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Theoretical Considerations for Geosynchronous, Earth-Based Gravity Wave Interferometer

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THEORETICAL CONSIDERATIONS FOR A GEOSYNCHRONOUS, EARTH-BASED GRAVITY WAVE INTERFEROMETER

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Abstract

We investigated theoretical considerations in the design of an Earth-based laser interferometer for detecting gravitational waves. Our design envisages a ground-based tracking station in communication with two geosynchronous satellites. We assumed linearized gravitational waves in a Schwarzschild spacetime geometry outside the Earth. Our initial calculations show that such a design is sufficiently sensitive to successfully detect gravitational waves near Earth.

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1. Introduction

a. Spacetime

Space and time have traditionally been viewed as independent parameters. When Maxwell calculated that the speed of light was a constant independent of the velocity of the observer's coordinate system, the traditional view could not make sense of it. Einstein then developed special relativity in order to alter the Newtonian view to accommodate the constant. He built this theory on the idea that identical experiments should provide identical results if carried out in inertial frames, the principle of relativity [1]. Frames are coordinate systems, inertial frames are those related by constant velocities, and rest frames are those related by the same velocity. This reformulation resulted in the dependence between space and time that is apparent at speeds significant compared to that of light and could be viewed in two ways. The first is that clocks moving relative to an observer run slower than clocks at rest in the observer's frame. This is called time dilation. The second, known as length contraction, is that objects moving relative to an observer shrink in their direction of motion, as measured by that observer. This connection between space and time resulted in the new view, spacetime.

The concept of spacetime introduced many adjustments in thinking. New terminology was defined like proper time, the time registered by a clock in its rest frame, and proper length, the length of an object as measured in it's rest frame. Mechanics adapted new notation to accommodate four-vectors, vectors that included the fourth dimension, time.

We will denote four-vectors in bold, for example $\mathbf{a}$, to avoid confusion with three-dimensional spatial vectors, denoted $\vec{a}$. By manipulation of four-momentum, it was discovered that matter and energy are related by the equation

$$E = mc^2 \quad (1.1)$$

where E is the energy of matter in it's rest frame with mass m. This result became an aspect of spacetime curvature.

b. Spacetime Curvature

Einstein's discovery of a new principle allowed him to uncover another property of spacetime. The equivalence principle states that inertial and gravitational mass are equivalent, that is, no experiment can differentiate a uniform acceleration in flat spacetime from that due to a uniform gravitational field [1]. This allowed a relation between spacetime and matter-energy to be formulated, the Einstein equation:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}. \quad (1.2)$$

The details of this are complicated. The main idea is that the left-hand side represents how the local geometry of spacetime is curved and the right-hand side represents how matter-energy density is related to this curvature. It means that gravity is not a force, it is spacetime curvature. In other words, objects that are acting under a gravitational "pull" are actually in an inertial frame in the geometry of spacetime.

It is important to recognize the difference between coordinate systems and the geometry they describe. Two coordinates systems, one in black and the other blue, that can be used to represent one geometrical object, a red line, are shown in Fig. (1.1).

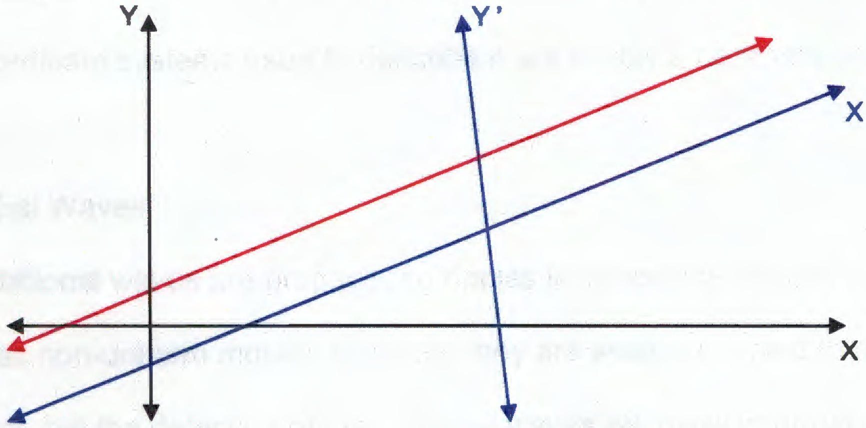

Figure 1.1

In the black coordinate system, the red line can be written in slope-intercept form, $y = mx + b$, with slope m and y-intercept b. In the blue coordinate system, the line is parallel to the x'-axis, so the slope is zero. Choosing the y'-intercept to be b' results in the simpler equation $y' = b'$. Another convenient coordinate system for a line would be polar coordinates centered anywhere on the red line. In this case, the line would be $\theta = \alpha$ where α is a constant that depends on the choice of the θ-axis. In these examples, the geometrical shape, the line, is independent of the coordinate systems, but it can be more simply described in some coordinate systems than others.

It is important to note that the results of any calculation should be independent of the coordinate system used. For example, if the line in Fig. (1.1) describes a plane near Earth with the x-axis parallel to the surface of Earth, a marble will roll down the plane in exactly the same way independent of the coordinate system chosen. Coordinate systems are merely convenient mathematical descriptions.

In relativity, the geometry of spacetime exists independent of the choice of coordinate system although some choices are convenient in describing it. Since mass-energy density causes curvature in the geometry of spacetime and the geometry exists independent of coordinate systems, any result of the mass-energy density should also be independent of coordinate systems. Thus the geometry of spacetime reflects reality while the coordinate systems used to describe it are simply a necessity of mathematics.

c. Gravitational Waves

Gravitational waves are propagating ripples in spacetime caused by matter in non-spherical, non-uniform motion. Because they are weakly coupled to matter they are hard to detect, but the detection of gravitational waves will allow information about sources such as supernovae, black holes, and the big bang to be gleaned that is qualitatively different from that of electromagnetic radiation. The weak coupling also means that only sources involving large matter-energy densities such as supernovae, black holes, and the big bang will be detectable. No direct detection of a gravity wave has been confirmed, but some projects are underway with this goal. Indirect experimental evidence for their existence has been observed in careful astronomical measurements of a binary pulsar. Before gravity waves can be examined more carefully, a better understanding of relativistic mechanics is required.

