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Research article

Fractional infinite time-delay evolution equations with non-instantaneous impulsive

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Abstract: This dissertation is regarded to investigate the system of infinite time-delay and non-instantaneous impulsive to fractional evolution equations containing an infinitesimal generator operator. It turns out that its mild solution is existed and is unique. Our model is built using a fractional Caputo approach of order lies between 1 and 2. To get the mild solution, the families associated with cosine and sine which are linear strongly continuous bounded operators, are provided. It is common to use Krasnoselskii's theorem and the Banach contraction mapping principle to prove the existence and uniqueness of the mild solution. To confirm that our results are applicable, an illustrative example is introduced.

Keywords: Caputo fractional derivative; infinite time-delay; mild solution; non-instantaneous impulsive; Krasnoselskii's theorem
Mathematics Subject Classification: 34A08, 34A12, 34A60, 34G99, 34K99

1. Introduction

Fractional differential equations are becoming a considerably more important and popular topic. In order to specify the so-called fractional differential equations, the conventional integer order derivative is generalized to arbitrary order. Fractional differential equations have been extensively employed to explain a variety of physical processes because of the effective memory function of the fractional derivative, such as seepage flow in porous media, fluid dynamic, and traffic models. Also, there are several applications of fractional differential equations in control theory, polymer rheology, aerodynamics, physics, chemistry, biology, and other exciting conceptual advancements (see [1–4] and their references).

In the real world, there are numerous processes and phenomena that are influenced by transient external factors as they evolve. When compared to the whole duration of the occurrences and processes being researched, their duration is tiny. As a result, it is reasonable to believe that these exterior impacts are "instantaneous", or take the shape of impulses. Differential equations including the impulse effect, or impulsive differential equations, seem to be a plausible explanation of the known evolution processes of various real-world issues. The impulsive differential equations have been studied as the subject of numerous excellent monographs [5, 6].

Differential equations are used as representations for many processes in applied sciences research. There are a variety of classical mechanics that experience abrupt changes in their states, such as biological systems (heartbeats and blood flow), mechanical systems with impact, radiophysics, pharmacokinetics, population dynamics, mathematical economics, ecology, industrial robotics, biotechnology processes, etc [7, 8]. The systems of differential equations with impulses are suitable mathematical models for such phenomena. Impulsive differential equations essentially have three parts: An impulse equation simulates an impulsive leap that is described by a jump function at the moment of impulse occurs, a continuous-time differential equation determines the state of the system between impulses, and jump criteria identifies a set of jump occurrences [9–11].

Furthermore, there are various models that have been developed in many fields like biology, economics, and materials science where the rate of change at time t depends not only on the system's current state but also on its history over a period of time $[t-\tau, t]$ [12–15]. These models have evolutionary equations with delay, which describe them mathematically. Equations with infinite delay are produced by the more generic if we take $\tau = \infty$.

Physics provides a compelling justification for studying the nonlocal partial differential equation. Fractional derivatives in space and time are used in abstract partial differential equations such as fractional diffusion equations. They may be used to simulate anomalous diffusion, in which a particle plume spreads differently than the traditional diffusion equation would suggest. By replacing the second-order space derivative in the classic diffusion equation by an infinitesimal generator operator of strongly continuous C_0 semi-group or cosine functions, the time fractional evolution equation is derived [16–18].

In [19], Kumar and Pandey attempted to examine the results of the existence of a solution to a class of FDEs (the fractional calculus due to Atangana-Baleanu) of the sort

$$\begin{cases} {}^{ABC}D^{\rho}\mathfrak{v}(r) = \mathcal{F}\mathfrak{v}(r) + \mathfrak{h}(r, \mathfrak{v}(r)), & r \in \cup_{j=0}^{m}(s_j, r_{j+1}], \\ \mathfrak{v}(r) = \delta_j(r, \mathfrak{v}(r)), & r \in \cup_{j=1}^{m}(r_j, s_j], \\ \mathfrak{v}(0) = \mathfrak{v}_0 - f(\mathfrak{v}), \end{cases}$$

for all $\mathfrak{v} \in D(\mathcal{F})$ (the domain of $\mathcal{F}$), where ${}^{ABC}D^{\rho}$ is the ABC fractional derivative of order $\rho \in (0, 1)$, $\mathcal{F}: D(\mathcal{F}) \subset \mathrm{X} \to \mathrm{X}$ is a generator of ρ-resolvent operator $\{S_{\rho}(r)\}_{r\geq 0}$ on a Banach space $(\mathrm{X}, \|\cdot\|)$, $J = [0, d]$, $0 = r_0 < r_1 < r_2 < \cdots < r_m < r_{m+1}$ and $s_j \in (r_j, r_{j+1})$ for all $j = 1, 2, \ldots, m$; $m \in \mathbb{N}$. The functions $\mathfrak{h}: \cup_{k=0}^{m}(s_j, r_{j+1}] \times \mathrm{X} \to \mathrm{X}$ and $f: \mathrm{X} \to \mathrm{X}$ are given continuous functions and $\delta_j: (r_j, s_j] \times \mathrm{X} \to \mathrm{X}$ are non-instantaneous impulsive functions for each $j = 1, 2, \ldots, m$; $m \in \mathbb{N}$ and $\mathfrak{v}_o \in \mathrm{X}$.

Recently, Kavitha et al. [20] examined the existence of solutions for a class of non-instantaneous impulses and infinite delay of fractional differential equations within the Mittag-Leffler kernel of the kind

$$\begin{cases} {}^{ABC}D^{\mathfrak{v}} p(\xi) = Bp(\xi) + \mathcal{F}(\xi, p_\xi), & \xi \in \cup_{l=0}^{m} (s_l, \xi_{l+1}], \\ p(\xi) = \kappa_l(\xi, p_\xi), & \xi \in \cup_{l=1}^{m} (\xi_l, s_l], \\ p(\xi) = \phi(\xi), & \xi \in (-\infty, 0], \end{cases}$$

where the fractional order $\mathfrak{v} \in (0,1)$, $B\colon D(B) \subset E \to E$ is infinitesimal generator of an ρ-resolvent operator $\{S_\rho(\xi)\}_{\xi \geq 0}$ on a Banach space $(E, \|\cdot\|$, $K = [0, b]$, $0 = \xi_0 = s_0 < \xi_1 \leq s_1 < \xi_2 < \cdots < \xi_q < \xi_{q+1} = b$ and $s_l \in (\xi_l, \xi_{l+1})$ for all $l = 1, 2, \ldots, q; q \in \mathbb{N}$. The function $\mathcal{F}\colon \cup_{l=0}^{q} (s_l, \xi_{l+1}] \times \mathcal{A} \to E$ satisfies Caratheodory conditions and the functions $\kappa_l\colon (\xi_l, s_l] \times \mathcal{A} \to E$ are non-instantaneous impulsive functions for each $l = 1, 2, \ldots, q$. They considered that $p_\xi\colon (-\infty, 0] \to E$ such that $p_\xi(x) = p(\xi + x)$ for all $x \leq 0$ and $\phi \in \mathcal{A}$ where $\mathcal{A}$ is an abstract phase space.

In light of the foregoing, in this publication, we examine the existence results for a class of fractional-order non-instantaneous impulses functional evolution equations.

Consider the fractional semilinear evolution of the form

$$\begin{cases} u(t) = \phi(t), & t \in (-\infty, 0], \\ {}^{c}D_t^{\alpha} u(t) = Au(t) + h(t, u_t), & t \in \cup_{k=0}^{m} (s_k, t_{k+1}], \\ u(t) = \mu_k(t, u(t)), & t \in \cup_{k=1}^{m} (t_k, s_k], \\ u'(t) = \xi_k(t, u(t)), & t \in \cup_{k=1}^{m} (t_k, s_k], \\ u'(0) = u_0, \end{cases} \tag{1.1}$$

where ${}^{c}D_t^{\alpha}$ is the fractional derivative due to Caputo of order $1 < \alpha < 2$ and $J = [0, a]$ is operational interval. Here, $h\colon J \times \mathcal{P}_{\mathfrak{h}} \to \mathbb{X}$ is a given function satisfying some assumptions that will be determined later, where $\mathcal{P}_{\mathfrak{h}}$ is an abstract phase space and $\mathbb{X}$ is a Banach space. The functions $\mu_k,\ \xi_k \in C((t_k, s_k] \times \mathbb{X}; \mathbb{X})$ for all $k = 1, 2, \ldots m;\ m \in \mathbb{N}$ reflect the impulsive circumstances and $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_m \leq s_m \leq t_{m+1} = a$ are pre-fixed numbers. The history function $u_t\colon (-\infty, 0] \to X$ is an element of $\mathcal{P}_{\mathfrak{h}}$ and defined by $u_t(\theta) = u(t + \theta), \theta \in (-\infty, 0]$.

