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Research article

A priori bounds and existence of smooth solutions to Minkowski problems for log-concave measures in warped product space forms

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Abstract: In the present paper, we prove the a priori bounds and existence of smooth solutions to a Minkowski type problem for the log-concave measure $e^{-f(|x|^2)}dx$ in warped product space forms with zero sectional curvature. Our proof is based on the method of continuity. The crucial factor of the analysis is the a priori bounds of an auxiliary Monge-Ampère equation on $\mathbb{S}^n$. The main result of the present paper extends the Minkowski type problem of log-concave measures to the space forms and it may be an attempt to get some new analysis for the log-concave measures.

Keywords: log-concave measure; Minkowski problem; Monge-Ampère equation; the continuous method; warped product space forms
Mathematics Subject Classification: 35B45, 35J96, 53C42

1. Introduction

In the present paper, we focus on *the geometry of log-concave measure* which is defined as follows:
Definition A.1 (Log-concave Measure(see [12, 34, 43])). *A measure μ is called log-concave if its density $\frac{d\mu(x)}{dx}$ is log-concave, that is, $\frac{d\mu(x)}{dx} = e^{-f(x)}$ for some convex function f which means that*

$$\mu(E) = \int_E e^{-f(x)}dx \tag{1.1}$$

for every Borel set $E \subseteq \mathbb{R}^{n+1}$ and some convex function f.

Now, we provide some examples of log-concave measures.
Examples A.2 (i) **Gauss measure.** The Gauss measure γ_n on $\mathbb{R}^n$ is defined as follows,

$$d\gamma_n = \frac{1}{(2\pi)^{\frac{n+1}{2}}}e^{-\frac{|x|^2}{2}}dx \tag{1.2}$$

which characterizes the Gaussian generalized random processes in stochastic analysis, see Bogachev [4].

(*ii*) **The weighted Bergman measure in Siegel domain.** The domain

$$\Omega_2 = \{z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : \mathrm{Im}(z_{n+1}) > |z'|^2\}. \tag{1.3}$$

is a pseudo-convex domain in $\mathbb{C}^{n+1}$. In order to analyze some potential theory on Ω_2, such as the estimates of Cauchy-Szegö kernel on Ω_2, the suitable Bergman space $X(\Omega_2)$ may be chosen provided the Bergman norm $\|\cdot\|_{X(\Omega_2)}$ is well-defined where

$$\|g\|_{X(\Omega_2)} = \int_{\mathbb{C}^{n+1}} |g(z)|^2 e^{-4\pi\lambda|z|^2} dV(z) \tag{1.4}$$

for any holomorphic function g in $\mathbb{C}^{n+1}$, see pp. 45–66 of Chang and Tie [9]. We may call the measure

$$d\bar{V}(z) = e^{-4\pi\lambda|z|^2} dV(z)$$

be the weighted Bergman measure associated with the Siegel domain Ω_2 and it is easy to see that $d\bar{V}$ is a log-concave measure for any fixed $\lambda > 0$.

(*iii*) **Gibbs measure of some nonlinear Schrödinger equation.** The Gibbs measure $\mathbb{P}(du)$ of some nonlinear Schrödinger equation is defined as follows:

$$\mathbb{P}(du) = e^{-H(u)} du \tag{1.5}$$

where

$$H(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx. \tag{1.6}$$

That is, $H(u)$ is the Hamilton functions for the following Schrödinger equation with unit mass,

$$i\partial_t u = -\Delta u, \tag{1.7}$$

(see a similar description of [17]). The Gibbs measures play an important role in quantum field theory and regularity and asymptotic behaviors of the Cauchy problem for some Schrödinger equations, see [17, 19, 35] and their references.

It may be interesting to mention that some of the classical concepts and results in integral geometry have been generalized to the log-concave measures, such as the support function and Steiner type formulas. Moreover, the convexity of f can be used to deduce some interesting geometric inequalities for the measure $e^{-f(x)}dx$, such as Brunn-Minkowski inequality, Prékopa-Leindler inequalities or Blaschke-Santaló inequalities and so on, see [4, 5, 7, 12, 14, 18, 34, 43]. Naturally, the prescribed log-concave measure problem has also been posed and studied which is called the L_p Minkowski problem of the log-concave measure in the present paper, see [12, 15, 30, 38]. The works of [12, 15, 30, 38] can be formulated in the following way:

Problem A.3. *For any fixed $n \geq 1$ and $p \in \mathbb{R}$, given any Borel measures $\frac{1}{\psi(x)}dx$, find a convex function u such that*

$$(\nabla u)_\sharp\left(\frac{u^{p-1}}{\psi(x)}dx\right) = e^{-f(|y|^2)}dy. \tag{1.8}$$

In particular, if the measure $d\mu$ and $e^{-f(x)}dx$ are both supported on the whole space $\mathbb{R}^{n+1}$, Problem A.3 is the so-called L_p Minkowski problem for log-concave measure and has been analyzed, see [12, 15, 43].

If $N = \mathbb{S}^n$ and the support set of the measure $e^{-f(|x|^2)}dx$ lies on the boundary of a hypersurface M, noting that in smooth case, the normal mapping ν and the support function u of a hypersurface M satisfies

$$\nu^{-1} = \nabla u, \tag{1.9}$$

Problem A.3 can be stated as follows,

Problem A.4 (A Minkowski problem for log-concave measure). *For any fixed $n \geq 1$ and $p \in \mathbb{R}$, given a non-negative, finite Borel measure $d\mu = \frac{1}{\psi(\xi)}d\xi$ defined on the unit sphere $\mathbb{S}^n$, find a convex hypersurface $M \subseteq \mathbb{R}^{n+1}$ such that*

$$\nu_\sharp(u^{1-p}e^{-f(|x|^2)}d\sigma(x)) = d\mu(\xi), \tag{1.10}$$

where ν, u and $d\sigma$ are the normal mapping, support function and surface measure of a convex hypersurface $M \subseteq \mathbb{R}^{n+1}$ respectively, f is convex.

In particular, if $f \equiv 0$, Problem A.4 becomes the following classical Minkowski problem for p-curvature function which is also called L_p Minkowski problem.

Problem A.5 (The classical Minkowski problem for p-curvature function). *For any fixed $n \geq 1$ and $p \in \mathbb{R}$, given a non-negative, finite Borel measure μ defined on the unit sphere $\mathbb{S}^n$, find a convex hypersurface $M \subseteq \mathbb{R}^{n+1}$ such that*

$$\nu_\sharp(u^{1-p}d\sigma(x)) = d\mu(\xi), \tag{1.11}$$

where ν, u and $d\sigma$ are the normal mapping, support function and surface measure of a convex hypersurface $M \subseteq \mathbb{R}^{n+1}$ respectively.

In particular, if $p = 1$, Problem A.5 was posed and analyzed by Minkowski for his wonderful construction of the Gaussian curvature (measure) via natural arguments in convex and integral geometry provided the measure $d\mu$ is the sum of the delta measure or the measure $d\mu$ is absolutely continuous with respect to the spherical Lebesgue measure whose density is continuous, see [44]. Later, Aleksandrov (see [44]) and Fenchel and Jensen (see [44]) generated the result of Minkowski for the general Borel measure on the unit sphere independently. Later, with the help of PDEs, Problem A.5 was resolved by Lewy, Nirenberg, Pogorelov, Cheng and Yau and so on, (see [44]).

Noting that Gaussian curvature is the Jocabian of normal mapping,

Problem A.6 (A prescribed Gaussian curvature problem). *For any fixed $n \geq 1$ and $p \in \mathbb{R}$, find a smooth convex hypersurface $M \subseteq \mathbb{R}^{n+1}$ such that*

$$\frac{u^{1-p}e^{-f(\rho^2)}}{\mathcal{K}} = \frac{1}{\psi(\xi)}, \tag{1.12}$$

where u, ρ and $\mathcal{K}$ are the support function, radial function and Gaussian curvature function of a convex hypersurface $M \subseteq \mathbb{R}^{n+1}$, f is convex.

