that “an oral narrative uses the word ‘but’ to mark the denial of an expectation not only of the preceding clause but of an entire preceding set of statements or of narrative coherence in general" (p. 44). So, Darlene's use of the word 'but' in the above statement can be taken as evidence of her expectation that conceptual understanding is not at all necessary in her problem solving approach. Since I categorize her approach as *Transliteration to Mathematics*, then it follows that she expects that conceptual understanding is not necessary in *Transliteration to Mathematics*; *i.e.* this game occurs in the *rote equation chasing* frame.

To sum up, in the rote equation chasing frame, students can play *Recursive Plug-and-Chug* or *Transliteration to Mathematics* without conceptual understanding of the mathematics used in the problem solving process. In these two games students simply use the syntactic structure of the mathematics as cues for how to generate an answer.

Qualitative sense-making frame

I identify two distinct epistemic games in the qualitative sense-making frame: *physical mechanism game* and *pictorial analysis*. I use the qualifier 'qualitative' when describing this frame, because the games do not rely on formal mathematical procedures or equations. However, these games may involve informal, intuitive mathematical reasoning.

There is evidence from Tony's comments on the Elevator Problem that the *Physical Mechanism Game* occurs in the *qualitative sense-making* frame. Lynn and Mary's initial approach to this problem involves numerous calculations and equations – they are not in a *qualitative sense-making* frame. Tony makes many hedges and qualifying statements in an attempt to nudge them into a different frame. Tannen (1993) states "by qualifying or modifying a word or statement, hedges measure the word or idea against what is expected" (p. 43).

> *Tony: Do we* **even** *need to do all that calculation?*
> *Lynn: I don't know.*
> *Tony: I don't know if they're asking for it.*
> *Lynn: They don't want numbers, but we couldn't really figure it out so we thought maybe numbers would help.*

Tony's initial comment is an indication that he doesn't expect that calculations are necessary for this problem – one indication that his approach to this problem (*Physical Mechanism Game*) occurs in the *qualitative sense-making* frame. A second interesting feature about Tony's comments is how he uses hedges to negotiate a frame shift with the other students.

> *Tony: Yeah.* ***Well, does um, let's see the the*** [normal force of the person on the scale] ***would—don't you think*** *that'd decrease? At--initially.*
> *Mary: When we're accelerating downward. The force of the...*
> *Tony: Right.* **You know, it's almost** *likes you can look at it and* **like** *exaggerate it--like the elevator pulls away from the person. And the person has to catch up to it.*
> *Lynn: OH! That makes sense. And that's why the person would weigh less.*
> *Tony: Right.*
> *Lynn: Which is what I remember from high school physics.*

Initially, Lynn and Mary attempt to use formal mathematics and physics principles (which they do not appear to understand) in their efforts to produce a solution. Tony's approach to this problem (the *Physical Mechanism Game*) stands in stark contrast to Lynn and Mary's collective approach. Lynn and Mary appear to have been in a rote equation chasing frame. Tony wants to play the *Physical Mechanism Game*, which is in the qualitative sense making frame. Tony's many hedges serve to mitigate the transition between the two frames. He makes many starts and stops and repetitions of words before offering his intuitive explanation: "Yeah. Well, does um, let's see the the…" These linguistic hedges are an indication that Tony intends to shift frames from *rote equation chasing*, the frame Lynn and Mary are initially operating in, to the *qualitative sense-making* frame.

Quantitative sense-making frame

I identify two different epistemic games in the quantitative sense-making frame: *Mapping Mathematics to Meaning* and *Mapping Meaning to Mathematics*. I use the qualifier 'quantitative' when describing this frame, because the games in this frame rely on formal mathematical procedures or equations. Students get nudged into this frame based on two expectations: the solution to the problem involves explicit calculations and the answer should make sense.

Mary and Emma's discussion while working on the Paper Towel Problem (Appendix A, #10) illustrates the point that *Mapping Mathematics to Meaning* occurs in the *quantitative sense-making* frame. Mary and Emma are initially playing the *Physical Mechanism Game* in an attempt to solve this problem.

> *Mary: If you pull it with one hand, so all the force is concentrated in one area of the towel, so it causes it to rip. You know. But, if you pull it with both hands, it's going to be a more equal distribution, maybe. So, you could (?), that's what I was thinking. But, if your hands are wet it makes the towel soggy, which makes it weak, so it's more likely to rip.*
> *Emma: It might make it more likely to rip, but still that's better than pulling it with one hand.*
> *Mary: Yeah, if both your hands--like (?) yeah.*

The students are attempting to develop a coherent, physical story without reference to formal physics principles or equations. Then, Emma voices her expectation that this type of approach is insufficient.

Emma: Is that all we're supposed to do with that? I ***feel like*** *(?). Like,* ***I feel like*** *it* ***should*** *have something to do with like* ***not just*** *force. I feel like it* ***should have*** *something to do with what we've learned like recently. Having to do with like water and pressure (?)*

The linguistic cues about Emma's expectations come from her use of negative statements and modals. Tannen (1993) states that "in general, a negative statement is made only when its affirmative was expected" (p. 44), and the modals "'must' and 'should'…reflect the speaker's judgment according to her own standards and experience" (p. 45). Emma's comments indicate she has the expectation that they need to use a concept that they have "learned like recently." It's not that Mary's explanation doesn't make sense; it's just that Emma has that expectation that the explanation should involve the concept of "pressure."

Emma parlays this expectation into an opportunity to play *Mapping Mathematics to Meaning*. (1) She identifies the target concept ('pressure'), and (2) finds an equation relating the target to other concepts:

Emma: Well pressure was force over surface area or something, right?
Mary: Pressure equals F over A, yeah.

Then, (3) she develops a story that uses the relationship between the target and the other concepts:

Emma: So like, you know, you could be exerting the same force but you're doing it over like a larger area, so it's one (?), less pressure on the towel it doesn't rip.
Mary: That's true. So, we could use that. So, what did you say, if you use the same amount of force each time, but over—you use a larger surface?
Emma: When you pull (?).
Tony: Fifty newtons of force and you apply it over five centimeters you have ten, ah, newtons or whatever in pressure. But, if you had fifty and you apply it over ten then you have five thingamabobers of pressure.
Mary: Oh, OK.

Finally, (4) Emma evaluates her story by referring to a particular physical example:

Mary: So, is that it? I'm just going to add a thing that says...
Emma: Yeah, because if you have a paper towel and you want to see how strong they are (?), you hold it at the sink and it's all wet and you put like a thing of grapes like in the middle of it, it's going to rip through the middle. But, if you put it over like all of it might not.
Mary: Uh-huh.
Emma: Spread out...

The new explanation that Emma generates by playing *Mapping Mathematics to Meaning* still makes sense to her, and it also fulfills her expectation that the answer should involve concepts they had learned more recently.

Discussion about epistemic games and frames

The astute reader may have noticed that there is considerable overlap between the moves in some games. For example, both *Mapping Meaning to Mathematics* and *Recursive Plug-and-Chug* involve mathematical manipulations. For this reason, some readers may contend that *Mapping Meaning to Mathematics* ends after the conceptual story is translated into mathematical entities (*i.e.* after move (3)), and that the manipulation of symbols (*i.e.* move (4)) is a different epistemic game. The basis for this contention is that mathematical manipulations are the essential component of the *Recursive Plug-and-Chug* game.

However, my assertion is that simply because a move is in one game (*e.g.* mathematical manipulation occurs in *Recursive Plug-and-Chug*) it doesn't mean that same move cannot appear in a different epistemic game (*e.g.* mathematical manipulations occur in *Mapping Meaning to Mathematics*). My reasons for this assertion are threefold:

1. *Empirical*. Epistemic games are an observational categorization of coherent units of activity. In order for two problem solving activities (epistemic

games) to be the same, they must contain all the same sub-activity (move), and the sub-activities must occur in the same order. In *Mapping Meaning to Mathematics*, the mathematical manipulations occur after the conceptual story is translated into mathematics; whereas, in *Recursive Plug-and-Chug* the mathematical manipulations occur after all the 'knowns' and 'unknowns' are identified. Although both games include mathematical manipulations, the moves before and after these manipulations are different in the two games. (See Figure 13 and Figure 21 to compare the moves in *Mapping Mathematics to Meaning* and *Recursive Plug-and-Chug*, respectively.)

2. *Pedagogical*. Allowing the same move to occur in different epistemic games can help educators and researchers distinguish between seemingly similar expert and novice problem solving behavior. Students often use the symbols in a physics equation without conceptual understanding (*e.g.* Valerie identifies R as the radius in the equation $PV = nRT$, p. 138). Experts' often have conceptual understanding of the symbols that they manipulate.[21]
3. *Theoretical*. Frames are a larger theoretical construct, than epistemic games – epistemic games occur within a particular frame. The mathematical manipulations in *Recursive Plug-and-Chug* occurs in the *rote equation chasing* frame; whereas, the mathematical manipulations in *Mapping Meaning to Mathematics* occur in the *quantitative sense-making* frame.

[21] There is additional discussion about how this framework helps distinguish between expert and novice problem solving behavior in Chapter 6.

Alternative frameworks that address *process component*

Any observational study of the process component of students' use of mathematics in physics leads to an obvious conclusion: the actual path that students follow during problem solving in physics varies from problem to problem and student to student. This fact is largely overlooked in many cognitive models of student problem solving (Larkin, 1983; Kintsch & Greeno, 1985; Nathan, Kintsch, & Young, 1992). Many of these cognitive models of mathematical problem solving are normative, not descriptive. That is, they adequately model *ideal* student problem solving approaches, but they do not describe all the different approaches that students *actually use* during problem solving in physics. I discuss the model of algebra word problem comprehension introduced by Nathan, Kintsch, and Young (1992). In particular, I show that the Nathan *et al* model can adequately model an ideal problem solving approach, but it can not be used to describe some non-ideal student problem solving activities.

Nathan *et al* (1992) argue that any model of problem solving must include the aspect of language comprehension. According to their model there are three components to algebra word problem comprehension: a textbase, a situation model, and a problem model. The *textbase* consists of a network of propositions, generated by the problem solver, which captures the meaning of the problem statement. The reader's mental representation of the actions in the text, described in terms of everyday terms and objects, is called the *situation model*. The *problem model* consists of quantitative algebraic relationships between entities, which are generated from "problem schema" or "templates for organizing problem-relevant information." According to Nathan *et al*, the "textbase is organized into a (qualitative) situation model and mapped into a (quantitative) problem model that captures the algebraic problem structure" (p. 332).

This model adequately describes an ideal student solution; however, students use mathematics in physics in ways that are less than ideal. Epistemic games and frames offer more descriptive language for analyzing students' actual use of mathematics in physics. I explicitly identify six different approaches, or games, that students attempt during problem solving in physics. Nathan *et al* acknowledge that there is not one path that students must follow during problem solving: "We do not propose a stage theory, however, in which situational understanding must precede the formation of the problem model" (p. 337). The go on to say, "we find a mutually supporting relationship in which situational understanding helps students realize the episodic meaning of a formal problem model, and reciprocally, sensitivity to the requirements of a problem schema aids in the construction of a suitable situation model" (p. 337). According to the language of epistemic games, this is simply the difference between *Mapping Meaning to Mathematics* and *Mapping Mathematics to Meaning*. Additionally, however, epistemic games offer a language to describe students' non-ideal use of mathematics – when they use mathematics without conceptual understanding.

Summary

Two theoretical constructs describe the process component of students' use of mathematics in physics: epistemic games and frames. Students play six different kinds of epistemic games while using mathematics in the context of physics: *Mapping Meaning to Mathematics*, *Mapping Mathematics to Meaning*, *Physical Mechanism Game*, *Pictorial Analysis*, *Recursive Plug-and-Chug*, and *Transliteration to Mathematics*. Three different frames correspond with students' expectations about problems and problem solving in physics: *quantitative sense-making*, *qualitative sense-making*, and *rote equation chasing*.

Chapter 6: A case study illustrating the use of *mathematical resources*, *epistemic games*, and *frames* in the analysis of students' mathematical thinking

Introduction

Constructivism (student construction of knowledge) is the dominant paradigm in modern educational theory. The educator's role in the constructivist paradigm is to help students construct new knowledge from their existing knowledge. In order to assist the students, the educator needs to be able to determine what the students are thinking and why they make the mistakes that they do. In chapters 4 and 5 I described the three major components used in my theoretical framework for analyzing and describing students' use of mathematics in physics: *mathematical resources*, *epistemic games*, and *frames*. In this chapter, through a detailed analysis of a one-hour student problem solving session, I show how my theoretical framework offers educators and researchers a technical language capable of describing students' (correct and incorrect) use of mathematics in physics. That is, this framework offers a vocabulary (definition of the relevant cognitive structures) and grammar (relationship between the cognitive structures) for understanding the nature and origin of students' mathematical thinking in physics.

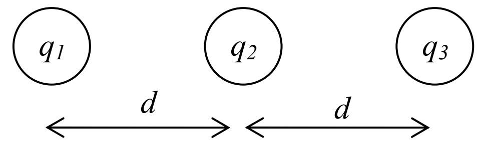
In the next section I give the specific context in which the case study is derived: I describe the students, the time and setting in which the students worked, and the particular homework problem under investigation. In section three I discuss how the problem solving session is parsed for analysis. In section four, I give detailed analysis of the problem solving session in terms of the theoretical framework that I have developed. Finally, in section five, I give a summary and some closing remarks.

The context of this case study: Three Charge Problem

The episode for this case study involves four students working on an electrostatics problem: The Three Charge Problem (Appendix A, #15). Three of the students are female (pseudonyms, Alisa, Bonnie, and Darlene) and one of the students is male (pseudonym, Edgar). Edgar very rarely speaks during the entire 60 minute video record of these students working on this problem. In fact, in the excerpts of the problem solving session that follow, he does not speak at all.

This episode occurs in the first week of the second semester in a two semester introductory, algebra-based physics course. (A more complete description of the particular introductory physics course appears in chapter 3.) All the students in the group had been in the reformed, non-traditional introductory course the first semester. Therefore, they were familiar with the 'peculiarities' of this course. In particular, they were familiar with the typical interaction style between students and teaching assistants in the course center, and the type of homework problems that were typically assigned in this course. Most importantly, they were cognizant of the fact that the instructor expected the students to spend about an hour on each homework problem – during which time they were expected to generate solutions to the questions that 'made sense to them.' The students' familiarity with these aspects of the course will become particular important when I discuss their *framing* of the problem solving episode.

The particular problem that the students work on in this episode is the Three Charge Problem, which reads as follows:



> In the figure above three charged particles lie on a straight line and are separated by distances d. Charges q_1 and q_2 are held fixed. Charge q_3 is free to move but happens to be in equilibrium (no net electrostatic force acts on it). If charge q_2 has the value Q, what value must the charge q_1 have?

An ideal solution to this problem involves straight-forward balancing of forces and Coulomb's Law. The parenthetic comment in the problem states there is "no net electrostatic force" acting on charge q_3. Symbolically, this becomes $\vec{F}_{q2\to q3} + \vec{F}_{q1\to q3} = 0$. Manipulating this equation, and defining the positive $\hat{i}$ direction to be to the right, yields:

$$\vec{F}_{q2\to q3} = -\vec{F}_{q1\to q3}$$
$$\frac{kq_2q_3}{d^2}\hat{i} = -\frac{kq_1q_3}{(2d)^2}\hat{i}$$

Canceling similar terms on both sides of the equation and setting $q_2 = Q$ yields the result: $q_1 = -4Q$.