The closed operator A is an infinitesimal generator of a uniformly bounded family of strongly continuous cosine operators $\{\Re(t)\}_{t \in \mathbb{R}}$, which is defined on a Banach space $\mathbb{X}$. The Banach space of continuous and bounded functions from $(-\infty, a]$ into $\mathbb{X}$ provided with the topology of uniform convergence is denoted by $C = C_a((-\infty, a], \mathbb{X})$ with the norm

$$||u||_C = \sup_{t \in (-\infty, a]} |u(t)|.$$

As $\{\Re(t)\}_{t \in \mathbb{R}}$ is a cosine family on $\mathbb{X}$, then there exists $\varpi \geq 1$ [21] such that

$$||\Re(t)|| \leq \varpi. \tag{1.2}$$

The rest of the text is structured as follows. In section 2, we give some fundamental concepts and lemmas related to our study. In section 3, we formulate the mild solution of (1.1) by considering that operator A is an infinitesimal generator of strongly continuous cosine functions $\{\Re(t)\}_{t \in \mathbb{R}}$. By using the fixed-point theorem, Section 4 presents our study outcomes. An instance is provided in Section 5 to be an application.

2. Preliminaries

In this section, we present some concepts and definitions related to the components of the research paper such as fractional calculus, cosine and sine families operators, and abstract phase space. Also, some lemmas that give helpful results to prove the main results of this contribution, are provided.

2.1. Fractional calculus

The definitions of R-L integral and Caputo derivative and the important related lemma are introduced as follow.

Definition 2.1. *(Caputo derivative [22]) Let* $\ell - 1 < \mathfrak{q} \leq \ell$; $\ell \in \mathbb{N}$ *and* $x\colon [a, b] \to \mathbb{R}$ $(-\infty < a < b < \infty)$ *be nth continuously differentiable function. Then, the left derivative of fractional order* $\mathfrak{q}$ *due to Caputo is presented as*

$$ {}_cD_a^{\mathfrak{q}}x(s) = \frac{1}{\Gamma(\ell - \mathfrak{q})}\int_a^s (s-t)^{\ell-\mathfrak{q}-1}x^{(\ell)}(t)dt, \quad s \in [a, b]. $$

Definition 2.2. *(Riemann-Liouville fractional integral [22]) The left R-L fractional integral of the integrable function x over the interval* $[a, b]$ *is derived as*

$$ I_a^{\mathfrak{q}}x(s) = \frac{1}{\Gamma(\mathfrak{q})}\int_a^s (s-t)^{\mathfrak{q}-1}\, x(t)dt, \quad \mathfrak{q} > 0,\ s \in [a, b]. $$

Lemma 2.1. *[23] Let* $\ell \in \mathbb{N}$, $\ell - 1 < \mathfrak{q} \leq \ell$ *and* $x(s)$ *be nth continuously differentiable function over the interval* $[a, b]$*. Then,*

$$ I_a^{\mathfrak{q}}\ {}_cD_a^{\mathfrak{q}}x(s) = x(s) + a_0 + a_1(s-a) + \cdots + a_{\ell-1}(s-a)^{\ell-1}, \quad s \in [a, b]. $$

Definition 2.3. *(Atangana-Baleanu fractional derivative in Caputo sense [19]) ABC-derivative for the order* $\alpha \in [0, 1]$ *and* $x(s) \in H^1(a, b), a < b$ *is given by*

$$ {}^{ABC}D^{\alpha}x(s) = \frac{M(\alpha)}{(1-\alpha)}\int_a^s E_{\alpha}\left[-\frac{\alpha(s-r)^{\alpha}}{1-\alpha}\right]x'(r)dr, $$

where $E_{\alpha}(\cdot)$ *and* H^1 *are the Mittag-Leffler function and the non-typical Banach space defined, respectively, as*

$$ E_{\alpha}(z) = \sum_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i + 1)}, \qquad \Re(\alpha) > 0, z \in \mathbb{C}, $$
$$ H^1(\Omega) = \left\{\eta(s)|\eta(s),\ D^{\rho}\eta(s) \in L^2(\Omega),\ \forall\rho \leq 1\right\}. $$

2.2. Cosine family operator

Let A be an infinitesimal generator of a uniformly bounded family of strongly continuous cosine operators $\{\Re(t)\}_{t\in\mathbb{R}}$ which is defined on a Banach space $\mathbb{X}$. We collect some basic properties of a cosine family and its relations with the operator A and the associated sine family.

Definition 2.4. *[24] Consider $\{\mathfrak{R}(t)\}_{t\in\mathbb{R}}$ is a one parameter family of bounded linear operators mapping the Banach space $\mathbb{X} \to \mathbb{X}$. It is referred to a strongly continuous cosine family if and only if*

(i) $\mathfrak{R}(0) = I$;

(ii) $\mathfrak{R}(s + t) + \mathfrak{R}(s - t) = 2\mathfrak{R}(s)\mathfrak{R}(t)$ *for all* $s, t \in \mathbb{R}$;

(iii) *The function* $t \mapsto \mathfrak{R}(t)x$ *is a continuous on* $\mathbb{R}$ *for any* $x \in \mathbb{X}$.

The sine family $\{\mathfrak{T}(t)\}_{t\in R}$ is correlated to the strongly continuous cosine family $\{\mathfrak{R}(t)\}_{t\in\mathbb{R}}$, it is characterized by

$$\mathfrak{T}(t)x = \int_0^t \mathfrak{R}(s)xds, \quad x \in \mathbb{X}, t \in \mathbb{R}.$$

Lemma 2.2. *[24] Consider A is an infinitesimal generator of a strongly continuous cosine family $\{\mathfrak{R}(t)\}_{t\in\mathbb{R}}$ on Banach space $\mathbb{X}$ such that $\|\mathfrak{R}(t)\| \leq Me^{\xi|t|}, t \in \mathbb{R}$. Then, for $\lambda > \xi$ and $(\xi^2, \infty) \subset \rho(A)$ (the resolvent set of A), we have*

$$\lambda R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t}\mathfrak{R}(t)xdt, \qquad R(\lambda^2; A)x = \int_0^\infty e^{-\lambda t}\mathfrak{T}(t)xdt, \quad x \in \mathbb{X},$$

where the operator $R(\lambda; A) = (\lambda I - A)^{-1}$ is the resolvent of the operator A and $\lambda \in \rho(A)$.

In this case, the operator A is defined by

$$Ax = \lim_{t\to 0} \frac{d^2}{dt^2}\mathfrak{R}(t)x, \qquad \forall \quad x \in \mathcal{D}(A),$$

where $\mathcal{D}(A) = \{x \in \mathbb{X} : \mathfrak{R}(t)x \in C^2(\mathbb{R}, \mathbb{X})\}$ is the domain of the operator A. Clearly the infinitesimal generator A is densely defined operator in $\mathbb{X}$ and closed.

In the sequel to present our results, we need the following:

Definition 2.5. *Suppose that $\tau > 0$, the Mainardi's Wright-type function is defined as*

$$M_\varrho(\tau) = \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!\Gamma(1 - \varrho(n + 1))}, \quad \varrho \in (0, 1), \quad \tau \in \mathbb{C},$$

and achieves

$$M_\varrho(\tau) \geq 0, \qquad \int_0^\infty \theta^\xi M_\varrho(\theta)d\theta = \frac{\Gamma(1 + \xi)}{\Gamma(1 + \varrho\xi)}, \qquad \xi > -1.$$

2.3. Abstract phase space $\mathcal{P}_{\mathfrak{h}}$

The abstract phase space $\mathcal{P}_{\mathfrak{h}}$ is demonstrated by convenient way [25,26]. Let $\mathfrak{h} = C((-\infty, 0], [0, \infty))$ with $\int_{-\infty}^0 \mathfrak{h}(t)dt < \infty$. Then, for any $c > 0$, we can define the set

$$\mathcal{P} = \{\mathfrak{A} : [-c, 0] \to \mathbb{X} \ , \quad \mathfrak{A} \text{ is bounded and measurable}\},$$

and establish the space $\mathcal{P}$ with the norm

$$\|\mathfrak{A}\|_{\mathcal{P}} = \sup_{s\in[-c,0]} |\mathfrak{A}(s)|, \quad \text{for all} \quad \mathfrak{A} \in \mathcal{P}.$$

Let us define the space

$$\mathcal{P}_{\mathfrak{h}} = \Big\{ \mathfrak{A} \colon (-\infty, 0] \to \mathbb{X} \text{ such that for any } \ c > 0, \mathfrak{A}|_{[-c,0]} \in \mathcal{P} \text{ and } \int_{-\infty}^{0} \mathfrak{h}(t) \sup_{t \le s \le 0} \mathfrak{A}(s) dt < \infty \Big\}.$$