It may be worth mention that the arguments of Minkowski are based on some convex analysis of volume under the Minkowski sum, such as Brunn-Minkowski inequality and Hadamard variational formula, see [8, 21, 44]. A natural generalization of Minkowski sum is the so-called p-sum posed by Firey [16] for any fixed $p \geq 1$ due to the convexity of the function $g(t) = t^p$ for $p \geq 1$. Based on the concept of p-sum posed by Firey, Lutwak [39] introduced to p-Gaussian curvature function and posed and studied Problem A.5 which is called L_p Minkowski problem later. More results on L_p Minkowski problem can be referred to [6, 11, 22, 26–28, 31, 32, 40, 41]

On the one hand, recently, more and more researchers have been focusing on prescribed curvature problem in Riemannian manifolds, see [10, 24, 37] and so on. In the point of view of development of geometric analysis, it is interesting to focus on geometric problems for log-concave measures in Riemannian manifolds.

On the other hand, recently, classical stochastic analysis has been developing in Riemannian manifolds [3, 13, 29, 45]. It follows from Example A.2 that the log-concave measures have their origin in stochastic analysis, it is also interesting to focus on the theory of integral geometry of log-concave measures in Riemannian manifolds.

The main focus of the present paper is on Problem A.3 when the support set of the log-concave measure $e^{-f(|x|^2)}dx$ enjoys a more interesting metric structure. Among them, one interesting object is the so-called warped product space forms.

In the polar coordinate system, the metric of the hypersurface M satisfies

$$ds^2 = d\rho^2(\xi) + \rho^2(\xi)d\xi^2. \tag{1.13}$$

where ρ is the so-called radial function defined in (1.2). As a generalization to (1.13), one may consider a hypersurface M in $\mathbb{R}^{n+1}$ whose metric satisfies

$$ds^2 = d\rho^2(\xi) + \varphi^2(\rho)(\xi)d\xi^2 \tag{1.14}$$

for any given function $\varphi : (0, \infty) \mapsto \mathbb{R}$, see [2]. In particular, if $\varphi(\rho) = \rho$, M is a hypersurface in Euclidean Space. Motivated by the work of Aleksandrov's construction of integral Gaussian curvature, Oliker [42] focused on the existence of hypersurface which was prescribed the so-called integral Gaussian curvature and zero sectional curvature in a smooth frame. The works of Aleksandrov and Oliker provided new motivations on the geometric analysis of the warped product space forms, see [2, 23, 25, 33, 36, 46] and so on.

It is worth mentioning that the target hypersurfaces of [30, 38] both lie in $\mathbb{R}^{n+1}$, that is, $\varphi(\rho) = \rho$ provided we suppose the metric of M with the form (1.14). It is natural to analyze Problem A.3 when the metric of the hypersurface M satisfies (1.14) for a given function φ.

In a smooth frame, if the sectional curvature of the hypersurface M is zero, we know that the Gaussian curvature $\mathcal{K}$ and the support function u of M can be written as follows:

$$\mathcal{K} = \frac{\det(-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij})}{\varphi^{n-2}(\rho)(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n}{2}+1}} \tag{1.15}$$

and

$$u = \frac{\varphi^2(\rho)}{\sqrt{\varphi^2(\rho) + |\nabla\rho|^2}} \tag{1.16}$$

(see Lemma A in Appendix). Therefore, we focus on the existence of smooth solutions to the Eq (1.1) on the unit sphere $\mathbb{S}^n$:

$$\frac{\det(-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij})}{(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+1+p}{2}}} = \psi(\xi)e^{f(\rho^2)}\varphi^{n-2p}(\rho). \tag{1.17}$$

Before stating the main result of the present paper, we assume the following conditions hold.
$(A.1.)$ f, φ and ψ are both C^2 positive functions,

$$\|f\|_{C^2(\mathbb{R})} + \|\psi\|_{C^2(\mathbb{R})} + \|\varphi\|_{C^2(\mathbb{R})} < \infty \tag{1.18}$$

and

$$\begin{cases} \lim\limits_{\rho\to\infty} e^{-f(\rho^2)} \frac{\varphi^{n+1-p}(\rho)}{(\varphi'(\rho))^n} = 0; \\ \lim\limits_{\rho\to 0} e^{-f(\rho^2)} \frac{\varphi^{n+1-p}(\rho)}{(\varphi'(\rho))^n} = \infty. \end{cases} \tag{1.19}$$

$(A.2.)$ The function $e^{-f(t^2)}\varphi^{n+1+p}(t)$ is non-increasing on $(0,\infty)$.
$(A.3.)$ $\inf\limits_{t>0} \varphi''(t) \geq 0$ and there exists a positive number γ such that

$$nt\varphi''(t) \leq \gamma\varphi'(t) \tag{1.20}$$

for any $t > 0$.

The main result of the present paper can be stated as follows,

Theorem 1.1. *For any fixed $n \geq 1$ and $p > -n-1$, suppose that the assumptions* $(A.1.) \sim (A.3.)$ *holds, then there exists a $\rho \in C^2(\mathbb{S}^n)$ to Eq (1.17) satisfying*

$$\|\rho\|_{C^2(\mathbb{S}^n)} \leq c, \tag{1.21}$$

where c is independent of ρ.

Remark 1.2. It follows from (1.1) that the equation is associated to the so-called prescribed Gauss curvature problem for the log-concave measure $e^{-f(|x|^2)}dx$ which may be an attempt on more differential geometric analysis for the log-concave measure $e^{-f(|x|^2)}dx$. In particular, if $f(t) = \frac{n+1}{2}\ln(2\pi) + \frac{t}{2}$, some of these topics have been focused on, see [4,5,7,8,18].

The rest of the paper is organized as follows: In Section 2, we get the a priori bounds of solutions. In Section 3, we prove Theorem 1.1. In the Appendix, we list some basic geometric quantity associated to the discussion.

2. A priori bounds

Section 2 devotes to the a priori bounds of solutions to the following equation on the unit sphere $\mathbb{S}^n$:

$$\frac{\det(-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij})}{(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+1+p}{2}}} = \psi(\xi)e^{f(\rho^2)}\varphi^{n-2p}(\rho). \tag{2.1}$$

We let the set of the positive continuous function on the unit sphere $\mathbb{S}^n$ be $C_+(\mathbb{S}^n)$ and

$$C = \{\rho \in C^{2,\sigma}(\mathbb{S}^n) : (-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij}) \text{ is positive definite}\}. \tag{2.2}$$

This main result of this section can be stated as follows,

Theorem 2.0. *For any fixed $n \geq 1$ and $p > -n-1$, let $\rho \in C \cap C_+(\mathbb{S}^n)$ be a solution to (2.1) and f, φ and ψ satisfy the condition* $(A.1.)$. *Then there exists a positive constant c, independent of ρ, such that*

$$0 < c^{-1} \leq \|\rho\|_{C^{2,\sigma}(\mathbb{S}^n)} \leq c < \infty, \tag{2.3}$$

where $\sigma \in (0,1)$.

Now, we divide the proof of Theorem 2.0 into following several of lemmas.

