Diameter-separation of chessboard graphs

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Abstract

We define the *queens (resp., rooks) diameter-separation number* to be the minimum number of pawns for which some placement of those pawns on an $n \times n$ board produces a board with a queens graph (resp., rooks graph) with a desired diameter d. We determine these numbers for some small values of d.

1 Introduction

In chess, a rook can move any number of squares horizontally or vertically, but cannot jump over or move through another piece. The queen in chess can move any number of spaces horizontally, vertically, or diagonally, but cannot jump over or move through another piece. A single rook or queen on an otherwise empty chessboard can reach any square in two moves. In this paper we discuss how many other pieces (call them "pawns") we need to put on the board so that some trips between squares require a larger number d of moves.

As other authors have done with problems like the n queens problem and knight tours [Wat04], we express our problem in terms of graph theory, starting with some basic terms. Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. For $a, b \in V$, a *walk from a to b* is a sequence of vertices, not necessarily distinct, $a = v_0, v_1, \ldots, v_k = b$ such that $(v_i, v_{i+1}) \in E$ for $1 \leqslant i \leqslant k-1$, and the *length* of that walk is k. A walk for which the vertices are distinct is a *path*. For $a, b \in V$, the *distance* $d(a, b)$ is the length of the shortest path from a to b. The *diameter* of G, denoted $diam(G)$, is the maximum distance between vertices of G, i.e. $diam(G) = \max\limits_{a,b \in V} d(a, b)$. For a graph $G = (V, E)$ with n numbered vertices, the *adjacency matrix* A is the $n \times n$ matrix with entries a_{ij} defined by

$$a_{ij} = \begin{cases} 1 & \text{for } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

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It is known that the entry in row i and column j of the k^{th} power of an adjacency matrix A of a graph G counts the number of walks from vertex i to vertex j and therefore the diameter of a connected graph is equal to the smallest nonnegative integer k for which the sum $\sum_{j=0}^{k} A^j$ has only nonzero entries [Wes00, Proposition 8.6.7],[Raz08]. Furthermore, in computing the diameter of a connected graph, we do not need to keep track of how many walks exist between vertices, just whether or not at least one walk exists for each pair of vertices, so we can replace the standard matrix sums and products with Boolean sums and products [Raz08].

We next define the various parts of the chessboard and our chessboard problem. A *square* is an ordered pair of integers. A *board* B is a set of squares. The $n \times n$ *board* $B_{n\times n}$ is the cartesian product of sets $\{0,\ldots,n-1\} \times \{0,\ldots,n-1\}$. For a board $B \subseteq B_{n\times n}$ we say that a square $(i,j) \in B_{n\times n}$ has a pawn placed on it if $(i,j) \in B_{n\times n} \setminus B$. Row i (respectively, column i) of a board B is the set of all squares of B with first element (resp., second element) i. The *sum diagonal* s (resp., *difference diagonal* d) of B is the set of all squares (i,j) of B with $i+j=s$ (resp., $i-j=d$). The *rooks graph* on a board B is the graph with $V=B$ and edges $((a,b),(c,d))$ where (1) $a=c$ and $(a,e) \in B$ for all $\min(b,d) \leqslant e \leqslant \max(b,d)$ or (2) $b=d$ and $(e,b) \in B$ for all $\min(a,c) \leqslant e \leqslant \max(a,c)$. The *queens graph* on a board B is the graph with $V=B$ and edges $((a,b),(c,d))$ where (1) $((a,b),(c,d))$ is an edge of the rooks graph on B, (2) $a+b=c+d=s$ and $(i,s-i) \in B$ for all $\min(a,c) \leqslant i \leqslant \max(a,c)$, or (3) $a-b=c-d=e$ and $(e+i,i) \in B$ for all $\min(b,d) \leqslant i \leqslant \max(b,d)$.