2. Relativistic Mechanics

a. Conventions

To understand the geometry of spacetime, some conventions simplify the mechanics. Since the speed of light, c, is the constant that relates space to time, mechanics can be written in units such that $c = 1$. The gravitational constant, G, similarly relates mass to space and time. Thus, a convention that is used for the rest of this paper, unless otherwise stated, is that mass and time will be measured in units of length such that $G = c = 1$, called geometrized units. Also, since different basis vectors are useful in different situations, a bold $\mathbf{e}_\alpha$ denotes basis vectors while the components of vectors are not bolded as in a^α. Another convention is that component indices, called summation indices, are written as numbers with $a^0 = a^t$, $a^1 = a^x$, etc., as seen in Eq. (2.1) and Eq. (2.2) below. The summation convention allows vectors to be compactly written by stating that superscript-subscript or subscript-superscript pairs imply summation from 0 to 3 for Greek indices and 1 to 3 for Roman as in the following examples:

$$\mathbf{a} = a^\alpha \mathbf{e}_\alpha = \sum_{\alpha=0}^{3} a^\alpha \mathbf{e}_\alpha = a^0\mathbf{e}_0 + a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3 = a^t\mathbf{e}_t + a^x\mathbf{e}_x + a^y\mathbf{e}_y + a^z\mathbf{e}_z \tag{2.1}$$

$$\mathbf{a} = a^i \mathbf{e}_i = \sum_{i=1}^{3} a^i \mathbf{e}_i = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3 = a^x\mathbf{e}_x + a^y\mathbf{e}_y + a^z\mathbf{e}_z . \tag{2.2}$$

As long as indices are consistent throughout an equation, the particular index used does not matter, so Eq. (2.1) could have instead been written with index β while maintaining the same meaning.

b. Mathematical Tools

Before the mechanics can be examined, a rough mathematical background is necessary. A rank r tensor is a linear map from r vectors into a real number. In relativistic mechanics, a useful tensor called the metric tensor, or metric for short, is a tensor of rank 2, typically written $g_{\alpha\beta}$, and defined by the basis vectors of the coordinate system used to describe the geometry of local spacetime through the equation

$$g_{\alpha\beta}(x) = \mathbf{e}_\alpha(x) \cdot \mathbf{e}_\beta(x) . \tag{2.3}$$

This scalar product forms a 4×4 matrix. In fact, any rank 2 tensor can be represented by a matrix, so from now on the term matrix will be used interchangeably with rank 2 tensor. It is important to note that in general relativity, tensors contain information directly related to the geometry of spacetime while metrics depend on the coordinate system used to describe spacetime. Eq. (2.3) can be used to show that a rank 2 tensor is a linear map from 2 vectors into a real number by showing how the metric is related to the scalar products of two vectors:

$$\mathbf{a}\cdot\mathbf{b}=\left(a^{\alpha}\mathbf{e}_{\alpha}\right)\cdot\left(b^{\beta}\mathbf{e}_{\beta}\right)=\left(\mathbf{e}_{\alpha}\cdot\mathbf{e}_{\beta}\right)a^{\alpha}b^{\beta}=g_{\alpha\beta}a^{\alpha}b^{\beta}. \tag{2.4}$$

A property of tensors is that if one is represented by a matrix then it's inverse is the inverse matrix of the original tensor:

$$g^{\alpha\beta}=\left(g_{\alpha\beta}\right)^{-1}. \tag{2.5}$$

In general relativity, even these new tensors are simply abstractions that will be related to the geometry of spacetime, but they allow more flexibility in complicated calculations.

Inverse tensors also allow another tool, dual vectors, to be used. Dual-basis vectors are written with a superscript index, $\mathbf{e}^{\alpha}$, to differentiate them from basis vectors written with a subscript, $\mathbf{e}_{\beta}$. These are also related by the inverse metric:

$$\mathbf{e}^{\alpha}=g^{\alpha\beta}\mathbf{e}_{\beta}. \tag{2.6}$$

Dual vector components are written with a subscript index, u_{α}, so that they can be distinguished from vector components written with a superscript, u^{α}. These components are related by the metric:

$$u_{\alpha}=g_{\alpha\beta}u^{\beta}. \tag{2.7}$$

Eq. (2.6) and Eq. (2.7) substituted into Eq. (2.1) imply

$$\mathbf{u}=u^{\alpha}\mathbf{e}_{\alpha}=u_{\alpha}\mathbf{e}^{\alpha}. \tag{2.8}$$

Substituting Eq. (2.7) into the right-hand side of Eq. (2.4) shows how dual vectors can be used to simplify the scalar product:

$$\mathbf{a}\cdot\mathbf{b}=a_{\beta}b^{\beta}. \tag{2.9}$$

Note the indices on tensors, like those on vectors, can also be raised and lowered by other tensors:

$$t_{\alpha}{}^{\beta} = g_{\alpha\gamma} t^{\gamma\beta} . \quad (2.10)$$

These properties will be useful in manipulating calculations in general relativity.

With these tools and conventions, relativistic mechanics can be understood. The Einstein equation showed how matter-energy density relates to curvature in spacetime. In the absence of matter-energy this reduces to the vacuum Einstein equation:

$$R_{\alpha\beta} = 0 . \quad (2.11)$$

The Ricci curvature, $R_{\alpha\beta}$, needs to be recast into a quantity that is more applicable to relativistic mechanics, the metric. The key to relating the Ricci curvature to the metric are Christoffel symbols. Christoffel symbols are related to the metric by the equation

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\delta\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}} \right) . \quad (2.12)$$

The Ricci curvature can then be defined in terms of Christoffel symbols as

$$R_{\alpha\beta} = \frac{\partial \Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\gamma}_{\alpha\gamma}}{\partial x^{\beta}} + \Gamma^{\gamma}_{\alpha\beta} \Gamma^{\delta}_{\gamma\delta} - \Gamma^{\gamma}_{\alpha\delta} \Gamma^{\delta}_{\beta\gamma} . \quad (2.13)$$

One important point is that some freedom still exists to choose a coordinate system for the metric. Killing vectors, denoted ξ, are vectors that characterize symmetry in any coordinate system and can be helpful in altering a coordinate system so that it still correctly describes the local spacetime curvature.

c. Flat Spacetime

A special metric that describes the geometry of spacetime in the absence of curvature is the flatspace metric ($G \neq c \neq 1$ units):

$$g_{\alpha\beta} \equiv \eta_{\alpha\beta} = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array}\!\!\begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{2.14}$$

This metric exhibits the non-Euclidean aspect of the geometry of spacetime because the time component is negative. This negative sign is the source of time dilation and length contraction. The flatspace metric also expresses the importance of the speed of light in the concept of spacetime. Another important quantity related to the metric is the line element, ds^2. A line element is related to the metric by the equation

$$ds^2 = g_{tt}dt^2 + g_{tx}dt\,dx + g_{ty}dt\,dy + \ldots + g_{zz}dz^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{2.15}$$

The line element for flat spacetime is

$$ds^2 = -(c\,dt)^2 + dx^2 + dy^2 + dz^2. \tag{2.16}$$

A line element allows the distance between any two points in spacetime, L, to be calculated by integration:

$$L = \int\sqrt{ds^2}\,. \tag{2.17}$$

The line element in Eq. (2.16) is written in Cartesian spatial coordinates, but space could also be described in spherical polar spatial coordinates:

$$ds^2 = -(c\,dt)^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\,. \tag{2.18}$$

Many other metrics exist for space that is not flat such as the linearized gravitational wave metric.