If $\mathcal{P}_{\mathfrak{h}}$ is furnished with the norm

$$\|\mathfrak{A}\|_{\mathcal{P}_{\mathfrak{h}}} = \int_{-\infty}^{0} \mathfrak{h}(t) \sup_{t \le s \le 0} \|\mathfrak{A}(s)\| dt, \quad \forall\, \mathfrak{A} \in \mathcal{P}_{\mathfrak{h}},$$

then $\left(\mathcal{P}_{\mathfrak{h}}, \|\cdot\|_{\mathcal{P}_{\mathfrak{h}}}\right)$ is a Banach space. Next, we introduce the available space

$$\overline{\mathcal{P}}_{\mathfrak{h}} = \Big\{ v \colon (-\infty, a] \to \mathbb{X} \ \text{ such that } v|_{[0,a]} \in C((t_k, t_{k+1}], \mathbb{X}), \ v|_{(-\infty,0]} = \phi \in \mathcal{P}_{\mathfrak{h}} \Big\},$$

which has the norm

$$\|x\|_{\overline{\mathcal{P}}_{\mathfrak{h}}} = \sup_{s \in [0,a]} \|v(s)\| + \|\phi\|_{\mathcal{P}_{\mathfrak{h}}}, \qquad x \in \overline{\mathcal{P}}_{\mathfrak{h}}.$$

Definition 2.6. *[27] If $v \colon (-\infty, a] \to \mathbb{X}$, $a > 0$, such that $\phi \in \mathcal{P}_{\mathfrak{h}}$. The situations listed below are accurate for all $\tau \in [0, a]$,*

1) $v_\tau \in \mathcal{P}_{\mathfrak{h}}$*;*

2) *Two functions, $\zeta_1(\tau), \zeta_2(\tau) > 0$, are such that $\zeta_1(\tau) \colon [0, \infty) \to [0, \infty)$ is a continuous function and $\zeta_2(\tau) \colon [0, \infty) \to [0, \infty)$ is a locally bounded function which are independent to $v(\cdot)$ whereas*

$$\|v_\tau\|_{\mathcal{P}_{\mathfrak{h}}} \le \zeta_1(\tau) \sup_{0 < s < \tau} \|v(s)\| + \zeta_2(\tau) \|\phi\|_{\mathcal{P}_{\mathfrak{h}}};$$

3) $\|v(\tau)\| \le H \|v_\tau\|_{\mathcal{P}_{\mathfrak{h}}}$*, where $H > 0$ is a constant.*

3. Structure of mild solution

Before introducing the mild solution of evolution Eq (1.1), we have to establish the following Lemmas.

Lemma 3.1. *Let I_s^α be the left R-L integral of order α and $f(t)$ is integrable function defined for $t \ge s \ge 0$. Then,*

$$\int_s^\infty e^{-\lambda t} I_s^\alpha f(t) dt = \lambda^{-\alpha} \int_s^\infty e^{-\lambda t} f(t) dt.$$

Proof. From the Definition 2.2 and the rule of converting double integral to single integral, we get

$$\begin{aligned}
\int_s^\infty e^{-\lambda t} I_s^\alpha f(t) dt &= \int_s^\infty e^{-\lambda t} \int_s^t (t-\tau)^{\alpha-1} f(\tau) d\tau dt \\
&= \frac{1}{\Gamma(\alpha)} \int_s^\infty f(\tau) d\tau \int_\tau^\infty e^{-\lambda t} (t-\tau)^{\alpha-1} dt \\
&= \frac{1}{\Gamma(\alpha)} \int_s^\infty f(\tau) e^{-\lambda \tau} d\tau \int_0^\infty t^{\alpha-1} e^{-\lambda t} dt \\
&= \lambda^{-\alpha} \int_s^\infty e^{-\lambda t} f(\tau) d\tau.
\end{aligned}$$

The proof is over.

Lemma 3.2. *Let $1 < \alpha \le 2$ and $h\colon J \to X$ be an integrable function. Then, the mild solution to our problem* (1.1) *possess the form*

$$u(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \Re_q(t)\phi(0) + \int_0^t \Re_q(s)u_o ds + \int_0^t (t-s)^{q-1}\mathfrak{T}_q(t,s)h(s)ds, & t \in [0, t_1], \\ \mu_k(t, u(t)), & t \in \cup_{k=1}^m (t_k, s_k], \\ \Re_q(t-s_k)\mu_k(s_k, u(s_k)) + \int_{s_k}^t \Re_q(y-s_k)\xi_k(s_k, u(s_k))dy & \\ + \int_{s_k}^t (t-y)^{q-1}\mathfrak{T}_q(t-y)\, h(y)dy, & t \in \cup_{k=1}^m (s_k, t_{k+1}], \end{cases}$$

where $1/2 < q = \frac{\alpha}{2} \le 1$,

$$\Re_q(t) = \int_0^\infty M_q(\theta)\Re(t^q\theta)\, d\theta, \qquad \mathfrak{T}_q(t,s) = q\int_0^\infty \theta M_q(\theta)\mathfrak{T}((t-s)^q\theta)\, d\theta,$$

and M_q is a probability density function defined by Definition 2.5.

Proof. Using Lemma 2.1 with operating by $I_{s_r}^\alpha$ on both sides to the fractional differential equation in (1.1), we arrive at

$$u(t) = I_{s_k}^\alpha[Au(t) + h(t)] + c_{1,k}(t-s_k) + c_{0,k}, \tag{3.1}$$

where $c_{1,k}, c_{0,k} \in \mathbb{R},\ k = 0, 1, \ldots, m$ are constants to be determined.

- For $t \in [0, t_1]$: By taking $\rho \to 1$ to the results given in Lemma 5 in [28], we have

$$u(t) = \Re_q(t)\phi(0) + \int_0^t \Re_q(s)u_o ds + \int_0^t (t-s)^{q-1}\mathfrak{T}_q(t,s)h(s)ds.$$

- For $t \in (t_1, s_1]$: We obtain

$$u(t) = \mu_1(t, u(t)) \quad \text{and} \quad u'(t) = \xi_1(t, u(t)).$$

- For $t \in (s_1, t_2]$: The problem (1.1) becomes

$$\begin{aligned} {}^cD_{s_1}^\alpha u(t) &= Au(t) + h(t), \\ u(s_1) &= \mu_1(s_1, u(s_1)), \\ u'(s_1) &= \xi_1(s_1, u(s_1)). \end{aligned}$$

  In this interval, Eq (3.1) becomes

$$u(t) = I_{s_1}^\alpha[Au(t) + h(t)] + c_{1,1}(t-s_1) + c_{0,1}.$$

  Considering the past impulsive conditions, we get

$$c_{0,1} = \mu_1(s_1, u(s_1)) \qquad \text{and} \qquad c_{1,1} = \xi_1(s_1, u(s_1)),$$

  which imply that

$$u(t) = I_{s_1}^\alpha[Au(t) + h(t)] + \xi_1(s_1, u(s_1))(t-s_1) + \mu_1(s_1, u(s_1)).$$

Multiplying both sides by $e^{-\lambda t}$ followed by integrating from s_1 to ∞, we achieve

$$U(\lambda) = \lambda^{-\alpha}\{AU(\lambda) + H(\lambda)\} + \lambda^{-1}e^{-\lambda s_1}\mu_1(s_1, u(s_1)) + \lambda^{-2}e^{-\lambda s_1}\xi_1(s_1, u(s_1)),$$

where

$$U(\lambda) = \int_{s_1}^{\infty} u(t)e^{-\lambda t}dt \quad \text{and} \quad H(\lambda) = \int_{s_1}^{\infty} h(t)e^{-\lambda t}dt.$$

Given that $(\lambda^{\alpha}I - A)^{-1}$ exists, then $\lambda^{\alpha} \in \rho(A)$. We obtain

$$\begin{aligned} U(\lambda) =&(\lambda^{\alpha}I - A)^{-1}\left\{\lambda^{\alpha-1}e^{-\lambda s_1}\mu_1(s_1, u(s_1)) + \lambda^{\alpha-2}e^{-\lambda s_1}\xi_1(s_1, u(s_1)) + H(\lambda)\right\} \\ =&\lambda^{q-1}e^{-\lambda s_1}\int_0^{\infty} e^{-\lambda^q t}\Re(t)\mu_1(s_1, u(s_1))dt \\ &+ \lambda^{q-2}e^{-\lambda s_1}\int_0^{\infty} e^{-\lambda^q t}\Re(t)\xi_1(s_1, u(s_1))dt + \int_0^{\infty} e^{-\lambda^q t}\mathfrak{T}(t)H(\lambda)dt. \end{aligned}$$

