Mathematics

Research article

Star edge coloring of $K_{2,t}$-free planar graphs

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Abstract: The star chromatic index of a graph G, denoted by $\chi'_{st}(G)$, is the smallest number of colors required to properly color $E(G)$ such that every connected bicolored subgraph is a path with no more than three edges. A graph is $K_{2,t}$-free if it contains no $K_{2,t}$ as a subgraph. This paper proves that every $K_{2,t}$-free planar graph G satisfies $\chi'_{st}(G) \leq 1.5\Delta + 20t + 20$, which is sharp up to the constant term. In particular, our result provides a common generalization of previous results on star edge coloring of outerplanar graphs by Bezegová et al.(2016) and of C_4-free planar graphs by Wang et al.(2018), as those graphs are subclasses of $K_{2,3}$-free planar graphs.

1. Introduction

The graphs discussed in this paper are simple. When possibly multiple edges occurs, we call it a multigraph. The edge set, vertex set, minimum degree and maximum degree of G are denoted by $E(G), V(G), \delta(G)$, and $\Delta(G)$, respectively. We sometimes use E, V, δ, and Δ for short if G is understood in the context. In a multigraph G, an edge coloring satisfying any two adjacent edges assigned distinct colors is called a *proper edge coloring*. The *chromatic index $\chi'(G)$* denotes the minimum integer s such that G admits a proper edge coloring using s colors. In 1964, Vizing [1] obtained a celebrated theorem stating that $\Delta + 1 \geq \chi'(G) \geq \Delta$ for every simple graph G.

In 1983, Fouquet and Jolivet [2] first introduced and studied the notion of strong edge coloring. Here strong edge coloring means proper edge coloring such that any two edges with a distance at most 2 have different colors. Similarly, we use $\chi'_s(H)$, called the *strong chromatic index*, to denote the minimum integer c such that H has a strong edge c-coloring. Introduced by Liu and Deng [3] in 2008, a *star edge coloring* of a graph H is a proper edge coloring such that the edges of all cycles and

paths of length 4 are colored by at least three distinct colors, i.e., there is neither a bicolored path nor cycle with length 4. The star chromatic index $\chi'_{st}(H)$ is the minimum integer s such that H admits a star edge coloring using color set $\{1, 2, \ldots, s\}$. According to the above definition, it is natural to obtain $\chi'(G) \le \chi'_{st}(G) \le \chi'_s(G)$.

From the computational complexity point of view, Lei, Shi, and Song [4] showed that determining whether $\chi'_{st}(G) \le 3$ or not for a given graph is NP-complete. Given the difficulty of computing $\chi'_{st}(G)$, it is natural to consider its valid upper bounds. Liu and Deng [3] first proved a general result that, when $\Delta \ge 7$,

$$\chi'_{st}(G) \le \lceil 16(\Delta - 1)^{\frac{3}{2}} \rceil.$$

For sufficiently large Δ, Dvořák et al. [5] further showed that

$$\chi'_{st}(G) \le \Delta \cdot 2^{O(1)\sqrt{\log \Delta}}.$$

In addition, the star edge coloring problem remains hard for many graph classes. It is difficult even for complete graph K_t, where in [5] it is proved that

$$2t(1 + o(1)) \le \chi'_{st}(K_t) \le t\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log t}}}{(\log t)^{\frac{1}{4}}}.$$

On the other hand, much better results are known for sparse graphs. In [5] it is obtained that every subcubic graph H satisfies $\chi'_{st}(H) \le 7$. For trees, Deng et al. [6] and Bezegová et al. [7] showed a sharp bound below, independently.

Theorem 1.1 ([7], [6]). *For any tree T with maximum degree Δ, we have*

$$\chi'_{st}(T) \le \lfloor 1.5\Delta \rfloor,$$

and the bound is sharp.

Planar graphs are graphs that can be drawn on a plane without any edge crossing. If a graph has a plane embedding such that all vertices of the graph belong to the unbounded face of the drawing, we call it an outerplanar graph. For the star edge coloring of outerplanar graphs, Bezegová et al. [7] showed the results blow.

Theorem 1.2. *[7] Suppose G is an outerplanar graph.*
  (i) If $\Delta \ge 4$, then $\chi'_{st}(G) \le \lfloor 1.5\Delta \rfloor + 12$;
  (ii) If $\Delta \le 3$, then $\chi'_{st}(G) \le 5$.