In [Cha09], given a chess piece C and a graph parameter π, the π-*separation number* $s_C(\pi,n,p)$ for C is defined as the minimum number of pawns for which some placement of those pawns on an $n \times n$ board will produce a board whose C graph has $\pi = p$. Following that pattern, we define the *rooks diameter-separation number* $s_R(diam,n,d)$ to be the minimum number of pawns for which some placement of those pawns on an $n \times n$ board will produce a board whose rooks graph has diameter d. We also define the *queens diameter-separation number* $s_Q(diam,n,d)$ to be the minimum number of pawns for which some placement of those pawns on an $n \times n$ board will produce a board whose queens graph has diameter d.

We note in passing that if $s_R(diam,n,d)$ or $s_Q(diam,n,d)$ exists for a given n and d, then that diameter-separation number is at most $n^2-(d+1)$, since a graph with diameter d must have a path of length d, which requires at least $d+1$ vertices.

In Section 2, we establish the rooks and queens diameter-separation numbers for $d=3,4$ and $n \geqslant 4$. In Section 3, we summarize results and pose several open questions.

$n\backslash d$	3	4	5	6	7	8	9	10
3	1	2						
4	1	2	3	3	4	5		
5	1	2	3	3	4	5	6	6
6	1	2	3	3	4	5	6	6

Table 1: Rooks diameter-separation numbers $s_R(diam, n, d)$ for some values of n and d. On blank cells, $s_R(diam, n, d)$ is undefined.

$n\backslash d$	3	4	5	6	7	8	9	10
3	2	3						
4	1	3	4	6				
5	1	3	5	6	7	8	9	10
6	1	3	5	6	7	9	9	10

Table 2: Queens diameter-separation numbers $s_Q(diam, n, d)$ for some values of n and d. On blank cells, $s_Q(diam, n, d)$ is undefined.

2 Results

In the Appendix we include Python 3 code for finding the queens diameter-separation number for a square board. The program for the rooks diameter-separation number is similar. Using this code we get the results listed in Tables 1 and 2. We can extend many of the results involving at most three pawns to square boards of arbitrary size, as we show in the subsections below.

2.1 Effect of one pawn

We can increase the diameter of the rooks and queens graph by placing one pawn on almost any square of the board.

Theorem 2.1. *Let $n \geqslant 3$ be an integer. If a pawn is placed on a square of an $n \times n$ board, but not on any of the four corner squares, then the rooks graph on the resulting board has diameter 3. If instead a pawn is placed on one of the four corner squares, then the rooks graph on the resulting board has diameter 2.*

Proof. Let (a, b) and (c, d) be distinct squares on an $n \times n$ board with one pawn placed on some other square. If $a \neq c$ and $b \neq d$, then either $(a, b) - (c, b) - (c, d)$ or $(a, b) - (a, d) - (c, d)$ is a path from (a, b) to (c, d) of length 2 not blocked by the pawn. So without loss of generality, suppose $a = c$. If the pawn is not between (a, b) and (a, d), then the squares attack each other, so suppose the pawn is between (a, b) and (a, d). We note in passing that in this case the pawn cannot be on one of the corner squares. Then the squares that are at distance 2 or less from (a, b) consist of all rows that are not row a and the part of row a on the side of the pawn that does not include (a, d). Then $(a, b) - (a + 1, b) - (a + 1, d) - (a, d)$ or $(a, b) - (a - 1, b) - (a - 1, d) - (a, d)$ is a path of minimum length from (a, b) to (c, d). A similar argument works for the case that $a \neq c$ and $b = d$. Therefore the diameter of the rooks graph for the board with a pawn on any square other than a corner square is 3. $\square$

Corollary 2.2. *For $n \geqslant 3$, $s_R(diam, n, 3) = 1$.*

Proof. Since the diameter of the rooks graph on an empty board is 2, at least 1 pawn is needed to obtain a graph of diameter 3. Theorem 2.1 shows that 1 pawn is sufficient. $\square$

Theorem 2.3. *Let $n \geqslant 4$ be an integer. If a pawn is placed on a square of an $n \times n$ board, but not on the first or last row or column, and, if n is odd, not in a square diagonally adjacent to a corner square, then the queens graph on the resulting board has diameter 3. If the pawn is placed on the first or last row or column, or n is odd and the pawn is placed in a square diagonally adjacent to a corner square, then the queens graph on the resulting board has diameter 2.*

Proof. Arguing as in the proof of Theorem 2.1, there is a path between any two squares of length 3 or less, so the diameter of the graph is at most 3.

