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Research article

Some common fixed-point and fixed-figure results with a function family on S_b-metric spaces

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Abstract: In this paper, we prove a common fixed-point theorem for four self-mappings with a function family on S_b-metric spaces. In addition, we investigate some geometric properties of the fixed-point set of a given self-mapping. In this context, we obtain a fixed-disc (resp. fixed-circle), fixed-ellipse, fixed-hyperbola, fixed-Cassini curve and fixed-Apollonious circle theorems on S_b-metric spaces.

Keywords: S_b-metric; fixed point; fixed figure
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1. Introduction and motivation

Fixed-point theory has been extensively researched in various areas, such as mathematics, engineering, and physics. Of particular importance is metric fixed-point theory, which is used in various branches of mathematics like topology, analysis, and applied mathematics. This theory was initiated with the famous Banach fixed-point theorem [1]. This theorem ensures that a self-mapping will have a unique fixed point. Despite this, there remain instances of self-mapping that have a fixed point but do not meet the criteria set by the Banach fixed point theorem, such as:

Let $\mathcal{X} = \mathbb{R}$, and $(\mathcal{X}, \zeta)$ be the usual metric space. Consider a self-mapping $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g\mu = 2\mu - 4,$$

for all $\mu \in \mathbb{R}$. Then g has a unique fixed point $\mu = 4$, but g does not meet the criteria of Banach contraction principle.

There are two popular methods used by researchers to generalize the Banach contraction principle. The first entails extending the utilized contractive condition while the second centers around generalizing the underlying metric space. For example, G_b-metric spaces, G-metric spaces [2], complex valued G_b-metric spaces [3,4], S-metric spaces, A-metric spaces [5], S_b-metric spaces, fuzzy cone metric spaces [6], modular metric spaces [7–9] et al. were defined for this purpose (for more details, see [10–13] and the references therein). Especially, we focus on the notion of S_b-metric spaces. To do this, we recall the following basic concepts:

Definition 1.1. *[14] Let $\mathcal{X}$ be a nonempty set and $s \geq 1$ a given real number. A function $S_b : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is said to be S_b-metric if and only if for all $\mu, \tau, \hbar, \rho \in \mathcal{X}$ the following conditions are satisfied*:

(S_b1) $S_b(\mu, \tau, \hbar) = 0$ *if and only if* $\mu = \tau = \hbar$,

(S_b2) $S_b(\mu, \tau, \hbar) \leq s[S_b(\mu, \mu, \rho) + S_b(\tau, \tau, \rho) + S_b(\hbar, \hbar, \rho)]$.

The pair $(\mathcal{X}, S_b)$ is called an S_b-metric space.

As every S-metric is a S_b-metric with $s = 1$, we observe that S_b-metric spaces are extensions of S-metric spaces. However, the converse statement is not always true (see [14] and [15] for more details).

Definition 1.2. *[15] Let $(\mathcal{X}, S_b)$ be an S_b-metric space and $s > 1$. An S_b-metric S_b is called symmetric if*

$$S_b(\mu, \mu, \tau) = S_b(\tau, \tau, \mu),$$

for all $\mu, \tau \in \mathcal{X}$.

Definition 1.3. *[14] Let $(\mathcal{X}, S_b)$ be an S_b-metric space, and $\{\hbar_n\}$ be a sequence in $\mathcal{X}$.*

1) *Then the sequence $\{\hbar_n\}$ converges to $\hbar \in \mathcal{X}$ if and only if $S_b(\hbar_n, \hbar_n, \hbar) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S_b(\hbar_n, \hbar_n, \hbar) < \varepsilon$. It is denoted by*

$$\lim_{n\to\infty} \hbar_n = \hbar.$$

2) *Then the sequence $\{\hbar_n\}$ is called a Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S_b(\hbar_n, \hbar_n, \hbar_m) < \varepsilon$ for each $n, m \geq n_0$.*
3) *The S_b-metric space $(\mathcal{X}, S_b)$ is called complete if every Cauchy sequence is convergent.*

On the other hand, in the literature, besides various fixed point theorems there are also common fixed point theorems on metric and some extended metric spaces (for example, see [14, 16–19] and the references therein).

Recently, as a geometric approach to the fixed-point theory, the fixed-circle problem (see [20]) and the fixed-figure problem (see [21]) have been introduced. When there are more than one fixed points, it is interesting to investigate for the possible solutions as follows:

Let us define a self-mapping, $g : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is with the usual metric

$$g\mu = \frac{\mu^2 - 1}{\sqrt{2}},$$

for all $\mu \in \mathbb{R}$. Then g has two fixed points
$\mu_1 = -1$ and $\mu_2 = 1$. We consider these fixed points as a unit circle $C_{0,1} = \{-1, 1\}$.

For this reason, there exist some studies related to these recent aspects (for example, see [22–28] and the references therein).

By the above motivation, in this paper, we prove a common fixed-point theorem and some fixed-figure results on S_b-metric spaces.

2. A common fixed-point result

In this section, we prove a common fixed-point result on S_b-metric spaces. To do this, we are inspired by the function family $\mathcal{F}_6$ introduced in [29] and the function family $\mathcal{M}$ defined in [30]. We modify these families as follows:

Let Ψ be the family of all lower semi-continuous functions $\psi : \mathbb{R}_+^6 \to \mathbb{R}$ that satisfy the following condition:

(ψ^*) . For all $\mu, \tau, \hbar \geq 0$ and $s \geq 1$, there exists a $k \in [0,1)$ such that

$$\mu \leq \psi(\mu, \tau, \tau, \mu, 0, \hbar)$$

with $\hbar \leq 2s\mu + s\tau$ then

$$\mu \leq k\tau.$$

If we define the function $\psi : \mathbb{R}_+^6 \to \mathbb{R}$ such as

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = k \max\{t_1, t_2\},$$

with $k \in [0,1)$. Then $\psi \in \Psi$.

Now, we give the following theorem.

Theorem 2.1. *Let $(\mathcal{X}, S_b)$ be a complete continuous S_b-metric space with the symmetric metric S_b. Let $g, h, G, H : \mathcal{X} \to \mathcal{X}$ be four self-mappings, where g, G and H are continuous and satisfying the following conditions:*

(i) $g(\mathcal{X}) \subset H(\mathcal{X})$ and $h(\mathcal{X}) \subset G(\mathcal{X})$,

(ii) For all $\mu, \tau \in \mathcal{X}$ and $\psi \in \Psi$,

$$S_b(g\mu, g\mu, h\tau) \leq \psi\begin{pmatrix} S_b(g\mu, g\mu, h\tau), S_b(\mu, \mu, \tau), S_b(\mu, \mu, g\mu), \\ S_b(\tau, \tau, h\tau), S_b(\tau, \tau, g\mu), S_b(\mu, \mu, h\tau) \end{pmatrix},$$

(iii) For all $\mu, \tau \in \mathcal{X}$ and $\psi \in \Psi$,

$$S_b(G\mu, G\mu, H\tau) \leq \psi\begin{pmatrix} S_b(G\mu, G\mu, H\tau), S_b(\mu, \mu, \tau), S_b(\mu, \mu, G\mu), \\ S_b(\tau, \tau, H\tau), S_b(\tau, \tau, G\mu), S_b(\mu, \mu, H\tau) \end{pmatrix},$$

holds, then g, h, G and H have a common fixed point in $\mathcal{X}$.