Wang et al. [8] improved the above bounds of outerplanar graphs and studied more general graphs without 4-cycles by introducing some novel edge-partition techniques. Their results are as below.

Theorem 1.3. *[8] (i) For a planar graph G, $\chi'_{st}(G) \le 2.75\Delta + 18$.*
  (ii) For a planar graph G having no 4-cycles, $\chi'_{st}(G) \le \lfloor 1.5\Delta \rfloor + 18$.
  (iii) For any outerplanar graph H, $\chi'_{st}(H) \le \lfloor 1.5\Delta \rfloor + 5$.

If a planar graph G has no 4-cycles or G is outerplanar, then G is certainly $K_{2,3}$-free, i.e., G contains no complete bipartite graph $K_{2,3}$ as a subgraph. By extending Theorem 1.3, we study the star edge coloring problem of more general planar graphs without $K_{2,t}$ for every $t \ge 2$ and prove the following main result.

Theorem 1.4. *Let $t \geq 2$ be an integer. Then every $K_{2,t}$-free planar graph satisfies*

$$\chi'_{st}(G) \leq 1.5\Delta + 20t + 20.$$

Note that, as constructed in [9], there exists a family of planar graphs with maximum degree Δ containing $K_{2,\Delta}$ as a subgraph such that their star chromatic indices are at least $\frac{13}{8}\Delta - \frac{3}{4}$. So the upper bound in Theorem 1.4 is essentially sharp up to the constant term.

The rest of this paper is divided into several steps for discussion. We first describe the properties of $K_{2,t}$-free graphs in Lemma 2.1. Then, using those properties and some edge-partition methods, we show that $K_{2,t}$-free planar graphs can be edge decomposed into several parts and colored with different strategies to obtain the desired star edge coloring, thus proving our main result Theorem 1.4. Finally, we end this paper with a few concluding remarks.

2. Star edge coloring of $K_{2,t}$-free planar graphs

For vertex $x \in V(G)$ in graph G, let $d_G(x)$(or $d(x)$ for short) be the degree of vertex x. If $d(x) = k$ ($d(x) \geq k$, or $d(x) \leq k$, respectively), then x is called a k-vertex(k^+-vertex, or k^--vertex, respectively). Let $N_G(x)$ denote the set of neighbors of x. A neighbor of vertex x in $N_G(x)$ with degree k (at most k, or at least k, respectively) is called a k-neighbor(k^--neighbor, or k^+-neighbor, respectively).

For a planar graph G with a fixed plane embedded, let $F(G)$ be its face set. For $f \in F(G)$, $d_G(f)$ is used to denote the number of edges contained in face f. If $d_G(f) \geq 4$, then f is called a 4^+-face. A planar embedding of a planar graph G is said an *elegant drawing* if it satisfies the following condition: for each pair of vertices $v, u \in V(G)$ with $\{w_1, w_2, \cdots, w_t\}$ being their common 2-neighbors, in such an embedding the 2-vertices $w_1, w_2, \cdots, w_t$ are all located successively in clockwise (or anticlockwise) order. That is, we require that each 4-cycle vw_iuw_{i+1} for $i \in \{1, 2, \cdots, t-1\}$ is not separating in such an embedding for any pair of vertices. For instance, in Figure 1 graph H_2 is an elegant drawing, while graph H_1 is not.
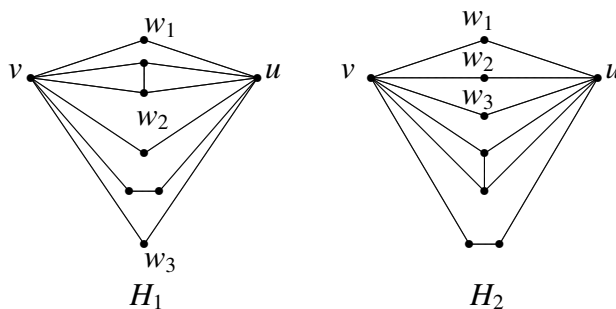

Figure 1. H_2 is an elegant drawing, while H_1 is not.

The key to our proof is the following lemma concerning the structure of planar graphs, which we prove below by discharging method.