Suppose the pawn is on square (e, f) with $2 \leqslant e \leqslant n-3$ and $1 \leqslant f \leqslant n-2$. We claim that either $(e, n - 1)$ or $(e, n - 2)$ is at distance more than 2 from $(e, 0)$. The squares at distance 1 from $(e, 0)$ consist of squares $(a, 0)$ on column 0, squares $(e - i, i)$ on the sum diagonal with sum e, and squares $(e + i, i)$ on the difference diagonal with difference e.

If $(a, 0)$ attacks $(e, n - 1)$ or $(e, n - 2)$, it must do so diagonally. If the attack is on a sum diagonal then $a = e + n - 1$ or $a = e + n - 2$, but $e \geqslant 2$, which implies $a \geqslant 2 + n - 2 = n$, a contradiction since the columns are numbered from 0 to $n - 1$. If the attack is on a difference diagonal, then $a = e - n + 1$ or $a = e - n + 2$, but $e \leqslant n - 3$ so $a \leqslant n - 3 - n + 2 = -1$, which again is a contradiction.

If a square (r, s) attacks another square (t, v) diagonally, then $(r+s) \equiv (t+v) \mod 2$. Since $(e + n - 1) \not\equiv (e + n - 2) \mod 2$, either $(e, n - 1)$ or $(e, n - 2)$ is not diagonally attacked by $(e + i, i)$ and $(e - i, i)$. If $(e + i, i)$ were on the same column as $(e, n - 2)$ or $(e, n - 1)$, then $e + i \geqslant e + n - 2 \geqslant n$, a contradiction. Therefore either $(e, n - 1)$ or $(e, n - 2)$ has distance at least 3 from $(e, 0)$.

By switching rows and columns, we can show a similar result for the case where the pawn is on square (f, e) with $2 \leqslant e \leqslant n - 3$ and $1 \leqslant f \leqslant n - 2$.

Next suppose the pawn is in a square diagonally adjacent to a corner square. Without loss of generality, suppose $e = f = 1$. Then $(1, 0) - (i, 0) - (1, i - 1)$ for $2 \leqslant i \leqslant n - 2$ is a path of length 2 connecting $(i, 0)$ to every other empty square in row 1 except $(1, n - 1)$.

The square $(1, n-1)$ is not attacked by any square in column 0. If $(1, n-1)$ were attacked by a square on the diagonals that $(1, 0)$ attacks (that is, $(1+i, i)$ or $(1-i, i)$), the attack would be along a diagonal. That diagonal would be the sum diagonal with sum $1 + n - 1 = n$. We would have $1 + i + i = 1 + n - 1$, so $2i + 1 = n$. If n is even, then we have a contradiction and thus there is no path of length 2 or less from $(1, 0)$ to $(1, n-1)$. If n is odd, then $(1, 0) - (1 + \frac{n-1}{2}, \frac{n-1}{2}) - (1, n-1)$ is a path of length 2 from $(1, 0)$ to $(1, n-1)$ and, by using arguments similar to those in the proof of Theorem 2.1, we can find a path of length at most 2 between every pair of squares of the board.

The remaining case is that where the pawn is on the first or last row or column. Without loss of generality, suppose the pawn is on row 0. By the arguments in Theorem 2.1, we need only show that the distance from $(0, a)$ to $(0, b)$ is 2 for $a < b$ where the pawn is between $(0, a)$ and $(0, b)$. The path of length 2 is $(0, a) - (b - a, b) - (0, b)$. $\square$

Corollary 2.4. *For $n \geqslant 4$, $s_Q(diam, n, 3) = 1$.*

Proof. Since the diameter of the queens graph on an empty board is 2, at least 1 pawn is needed to obtain a graph of diameter 3. Theorem 2.3 shows that 1 pawn is sufficient. $\square$

2.2 Effect of two pawns

To increase the diameter of the rooks graph to 4, two pawns are sufficient.