Let $\Psi_q(\theta) = \frac{q}{\theta^{q+1}}M_q(\theta^{-q})$ be defined for $\theta \in (0, \infty)$ and $q \in (\frac{1}{2}, 1)$. Then,

$$\int_0^{\infty} e^{-p\theta}\Psi_q(\theta)d\theta = e^{-p^q},$$

which can be used to calculate the first term with replacing t by s^q as

$$\begin{aligned} &\lambda^{q-1}e^{-\lambda s_1}\int_0^{\infty} e^{-\lambda^q t}\Re(t)\mu_1(s_1, u(s_1))dt \\ &\qquad =q\int_0^{\infty}(\lambda s)^{q-1}e^{-(\lambda s)^q}\Re(s^q)e^{-\lambda s_1}(\mu_1 s_1, u(s_1))ds \\ &\qquad =\frac{-1}{\lambda}\int_0^{\infty}\frac{d}{ds}\left(e^{-(\lambda s)^q}\right)\Re(s^q)e^{-\lambda s_1}(\mu_1 s_1, u(s_1))ds \\ &\qquad =\int_0^{\infty}\int_0^{\infty}\theta\Psi_q(\theta)e^{-\lambda s\theta}\Re(s^q)e^{-\lambda s_1}(\mu_1 s_1, u(s_1))d\theta ds \\ &\qquad =\int_0^{\infty} e^{-\lambda(x+s_1)}\left\{\int_0^{\infty}\Psi_q(\theta)\Re\left(\left(\frac{x}{\theta}\right)^q\right)\mu_1(s_1, u(s_1))d\theta\right\}dx \\ &\qquad =\int_0^{\infty} e^{-\lambda(x+s_1)}\left\{\int_0^{\infty} M_q(\theta)\Re\left(x^q\theta\right)\mu_1(s_1, u(s_1))d\theta\right\}dx \\ &\qquad =\int_0^{\infty} e^{-\lambda(x+s_1)}\Re_q(x)\mu_1(s_1, u(s_1))dx \\ &\qquad =\int_{s_1}^{\infty} e^{-\lambda t}\Re_q(t - s_1)\mu_1(s_1, u(s_1))dt. \end{aligned}$$

By using Lemma 3.1 with $\alpha = 1$, we get

$$\lambda^{q-2}\int_0^{\infty} e^{-\lambda^q t}\Re_q(t)e^{-\lambda s_1}\xi_1(s_1, u(s_1))dt = \int_{s_1}^{\infty} e^{-\lambda t}\left\{\int_{s_1}^{t}\Re_q(y - s_1)\xi_1(s_1, u(s_1))dy\right\}dt.$$

Finally, we can write

$$\begin{aligned}\int_0^\infty e^{-\lambda^q t}\mathfrak{T}(t)H(\lambda)dt &= q\int_0^\infty e^{-(\lambda s)^q}\mathfrak{T}(s^q)s^{q-1}H(\lambda)ds\\ =&q\int_0^\infty\int_0^\infty e^{-\lambda s\theta}\Psi_q(\theta)\mathfrak{T}(s^q)s^{q-1}H(\lambda)d\theta ds\\ =&q\int_0^\infty\int_{s_1}^\infty\int_0^\infty \theta^{-q}e^{-\lambda x}\Psi_q(\theta)\mathfrak{T}\left(\left(\frac{x}{\theta}\right)^q\right)x^{q-1}e^{-\lambda y}h(y)d\theta dydx\\ =&q\int_0^\infty\int_{s_1}^\infty\int_0^\infty e^{-\lambda(x+y)}\theta M_q(\theta)\mathfrak{T}\left(x^q\theta\right)x^{q-1}h(y)d\theta dydx\\ =&q\int_{s_1}^\infty\int_y^\infty\int_0^\infty e^{-\lambda t}\theta M_q(\theta)\mathfrak{T}\left((t-y)^q\theta\right)(t-y)^{q-1}h(y)d\theta dtdy\\ =&\int_{s_1}^\infty\int_y^\infty e^{-\lambda t}(t-y)^{q-1}\mathfrak{T}_q\left(t-y\right)h(y)dtdy\\ =&\int_{s_1}^\infty e^{-\lambda t}\left\{\int_{s_1}^t(t-y)^{q-1}\mathfrak{T}_q\left(t-y\right)h(y)dy\right\}dt.\end{aligned}$$

In conclusion, we can write

$$\begin{aligned}\int_{s_1}^\infty e^{-\lambda t}u(t)dt = \int_{s_1}^\infty e^{-\lambda t}\Big\{&\mathfrak{R}_q(t-s_1)\mu_1(s_1,u(s_1))\\ &+\int_{s_1}^t\mathfrak{R}_q(y-s_1)\xi_1(s_1,u(s_1))dy+\int_{s_1}^t(t-y)^{q-1}\mathfrak{T}_q\left(t-y\right)h(y)dy\Big\}dt.\end{aligned}$$

Therefore, by taking the inverse Laplace transform, we have

$$\begin{aligned}u(t) = \mathfrak{R}_q(t-s_1)\mu_1(s_1,u(s_1)) &+ \int_{s_1}^t\mathfrak{R}_q(y-s_1)\xi_1(s_1,u(s_1))dy\\ &+\int_{s_1}^t(t-y)^{q-1}\mathfrak{T}_q\left(t-y\right)h(y)dy.\end{aligned}$$

- For $t\in(s_k,t_{k+1}],\ k=2,3,\ldots,m$: In a similar manner, we can write

$$\begin{aligned}u(t) = \mathfrak{R}_q(t-s_k)\mu_k(s_k,u(s_k)) &+ \int_{s_k}^t\mathfrak{R}_q(y-s_k)\xi_k(s_k,u(s_k))dy\\ &+\int_{s_k}^t(t-y)^{q-1}\mathfrak{T}_q\left(t-y\right)h(y)dy.\end{aligned}$$

Consequently, we get the solution from the earlier (1.1). Direct calculations show that the opposite results are true. The proof is completed.

Remark 3.1. *[28] From linearity of* $\mathfrak{R}(t)$ *and* $\mathfrak{T}(t)$ *for all* $t\geq 0$*, it is clearly to deduce that* $\mathfrak{R}_q(t)$ *and* $\mathfrak{T}_q(t,s)$ *are also linear operators where* $0<s<t$*. Therefore, the proofs of all next Lemmas are same when taking* ρ *approaches* 1.

Lemma 3.3. *[28] The following estimates for $\Re_q(t)$ and $\mathfrak{T}_q(t,s)$ are verified for any fixed $t \geq 0$ and $0 < s < t$*

$$\left|\Re_q(t)x\right| \leq \varpi|x| \qquad \text{and} \qquad \left|\mathfrak{T}_q(t,s)x\right| \leq \frac{\varpi a^q}{\Gamma(2q)}|x|.$$

Lemma 3.4. *[28] The operators $\Re_q(t)$ and $\mathfrak{T}_q(s,t)$ are strongly continuous for every $0 < s < t$ and $t > 0$.*

Lemma 3.5. *[28] Assume that $\Re(t)$ and $\mathfrak{T}(t,s)$ are compact for every $0 < s < t$. Then, the operators $\Re_q(t)$ and $\mathfrak{T}_q(s,t)$ are compact for every $0 < s < t$.*

4. The main results

Define the operator $\Re\colon \overline{\mathcal{P}}_{\mathfrak{h}} \to \overline{\mathcal{P}}_{\mathfrak{h}}$ as follows

$$\Re(u)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \Re_q(t)\phi(0) + \int_0^t \Re_q(y)u_o dy + \int_0^t (t-y)^{q-1}\mathfrak{T}_q(t,y)h(y,u_y)dy, & t \in [0,t_1], \\ \mu_k(t,u(t)), & t \in \cup_{k=1}^m (t_k, s_k], \\ \Re_q(t-s_k)\mu_k(s_k,u(s_k)) + \int_{s_k}^t \Re_q(y-s_k)\xi_k(s_k,u(s_k))dy, & \\ + \int_{s_k}^t (t-y)^{q-1}\mathfrak{T}_q\,(t-y)\,h(y,u_y)dy, & t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases}$$

Let $\varkappa(\cdot)\colon (-\infty, a] \to \mathbb{X}$ be the function denoted by

$$\varkappa(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in (0, a]. \end{cases}$$

Plainly, $\varkappa(0) = \phi(0)$. For each $z \in C([0,a],\mathbb{X})$ with $z(0) = 0$, we indicate by ϑ to the function defined as

$$\vartheta(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ z(t), & t \in [0, a]. \end{cases}$$

If $u(\cdot)$ satisfies that $u(t) = \Re(u)(t)$ for all $t \in (-\infty, a]$, we can decompose that $u(t) = \vartheta(t) + \varkappa(t)$, $t \in (-\infty, a]$, it denotes $u_t = \vartheta_t + \varkappa_t$ for every $t \in (-\infty, a]$ and the function $z(\cdot)$ satisfies

$$z(t) = \begin{cases} \Re_q(t)\phi(0) + \int_0^t \Re_q(y)u_o dy + \int_0^t (t-y)^{q-1}\mathfrak{T}_q(t,y)h(y,\vartheta_y + \varkappa_y)dy, & t \in [0,t_1], \\ \mu_k(t,\vartheta + x), & t \in \cup_{k=1}^m (t_k, s_k], \\ \Re_q(t-s_k)\mu_k(s_k,\vartheta + \varkappa) + \int_{s_k}^t \Re_q(y-s_k)\xi_k(s_k,\vartheta + \varkappa)dy, & \\ + \int_{s_k}^t (t-y)^{q-1}\mathfrak{T}_q\,(t-y)\,h(\vartheta_y + \varkappa_y)dy, & t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases}$$