Lemma 2.1. *Given an integer $t \geq 2$, let G be a simple planar graph with $\delta(G) \geq 2$. Then either there exists an edge $xy \in E(G)$ with $\max\{d(x), d(y)\} \leq 5t + 5$ or there exist t 2-vertices with two common neighbors.*

Proof. By contradiction, suppose that such counterexamples to Lemma 2.1 exist. Let $\mathcal{P}$ denote the set of graphs G satisfying the following two conditions:

  (i) G is a counterexample to Lemma 2.1,
  (ii) for every 2-vertex u with $N_G(u) = \{u_1, u_2\}$, u_1 and u_2 are adjacent in G, i.e., $u_1 u_2 \in E(G)$.

We first claim that set $\mathcal{P}$ is nonempty.

Claim 1. *We have $\mathcal{P} \neq \emptyset$.*

Proof. Let G be an arbitrary counterexample. That is, we have $\max\{d(x), d(y)\} \geq 5t + 6$ for every edge $xy \in E(G)$ and there do not exist t 2-vertices with two common neighbors in graph G.

If condition (ii) is not satisfied, let u be a 2-vertex with two neighbors a, b which are non-adjacent. Since G is a counterexample, considering the edges ua, ub, we must have $d(a) \geq 5t+6$ and $d(b) \geq 5t+6$. Now we obtain a new graph from G by connecting b and a. Since the degrees of b and a are at least $5t+7$ now, the obtained graph is still a counterexample. So by a finite number of edge addition operations, the condition *(ii)* can be satisfied. Note that the resulting graph is still a simple graph. Hence $\mathcal{P} \neq \emptyset$. ∎

The rest of the proof will involve multigraphs, in which multiple edges may exist. Note that a simple graph itself is also a multigraph. Let Q be the collection of multigraphs H satisfying the following three conditions:

(α-i) *there do not exist t 2-vertices with two common neighbors in H, and for each edge $yx \in E(H)$ we have $\max\{d_H(y), d_H(x)\} \geq 5t + 6$,*
(α-ii) *for every 2-vertex u with $N_H(u) = \{u_1, u_2\}$, u_1 and u_2 are adjacent in H;*
(β) *the multigraph $G' = H - V_2(H)$ is a planar triangulation, where G' is obtained from H by eliminating all 2-vertices of H, denoted by $V_2(H)$.*

We now prove that the set Q is nonempty.

Claim 2. *We have $Q \neq \emptyset$.*

Proof. By Claim 1, $\mathcal{P} \neq \emptyset$, and so there exists a graph or multigraph $H \in \mathcal{P}$ satisfying conditions (α-i) and (α-ii). Choose a multigraph H satisfying conditions (α-i) and (α-ii) such that the total length of all the 4^+-faces in $G' = H - V_2(H)$ is minimized. We shall show below that $G' = H - V_2(H)$ is a triangulation.

By contradiction, fix arbitrary 4^+-face f in G', and denote the vertices in the boundary of f by $v_1, v_2, v_3, v_4, \ldots, v_p$ in a clockwise order. If $\max\{d_H(v_1), d_H(v_3)\} \geq 5t + 6$, we add a new edge $v_1 v_3$ in H to obtain a new multigraph H_1. If $\max\{d_H(v_1), d_H(v_3)\} \leq 5t + 5$, then we must have $d_H(v_2) \geq 5t + 6$ as $v_1 v_2 \in E(H)$ and by (α-i). So we add a new edge $v_2 v_4$ in H to obtain a new multigraph H_2. Let $J = H_1$ or H_2. Then as

$$\min\{d_J(v_1), d_J(v_2), d_J(v_3), d_J(v_4)\} \geq \min_{1 \leq i \leq 4}\{d_H(v_i)\} \geq 3,$$

the multigraph J still satisfies conditions (α-i) and (α-ii). But by the construction of adding a new edge in a 4^+-face, the total length of all the 4^+-faces in J is strictly smaller than that of H. This is a contradiction to the minimality of H. Thus there exists a multigraph $H \in Q$ satisfying conditions (α-i), (α-ii), and (β), proving this lemma. Note that H may be a multigraph. ∎

Now by Claim 2 we choose a multigraph $G \in Q$. Denote $G' = G - V_2(G)$ as a planar triangulation by Claim 2. Then we provide some restrictions on the degrees in G' and G.