Lemma 2.5. *If pawns are placed on squares $(0, 0)$ and $(1, 1)$ of an $n \times n$ board, with $n \geqslant 3$, the diameter of the rooks graph on the resulting board is 4.*

Proof. Consider the square $(1, 0)$. Since there is a pawn at $(1, 1)$, only the squares on column 0 are attacked by $(1, 0)$. So, the squares of distance 2 or less from $(1, 0)$ consists of column 0 and all rows except rows 0 and 1. The squares of distance 3 or less include all squares except $(0, 1)$. So the distance between $(1, 0)$ and $(0, 1)$ is 4. For any pair of squares that does not include $(1, 0)$ or $(0, 1)$, the distance is at most 2. Hence the diameter of the rooks graph is 4. $\square$

Lemma 2.6. *If two pawns are placed on an $n \times n$ board with $n \geqslant 3$, the diameter of the rooks graph on the resulting board is either infinite or at most 4.*

Proof. Suppose we have an $n \times n$ board with $n \geqslant 3$ with two pawns placed on it. If the pawns are placed so that the rooks graph of the board is disconnected (e.g., at $(1, 0)$ and $(0, 1)$, then the diameter is infinite and we are done.

Suppose the diameter is finite. Let (a, b) and (c, d) be two distinct empty squares of the board. Without loss of generality $a \leqslant c$ and $b \leqslant d$.

Suppose that $c \geqslant a + 2$ and $d \geqslant b + 2$. If the path $(a, b) - (c, b) - (c, d)$ or the path $(a, b) - (a, d) - (c, d)$ are not blocked by a pawn, then we have a path of length 2. Suppose each path has a pawn on it. Since there are only two pawns and the rooks graph on the

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board is connected, either $(a, b+1)$ is empty, $(a+1, b)$ is empty, or both $(a, b+1)$ and $(a+1, b)$ have a pawn and the board has an empty row $a-1$ or column $b-1$. Similarly, either $(c-1, d)$ is empty, $(c, d-1)$ is empty, or both of those square have a pawn and the board has an empty row $c+1$ or column $d+1$.

If $(a, b+1)$ and $(c-1, d)$ are empty, then $(a, b)-(a, b+1)-(c-1, b+1)-(c-1, d)-(c, d)$ is a path of length 4. If $(a+1, b)$ and $(c, d-1)$ are empty, then $(a, b)-(a+1, b)-(a+1, d-1)-(c, d-1)-(c, d)$ is a path of length 4. If $(a, b+1)$ and $(c, d-1)$ are empty, then if $(a+1, b)$ or $(c-1, d)$ are empty we can use one of the previous cases. If both $(a+1, b)$ and $(c-1, d)$ have pawns, then $(a, b)-(a, b+1)-(c, b+1)-(c, d)$ is a path of length 3. The case where $(a+1, b)$ and $(c-1, d)$ are both empty can be handled similarly.

If both $(a, b+1)$ and $(a+1, b)$ have pawns, then there is either an empty row $a-1$ or column $b-1$. Suppose we have a column $b-1$. Then $(a, b)-(a, b-1)-(c, b-1)-(c, d)$ is a path of length 3. The subcase with empty row $a-1$ is similar.

Next, suppose $c = a+1$. If both $(a, b)-(c, b)-(c, d)$ and $(a, b)-(a, d)-(c, d)$ are blocked by pawns, then we have one pawn in row a and one pawn in row c. Since $n \geqslant 3$, the board contains either an empty row $a-1$ or an empty row $a+2$. Without loss of generality, suppose it is row $a+2$. Either $(a+1, b)$ or $(a, b+1)$ has no pawn or there is an empty column $b-1$ or row $a-1$. If $(a+1, b)$ is empty, then $(a, b)-(a+2, b)-(a+2, d)-(c, d)$ is a path of length at most 3. If $(a+1, b)$ has a pawn and $(a, b+1)$ is empty, then $(a, b)-(a, b+1)-(a+2, b+1)-(a+2, d)-(c, d)$ is a path of length at most 4. If both $(a+1, b)$ and $(a, b+1)$ have pawns and there is an empty column $b-1$, then $(a, b)-(a, b-1)-(a+2, b-1)-(a+2, d)-(c, d)$ is a path of length at most 4. If both $(a+1, b)$ and $(a, b+1)$ have pawns and there is an empty row $a-1$, then $(a, b)-(a-1, b)-(a-1, d)-(c, d)$ is a path of length at most 3.