Proof. Let $\hbar_0 \in \mathcal{X}$, $\hbar_1 = g\hbar_0$ and $\hbar_2 = h\hbar_1$. Using the condition (ii), we get

$$\begin{aligned} S_b(g\hbar_0, g\hbar_0, h\hbar_1) &= S_b(\hbar_1, \hbar_1, \hbar_2) \\ &\leq \psi\begin{pmatrix} S_b(g\hbar_0, g\hbar_0, h\hbar_1), S_b(\hbar_0, \hbar_0, \hbar_1), \\ S_b(\hbar_0, \hbar_0, g\hbar_0), S_b(\hbar_1, \hbar_1, h\hbar_1), \\ S_b(\hbar_1, \hbar_1, g\hbar_0), S_b(\hbar_0, \hbar_0, h\hbar_1) \end{pmatrix} \\ &= \psi\begin{pmatrix} S_b(\hbar_1, \hbar_1, \hbar_2), S_b(\hbar_0, \hbar_0, \hbar_1), \\ S_b(\hbar_0, \hbar_0, \hbar_1), S_b(\hbar_1, \hbar_1, \hbar_2), \\ S_b(\hbar_1, \hbar_1, \hbar_1), S_b(\hbar_0, \hbar_0, \hbar_2) \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} S_b(\hbar_1,\hbar_1,\hbar_2), S_b(\hbar_0,\hbar_0,\hbar_1), \\ S_b(\hbar_0,\hbar_0,\hbar_1), S_b(\hbar_1,\hbar_1,\hbar_2), \\ 0, S_b(\hbar_0,\hbar_0,\hbar_2) \end{pmatrix}. \tag{2.1}$$

By (S_b2) and the symmetry property of S_b, we have

$$\begin{aligned} S_b(\hbar_0,\hbar_0,\hbar_2) &= S_b(\hbar_2,\hbar_2,\hbar_0) \\ &\leq s[2S_b(\hbar_2,\hbar_2,\hbar_1) + S_b(\hbar_0,\hbar_0,\hbar_1)] \\ &= 2sS_b(\hbar_1,\hbar_1,\hbar_2) + sS_b(\hbar_0,\hbar_0,\hbar_1). \end{aligned} \tag{2.2}$$

Using (2.1), (2.2) and (ψ^*), there exists a $k \in [0,1)$ such that

$$S_b(\hbar_1,\hbar_1,\hbar_2) \leq kS_b(\hbar_0,\hbar_0,\hbar_1).$$

Continuing this process with induction with the condition (i), we can define the sequence $\{\hbar_n\}$ as follows:

$$\hbar_{2n+1} = g\hbar_{2n} = H\hbar_{2n}$$

and

$$\hbar_{2n} = h\hbar_{2n-1} = G\hbar_{2n-1}.$$

Using (ii), for $\mu = \hbar_{2n}$ and $\tau = \hbar_{2n+1}$, we find

$$\begin{aligned} S_b(g\hbar_{2n}, g\hbar_{2n}, h\hbar_{2n+1}) &= S_b(\hbar_{2n+1},\hbar_{2n+1},\hbar_{2n+2}) \\ &\leq \psi\begin{pmatrix} S_b(g\hbar_{2n}, g\hbar_{2n}, h\hbar_{2n+1}), S_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+1}), \\ S_b(\hbar_{2n},\hbar_{2n},g\hbar_{2n}), S_b(\hbar_{2n+1},\hbar_{2n+1},h\hbar_{2n+1}), \\ S_b(\hbar_{2n+1},\hbar_{2n+1},g\hbar_{2n}), S_b(\hbar_{2n},\hbar_{2n},h\hbar_{2n+1}) \end{pmatrix} \\ &= \psi\begin{pmatrix} S_b(\hbar_{2n+1},\hbar_{2n+1},\hbar_{2n+2}), S_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+1}), \\ S_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+1}), S_b(\hbar_{2n+1},\hbar_{2n+1},\hbar_{2n+2}), \\ S_b(\hbar_{2n+1},\hbar_{2n+1},\hbar_{2n+1}), S_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+2}) \end{pmatrix} \\ &= \psi\begin{pmatrix} S_b(\hbar_{2n+1},\hbar_{2n+1},\hbar_{2n+2}), S_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+1}), \\ S_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+1}), S_b(\hbar_{2n+1},\hbar_{2n+1},\hbar_{2n+2}), \\ 0, S_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+2}) \end{pmatrix} \end{aligned} \tag{2.3}$$

By (S_b2) and the symmetry property of S_b, we have

$$\begin{aligned} S_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+2}) &= S_b(\hbar_{2n+2},\hbar_{2n+2},\hbar_{2n}) \\ &\leq s[2S_b(\hbar_{2n+2},\hbar_{2n+2},\hbar_{2n+1}) + S_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+1})] \\ &= 2sS_b(\hbar_{2n+1},\hbar_{2n+1},\hbar_{2n+2}) + sS_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+1}). \end{aligned} \tag{2.4}$$

Using (2.3), (2.4) and (ψ^*), there exists a $k \in [0,1)$ such that

$$S_b(\hbar_{2n+1},\hbar_{2n+1},\hbar_{2n+2}) \leq kS_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+1}). \tag{2.5}$$

Using (iii), for $\mu = \hbar_{2n-1}$ and $\tau = \hbar_{2n}$, we get

$$S_b(G\hbar_{2n-1}, G\hbar_{2n-1}, H\hbar_{2n}) = S_b(\hbar_{2n},\hbar_{2n},\hbar_{2n+1})$$

$$
\begin{aligned}
&\leq \psi\left(\begin{array}{c} S_b\left(G\hbar_{2n-1}, G\hbar_{2n-1}, H\hbar_{2n}\right), S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n}\right), \\ S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, G\hbar_{2n-1}\right), S_b\left(\hbar_{2n}, \hbar_{2n}, H\hbar_{2n}\right), \\ S_b\left(\hbar_{2n}, \hbar_{2n}, G\hbar_{2n-1}\right), S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, H\hbar_{2n}\right) \end{array}\right) \\
&= \psi\left(\begin{array}{c} S_b\left(\hbar_{2n}, \hbar_{2n}, \hbar_{2n+1}\right), S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n}\right), \\ S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n}\right), S_b\left(\hbar_{2n}, \hbar_{2n}, \hbar_{2n+1}\right), \\ S_b\left(\hbar_{2n}, \hbar_{2n}, \hbar_{2n}\right), S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n+1}\right) \end{array}\right) \\
&= \psi\left(\begin{array}{c} S_b\left(\hbar_{2n}, \hbar_{2n}, \hbar_{2n+1}\right), S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n}\right), \\ S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n}\right), S_b\left(\hbar_{2n}, \hbar_{2n}, \hbar_{2n+1}\right), \\ 0, S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n+1}\right) \end{array}\right) \qquad (2.6)
\end{aligned}
$$