Lemma 2.1. *For any fixed* $n \geq 1$, *let* $\rho \in C \cap C_+(\mathbb{S}^n)$ *be a solution to (2.1) and* f, φ *and* ψ *satisfy the condition* (*A*.1.). *Then there exists a positive constant c, independent of* ρ, *such that*

$$0 < c^{-1} \leq \rho(\xi) \leq c < \infty, \forall \xi \in \mathbb{S}^n. \tag{2.4}$$

Proof. We consider the following extremal problem,

$$R = \max_{\xi \in \mathbb{S}^n} \rho(\xi). \tag{2.5}$$

It follows from the compactness of $\mathbb{S}^n$ and the continuity of ρ that there exists $\xi_1 \in \mathbb{S}^n$ such that

$$R = \rho(\xi_1). \tag{2.6}$$

It follows from (2.1) that at the point $\xi = \xi_1$,

$$\begin{aligned} \frac{(\varphi'(R))^n}{\varphi^{1+p}(R)} &= \frac{(\varphi'(R)\varphi(R))^n}{\varphi^{n+1+p}(R)} \\ &\leq \frac{\det(-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij})}{(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+1+p}{2}}} = \psi(\xi_1)e^{f(R^2)}\varphi^{n-2p}(R), \end{aligned} \tag{2.7}$$

that is,

$$e^{f(R^2)}\frac{\varphi^{n+1-p}(R)}{(\varphi'(R))^n} \geq \frac{1}{\psi(\xi_1)} \geq \frac{1}{\max\limits_{\xi \in \mathbb{S}^n} \psi(\xi)} > 0. \tag{2.8}$$

However, there exists a contradiction between (2.8) and (1.19) provided R is sufficiently large. This implies there exists a positive constant $c > 0$ such that

$$R \leq c < \infty. \tag{2.9}$$

Adopting a similar argument, we also get

$$r \geq \frac{1}{c} > 0. \tag{2.10}$$

(2.9) and (2.10) yield the desired conclusion of lemma 2.1.

The following lemma can be referred to [2]. For the sake of the completeness, we give the proof here.

Lemma 2.2. *We let* ρ *be a solution of* (2.1) *and* $v(\xi) = \frac{\varphi'(\rho(\xi))}{\varphi(\rho(\xi))}$, *then v solves the following equation,*

$$\frac{\det(v_{ij} + v\delta_{ij})}{(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+1+p}{2}}} = \psi(\xi)e^{f(\rho^2)}\varphi^{-n-2p}(\rho). \tag{2.11}$$

Proof. By the definition of v, we have,

$$v_i = -\frac{\rho_i}{\varphi^2(\rho)}, v_{ij} = -\frac{\rho_{ij}}{\varphi^2(\rho)} + \frac{2\varphi'(\rho)}{\varphi^3(\rho)}\rho_i\rho_j \tag{2.12}$$

and

$$v_{ij} + v\delta_{ij} = \frac{1}{\varphi^2(\rho)}(-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij}). \tag{2.13}$$

Therefore,

$$\begin{aligned} u^{1-p}e^{f(\rho^2)}\psi(\xi) = \mathcal{K} &= \frac{\det(-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij})}{\varphi^{n-2}(\rho)(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+2}{2}}} \\ &= \frac{\det(v_{ij} + v\delta_{ij})}{\varphi^{-n-2}(\rho)(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+2}{2}}}. \end{aligned} \tag{2.14}$$

Noting that

$$u = \frac{\varphi^2(\rho)}{\sqrt{\varphi^2(\rho) + |\nabla\rho|^2}} \tag{2.15}$$

we have,

$$\frac{\det(v_{ij} + v\delta_{ij})}{(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+1+p}{2}}} = \psi(\xi)e^{f(\rho^2)}\varphi^{-n-2p}(\rho). \tag{2.16}$$

This is the desired conclusion of Lemma 2.2.

In the rest of this section, we will consider the a priori bounds of solutions to Eq (2.11).

It is easy to see that

$$\rho \in C \Leftrightarrow v \in \{v \in C^{2,\sigma}(\mathbb{S}^n) : (v_{ij} + \delta_{ij}v) \text{ is positive definite}\} \triangleq \bar{C}. \tag{2.17}$$

Lemma 2.3. *For any fixed $n \geq 1$ and $p > -n-1$, we let $v \in \bar{C} \cap C_+(\mathbb{S}^n)$ be the solution of* (2.11). *Suppose that f, φ and ψ satisfy the condition* (*A*.1.). *Then there exists a positive constant c, independent of ρ, such that*

$$0 \leq |\nabla v(\xi)| \leq c, \forall \xi \in \mathbb{S}^n. \tag{2.18}$$

Thus,

$$0 \leq |\nabla \rho(\xi)| \leq c, \forall \xi \in \mathbb{S}^n. \tag{2.19}$$

Proof. The proof follows from the argument of Oliker [42]. Let $\{\xi_i\}_{i=1}^n$ be a system of smooth local orthogonal coordinates on $\mathbb{S}^n$, we therefore get

$$|d\xi|^2 = \delta_{ij}d\xi_i d\xi_j, \tag{2.20}$$

where δ_{ij} is the Dirac notation. It is easy to see that

$$|\nabla v|^2 = \delta_{ij}\frac{\partial v}{\partial \xi_i}\frac{\partial v}{\partial \xi_j} = \sum_{i=1}^{n}|\frac{\partial v}{\partial \xi_i}|^2, \tag{2.21}$$

(see pp. 812 of Oliker [42]). We let

$$u = \frac{v^2 + |\nabla v|^2}{2}.$$

Suppose that there exists $\xi_2 \in \mathbb{S}^n$ such that

$$u(\xi_2) = \max_{\xi \in \mathbb{S}^n} u(\xi).$$

Then, for any fixed $i \in \{1, 2, \cdots, n\}$, at the point ξ_2,

$$0 = u_i = \sum_{j=1}^{n} (v_{ij} + v\delta_{ij}) \frac{\partial v}{\partial \xi_j}. \tag{2.22}$$

It follows from Lemma 2.1 that there exists a positive constant c such that

$$\det(v_{ij} + v\delta_{ij}) = \psi(\xi) e^{f(\rho^2)} \varphi^{-n-2p}(\rho)(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+1+p}{2}} \geq c > 0 \tag{2.23}$$

at the point ξ_0. This means that the matrix $(v_{ij} + v\delta_{ij})_{n\times n}$ is nonsingular at the point ξ_2. Therefore, combining with (2.22), we get

$$v_k(\xi_2) = 0$$

for any fixed $k \in \{1, 2, \cdots, n\}$. Therefore, it follows from Lemma 2.1 that there exists a positive constant c such that

$$\frac{1}{2}|\nabla v|^2(\xi) \leq u(\xi) \leq u(\xi_2) = \frac{1}{2} \max_{\xi} v(\xi) \leq c, \forall \xi \in \mathbb{S}^n.$$

This completes the proof of Lemma 2.3.

We first let $W_{ij} = (v_{ij} + \delta_{ij} v)$, $\mathcal{G}(W_{ij}) = (\det W_{ij})^{\frac{1}{n}}$ and

$$\Psi(\xi) = (\psi(\xi) e^{f(\rho^2)} \varphi^{-n-2p}(\rho)(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+1+p}{2}})^{\frac{1}{n}},$$

then Eq (2.11) becomes

$$\mathcal{G}(W_{ij}) = \Psi. \tag{2.24}$$

Lemma 2.4. *For any fixed $n \geq 1$ and $p > -n-1$, let $\rho \in C \cap C_+(\mathbb{S}^n)$ be a solutions of (2.1). Suppose that f, φ and ψ satisfy the condition $(A.1.)$. Then there exists a positive constant c, independent of ρ, such that*

$$-\Delta\rho \leq c. \tag{2.25}$$

Proof. Let $H = \sum_i W_{ii} = \Delta v + nv$. By the commutator identity, we have,

$$H_{ii} = \Delta W_{ii} - nW_{ii} + H. \tag{2.26}$$

Suppose that H achieves it maximum at the point $\xi = \xi_3$. Without loss of generality, we may $(H_{ij})_{n\times n}$ is diagonal at the point $\xi = \xi_3$. Therefore, at the point $\xi = \xi_3$,

$$0 \geq \mathcal{G}^{ij} H_{ij} = \mathcal{G}^{ii}(\Delta W_{ii}) - n\mathcal{G}^{ii} + H\Sigma_i \mathcal{G}^{ii}. \tag{2.27}$$