Finally, suppose $c = a$. If there is no pawn between (a, b) and (a, d), then $(a, b)-(a, d)$ is a path of length 1. Otherwise, since $n \geqslant 3$, either the board contains rows $a-1$ and $a+1$, rows $a-1$ and $a-2$, or rows $a+1$ and $a+2$. If the rows are $a-1$ and $a+1$, one of those rows is empty, say $a-1$. In that case $(a, b)-(a-1, b)-(a-1, d)-(a, d)$ is a path of length at most 3. If the rows are $a-1$ and $a-2$, then either row $a-1$ or row $a-2$ is empty. If row $a-1$ is empty, then $(a, b)-(a-1, b)-(a-1, d)-(a, d)$ is a path of length at most 3. Suppose row $a-1$ has a pawn and row $a-2$ is empty. Either $(a-1, b)$ or $(a-1, b+1)$ is empty. Also, either $(a-1, b)$ or $(a-1, d)$ is empty. If $(a-1, b)$ and $(a-1, d)$ are both empty, then $(a, b)-(a-2, b)-(a-2, d)-(a, d)$ is a path of length at most 3. If $(a-1, b)$ has a pawn, then either $(a, b+1)$ is empty (and $(a, b)-(a, b+1)-(a-2, b+1)-(a-2, d)-(a, d)$ is a path of length at most 4) or there is an empty column $b-1$ or an empty row $a+1$ or the graph is disconnected. The other cases can be handled similarly.

In all cases, either the graph is disconnected or there is a path of length at most 4 between any two empty squares. □

Theorem 2.7. *For $n \geqslant 3$, $s_R(diam, n, 4) = 2$.*

Proof. By Theorem 2.1, placing one pawn on the board produces a rooks graph of diameter at most 3, so at least 2 pawns are needed to make a rooks graph of diameter 4. Lemma 2.5 shows there is a placement of 2 that produces a rooks graph of diameter 4. $\square$

On a 3×3 board with pawns at $(0,1)$ and $(1,1)$, the diameter of the queens graph is 4 since the distance from $(0,0)$ to $(0,2)$ is 4. For larger boards, adding two pawns produces a queens graph with diameter at most 3.

On the other hand, two pawns are not sufficient to raise the diameter of the queens graph to 4.

Lemma 2.8. *If two pawns are placed on a $n \times n$ with $n \geqslant 4$, the diameter of the queens graph on the resulting board is at most* 3.

Proof. Consider an $n \times n$ board with $n \geqslant 4$ with two pawns placed on it. Let (a,b) and (c,d) be two distinct squares on this board. Without loss of generality, $a \leqslant c$ and $b \leqslant d$. Suppose $c \geqslant a+2$ and $d \geqslant b+2$, so there is at least one row between rows a and c and at least one column between columns b and d. Then one of the following paths has no pawns on it and is a path of length at most 3 between (a,b) and (c,d):

- $(a,b)-(c,b)-(c,d)$
- $(a,b)-(a,d)-(c,d)$
- $(a,b)-(c-1,b+(c-1)-a)-(c-1,d-1)-(c,d)$ (if $d-b \geqslant c-a$)
- $(a,b)-(a+(d-1)-b,d-1)-(c-1,d-1)-(c,d)$ (if $d-b \leqslant c-a$)

Suppose $c = a+1$. If both $(a,b)-(c,b)-(c,d)$ and $(a,b)-(a,d)-(c,d)$ are blocked by pawns, then we have one pawn in row a and one pawn in row c. Since $n \geqslant 3$, the board contains either an empty row $a-1$ or an empty row $a+2$. Without loss of generality, suppose it is row $a+2$. Either (c,b) or $(c,b+1)$ is empty. If (c,b) is empty, then $(a,b)-(a+2,b)-(a+2,d)-(c,d)$ is a path of length at most 3. If (c,b) has a pawn and $(c,b+1)$ is empty, then $(a,b)-(c,b+1)-(c,d)$ is a path of length at most 2.