By (S_b2) and the symmetry property of S_b, we find

$$
\begin{aligned}
S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n+1}\right) &= S_b\left(\hbar_{2n+1}, \hbar_{2n+1}, \hbar_{2n-1}\right) \\
&\leq s\left[2S_b\left(\hbar_{2n+1}, \hbar_{2n+1}, \hbar_{2n}\right) + S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n}\right)\right] \\
&= 2sS_b\left(\hbar_{2n}, \hbar_{2n}, \hbar_{2n+1}\right) + sS_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n}\right). \qquad (2.7)
\end{aligned}
$$

Using (2.6), (2.7) and (ψ^*), there exists a $k \in [0, 1)$ such that

$$
S_b\left(\hbar_{2n}, \hbar_{2n}, \hbar_{2n+1}\right) \leq kS_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n}\right). \qquad (2.8)
$$

Using the inequalities (2.5) and (2.8), we get

$$
S_b\left(\hbar_{2n+1}, \hbar_{2n+1}, \hbar_{2n+2}\right) \leq kS_b\left(\hbar_{2n}, \hbar_{2n}, \hbar_{2n+1}\right) \leq k^2S_b\left(\hbar_{2n-1}, \hbar_{2n-1}, \hbar_{2n}\right)
$$

and so, using similar arguments, we have

$$
S_b\left(\hbar_n, \hbar_n, \hbar_{n+1}\right) \leq k^nS_b\left(\hbar_0, \hbar_0, \hbar_1\right). \qquad (2.9)
$$

Now we show that the sequence $\{\hbar_n\}$ is Cauchy. Using (S_b2), (2.9) and the symmetry property of S_b, for all $n, m \in \mathbb{N}$ with $m > n$, we get

$$
\begin{aligned}
S_b\left(\hbar_n, \hbar_n, \hbar_m\right) &\leq s\left[2S_b\left(\hbar_n, \hbar_n, \hbar_{n+1}\right) + S_b\left(\hbar_m, \hbar_m, \hbar_{n+1}\right)\right] \\
&= s\left[2S_b\left(\hbar_n, \hbar_n, \hbar_{n+1}\right) + S_b\left(\hbar_{n+1}, \hbar_{n+1}, \hbar_m\right)\right] \\
&\leq 2sS_b\left(\hbar_n, \hbar_n, \hbar_{n+1}\right) \\
&\quad + s^2\left[2S_b\left(\hbar_{n+1}, \hbar_{n+1}, \hbar_{n+2}\right) + S_b\left(\hbar_m, \hbar_m, \hbar_{n+2}\right)\right] \\
&= 2sS_b\left(\hbar_n, \hbar_n, \hbar_{n+1}\right) \\
&\quad + s^2\left[2S_b\left(\hbar_{n+1}, \hbar_{n+1}, \hbar_{n+2}\right) + S_b\left(\hbar_{n+2}, \hbar_{n+2}, \hbar_m\right)\right] \\
&\leq 2sS_b\left(\hbar_n, \hbar_n, \hbar_{n+1}\right) + 2s^2S_b\left(\hbar_{n+1}, \hbar_{n+1}, \hbar_{n+2}\right) + \cdots \\
&\leq 2sk^nS_b\left(\hbar_0, \hbar_0, \hbar_1\right) + 2s^2k^{n+1}S_b\left(\hbar_0, \hbar_0, \hbar_1\right) + \cdots \\
&\leq \frac{2sk^n}{1 - sk}S_b\left(\hbar_0, \hbar_0, \hbar_1\right).
\end{aligned}
$$

Since $s \geq 1$ and $k \in [0, 1)$, taking $n, m \to \infty$, we get

$$
S_b\left(\hbar_n, \hbar_n, \hbar_m\right) \to 0
$$

and so $\{\hbar_n\}$ is Cauchy. Since $(\mathcal{X}, S_b)$ is a complete S_b-metric space, $\{\hbar_n\}$ is convergent to a point $\hbar \in \mathcal{X}$, that is,

$$\lim_{n\to\infty} S_b(\hbar_n, \hbar_n, \hbar) = 0.$$

Next, we establish that $\hbar$ is a common fixed point of g, h, G and H. Using (*ii*), for $\mu = \hbar_{2n}$ and $\tau = \hbar$, we get

$$\begin{aligned} S_b(g\hbar_{2n}, g\hbar_{2n}, h\hbar) &= S_b(\hbar_{2n+1}, \hbar_{2n+1}, h\hbar) \\ &\leq \psi\begin{pmatrix} S_b(g\hbar_{2n}, g\hbar_{2n}, h\hbar), S_b(\hbar_{2n}, \hbar_{2n}, \hbar), \\ S_b(\hbar_{2n}, \hbar_{2n}, g\hbar_{2n}), S_b(\hbar, \hbar, h\hbar), \\ S_b(\hbar, \hbar, g\hbar_{2n}), S_b(\hbar_{2n}, \hbar_{2n}, h\hbar) \end{pmatrix} \\ &= \psi\begin{pmatrix} S_b(\hbar_{2n+1}, \hbar_{2n+1}, h\hbar), S_b(\hbar_{2n}, \hbar_{2n}, \hbar), \\ S_b(\hbar_{2n}, \hbar_{2n}, \hbar_{2n+1}), S_b(\hbar, \hbar, h\hbar), \\ S_b(\hbar, \hbar, \hbar_{2n+1}), S_b(\hbar_{2n}, \hbar_{2n}, h\hbar) \end{pmatrix} \end{aligned}$$

and taking $n \to \infty$, we obtain

$$\begin{aligned} S_b(\hbar, \hbar, h\hbar) &\leq \psi\begin{pmatrix} S_b(\hbar, \hbar, h\hbar), S_b(\hbar, \hbar, \hbar), \\ S_b(\hbar, \hbar, \hbar), S_b(\hbar, \hbar, h\hbar), \\ S_b(\hbar, \hbar, \hbar), S_b(\hbar, \hbar, h\hbar) \end{pmatrix} \\ &= \psi\begin{pmatrix} S_b(\hbar, \hbar, h\hbar), 0, 0, \\ S_b(\hbar, \hbar, h\hbar), 0, S_b(\hbar, \hbar, h\hbar) \end{pmatrix} \end{aligned} \tag{2.10}$$