Set the space $\Upsilon = \{z \in C([0,a],\mathbb{X}), z(0) = 0\}$ equipped the norm

$$\|z\|_\Upsilon = \sup_{t\in[0,a]} \|z(t)\|.$$

Therefore, $(\Upsilon, \|\cdot\|_\Upsilon)$ is a Banach space. Assume that the operator $\mathscr{G}\colon \Upsilon \to \Upsilon$ is formulated as follows:

$$\mathscr{G}(z)(t) = \begin{cases} \Re_q(t)\phi(0) + \int_0^t \Re_q(y)u_o dy + \int_0^t (t-y)^{q-1}\mathfrak{T}_q(t,y)h(y,\vartheta_y + \varkappa_y)dy, & t \in [0,t_1], \\ \mu_k(t,\vartheta + x), & t \in \cup_{k=1}^m (t_k, s_k], \\ \Re_q(t-s_k)\mu_k(s_k,\vartheta + \varkappa) + \int_{s_k}^t \Re_q(y-s_k)\xi_k(s_k,\vartheta + \varkappa)dy, & \\ + \int_{s_k}^t (t-y)^{q-1}\mathfrak{T}_q\,(t-y)\,h(\vartheta_y + \varkappa_y)dy, & t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases}$$

The operator $\mathfrak{N}$ seems to have a fixed point is equivalent to $\mathscr{G}$ has a fixed point. Thus, we proceed to prove that $\mathscr{G}$ has a fixed point.

Now, we make the following assumptions:

($\mathcal{E}_1$) The function $h\colon [0,a]\times\mathcal{P}_{\mathfrak{h}} \to \mathbb{X}$ is a continuous and $\mu_k, \xi_k\colon [t_k, s_k]\times\mathbb{X} \to \mathbb{X}$ are continuous functions for all $k = 1, 2, \ldots, m;\ m \in \mathbb{N}$.

($\mathcal{E}_2$) There is a constant $\Omega > 0$ satisfying

$$\|h(t,u_t) - h(t,v_t)\| \leq \Omega\|u_t - v_t\|_{\mathcal{P}_{\mathfrak{h}}}.$$

($\mathcal{E}_3$) There exist $\delta_k, \delta_k^* > 0;\ k = 1, 2, \ldots, m;\ m \in \mathbb{N}$ such that

$$\|\mu_k(t,u)\| \leq \delta_k \quad \text{and} \quad \|\xi_k(t,u)\| \leq \delta_k^*.$$

($\mathcal{E}_4$) There are positive constants $D_k, D_k^*,\ k = 1, 2, \ldots, m;\ m \in \mathbb{N}$ such that

$$\|\mu_k(t,u_1) - \mu_k(t,u_2)\| \leq D_k\|u_1 - u_2\|,$$

$$\|\xi_k(t,u_1) - \xi_k(t,u_2)\| \leq D_k^*\|u_1 - u_2\|.$$

($\mathcal{E}_5$) There exists a continuous function $\mathfrak{g}(t)\colon [0,a] \to [0,\infty)$ such that, for any $(t,u_t) \in [0,a]\times\mathcal{P}_{\mathfrak{h}}$, it satisfies

$$\|h(t,u_t)\| \leq \mathfrak{g}(t)\|u_t\|_{\mathcal{P}_{\mathfrak{h}}}.$$

The brief constants that will be utilized later to streamline handling, are listed as follow

$$\begin{aligned}
\mathscr{E}(q) &= \frac{t_1^{q+1}}{q\Gamma(2q)},\\
\mathscr{E}_k(q) &= \frac{a(a-s_k)^q}{q\Gamma(2q)},\\
\mathscr{B} &= \varpi\Omega\zeta_1^*,\\
\overline{\mathscr{B}} &= \varpi\mathfrak{g}\zeta_1^*\mathscr{E}(q),\\
\mathscr{B}_k &= \varpi H\zeta_1^*[D_k + D_k^*(a-s_k)],\\
\overline{\mathscr{B}}_k &= \varpi\mathfrak{g}\zeta_1^*\mathscr{E}_k(q),\\
\mathscr{O} &= \varpi\left(\|\phi(0)\| + \|u_o\|t_1 + \mathscr{E}(q)\Omega\zeta_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}} + \mathscr{E}(q)c\right),\\
\mathscr{O}_k &= \varpi\left(\delta_k + \delta_k^*(a-s_k) + \mathscr{E}_k(q)\Omega\zeta_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}} + \mathscr{E}_k(q)c\right),\\
\mathscr{Q} &= \varpi\left(\|\phi(0)\| + \|u_o\|t_1 + \mathscr{E}(q)\zeta_2^*\mathfrak{g}\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right),\\
\mathscr{Q}_k &= \varpi\left(\delta_k + \delta_k^*(a-s_k) + \mathscr{E}_k(q)\zeta_2^*\mathfrak{g}\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right)
\end{aligned}$$

where $k = 1, 2, \ldots, m;\ m \in \mathbb{N}$.

Lemma 4.1. *Assume that the requirement* $(\mathcal{E}_2)$ *is met by* $c = \max_{t\in[0,a]} |h(t, 0)|$. *Ponder about the expressions* $\zeta_1^* = \sup_{t\in[0,a]} \zeta_1(t)$ *and* $\zeta_2^* = \sup_{t\in[0,a]} \zeta_2(t)$ *where* $\zeta_1(\cdot)$ *and* $\zeta_2(\cdot)$ *are established in Definition 2.6. Then,*

$$\|h(t, \vartheta_t + \varkappa_t)\| \leq \Omega \left(\zeta_1^* \|z\|_\Upsilon + \zeta_2^* \|\phi\|_{\mathcal{P}_\mathfrak{h}}\right) + c.$$

Proof. Regarding Definition 2.6 and the presumption $(\mathcal{E}_2)$. Then,

$$\begin{aligned}
\|h(t, \vartheta_t + \varkappa_t)\| &= \|h(t, \vartheta_t + \varkappa_t) - h(t, 0) + h(t, 0)\| \\
&\leq \|h(t, \vartheta_t + \varkappa_t) - h(t, 0)\| + \|h(t, 0)\| \\
&\leq \Omega \|\vartheta_t + \varkappa_t\|_\Upsilon + c \\
&\leq \Omega \left(\zeta_1(t) \sup_{t\in[0,a]} \|\vartheta(t)\| + \zeta_2(t) \|\phi\|_{\mathcal{P}_\mathfrak{h}}\right) + c \\
&\leq \Omega \left(\zeta_1(t) \|z\|_\Upsilon + \zeta_2(t) \|\phi\|_{\mathcal{P}_\mathfrak{h}}\right) + c \\
&\leq \Omega \left(\zeta_1^* \|z\|_\Upsilon + \zeta_2^* \|\phi\|_{\mathcal{P}_\mathfrak{h}}\right) + c.
\end{aligned}$$

This ends the proof.

Lemma 4.2. *Suppose that the statement* $(\mathcal{E}_5)$ *is satisfied with* $\mathfrak{g} = \sup_{t\in[0,a]} \mathfrak{g}(t)$. *Let* $\zeta_1^* = \sup_{t\in[0,a]} \zeta_1(t)$ *and* $\zeta_2^* = \sup_{t\in[0,a]} \zeta_2(t)$ *where* $\zeta_1(\cdot)$ *and* $\zeta_2(\cdot)$ *are outlined in Definition 2.6. Then,*

$$\|h(t, \vartheta_t + \varkappa_t)\| \leq \ell(t) \leq \ell,$$

where

$$\ell = \sup_{t\in[0,a]} \ell(t) = \sup_{t\in[0,a]} \left\{\mathfrak{g}(t) \left(\zeta_1(t) \|z\|_\Upsilon + \zeta_2(t) \|\phi\|_{\mathcal{P}_\mathfrak{h}}\right)\right\} = \mathfrak{g} \left(\zeta_1^* \|z\|_\Upsilon + \zeta_2^* \|\phi\|_{\mathcal{P}_\mathfrak{h}}\right).$$

Proof. By the same way in Lemma 4.1, we can easily reach the desired result.

Theorem 4.1. *Consider the assertions* $(\mathcal{E}_1) - (\mathcal{E}_4)$ *hold and*

$$\Lambda = \max_k \left\{\mathscr{B}\mathscr{E}(q), D_k H \zeta_1^*, \mathscr{B}_k + \mathscr{B}\mathscr{E}_k(q)\right\}.$$

Then, the fractional evolution equation with non-instantaneous impulsive (1.1) *has a unique mild solution on* $(-\infty, a]$ *if* $\Lambda < 1$.

Proof. To show that the operator $\mathscr{G}$ maps bounded subset of Υ into bounded subset in Υ, we set

$$\Upsilon_r = \{z \in \Upsilon \colon \|z\|_\Upsilon \leq r\},$$

where

$$r \geq \max_k \left\{\frac{\mathscr{O}}{1 - \mathscr{B}\mathscr{E}(q)}, \delta_k, \frac{\mathscr{O}_k}{1 - \mathscr{B}\mathscr{E}_k(q)}\right\}.$$

Then, for any $z \in \Upsilon_r$ and in spite of $(\mathcal{E}_2)$ and $(\mathcal{E}_3)$ and Lemma 4.1. Correspondingly, three situations are taken into consideration.