I went through the details of the solution to illustrate that there are several inferences and steps involved in generating this solution. However, in spite of the multiple steps involved, most readers would solve this problem in less than fifteen seconds. An interesting aspect about the students' problem solving approach is that it takes so long. The students work for nearly 60 minutes before arriving at a solution – 240 times longer than the typical reader! Does this mean that the typical reader is 240 times *smarter* than these students? To boost my own ego, I'd like to say 'yes'; however, I don't believe this is the case. Rather, according to the theoretical framework developed in this dissertation, the typical reader probably has a broader mathematical knowledge base (*i.e.* a larger collection of compiled mathematical resources) and richer collection of problem solving strategies (*i.e.* an assortment of epistemic games for solving problems in physics). For the typical reader, the problem statement immediately cues the appropriate resources and epistemic game; whereas, the students' mathematical resources do not exist in compiled form. The difference in the reader and the students' knowledge structure could account for the difference in the speed of the problem solution.

Analysis of phenomena at different grain-sizes

Often researchers distinguish between different scales in order to 'chunk' phenomena into manageable pieces. I discuss an example from particle physics (mass scale of the unobserved right-handed neutrino) and an example from educational theory (time scales of interest for understanding mathematical thinking and problem solving).

Mass scales of interest in explanation of neutrino mass

In particle physics the heavy mass scale of the unobserved right-handed neutrino field is used to explain the light, but non-zero mass of the observed left-handed neutrino field. The conventional mechanism to explain the apparent lightness of the mass of the observed left-handed neutrino is the see-saw mechanism of *SO(10)* grand unified theories (GUTs). In these models there is a neutrino doublet, $\mathbf{N}$, consisting of the left-handed

(ν_L) and right-handed (N) neutrino fields; *i.e.* $\mathbf{N} = (\nu_L, N)^T$. The mass term for this neutrino doublet is of the form $\mathbf{N}^T \mathbf{M} \mathbf{N}$, with the mass matrix given by $\mathbf{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$.

The entry m arises from the standard Yukawa coupling which appears due to the electroweak breaking, so $m \sim \left(10^2 \text{ GeV}\right)$. However, M comes from the Majorana mass of the right-handed singlet N, and arises from the breaking of the *SO(10)* GUT symmetry. Due to the scale of the GUT symmetry breaking it is believed that $M \sim 10^{16}$ GeV. Diagonalizing the mass matrix, the two mass eigenvalues are obtained $\lambda_{\pm} = \frac{1}{2}\left(M \pm \sqrt{M^2 + 4m}\right)$. Since $m << M$ the two mass eigenvalues can be written as $\lambda_{-} \approx -\frac{m^2}{M}$ and $\lambda_{+} \approx M$. The linear combination corresponding to the light mass eigenvalue, λ_{-}, is taken to be the physical light neutrino. So the presence of the heavy mass scale M serves to explain the small but non-zero mass of the physical light neutrino.

Time scales of interest in understanding students' use of mathematics

In education research distinguishing between different time scales can be instructive. Time scales of interest for understanding students' use of mathematics are (adapted from Sherin, 1996):

1. *The learning time scale* (~ 1 year). During a two semester course we would hope that some of our students would learn some physics.

2. *The problem solving time scale* (~ 1 hour). Students were expected to spend about an hour on the homework problems in this study. In fact, it was often the case that students spent *at least* an hour on these problems.
3. *The problem heuristics time scale* (~ 10 minutes). As shown by Schoenfeld (1985), students (and experts) engage in different problem solving strategies, or heuristics, during the course of solving a single problem.
4. *The thought time scale* (~ 1 second). This time scale is associated with the time it takes a student to look at an equation (or graph) and then say something about it.

As discussed in chapter 3, I do not systematically examine the practices of mathematics use in physics across different classrooms, and I do not perform a longitudinal study of how students change during the course of the semester. Therefore, I do not have anything to say about time scale (1). However, time scales (2), (3), and (4) are nicely accommodated by the theoretical framework I have described. A students' *frame* (and moment to moment *framing*) will shape the problem solving process – *i.e.* time scale (2). The particular *epistemic games* the students play will partially determine the problem solving heuristic the students employ – *i.e.* time scale (3). Lastly, the particular *mathematical resources* that are activated at a given moment help us understand the students' mathematical thinking – *i.e.* time scale (4). So, the three different aspects of the theoretical framework can be seen as parsing the students' use and understanding of mathematics at different time scales. In the remainder of this chapter, I use the notion of these different time scales to parse the students' problem solving process in the Three Charge Problem.

Analysis of the Three Charge Problem

In my analysis that follows, I start with time scale (2): I examine how the students frame this problem solving process. Then, I probe the students' mathematics use on the problem heuristics time scale: I break the students' problem solving process into different epistemic games. Lastly, during my discussion of the different epistemic games that the students play, I examine the students' mathematics use on the thoughts time scale.

Analysis in terms of Frames

Alisa, Bonnie, and Darlene had all been in the non-traditional, reformed introductory physics course in the first semester. They were familiar with the types of homework problems, and the typical kinds of interactions between the students and the teaching assistant in the course center.

I suggest that the students' familiarity with these aspects of the course caused them to *frame* this problem solving session in a particular manner, which is evidenced by their behavior during the problem solving process. First, the students spend nearly an hour working on this problem. Many typical introductory students expect to spend less than 10 minutes on a problem. If they don't find a solution in this time they either give up trying, or ask for assistance. This leads into the second piece of evidence of the students' *framing* of this problem solving session: Alisa, Bonnie, and Darlene proceed with very little guidance or assistance from me (the teaching assistant). The majority of the work and progress comes from the students. The only direction I offer these students is to draw a picture.

These two points taken together suggest that the students seem to be in a *sense making frame*. In particular, they start in a *qualitative sense making frame* and end in a *quantitative sense making frame*. This is not to say that at any one time that all the students are in the same frame. For example, Darlene makes a digression at one point during the problem solving session and slips into a *rote equation chasing frame*.

Analysis in terms of epistemic games *and* mathematical resources

The students do not follow a straight forward approach to solving this problem. However, these students' various problem solving approaches are easily understood in terms of epistemic games. I identify five different epistemic games that are played during this problem solving session: *Physical Mechanism Game*, *Pictorial Analysis*, *Mapping Mathematics to Meaning*, *Transliteration to Mathematics*, and *Mapping Meaning to Mathematics*.

Physical Mechanism Game

The students' initial attempt to solve this problem follows a less formal path than the ideal solution outlined above. Throughout this entire clip the students are drawing on intuitive reasoning primitives to explain and support their conclusions. The students do not draw on any formal mathematics or physics principles to support their claims. They use reasoning that makes sense to them. This first clip occurs about 7 minutes into the problem solving process.

> *Darlene: I'm thinking that the charge q_1 must have it's...negative Q.*
> *Alisa: We thought it would be twice as much, because it can't repel q_2, because they're fixed. But, it's repelling in such a way that it's keeping q_3 there.*
> *Bonnie: Yeah. It has to--*
> *Darlene: Wait say that.*
> *Alisa: Like— q_2 is— q_2 is pushing this way, or attracting—whichever. There's a certain force between two Q, or q_2 that's attracting.*
> *Darlene: q_3.*
> *Alisa: But at the same time you have q_1 repelling q_3.*

Darlene initiates the conversation by asserting that the charge on q_1 must be 'negative Q'; the negative sign in this case signifies that q_1 will have the opposite effect on q_3 than q_2. Alisa elaborates on this point by articulating that q_2 exerts an influence on q_3, which she identifies as a force, that is either repelling or attracting, and that q_1 exerts the opposite influence on q_3. The semantic content contained in Alisa's explanation can be summarized in the following facet: 'the attractive effect of q_2 on q_3 cancels the repulsive effect of q_1 on q_3.' The abstract reasoning primitive underlying this facet is *canceling*. That is, the influences in this problem get mapped onto the abstract reasoning primitive of *canceling* resulting in the facet articulated above. In this case, *canceling* is an appropriately mapped primitive, because in fact the two forces acting on q_3 do cancel, which results in there being no net force on q_3.

In addition, from Alisa's initial cursory comment ("we thought [the charge on q_1] would be twice as much [than the charge on q_2]") it appears that she has the reasoning primitives *more is more* and *balancing* activated. That is, since the two influences acting on q_3 balance, then q_1 must have more charge because there is more distance between q_1 and q_3 then there is between q_2 and q_3.

It cannot be confirmed whether Alisa has *more is more* and *balancing* activated, because the direction of the conversation changes. Darlene contends with the other students, because it appears she has a different reasoning primitive activated: *blocking*.

Darlene: How is it repelling when it's got this charge in the middle?
Alisa: Because it's still acting. Like if it's bigger, than q_2 it can still, because they're fixed. This isn't going to move to its equilibrium point. So, it could be being pushed this way.
Darlene: Oh, I see what you're saying.
Alisa: Or, pulled. You know, it could be being pulled more, but it's not moving.

Darlene: Um-huh.

The orientation of the charges cues the reasoning primitive of *blocking*, because q_2 is in between q_1 and q_3. That is, the presence of q_2 "blocks" the effect of q_1 on q_3. From the superposition principle we know the effect of q_1 on q_3 does not get blocked by the presence of q_2, so the activation of *blocking* is an unnecessary distraction for these students. In contrast to the reasoning primitive of *canceling* that was activated earlier in this clip, *blocking* does not get mapped into a productive facet for solving this problem. This is not to say that *blocking* is 'wrong'; rather, in this particular instance the activation of *blocking* does not lead to a productive facet.

Bonnie continues Alisa's line of reasoning by explaining why the value of q_1 has to be twice as big as that of q_2.

Alisa: So, we—we were thinking it was like negative two Q or something like that.
Bonnie: Yeah. Cause it has to be like big enough to push away.
Darlene: Push away q_3.
Bonnie: Yeah, which we—which I figured out negative two.
Darlene: Cause it's twice the distance away than q_2 is?
Bonnie: Yeah.
Darlene: I agree with that.

It appears that Bonnie draws on *overcoming* when she explains that '[q_1] has to be like big enough to push away [q_3].' That is, q_1 has to have enough charge to overcome the influence of q_2. The tacit conclusion from this assertion is that the charge of q_1 must have a larger magnitude than that of q_2. Bonnie and Darlene quantify this conclusion by using the reasoning primitives of *more is more* and *dependence* (which has the symbol template □ = […*x*…]) to assert that the charge on q_1 has to be twice the magnitude of q_2. *More is more* and *dependence* get mapped into the facet *twice the distance is twice the charge*.

The students' problem solving activities during this entire clip have the ontology of the *Physical Mechanism* epistemic game. The ontological components of the *Physical Mechanism Game* are the knowledge base and the epistemic form. While playing this game the students are drawing on an intuitive knowledge base rather than formal knowledge to support their claims. There is evidence, as I tried to indicate above, that the students use various reasoning primitives during this clip. And, at no point during their discussion do they mention any formal mathematics or physics principles. The epistemic form in the *Physical Mechanism Game* involves a coherent, physical description that is either verbal or imagistic. These students are actively seeking physical causes for the effects that are described in the problem.

The structural aspects of the students' problem solving activities are also consistent with the *Physical Mechanism Game*. The fact that these students engage in this activity to solve a problem sets it apart from other "everyday" activities. This discussion has a beginning and an end, which makes it distinguishable from everyday activities. In addition, there are certain "moves" in this game. For one, all assertion must be supported with reasons. For example, Alisa makes the assertion that q_1 is "like negative two Q." The support for this assertion is that "it's twice the distance away than q_2."

In this clip, the ontology and structure of the students' problem solving activity suggest that they are playing the epistemic game of *Physical Mechanism*. Playing this game helps the students become oriented to this problem, but the solution to this problem necessarily involves physics equations (in particular Coulomb's Law). Since *Physical Mechanism* does not include mathematical expressions or equations (like Coulomb's Law), it cannot ultimately lead them to the correct answer. In the next clip, I help them reframe this problem, in an attempt to activate other resources they have and epistemic games they already know how to play.

Pictorial Analysis

In the last clip we saw the students making sense of the problem by using their intuitive reasoning primitives in the context of the *Physical Mechanism* epistemic game. It appears that the students have difficulty focusing their collective attention. To assist the students I offer a suggestion.

> *Darlene: I think they all have the same charge.*
> *Bonnie: You think they all have the same charge? Then they don't repel each other.*
> *Darlene: Huh?*
> *Bonnie: Then they would all repel each other.*
> *Darlene: That's what I think is happening.*
> *Bonnie: Yeah, but* q_3 *is fixed. If it was being repelled—*
> *Alisa: No, it's not.* q_3 *is free to move.*
> *Bonnie: I mean,* q_3 *is not fixed. That's what I meant.*
> *Darlene: Right.*
> *Bonnie: So, like...*
> *Darlene: So, the force of* q_2 *is pushing away with is only equal to d.*
> *Bonnie: Yeah, but then...*
> *Darlene: These two aren't moving.*
> *Bonnie: Wouldn't this push it somewhat?*
> *Alisa: Just because they're not moving doesn't mean they're not exerting forces.*
> *Darlene: I know.*
> *Alisa: What do you think?*
> *Tuminaro: Can I make a suggestion?*
> *Darlene: Uh-huh.*
> *Tuminaro: You guys are talking about like a lot of forces and stuff. And, one thing I've suggested in previous semesters, if you write it down and say, what forces do you think are acting here, you can all talk about it.*
> *Darlene: Where did the marker go?*
> *Tuminaro: That's a suggestion—a general suggestion—that I might make.*

In the first few lines above, it seems as though the students take a step back in terms of progress on this problem. Earlier the students appeared to have established the major aspect of the problem: two influences act on q_3, which exactly cancel each other. In this clip, the students restate the set up of the problem ("these two are moving") and recite remembered facts ("just because they're not moving doesn't mean they're not exerting forces"). While these things are important to keep straight, this discussion does not appear to push the problem solving process forward.

To assist the students I offer a suggestion, which has two effects. First, it nudges the students into playing a different epistemic game: *Pictorial Analysis*.[22] Second, the introduction of this new epistemic game reframes the students' interactions and helps them focus their collective attention on one external representation.

Alisa attempts to make an external representation of this problem on the white board while Bonnie and Darlene offer their assistance:

> *Darlene: You're trying to figure out what q_1 is, right?*
> *Bonnie: Oh, yeah.*
> *Alisa: Because this is in equilibrium, there's some force...*
> *Darlene: Pulling it that way and some force pulling ex—equally back on it.*
> *Bonnie: Yeah.*
> *Alisa: And, they're equal?*
> *Bonnie: Yes.*
> *Darlene: Same with up and down. Not that that matters, really.*
> *Bonnie: We'll just stick with...*
> *Darlene: Horizontal.*
> *Bonnie: Yeah, one dimension.*

In this clip the students are deciding which features mentioned in the problem should be included in their diagram. That is, the students are playing the *Pictorial Analysis* epistemic game. The structure of this game is similar to *Physical Mechanism*; however, the ontological components of *Physical Mechanism* and *Pictorial Analysis* are slightly

[22] At the time of the instructional intervention, I was not consciously attempting to nudge "the students into playing a different epistemic game." It is only in the analysis, not in the actual event, that I used the concept of epistemic games to describe this episode.

different. The epistemic form in *Pictorial Analysis* involves a coherent, physical description *and* an external representation; the epistemic form for *Physical Mechanism* only involves a coherent, physical description.