and

$$S_b(\hbar, \hbar, h\hbar) \leq 2sS_b(\hbar, \hbar, h\hbar) + s.0. \tag{2.11}$$

Using (2.10), (2.11) and (ψ^*), there exists a $k \in [0, 1)$ such that

$$S_b(\hbar, \hbar, h\hbar) \leq k.0 = 0,$$

that is,

$$h\hbar = \hbar.$$

Using the continuity hypothesis of g, we have

$$\begin{aligned} \lim_{n\to\infty} S_b(\hbar_{2n}, \hbar_{2n}, \hbar) &= 0 \\ &\Longrightarrow \lim_{n\to\infty} S_b(g\hbar_{2n}, g\hbar_{2n}, g\hbar) = 0 \\ &\Longrightarrow \lim_{n\to\infty} S_b(\hbar_{2n+1}, \hbar_{2n+1}, g\hbar) = 0 \\ &\Longrightarrow S_b(\hbar, \hbar, g\hbar) = 0 \\ &\Longrightarrow g\hbar = \hbar. \end{aligned}$$

Hence $\hbar$ is a common fixed point g and h. Using (*iii*), for $\mu = \hbar_{2n}$ and $\tau = \hbar$, we get

$$S_b(G\hbar_{2n}, G\hbar_{2n}, H\hbar) = S_b(\hbar_{2n+1}, \hbar_{2n+1}, H\hbar)$$

$$\leq \psi\begin{pmatrix} S_b(G\hbar_{2n}, G\hbar_{2n}, H\hbar), S_b(\hbar_{2n}, \hbar_{2n}, \hbar), \\ S_b(\hbar_{2n}, \hbar_{2n}, G\hbar_{2n}), S_b(\hbar, \hbar, H\hbar), \\ S_b(\hbar, \hbar, G\hbar_{2n}), S_b(\hbar_{2n}, \hbar_{2n}, H\hbar) \end{pmatrix}$$

$$= \psi\begin{pmatrix} S_b(\hbar_{2n+1}, \hbar_{2n+1}, H\hbar), S_b(\hbar_{2n}, \hbar_{2n}, \hbar), \\ S_b(\hbar_{2n}, \hbar_{2n}, \hbar_{2n+1}), S_b(\hbar, \hbar, H\hbar), \\ S_b(\hbar, \hbar, \hbar_{2n+1}), S_b(\hbar_{2n}, \hbar_{2n}, H\hbar) \end{pmatrix}$$

and taking $n \to \infty$, we obtain

$$\begin{aligned} S_b(\hbar, \hbar, H\hbar) &\leq \psi\begin{pmatrix} S_b(\hbar, \hbar, H\hbar), S_b(\hbar, \hbar, \hbar), \\ S_b(\hbar, \hbar, \hbar), S_b(\hbar, \hbar, H\hbar), \\ S_b(\hbar, \hbar, \hbar), S_b(\hbar, \hbar, H\hbar) \end{pmatrix} \\ &= \psi\begin{pmatrix} S_b(\hbar, \hbar, H\hbar), 0, 0, \\ S_b(\hbar, \hbar, H\hbar), 0, S_b(\hbar, \hbar, H\hbar) \end{pmatrix} \end{aligned} \tag{2.12}$$

and

$$S_b(\hbar, \hbar, H\hbar) \leq 2sS_b(\hbar, \hbar, H\hbar) + s.0. \tag{2.13}$$

Using (2.12), (2.13) and (ψ^*), there exists a $k \in [0, 1)$ such that

$$S_b(\hbar, \hbar, H\hbar) \leq k.0 = 0,$$

that is,

$$H\hbar = \hbar.$$

Using the continuity hypothesis of G, we have

$$\begin{aligned} \lim_{n\to\infty} S_b(\hbar_{2n}, \hbar_{2n}, \hbar) &= 0 \\ &\Longrightarrow \lim_{n\to\infty} S_b(G\hbar_{2n}, G\hbar_{2n}, G\hbar) = 0 \\ &\Longrightarrow \lim_{n\to\infty} S_b(\hbar_{2n+1}, \hbar_{2n+1}, G\hbar) = 0 \\ &\Longrightarrow S_b(\hbar, \hbar, G\hbar) = 0 \\ &\Longrightarrow G\hbar = \hbar. \end{aligned}$$

Consequently, we obtain

$$\hbar = h\hbar = g\hbar = H\hbar = G\hbar,$$

that is, $\hbar$ is a common fixed point of four self-mappings g, h, G and H. □

3. Some fixed-figure results

In this section, we investigate some fixed-figure results on S_b-metric spaces. At first, we recall the following notions:

Definition 3.1. *[22, 31] Let $(\mathcal{X}, S_b)$ be an S_b-metric space with $s \geq 1$ and $\hbar_0, \hbar_1, \hbar_2 \in \mathcal{X}$, $r \in [0, \infty)$.*

- *The circle is defined by*

$$C^{S_b}_{\hbar_0, r} = \{\mu \in \mathcal{X} : S_b(\mu, \mu, \hbar_0) = r\}.$$

- The disc is defined by

$$D^{S_b}_{\hbar_0,r} = \{\mu \in \mathcal{X} : S_b(\mu,\mu,\hbar_0) \leq r\}.$$

- The ellipse is defined by

$$E^{S_b}_r(\hbar_1,\hbar_2) = \{\mu \in \mathcal{X} : S_b(\mu,\mu,\hbar_1) + S_b(\mu,\mu,\hbar_2) = r\}.$$

- The hyperbola is defined by

$$H^{S_b}_r(\hbar_1,\hbar_2) = \{\mu \in \mathcal{X} : |S_b(\mu,\mu,\hbar_1) - S_b(\mu,\mu,\hbar_2)| = r\}.$$

- The Cassini curve is defined by

$$C^{S_b}_r(\hbar_1,\hbar_2) = \{\mu \in \mathcal{X} : S_b(\mu,\mu,\hbar_1)S_b(\mu,\mu,\hbar_2) = r\}.$$

- The Apollonious circle is defined by

$$A^{S_b}_r(\hbar_1,\hbar_2) = \left\{\mu \in \mathcal{X} - \{\hbar_2\} : \frac{S_b(\mu,\mu,\hbar_1)}{S_b(\mu,\mu,\hbar_2)} = r\right\}.$$

Definition 3.2. *[22] Let $g : \mathcal{X} \to \mathcal{X}$ be a self-mapping where $(\mathcal{X}, S_b)$ is a S_b-metric space with $s \geq 1$. Let $Fix(g)$ be set of all fixed points of g, then a geometric figure $\mathcal{F}$ is said to be a fixed figure of g if $\mathcal{F}$ is contained in $Fix(g)$.*