- **Case I.** Whenever $t \in [0, t_1]$, we have

$$\begin{aligned}
\|\mathscr{G}(z)(t)\|_{\Upsilon} &\leq \varpi\left(\|\phi(0)\| + \|u_o\|t_1\right) + \frac{\varpi t_1}{\Gamma(2q)}\int_0^t (t-s)^{q-1}\|h(y, \vartheta_y + \varkappa_y)\|dy \\
&\leq \varpi\left[\|\phi(0)\| + \|u_o\|t_1 + \frac{t^q t_1}{q\Gamma(2q)}\{\Omega\left(\zeta_1^*\|z\|_{\Upsilon} + \zeta_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right) + c\}\right] \\
&\leq \varpi\left[\|\phi(0)\| + \|u_o\|t_1 + \mathscr{E}(q)\{\Omega\left(\zeta_1^*\|z\|_{\Upsilon} + \zeta_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right) + c\}\right] \\
&\leq \mathscr{O} + \mathscr{E}(q)\mathscr{B}\|z\|_{\Upsilon} \\
&\leq \mathscr{O} + \mathscr{E}(q)\mathscr{B}r \leq r.
\end{aligned}$$

- **Case II.** Whenever $t \in (t_k, s_k],\ k = 1, \ldots, m;\ m \in \mathbb{N}$, we have

$$\|\mathscr{G}(z)(t)\|_{\Upsilon} = \|\mu_k(t, \vartheta + \varkappa)\| \leq \delta_k.$$

- **Case III.** Whenever $t \in (s_k, s_{k+1}],\ k = 1, \ldots, m;\ m \in \mathbb{N}$, we have

$$\begin{aligned}
\|\mathscr{G}(z)(t)\|_{\Upsilon} &\leq \varpi\left[\delta_k + \delta_k^*(t - s_k) + \frac{a}{\Gamma(2q)}\int_{s_k}^t (t-y)^{q-1}\|h(y, \vartheta_y + \varkappa_y)\|dy\right] \\
&\leq \varpi\left[\delta_k + \delta_k^*(a - s_k) + \frac{a(a-s_k)^q}{q\Gamma(2q)}\{\Omega\left(\zeta_1^*\|z\|_{\Upsilon} + \zeta_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right) + c\}\right] \\
&\leq \varpi\left[\delta_k + \delta_k^*(a - s_k) + \mathscr{E}_k(q)\{\Omega\left(\zeta_1^*\|z\|_{\Upsilon} + \zeta_2^*\|\phi\|_{\mathcal{P}_{\mathfrak{h}}}\right) + c\}\right] \\
&\leq \mathscr{O}_k + \mathscr{E}_k(q)\mathscr{B}r \leq r.
\end{aligned}$$

For the aforementioned, we acquire $\|\mathscr{G}(z)(t)\|_{\Upsilon} \leq r$. Thus, the operator $\mathscr{G}$ maps bounded subset into bounded subset in Υ.

Now, we prove that the operator $\mathscr{G}$ is a contraction mapping. Certainly, consider $z, z^* \in \Upsilon$. Then, there still are the subsequent situations.

- **Case I.** For any $t \in [0, t_1]$, we have

$$\begin{aligned}
\|\mathscr{G}(z)(t) - \mathscr{G}(z^*)(t)\| &\leq \frac{\varpi t_1}{\Gamma(2q)}\int_0^t (t-s)^{q-1}\|h(y, \vartheta_y + \varkappa_y) - h(y, \vartheta_y^* + \varkappa_y)\|dy \\
&\leq \frac{\Omega\varpi t_1}{\Gamma(2q)}\int_0^t (t-s)^{q-1}\|\vartheta_y - \vartheta_y^*\|_{\mathcal{P}_{\mathfrak{h}}}dy \\
&\leq \frac{\Omega\varpi t_1}{\Gamma(2q)}\zeta_1^*\|z - z^*\|_{\Upsilon}\int_0^t (t-s)^{q-1}dy \\
&\leq \frac{\Omega\varpi t_1^{q+1}}{q\Gamma(2q)}\zeta_1^*\|z - z^*\|_{\Upsilon} \\
&= \mathscr{B}\mathscr{E}(q)\|\vartheta - \vartheta^*\|_{\Upsilon}.
\end{aligned}$$

- **Case II.** For any $t \in (t_k, s_k],\ k = 1, \ldots, m;\ m \in \mathbb{N}$, we have

$$\begin{aligned}
\|\mathscr{G}(z)(t) - \mathscr{G}(z^*)(t)\| &= \|\mu_k(t, \vartheta + \varkappa) - \mu_k(t, \vartheta^* + \varkappa)\| \\
&\leq D_k\|\vartheta - \vartheta^*\|_{\Upsilon} \\
&\leq D_k H\|z_t - z_t^*\|_{\mathcal{P}_{\mathfrak{h}}} \\
&\leq D_k H\zeta_1^*\|z - z^*\|_{\Upsilon} \\
&= D_k H\zeta_1^*\|\vartheta - \vartheta^*\|_{\Upsilon}.
\end{aligned}$$

- **Case III.** For any $t \in (s_k, s_{k+1}],\ k = 1, \dots, m;\ m \in \mathbb{N}$, we have

$$
\begin{aligned}
&\|\mathscr{G}(z)(t) - \mathscr{G}(z^*)(t)\| \le \|\Re_q(t - s_k)\|\|\mu_k(s_k, \vartheta + \varkappa) - \mu_k(s_k, \vartheta^* + \varkappa)\|\\
&+ \int_{s_k}^{t} \|\Re_q(y - s_k)\|\|\xi_k(s_k, \vartheta + \varkappa) - \xi_k(s_k, \vartheta^* + \varkappa)\|dy\\
&+ \int_{s_k}^{t} (t - y)^{q-1}\|\mathfrak{T}_q(t - y)\|\|h(\vartheta_y + \varkappa_y) - h(\vartheta_y^* + \varkappa_y)\|dy\\
&\le \varpi D_k\|\vartheta - \vartheta^*\|_\Upsilon + \varpi D_k^* \int_{s_k}^{t} \|\vartheta(y) - \vartheta^*(y)\|_\Upsilon dy\\
&+ \frac{\varpi a}{\Gamma(2q)}\Omega \int_{s_k}^{t} (t - y)^{q-1}\|\vartheta_y - \vartheta_y^*\|_{\mathcal{P}_\mathfrak{h}} dy\\
&\le \varpi D_k H\|z_y - z_y^*\|_{\mathcal{P}_\mathfrak{h}} + \varpi D_k^* H \int_{s_k}^{t} \|z_y - z_y^*\|_{\mathcal{P}_\mathfrak{h}} dy\\
&+ \frac{\varpi a}{\Gamma(2q)}\Omega \int_{s_k}^{t} (t - y)^{q-1}\|\vartheta_y - \vartheta_y^*\|_{\mathcal{P}_\mathfrak{h}} dy\\
&\le \varpi H\zeta_1^* \left[D_k + D_k^*\mathscr{E}_k(q)\right]\|z - z^*\|_\Upsilon + \frac{\varpi a}{\Gamma(2q)}\Omega\zeta_1^* \int_{s_k}^{t} (t - y)^{q-1}\|z - z^*\|_\Upsilon dy\\
&\le \varpi H\zeta_1^* \left[D_k + D_k^*(a - s_k) + \frac{a(a - s_k)^q\Omega}{q\Gamma(2q)H}\right]\|z - z^*\|_\Upsilon\\
&= (\mathscr{B}_k + \mathscr{B}\mathscr{E}_k(q))\|z - z^*\|_\Upsilon = (\mathscr{B}_k + \mathscr{B}\mathscr{E}_k(q))\|\vartheta - \vartheta^*\|_\Upsilon.
\end{aligned}
$$

For the aforementioned, we may write

$$
\|\mathscr{G}(z)(t) - \mathscr{G}(z^*)(t)\|_\Upsilon \le \Lambda\|\vartheta - \vartheta^*\|_\Upsilon.
$$

Amid the existing circumstances, $\Lambda < 1$ shows that the operator $\mathscr{G}$ is a contraction. This suggests that the problem (1.1) has a unique solution on $(-\infty, a]$ relying on the Banach contraction mapping principle.