3. Linearized Gravitational Waves

a. Purpose of Linearization

No general technique is capable of solving the vacuum Einstein equation in every situation, but for nearly flat spacetimes an approximation to it exists that is completely solvable, the linearized vacuum Einstein equation. Gravitational waves are perturbations to flat spacetime, so they can be well approximated using this technique. Linearized gravitational waves can then be completely analyzed in ways that non-linearized waves could not.

b. Metric Perturbations

This derivation follows from Hartle, section 21.5 – "Linearized Gravity" [1]. In order to linearize the Einstein equation, some quantities must be introduced. In particular, a short hand for partial derivatives:

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha}. \tag{3.1}$$

Also, the flatspace wave operator, known as the d'Alembertian:

$$\Box \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2. \tag{3.2}$$

The derivation begins with writing a perturbation to the flatspace metric as

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + h_{\alpha\beta}(x) \tag{3.3}$$

where $h_{\alpha\beta}(x)$ that are linearizable, such as that of a gravity wave far from it's source, have amplitudes much less than 1 and are called metric perturbations.

Since linear corrections to $g^{\alpha\beta}$ result in negligible quadratic corrections to Eq. (3.3), when raising or lowering an index on the perturbation tensor, as in Eq. (2.10), only the flat space metric is required:

$$h_\alpha^\gamma = \eta^{\gamma\beta} h_{\beta\alpha}. \quad (3.4)$$

To get the linearized vacuum Einstein equation, use Eq. (3.3) in the vacuum Einstein equation, Eq. (2.11), and expand to first order in $h_{\alpha\beta}(x)$. The first term vanishes, leaving the Ricci curvature term that is linear in the perturbation:

$$\delta R_{\alpha\beta} = 0. \quad (3.5)$$

In the linear order approximations of the Christoffel symbols, using Eq. (2.12), the first term disappears because the flatspace metric is constant, resulting in

$$\delta\Gamma_{\alpha\beta}^\gamma = \frac{1}{2}\eta^{\gamma\delta}\left(\frac{\partial h_{\delta\alpha}}{\partial x^\beta} + \frac{\partial h_{\delta\beta}}{\partial x^\alpha} + \frac{\partial h_{\alpha\beta}}{\partial x^\delta}\right). \quad (3.6)$$

Referring to Eq. (2.13), the last two terms in the Ricci curvature approximation are negligible because they are quadratic in $h_{\alpha\beta}$:

$$\delta R_{\alpha\beta} = \frac{\partial\delta\Gamma_{\alpha\beta}^\gamma}{\partial x^\gamma} - \frac{\partial\delta\Gamma_{\alpha\gamma}^\gamma}{\partial x^\beta} = \frac{1}{2}\left(-\Box h_{\alpha\beta} + \partial_\alpha V_\beta + \partial_\beta V_\alpha\right) = 0 \quad (3.7)$$

The third part of Eq. (3.7), the result of substituting Eq. (3.6) into the second part, was simplified by introducing the vector:

$$V_\alpha \equiv \partial_\gamma h_\alpha^\gamma - \frac{1}{2}\partial_\alpha h_\gamma^\gamma. \quad (3.8)$$

c. The Gauge Transformation

The choice of coordinates used is arbitrary, so the coordinates can be changed to

$$x'^\alpha = x^\alpha + \xi^\alpha(x), \quad (3.9)$$

assuming $\xi^\alpha(x)$ is as small as $h_{\alpha\beta}(x)$. To the first order in ξ^α, x'^α can substitute for x^α, so Eq. (3.9) works into

$$\frac{\partial x^\alpha}{\partial x'^\beta} = \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial x'^\beta} = \delta^\alpha_\beta - \frac{\partial \xi^\alpha}{\partial x^\beta}. \tag{3.10}$$

Since the transformation of the matrix representation of a metric under a change in coordinates is

$$g'_{\alpha\beta}(x') = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\delta}{\partial x'^\beta} g_{\gamma\delta}(x), \tag{3.11}$$

the gauge transformation in linearized gravity can be calculated to be

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha. \tag{3.12}$$

Choosing $\xi^\alpha(x)$ such that $V'_\alpha(x) = 0$ causes Eq. (3.7) to simplify to the linearized Einstein Equation:

$$\Box h_{\alpha\beta}(x) = 0. \tag{3.13}$$

The prime has been dropped because we can assume we are already in a coordinate system in which our conditions are satisfied. The condition

$$V_\alpha(x) \equiv \partial_\gamma h^\gamma_\alpha(x) - \frac{1}{2}\partial_\alpha h^\gamma_\gamma(x) = 0 \tag{3.14}$$

is known as the Lorentz gauge condition.

d. The Wave Equation

To solve the wave equation, (3.13), it is easier to simplify it by considering a flatspace, scalar wave equation. Let scalar $f(x) = ae^{i\mathbf{k}\cdot\mathbf{x}}$ where $\mathbf{k}\cdot\mathbf{x} = -k^t t + \vec{k}\cdot\vec{x}$. The wave equation then becomes

$$\Box f(x) = -\frac{\partial^2 f}{\partial t^2} + \vec{\nabla}^2 f = -\mathbf{k}\cdot\mathbf{k} f = 0, \tag{3.15}$$

where $\mathbf{k}\cdot\mathbf{k}=\eta_{\alpha\beta}k^{\alpha}k^{\beta}$. The last equality in Eq. (3.15) is only true if $\mathbf{k}$ is a null vector, a four-vector of magnitude zero. Letting $\left|\vec{k}\right|=\omega_{\vec{k}}$ the null vector can be written as $\mathbf{k}=(\left|\vec{k}\right|,\vec{k})=(\omega_{\vec{k}},\vec{k})$. The physical solution is the real part of $f(x)$:

$$\left|a\right|\cos(\mathbf{k}\cdot\mathbf{x}+\delta)=\left|a\right|\cos(-\omega_{\vec{k}}t+\vec{k}\cdot\vec{x}+\delta). \qquad (3.16)$$

In this form, it is clear that the wave frequency is $\omega_{\vec{k}}$, the wavelength is $\frac{2\pi}{\left|\vec{k}\right|}$, and the wave speed is $\frac{\omega_{\vec{k}}}{\left|\vec{k}\right|}=1$, the speed of light in $c=1$ units, in the $\vec{k}$ direction. The general solution to the wave equation is the real part of the following integral over all $\vec{k}$, a superposition of waves with definite wave vectors:

$$f(x)=\int d^{3}k\,a(\vec{k})e^{i\,\mathbf{k}\cdot\mathbf{x}}. \qquad (3.17)$$

e. Transverse-Traceless Gauge

Each component of the perturbation must satisfy the flat-space, scalar wave equation:

$$h_{\alpha\beta}(x)=a_{\alpha\beta}e^{i\,\mathbf{k}\cdot\mathbf{x}}. \qquad (3.18)$$

The amplitudes of the wave components are the components of $a_{\alpha\beta}$. Transformations of the form in Eq. (3.12) that preserve the Lorentz gauge condition, Eq. (3.14), are allowed and can be used to simplify $a_{\alpha\beta}$. Since $h_{\alpha\beta}(x)$ already satisfies the gauge condition, it can be shown that substituting a form of Eq. (3.12) into the gauge condition results in

$$\Box\xi_\alpha(x) = 0. \tag{3.19}$$

Thus, any four $h_{\alpha\beta}$ may vanish in the four possible transformations. Choosing the conditions $h_{ti} = 0$ and $h^\beta_\beta = 0$, the equations of the gauge condition imply:

$$V_t = \frac{\partial h^t_t}{\partial t} = i\omega_{\vec{k}} a_{tt} e^{i\mathbf{k}\cdot\mathbf{x}} = 0 \tag{3.20}$$

$$V_i = \frac{\partial h^j_i}{\partial x^j} = ik^j a_{ij} e^{i\mathbf{k}\cdot\mathbf{x}} = 0. \tag{3.21}$$