It follows from (2.25) that

$$\mathcal{G}^{ij}W_{ij\alpha} = \Psi_\alpha, \mathcal{G}^{ij,rs}W_{ij\alpha}W_{rs\alpha} + \mathcal{G}^{ij}\Delta W_{ij} = \Delta\Psi \tag{2.28}$$

By the concavity of $\mathcal{G}$, we have

$$\mathcal{G}^{ij,rs}W_{ij\alpha}W_{rs\alpha} \leq 0. \tag{2.29}$$

This implies that

$$\mathcal{G}^{ii}\Delta W_{ii} \geq \mathcal{G}^{ij,rs}W_{ij\alpha}W_{rs\alpha} + \mathcal{G}^{ij}\Delta W_{ij} = \Delta\Psi. \tag{2.30}$$

Putting (2.30) into (2.27), we have, at the point $\xi = \xi_3$,

$$0 \geq \Delta\Psi - n\Psi + H\Sigma_i\mathcal{G}^{ii}. \tag{2.31}$$

It follows from Newton-MacLaurin inequality that

$$\Sigma_i\mathcal{G}^{ii} \geq 1, \tag{2.32}$$

see [26].

Now, we claim that at the point $\xi = \xi_3$,

$$\frac{\Delta\Psi}{\Psi} \geq \frac{n+1+p}{2n} \min_{\xi\in\mathbb{S}^n} \frac{\varphi^2(\rho(\xi))}{\varphi^2(\rho(\xi)) + |\nabla\rho(\xi)|^2} \sum_{k\alpha} \rho_{k\alpha}^2 - c. \tag{2.33}$$

Indeed, it follows from the definition of Ψ that

$$\begin{aligned} \log\Psi = \frac{1}{n}\log\psi(\xi) - \frac{n+2p}{n}\log\varphi(\rho) + \frac{1}{n}f(\rho^2) \\ + \frac{n+1+p}{2n}\log(\varphi^2(\rho) + |\nabla\rho|^2). \end{aligned} \tag{2.34}$$

For any fixed $\alpha \in \{1, 2, \cdots, n\}$, taking α-th partial derivatives on both sides of (2.34) twice, we have

$$\begin{aligned} \frac{\Psi_\alpha}{\Psi} = (\frac{1}{n}(\log\psi(\xi))' - \frac{n+2p}{n}(\log\varphi)')\rho_\alpha + \frac{2}{n}f'(\rho^2)\rho\rho_\alpha \\ + \frac{n+1+p}{n}\frac{(\varphi(\rho)\varphi'(\rho)\rho_\alpha + \rho_k\rho_{k\alpha})}{\varphi^2(\rho) + |\nabla\rho|^2} \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} \frac{\Delta\Psi}{\Psi} - \frac{|\nabla\Psi|^2}{\Psi^2} &= \Sigma_\alpha\frac{\Psi_{\alpha\alpha}}{\Psi} - \frac{\Psi_\alpha^2}{\Psi^2} \\ &= \Sigma_\alpha(\frac{1}{n}(\log\psi)'' - \frac{n+2p}{n}(\log\varphi)'\rho_{\alpha\alpha} \\ &- \frac{n+2p}{n}(\log\varphi)''\rho_\alpha^2 + \frac{2f'(\rho^2)}{n}(\rho_\alpha^2 + \rho\rho_{\alpha\alpha}) \\ &+ \frac{4f''(\rho^2)}{n}\rho^2\rho_\alpha^2 + \frac{n+1+p}{n}\frac{(\varphi(\rho)\varphi''(\rho) + (\varphi'(\rho))^2)\rho_\alpha^2 + \varphi(\rho)\varphi'(\rho)\rho_{\alpha\alpha}}{\varphi^2(\rho) + |\nabla\rho|^2} \\ &+ \frac{n+1+p}{n}\frac{\sum_k\rho_{k\alpha}^2 + \sum_k\rho_{k\alpha\alpha}\rho_k}{\varphi^2(\rho) + |\nabla\rho|^2} \\ &- \frac{n+1+p}{n}\frac{(\varphi(\rho)\varphi'(\rho)\rho_\alpha + \sum_k\rho_k\rho_{k\alpha})^2}{(\varphi^2(\rho) + |\nabla\rho|^2)^2}) = \sum_{j=1}^4 I_j. \end{aligned} \tag{2.36}$$

where

$$I_1 = (-\frac{n+2p}{n}(\log\varphi)' + \frac{n+1+p}{n}\frac{\varphi(\rho)\varphi'(\rho)}{\varphi^2(\rho)+|\nabla\rho|^2} + \frac{2}{n}f'(\rho^2)\rho)\Delta\rho, \tag{2.37}$$

$$\begin{aligned} I_2 = &(\log\psi)'' + (-\frac{n+2p}{n}(\log\varphi)'' + \frac{2}{n}f'(\rho^2) + \frac{4}{n}f''(\rho^2)\rho^2 \\ &+ \frac{n+1+p}{n}\frac{(\varphi(\rho)\varphi''(\rho) + (\varphi'(\rho))^2)}{\varphi^2(\rho)+|\nabla\rho|^2} \\ &- \frac{n+1+p}{n}\frac{(\varphi(\rho)\varphi'(\rho))^2}{(\varphi^2(\rho)+|\nabla\rho|^2)^2})|\nabla\rho|^2, \end{aligned} \tag{2.38}$$

$$I_3 = -\frac{n+1+p}{n}(\frac{\sum_\alpha(\sum_k\rho_k\rho_{k\alpha})^2}{(\varphi^2(\rho)+|\nabla\rho|^2)^2} + \frac{2\sum_{k\alpha}\rho_k\rho_\alpha\rho_{k\alpha}\varphi(\rho)\varphi'(\rho)}{(\varphi^2(\rho)+|\nabla\rho|^2)^2}) \tag{2.39}$$

and

$$I_4 = \frac{n+1+p}{n}(\frac{\sum_{k\alpha}\rho_{k\alpha}^2}{\varphi^2(\rho)+|\nabla\rho|^2} + \frac{\nabla\rho\cdot\nabla\Delta\rho}{\varphi^2(\rho)+|\nabla\rho|^2}). \tag{2.40}$$

We first estimate the term I_1. It follows from Hölder inequality that

$$\begin{aligned} |I_1| \le c\Delta\rho = c\Sigma_i\rho_{ii} \\ &\le \frac{(n+1+p)\varepsilon}{n}\Sigma_i\rho_{ii}^2 + \frac{c}{2\varepsilon} \\ &\le \frac{(n+1+p)\varepsilon}{n}\Sigma_{k\alpha}\rho_{k\alpha}^2 + \frac{c}{2\varepsilon}. \end{aligned} \tag{2.41}$$

for some ε to be chosen later. Therefore,

$$I_1 \ge -\frac{(n+1+p)\varepsilon}{n}\Sigma_{k\alpha}\rho_{k\alpha}^2 + \frac{c}{2\varepsilon}. \tag{2.42}$$