The external representation generated in the *Pictorial Analysis* epistemic game cues additional resources in the students, which help them better understand this problem. In particular, the students draw on the interpretive device of *physical change* to conclude that q_1 and q_2 have to have opposite charges.

> *Alisa: So, maybe this is pushing...*
> *Darlene: That's [q_2] repelling and q_1's attracting?*
> *Bonnie: Yeah, it's just that whatever q_2 is, q_1 has to be the opposite. Right?*
> *Alisa: Not necessarily.*
> *Darlene: Yeah.*
> *Bonnie: OK, like what if they were both positive?*
> *Alisa: Well, I guess you're right, they do have to be different, because if they were both positive...*
> *Bonnie: Then, they'd both push the same way.*
> *Alisa: And, this were positive it would go zooming that way.*
> *Darlene: They would both push.*
> *Alisa: And, if this were negative it would go there.*
> *Bonnie: It would go zooming that way.*
> *Alisa: And, if they were negative...*
> *Darlene: It would still—they'd all go that way.*
> *Alisa: It would be the same thing.*

Bonnie makes a claim that the charge on q_1 has to be the opposite of q_2, but the others don't initially agree. Bonnie's suggestion to verify, or falsify, her claim involves the interpretive strategy of *physical change*. That is, she considers the affect of an actual physical alteration to the system ("OK, like what if they were both positive?"). From this move the students almost immediately conclude that the charges on q_1 and q_2 must be different, or else q_3 would go 'zooming' away.

Switching to *Pictorial Analysis* turns out to be a very effective problem solving strategy. By decomposing the forces in space and creating on external representation, the students are able to physically justify why q_1 and q_2 have to have opposite charge. This clip also illustrates that the students' problem does not stem from *lack* of knowledge or skills; rather, the epistemic game the students play in their initial approach (*Physical Mechanism*) does not help adequately articulate the physical relationship between the charges. The external representation they collectively generate in *Pictorial Analysis* cues resources they already possess (*physical change*), which helps them make progress on this problem (i.e. conclude that q_1 and q_2 have opposite charges).

Although the students' external representation and conclusion marks progress, they have yet to solve the problem. In fact, they have not even identified the necessary physics principle: *Coulomb's Law*. That's exactly what happens in the next clip.

Mapping Mathematics to Meaning

So far the students have drawn a diagram representing which forces act and in what direction, and they have concluded that q_1 and q_2 have opposite charges; however, they have not yet solved this problem. In this clip we see Alisa spontaneously reframe the problem solving process by drawing on a new set of resources: *formal mathematics knowledge*.

Alisa: Are we going to go with that?
Bonnie: I think it makes sense.
Darlene: That makes...
Alisa: Well, I don't know, because when you're covering a distance you're using it in the denominator as the square.
Bonnie: Oh! Is that how it works?
Alisa: And (?) makes a difference.
Bonnie: Yeah, you're right.
Tuminaro: So, how do you know that?
Darlene: From the Coulomb's Law.
Bonnie: So, it should actually be negative four q? Or what? Since it has…
Alisa: Cause we were getting into problems in the beginning of the problem with two A A, because I thought that like if you move this a little bit

to the right the decrease for this would make up for the increase for this. But, then we decided it didn't. So, that's how I know that I don't think it would just increase it by a factor of two.

Alisa is not only attempting to introduce a new epistemic game, she is negotiating a frame shift. All the previous reasoning relied on intuitive reasoning primitives, without any explicit reference to physics principles or equations. The students played *Physical Mechanism* and *Pictorial Analysis* within the *qualitative sense making frame*. Alisa's introduction of Coulomb's Law is the first mention of a physics principle during this entire problem solving process. In addition, it's the first time any one explicitly makes reference to an equation ("when you cover a distance you use it in the denominator as the square"). Alisa's use of formal physics principles and explicit reference to equations is an attempt to nudge the other students into the *quantitative sense making frame*. In particular, she is (tacitly) asking them to play *Mapping Mathematics to Meaning*.[23]

Alisa's discussion follows all the moves within *Mapping Mathematics to Meaning* (see

Figure 24). *One*, the distance and force are identified as the relevant concepts in this problem. *Two*, she identifies Coulomb's Law $\left(F = \frac{kq_1q_2}{r^2}\right)$ as an equation that relates the target concept to other concepts. *Third*, she develops a story using this relationship between concepts: "When you're covering a distance you're using it in the denominator as the square." *Fourth*, she evaluates the validity of her story by referencing a previous problem. She acknowledges that her intuitive reasoning had failed her on the previous problem, which justifies the need for Coulomb's Law on this problem: "I thought that

[23] Admittedly, Alisa would not describe her comments as an invitation to play *Mapping Mathematics to Meaning*.

like if you move this a little bit to the right the decrease for this would make up for the increase for this. But, then we decided it didn't."

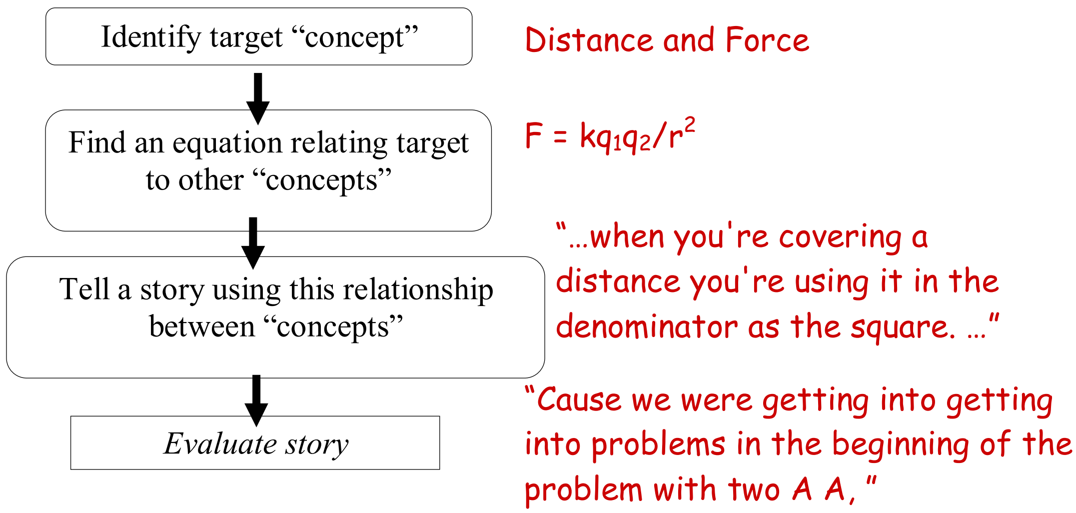


Figure 24. Schematic map of Alisa's moves within *Mapping Mathematics to Meaning*.

Alisa's use of Coulomb's Law is significant progress on this problem, but all the other students don't know how to apply this new piece of information. In fact, the introduction of Coulomb's Law cues Darlene to play a new epistemic game.

Transliteration to Mathematics

Although it appears the students are making progress on this problem, they take a detour and attempt to use another problem as a prototype for solving this problem. Alisa has suggested that Coulomb's Law is an important concept. It appears that Darlene does not initially know how to apply this new information. She attempts to find a different problem that uses Coulomb's Law in its solution, and then map the solution pattern from the other problem to the Three Charge Problem. The problem that Darlene identifies as using Coulomb's Law in the solution is the Force-Distance Two-Charge Problem (Appendix A, # 7).

Darlene: Where is that other problem? Three times as far apart as they were now what is the magnitude of the force?
Bonnie: I think it should be four times.
Darlene: If it's three times as far apart it's...you divide...uh! I think it's q over two.
Bonnie: Q over two? So, if you think of it as half the force of q_2.
Darlene: Look at this one.
Bonnie: Is this one you're talking about?
Darlene: Uh-huh. If you increase the distance that they are from each other it's decreasing by the same amount. I thought it was four (?), but they said it was (?). I don't know why. Just three times...does it matter? I'm looking at this one. Number three, isn't that like the same thing?
Alisa: Three was an estimation problem.
Darlene: No, no with the q and four q and all that, you know how there was this question that asked when you move the charges three times further apart than they originally were, what the resulting force is.
Alisa: OK.
Darlene: And, you said it was—we said it was four—the charge would be like q, or nine, but it would get three times as far apart. Why it's not three I don't understand, but that's all right. So—
Alisa: Well, 'cause in the equation you square this—the distance between them. Like if you're multiplying by three...
Darlene: Oh! So, I would think this one would be q over four—negative q over four. Cause it's twice as far away, opposite charge. Does that make sense?
Alisa: But, then it's a smaller charge than this.
Bonnie: Yeah.
Alisa: So, I don't understand how it would be pushing three or pulling three whatever it's doing.

In the Force-Distance Two-Charge Problem, the students had found that if the force between two charges for a given distance is *F*, tripling the distance results in a force between the two charges that is decreased by a factor of nine (see Appendix A, #7), in compliance with Coulomb's Law. Darlene is attempting to match the quantities in the Three Charge Problem with quantities from the Force-Distance Charge Problem, so the solution pattern can be transferred; *i.e.* she is playing the *Transliteration to Mathematics* epistemic game.

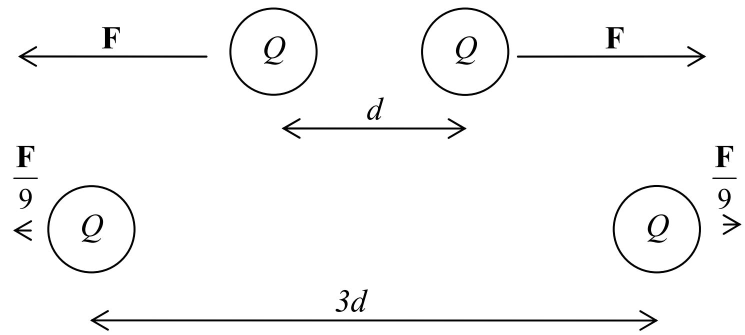


Figure 25. Displays the difference between the forces on two charges when the distance between the charges is tripled.

One obvious piece of evidence that Darlene is playing *Transliteration to Mathematics* comes when she says, “Why it’s not three I don’t understand, but that’s all right.” Darlene is explicitly meta-cognitive indicating that she doesn’t understand the previous problem, but conceptual understanding is not terribly important in the *Transliteration to Mathematics* epistemic game. All that is important in this game is that the problems have enough similar features that the solution from one problem can be transferred to the other.

Darlene’s metacognitive statement (“Why it’s not three I don’t understand, but that’s all right.”) stands in stark contrast to Alisa’s meta-cognitive statement (“I thought that like if you move this a little bit to the right the decrease for this would make up for the increase for this.”). Darlene simply admits she doesn’t understand and slavishly transfers the solution pattern from the previous problem anyway. Alisa's metacognitive statement leads to her justification for using Coulomb's Law.

This difference between Darlene and Alisa's metacognitive statements is an iconic example of the difference between the two frames in which these statements are couched. Darlene's statement occurs while playing *Transliteration to Mathematics* in the *rote equation chasing frame*. Conceptual understanding is not a necessary component of the *rote equation chasing frame*. Alisa's comment occurred while playing *Mapping Mathematics to Meaning* in the *quantitative sense making frame*. Conceptual understanding is a necessary component of the *quantitative sense making frame*.

Darlene's *Transliteration to Mathematics* approach doesn't help her with the Three Charge Problem. She says, "If you increase the distance that they are from each other it's decreasing by the same amount." The problem with Darlene's approach is that she is unaware of the two meanings that she attributes to the pronoun 'it.' In the previous problem the pronoun stands for 'force,' so that the statement would read, "If you increase the distance that they are from each other, then *the force* is decreasing by the same amount." However, Darlene tacitly maps this into the statement, "If you increase the distance that they are from each other, then *the charge* is decreasing by the same amount." The *Transliteration to Mathematics* game is not helpful in this case because force and charge are not related to distance in the same way in Coulomb's Law. This is not to say that the *Transliteration to Mathematics* game is wrong; it doesn't work in this situation because of Darlene's inappropriate mapping of force and charge.

<u>Mapping Meaning to Mathematics</u>

In this clip the students finally come to the solution of the problem. Alisa summarizes her final solution as the other students and I listen.

Alisa's problem solving activities follow the *Mapping Meaning to Mathematics* epistemic game (see Figure 11). *First*, she develops a conceptual story describing the physical situation. This conceptual story relies heavily on the reasoning primitives of *balancing*.

> *Tuminaro: What did you do there?*
> *Alisa: What did I do there?*
> *Tuminaro: Yeah, can I ask?*
> *Alisa: All right, so* ***because this isn't moving the two forces that are acting on it are equal****: the push and the pull.*

Alisa correctly maps 'force' as the two influences that balance in this physical situation.

Second, Alisa uses the *identity* symbolic form, which has the symbol template □ = …, to translate her conceptual story into mathematical expressions:

> *So, the F—I don't know if this is the right F symbol—but, the F* q_2 *on* q_3 ***is equal to*** *this (see Equation 1). And, then the F* q_1 *on* q_3 ***is equal to*** *this (see Equation 2), because the distance is twice as much, so it would be four d squared instead of d squared.*

$$F_{q2\rightarrow q3} = \frac{kQq_3}{d^2}$$

Equation 1

$$F_{q1\rightarrow q3} = \frac{kxQq_3}{4d^2}$$

Equation 2

Alisa explains why she wrote the charge on q_1 as 'xQ,' by drawing on the reasoning primitive of *scaling*, which has the syntax *x*□.

> *And, then I used x Q like or you can even do—yeah—x Q for the charge on q_1, because we know in some way* ***it's going to be related to Q*** *like the big Q we just got to find* ***the factor that relates*** *to that.*

The *third* step in the *Mapping Meaning to Mathematics*, Alisa relates the mathematical entities that she derived in step 2 with her conceptual story that she developed in step 1:

> *Then, I set them equal to each other…*

Fourth, she manipulates the mathematical expression to arrive at the desired solution:

> *… and I crossed out like the q_2 and the k and the d squared and that gave me Q equals x Q over four. And, then x Q equals four Q, so x would have to be equal to four. That's how you know it's four Q.*

Fifth, the other students evaluate Alisa's problem solving approach and conclusion.

> *Bonnie: Well, shouldn't it be—well equal and opposite, but...*
> *Alisa: Yeah, you could stick the negative.*
> *Bonnie: Yeah.*
> *Darlene: I didn't use Coulomb's equation, I just—but it was similar to that.*
> *Bonnie: That's a good way of proving it.*
> *Darlene: Uh-huh.*
> *Bonnie: Good explanation.*
> *Alisa: Can I have my A now?*

Darlene admits that is not the way she arrived at a solution, but acknowledges that Alisa's approach is consistent with her own. Bonnie makes a single critique ("shouldn't it be…equal and opposite"), yet admits Alisa's approach is "a good way of proving it." In fact, Alisa must realize that this is a good way to prove this, since she audaciously asks for an "*A* now."