Suppose $c = a$. If there is no pawn between (a,b) and (a,d), then $(a,b)-(a,d)$ is a path of length 1. Otherwise, since $n \geqslant 4$, either the board contains rows $a-1$ and $a-2$ or rows $a+1$ and $a+2$. Without loss of generality, suppose we have rows $a+1$ and $a+2$. If row $a+1$ is empty, then $(a,b)-(a+1,b)-(a+1,d)-(a,d)$ is a path of length at most 3. If row $a+1$ contains a pawn, then row $a+2$ is empty. Either $(a+1,b)$ or $(a+1,b+1)$ is empty. If $(a+1,b)$ is empty, let $A=(a+2,b)$; otherwise, let $A=(a+2,b+2)$. Either $(a+1,d-1)$ or $(a+1,d-1)$ is empty. If $(a+1,d-1)$ is empty, let $B=(a+2,d-2)$; otherwise, let $B=(a+2,d)$. Then $(a,b)-A-B-(a,d)$ is a path of length at most 3 between (a,b) and (a,d). Using reflections and rotations of the above arguments to deal with the remaining cases, we can find paths of length at most 3 between any two squares of the board. Therefore the diameter of the queens graph on an $n \times n$ board with 2 pawns is at most 3. $\square$

2.3 Effect of three pawns

There is a placement of three pawns that increases the diameter of the rooks graph to 5 and a different placement that raises the diameter to 6.

Lemma 2.9. *For $n \geqslant 4$, the rooks graph of the $n \times n$ board with pawns placed at $(0,1),(1,2)$ and $(2,1)$ has diameter 5.*

Proof. If (a,b) and (c,d) are squares for which $\max\{a,b\} \geqslant 2$ and $\max\{c,d\} \geqslant 2$, then either $(a,b)-(a,d)-(c,d)$ or $(a,b)-(c,b)-(c,d)$ is an unobstructed path of length at most 2 from (a,b) to (c,d).

If $a=0$, $b=0,1$, or 2, and $\max\{c,d\} \geqslant 2$, then $(3,0)$ is a neighbor of (a,b), so there is an unobstructed path of length at most 3 from (a,b) to (c,d).

So to conclude the diameter of the rooks graph is 5, it suffices to check the length of paths from $(0,2)$, $(1,1)$, and $(2,2)$.

Consider the square $(0,2)$. Its neighborhood is the set of squares $(0,k)$ for which $k>2$. At distance 2 from $(0,2)$ are the other squares of columns $2,3,\dots n-1$. At distance 3 is the rest of columns 0, 1 and 2, except squares $(0,0),(1,0),(1,1)$ and $(2,0)$ At distance 4 squares $(0,0),(1,0)$, and $(2,0)$. So the only square at distance 5 from $(0,2)$ is $(1,1)$.

Next consider the square $(1,1)$. Its only neighbor is $(1,0)$. At distance 2 from $(1,1)$ are the rest of the squares of column 0. At distance 3 from $(1,1)$ are the squares (c,d) for which $c \geqslant 3$ and $d \geqslant 1$. At distance 4 from $(1,1)$ are $(2,2)$ and (c,d) for which $c=0,1$ or 2 and $d \geqslant 3$. That leaves $(0,2)$ as the only square at distance 5 from $(1,1)$.

Finally, consider the square $(2,2)$. Its neighbors are $(2,d)$ for $d \geqslant 3$ and $(c,2)$ for $c \geqslant 3$. At distance 2 from $(2,2)$ are the squares other than $(0,0)$, $(0,2)$, $(1,0)$, $(1,1)$, $(2,0)$ and the squares at distances 0 and 1. Squares $(0,0),(0,2),(1,0)$ and $(2,0)$ are at distance 3 from $(2,2)$, leaving $(1,1)$ as the only square at distance 4 from $(2,2)$.