These imply four more conditions: $a_{tt} = 0$ and $k^j a_{ij} = 0$. Combined with the prior conditions, the time components of the matrix disappear. Choosing the z-axis to be the direction of propagation, called the longitudinal direction, means that $\vec{k} = (0,0,\omega_{\vec{k}})$ and, by the last three conditions mentioned, $k^j a_{ij} = 0$, all of the z-components disappear. It is also worth mentioning that these three conditions, in the context of Eq. (3.21), mean the waves are transverse, dependent only on the motion in one spatial direction. Since there are now eight conditions and the matrix is symmetric, there are now only two independent components, both in the x-y sub-matrix. Since the second condition was that their trace must vanish, the most general form of the linearized gravitational wave perturbation is:

$$h_{\alpha\beta}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_\times(t-z) & 0 \\ 0 & f_\times(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.22}$$

This solution is known as the transverse-traceless gauge, or TT-gauge for short. The last part of the equation shows a common form that the two polarizations of the gravitational wave are represented by, generally known as the plus and cross polarizations.

4. Gravity Wave Detection

a. Test Masses

Spacetime curvature is detected by motion of test masses. These are objects of little mass so they don't produce significant spacetime curvature. According to the variational principle for free test mass motion, if no forces are acting on a test mass, keeping in mind gravity is not a force, it follows a path of extremal proper time, called a geodesic [1]. They travel straight lines in the curved geometry of spacetime.

Instead of changing the coordinate positions of test masses, a passing gravity wave actually changes the coordinate system. This follows from the fact that gravity is not a force, it is curvature in spacetime. The change in coordinate systems means the distance between test masses will change even though their coordinate separation won't. To measure this change in distance, the metric used must be calculated by combining Eq. (2.14) and Eq. (3.3) with the result in Eq. (3.22) to get

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+f_+(t-z) & f_\times(t-z) & 0 \\ 0 & f_\times(t-z) & 1-f_+(t-z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.1}$$

For simplicity, choose two test masses to be on the x-axis at the origin and $x = L_*$ where L_* is the distance between the points in flatspace. Since $\Delta t = \Delta y = \Delta z = 0$ in this situation, the line element corresponding to the metric simplifies and Eq. (2.17) becomes

$$L(t) = \int_0^{L_*} \sqrt{g_{xx}}\,dx = \left(1+f_+(t-0)\right)^{1/2} \int_0^{L_*} dx\,. \tag{4.2}$$

The last part of this equation follows from the metric, Eq. (4.1), while the next approximation is a result of the binomial approximation [2]:

$$L(t) \approx \left(1 + \frac{1}{2} f_+(t)\right) L_* . \qquad (4.3)$$

Letting the change in distance be $\delta L(t)$ allows the fractional strain, $\frac{\delta L(t)}{L_*}$, to be

$$\frac{\delta L(t)}{L_*} \approx \frac{1}{2} f_+(t) = \frac{1}{2} a \sin(\omega t + \delta) , \qquad (4.4)$$

where a is the amplitude, ω is the frequency, and δ is the phase constant of the wave. The choice of $f_+(t) = a \sin(\omega t + \delta)$ makes sense based on the solution to the wave equation found in section 3. This solution can be generalized to a test mass anywhere in the xy- plane by letting unit vector $\vec{n}$ point from the origin to the test mass:

$$\frac{\delta L(t)}{L_*} = \frac{1}{2} h_{ij}(t) n^i n^j , \qquad (4.5)$$

where $h_{ij}(t)$ is the spatial sub-matrix of the metric perturbation when $z = 0$. Fig. (4.1), found in Hartle, section (16.3), shows test masses arranged in a circle around a central test mass [1].

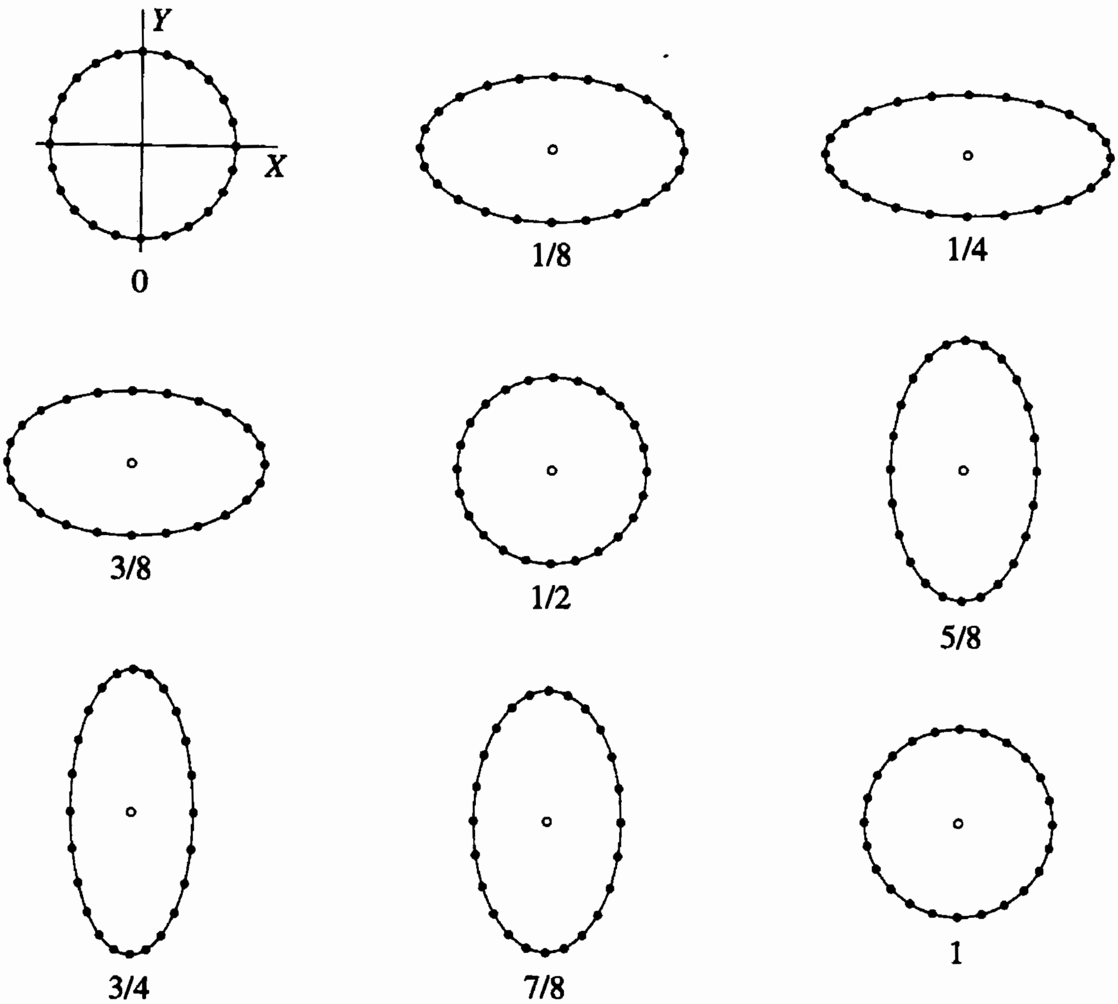

Figure 4.1

It then shows the evolution of the distance between masses, not their coordinate positions, of one period of the plus polarization of a gravity wave with amplitude $a = 0.8$. The cross polarization would be the same, but rotated $45°$. Gravity waves that are expected to be detected on Earth have amplitudes of order of magnitude 10^{-21} [1].

b. Concept of Interferometry

The small amplitudes of gravity waves make their detection challenging. One potential method is to use interferometers. A schematic of one is shown in Fig. (4.2).