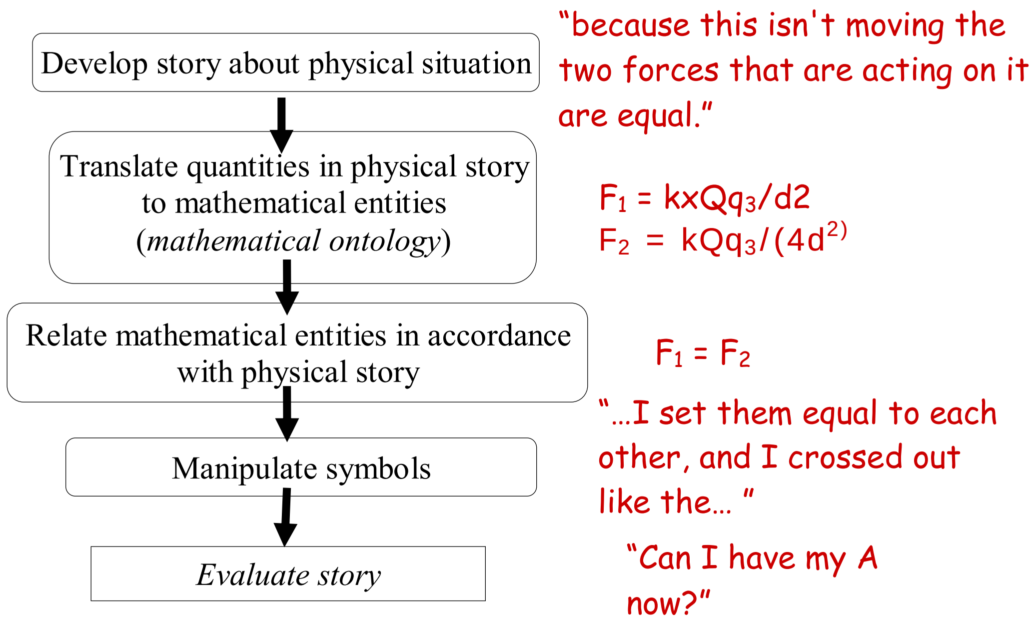


Figure 26. Schematic map of Alisa's moves within the *Mapping Meaning to Mathematics* epistemic game.

Instructional implications

Earlier in this chapter, I made the assertion that the typical reader would probably solve the Three Charge Problem in about fifteen seconds – 240 times faster than these students. This lead to a slightly whimsical question: Does this mean that the typical reader is 240 times *smarter* than these students? My answer to this question was, and still is, no. I made the claim then that the difference in the reader and the students' knowledge structure could account for the difference in the speed of the problem solution. That is, the reader's knowledge exists in compiled form; whereas, the students knowledge does not. Therefore, it takes the student a longer amount of time to execute the same operations as the reader.

The typical reader may not be aware of all the knowledge and reasoning that goes into solving this problem, since the solution comes so easily and quickly. Decomposing the students' problem solving session in terms of frames, epistemic games, and resources allows us to 'see' and examine all the knowledge and reasoning that is involved in this problem. With increased understanding of the knowledge and reasoning involved in such a seemingly simple problem, instructors and educators can begin to develop teaching environments and interventions that more effectively and efficiently cue the appropriate resources and epistemic games. This in turn could help students become better and more efficient problem solvers.

Conclusion

One can use the theoretical framework that I have developed in this dissertation to make sense of students' use of mathematics in physics. In particular, this framework introduces the relevant cognitive structures (mathematical resources) and the relationship between these structures (epistemic games and frames) for describing and analyzing mathematical thinking and problem solving. Students' use of mathematics in physics can be broken into the problem solving time scale (~ 1 hour), the problem heuristic time scale (~ 10 minutes), and the thought time scale (~ 1 second), which are described by frames, epistemic games, and mathematical resources, respectively.

Chapter 7: Understanding Student Mathematical Errors in Terms of *Resources*, *Epistemic Games*, and *Frames*

Introduction

Galileo wrote that "the book of nature is written in the language of mathematics." So, it seems natural that in order for students to understand physics they must be fluent in this mathematical language. However, it's often the case that students perform poorly on mathematical problem solving tasks in the context of physics. There are at least two possible, distinct reasons for this poor performance: (1) students simply lack the mathematical knowledge and skills needed to solve problems in physics, or (2) students do not know how to apply the mathematical skills they have to particular problem situations in physics. While many students do lack the requisite mathematical skills, research in mathematics education suggests that many of students' mathematics errors arise from erroneously learned rules – not simply lack of mathematics knowledge (Ben-Zeev, 1996, 1998; Matz, 1982; Silver 1986; VanLehn, 1983, 1986). Analyzing students' mathematical errors in physics in terms of *resources*, *epistemic games*, and *frames* suggests that some students' errors arise because they fail to use or interpret their mathematics knowledge and skills correctly in the context of physics – in accordance with reason (2).

In this chapter I discuss students' mathematical errors in the context of physics. First, I discuss Ben-Zeev's taxonomy of rational mathematical errors that can be used to classify student mathematical errors in the context of mathematics. In particular, Ben-Zeev identifies three classes of rational errors: critic-related failures, syntactic errors, and semantic errors. In section three I review previous research about student mathematical errors, and I discuss how this research can help make sense of students' inappropriate use of mathematical symbolism. That is, this previous research helps make sense of students' critic-related failures and syntactic errors. In section four I analyze students' mathematical errors in physics in terms of *resources*, *epistemic games*, and *frames*. In particular, I show that resources, epistemic games, and frames help make sense of students' semantic math errors in the context of physics.

A taxonomy of rational mathematical errors: REASON

Erroneous symbolic manipulations are not the only types of errors that students produce while using mathematics. To get a handle on the many different kinds of errors that students produce while using mathematics, Ben-Zeev (1996, 1998) developed a taxonomy of rational mathematical errors, which she calls ***R**ational **E**rrors **A**s **S**ources **O**f **N**ovelty* (or **REASON**). The word 'rational' indicates that these errors do not arise out of carelessness on the students' parts; rather, these errors arise from (often times sophisticated) mathematical thinking that is applied in an inconsistent or inappropriate manner. For example, the students that produce the error $\frac{1}{3}+\frac{1}{2}=\frac{2}{5}$ are not simply being lazy or careless; they are systematically applying a rule in which they add the numerator and denominator (Silver, 1986).

Ben-Zeev identifies three major categories of rational errors: *critic-related failures*, *syntactic induction*, and *semantic induction*. Critic-related failures and syntactic induction are errors associated with erroneous symbolic manipulations. Semantic inductive errors arise from inappropriate conceptualization of the mathematical symbolism. I discuss each of these classes of errors in turn.

Critic-related failures in REASON

A *critic* is a meta-cognitive knowledge structure. Critic-related failures arise from a lack of meta-cognitive monitoring during the mathematical problem solving process. An informal definition of a critic is that it is a metacognitive resource, which monitors the current problem state and fires when a violation occurs. Ben-Zeev formally defines a critic in terms of production rules (see Anderson, 1983; Anderson and Thompson, 1989; Anderson, 1993). In short, a *production rule* is an algorithm for solving problems. A production rule has the form "*If B, then A*," where *B* is a particular problem state and *A* is the algorithm that can be implemented to arrive at a solution for the problem in state *B*. Critics are associated with production rule that have the form "*If C, then ?*". If state *C* is reached a critic will fire, because there is no algorithm that can be implemented to arrive at a solution for problem state *C*.

Ben-Zeev articulates three mechanisms by which this class of error can occur: absent critic, weak critic, and constraint satisfaction. An *absent critic* is a critic that simply doesn't exist. A *weak critic* is a critic that is in competition with a previously learned rule. *Constraint satisfaction* occurs if a 'fix' is spontaneously generated to stop the critic from firing. One type of 'fix' involves altering the state *C*, so the critic associated with the production rule "*if C, then ?*" stops firing. Ben-Zeev calls this type of fix a *negation*.

Figure 27 shows a schematic diagram showing the different kinds of critic-related failures.

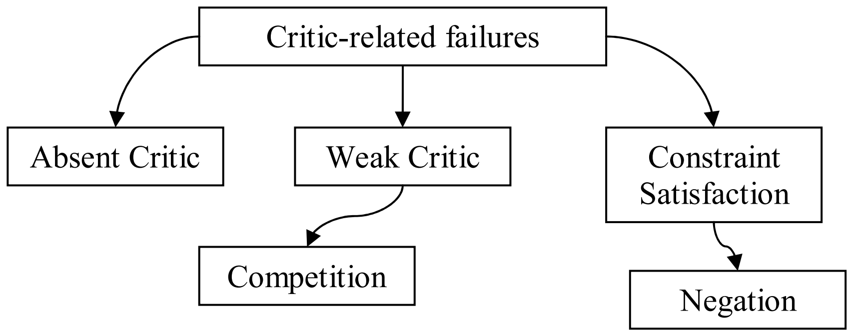


Figure 27: Schematic diagram of the kinds of *critic-related* failures (Ben-Zeev, 1996, p. 70).

Absent critic

Many introductory physics students don't differentiate between symbols that look the same but represent different physical quantities. For example, many students in my study did not distinguish between Δv (the change in velocity) and $\langle v \rangle$ (the average velocity) when solving problems, even though the symbols represent distinct physical quantities. According to Ben-Zeev, if the students simply ignore the difference between the symbols Δv and $\langle v \rangle$ while solving problems, and if they are not immediately corrected, then they may fail to develop the appropriate critic to signal a difference between Δv and $\langle v \rangle$. In this case the students would have an *absent critic*.

Weak critic

Alternatively the critic to distinguish between Δv and $\langle v \rangle$ could present, but it is competition with a prior knowledge rule – *i.e.* it's a *weak critic*. That is, the critic to signal a difference between Δv and $\langle v \rangle$ may not be strong enough to take precedence over a previously learned rule for manipulating symbols. "The strength of the rule is primarily affected by how successfully the rule has performed in the past problem-solving episodes" (Ben-Zeev, 1998, p.372).

<u>Constraint satisfaction</u>

A third mechanism for a critic-related error is that the students *negate* or alter the situation that caused the critic to fire in the first place. Recall that a critic fires when a production rule of the form "*if C, then ?*" is reached. If the situation *C* is negated to ~*C* or altered in some way, then the critic will stop firing and the algorithm associated with ~*C* can be implemented. Considering the example with Δv and $\langle v \rangle$ again, it may be the case that students have a critic that fires when they see the symbols 'Δ' and '$\langle\ \rangle$.' By a tacit mental removal of these symbols, the students may be able to adequately alter the situation so that the critic no longer fires. Therefore, they don't *need* to distinguish between Δv and $\langle v \rangle$ anymore, because they have tacitly removed 'Δ' and '$\langle\ \rangle$' – so, they are left with v and v, which are obviously the same.

Syntactic Induction in REASON

According to REASON, inductive failures arise when a student over-generalizes or over-specializes a rule or worked example. *Syntactic induction* is a type of inductive error that arises from inappropriate use of mathematical symbolism. Figure 28 shows a schematic overview of the mechanisms by which students may generate inductive errors when faced with an unfamiliar problem situation: partial matching, mis-specification, and spurious correlation. Each of these mechanisms is discussed below.

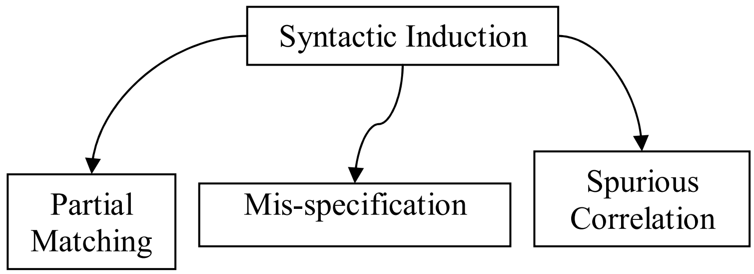


Figure 28: Schematic diagram showing the mechanisms by which syntactic inductive errors occur (Ben-Zeev, 1996, p.70).

Partial Matching

The partial matching mechanism arises when students focus on surface feature similarities between two examples. Research by Hinsley and Hayes (1977) indicates that experts tend categorize physics problems according to the physics principles used to solve the problem, whereas students tend to categorize problems by the objects described in the problem statement. Students' attention to surface feature similarities can be translated into the language of production rules. For a given production rule of the form "*If C, then ?*", if C can be thought of as $C = C_1$ and C_2 and … and C_n, then the students may search for a C_i that partially matches the current problem state, C, and execute the production rule associated with C_i.

Misspecification

The *misspecification* of the constraints of the problem is the second mechanism by which syntactic rational errors occur. In this case, students either use an under- or overspecialized schema for solving problems. For example, students in my study made the following error:

$$F_{el} = k\frac{qq}{r^2} \Rightarrow k\frac{2q}{r^2}.$$

(The "$\Rightarrow$" symbol is used to denote an invalid equality statement.) This error can be understood as the over-generalization of a schema. Matz (1982) says that students generalize the distributive law of multiplication, $A(B+C) = AB + AC$, into the following schema: $\square(x \; \Delta \; y) = \square x \; \Delta \; \square y$, where the symbol Δ can stand for any binary operation. If the students map $\sqrt{\;}$ into $\square$ it leads to the error that $\sqrt{A+B} = \sqrt{A} + \sqrt{B}$. In the example given above, the students generalize the expression $A + A = 2A$ into $A \; \Delta \; A = 2A$, and then map $\times \rightarrow \Delta$ yielding the incorrect conclusion that $A \times A = 2A$.

Spurious correlation

The spurious correlation error occurs when students focus on a particular feature of a problem situation and correlate that feature with a specific algorithm.

An example for the case of subtraction comes from Brown and VanLehn's (1980) repair theory for describing students' erroneous symbolic manipulations. Brown and VanLehn argue that students do not quit when faced with a subtraction problem they don't know how to solve; rather, students create algorithms that help them solve the problems. Often times these algorithms are filled with "bugs" – *i.e.* the algorithms are spuriously correlated with a particular feature of a problem situation. The implementation of these "buggy" algorithms results in erroneous solutions.

VanLehn (1986) identifies the bug N-N-Causes-Borrow, which can be used to explain the following subtraction error:

$$\begin{array}{r} \overset{5}{\not{6}}\,{}^{1}2 \\ -3\;\;2 \\ \hline 2\;1\;0 \end{array}$$

In VanLehn's explanation, a student that commits the N-N-Causes-Borrow bug has correctly learned to borrow when the top digit is less than the bottom digit, and not when the bottom is less than the top. However, when the top and bottom are equal the student incorrectly implements the borrowing procedure, which results in the type of error illustrated above.

Semantic Induction in REASON

Critic-related failures and syntactic inductive errors are the result of erroneous symbolic manipulations. In contrast, semantic inductive errors result from erroneous performance based on conceptual aspects of a problem situation. Ben-Zeev offers two mechanisms by which semantic inductive failures occur (see Figure 29).