Claim 3. *We have $td_{G'}(v) \geq d_G(v) \geq d_{G'}(v)$ for any vertex v.*

Proof. Let $v \in V(G')$ be a vertex in G'. Denote $N_{G'}(v) = \{u_1, u_2, \cdots, u_{d_{G'}(v)}\}$. Note that $G' = G - V_2(G)$ is constructed from G by eliminating all 2-vertices. By condition (α-i), for any edge $vu_i \in E(G')$, where $1 \leq i \leq d_{G'}(v)$, there exists at most $t-1$ 2-vertices in G with v, u_i as common neighbors. In other words, the edge vu_i corresponds to at most $t-1$ 2-vertices in an elegant drawing of G. Hence we obtain the desired inequality

$$td_{G'}(v) = d_{G'}(v) + (t-1)d_{G'}(v) \geq d_G(v) \geq d_{G'}(v).$$

∎

Claim 4. *We have $\delta(G') \geq 3$.*

Proof. Assume $\delta(G') \leq 2$ and a minimum degree vertex of G' is denoted by v. By the construction of G' and by $\delta(G) \geq 2$, we have $d_G(v) \geq 3$, which implies that v is adjacent to no less than one 2-vertex u in G. By Claim 3, $d_G(v) \leq 2t$ and we obtain an edge $uv \in E(G)$ satisfying $\max\{d_G(u), d_G(v)\} \leq 5t + 5$, which is a contradiction to condition (α-i) and Claim 2. ∎

Claim 5. *Each of the following holds.*
   (a) For each $u \in V(G')$ with $d_{G'}(u) \leq 5$, $d_G(u) = d_{G'}(u)$.
   (b) Each 5^--vertex in G' is neither adjacent to any 5^--vertex in G' nor any 5^--vertex in G.
   (c) Each 5^--vertex in G' can not be adjacent to any 6-vertex in G'.

Proof. If (a) is not true, then $d_G(v) > d_{G'}(v)$. Hence the vertex v is adjacent to at least one 2-vertex w in G. By Claim 3, it holds $d_G(v) \leq td_{G'}(v) \leq 5t$. Now we get an edge $vw \in E(G)$ with $\max\{d_G(v), d_G(w)\} \leq 5t + 5$, which is a contradiction to condition (α-i) and Claim 2.

To prove (b), suppose that a 5^--vertex is adjacent to a 5^--vertex in G' or G. This implies that a $(5t)^-$-vertex is adjacent to a $(5t)^-$-vertex in G by Claim 4, which is a contradiction to condition (α-i).

To prove (c), suppose that a 5^--vertex v is adjacent to a 6-vertex u. By Claim 5 (a), we obtain $d_G(v) = d_{G'}(v)$ which implies that the edge uv does not correspond to any 2-vertex of G in an elegant drawing by condition (α-i). For each other edge incident to u, by condition (α-i) it corresponds to at most $t-1$ 2-vertices of G in an elegant drawing. Hence the vertex u has degree $d_G(u) \leq 6 + 5(t-1) < 5t + 5$. Now $vu \in E(G)$ satisfies $\max\{d_G(v), d_G(u)\} \leq 5t + 5$, which is a contradiction to condition (α-i) and Claim 2. ∎

Claim 6. *For any $u \in V(G')$, if u is adjacent to at most five 6^+-vertices in G', then any neighbor v of vertex u has $d_{G'}(v) \geq 6$.*

Proof. By contradiction, suppose that a neighbor v of vertex u has $d_{G'}(v) \leq 5$. By Claim 5(a), we have $d_G(v) = d_{G'}(v) \leq 5$. By Claim 5 (b) again, each 5^--neighbor of u have no 2-vertex of G as their common neighbor in G. For each 6^+-neighbor of u, it has at most $t-1$ common 2-neighbors with u in G. Thus we have

$$d_G(u) \leq d_{G'}(u) + 5(t-1).$$

Notice that G' is a planar triangulation. By Claim 5 (b), considering the embedding of 6^+-vertices neighbors around vertex u in an elegant drawing, there is at most one 5^--neighbor of u between two consecutive 6^+-neighbors. Hence $d_{G'}(u) \leq 10$ as u is adjacent to at most five 6^+-neighbors in G'.

Therefore, we have $d_G(u) \leq d_{G'}(u) + 5(t-1) \leq 5t + 5$, and thus $vu \in E(G)$ satisfies $\max\{d_G(v), d_G(u)\} \leq 5t + 5$, contrary to condition (α-i) and Claim 2. ∎

Finally, we are able to complete the proof by a discharging method on G'. Assign initial charges by setting $m(x) = d_{G'}(x) - 6$ for each vertex $x \in V(G')$ and $m(f) = 2(d_{G'}(f) - 3)$ for each face $f \in F(G')$. Since G' is a planar triangulation, we have $m(f) = 0$ for every face $f \in F(G')$. By rewriting Euler formula, it follows that

$$\sum_{x \in V(G') \cup F(G')} m(x) = \sum_{u \in V(G')} (d_{G'}(u) - 6) + 2 \sum_f (d_{G'}(f) - 3) = -12. \tag{2.1}$$

We shall use the following discharging rule.