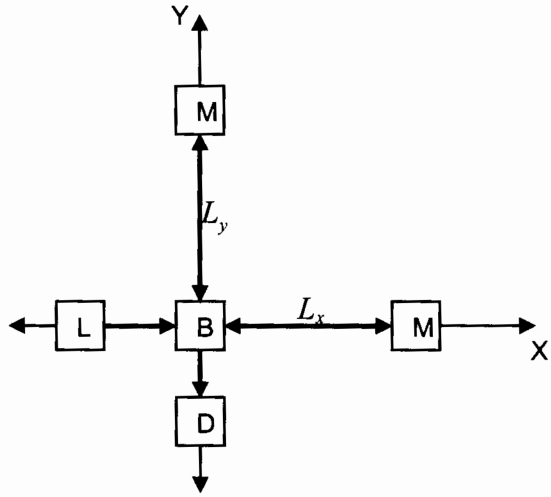

Figure 4.2

A laser, in red, is shot from L to a beam splitter, B, where it travels to the mirrors that are acting as test masses, M, at the end of each arm. The reflected beams are then recombined and because of a property of light, when light waves are combined the sum of the light waves is the result. The intensity of the resulting light wave can be measured by the detector, D. The conditions for constructive and destructive interference are, respectively,

$$\Delta L \equiv L_x - L_y = n\lambda \tag{4.6}$$

$$\Delta L \equiv L_x - L_y = (n + \frac{1}{2})\lambda \tag{4.7}$$

for $n = 0,1,2,...$ and wavelength of light λ. Since the speed of light is constant, if the arms change length the path of the light will lengthen or shorten depending on the change. This change will cause the sum of the light waves and the amount of interference to change. If arms of equal flatspace length, L_*, are lined up along the x-and y-axes in the $z = 0$ plane and the initial time is chosen such that $\delta = 0$, then Eq. (4.3) with $f_+(t) = a\sin(\omega t + \delta)$ can be used to determine ΔL:

$$\Delta L = \left[\left(1+\frac{1}{2}a\sin(\omega t)\right)L_*\right]-\left[\left(1-\frac{1}{2}a\sin(\omega t)\right)L_*\right]=L_* a\sin(\omega t)\,. \tag{4.8}$$

This equation shows that longer arms will cause greater changes in the interference pattern allowing the interferometer to be more sensitive.

c. Current Interferometers

Several projects with the intent of detecting gravity waves are in various stages of completion. An interferometer similar to the one described above called LIGO, short for Laser Interferometer Gravitational (Wave) Observatory, is currently in its initial operational stages [3]. The disadvantage of this interferometer is that the arms are relatively short, although this has been improved upon by adding partially reflecting mirrors that effectively lengthen the arms of the interferometer. Another gravitational wave observatory based on interferometric principles that solves this problem called LISA, short for Laser Interferometer Space Antenna, has been proposed for launch into space in 2011 [3]. In space, the lengths of the arms are only limited by the power of the laser. The disadvantage of this interferometer is that once it is launched, it will be very hard to reach if problems occur or it needs technological updates. The interferometer proposed in this paper was intended to solve both of these issues. This interferometer would be based on the interferometer in Fig. (4.2), but the test masses at the ends of the arms would be satellites with mirrors in geosynchronous orbits. This allows the arms to be about 40,000 kilometers long while keeping most of the equipment on or relatively near Earth where it can be easily repaired or updated.

5. Theoretical Considerations

a. Earth's Spacetime Curvature

The reason the spacetime curvature produced by Earth won't prevent the operation of the interferometer is that the paths of the lasers are equivalent. To understand this, the design of the interferometer must be more clearly defined. Let Earth be centered on the origin of a Cartesian coordinate system with the equator on the $z=0$ plane as shown in Fig. (5.1).

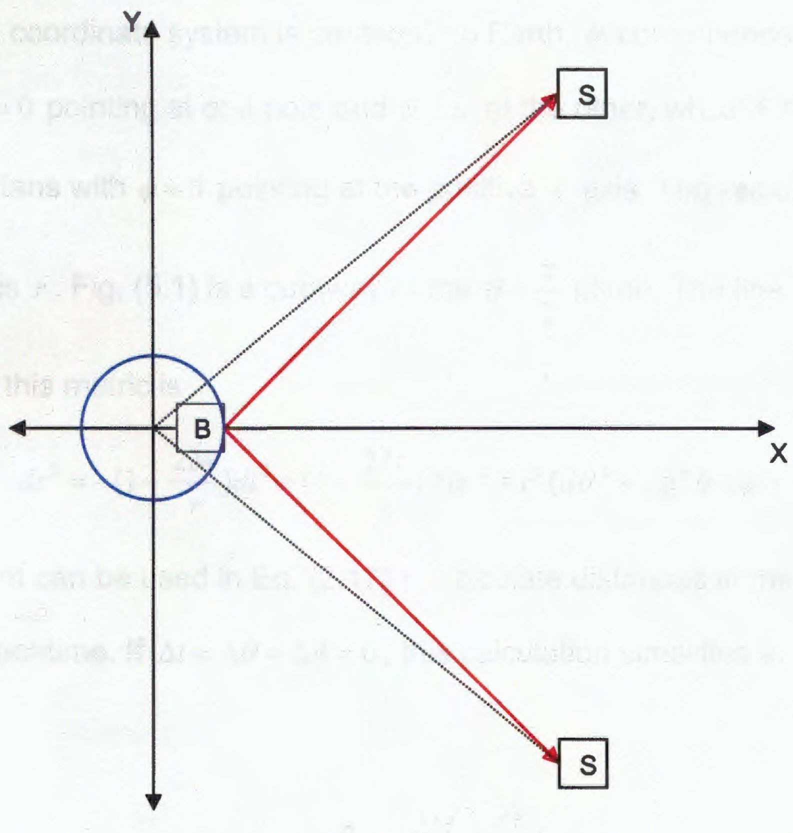

Figure 5.1

The point where the surface of Earth's equator (blue) coincides with the x-axis was chosen to be the beam splitter (B). Two points in the plane that are equidistant from the origin, equidistant from the beam splitter, and have arms (red) that form a right angle at the beam splitter were chosen to be the satellites (S).