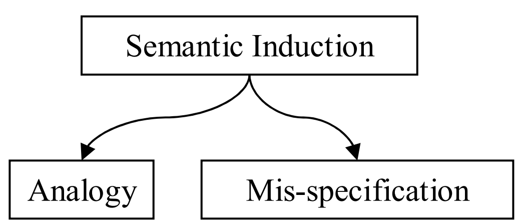


Figure 29: Schematic diagram showing the mechanism for semantic inductive errors (Ben-Zeev, 1996, p.70).

Analogical Breakdown Due to Linguistic Effects

In analogical breakdown the analogy that the student uses to generate a solution may simply lack the necessary features to help the student arrive at the correct answer. An example that Ben-Zeev offers comes from the following algebraic error: $n + mX \Rightarrow (n + m)X$. She claims that students may generate this error by making an analogy from linguistics statements, such as "three apples plus four gives seven apples." In this case the analogy to the linguistic statement breaks down, and leads to an erroneous conclusion.

Analogical Breakdown Generated from Real-World Situations

A second mechanism that Ben-Zeev offers as a semantic inductive error arises from analogy from real-world situations. A common error that students make in mathematics is to conclude that $n^0 = 0$. One reason why students might make this conclusion is that they erroneously conceptualize n^0 as "*n* multiplied by itself zero times, so it has to be zero," since *doing nothing equals nothing*. *Doing nothing equals nothing* is a natural argument that stems from real-world experience, however it leads to erroneous conclusions when used in the conceptualization of n^0.

Discussion about previous research and REASON

Research on understanding students' mathematical errors spans across many different domains of mathematics: addition and subtraction (Carpenter and Moser, 1983; Riley, Greeno, and Heller, 1983; VanLehn, 1983, 1986; Kintsch and Greeno, 1985; Fuson, 1992), multiplication and division (Greer, 1992; Vergnaud, 1983, 1988; Schwartz, 1988), the equals symbol (Herscovics and Kieran, 1980; Kieran, 1981), and algebraic equations (Clement, Lochhead, and Monk, 1981; Matz, 1982; Nathan, Kintsch, and Young, 1992).

In particular, there has been significant progress on understanding students' errors associated with incorrect symbolic manipulations. For instance, I have already discussed two examples from the literature that help us make sense of students' syntactic mathematics errors: Brown and VanLehn's (1980) repair theory for describing students' subtraction errors in terms of "buggy" algorithms (*e.g.* the N-N-Causes-Borrow bug identified by VanLehn, 1986); and, Matz's (1982) explanation of the square root error $\left(i.e.\ \sqrt{A+B} = \sqrt{A} + \sqrt{B}\right)$ in terms of an underspecified schema $\left(i.e.\ \square(x \,\Delta\, y) = \square x \,\Delta\, \square y\right)$ that is incorrectly generalized from the distributive law of multiplication: $A(B+C) = AB + AC$.

These two examples of symbolic mathematical errors are in no way an exhaustive list. I simply include them as representatives of the kinds of explanations that exist in the research literature for describing students' syntactic mathematics errors. For a more thorough review of research on symbolic mathematics errors see Ben-Zeev (1996), and for a general overview of mathematics education research see Reed (1998).

Although there has been significant progress on understanding students' syntactic mathematics errors (*i.e.* the critic-related and syntactic inductive errors in REASON), a comparable understanding of students' conceptual mathematics errors has not been realized (*i.e.* semantic inductive errors in REASON). More germane to the issue of mathematics in physics is the fact that the conceptual mathematics errors in the context of mathematics are different from those in the context of physics. One virtue of the theoretical framework that I propose in this dissertation is that it helps make sense of students' conceptual mathematics errors in the context of physics. In particular, analyzing students' mathematics errors in physics in terms of *resources*, *epistemic games*, and *frames* suggests that most students' errors arise because they fail to use or interpret their mathematics knowledge and skills correctly in the context of physics.

Analysis of students' mathematical errors in terms of *Resources*, *Epistemic Games*, and *Frames*

The framework that I propose has three major theoretical components: *resources*, *epistemic games*, and *frames*. Students' conceptual-mathematical errors usually arise through a complex interplay of all these theoretical constructs. I discuss each of these kinds of errors below.

Errors associated with resources

Students' knowledge base for mathematical thinking and problem solving can be modeled as collections of resources (see chapter 4). In particular, there are four classes of resources that are germane to the issue of mathematics in the context of physics: intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices. There are two mechanisms by which errors associated with resources can arise: (1) the appropriate resource is cued, but the entities in the problem situation are inappropriately mapped into the problem situation; or, (2) an inappropriate resource is cued.

Appropriate resource, but inappropriate mapping

Resources are abstract cognitive structures that are neither right nor wrong. It is not until a resource is mapped into a particular problem situation that the correctness of its usage can be determined. Therefore, errors can occur in which an appropriate resource is activated, yet it is inappropriately mapped into a particular problem situation.

For example, the situation of an object in motion may cue the abstract reasoning primitive of *agent causes effect*. If 'agent' is mapped onto 'force' and 'effect' is mapped onto 'velocity,' then the resulting facet is *force causes velocity* – which is incorrect. However, if 'agent' is mapped onto 'force' and 'effect' is mapped onto 'change in velocity,' then the resulting facet is *force causes changes in velocity* – which is correct. In this example the same abstract reasoning primitive can be mapped into an incorrect (*force causes velocity*) or a correct (*force causes changes in velocity*) facet (see Figure 30).[24]

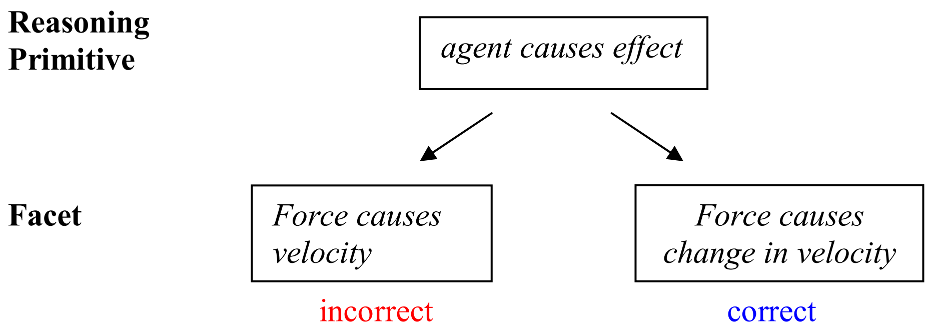


Figure 30. Two possible instantiations of the same abstract reasoning primitive (see Elby, 2001).

An example of an appropriately cued resource, but an inappropriate mapping occurs while Alisa, Bonnie, and Darlene work on the Three Charge Problem (Appendix A, #15). These students correctly realize that the "effect" (force) of q_1 on q_3 must cancel the

[24] The idea that one reasoning primitive can be mapped into a correct or incorrect facet can be used in instruction: Andy Elby uses this idea when developing curriculum (Elby, 2001). Redish calls two different facets that result from the same underlying reasoning primitive *Elby-Pairs*.

"effect" (force) of q_2 on q_3. Therefore, they correctly conclude that since q_1 is farther away that is must have more charge, but they quantify this conclusion incorrectly.

> *Alisa: So, we—we were thinking it was like negative two Q or something like that.*
> *Bonnie: Yeah. Cause it has to be like big enough to push away.*
> *Darlene: Push away q_3.*
> *Bonnie: Yeah, which we—which I figured out negative two.*
> *Darlene: Cause it's twice the distance away than q_2 is?*
> *Bonnie: Yeah.*
> *Darlene: I agree with that.*

It seems these students use the abstract reasoning primitives of *closer means stronger* (or *farther means weaker*) and *prop+*, along with the fact that the distance between q_1 and q_3 is twice as big as the distance between q_2 and q_3, to conclude that the charge on q_1 must be twice as big (and opposite in sign) as the charge on q_2. Although this is a great piece of intuitive reasoning, this is an example of appropriately cued resources that are mapped incorrectly. According to Coulomb's Law $\left(F = \frac{kq_1q_2}{r^2}\right)$ there is not a linear relationship between distance and force. Therefore, while *farther means weaker* and *prop+* are appropriately cued resources in this case, the distance is inappropriately mapped into a linear relationship with the force leading the students to an incorrect (albeit intuitively appealing) conclusion.

<u>Inappropriate resource</u>

Students' conceptual mathematics errors can also occur from an inappropriately cued resource. That is, the resource that is cued cannot be mapped into a useful facet for the particular problem situation under investigation.

An example of an inappropriately cued resource occurs while Alisa and Darlene discuss the Three Charge Problem (Appendix A, #15). Alisa is explaining to Darlene her interpretation of the physical situation, but the activation of the reasoning primitive of *blocking* distracts Darlene:

> *Alisa: Like—q_2 is—q_2 is pushing this way, or attracting--whichever. There's a certain force between two Q, or q_2 that's attracting.*
> *Darlene: q_3.*
> *Alisa: But at the same time you have q_1 repelling q_3.*
> *Darlene: How is it repelling when it's got* **this charge in the middle***?*

In this particular problem situation the activation of *blocking* does not help the students make progress on this problem. The Coulomb force of q_1 on q_3 is not blocked by the presence of q_2. The resource of *blocking* is inappropriately cued. Such an inappropriately cued resource can lead to an error.

Errors associated with epistemic games and frames

Epistemic games and frames can be used to model the process component of students' use of mathematics in physics. Therefore, I call errors associated with epistemic games and frames *process errors*. There are two mechanisms by which process errors may occur: (1) students play the appropriate epistemic game, but make inappropriate move within that game; or, (2) students frame the problem situation inappropriately, and therefore play an inappropriate epistemic game.

<u>Appropriate epistemic game, but wrong move within that game</u>

The major structural component of an epistemic game is the moves. The moves in a particular epistemic game are always the same. For example, there are four moves in *Mapping Mathematics to Meaning*: (1) identify target concept(s), (2) find an equation relating target to other concepts, (3) tell a story using this relationship between concepts, and (4) evaluate story. Although the moves in a particular game are always the same, the particular problem situation and resources that are active can vary from problem to problem resulting different instantiations of a particular epistemic game. Students can play an appropriate epistemic game, but make an inappropriate move along the way because of an inappropriately activated resource. For instance, a student can play an epistemic game that is appropriate for solving a particular problem, but use an inappropriate interpretive device (i.e. make an inappropriate move within an epistemic game), resulting in a process error.

An example of this type of error occurs while Arielle and Tommy work on the Colliding Blocks Problem (Appendix A, #3). At first Arielle plays *Mapping Mathematics to Meaning* to arrive at the correct conclusion:

> *Arielle: So then the Fnet for A, the Fnet for M. This is a big mass and this is a little mass and [the forces] are equal, so this has got to be a big, what is it, a big velocity and this has got to be a small velocity. So, p for A and p for m—the change in velocity here has got to be sort of bigger. Big velocity little mass, big mass little velocity. But these are equal.*
> *Tommy: Right.*
> *Arielle: So the momentums got to be the same right?*

In this case Arielle plays the *Mapping Mathematics to Meaning* epistemic game. She (1) identifies the target concept (the momentum) and (2) finds an equation relating the target quantity to other concepts ($F_{net} = m\frac{\Delta v}{\Delta t}$). Then, (3) she tells the story that since block *m* has a larger mass it must have a smaller velocity. In particular, it appears that Arielle uses the interpretive device of *changing parameters* by considering how the expression for the momenta would change if the velocity and the mass varied. That is, she has the mathematical expression $m^A \Delta v^A = m^B \Delta v^B$ and she considers how changing the value of the parameters on the left will affect the value of the parameters on the right.

Using the *changing parameters* interpretive device within the *Mapping Mathematics to Meaning* epistemic game Arielle correctly concludes that the momenta would have to be the same. Consistent with the fourth move in *Mapping Mathematics to Meaning*, she evaluates her story and is not satisfied with its conclusion. So, she continues to discuss this problem:

> *How could [the momenta] be the same? If the masses are different and the change in velocities are different the momentums can't be the same.*

In this case Arielle is again playing the *Mapping Mathematics to Meaning* epistemic game, but this time it appears that she uses the interpretive device of *feature analysis*. That is, she considers the features of momentum (namely, the velocity and the mass) and concludes that if two momenta have different features than they can't be the same (in much the same way that two faces with different features can't be the same face).

This example illustrates that in the context of an appropriate epistemic game (*Mapping Mathematics to Meaning*) the same student uses an inappropriate interpretive devices (*feature analysis*) leading to the incorrect conclusion ("…the momentums can't be the same."). In the first instantiation of *Mapping Mathematics to Meaning* it appears that *changing parameters* is activated, which leads her to the correct conclusion that the momenta are the same. However, in the second instantiation it appears that *feature analysis* is activated, which leads to the incorrect conclusion that the momenta are different. The epistemic game does not lead to Arielle's incorrect conclusion; it is the particular resource that is activated during that game that leads to the incorrect conclusion.

Inappropriate framing leading to an inappropriate epistemic game

In chapter 5 I discussed how a students' expectations, or *framing*, determine which epistemic game they tacitly choose to play. That is, the entry conditions for a particular epistemic game are determined by how a student frames the particular problem situation. If the student inappropriately frames the problem situation, then it can lead him to play an inappropriate epistemic game.

An example of this type of error occurs while Valerie and Sarah work on the Dorm-Room Pressure Problem (Appendix A, #5).

> *Valerie: Pressure's equal to the radius times the moles of the gas times the temperature divided by the volume. So, what we need to do, we know the pressure find the volume from this. Density is equal to...*
> *Sarah: Are you using PV equals N R T?*
> *Valerie: Huh?*
> *Sarah: Are you using P V equals N R T?*
> *Valerie: Yeah, or yeah.*
> *Sarah: Or.*
> *Valerie: Or P equals R times N T...*
> *Sarah: Over V.*
> *Valerie: Over V.*
> *Sarah: We know the pressure.*
> *Valerie: We know the pressure. But we need to take the density to volume. Density is equal to...*
> *Sarah: Oh, we have the density.*
> *Valerie: Yeah, yeah, but that doesn't matter we need the volume.*

As I discussed in chapter 5, it appears that these students are playing the *Recursive Plug-and-Chug* epistemic game. There are two errors that the students commit in solving this problem. The first, and most obvious, error is that they choose an equation (or relationship) that simply cannot help them solve this problem. The ideal gas law ($PV = nRT$) could help them determine the pressure of the air in the dorm room, but not the *difference* in pressure between the floor and the ceiling. So, we could say that these students are not playing the *Recursive Plug-and-Chug* game well, because they pick an inappropriate equation. The second error is that this is the wrong epistemic game to be playing to solve this problem; slavishly playing this game will not lead to the correct answer. Even if the students had chosen an appropriate equation $\left(e.g.\ P_1 = P_0 + \rho gh\right)$, they could not simply determine the unknowns and solve for the target quantity. At some point the students need to estimate the height of a dorm room, which is not a move within the *Recursive Plug-and-Chug* game.

It's not just the case that the students chose the wrong equation; the problem is worse than that. In fact, these students are stuck in the wrong process – they are playing the wrong epistemic game. The only way these students can solve this problem is if they play a different epistemic game (like *Mapping Mathematics to Meaning*). The reason these students are stuck is that they framed this problem situation inappropriately, which lead them to play an inappropriate epistemic game. This is not to same that *Recursive Plug-and-Chug* is necessarily wrong; it just happens to be the case that in this instance this game does not lead the students to the correct answer.