Let us define the number r as

$$r = \inf\{S_b(\mu,\mu,g\mu) : \mu \notin Fix(g)\}. \tag{3.1}$$

Theorem 3.1. *Let $(\mathcal{X}, S_b)$ be an S_b-metric space with $s \geq 1$, $g : \mathcal{X} \to \mathcal{X}$ be a self-mapping, S_b be symmetric and r be defined as in (3.1). If there exist $\hbar_0 \in \mathcal{X}$ and $\psi \in \Psi$ for all $\mu \in \mathcal{X} - \{\hbar_0\}$ such that*

$$\mu \notin Fix(g) \Longrightarrow S_b(g\mu, g\mu, \mu) < \psi\begin{pmatrix} S_b(g\mu,g\mu,\mu), S_b(\mu,\mu,\hbar_0), \\ S_b(\mu,\mu,g\hbar_0), S_b(g\mu,g\mu,\mu), \\ S_b(g\hbar_0,g\hbar_0,\hbar_0), S_b(g\mu,g\mu,\hbar_0) \end{pmatrix}$$

and $g\hbar_0 = \hbar_0$, then $D^{S_b}_{\hbar_0,r} \subset Fix(g)$. Especially, we have $C^{S_b}_{\hbar_0,r} \subset Fix(g)$.

Proof. Let $r = 0$. Then we have $D^{S_b}_{\hbar_0,r} = \{\hbar_0\}$. By the hypothesis $g\hbar_0 = \hbar_0$, we obtain

$$D^{S_b}_{\hbar_0,r} \subset Fix(g).$$

Let $r > 0$ and $\mu \in D^{S_b}_{\hbar_0,r}$ such that $\mu \notin Fix(g)$. Using the hypothesis, we get

$$\begin{aligned} S_b(g\mu,g\mu,\mu) &< \psi\begin{pmatrix} S_b(g\mu,g\mu,\mu), S_b(\mu,\mu,\hbar_0), \\ S_b(\mu,\mu,g\hbar_0), S_b(g\mu,g\mu,\mu), \\ S_b(g\hbar_0,g\hbar_0,\hbar_0), S_b(g\mu,g\mu,\hbar_0) \end{pmatrix} \\ &= \psi\begin{pmatrix} S_b(g\mu,g\mu,\mu), S_b(\mu,\mu,\hbar_0), \\ S_b(\mu,\mu,\hbar_0), S_b(g\mu,g\mu,\mu), \\ S_b(\hbar_0,\hbar_0,\hbar_0), S_b(g\mu,g\mu,\hbar_0) \end{pmatrix} \end{aligned}$$

$$= \psi \begin{pmatrix} S_b(g\mu, g\mu, \mu), S_b(\mu, \mu, \hbar_0), \\ S_b(\mu, \mu, \hbar_0), S_b(g\mu, g\mu, \mu), \\ 0, S_b(g\mu, g\mu, \hbar_0) \end{pmatrix}. \tag{3.2}$$

By (S_b2) and the symmetry property of S_b, we have

$$\begin{aligned} S_b(g\mu, g\mu, \hbar_0) &\leq s[2S_b(g\mu, g\mu, \mu) + S_b(\hbar_0, \hbar_0, \mu)] \\ &= 2sS_b(g\mu, g\mu, \mu) + sS_b(\mu, \mu, \hbar_0). \end{aligned} \tag{3.3}$$

Using (3.2), (3.3) and (ψ^*), there exists a $k \in [0, 1)$ such that

$$S_b(g\mu, g\mu, \mu) \leq kS_b(\mu, \mu, \hbar_0) \leq kr \leq kS_b(g\mu, g\mu, \mu) < S_b(g\mu, g\mu, \mu),$$

a contradiction. Hence it should be $\mu \in Fix(g)$. Consequently, we get

$$D^{S_b}_{\hbar_0,r} \subset Fix(g).$$

Using the similar arguments, it can be easily see that

$$C^{S_b}_{\hbar_0,r} \subset Fix(g).$$

□

Theorem 3.2. *Let (X, S_b) be an S_b-metric space with $s \geq 1$, $g : X \to X$ be self-mapping, S_b be a symmetric and r be defined as in (3.1). If there exist $\hbar_1, \hbar_2 \in X$ and $\psi \in \Psi$ for all $\mu \in X - \{\hbar_1, \hbar_2\}$ such that*

$$\mu \notin Fix(g) \Longrightarrow S_b(g\mu, g\mu, \mu) < \psi \begin{pmatrix} S_b(g\mu, g\mu, \mu), S_b(\mu, \mu, \hbar_1) + S_b(\mu, \mu, \hbar_2), \\ S_b(\mu, \mu, g\hbar_1) + S_b(\mu, \mu, g\hbar_2), S_b(g\mu, g\mu, \mu), \\ S_b(g\hbar_1, g\hbar_1, \hbar_1) + S_b(g\hbar_2, g\hbar_2, \hbar_2), \\ S_b(g\mu, g\mu, \hbar_1) + S_b(g\mu, g\mu, \hbar_2) \end{pmatrix}$$

and $g\hbar_1 = \hbar_1$, $g\hbar_2 = \hbar_2$ with $g(\mu) \in E^{S_b}_r(\hbar_1, \hbar_2)$, then $E^{S_b}_r(\hbar_1, \hbar_2) \subset Fix(g)$.

Proof. Let $r = 0$. Then we have $E^{S_b}_r(\hbar_1, \hbar_2) = \{\hbar_1\} = \{\hbar_2\}$. By the hypothesis $g\hbar_1 = \hbar_1$ and $g\hbar_2 = \hbar_2$, we obtain

$$E^{S_b}_r(\hbar_1, \hbar_2) \subset Fix(g).$$

Let $r > 0$ and $\mu \in E^{S_b}_r(\hbar_1, \hbar_2)$ such that $\mu \notin Fix(g)$. Using the hypothesis, we get

$$\begin{aligned} S_b(g\mu, g\mu, \mu) &< \psi \begin{pmatrix} S_b(g\mu, g\mu, \mu), S_b(\mu, \mu, \hbar_1) + S_b(\mu, \mu, \hbar_2), \\ S_b(\mu, \mu, g\hbar_1) + S_b(\mu, \mu, g\hbar_2), S_b(g\mu, g\mu, \mu), \\ S_b(g\hbar_1, g\hbar_1, \hbar_1) + S_b(g\hbar_2, g\hbar_2, \hbar_2), \\ S_b(g\mu, g\mu, \hbar_1) + S_b(g\mu, g\mu, \hbar_2) \end{pmatrix} \\ &= \psi \begin{pmatrix} S_b(g\mu, g\mu, \mu), S_b(\mu, \mu, \hbar_1) + S_b(\mu, \mu, \hbar_2), \\ S_b(\mu, \mu, \hbar_1) + S_b(\mu, \mu, \hbar_2), S_b(g\mu, g\mu, \mu), \\ S_b(\hbar_1, \hbar_1, \hbar_1) + S_b(\hbar_2, \hbar_2, \hbar_2), \\ S_b(g\mu, g\mu, \hbar_1) + S_b(g\mu, g\mu, \hbar_2) \end{pmatrix} \end{aligned}$$