Now, we turn to the estimate of the term I_2. It follows from Lemma 2.2 that

$$I_2 \ge -c. \tag{2.43}$$

Now, we estimate the term I_3. It follows from Hölder inequality and Lemma 2.2 that

$$\begin{aligned} |\frac{2\sum_{k\alpha}\rho_k\rho_\alpha\rho_{k\alpha}\varphi(\rho)\varphi'(\rho)}{(\varphi^2(\rho)+|\nabla\rho|^2)^2}| &\le c|\nabla\rho|^2(\sum_{k\alpha}\rho_{k\alpha})^2)^{\frac{1}{2}} \\ &\le c(\sum_{k\alpha}\rho_{k\alpha})^2)^{\frac{1}{2}} \\ &\le \frac{(n+1+p)\varepsilon}{n}\sum_{k\alpha}\rho_{k\alpha}^2 + \frac{c}{4\varepsilon} \end{aligned} \tag{2.44}$$

for the same ε as in (2.41) and

$$\frac{\sum_\alpha(\sum_k\rho_k\rho_{k\alpha})^2}{(\varphi^2(\rho)+|\nabla\rho|^2)^2} \le \frac{|\nabla\rho|^2}{(\varphi^2(\rho)+|\nabla\rho|^2)^2}\sum_{k\alpha}\rho_{k\alpha}^2 \tag{2.45}$$

Putting (2.44) and (2.45) into (2.36), we have

$$I_3 \geq -\frac{n+1+p}{n}(\frac{|\nabla\rho|^2}{(\varphi^2(\rho)+|\nabla\rho|^2)^2}+\varepsilon)\sum_{k\alpha}\rho_{k\alpha}^2 - c. \tag{2.46}$$

Now, we estimate the term I_4. Since

$$v_i = -\frac{\rho_i}{\varphi^2(\rho)}, \tag{2.47}$$

we have

$$v_{ii} = -\frac{\rho_{ii}}{\varphi^2(\rho)} + \frac{2\varphi'(\rho)}{\varphi^3(\rho)}\rho_i^2 \tag{2.48}$$

and thus,

$$\Delta\rho = \frac{2\varphi'(\rho)}{\varphi(\rho)}|\nabla\rho|^2 - \varphi^2(\rho)\Delta v. \tag{2.49}$$

Moreover, since

$$H = \Delta v + nv, \tag{2.50}$$

we have,

$$\Delta\rho = (-H+nv)\varphi^2(\rho) + \frac{2\varphi'(\rho)}{\varphi(\rho)}|\nabla\rho|^2. \tag{2.51}$$

Therefore, for any $i \in \{1, 2, \cdots, n\}$, we have

$$\begin{aligned}(\Delta\rho)_i &= -H_i\varphi^2(\rho) + n\varphi^2(\rho)v_i - 2\varphi(\rho)(H-nv)\varphi'(\rho)\rho_i \\ &\quad + (\frac{2\varphi''(\rho)}{\varphi(\rho)} - \frac{2(\varphi'(\rho))^2}{\varphi^2(\rho)})\rho_i|\nabla\rho|^2 + \frac{2\varphi'(\rho)}{\varphi(\rho)}\sum_l \rho_l\rho_{li} \\ &= -H_i\varphi^2(\rho) - n\rho_i - 2\varphi(\rho)(H-nv)\varphi'(\rho)\rho_i \\ &\quad + (\frac{2\varphi''(\rho)}{\varphi(\rho)} - \frac{2(\varphi'(\rho))^2}{\varphi^2(\rho)})\rho_i|\nabla\rho|^2 + \frac{2\varphi'(\rho)}{\varphi(\rho)}\sum_l \rho_l\rho_{li}\end{aligned} \tag{2.52}$$

and

$$\begin{aligned}\nabla\rho\cdot\nabla\Delta\rho &= -\nabla\rho\cdot\nabla H\varphi^2(\rho) - n|\nabla\rho|^2 - 2\varphi(\rho)(H-nv)\varphi'(\rho)|\nabla\rho|^2 \\ &\quad + (\frac{2\varphi''(\rho)}{\varphi(\rho)} - \frac{2(\varphi'(\rho))^2}{\varphi^2(\rho)})|\nabla\rho|^4 + \frac{2\varphi'(\rho)}{\varphi(\rho)}\sum_{li}\rho_l\rho_i\rho_{li} \\ &= -n|\nabla\rho|^2 - 2\varphi(\rho)(H-nv)\varphi'(\rho)|\nabla\rho|^2 \\ &\quad + (\frac{2\varphi''(\rho)}{\varphi(\rho)} - \frac{2(\varphi'(\rho))^2}{\varphi^2(\rho)})|\nabla\rho|^4 + \frac{2\varphi'(\rho)}{\varphi(\rho)}\sum_{li}\rho_l\rho_i\rho_{li}\end{aligned} \tag{2.53}$$

at the point $\xi = \xi_3$ since ξ_3 is a critical point of H. It follows from Lemma 2.2 that

$$-n|\nabla\rho|^2 - 2\varphi(\rho)(H-nv)\varphi'(\rho)|\nabla\rho|^2 + (\frac{2\varphi''(\rho)}{\varphi(\rho)} - \frac{2(\varphi'(\rho))^2}{\varphi(\rho)^2})|\nabla\rho|^4 \geq -c \tag{2.54}$$

at the point $\xi = \xi_3$. It follows from Hölder inequality that

$$\frac{2\varphi'(\rho)}{\varphi(\rho)}\sum_{li}\rho_l\rho_i\rho_{li} \leq \frac{(n+1+p)\varepsilon}{n}\sum_{li}\rho_{li}^2 + \frac{c}{4\varepsilon} \tag{2.55}$$

for the same ε as in (2.41). Therefore, at the point $x = x_3$, we have,

$$\frac{n+2-p}{n}\frac{\nabla\rho\cdot\nabla\Delta\rho}{\varphi^2(\rho)+|\nabla\rho|^2} \geq -\frac{(n+1+p)\varepsilon}{n}\sum_{li}\rho_{li}^2 - \frac{c}{4\varepsilon} - c. \tag{2.56}$$

Putting (2.56) into (2.40), we have,

$$I_4 \geq \frac{n+1+p}{n}\left(\frac{1}{\varphi^2(\rho)+|\nabla\rho|^2} - \varepsilon\right)\sum_{li}\rho_{li}^2 + \frac{c}{2\varepsilon} - c. \tag{2.57}$$

Therefore,

$$\begin{aligned}\frac{\Delta\Psi}{\Psi} &\geq \sum_{j=1}^{4} I_j \\ &\geq \frac{n+1+p}{n}\left(\frac{1}{\varphi^2(\rho)+|\nabla\rho|^2} - \frac{|\nabla\rho|^2}{(\varphi^2(\rho)+|\nabla\rho|^2)^2} - 2\varepsilon\right)\sum_{li}\rho_{li}^2 + \frac{c}{2\varepsilon} - c \\ &\geq \frac{n+1+p}{n}\left(\frac{\varphi^2(\rho)}{\varphi^2(\rho)+|\nabla\rho|^2} - 2\varepsilon\right)\sum_{li}\rho_{li}^2 - \frac{c}{2\varepsilon} - c.\end{aligned} \tag{2.58}$$

Let $\varepsilon_0 = \min\limits_{\xi\in\mathbb{S}^n}\frac{\varphi^2(\rho(\xi))}{\varphi^2(\rho(\xi))+|\nabla\rho(\xi)|^2}$, for any $\varepsilon \in (0, 4\varepsilon_0)$, we have,

$$\frac{\Delta\Psi}{\Psi} \geq \frac{n+1+p}{2n}\min_{\xi\in\mathbb{S}^n}\frac{\varphi^2(\rho(\xi))}{\varphi^2(\rho(\xi))+|\nabla\rho(\xi)|^2}\sum_{li}\rho_{li}^2 - c. \tag{2.59}$$

(2.31), (2.32) and (2.59) yields that there exists a positive constant c such that

$$\Sigma_{li}\rho_{li}^2 \leq c \tag{2.60}$$

at the point $\xi = \xi_3$. Therefore, it follows from Hölder inequality that

$$-\Delta\rho = -\Sigma_l\rho_{ll} \leq \sqrt{n}\sqrt{\Sigma_l\rho_{ll}^2} \leq \sqrt{n}\sqrt{\Sigma_{li}\rho_{li}^2} \leq c \tag{2.61}$$

at the point $\xi = \xi_3$. This completes the proof of Lemma 2.4.