Remark 4.1. *In viewing our problem, it is very difficult to obtain the exact solution and so it is useful to investigate some properties of the solutions, especially the uniqueness. The previous theorem shaw that the mild solution of the problem* (1.1) *is unique under the assumptions* $(\mathcal{E}_1) - (\mathcal{E}_4)$ *and* $\Lambda < 1$. *This enables us to apply our results to real-life problem or phenomena as in the last section.*

Assume that the operator $\mathscr{G}$ is divided as a sum of the two operators $\mathscr{G}_i\colon \Upsilon \to \Upsilon,\ i = 1, 2$ as

$$
\mathscr{G} = \mathscr{G}_1(z) + \mathscr{G}_2(z) \tag{4.1}
$$

where,

$$
\mathscr{G}_1(z)(t) = \begin{cases} \Re_q(t)\phi(0) + \int_0^t \Re_q(y)u_o dy \\ + \int_0^t (t - y)^{q-1}\mathfrak{T}_q(t, y)h(y, \vartheta_y + \varkappa_y)dy, & t \in [0, t_1],\\ 0, & t \in \cup_{k=1}^m (t_k, s_k],\\ \int_{s_k}^t (t - y)^{q-1}\mathfrak{T}_q(t - y)\, h(\vartheta_y + \varkappa_y)dy, & t \in \cup_{k=1}^m (s_k, t_{k+1}] \end{cases}
$$

and

$$\mathscr{G}_2(z)(t) = \begin{cases} 0, & t \in [0, t_1], \\ \mu_k(t, \vartheta + x), & t \in \cup_{k=1}^{m}(t_k, s_k], \\ \Re_q(t - s_k)\mu_k(s_k, \vartheta + \varkappa) & \\ + \int_{s_k}^{t} \Re_q(y - s_k)\xi_k(s_k, \vartheta + \varkappa)dy, & t \in \cup_{k=1}^{m}(s_k, t_{k+1}]. \end{cases}$$

Theorem 4.2. *Suppose the hypotheses* $(\mathcal{E}_1)$ *and* $(\mathcal{E}_3) - (\mathcal{E}_5)$ *are correct. Then the fractional evolution equation with non-instantaneous impulsive* (1.1) *has at least one mild solution on* $(-\infty, a]$ *if* $\Delta < 1$ *where* Δ *is given by*

$$\Delta = \max_k \left\{ \overline{\mathscr{B}}, \overline{\mathscr{B}}_k \right\}.$$

Proof. Let the operators $\mathscr{G}_1$ and $\mathscr{G}_2$ be defined as (4.1). Setting $\mathfrak{g} = \sup_{t\in[0,a]} |\mathfrak{g}(t)|$. Let us define the closed ball $\Upsilon_\rho = \{z \in \Upsilon : \|z\|_\Upsilon \le \rho\}$ with radius

$$\rho \ge \max_k \left\{ \frac{\mathscr{Q}}{1 - \overline{\mathscr{B}}}, \delta_k, \frac{\mathscr{Q}_k}{1 - \overline{\mathscr{B}}_k} \right\}.$$

Then, for $u, v \in \Upsilon_\rho$, we claim that $\|\mathscr{G}_1(z)(u) + \mathscr{G}_2(z)(v)\| \le \rho$ which concludes that $\mathscr{G}_1(u) + \mathscr{G}_1(v) \in \Upsilon_\rho$. To verify our claiming, we show that $\mathscr{G}$ maps bounded sets of Υ into bounded sets in Υ, for any $\rho \ge 0$. Then for any $z \in \Upsilon_\rho$ and in light of $(\mathcal{E}_3)$, $(\mathcal{E}_5)$ and Lemma 4.2, we have three cases

- **Case I.** For any $t \in [0, t_1]$, we have

$$\begin{aligned} \|\mathscr{G}(z)(t)\|_\Upsilon &\le \varpi\left(\|\phi(0)\| + \|u_o\|t_1\right) + \frac{\varpi t_1}{\Gamma(2q)} \int_0^t (t-s)^{q-1} \|h(y, \vartheta_y + \varkappa_y)\| dy \\ &\le \varpi\left[\|\phi(0)\| + \|u_o\|t_1 + \frac{t_1}{\Gamma(2q)} \int_0^t (t-s)^{q-1} \left\{\mathfrak{g}(y)\left(\zeta_1(y)\|z\|_\Upsilon + \zeta_2(y)\|\phi\|_{\mathcal{P}_\mathfrak{h}}\right)\right\} dy\right] \\ &\le \varpi\left[\|\phi(0)\| + \|u_o\|t_1 + \mathscr{E}(q)\left\{\mathfrak{g}\left(\zeta_1^*\|z\|_\Upsilon + \zeta_2^*\|\phi\|_{\mathcal{P}_\mathfrak{h}}\right)\right\}\right] \\ &\le \mathscr{Q} + \mathscr{E}(q)\overline{\mathscr{B}}\|z\|_\Upsilon \\ &\le \mathscr{Q} + \overline{\mathscr{B}}\rho \le \rho. \end{aligned}$$

- **Case II.** For any $t \in (t_k, s_k],\ k = 1, \ldots, m;\ m \in \mathbb{N}$, we have

$$\|\mathscr{G}(z)(t)\|_\Upsilon = \|\mu_k(t, \vartheta + \varkappa)\| \le \delta_k \le \rho.$$

- **Case III.** For any $t \in (s_k, s_{k+1}],\ k = 1, \ldots, m;\ m \in \mathbb{N}$, we have

$$\begin{aligned} \|\mathscr{G}(z)(t)\|_\Upsilon &\le \varpi\left[\delta_k + \delta_k^*(t - s_k) + \frac{a}{\Gamma(2q)} \int_{s_k}^t (t-y)^{q-1} \|h(y, \vartheta_y + \varkappa_y)\| dy\right] \\ &\le \varpi\left[\delta_k + \delta_k^*(a - s_k) + \frac{a}{\Gamma(2q)} \int_{s_k}^t (t-y)^{q-1} \left\{\mathfrak{g}(y)\left(\zeta_1(y)\|z\|_\Upsilon + \zeta_2(y)\|\phi\|_{\mathcal{P}_\mathfrak{h}}\right)\right\} dy\right] \\ &\le \varpi\left[\delta_k + \delta_k^*(a - s_k) + \frac{a(a-s_k)^q}{q\Gamma(2q)}\{\mathfrak{g}\left(\zeta_1^*\|z\|_\Upsilon + \zeta_2^*\|\phi\|_{\mathcal{P}_\mathfrak{h}}\right)\}\right] \\ &\le \varpi\left[\delta_k + \delta_k^*(a - s_k) + \mathscr{E}_k(q)\{\mathfrak{g}\left(\zeta_1^*\|z\|_\Upsilon + \zeta_2^*\|\phi\|_{\mathcal{P}_\mathfrak{h}}\right)\}\right] \\ &\le \mathscr{Q}_k + \overline{\mathscr{B}}_k\rho \le \rho. \end{aligned}$$

By virtue of the above, we obtain $\|\mathscr{G}(z)(t)\|_{\Upsilon} \leq \rho$. Thus the operator $\mathscr{G}$ maps bounded sets into bounded sets in Υ.

The following step is to confirm that the operator $\mathscr{G}_2$ maps bounded sets into equicontinuous sets in Υ. In light of the situation $(\mathcal{E}_1)$, $\mathscr{G}_2$ is continuous. The following scenarios are therefore possible.

- **Case I.** For each $t_k \leq \gamma_1 < \gamma_2 \leq s_k$ and $z \in \Upsilon\rho$, we have

$$\|\mathscr{G}_2(z)(\gamma_2) - \mathscr{G}_2(z)(\gamma_1)\| \leq \|\mu_k(\gamma_2, \vartheta + \varkappa) - \mu_k(\gamma_1, \vartheta + \varkappa)\|.$$

  Due to the continuity of $\mu(t, u(t))$. It is clear that the above inequality approaches zero when letting $\gamma_2 \to \gamma_1$.
- **Case II.** For any $s_k \leq \gamma_1 < \gamma_2 \leq t_{k+1},\ k = 1, \ldots, m;\ m \in \mathbb{N}$ and $z \in \Upsilon\rho$, we have

$$\begin{aligned}\|\mathscr{G}_2(z)(\gamma_2) - \mathscr{G}_2(z)(\gamma_1)\| &\leq \delta_k\|\Re_q(\gamma_2 - s_k) - \Re_q(\gamma_1 - s_k)\| + \delta_k^* \int_{\gamma_1}^{\gamma_2} \|\Re_q(y - s_k)\| dy \\ &\leq \delta_k\|\Re_q(\gamma_2 - s_k) - \Re_q(\gamma_1 - s_k)\| + \delta_k^* \varpi(\gamma_2 - \gamma_1).\end{aligned}$$

Due to compactness of operator $\Re_q(y)$ and $\mathfrak{T}_q(t, y)$ (see Lemma 3.5), we infer that $\|\mathscr{G}_2(z)(\gamma_1) - \mathscr{G}_2(z)(\gamma_2)\| \to 0$ as $\gamma_2 \to \gamma_1$. Thus, $\mathscr{G}_2$ is a relatively compact on Υ. By Arezela Ascoli Theorem the operator $\mathscr{G}_2$ is completely continuous on Υ_ρ. The only thing left to do is provide evidence that $\mathscr{G}_1$ is a contraction mapping. Thus, two cases are thought about.