$$= \psi\begin{pmatrix} S_b(g\mu, g\mu, \mu), S_b(\mu, \mu, \hbar_1) + S_b(\mu, \mu, \hbar_2), \\ S_b(\mu, \mu, \hbar_1) + S_b(\mu, \mu, \hbar_2), S_b(g\mu, g\mu, \mu), \\ 0, S_b(g\mu, g\mu, \hbar_1) + S_b(g\mu, g\mu, \hbar_2) \end{pmatrix}$$
$$= \psi(S_b(g\mu, g\mu, \mu), r, r, S_b(g\mu, g\mu, \mu), 0, r).$$

Since

$$r \leq 2sS_b(g\mu, g\mu, \mu) + sr,$$

using (ψ^*), there exists a $k \in [0, 1)$ such that

$$S_b(g\mu, g\mu, \mu) \leq kr \leq kS_b(g\mu, g\mu, \mu) < S_b(g\mu, g\mu, \mu),$$

a contradiction. Hence it should be $\mu \in Fix(g)$. Consequently, we get

$$E_r^{S_b}(\hbar_1, \hbar_2) \subset Fix(g).$$

□

Theorem 3.3. *Let $(\mathcal{X}, S_b)$ be an S_b-metric space with $s \geq 1$, $g : \mathcal{X} \to \mathcal{X}$ be self-mapping, S_b be a symmetric and r be defined as in (3.1). If $r > 0$ and there exist $\hbar_1, \hbar_2 \in \mathcal{X}$, $\psi \in \Psi$ for all $x \in \mathcal{X} - \{\hbar_1, \hbar_2\}$ such that*

$$\mu \notin Fix(g) \Longrightarrow S_b(g\mu, g\mu, \mu) < \psi\begin{pmatrix} S_b(g\mu, g\mu, \mu), |S_b(\mu, \mu, \hbar_1) - S_b(\mu, \mu, \hbar_2)|, \\ |S_b(\mu, \mu, g\hbar_1) - S_b(\mu, \mu, g\hbar_2)|, S_b(g\mu, g\mu, \mu), \\ |S_b(g\hbar_1, g\hbar_1, \hbar_1) - S_b(g\hbar_2, g\hbar_2, \hbar_2)|, \\ |S_b(g\mu, g\mu, \hbar_1) - S_b(g\mu, g\mu, \hbar_2)| \end{pmatrix}$$

and $g\hbar_1 = \hbar_1$, $g\hbar_2 = \hbar_2$ with $g(\mu) \in H_r^{S_b}(\hbar_1, \hbar_2)$, then $H_r^{S_b}(\hbar_1, \hbar_2) \subset Fix(g)$.

Proof. Let $r > 0$ and $\mu \in H_r^{S_b}(\hbar_1, \hbar_2)$ such that $\mu \notin Fix(g)$. Using the hypothesis, we get

$$S_b(g\mu, g\mu, \mu) < \psi\begin{pmatrix} S_b(g\mu, g\mu, \mu), |S_b(\mu, \mu, \hbar_1) - S_b(\mu, \mu, \hbar_2)|, \\ |S_b(\mu, \mu, g\hbar_1) - S_b(\mu, \mu, g\hbar_2)|, S_b(g\mu, g\mu, \mu), \\ |S_b(g\hbar_1, g\hbar_1, \hbar_1) - S_b(g\hbar_2, g\hbar_2, \hbar_2)|, \\ |S_b(g\mu, g\mu, \hbar_1) - S_b(g\mu, g\mu, \hbar_2)| \end{pmatrix}$$
$$= \psi\begin{pmatrix} S_b(g\mu, g\mu, \mu), |S_b(\mu, \mu, \hbar_1) - S_b(\mu, \mu, \hbar_2)|, \\ |S_b(\mu, \mu, \hbar_1) - S_b(\mu, \mu, \hbar_2)|, S_b(g\mu, g\mu, \mu), \\ |S_b(\hbar_1, \hbar_1, \hbar_1) - S_b(\hbar_2, \hbar_2, \hbar_2)|, \\ |S_b(g\mu, g\mu, \hbar_1) - S_b(g\mu, g\mu, \hbar_2)| \end{pmatrix}$$
$$= \psi\begin{pmatrix} S_b(g\mu, g\mu, \mu), |S_b(\mu, \mu, \hbar_1) - S_b(\mu, \mu, \hbar_2)|, \\ |S_b(\mu, \mu, \hbar_1) - S_b(\mu, \mu, \hbar_2)|, S_b(g\mu, g\mu, \mu), \\ 0, |S_b(g\mu, g\mu, \hbar_1) - S_b(g\mu, g\mu, \hbar_2)| \end{pmatrix}$$
$$= \psi(S_b(g\mu, g\mu, \mu), r, r, S_b(g\mu, g\mu, \mu), 0, r).$$

Since

$$r \leq 2sS_b(g\mu, g\mu, \mu) + sr,$$

using (ψ^*), there exists a $k \in [0,1)$ such that

$$S_b(g\mu, g\mu, \mu) \leq kr \leq kS_b(g\mu, g\mu, \mu) < S_b(g\mu, g\mu, \mu),$$

a contradiction. Hence it should be $\mu \in Fix(g)$. Consequently, we get

$$H_r^{S_b}(\hbar_1, \hbar_2) \subset Fix(g).$$

□

Theorem 3.4. *Let $(\mathcal{X}, S_b)$ be an S_b-metric space with $s \geq 1$, $g : \mathcal{X} \to \mathcal{X}$ be a self-mapping, S_b be symmetric and r be defined as in (3.1). If there exist $\hbar_1, \hbar_2 \in \mathcal{X}$ and $\psi \in \Psi$ for all $\mu \in \mathcal{X} - \{\hbar_1, \hbar_2\}$ such that*

$$\mu \notin Fix(g) \Longrightarrow S_b(g\mu, g\mu, \mu) < \psi \begin{pmatrix} S_b(g\mu, g\mu, \mu), S_b(\mu, \mu, \hbar_1) S_b(\mu, \mu, \hbar_2), \\ S_b(\mu, \mu, g\hbar_1) S_b(\mu, \mu, g\hbar_2), S_b(g\mu, g\mu, \mu), \\ S_b(g\hbar_1, g\hbar_1, \hbar_1) S_b(g\hbar_2, g\hbar_2, \hbar_2), \\ S_b(g\mu, g\mu, \hbar_1) S_b(g\mu, g\mu, \hbar_2) \end{pmatrix}$$

and $g\hbar_1 = \hbar_1$, $g\hbar_2 = \hbar_2$ with $g(\mu) \in C_r^{S_b}(\hbar_1, \hbar_2)$, then $C_r^{S_b}(\hbar_1, \hbar_2) \subset Fix(g)$.