Discharging Rule: *Every 5^--vertex v takes charges $\frac{6 - d_{G'}(v)}{d_{G'}(v)}$ from each 7^+-neighbor in G'.*

We will derive the final contradiction from the next claim.

Claim 7. *Each vertex in G' has a non-negative charge after discharging.*

Proof. By Claim 5(c), every neighbor of each 5^--vertex in G' has a degree of at least 7. Hence the final charge of each 5^--vertex is at least zero by the discharging rule.

Assume that $u \in V(G')$ has degree $d_{G'}(u) = k \in \{7, 8, 9, 10\}$. We claim that u is adjacent to at most $k - 6$ 5^--vertex in G'. Otherwise, u is adjacent to at most five 6^+-vertices. Let a be any 5^--neighbor of u. Then we have $d_G(a) = d_{G'}(a) \leq 5$ by Claim 5(a). Since $\max\{d_G(u), d_G(a)\} > 5t + 5$, we have $d_G(u) > 5t + 5 > d_{G'}(u)$. But as u has at most five 6^+-neighbors, and for each 6^+-neighbor at most $t - 1$ 2-vertices are added from G' to G, and thus $d_G(u) \leq 5(t-1) + k \leq 5t + 5$, a contradiction. This verifies our claim that u is adjacent to at most $k - 6$ 5^--vertex in G'. By the discharging rule, the vertex u loses at most $d_{G'}(u) - 6$ charges and remains non-negative.

Assume instead that $u \in V(G')$ has degree $d_{G'}(u) \geq 11$. If u is adjacent to at least six 6^+-vertices, then u is adjacent to at most $d_{G'}(u) - 6$ 5^--vertices in G', and hence the final charge is non-negative by our discharging rule. If u is adjacent to at most five 6^+-vertices, then it follows from Claim 6 that the final charge of u is $d_{G'}(u) - 6 > 0$. Therefore, we have proved that each vertex in G' has a non-negative charge after discharging. ∎

As G' is a planar triangulation, the final charge of each face in G' is zero. By Claim 7, each vertex has a non-negative final charge. Hence total final charge is non-negative, which is a contradiction to Equation (2.1). This contradiction proves Lemma 2.1.

□

Lemma 2.1 leads to the following edge-partition result on $K_{2,t}$-free planar graphs.

Lemma 2.2. *Let G be a $K_{2,t}$-free planar graph, where $t \geq 2$ is an integer. Then we can edge-partition G into a forest T and a subgraph $H \subseteq G$ with $\Delta(H) \leq 5t + 5$.*

Proof. Let G be a minimal counterexample with $|E(G)| + |V(G)|$ as small as possible. If $\delta(G) = 1$, then we denote a 1-vertex by u and its unique neighbor by v. By the minimal counterexample property, we can edge-partition the graph $G' = G - \{uv\}$ into a forest T' and a subgraph $H' \subseteq G'$ with $\Delta(H') \leq 5t + 5$. Denote $T = T' + uv$ and $H = H'$. By $d_G(u) = 1$, T has no cycle, and thus it is a forest with $\Delta(T) \leq \Delta(G)$, and we have further $\Delta(H) \leq 5t + 5$, which is a contradiction. Hence we obtain $\delta(G) \geq 2$. It follows from Lemma 2.1 that we have an edge $xy \in E(G)$ with $\max\{d_G(x), d_G(y)\} \leq 5t + 5$. Let $G^* = G - \{xy\}$.

So $E(G^*)$ can be decomposed into a forest T^* and a subgraph $H^* \subseteq G^*$ with $\Delta(H^*) \leq 5t + 5$. Let $H = H^* + \{xy\}$. Since $\Delta(H) \leq \max\{\Delta(H^*), d_G(x), d_G(y)\}$, $T = T^*$ and $H = H^* + \{xy\}$ are the desired partition of G, which is a contradiction. $\qquad\square$

We need one more result on edge coloring as part of the prerequisites.