To a very good approximation, the Earth can be treated as spherically symmetric. A special metric exists for spacetime near spherically symmetric matter of mass M called the Schwarzschild metric:

$$g_{\alpha\beta} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \end{array}. \quad (5.1)$$

This metric is similar to spherical polar coordinates in flatspace, Eq. (2.18). When the origin of such a coordinate system is centered on Earth, θ corresponds to latitude in radians with $\theta = 0$ pointing at one pole and $\theta = \pi$ at the other, while ϕ corresponds to longitude in radians with $\phi = 0$ pointing at the positive x-axis. The radius measured from the origin is r. Fig. (5.1) is a cutaway of the $\theta = \frac{\pi}{2}$ plane. The line element that corresponds to this metric is

$$ds^2 = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (5.2)$$

This line element can be used in Eq. (2.17) to calculate distances in the Schwarzschild geometry of spacetime. If $\Delta t = \Delta\theta = \Delta\phi = 0$, this calculation simplifies to a function of the radial distance:

$$L(r) = \int\left(1-\frac{2M}{r}\right)^{-1/2} dr. \quad (5.3)$$

Since the flatspace distance from the origin to each satellite is the same, Eq. (5.3) implies the distances in Schwarzschild space must also be the same. The light reflected from the satellites also ends at the same point, the beam splitter. Because of the symmetry of this design, $\Delta\phi$ along each arm must also be exactly the same, but it should also be clear that distance in the Schwarzschild metric is only dependant on the change in ϕ, not the specific values of ϕ. This implies the arms must be of exactly the same length and beams traveling down each arm must be traveling exactly equivalent paths. This means that the spacetime curvature produced by Earth will not effect the operation of this interferometer.

b. Lengths of Arms in Flatspace

To determine the sensitivity of the interferometer, it is necessary to find the length of the arms in flatspace. The radius of a geosynchronous orbit, R_0, must first be found by setting the centripetal force [4] acting on a satellite of mass m traveling at a tangential velocity v equal to the Newtonian gravitational force [4] caused on it by the mass of Earth, M:

$$\frac{mv^2}{R_0} = \frac{GMm}{(R_0)^2}. \tag{5.4}$$

Relativistic mechanics are not necessary for this calculation because space is flat and speeds involved are not significant compared to the speed of light. The satellites are geosynchronous, so they orbit Earth once per sidereal day, a day with respect to the distant stars instead of the sun. A sidereal day is about four minutes shorter than a solar day [5]. This information can be used to calculate the angular velocity of the satellite, $\omega = \frac{\pi}{43080}$ radians/s. The tangential velocity is simply the angular velocity multiplied by the radius of orbit [4]. Substituting for v into Eq. (5.4) allows the orbital radius to be solved for:

$$R_0 = \left(\frac{43080}{\pi}\right)^{2/3} (GM)^{1/3} \approx 4.22 \cdot 10^7 \text{ m}. \tag{5.5}$$

This result is obtained by letting $M = 5.97 \cdot 10^{24}$ kg [1] and $G = 6.6726 \cdot 10^{-11}$ N•m^2/kg^2 [4].

Now the length of the arms can be calculated geometrically by choosing the origin of a Cartesian coordinate system to be at the beam splitter with the arms along the x- and y-axes in the $z=0$ plane, as shown in Fig. (5.2).

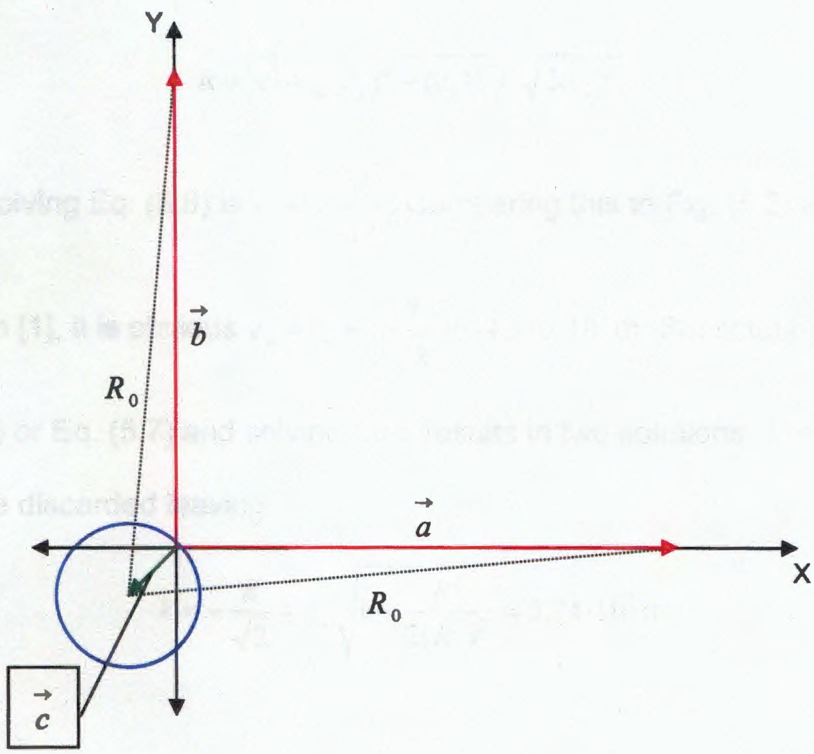

Figure 5.2

Let $\vec{c}=(c_x, c_y, 0)$ point from the beam splitter to the center of Earth. The lengths of the arms were defined to be equal, so let constant k be this length so that $\vec{a}=(k,0,0)$ and $\vec{b}=(0,k,0)$ represent the arms. Next, the following quantities were found:

$$R_0=\left|\vec{a}-\vec{c}\right|=\sqrt{(k-c_x)^2+(0-c_y)^2} \tag{5.6}$$

$$R_0=\left|\vec{b}-\vec{c}\right|=\sqrt{(0-c_x)^2+(k-c_y)^2}\,. \tag{5.7}$$

Setting these equations equal to each other and simplifying gives the result $c_x = c_y$.

These components can be calculated by setting the equatorial radius of Earth, R, equal to $\left|\vec{c}\right|$ and solving:

$$R = \left|\vec{c}\right| = \sqrt{(c_x)^2 + (c_y)^2} = \sqrt{2(c_x)^2}\,. \tag{5.8}$$

The result of solving Eq. (5.8) is $c_x = \pm\frac{R}{\sqrt{2}}$. Comparing this to Fig. (5.2) and letting $R = 6.378\cdot10^6$ m [1], it is obvious $c_x = c_y = -\frac{R}{\sqrt{2}} \approx -4.510\cdot10^7$ m. Substituting this result into either Eq. (5.6) or Eq. (5.7) and solving for k results in two solutions. The negative solution can be discarded leaving

$$k = -\frac{R}{\sqrt{2}} + R_0\sqrt{1 - \frac{R^2}{2(R_0)^2}} \approx 3.74\cdot10^7 \text{ m}. \tag{5.9}$$

c. Length of Arms in Schwarzschild Geometry

To see how the mass of Earth affects the lengths of the arms, these lengths will be calculated in the Schwarzschild geometry. This calculation will be eased by allowing the center of a Cartesian coordinate system to coincide with the center of Earth and one of the satellites to be on the x-axis, as shown in Fig. (5.3). Then let vector $\vec{B}$ of magnitude R point from the origin to the beam splitter and vector $\vec{S}$ of magnitude R_0 point from the origin to the satellite on the x-axis. Another useful tool in this calculation is a line of the form $y = mx + b$ that describes the straight path through beam splitter and the satellite on the x-axis. For reference, the flatspace lengths of the arms, k, have also been included in Fig. (5.3).