Proof. Let $r = 0$. Then we have $C_r^{S_b}(\hbar_1, \hbar_2) = \{\hbar_1\}$ or $\{\hbar_2\}$. By the hypothesis $g\hbar_1 = \hbar_1$ and $g\hbar_2 = \hbar_2$, we obtain

$$C_r^{S_b}(\hbar_1, \hbar_2) \subset Fix(g).$$

Let $r > 0$ and $\mu \in C_r^{S_b}(\hbar_1, \hbar_2)$ such that $\mu \notin Fix(g)$. Using the hypothesis, we get

$$\begin{aligned} S_b(g\mu, g\mu, \mu) &< \psi \begin{pmatrix} S_b(g\mu, g\mu, \mu), S_b(\mu, \mu, \hbar_1) S_b(\mu, \mu, \hbar_2), \\ S_b(\mu, \mu, g\hbar_1) S_b(\mu, \mu, g\hbar_2), S_b(g\mu, g\mu, \mu), \\ S_b(g\hbar_1, g\hbar_1, \hbar_1) S_b(g\hbar_2, g\hbar_2, \hbar_2), \\ S_b(g\mu, g\mu, \hbar_1) S_b(g\mu, g\mu, \hbar_2) \end{pmatrix} \\ &= \psi \begin{pmatrix} S_b(g\mu, g\mu, \mu), S_b(\mu, \mu, \hbar_1) S_b(\mu, \mu, \hbar_2), \\ S_b(\mu, \mu, \hbar_1) S_b(\mu, \mu, \hbar_2), S_b(g\mu, g\mu, \mu), \\ S_b(\hbar_1, \hbar_1, \hbar_1) S_b(\hbar_2, \hbar_2, \hbar_2), \\ S_b(g\mu, g\mu, \hbar_1) S_b(g\mu, g\mu, \hbar_2) \end{pmatrix} \\ &= \psi \begin{pmatrix} S_b(g\mu, g\mu, \mu), S_b(\mu, \mu, \hbar_1) S_b(\mu, \mu, \hbar_2), \\ S_b(\mu, \mu, \hbar_1) S_b(\mu, \mu, \hbar_2), S_b(g\mu, g\mu, \mu), \\ 0, S_b(g\mu, g\mu, \hbar_1) S_b(g\mu, g\mu, \hbar_2) \end{pmatrix} \\ &= \psi(S_b(g\mu, g\mu, \mu), r, r, S_b(g\mu, g\mu, \mu), 0, r). \end{aligned}$$

Since

$$r \leq 2sS_b(g\mu, g\mu, \mu) + sr,$$

using (ψ^*), there exists a $k \in [0,1)$ such that

$$S_b(g\mu, g\mu, \mu) \leq kr \leq kS_b(g\mu, g\mu, \mu) < S_b(g\mu, g\mu, \mu),$$

a contradiction. Hence it should be $\mu \in Fix(g)$. Consequently, we get

$$C_r^{S_b}(\hbar_1, \hbar_2) \subset Fix(g).$$

□

Theorem 3.5. *Let $(\mathcal{X}, S_b)$ be an S_b-metric space with $s \geq 1$, $g : \mathcal{X} \to \mathcal{X}$ be a self-mapping, S_b be symmetric and r be defined as in (3.1). If there exist $\hbar_1, \hbar_2 \in \mathcal{X}$ and $\psi \in \Psi$ for all $\mu \in \mathcal{X} - \{\hbar_1, \hbar_2\}$ such that*

$$\mu \notin Fix(g) \Longrightarrow S_b(g\mu, g\mu, \mu) < \psi\left(\begin{array}{c} S_b(g\mu, g\mu, \mu), \frac{S_b(\mu,\mu,\hbar_1)}{S_b(\mu,\mu,\hbar_2)}, \frac{S_b(\mu,\mu,g\hbar_1)}{S_b(\mu,\mu,g\hbar_2)}, \\ S_b(g\mu, g\mu, \mu), 0, \frac{S_b(g\mu,g\mu,\hbar_1)}{S_b(g\mu,g\mu,\hbar_2)} \end{array}\right)$$

and $g\hbar_1 = \hbar_1$, $g\hbar_2 = \hbar_2$ with $g(\mu) \in A_r^{S_b}(\hbar_1, \hbar_2)$, then $A_r^{S_b}(\hbar_1, \hbar_2) \subset Fix(g)$.

Proof. Let $r = 0$. Then we have $A_r^{S_b}(\hbar_1, \hbar_2) = \{\hbar_1\}$. By the hypothesis $g\hbar_1 = \hbar_1$, we obtain

$$A_r^{S_b}(\hbar_1, \hbar_2) \subset Fix(g).$$

Let $r > 0$ and $\mu \in A_r^{S_b}(\hbar_1, \hbar_2)$ such that $\mu \notin Fix(g)$. Using the hypothesis, we get

$$\begin{aligned} S_b(g\mu, g\mu, \mu) &< \psi\left(\begin{array}{c} S_b(g\mu, g\mu, \mu), \frac{S_b(\mu,\mu,\hbar_1)}{S_b(\mu,\mu,\hbar_2)}, \frac{S_b(\mu,\mu,g\hbar_1)}{S_b(\mu,\mu,g\hbar_2)}, \\ S_b(g\mu, g\mu, \mu), 0, \frac{S_b(g\mu,g\mu,\hbar_1)}{S_b(g\mu,g\mu,\hbar_2)} \end{array}\right) \\ &= \psi\left(\begin{array}{c} S_b(g\mu, g\mu, \mu), \frac{S_b(\mu,\mu,\hbar_1)}{S_b(\mu,\mu,\hbar_2)}, \frac{S_b(\mu,\mu,\hbar_1)}{S_b(\mu,\mu,\hbar_2)}, \\ S_b(g\mu, g\mu, \mu), 0, \frac{S_b(g\mu,g\mu,\hbar_1)}{S_b(g\mu,g\mu,\hbar_2)} \end{array}\right) \\ &= \psi(S_b(g\mu, g\mu, \mu), r, r, S_b(g\mu, g\mu, \mu), 0, r). \end{aligned}$$

Since

$$r \leq 2sS_b(g\mu, g\mu, \mu) + sr,$$

using (ψ^*), there exists a $k \in [0, 1)$ such that

$$S_b(g\mu, g\mu, \mu) \leq kr \leq kS_b(g\mu, g\mu, \mu) < S_b(g\mu, g\mu, \mu),$$

a contradiction. Hence it should be $\mu \in Fix(g)$. Consequently, we get

$$A_r^{S_b}(\hbar_1, \hbar_2) \subset Fix(g).$$

□

Now we give the following illustrative example of above proved geometric results.