Conclusions

In this chapter I discussed students' mathematical errors in the context of physics. I introduced Ben-Zeev's taxonomy of mathematical errors, called REASON, and I showed how previous research on mathematical errors helps us understand students' syntactic mathematics errors. Then I showed how the theoretical framework that I propose in this dissertation helps us understand students' semantic mathematics errors in the context of physics. Semantic mathematics errors can be associated with resources, epistemic games, or frames. In particular, I identify four different kinds of errors: (1) inappropriate resource, (2) appropriate resource, but inappropriate mapping, (3) appropriate epistemic game, and (4) inappropriate framing leading to an inappropriate epistemic game.

Chapter 8: Summary and speculations for future research

Summary of cognitive framework

Physics is a difficult subject. Mathematical problem solving in the context of physics has proven to be a considerable challenge for students attempting to learn physics. From this general notion that students do not perform well on mathematical problem solving tasks in physics, I attempt to answer two specific questions: (1) *What are the cognitive tools involved in formal mathematical thinking in physics?* And: (2) *why do students make the kinds of mistakes they do when using mathematics in physics?*

Through observation and analysis of students solving homework problems, I develop a cognitive framework that can be used to analyze and describe students' use and understanding of mathematics in the context of physics. In particular, this cognitive framework can be used to answer questions (1) and (2) from above. That is, this framework helps us understand the *ontological* and *process* components of students' use of mathematics in physics.

The ontological component of students' use of mathematics in physics

<u>Mathematical Resources</u>

In chapter 4 I introduce the notion of mathematical resources (*e.g.* intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices) to address the ontological component of students' use and understanding of mathematics in the context of physics. Mathematical resources are the cognitive tools that are activated in formal mathematical thinking in physics.

Intuitive mathematics knowledge: a collection of primitive cognitive capacities that are required for and involved in advanced and abstract mathematical thinking. I identify four different pieces of intuitive mathematics knowledge from my data: subitizing, counting, pairing, and ordering. *Subitizing* is the ability to distinguish between sets of one, two, and three objects. *Counting* is the ability to enumerate a series of objects. *Pairing* is the ability to group two objects for collective consideration. And lastly, *ordering* is the ability to rank relative magnitudes of mathematical objects.

Reasoning primitives: abstract cognitive elements that describe students' intuitive sense of physical mechanism. Reasoning primitives are abstracted from the notion of *phenomenological primitives* (diSessa, 1993). The appropriate coordination of these abstract cognitive elements can lead to expert understanding. In my data set I identify and discuss four different abstract reasoning primitives: blocking, overcoming, balancing, and more is more. *Blocking* is the abstract notion that inanimate objects are not active agents in any physical scenario. *Overcoming* is the abstract notion that two opposing influences attempt to achieve mutually exclusive results, with one of these influences beating out the other. *Balancing* is the abstract notion that two opposing influences exactly cancel each other out to produce no apparent result. *More is more* is the abstract notion that more of one quantity implies more of a related quantity. This is not an exhaustive list; it represents a sample of a large set of reasoning primitives. There may be dozens of reasoning primitives, but not thousands.

Symbolic forms: are cognitive elements that describe students' intuitive understanding of physics equations. Symbolic forms were introduced by Bruce Sherin (1996). In his dissertation, Sherin identifies 21 different symbolic forms; I discuss three of these symbolic forms that are prevalent in my data set: proportionality plus, balancing, and canceling. *Proportionality plus* is the combination of the abstract notion *more is more* combined with the symbol template of $\square = [\ldots x \ldots]$. The symbolic form of *balancing* combines the abstract notion that two opposing influences are exactly equal with the symbol template $\square = \square$. Lastly, the symbolic form of *canceling* combines the abstract notion that two opposing influences exactly cancel out with the symbol template $\square = \square$.

Interpretive devices: resources that when activated determine how students interpret physics equations. Interpretive devices were also introduced by Bruce Sherin (1996). He identifies three different classes of interpretive devices: narrative, static, and special case. Interpretive devices in the *narrative class* project the physics equation in an imaginary process in which some type of change occurs. The *static class* consists of interpretive devices that map the physics equation into a static situation. Conclusions drawn from interpretive devices in the *special case class* are based on the values of the physics equations being somehow restricted.

I identify a class of interpretive devices that Sherin did not: intuitive class. Interpretive devices in the *intuitive class* are reasoning strategies that are abstracted from everyday reasoning and applied to physics equations. In particular, I identify *feature analysis* and *ignoring* as belonging to the class of intuitive interpretive devices. *Feature analysis* is a reasoning strategy in which one analyzes the features in a physics equation (*e.g.* symbols or terms) – in much the same way that one could analyze the features of two faces (*e.g.* eyes or noses). *Ignoring* is an interpretive strategy in which some aspects of the physics equation are simply ignored.

These four classes of mathematical resources (intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices) represent the cognitive tools that students use during mathematical thinking and problem solving in physics. That is, mathematical resources are my answer to the first research question: W*hat are the cognitive tools involved in formal mathematical thinking in physics?*

The process component of students' use of mathematics in physics

In chapter 5 I introduce epistemic games and frames, which, taken together, help us understand the process component of students' mathematical thinking and problem solving in the context of physics.

<u>Epistemic games</u>

Epistemic games were introduced by Collins and Ferguson (1993) to describe expert scientific inquiry across all scientific disciplines. I generalize Collins' and Ferguson's notion of an epistemic game to be descriptive rather than normative; *i.e.* the epistemic games I identify describe how students actually use mathematics in physics, in contrast to how we would want them to use mathematics in physics. I use the main characteristics that Collins and Ferguson attribute to epistemic games to identify a set of games that introductory, algebra-based physics students play while solving problems in physics.

Epistemic games have ontological and structural components. The ontological components are the knowledge base and the epistemic form. The *knowledge base* is the collection of mathematical resources (*e.g.* intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices) that students use during the epistemic game. The *epistemic form* is a target structure that guides the inquiry in the epistemic game. The structural components of an epistemic game are the entry conditions and the moves. The *entry conditions* are the reasons and conditions that lead students to play a particular epistemic game. (The entry conditions are based on the students' *frames* and *framing*.) The *moves* are the activities that occur during the course of an epistemic game.

I identify six different epistemic games that introductory students play while using mathematics in the context of physics: Mapping Meaning to Mathematics, Mapping Mathematics to Meaning, Physical Mechanism Game, Pictorial Analysis, Recursive Plug-and-Chug, and Transliteration to Mathematics. *Mapping Meaning to Mathematics* is an epistemic game in which students start with a conceptual understanding of a physical situation that they then translate into physics equations. In *Mapping Mathematics to Meaning*, students begin with a physics equation, which they use to make sense of a particular physical situation or physics problem. In the *Physical Mechanism Game* students develop a physical sense of mechanism for a particular physical situation or physics problem based on their intuitive conceptual understanding. *Pictorial Analysis* is an epistemic game in which students create an external representation that captures the spatial relationship between the various (relevant) entities in a physics problem. *Recursive Plug-and-Chug* is an epistemic game that does not involve conceptual understanding; rather, students simply plug numbers or symbols into physics equations, in a recursive manner, to calculate an answer. And lastly, *Transliteration to Mathematics* is an epistemic game in which students use worked examples to generate a solution;

however, they do so without developing a conceptual understanding of the worked example.

Frames

It's not just the elements of a student's knowledge structure that are relevant to understanding their behavior; it's how that knowledge is organized and accessed. A useful structure in helping us understand these issues is framing. A *frame* is an individual's interpretation of a situation or event based on her expectations of the situation or event. That is, frames help the individual answer the question, "what kind of activity is this?"

Frames help us understand why a student plays a particular epistemic game in a particular situation. A student's real-time assessment of a particular problem and/or preconceived epistemological beliefs about physics problem solving in general determine how she interprets that problem – *i.e.* how she frames the problem. For example, if a student reads a problem and determines that the problem is about forces, then she may decide to play *Pictorial Analysis* (*i.e.* draw a free-body diagram) based on her real-time assessment of the problem. Alternatively, if the student has the epistemological belief that problem solving in physics involves plugging in numbers into memorized equations, then she may choose to play *Recursive Plug-and-Chug* – without attempting to understand the problem conceptually.

The ontological and process components help us understand students' mathematical thinking and problem solving

Mathematical resources, epistemic games, and frames, taken together, represent my attempt to answer the second research question: *Why do students make the kinds of mistakes they do when using mathematics in physics?* In chapter 6 I offer an in-depth analysis in terms of mathematical resources, epistemic games, and frames of a one-hour problem solving session. This analysis shows how mathematical resources, epistemic games, and frames offer educators and researchers a technical language capable of describing students' (correct and incorrect) use of mathematics in physics.

In chapter 7 I show how this framework can be used to understand students' semantic math errors in the context of physics. In particular, I identify four different kinds of semantic math errors: (1) an appropriate resource with an inappropriate mapping, (2) an inappropriately cued resource, (3) an appropriate epistemic game with an inappropriate move within that game, and (4) an inappropriate framing leading to an inappropriate epistemic game.

Results of this study

The major result of this dissertation is the construction of a theoretical framework that offers educators and researchers a vocabulary (ontological classification of cognitive structures) and grammar (relationship between the cognitive structures) for understanding the nature and origin of students' use of mathematics in the context physics. The cognitive structures are mathematical resources, and epistemic games and frames describe how students associate and coordinate these mathematical resources when using mathematics in physics.

In addition to offering educators a more thorough understanding of students' use of mathematics, this dissertation synthesizes previous research. The theoretical framework presented here pulls together *phenomenological primitives* (diSessa, 1993), *symbolic forms* and *interpretive devices* (Sherin, 1996), *epistemic games* (Collins and Ferguson, 1993), and *frames* (Goffman, 1974; Tannen, 1993) into one coherent theoretical framework for describing how students' understand and use mathematics in physics.

Instructional implications

Since this theoretical framework offers researchers and educators a more thorough understanding of mathematical thinking and problem solving it has instructional implications. In particular, this theoretical framework can be used as a diagnostic tool, a guide for instructional intervention, or a guide for curriculum development.

A diagnostic tool

One important result of this theoretical framework is that it can help researchers and educators distinguish between seemingly similar expert and novice problem solving behavior. As an example, consider the similarities and differences between *Mapping Mathematics to Meaning* and *Recursive Plug-and-Chug*. Experts often play *Mapping Mathematics to Meaning* while solving problems in the context of physics. There are five moves in this epistemic game: (1) identify the target concept, (2) identify a physics equation relating the target concept to other concepts, (3) develop a conceptual story relating the physical objects in accordance with the physics equation, (4) manipulate the symbols in the equation and solve for the target, and then (5) evaluate the solution. Students often play *Recursive Plug-and-Chug*, which, at first glance, may appear to be the same as *Mapping Mathematics to Meaning*; but, there are some important, subtle differences between the two games. Some of the moves in *Recursive Plug-and-Chug* are similar to the moves in *Mapping Mathematics to Meaning*: (1) identify the target quantity, (2) identify an equation that relates the target quantity to other quantities, and then (3) identify which quantities are known and which quantities are unknown. If the target is the only unknown quantity, then the student can proceed to calculate the target quantity; however, if there are other unknowns, then the student must choose a sub-target and loop back to move (2) mentioned above.

Although some moves in the two games are similar (*i.e.* the structural components of the two games are similar), the mathematical resources activated in the two games are different (*i.e.* the ontological components of the two games are different). While experts play *Mapping Mathematics to Meaning* there are conceptual and epistemological resources that are active that help the experts make sense of the symbolic equations – the experts use the equations to organize and coordinate their conceptual knowledge. In contrast, while playing *Recursive Plug-and-Chug*, the students don't have the conceptual and epistemological resources active that would help them make sense of the mathematical symbolism involved in the physics equations – the students use the equations without making sense of the mathematical symbolism. So, while some of the moves are similar in *Mapping Mathematics to Meaning* and *Recursive Plug-and-Chug* (they both involve the identification of a target concept and an equation, and mathematical manipulations), the cognitive and epistemological resources that are active in the two games are different.

To summarize, the theoretical framework offers educators and researchers a language to help tease apart these seemingly similar behaviors.

A guide for instructional interventions

An investigation of the instructional practices used to teach mathematical problem solving in physics was not the central theme of my dissertation research; however, the cognitive framework that I have developed does have implications for instructional interventions.

For example, since *Recursive Plug-and-Chug* and *Mapping Mathematics to Meaning* have similar structures, instructors of physics may mistake students' novice behavior (*e.g.* playing *Recursive Plug-and-Chug*) with expert-like behavior (*e.g.* playing *Mapping Mathematics to Meaning*) – even though the two behaviors involve different underlying cognitive structures. This mis-diagnosis can have negative ramifications for the students' learning.

If the instructors do not realize that the students are using the mathematical symbolism without conceptual understanding, then they may encourage the students' rote problem solving behavior. That is, research suggests that if students are not corrected early and often enough after making mathematical errors, then they might not develop the appropriate (internal) critics to adequately monitor their own problem solving behavior (Ben-Zeev, 1996). If the instructors don't realize that the students are engaging in rote problem solving behavior, then they won't be able to correct the students appropriately. Without correction from the instructor, the students might not develop the appropriate monitoring skills (or critics) to distinguish between rote symbol manipulation and mathematical problem solving that includes conceptual understanding. Therefore, they may continue using mathematics in physics without conceptual understanding.

Conversely, if an instructor is aware of the difference between these two problem solving behaviors, then she can potentially help those students who engage in rote symbolic manipulations (*e.g.* play *Recursive Plug-and-Chug*) shift to a more conceptually meaningful problem solving approach (*e.g. Mapping Mathematics to Meaning*). Since, according to this theoretical framework, this two problem solving behaviors occur in different frames, the instructor must be able to affect the students' expectations about problem solving in physics – *i.e.* shift the students from a *rote equation chasing* frame to a *quantitative sense making* frame. Exactly how this can be accomplished becomes a research question for a future research project; however, this theoretical framework could serve as a guide for such a research project – which leads into a discussion about physics curriculum.

A guide for physics curriculum

As discussed in chapter 3, understanding the instructional and learning practices is important in any inquiry about mathematical thinking and problem solving. The theoretical framework developed in this dissertation can be used as a guide for researchers and educators attempting to create physics curriculum that could improve students' use of mathematics in physics.

As discussed above, one possible approach to improve students' use of mathematics in physics may be to nudge them from a *rote equation chasing frame* into a *quantitative sense-making* frame. In attempts to achieve this frame shift, Professor Redish implemented two different instructional methods. First, he used class time to model how conceptual information can be interpreted from physics equations. For example, he has written the equations in "idea form." That is, he replaced the algebraic symbols with the semantic content that they represent, so that an equation like $\langle v \rangle = \frac{\Delta x}{\Delta t}$ becomes $(\text{average velocity}) = \frac{(\text{change in position})}{(\text{change in time})}$. Second, he asks both conceptual and quantitative questions, about a given physical situation, in attempts to get the students to coordinate their conceptual and quantitative knowledge while solving physics problems – *i.e.* he attempts to nudge them into a *quantitative sense-making* frame. Preliminary, anecdotal evidence suggests that these changes in the teaching style and curriculum have led to modest improvements.