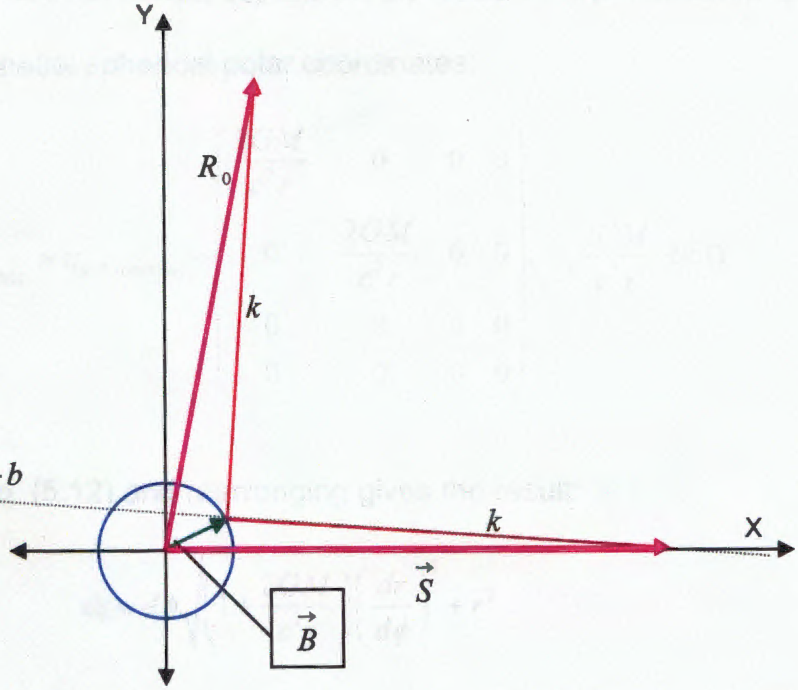

Figure 5.3

It has already been noted that the laser paths are identical in the Schwarzschild coordinates, so finding the length in this geometry of the arm near the x axis will be sufficient for both.

Since $dt = d\theta = 0$, the Schwarzschild line element in Eq. (5.2) simplifies to

$$ds^2 = (1 - \frac{2M}{r})^{-1} dr^2 + r^2 \sin^2\theta \, d\phi^2 = (1 - \frac{2GM}{c^2 r})^{-1} dr^2 + r^2 \sin^2\theta \, d\phi^2 . \qquad (5.10)$$

where the right-hand side is written in $G \neq c \neq 1$ units, as will be used from here on. The limits $R \le r \le R_0$ imply

$$2.10 \cdot 10^{-10} = \frac{2GM}{c^2 R_0} \le \frac{2GM}{c^2 r} \le \frac{2GM}{c^2 R} = 1.39 \cdot 10^{-9} << 1 . \qquad (5.11)$$

This means the right-hand side of Eq. (5.10) can be linearly approximated in $\frac{2GM}{c^2 r}$:

$$ds^2 \approx (1 + \frac{2GM}{c^2 r}) dr^2 + r^2 \sin^2\theta \, d\phi^2 . \qquad (5.12)$$

If the dt and $d\theta$ terms had not cancelled, this metric would be a perturbation to the flatspace geometry in spatial spherical polar coordinates:

$$g_{\alpha\beta,Schwarzschild} \approx \eta_{\alpha\beta,spherical} + \begin{pmatrix} \frac{2GM}{c^2 r} & 0 & 0 & 0 \\ 0 & \frac{2GM}{c^2 r} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\frac{2GM}{c^2 r} << 1) \tag{5.13}$$

Taking the square of Eq. (5.12) and rearranging gives the result:

$$ds \approx d\phi \sqrt{\left(1 + \frac{2GM}{c^2 r}\right)\left(\frac{dr}{d\phi}\right)^2 + r^2} \, . \tag{5.14}$$

Although this is not immediately solvable, if the line of the beam can be parameterized in terms of r and ϕ then it will be possible to integrate the equation.

Before the line can be written in spatial spherical polar coordinates, it must be found in spatial Cartesian coordinates. One point on the line, $(R_0, 0, 0)$, is the position of a satellite. The coordinates of another point correspond to the components of $\vec{B} = (B_x, B_y, 0)$. Since the magnitude of this vector is known,

$$R = \left|\vec{B}\right| = \sqrt{(B_x)^2 + (B_y)^2} \, , \tag{5.15}$$

B_y can be written in terms of B_x:

$$(B_y)^2 = R^2 + (B_x)^2 \, . \tag{5.16}$$

It is also clear that

$$k = \left|\vec{S} - \vec{B}\right| = \sqrt{(R_0 - B_x)^2 + (0 - B_y)^2} = \sqrt{(R_0 - B_x)^2 + (B_y)^2} \tag{5.17}$$

Substituting Eq. (5.16) into this equation and solving for B_x results in

$$B_x = \frac{(R_0)^2 + R^2 - k^2}{2R_0} \approx 5.01 \cdot 10^6 \text{ m.} \tag{5.18}$$

Substituting this result into Eq. (5.16), solving for B_y, and choosing the positive solution gives the result

$$B_y = \sqrt{R^2 + (B_x)^2} \approx 3.95 \cdot 10^6 \text{ m.} \tag{5.19}$$

Since the coordinates of the beam splitter and the satellite are now known, the slope of the line that they are on, m, can be calculated:

$$m = \frac{0 - B_y}{R_0 - B_x} = \frac{B_y}{B_x - R_0} \approx -0.106\,. \tag{5.20}$$

It is also known that $0 = y(R_0) = mR_0 + b$, so the y-intercept, b, can be solved for:

$$b = -mR_0 \approx 4.47 \cdot 10^6 \text{ m.} \tag{5.21}$$

The coordinate transformations from spherical polar to Cartesian coordinates are calculated geometrically:

$$x = r\cos\phi \tag{5.22}$$

$$y = r\sin\phi\,. \tag{5.23}$$

The line can be written in spatial spherical polar coordinates by substituting the coordinate transformations into the equation of the line. Solving this equation for r and taking the derivative with respect to ϕ gives the results:

$$r = \frac{b}{\sin\phi - m\cos\phi} \tag{5.24}$$

$$\frac{dr}{d\phi} = \frac{b(\cos\phi + m\sin\phi)}{(\sin\phi - m\cos\phi)^2}\,. \tag{5.25}$$