- **Case I.** For any $t \in [0, t_1],\ k = 1, \ldots, m;\ m \in \mathbb{N}$ and $z \in \Upsilon\rho$, we have

$$\begin{aligned}\|\mathscr{G}_1(z)(t) - \mathscr{G}_1(z^*)(t)\| &\leq \frac{\varpi t_1}{\Gamma(2q)} \int_0^t (t - y)^{q-1} \|h(y, \vartheta_y + \varkappa_y) - h(y, \vartheta_y^* + \varkappa_y)\| dy \\ &\leq \frac{\varpi t_1}{\Gamma(2q)} \int_0^t (t - y)^{q-1} \mathfrak{g}(y) \|\vartheta_y - \vartheta_y^*\|_{\mathcal{P}_\mathfrak{h}} dy \\ &\leq \frac{\mathfrak{g}\varpi t_1}{\Gamma(2q)} \zeta_1^* \|z - z^*\|_\Upsilon \int_0^t (t - s)^{q-1} dy \\ &\leq \frac{\mathfrak{g}\varpi t_1^{q+1}}{q\Gamma(2q)} \zeta_1^* \|z - z^*\|_\Upsilon \\ &= \overline{\mathscr{B}} \|\vartheta - \vartheta^*\|_\Upsilon.\end{aligned}$$

- **Case II.** For any $t \in (s_k, t_{k+1}],\ k = 1, \ldots, m;\ m \in \mathbb{N}$, we have

$$\begin{aligned}\|\mathscr{G}_1(z)(t) - \mathscr{G}_1(z^*)(t)\| &\leq \int_{s_k}^t (t - y)^{q-1} \|\mathfrak{T}_q(t - y)\| \|h(\vartheta_y + \varkappa_y) - h(\vartheta_y^* + \varkappa_y)\| dy \\ &\leq \frac{\varpi a}{\Gamma(2q)} \int_{s_k}^t (t - y)^{q-1} \mathfrak{g}(y) \|\vartheta_y - \vartheta_y^*\|_{\mathcal{P}_\mathfrak{h}} dy \\ &\leq \frac{a}{\Gamma(2q)} \varpi \mathfrak{g} \zeta_1^* \|z - z^*\|_\Upsilon \int_{s_k}^t (t - y)^{q-1} dy \\ &\leq \frac{a(a - s_k)^q}{q\Gamma(2q)} \varpi \mathfrak{g} \zeta_1^* \|z - z^*\|_\Upsilon = \overline{\mathscr{B}}_k \|z - z^*\|_\Upsilon.\end{aligned}$$

As a sense, the fractional evolution equation with non-instantaneous impulsive (1.1) has at least one mild solution on Υ, according to the Krasnoselskii Theorem. The evidence is now complete.

Remark 4.2. *Also, it is useful to investigate the existence of the solution instead of the its uniqueness. The theorem above shaw that the mild solution of the problem* (1.1) *exists under the assumptions* $(\mathcal{E}_1)-(\mathcal{E}_3)$ *and* $(\mathcal{E}_5)$ *with* $\Delta < 1$.

5. An application

Presume the following fractional wave equation with impulsive effect and infinite delay

$$\begin{cases} u(t,x) = \dfrac{1}{3}\sin t, & t \in (-\infty, 0],\ x \in [0,\pi], \\ {}^cD_t^\alpha u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + h(t,u_t), & t \in (0, \frac{2}{5}] \cup (\frac{4}{5}, 1],\ x \in [0,\pi], \\ u(t,x) = \frac{1}{7}t^{\frac{3}{2}} + \frac{1}{5}\sin u(t), & t \in (\frac{2}{5}, \frac{4}{5}]\ x \in [0,\pi], \\ u'(t) = \frac{3}{14}t^{\frac{1}{2}} + \frac{1}{8}\cos u(t), & t \in (\frac{2}{5}, \frac{4}{5}]\ x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \in [0,1], \\ u'(0,x) = \frac{3}{2}e^{\frac{-x}{3}}, & x \in [0,\pi]. \end{cases}$$

Consider that

$$\begin{aligned} & J = [0,1],\ 0 = t_0 = s_0 < t_1 = \frac{2}{5} < s_1 = \frac{4}{5} < t_2 < 1 = a, \\ & \alpha = \frac{3}{2} \quad \Rightarrow \quad q = \frac{3}{4},\ u_0 = \frac{3}{2}e^{\frac{-x}{3}}, \quad A = \frac{\partial^2}{\partial x^2}, \quad x \in [0,\pi], \quad H = \frac{1}{16}. \\ & \text{While} \quad \zeta_1(t) = \frac{3}{5}t^{\frac{3}{2}} \quad \Rightarrow \quad \zeta_1^* = \frac{3}{5}\left\{\sup_{t\in(0,1]} t^{\frac{3}{2}}\right\} \le \zeta_1(1) = \frac{3}{5}. \\ & \text{If we take} \quad \varpi = 1 \quad \Rightarrow \quad \|\mathfrak{T}_q(t,s)\| \le \frac{1}{\Gamma(\frac{3}{4})}, \quad 0 < s < t \le 1. \end{aligned}$$

Case I. Banach fixed point theorem.

In order to explain Theorem 4.1, we obtain:

$$h(t,u_t) = \frac{t^3}{8\sqrt{t+1}} + \frac{1}{9}u_t. \tag{5.1}$$

Clearly, $h\colon [0,1]\times\mathcal{P}_{\mathfrak{h}} \to \mathbb{R}$ is continuous and satisfying, for $u_t, v_t \in \mathcal{P}_{\mathfrak{h}}$, that

$$\|h(t,u_t) - h(t,v_t)\| \le \frac{1}{9}\|u_t - v_t\|_{\mathcal{P}_{\mathfrak{h}}},$$

it suggests that $\Omega = \frac{1}{9}$. For all $t \in (\frac{2}{5}, \frac{4}{5}]$ and $u, v \in \mathbb{R}$, we get

$$\begin{aligned} \|\mu(t,u) - \mu(t,v)\| &\le \frac{1}{5}\|\sin u - \sin v\| \le \frac{1}{5}\|u - v\|, \\ \|\xi(t,u) - \xi(t,v)\| &\le \frac{1}{8}\|\cos u - \cos v\| \le \frac{1}{8}\|u - v\|. \end{aligned}$$

As you can see, the Theorem 4.1 condition ($\mathcal{E}_4$) is satisfied with

$$D_k = \frac{1}{5} \qquad \text{and} \qquad D_k^* = \frac{1}{8}.$$

In summary, we have

$$\Lambda = \max_k \{\mathscr{B}\mathscr{E}(q), D_k H\zeta_1^*, \mathscr{B}_k + \mathscr{B}\mathscr{E}_k(q)\} = \{0.0202, 0.0075, 0.0384\} = 0.0384 < 1.$$

Thus all assumptions of this theorem are verified. Therefore, the problem (1.1) has a unique mild solution on $(-\infty, 1]$.

Case II. Krasnoselskii's theorem.

To realize Theorem 4.2, take $h(t, u_t)$ as given in (5.1). Therefore, $\mathfrak{g}(t) = \frac{t^3}{8\sqrt{t+1}}$ is increasing function which admits the hypothesis ($\mathcal{E}_5$) with

$$\|\mathfrak{g}\| \leq \mathfrak{g}(1) = \frac{1}{8\sqrt{2}}.$$

These calculate that

$$\Delta = \max_k \left\{\overline{\mathscr{B}}, \overline{\mathscr{B}}_k\right\} = \max_k \{0.0161, 0.0239\} = 0.0239 < 1.$$

Since every requirements of Theorem 4.2 are met, it follows that there exists at least one mild solution of (1.1) on $(-\infty, 1]$.

6. Conclusions

We analyzed a set of impulsive fractional evolution equations with infinite delay in the current work. Current functional analysis methodologies serve as the foundation for our conclusions. By using the unbounded operator A as the generator of the strongly continuous cosine family, we were able to suggest a mild solution for the suggested problem. In the instance of problem (1.1), we had two successful outcomes: While the second argument focuses on whether there are solutions for the given problem, the first argument concentrated on the existence and uniqueness of the solution.

The first result, which is built on a Banach fixed point theorem, provides criteria for ensuring that the problem at hand has no prior solutions by requiring the usage of $h(t, u_t)$ to satisfy the classic Lipschitz condition.

The second argument was based on a Krasnoselskii's theorem, which allows $h(t, u_t)$ to behave as $\|h(t, u_t)\| \leq \mathfrak{g}(t)\|u_t\|_{\mathcal{P}_\mathfrak{h}}$. The instruments used by fixed point theory in the scenario with simple assumptions. Finally, a numerical example that examines a function that satisfies all the prerequisites was provided to illustrate our conclusion.

In the next paper, we will study the controllability of mild solution to fractional evolution equations with an infinite time-delay and nonlocal condition by applying Krasnoselskii's theorem in the compactness case and the Sadvskii and Kuratowski measure of noncompactness.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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