Future research

Three possible research projects that could derive from this dissertation are extensions of this framework to larger populations, investigations of the cognitive ontology involved in understanding different mathematical objects, and the development of computer models based on this theoretical framework.

Extensions to larger populations

The theoretical framework derives from an investigation of how students in an introductory, algebra-based physics course use and understand mathematics in physics. If I had investigated a different population of students the cognitive structures would most certainly be different. For example, many physics majors have trouble with the transition from introductory physics course, which are generally taken in the freshmen and sophomore years, to the more advanced physics courses that are take in the junior and senior years.   A common belief held by many physics faculty is that the students' difficulties stem from the level of mathematical sophistication required in the junior and senior level physics courses. It is this belief that has led many physics departments to offer mathematical physics courses. For example, the physics department at the University of Maryland offers PHYS 374 entitled "Intermediate Theoretical Methods." Does the students' problem solely lie in the mathematical formalism? Or, could the problem stem from the students' difficulty with mapping physical meaning onto sophisticated mathematics? The work in this dissertation can serve as a potential starting point for investigations to improve the intermediate physics majors' mathematical skills and understanding.

Investigations of the cognitive ontology of different mathematical objects

Physics is a subject matter that uses many different kinds of mathematical objects. There are numbers, variables, vectors, operators, tensors, and matrices to name a few. I call these different mathematical objects the *mathematical ontology* of physics. A possible extension of the theoretical framework developed in this dissertation could include the aspect of the many different mathematical objects used in physics. Since most introductory, algebra-based physics courses only use numbers and variables (and sometimes vectors) this is not the ideal population for examining differences in mathematical ontology. In fact, my data was not rich in student discussions about the mathematical ontology of physics; however, I imagine a quantum mechanics or theoretical dynamics course would have extensive discussions of such topics. These courses could provide sufficient data to extend the theoretical framework I developed in this dissertation, so that it incorporates the mathematical ontology of physics.

Development of computer models based on theoretical framework

In addition to the instructional implications, the cognitive framework developed in this dissertation can be used to create computer models, which could help us better understand students' use of mathematics. I have identified the relevant cognitive structures (mathematical resources) and how they associated and coordinated during mathematical problem solving in physics (epistemic games and frames). My classification of the *ontological* and *process* component of mathematical thinking – in terms of mathematical resources, epistemic games, and frames – can be used as a blueprint in the construction of such computer models.

In particular, these models could be programmed in an object-oriented programming language like C++. The mathematical resources could be created as a class – one class for each kind of mathematical resources (intuitive mathematics knowledge, reasoning primitives, symbolic forms, and interpretive devices). Then epistemic games and frames could become a *derived* class that *inherits* some of the characteristics of the *base* class of mathematical resources. For example, *Mapping Mathematics to Meaning* is an epistemic game that should inherit all the different mathematical resources, whereas *Recursive Plug-and-Chug* should not inherit these mathematical resources – since this game doesn't involve conceptual understanding as discussed above.

These ideas for creating computer programs are in the preliminary stages, however the theoretical framework seems to be a promising place to start in the construction of such models.

Closing remarks

In this dissertation I develop a cognitive framework consisting of three major theoretical constructs: mathematical resources, epistemic games, and frames. I try to show that mathematical resources, epistemic games, and frames give educators and researchers a better understanding of the cognitive structures and processes involved in mathematical thinking and problem solving. Hopefully this increased understanding can lead to instructional interventions, classroom environments, and curriculum that will improve introductory students' use and understanding of mathematics in the context of physics.

Appendix A: Homework Problems

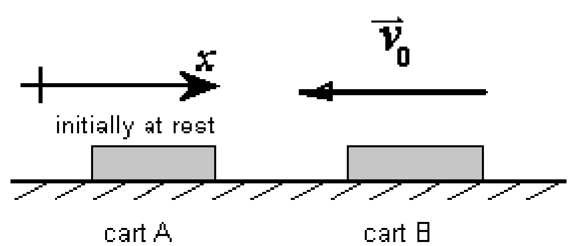
1. Air Drag Problem

For the first part of the problem, let's figure out what the drag force has to look like as a function of the possible variables using dimensional analysis. Consider a sphere of radius R and mass m moving through the air at a speed v. Assume the air has a density ρ (measured in kg/m^3)

- The force the air exerts on the sphere is independent of the sphere's mass. Discuss why this is plausible. (*Hint:* consider the case of the sphere held fixed and the air blowing past it at a speed v.)
- From the quantities R, ρ, and v use dimensional analysis to show that there is only one possible combination of these variables that produces a quantity with the dimension of force.

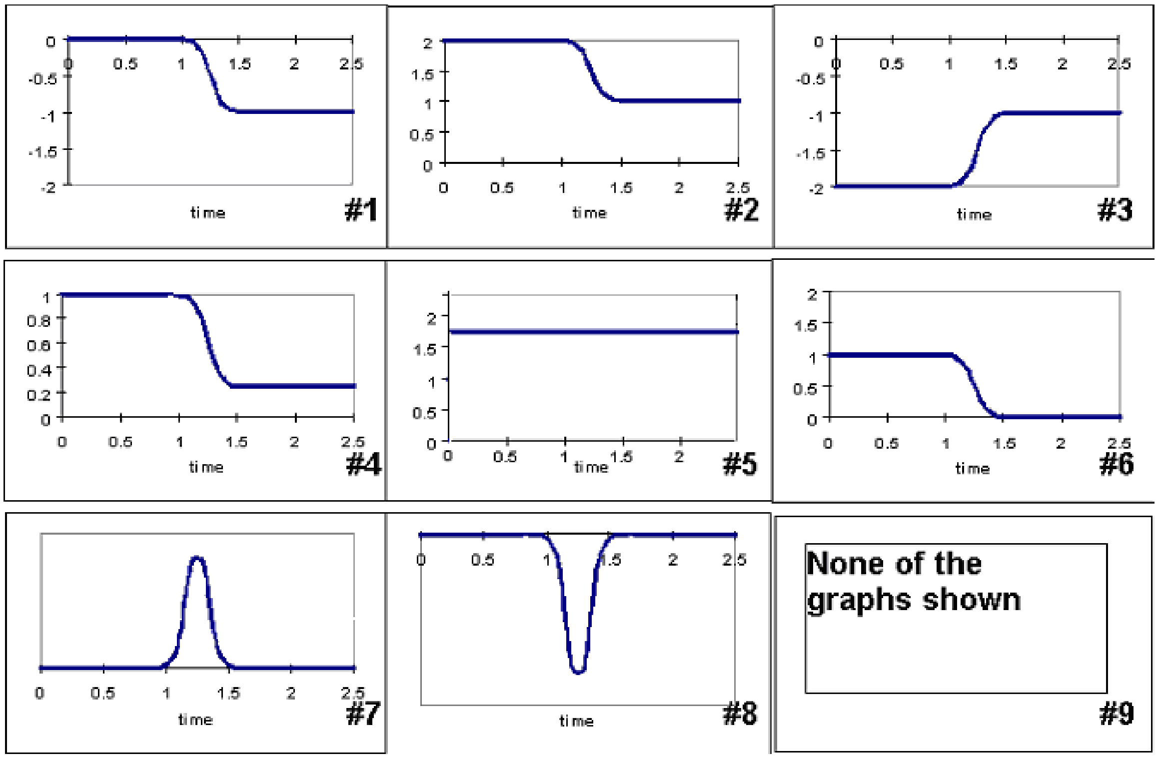
2. Colliding Carts (Representation Translation) Problem

Two identical carts labeled A and B are initially resting one thee air track. The coordinate system for describing the system is shown. The cart on the right, cart B is given a push to the left and is released. The clock is then started. At $t = 0$, cart B moves in the direction shown with a speed v_0. They hit and stick to each other.



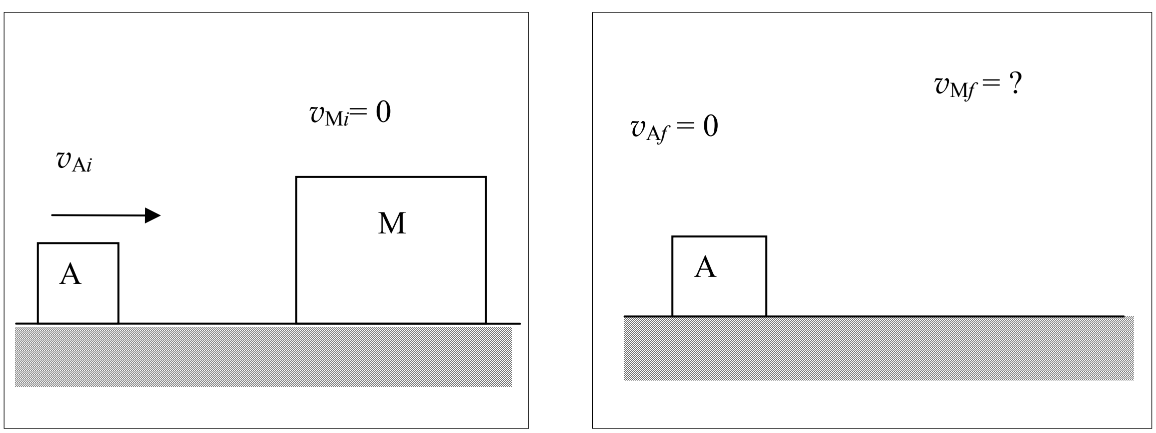
The graphs below describe some of the variables associated with the motion as a function of time. For the experiment described and for each item in the list below, identify which graph is possible display of the variable as a function of time assuming a proper scale. “The system” refers to carts A and B together.

1. the momentum of cart B _______
2. the force of cart A _______
3. the total momentum of the system _______
4. the kinetic energy of cart B _______
5. the total kinetic energy of the system _______



3. Colliding Gliders (Algebraic) Problem

The mass of glider A is one-half that of glider M (i.e. $m_M = 2m_A$). Apply Newton's second law ($F_{net} = m\Delta v/\Delta t$) to each of the colliding gliders to compare the *change in momentum* ($\Delta p = m\Delta v$) of gliders A and M during the collision. Discuss both magnitude and direction. Explain.



4. Conversion Problem

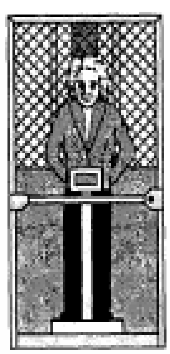
Discuss the question: "Is 500 feet big or small?" Before you do so, carry out the following estimates.

a) You are on the top floor of a 500 ft tall building. A fire breaks out in the building and the elevator stops working. You have to walk down to the ground floor. Estimate how long this would take you. (Your stairwell is on the other side of the building from the fire.)
b) You are hiking the Appalachian Trail on a beautiful Fall morning as part of a 10 mile hike with a group of friends. You are walking along a well-tended, level part of the trail. Estimate how long it would take you to walk 500 feet.
c) You are driving on the New Jersey Turnpike at 65 mi/hr. You pass a sign that says "Lane ends 500 feet." How much time do you have in order to change lanes?

5. Dorm Room Pressure Problem

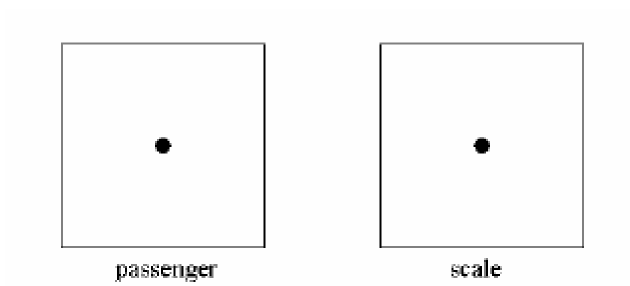
Estimate the difference in pressure between the floor and the ceiling in your dorm room.

6. Elevator Problem



A passenger is standing on a scale in an elevator. The building has a height of 500 feet, the passenger has a mass of 80 kg, and the scale has a mass of 7 kg. The scale sits on the floor of the elevator. You may take g = 10 N/kg.

(a) Draw free-body diagrams for the passenger and the scale while the elevator is sitting at rest on the 33rd floor. Be sure to identify:



(1) the type of force, (2) the object causing the force, and (3) the object feeling the force somewhere in your diagram or labeling. Indicate which (if any) two individual forces in these diagrams have the same magnitude.

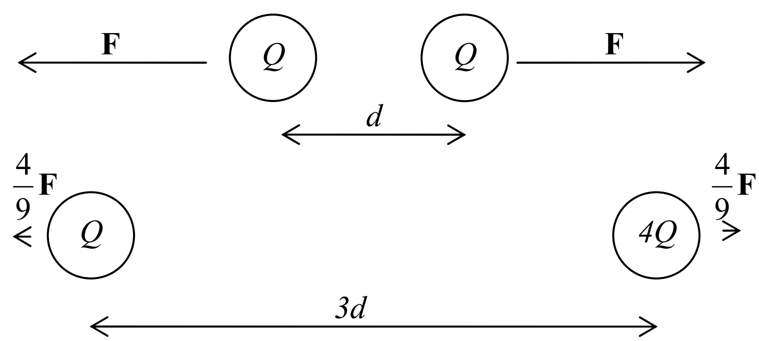
(b) The elevator now begins to descend. Starting from rest, it takes the elevator 6 seconds to get up to its downward speed of 8 m/s. Assuming that it is accelerating downward at a uniform rate during these 6 seconds, which of the forces in your diagram for (a) will change? For each force that changes, specify whether it will become bigger or smaller.

(c) While it is accelerating downward, which of the forces in your diagrams have the same magnitude? For each equality you claim, explain why you think they are equal.

(d) While it is accelerating downward, what does the scale read?

7. Force-Distance Two-Charge Problem

Two small objects each with a net charge of Q (where Q is a positive number) exert a force of magnitude F on each other. We replace one of the objects with another whose net charge is $4Q$. If we move the Q and $4Q$ charges to be 3 times as far apart as they were. Now what is the magnitude of the force on the $4Q$?
(a) F/9 (b) F/3 (c) 4F/9 (d) 4F/3 (e) other



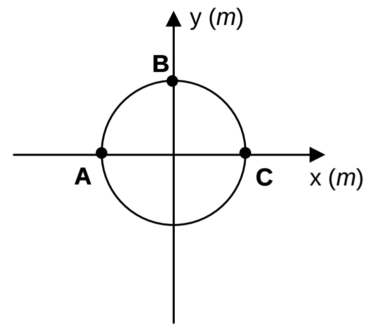
8. Fuel Efficiency Problem

In America, we measure fuel efficiency of our cars by citing the number of miles you can drive on one gallon of gas (mi/gal). In Europe, the same information is given by quoting how many liters of gas it takes to go 100 km (l/100 km).

a. My current car gets 21 mi/gal in highway travel. What number (in li/100 km) should I give to my Swedish friend so that he can compare it to his Volvo?
b. The car I drove in England last summer needed 6 liters of gas to go 100 km. How many mi/gal did it get?
c. If my car has a fuel efficiency, f, in miles/gallon, what is its European efficiency, e, in liters/100 km? (Write an equation that would permit an easy conversion.)