Example 3.1. *Let us consider Example 2.2 given in [22]. Let $\mathcal{X} = [-1, 1] \cup \left\{-7, -\sqrt{2}, \sqrt{2}, \frac{7}{3}, 7, 8, 21\right\}$ and the S-metric defined as*

$$S(\mu, \tau, \hbar) = |\mu - \hbar| + |\mu + \hbar - 2\tau|,$$

for all $\mu, \tau, \hbar \in \mathcal{X}$ [32]. Then the function S is also an S_b-metric with $s = 1$. Let us define the function $g : \mathcal{X} \to \mathcal{X}$ as

$$g\mu = \begin{cases} 7 & , \quad \mu = 8 \\ \mu & , \quad \mu \in \mathcal{X} - \{8\} \end{cases},$$

for all $\mu \in \mathcal{X}$ and the function $\psi : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = kt_2,$$

with $k \in [0,1)$. *Under these assumptions, we get*

$$\begin{aligned} r &= \inf\{S(\mu,\mu,g\mu) : \mu \notin Fix(g)\} \\ &= \inf\{S(\mu,\mu,g\mu) : \mu = 8\} = 2. \end{aligned}$$

★ *If we take* $\hbar_0 = 0$ *and* $k = \frac{1}{2}$, *then the function* g *satisfies the conditions of Theorem 3.1. Therefore, we obtain*

$$D_{0,2}^{S_b} = [-1,1] \subset Fix(g) = \mathcal{X} - \{8\}$$

and

$$C_{0,2}^{S_b} = \{-1,1\} \subset Fix(g) = \mathcal{X} - \{8\}.$$

★ *If we take* $\hbar_1 = -\frac{1}{2}$, $\hbar_2 = \frac{1}{2}$ *and* $k = \frac{1}{2}$, *then the function* g *satisfies the conditions of Theorem 3.2. Therefore, we obtain*

$$E_2^{S_b}\left(-\frac{1}{2},\frac{1}{2}\right) = \left[-\frac{1}{2},\frac{1}{2}\right] \subset Fix(g) = \mathcal{X} - \{8\}.$$

★ *If we take* $\hbar_1 = -1$, $\hbar_2 = 1$ *and* $k = \frac{3}{4}$, *then the function* g *satisfies the conditions of Theorem 3.3. Therefore, we obtain*

$$H_2^{S_b}(-1,1) = \left\{-\frac{1}{2},\frac{1}{2}\right\} \subset Fix(g) = \mathcal{X} - \{8\}.$$

★ *If we take* $\hbar_1 = -1$, $\hbar_2 = 1$ *and* $k = \frac{3}{4}$, *then the function* g *satisfies the conditions of Theorem 3.4. Therefore, we obtain*

$$C_2^{S_b}(-1,1) = \left\{-\sqrt{2},0,\sqrt{2}\right\} \subset Fix(g) = \mathcal{X} - \{8\}.$$

★ *If we take* $\hbar_1 = -7$, $\hbar_2 = 7$ *and* $k = \frac{1}{5}$, *then the function* g *satisfies the conditions of Theorem 3.5. Therefore, we obtain*

$$A_2^{S_b}(-7,7) = \left\{\frac{7}{3},21\right\} \subset Fix(g) = \mathcal{X} - \{8\}.$$

4. An application to parametric rectified linear unit activation functions

Recently, activation functions have been used in applicable areas. Especially, these functions are used in neural network. For example, for state-of-the-art neural networks, rectified activation units are essential. Therefore, in this section, we focus on the parametric rectified linear unit (*PReLU*) activation functions [33]. *PReLU* is defined as

$$PReLU(\mu) = \begin{cases} \alpha\mu & , \quad \mu < 0 \\ \mu & , \quad \mu \geq 0 \end{cases},$$

where α is a coefficient.

Let $\mathcal{X} = \mathbb{R}^+ \cup \{-2,-1,0\}$, the S-metric defined as in Example 3.1 and the function $\psi : \mathbb{R}_+^6 \to \mathbb{R}$ defined as in Example 3.1. Let us consider $\alpha = 0.8$, then we obtain the function *PReLU* as

$$PReLU(\mu) = \begin{cases} 0.8\mu & , \quad \mu < 0 \\ \mu & , \quad \mu \geq 0 \end{cases},$$

for all $\mu \in \mathbb{R}$. If we take $\hbar_0 = 0$ and $k = \frac{1}{2}$, then the function *PReLU* satisfies the conditions of Theorem 3.1. Indeed, for $\mu \in (-\infty, 0)$, we get

$$\begin{aligned} S\left(PReLU(\mu), PReLU(\mu), \mu\right) &= 2\left|0.8\mu - \mu\right| = 0.4\left|\mu\right| \\ &< \left|\mu\right| = 2k\left|\mu\right| \\ &= kS\left(\mu, \mu, \hbar_0\right) \\ &= k\psi\begin{pmatrix} S_b\left(PReLU\mu, PReLU\mu, \mu\right), \\ S_b\left(\mu, \mu, \hbar_0\right), \\ S_b\left(\mu, \mu, PReLU\hbar_0\right), \\ S_b\left(PReLU\mu, PReLU\mu, \mu\right), \\ S_b\left(PReLU\hbar_0, PReLU\hbar_0, \hbar_0\right), \\ S_b\left(PReLU\mu, PReLU\mu, \hbar_0\right) \end{pmatrix}. \end{aligned}$$

Also, we obtain

$$\begin{aligned} r &= \inf\left\{S\left(\mu, \mu, PReLU\mu\right) : \mu \notin Fix(PReLU)\right\} \\ &= \inf\left\{0.2\left|\mu\right| : \mu < 0\right\} = \inf\left\{0.2, 0.4\right\} = 0.2. \end{aligned}$$

Consequently, we have

$$D_{0,0.2}^{S_b} = [0, 0.1] \subset Fix(PReLU) = \mathbb{R}^+ \cup \{0\}$$

and similarly

$$C_{0,0.2}^{S_b} = \{0.1\} \subset Fix(PReLU) = \mathbb{R}^+ \cup \{0\}.$$

Finally, we say that the parametric rectified linear unit (*PReLU*) activation function fixes the disc $D_{0,0.2}^{S_b}$ and $C_{0,0.2}^{S_b}$, that is, *PReLU* has at least two fixed figure. In this way, the learning capacity of the activation function *PReLU* increases.

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Conflict of interest

The authors declare no conflicts of interest.

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