These results can be substituted into Eq. (5.14). Algebraically manipulating this result and integrating it over the values $0 \le \phi \le B_\phi$, where B_ϕ is the ϕ coordinate of the beam splitter, gives a formula for the length of the arm in the Schwarzschild geometry, L_S:

$$L_S \approx \int_0^{B_\phi} \frac{b}{\sin\phi - m\cos\phi}\sqrt{1+\left(\frac{\cos\phi + m\sin\phi}{m\cos\phi - \sin\phi}\right)^2\left[1+\frac{2GM(\sin\phi - m\cos\phi)}{c^2 b}\right]}\, d\phi . \quad (5.26)$$

The coordinate, B_ϕ, can be found by taking the Eq. (5.23) divided by Eq. (5.22), solving for ϕ using the trigonometric identity $\tan\phi = \frac{\sin\phi}{\cos\phi}$, and transforming the spatial Cartesian coordinates of the beam splitter using this result:

$$B_\phi = \tan^{-1}\left(\frac{B_y}{B_x}\right) \approx 0.668 \text{ radians.} \quad (5.27)$$

Eq. (5.26) is not easily solvable by analytic techniques. It is easier to numerically solve when it is separated into a term that describes the flatspace length and one that describes the additional length caused by spacetime curvature due to Earth's mass as in Eq. (5.13). The second term can be identified by the presence of the gravitational constant. This only appears in one term in Eq. (5.26), so if the equation could be separated into a part with G and a part without it, the term without G would represent the flatspace length of the arm, k, and the second would be the additional length due to spacetime curvature. This can be done if the equation is rearranged into the form

$$L_S \approx \int_0^{B_\phi} \frac{b}{\sin\phi - m\cos\phi}\sqrt{1+\left(\frac{\cos\phi + m\sin\phi}{m\cos\phi - \sin\phi}\right)^2}\sqrt{1+q}\, d\phi \quad (5.28)$$

where

$$q = \frac{2GM(\sin\phi - m\cos\phi)\left(\frac{\cos\phi + m\sin\phi}{m\cos\phi - \sin\phi}\right)^2}{c^2 b\left[1+\left(\frac{\cos\phi + m\sin\phi}{m\cos\phi - \sin\phi}\right)^2\right]} . \quad (5.29)$$

The limits $0 \le \phi \le B_\phi = 0.668$ radians imply $2.08 \cdot 10^{-10} \le q \le 7.12 \cdot 10^{-10} << 1$, so the binomial approximation $\sqrt{1+q} \approx 1 + \frac{1}{2} q$ would be accurate [2]. Substituting this approximation into Eq. (5.28), separating the result into two terms, and letting the term without G be equal to k because it is the flatspace term gives the result

$$\Delta L_S = L_S - k \approx \int_0^{B_\phi} \frac{GM\left(\frac{\cos\phi + m\sin\phi}{m\cos\phi - \sin\phi}\right)^2}{c^2\left[1+\left(\frac{\cos\phi + m\sin\phi}{m\cos\phi - \sin\phi}\right)^2\right]} \sqrt{1+\left(\frac{\cos\phi + m\sin\phi}{m\cos\phi - \sin\phi}\right)^2}\, d\phi \,. \qquad (5.30)$$

This equation is manageable by a computer, so it was programmed into the software, MATLAB, and solved numerically using the "quadl" command [6] with the result $\Delta L_S \approx 0.00442$ m $<< k$. The length of the arm in the Schwarzschild geometry was then calculated to be $L_S \approx \Delta L_S + k \approx k \approx 3.74 \cdot 10^7$ m.

d. Sensitivity of the Interferometer

To find out if the interferometer can detect gravity waves, its sensitivity must be calculated. It has been shown the spacetime curvature caused by Earth is insignificant in the lengths of the arms. Another consideration is what affect the rotation of Earth would have on this length. It can be shown rotation's effects on curvature are smaller than those found using the Schwarzschild metric, so these effects can be ignored. This means the spacetime curvature near Earth is small enough that spacetime can be approximated to be flat. Thus the gravitational wave equations already calculated in section four are accurate in this situation.

To calculate the sensitivity of the interferometer, return to the arrangement of the interferometer with respect to the coordinate system in Fig. (4.2). Assume that this is the $z=0$ plane and that a gravity wave is propagating in the z-direction. Choose the initial time, $t=0$, such that $\delta=0$ and let the flatspace length of the arms be $L_* \approx k$. Let $a \approx 10^{-21}$ since that is the expected amplitude of gravity waves that will be detectable on Earth [1]. In this situation, Eq. (4.8) can be used to calculate the maximum difference in the lengths of the arms, $\Delta L_{\max}$, when a gravity wave passes with the maximum occuring when $\sin(\omega t)=1$. It is more useful to use the wavelength of the laser, λ, to calculate the fraction of a wavelength the lengths of the arms will change, f. For a typical Helium-Neon laser, $\lambda \approx 633 \cdot 10^{-9}$ m, the fraction is

$$f = \frac{\Delta L_{\max}}{\lambda} = \frac{L_* a}{\lambda} \approx 5.91 \cdot 10^{-8}. \tag{5.31}$$

Because of the partially reflecting mirrors, the effective lengths of LIGO's arms are $L_* \approx 8 \cdot 10^5$ m [7]. Substituting this value into Eq. (5.31) gives the result $f_{LIGO} \approx 1.26 \cdot 10^{-9}$. It is reported [1] that the initial LIGO detector will be able to detect $f_{LIGO,reported} \approx 10^{-9}$, so these values are consistent. If the proposed interferometer were operating with equipment similar to LIGO's such that it was capable of detecting $f \approx 10^{-9}$, then it would be able to detect gravity waves of $a \sim 10^{-21}$. Table (5.1) is a partial reproduction of the table found on page 171 of Bartusiak [7] of expected rates of gravitational wave detections that LIGO will make.

Table (5.1) : LIGO's Expected Rates of Gravitational Wave Detections

Event	Region of Space	Detections
Supernova	Within Milky Way	1 to 3 per century
Black Hole/Black Hole Merger	300 million light-years	1 per 1,000 years to 1 per year
Neutron Star/ Neutron Star Merger	60 million light-years	1 per 10,000 years to 10 per century
Neutron Star/ Neutron Star Merger	130 million light-years	1 per 10,000 years to 10 per century

6. Conclusion

The proposed interferometer can be expected to successfully make a gravitational wave detection given enough time according to the theoretical considerations made. Since it is more sensitive than LIGO, it would also make more detections than those expected in Table (5.1). It is also a safer expenditure than LISA because it is repairable and updatable. Thus the interferometer is at least worth further investigation.

The next step in making this interferometer a reality would be to consider the experimental issues. Many of these have already been solved in the process of creating LIGO and LISA, but it will also provide new challenges. LIGO's arms are contained in tubes that are not practical for this interferometer. On the other hand, LISA's arms will be in the vacuum of space. Interactions with Earth's atmosphere will introduce difficulties that neither of these other two interferometers encountered. Solving the experimental difficulties such as this one could make this interferometer a practical solution for a gravitational wave observatory of the future.

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