Now, we are in a position the prove Theorem 2.0.

Final proof of Theorem 2.0. It follows from (2.24) that Eq (2.1) becomes

$$\mathcal{F}(W_{ij}) = 0 \tag{2.62}$$

provided $\mathcal{F}(W_{ij}) = \mathcal{G}(W_{ij}) - \psi$. We let $\mathcal{F}_{ij} = \frac{\partial\mathcal{F}}{\partial W_{ij}}$. It follows from Lemmas 2.1–2.4 that there exist positive constants λ and Λ, independent of W_{ij}, such that

$$0 < \lambda\zeta^2 \leq \mathcal{F}_{ij}\zeta_i\zeta_j \leq \Lambda\zeta^2, \tag{2.63}$$

for any $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in \mathbb{R}^n$. That is,

(i) (2.62) is elliptic uniformly.

Moreover, it is easy to see that $\mathcal{G} = \det^{\frac{1}{n}}$ is concave with respect to W_{ij} and therefore,
(ii) $\mathcal{F}$ is concave with respect to W_{ij}.

Then, it follows from Theorem 17.14 of Gilbarg and Trudinger [20] that there exist $\tau_1 \in (0,1)$ and positive constant c, independent of W, such that

$$\|W\|_{C^{2,\tau_1}(\mathbb{S}^n)} \leq c, \tag{2.64}$$

and therefore there exist $\tau \in (0,1)$ and positive constant c, independent of ρ, such that

$$\|\rho\|_{C^{2,\tau}(\mathbb{S}^n)} \leq c, \tag{2.65}$$

(see pp. 457–461 of Gilbarg and Trudinger [20]). This is the desired conclusion of Theorem 2.0.

3. The proof of Theorem 1.1.

This section devotes to the proof of Theorem 1.1.

Motivated by [42], we consider the following auxiliary problem with a parameter $t \in [0,1]$ on the unit sphere $\mathbb{S}^n$,

$$M(\rho) = \frac{\det(-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij})}{(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+1+p}{2}}} = t\psi(\xi)K(\rho) + (1-t)g(\rho) \triangleq K_t \tag{3.1}$$

where $K(\rho) = e^{-f(\rho^2)}\varphi^{n-2p}(\rho)$ and $g(\rho) = \frac{(\varphi'(\rho))^n}{\varphi^{1+p}(\rho)}\rho^{-\gamma}$ with $\gamma > 0$.

By $(A.1.)$ and the definition of K_t, we have

$$\begin{cases} \lim\limits_{\rho\to\infty} K_t \frac{\varphi^{1+p}(R)}{(\varphi'(R))^n} = 0; \\ \lim\limits_{\rho\to 0} K_t \frac{\varphi^{1+p}(R)}{(\varphi'(R))^n} = \infty. \end{cases} \tag{3.2}$$

for any $t \in [0,1]$.

We let the set of the positive continuous function on the unit sphere $\mathbb{S}^n$ be $C_+(\mathbb{S}^n)$ and

$$C = \{\rho \in C^{2,\sigma}(\mathbb{S}^n) : (-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij}) \text{ is positive definite}\} \tag{3.3}$$

and

$$\mathcal{I} = \{t \in [0,1] : \rho \in C \cap C_+(\mathbb{S}^n),\ (3.1) \text{ is solvable}\}. \tag{3.4}$$

Adopting a similar argument in Section 2, we get

Lemma 3.1. *For any fixed $n \geq 1$, $p > -n-1$ and $t \in [0,1]$, we let $\rho_t \in C \cap C_+(\mathbb{S}^n)$ be a solution of (3.1). Suppose that the condition $(A.1.)$ holds, then there exists a constant c, independent on t, such that*

$$0 < c^{-1} \leq |\rho_t|_{C^{2,\sigma}(\mathbb{S}^n)} \leq c,$$

for any $t \in [0,1]$ and some $\sigma \in (0,1)$.

As a corollary of Lemma 3.1, we have,

Corollary 3.2. *For any fixed* $n \geq 1$, $p > -n-1$ *and* $t \in [0,1]$, *we let* $\mathcal{I}$ *is the set defined in* (3.4). *Suppose that* f, φ *and* ψ *satisfy the condition* (A.1). *Then* $\mathcal{I}$ *is closed.*

Proof. It suffices to show that for any sequence $\{t_j\}_{j=1}^{\infty} \subseteq \mathcal{I}$ satisfying

$$t_j \to t_0,$$

as $j \to \infty$ for some $t_0 \in [0,1]$, we need to prove $t_0 \in \mathcal{I}$.

We let ρ_j be a solutions of problem (3.1) at $t = t_j$. It follows from the conclusion of Lemma 3.1 that there exists a positive constant c, independent of j such that

$$\|\rho_j\|_{C^{2,\sigma}(\mathbb{S}^n)} \leq c.$$

By Ascoli-Arzela Theorem, we see that, up to a subsequence, there exists a $\rho_0 \in C^2(\mathbb{S}^n)$

$$\|\rho_j - \rho_0\|_{C^2(\mathbb{S}^n)} \to 0$$

as $j \to \infty$. It is easy to see that

$$M(\rho_j) \to M(\rho_0), K(\rho_j) \to K(\rho_0), g(\rho_j) \to g(\rho_0) \tag{3.5}$$

uniformly on $\mathbb{S}^n$ as $j \to \infty$. Letting $j \to \infty$, we can see that (t_0, ρ_0) is a solution to the following problem:

$$\frac{\det(-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij})}{(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+1+p}{2}}} = t\psi(\xi)K(\rho) + (1-t)g(\rho) \tag{3.6}$$

This implies that $t_0 \in \mathcal{I}$. This is the desired conclusion of Corollary 3.2.

Lemma 3.3. *For any fixed* $n \geq 1$, $p > -n-1$ *and* $t \in [0,1]$, *we let* $\mathcal{I}$ *is the set defined in* (3.4). *Suppose that* f, φ *and* ψ *satisfy the conditions* (A.1.), (A.1.) *and* (A.3.). *Then* $\mathcal{I}$ *is open.*

Proof. Suppose that there exists a $\bar{t} \in \mathcal{I}$ and a $\delta > 0$, for any $t_1 \in B_\delta(\bar{t}) \cap [0,1]$, we need to prove that $t_1 \in \mathcal{I}$. To achieve this goal, joint with Implicit Function Theorem, we need to analyze the kernel of linearized equation associated to (3.1). We assume that $\bar{\rho}$ is a solution to equation (3.1) at $t = \bar{t}$. For any $\zeta \in \mathbb{S}^n$, we let $M[\bar{\rho}](\zeta) = \frac{d}{d\varepsilon}M(\bar{\rho} + \varepsilon\zeta)|_{\varepsilon=0}$, $K_t[\bar{\rho}](\zeta) = \frac{d}{d\varepsilon}K_t(\bar{\rho} + \varepsilon\zeta)|_{\varepsilon=0}$ and

$$G_t(\bar{\rho}) = M(\bar{\rho}) - K_t(\bar{\rho}). \tag{3.7}$$

It is easy to see that

$$\begin{aligned} G_t[\bar{\rho}](\zeta) = \frac{d}{d\varepsilon}G_t(\bar{\rho} + \varepsilon\zeta)|_{\varepsilon=0} &= \frac{d}{d\varepsilon}M(\bar{\rho} + \varepsilon\zeta)|_{\varepsilon=0} - \frac{d}{d\varepsilon}K_t(\bar{\rho} + \varepsilon\zeta)|_{\varepsilon=0} \\ &= M[\bar{\rho}](\zeta) - K_t[\bar{\rho}](\zeta). \end{aligned} \tag{3.8}$$