Theorem 2.3 ([10],[11]). *Let H be a planar graph with maximum degree Δ. If $k = \max\{\Delta, 7\}$, then H is k-edge-colorable.*

Now it is ready to demonstrate the main result of Theorem 1.4 restated below.

Theorem 2.4. *Let G be a $K_{2,t}$-free planar graph, where $t \geq 2$ is an integer. We have*

$$\chi'_{st}(G) \leq \lfloor 1.5\Delta \rfloor + 4(5t + 5).$$

Proof. By Lemma 2.2, we can edge-partition G into a forest T with $\Delta(T) \leq \Delta(G)$ and a planar subgraph $H \subseteq G$ with $\Delta(H) \leq 5t + 5$. By Theorem 1.1, $\chi'_{st}(T) = k \leq \lfloor 1.5\Delta \rfloor$, thus we have a star k-edge-coloring $\phi : E(T) \mapsto \{f_1, f_2, \ldots, f_k\}$ for T. By Theorem 2.3 and $\Delta(H) \leq 5t + 5$, H is $(5t + 5)$-edge-colorable, and consequently H has an edge-partition $E_1, E_2, \ldots, E_{5t+5}$, where E_i denotes the collection of edges of color-i for each $1 \leq i \leq 5t + 5$. For fixed $i \in \{1, \ldots, 5t + 5\}$, consider the graph G_i constructed from G by contracting edge set E_i. By the planarity of G_i, it follows from the Four Color Theorem that G_i is 4-vertex colorable with colors saying $\{a_i, b_i, c_i, d_i\}$. Note that every edge e in E_i of G corresponds a contracted vertex in G_i, and we use its color to color the edge e. After performing this coloring process for all $i \in \{1, \ldots, 5t + 5\}$, we obtain a $4(5t + 5)$-edge-coloring $\psi : E(H) \mapsto \cup_{i=1}^{5t+5}\{a_i, b_i, c_i, d_i\}$. Furthermore, each pair of edges $e_1, e_2 \in E(H)$ with distance at most 2 receive different colors in the coloring ψ by its construction.

Then we show below that $\phi \cup \psi$ is a star edge coloring of G using $k + 4(5t + 5) \leq \lfloor 1.5\Delta \rfloor + 4(5t + 5)$ colors. Clearly, $\phi \cup \psi$ is a proper edge coloring of G, i.e., any adjacent edges get distinct colors. Let $P = x_1x_2x_3x_4x_5$ denote a path or cycle with 4 edges. Note that each edge in $E(T)$ receives a color in $\{f_1, f_2, \ldots, f_k\}$, and each edge in $E(H)$ receives a color in $\cup_{i=1}^{5t+5}\{a_i, b_i, c_i, d_i\}$. If x_1x_2 and x_3x_4 receive the same color, then by the construction of coloring $\phi \cup \psi$, we must have both $x_1x_2 \in E(T)$ and $x_3x_4 \in E(T)$, as any two edges with distance two in $E(H)$ receive distinct colors. If x_2x_3 and x_4x_5 receive the same color, then we also have $x_2x_3, x_4x_5 \in E(T)$ for the same reason. But since ϕ is a star edge coloring of T, $\phi(x_2x_1) = \phi(x_4x_3)$ and $\phi(x_3x_2) = \phi(x_5x_4)$ can not occur simultaneously. Therefore, $\phi \cup \psi$ is a star edge coloring of G by definition. $\qquad\square$

3. Conclusions

This paper has shown that every $K_{2,t}$-free planar graph admits a star $(\lfloor 1.5\Delta \rfloor + 20t + 20)$-edge coloring, which is sharp up to the constant term. This can be evident by the planar graph constructed in [9], whose star chromatic index is at least $\frac{13}{8}\Delta - \frac{3}{4}$ and contains $K_{2,\Delta}$ as a subgraph. On the other hand, it would be interesting to push further on the constant term to determine the optimal upper bound. In [7], a similar problem is proposed for outerplanar graphs by the authors, who conjectured that every outplanar graph is star $(\lfloor 1.5\Delta \rfloor + 1)$-edge colorable. Another related research direction is to consider the list version of star edge coloring. The edge-partition techniques in this paper (and in the literature) could not be generalized to list star edge coloring. There are some other techniques, such as that of [9],

which may be helpful in handling list star edge coloring. It should be of interest to search for a similar list version of Theorem 1.4, or its subclass of graphs like outerplanar graphs and C_4-free planar graphs.

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Conflict of interest

The authors declare there is no conflict of interest.

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