9. Jogger Problem

A jogger runs around a circular track of 30 m radius shown in the figure at the right. She runs at a constant speed in a clockwise direction and completes one lap in 40 seconds. What is her average velocity from A to C?



10. Melting Ice Problem

An experiment on the melting of ice is being done in an insulated calorimeter set up so no heat is exchanged with the outside environment. The calorimeter contains a mass m1 of water and a block of ice having a mass m2, is floating in the water. Both the ice and the water are at a temperature of 0 ºC.

a) What mass of boiling water, m3, must be added to the system to produce only water at 0 ºC? Express your answer in terms of symbols, defining symbols for any heat capacities you require.

b) Suppose m1 = 100 grams, m2 = 25 grams, and I add 50 grams of boiling water. When the system comes to thermal equilibrium, will there be any ice left? If there is none, what will the final temperature of the water be?

The following numbers may be of some use:1 cal/gram-ºC, 80 cal/gram, 540 cal/gram.

11. Paper Towel Problem

In public restrooms there are often paper towel dispensers that require you to pull downward on the towel to extract it. If your hands are wet and you are pulling with one hand, the towel often rips. When you pull with both hands, the towel can be extracted without tearing. Explain why.

12. Paramecium Problem

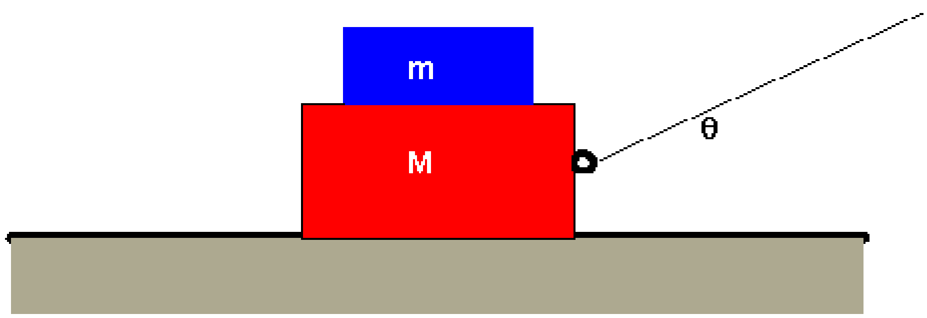
Unicellular organisms such as bacteria and protists are small objects that live in dense fluids. As a result, the resistive force they feel is large and viscous. Since their masses are small their motion looks very different from motion in a medium with little resistance. Paramecia move by pushing their cilia (little hairs on their surface) through the fluid. The fluid (of course) pushes

back on them. We will call this back force of the fluid on the cilia of the paramecium "the applied force," F (since it wouldn't happen if the paramecium didn't try to move its cilia).

- Write Newton's second law for a paramecium feeling two forces: the applied force and the viscous force.
- If the mass is small enough, for most of the time the term "ma" can be much smaller than the two forces, which are large and nearly cancel. Write what the equation for N2 turns into if we ignore the "ma" term. Describe what the motion would be like and how it would appear different from a low or no resistance example.

Suppose the paramecium is starting from rest and starts to move, coming quickly to a constant velocity. Describe how the three terms in the full N2 equation behave, illustrating your discussion with graphs of x, v, a, F_{net}, F, and F_{viscous}.

13. Pulling Two Boxes



(a) A worker is pulling a pair of heavy crates along the floor with a rope. The rope is attached to the lower crate, which has a mass M. The upper crate has a mass m and the coefficient of friction between the crate and the floor is μ. If the rope is held at an angle θ, what is the maximum force F that can be exerted without the box beginning to slide?

(b) The worker knows that the lower crate has a mass of 50 kg and the upper a mass of 10 kg. She finds that if she pulls with a force of 120 N at an angle of 60º she can keep the crates sliding at a constant speed. Can you use this information to find the coefficient of friction between the lower crate and the floor? If you can, do it. If you can't, explain why not.

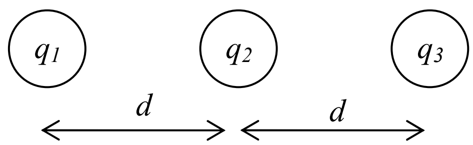
(c) In a different situation, she finds that she can pull a lower crate of mass 30 kg and an upper crate of mass 7.5 kg with a constant velocity of 50 cm/s pulling at an angle of 45º. Can you use this information to find the coefficient of friction between the lower crate and the floor? If you can, do it. If you can't, explain why not.

14. Speed versus Pace Problem

When we drive a car we usually describe our motion in terms of a speed or velocity. A speed limit, such as 60 miles/hr, is a speed. When runners or joggers describe their motion, they often do so in terms of a *pace* — how long it takes to go a given distance. A 4-minute mile (or better, "4 minutes / mile") is an example of a pace.

a) Express the speed 60 mi/hr as a pace in minutes/mile.
b) I walk on my treadmill at a pace of 17 minutes/mile. What is my speed in miles/hour?
c) If I travel at a speed, v, given in miles/hr, what is my pace, p, given in minutes/mile? (Write an equation that would permit easy conversion.)

15. Three Charge Problem



In the figure above three charged particles lie on a straight line and are separated by distances d. Charges q_1 and q_2 are held fixed. Charge q_3 is free to move but happens to be in equilibrium (no net electrostatic force acts on it). If charge q_2 has the value Q, what value must the charge q_1 have?

Appendix B: List of Epistemic Games

QUANTITATIVE SENSE MAKING FRAME

Mapping Meaning to Mathematics (p. 106)

Description: Translation of conceptual understanding of a situation into rigorous, quantitative mathematical expression. This game involves identifying and naming mathematical entities (numbers, constants, variables, etc.) and expressing their relations to each other mathematically.

Identification: Usually occurs after PICTORIAL ANALYSIS or PHYSICAL MECHANISM GAME, since the students need to have a conceptual understanding of the physical situation before they can play this game.

Moves: See Figure 31.

Knowledge Base: All mathematical resources.

Epistemic form: Mathematical expressions relating physical quantities.

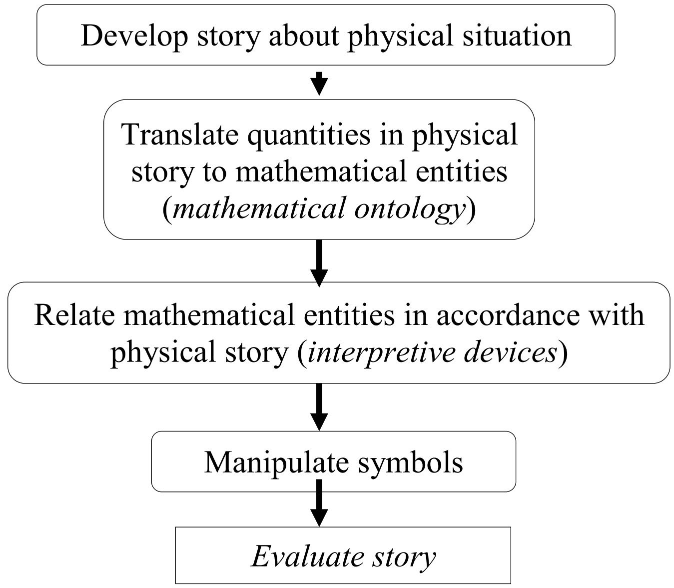


Figure 31. Schematic diagram of students' moves with

Mapping Meaning to Mathematics.

Mapping Mathematics to Meaning (p. 114)

Description: Students begin with a physics equation, and then develop a conceptual story.

Identification: Analysis involves formal mathematical expressions. The mathematical expression is usually written, however explicit verbal reference to a formal mathematical expression can occur.

Moves: See Figure 32.

Knowledge Base: All mathematical resources.

Epistemic forms: The equation, which is found in move (2), that relates the target to other concepts.

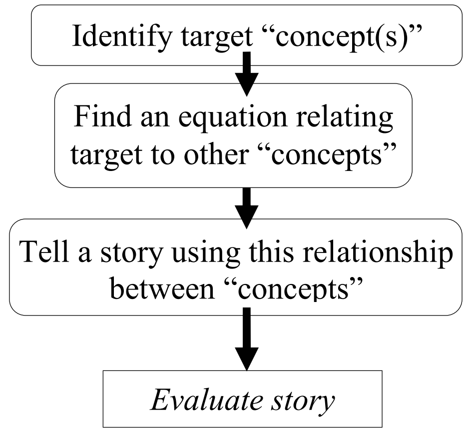


Figure 32. Schematic diagram of students' moves with

Mapping Mathematics to Meaning.

QUALITATIVE SENSE MAKING FRAME

Physical Mechanism Game (p. 120)

Description: Analysis of physical phenomena involving only "common-sense" reasoning.

Identification: Utterances involve common speech, without any reference to formal mathematics/physics principles or reference to formal mathematical machinery to support conclusions.

Moves: See Figure 33.

Knowledge Base: Reasoning primitives and intuitive mathematics knowledge.

Epistemic forms: A coherent, physical description; verbal or imagistic.

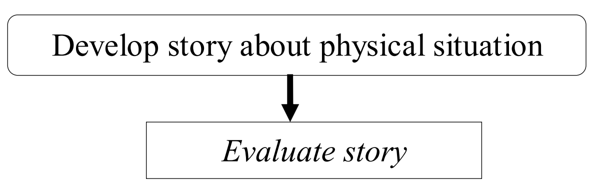


Figure 33. Schematic diagram of students' moves within the

Physical Mechanism Game.

Pictorial Analysis Game (p. 124)

Description: Analysis of quantities or entities in a problem in terms of their spatial relation to each other, involving explicit generation or use of an external representation. Similar to PHYSICAL MECHANISM, however the spatial relation of the quantities *must* be specified. Physical situation must be simplified, resulting in some information being ignored. This game does not include mathematical expressions.

Identification: Students will identify the entities, by either gesturing or making reference to an external representation, and then articulate how these entities are spatially related to each other.

Moves: See **Error! Reference source not found.**.

Knowledge Elements: Reasoning primitives, intuitive mathematics knowledge, and syntactic knowledge.

Epistemic forms: The external representation that is generated or referenced (*e.g.* a free-body diagram, a circuit diagram, or a cartoon picture).

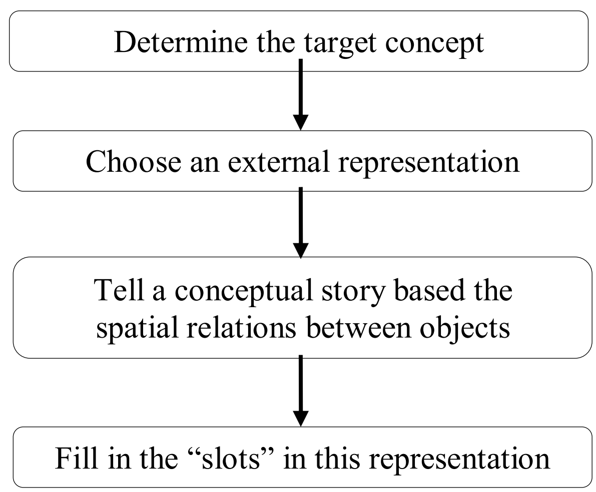


Figure 34. Schematic diagram of the moves in *Pictorial Analysis*.

ROTE EQUATION CHASING FRAME

Recursive Plug-and-Chug (p. 127)

Description: Rote application of the "plug-and-chug" problem solving method.

Identification: Students use the mathematical machinery without understanding of concepts that the symbols represent.

Moves: See Figure 35.

Knowledge Elements: Intuitive mathematics knowledge and symbol template.

Epistemic form: Standard solution form found at the back of most textbooks.

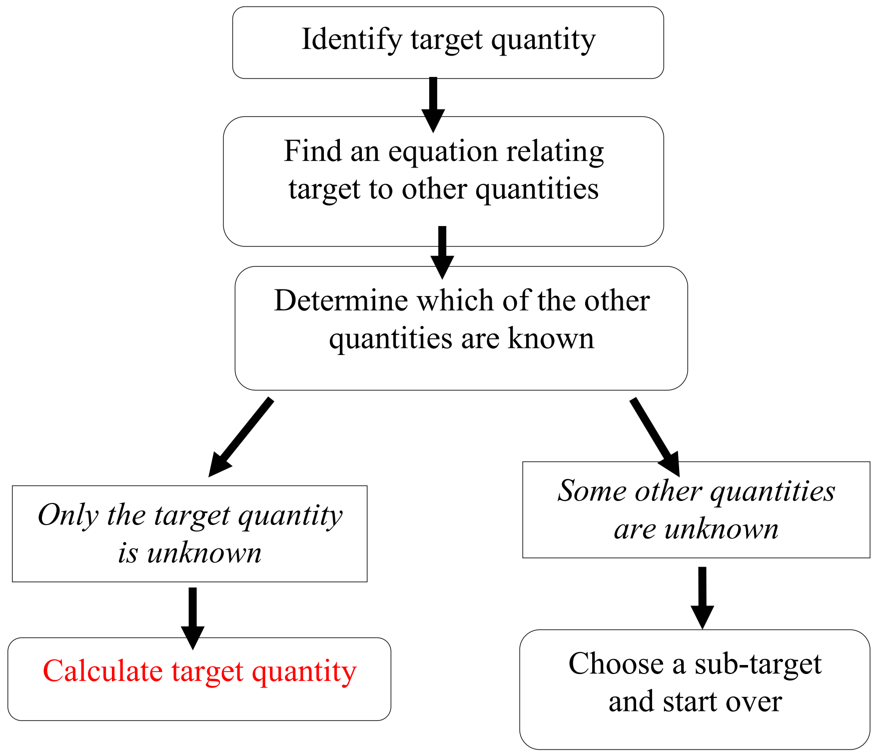


Figure 35. Schematic diagram of students' moves within

Recursive Plug-and-Chug.

Transliteration to Mathematics (p. 131)

Description: Directly mapping solution from reference example to a target example. This game works very well when reference and target examples are isomorphic.

Identification: Often students will play this game without conceptual understanding; instead they focus attention on syntactic similarities between the reference and target examples.

Moves: See Figure 36.

Knowledge Elements: Intuitive mathematics knowledge and symbol template.

Epistemic form: The solution pattern from the reference example.

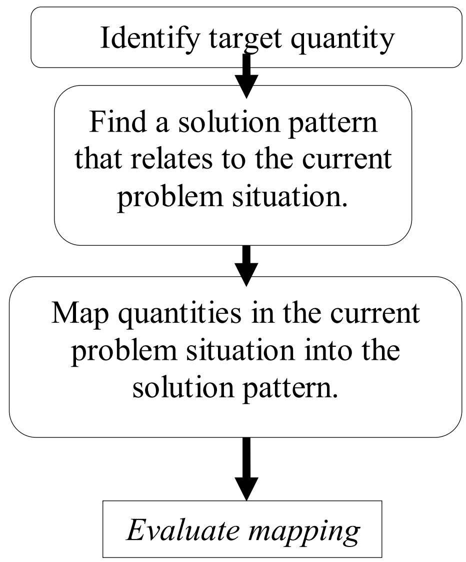


Figure 36. Schematic diagram of the moves in

Transliteration to Mathematics.

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