We first calculate $M[\rho](\zeta)$. Taking logarithm on the left hand side of (3.1), we get

$$\frac{M[\bar{\rho}](\zeta)}{M(\bar{\rho})} = \bar{P}_{ij}B(\zeta) - (n+1+p)\frac{\varphi(\bar{\rho})\varphi'(\bar{\rho})\zeta + \nabla\bar{\rho} \cdot \nabla\zeta}{\varphi^2(\bar{\rho}) + |\nabla\bar{\rho}|^2} \tag{3.9}$$

where $(\bar{P}_{ij})_{n\times n}$ is the inverse of the matrix $(-\bar{\rho}_{ij} + \frac{2\varphi'(\bar{\rho})}{\varphi(\bar{\rho})}\bar{\rho}_i\bar{\rho}_j + \varphi(\bar{\rho})\varphi'(\bar{\rho})\delta_{ij})_{n\times n}$ and

$$B(\zeta) = -\zeta_{ij} + \frac{2\varphi'(\bar{\rho})}{\varphi(\bar{\rho})}(\bar{\rho}_i\zeta_j + \bar{\rho}_j\zeta_i) + ((\frac{2\varphi'(\bar{\rho})}{\varphi(\bar{\rho})})'\bar{\rho}_i\bar{\rho}_j + (\varphi(\bar{\rho})\varphi'(\bar{\rho}))'\delta_{ij})\zeta. \tag{3.10}$$

We first analyze the term $-(n+1+p)\frac{\varphi(\bar\rho)\varphi'(\bar\rho)\zeta+\nabla\bar\rho\cdot\nabla\zeta}{\varphi^2(\bar\rho)+|\nabla\bar\rho|^2}$. We let $\zeta=\varphi(\bar\rho)\eta$. Direct Calculation shows that

$$\zeta_i=\varphi(\bar\rho)\eta_i+\varphi'(\bar\rho)\bar\rho_i\eta \tag{3.11}$$

and

$$\zeta_{ij}=\varphi(\bar\rho)\eta_{ij}+\varphi'(\bar\rho)(\bar\rho_i\eta_j+\bar\rho_j\eta_i)+(\varphi''(\bar\rho)\bar\rho_i\bar\rho_j+\varphi'(\bar\rho)\bar\rho_{ij})\eta. \tag{3.12}$$

We can see that

$$(n+1+p)\frac{\varphi(\bar\rho)\varphi'(\bar\rho)\zeta+\nabla\bar\rho\cdot\nabla\zeta}{\varphi^2(\bar\rho)+|\nabla\bar\rho|^2}=(n+1+p)\frac{\varphi'(\bar\rho)}{\varphi(\bar\rho)}\zeta+(n+1+p)\frac{\varphi(\bar\rho)\nabla\bar\rho\cdot\nabla\eta}{\varphi^2(\bar\rho)+|\nabla\bar\rho|^2}. \tag{3.13}$$

This implies that

$$-(n+1+p)\frac{\varphi(\bar\rho)\varphi'(\bar\rho)\zeta+\nabla\bar\rho\cdot\nabla\zeta}{\varphi^2(\bar\rho)+|\nabla\bar\rho|^2}=-(n+1+p)\frac{\varphi(\bar\rho)\nabla\bar\rho\cdot\nabla\eta}{\varphi^2(\bar\rho)+|\nabla\bar\rho|^2}-(n+1+p)\varphi'(\bar\rho)\eta. \tag{3.14}$$

Now, we move the term $\bar P_{ij}B(\zeta)$. It follows from (3.11) and (3.12) that

$$\begin{aligned}-\zeta_{ij}+\frac{2\varphi'(\bar\rho)}{\varphi(\bar\rho)}(\bar\rho_i\zeta_j+\bar\rho_j\zeta_i)&=-\varphi(\bar\rho)\eta_{ij}+\varphi'(\bar\rho)(\bar\rho_i\zeta_j+\bar\rho_j\zeta_i)\\&\quad+(-\varphi'(\bar\rho)\bar\rho_{ij}+(\frac{4(\varphi'(\bar\rho))^2}{\varphi(\bar\rho)}-\varphi''(\bar\rho))\bar\rho_i\bar\rho_j)\eta.\end{aligned} \tag{3.15}$$

Noting $\frac{4(\varphi'(\bar\rho))^2}{\varphi(\bar\rho)}-\varphi''(\bar\rho)+2\varphi(\bar\rho)(\frac{\varphi'(\bar\rho)}{\varphi(\bar\rho)})'=\frac{2(\varphi'(\bar\rho))^2}{\varphi(\bar\rho)}+\varphi''(\bar\rho)$ and

$$\varphi(\bar\rho)(\varphi(\bar\rho)\varphi'(\bar\rho))'=\varphi(\bar\rho)(\varphi'(\bar\rho))^2+\varphi^2(\bar\rho)\varphi''(\bar\rho),$$

we get

$$\begin{aligned}B(\zeta)&=-\varphi(\bar\rho)\eta_{ij}+\varphi'(\bar\rho_iv_j+\bar\rho_j\eta_i)\\&\quad+(-\varphi'(\bar\rho)\bar\rho_{ij}+(\frac{4(\varphi'(\bar\rho))^2}{\varphi(\bar\rho)}-\varphi''(\bar\rho)+2\varphi(\bar\rho)(\frac{\varphi'(\bar\rho)}{\varphi(\bar\rho)})')\bar\rho_i\bar\rho_j+\varphi(\bar\rho)(\varphi(\bar\rho)\varphi'(\bar\rho))'\delta_{ij})\eta\\&=-\varphi(\bar\rho)\eta_{ij}+\varphi'(\bar\rho)(\bar\rho_i\eta_j+\bar\rho_j\eta_i)\\&\quad+\varphi'(\bar\rho)(-\bar\rho_{ij}+\frac{2\varphi'(\bar\rho)}{\varphi(\bar\rho)}\bar\rho_i\bar\rho_j+\varphi(\bar\rho)\varphi'(\bar\rho)\delta_{ij})\eta+\varphi''(\bar\rho)(\bar\rho_i\bar\rho_j+\varphi^2(\bar\rho)\delta_{ij})\eta.\end{aligned} \tag{3.16}$$

Multiplying the matrix $(\bar P_{ij})_{n\times n}$ on both sides of (3.16), we get

$$\bar P_{ij}B(\zeta)=-\varphi(\bar\rho)\bar P_{ij}\eta_{ij}+2\varphi'(\bar\rho)\bar P_{ij}\bar\rho_i\eta_j+n\varphi'(\bar\rho)+\varphi''(\bar\rho)\bar P_{ij}(\bar\rho_i\bar\rho_j+\varphi^2(\bar\rho)\delta_{ij})\eta. \tag{3.17}$$

Putting (3.17) and (3.14) into (3.9), we have,

$$\begin{aligned}M[\bar\rho](\zeta)&=-\varphi(\bar\rho)M(\bar\rho)\bar P_{ij}\eta_{ij}-(n+2)\frac{\varphi(\bar\rho)(\bar\rho)M(\bar\rho)\nabla\bar\rho\cdot\nabla\eta}{\varphi^2(\bar\rho)+|\nabla\bar\rho|^2}+2M(\bar\rho)\varphi'(\bar\rho)\bar P_{ij}\bar\rho_i\eta_j\\&\quad+M(\bar\rho)\bar P_{ij}\varphi''(\bar\rho)(\bar\rho_i\bar\rho_j+\varphi^2(\bar\rho)\delta_{ij})\eta-(1+p)K_t(\bar\rho)\varphi'(\bar\rho)\eta.\end{aligned} \tag{3.18}$$

By the definition of K_t, we have

$$K_t[\bar{\rho}](\zeta) = K_t'(\bar{\rho})\zeta = t\psi(\xi)K'(\bar{\rho})\varphi(\bar{\rho})\eta + (1-t)g'(\bar{\rho})\varphi(\bar{\rho})\eta \tag{3.19}$$

Combining (3.19), (3.18) and (3.8), we have

$$\begin{aligned}\mathcal{L}(\eta) = G'[\bar{\rho}](\zeta) &= -\varphi(\bar{\rho})M(\bar{\rho})\bar{P}_{ij}\eta_{ij} - (n+1+p)\frac{\varphi(\bar{\rho})M(\bar{\rho})\nabla\bar{\rho}\cdot\nabla\eta}{\varphi^2(\bar{\rho})+|\nabla\bar{\rho}|^2} + 2M(\bar{\rho})\varphi'(\bar{\rho})\bar{P}_{ij}\bar{\rho}_i\eta_j \\ &+ M(\bar{\rho})(\varphi''(\bar{\rho})\bar{P}_{ij}(\bar{\rho}_i\bar{\rho}_j + \varphi^2(\bar{\rho})\delta_{ij}) \\ &- t\psi(\xi)(\varphi(\bar{\rho})\frac{\partial K}{\partial\bar{\rho}} + (1+p)K\varphi(\bar{\rho})') - (1-t)(g'(\bar{\rho})\varphi(\bar{\rho}) + (1+p)g(\bar{\rho})\varphi'(\bar{\rho}))\eta \\ &\triangleq a_{ij}\eta_{ij} + b_i\eta_i + N\eta,\end{aligned} \tag{3.20}$$

where

$$a_{ij} = -\varphi(\bar{\rho})M(\bar{\rho})\bar{P}_{ij}, \tag{3.21}$$

$$b_i = -(n+1+p)\frac{\varphi(\bar{\rho})M(\bar{\rho})\bar{\rho}_i}{\varphi^2(\bar{\rho})+|\nabla\bar{\rho}|^2} + 2M(\bar{\rho})\varphi'(\bar{\rho})\bar{P}_{ji}\bar{\rho}_i \tag{3.22}$$

and

$$\begin{aligned}N =&M(\bar{\rho})(\varphi''(\bar{\rho})\bar{P}_{ij}(\bar{\rho}_i\bar{\rho}_j + \varphi^2(\bar{\rho})\delta_{ij}) \\ &- t\psi(\xi)(\varphi(\bar{\rho})\frac{\partial K}{\partial\bar{\rho}} + (1+p)K\varphi'(\bar{\rho})) - (1-t)(g'(\bar{\rho})\varphi(\bar{\rho}) + (1+p)g(\bar{\rho})\varphi'(\bar{\rho})).\end{aligned} \tag{3.23}$$

Since $\varphi(\bar{\rho}), K_t(\bar{\rho}) > 0$, $(\bar{P}_{ij})_{n\times n}$ is positive, we see that a_{ij} is non-positive. It follows from Lemma 3.1 that b_i is bounded. Now, we claim that

$$N > 0. \tag{3.24}$$

Indeed, it is easy to see that the matrix $(\bar{\rho}_i\bar{\rho}_j + \varphi^2(\bar{\rho})\delta_{ij})$ is positive. Since $(\bar{P}_{ij})$ is positive, we have,

$$\varphi''(\bar{\rho})M(\bar{\rho})\bar{P}_{ij}(\bar{\rho}_i\bar{\rho}_j + \varphi^2(\bar{\rho})\delta_{ij}) \geq 0 \tag{3.25}$$

provided $\varphi''(\bar{\rho}) \geq 0$. It follows from assumption $(A.2.)$ and $(A.3.)$ that

$$\varphi(\bar{\rho})K'(\bar{\rho}) + (1+p)K(\bar{\rho})\varphi'(\bar{\rho}) = \varphi^{-p}(\bar{\rho})(K(\bar{\rho})\varphi^{p+1}(\bar{\rho}))' < 0 \tag{3.26}$$

and

$$\begin{aligned}&\varphi(\bar{\rho})g'(\bar{\rho}) + (1+p)g(\bar{\rho})\varphi'(\bar{\rho}) \\ &= \varphi^{-p}(\bar{\rho})(g(\bar{\rho})\varphi^{p+1}(\bar{\rho}))' \\ &= (\varphi'(\bar{\rho}))^{n-1}\varphi^{-p}(\bar{\rho})\bar{\rho}^{-\gamma-1}(n\varphi''(\bar{\rho})\bar{\rho} - \gamma\varphi'(\bar{\rho})) \leq 0.\end{aligned} \tag{3.27}$$

noting that $\min_{\xi\in\mathbb{S}^n}\psi(\xi) \geq 0$, (3.25), (3.26) and (3.28) imply (3.24). By Strong Maximum Principle for elliptic equations of second order, we see that

$$\eta \equiv 0 \tag{3.28}$$

(see pp. 35 of Gilbarg and Trudinger [20]) and thus,

$$\zeta \equiv 0 \tag{3.29}$$

since $\varphi(\bar{\rho}) > 0$. Then by the standard Implicit Function Theorem, for any $t \in B_\gamma(\bar{t}) \cap [0, 1]$, there exists a $\rho \in C^{2,\sigma_1}(\mathbb{S}^n)$ such that $G_t(\rho) = 0$ for some $\sigma_1 \in (0, 1)$. This means that $t \in \mathcal{I}$ and completes the proof of Lemma 3.3.

Now, we are in a position to prove Theorem 1.1.

Final proof of Theorem 1.1. It is easy to see that $\rho \equiv 1$ is a solution to equation (3.1) at $t = 0$, this means that $0 \in \mathcal{I}$ and thus, $\mathcal{I}$ is not empty. Combining this and Corollary 3.2 and Lemma 3.3, we see that $\mathcal{I} = [0, 1]$. Taking $t = 1$, we get the desired conclusion of Theorem 1.1.

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Conflict of interest

The author declares no conflict of interest.

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A. Appendix

In this section, we list some basic geometric quantity which has been used in the present paper and can be referred to [2].

Lemma A. *Suppose M is a hypersurface in $\mathbb{R}^{n+1}$ with the metric $ds^2 = d\rho^2 + \varphi^2(\rho)d\xi^2$ and with zero sectional curvature, then the following statements hold.*

(a) *The components of the metric g and its inverse g^{-1} can be expressed as follows:*

$$g_{ij} = \varphi^2(\rho)\delta_{ij} + \rho_i\rho_j, g^{ij} = \frac{1}{\varphi^2(\rho)}(\delta^{ij} - \frac{\rho^i\rho^j}{\varphi^2(\rho) + |\nabla\rho|^2}) \tag{A.1}$$

respectively and thus, $\det(g_{ij}) = \varphi^{2n-2}(\rho)(\varphi^2(\rho) + |\nabla\rho|^2)$.

(b) *The coefficients of the second fundamental form b_{ij} is given by:*

$$b_{ij} = \frac{\varphi(\rho)}{\sqrt{\varphi^2(\rho) + |\nabla\rho|^2}}(-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij}). \tag{A.2}$$

(c) *The Gaussian curvature $\mathcal{K}$ was given by:*

$$\mathcal{K}(\xi) = \frac{\det b_{ij}}{\det g_{ij}} = \frac{\det(-\rho_{ij} + \frac{2\varphi'(\rho)}{\varphi(\rho)}\rho_i\rho_j + \varphi(\rho)\varphi'(\rho)\delta_{ij})}{\varphi^{n-2}(\rho)(\varphi^2(\rho) + |\nabla\rho|^2)^{\frac{n+2}{2}}}. \tag{A.3}$$