Therefore the diameter of the rooks graph is 5. $\square$

Lemma 2.10. *For $n \geqslant 4$, the rooks graph of the $n \times n$ board with pawns placed at $(0,1),(1,2)$ and $(2,0)$ has diameter 6.*

Proof. As argued in the proof of Lemma 2.9, if (a,b) and (c,d) are squares for which $\max\{a,b\} \geqslant 2$ and $\max\{c,d\} \geqslant 2$, then there is an unobstructed path of length at most 2 from (a,b) to (c,d). Therefore, to conclude that the diameter of the rooks graph is 6, it suffices to consider the lengths of paths from $(0,0)$, $(0,2)$, $(1,0)$, $(1,1)$, $(2,1)$, and $(2,2)$.

Consider the square $(0,0)$. Its only neighbor is $(1,0)$. Only $(1,1)$ is at distance 2 from $(0,0)$. At distance 3 is the rest of column 1. At distance 4 are the rest of rows

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$2, 3 \ldots n-1$. At distance 5 are the remaining squares of rows 0 and 1, except $(0, 2)$, which is at distance 6 from $(0, 0)$.

Paths from the remaining squares can be found with similar arguments. No such path has length greater than 6. So the diameter of the rooks graph is 6. □

Theorem 2.11. *For $n \geqslant 4$, $s_R(diam, n, 5) = s_R(diam, n, 6) = 3$.*

Proof. Lemmas 2.9 and 2.10 show that 3 pawns are sufficient. Theorem 2.1 and Lemma 2.6 show that fewer pawns are insufficient. □

Three pawns are sufficient to raise the diameter of the queens graph to 4.

Lemma 2.12. *If pawns are placed on squares $(0, 1), (1, 1)$, and $(2, 0)$ of an $n \times n$ board with $n \geqslant 4$, the diameter of the queens graph on the resulting board is 4.*

Proof. Let (a, b) and (c, d) be empty squares that are not $(0, 0), (1, 0)$, or $(2, 1)$. Then there is an unblocked path of length at most 2 from (a, b) to (c, d).

Consider the square $(2, 1)$. Neighbors of this square include $(1, 0)$, $(a, 1)$ for all $a \geqslant 1$, and $(1, b)$ for all $b \geqslant 1$. So all empty squares have distance at most 2 from $(2, 1)$.

Consider the square $(1, 0)$. The neighbors of this square are $(0, 0)$ and $(1 + a, a)$ for $1 \leqslant a \leqslant n-2$. All other squares except $(1, n-1)$ and $(0, n-1)$ are on the same row or column as some $(1 + a, a)$. Each $(1 + a, a)$ is the sum diagonal with sum $2a + 1$, which is odd. If n is even, $(0, n-1)$ is on the same sum diagonal as $(\frac{n}{2}, \frac{n}{2} - 1)$ and so has distance 2 from $(1, 0)$, and $(1, n-1)$ is not diagonally attacked by any $(1 + a, a)$ and therefore has distance 3 from $(1, 0)$. If n is odd, then $(0, n-1)$ is on the same sum diagonal as $(\frac{n+1}{2}, \frac{n-1}{2})$ and thus has distance 2 from $(1, 0)$, and $(1, n-1)$ has distance 3 from $(1, 0)$. So all empty squares have distance at most 3 from $(1, 0)$.

Finally, consider the square $(0, 0)$. The only neighbor of $(0, 0)$ is $(1, 0)$, which has distance 3 from either $(0, n-1)$ or $(1, n-1)$. Thus $(0, 0)$ is at distance 4 from $(0, n-1)$ or $(1, n-1)$. Therefore the diameter of the queens graph on this board is 4. □

Theorem 2.13. *For $n \geqslant 4$, $s_Q(diam, n, 4) = 3$.*

Proof. Lemma 2.12 shows that $s_Q(diam, n, 4) \leqslant 3$. Theorem 2.3 and Lemma 2.8 show that $s_Q(diam, n, 4) \geqslant 3$. □

3 Conclusions

In this paper we have looked for the smallest number of pawns that we need to place on a square board to produce a board whose rooks and queens graph have some desired diameter. We have shown that one pawn is necessary and sufficient to produce a board with a rooks or queens graph of diameter 3. Also, two pawns are necessary and sufficient to produce a board with a rooks graph of diameter 4. We also showed that three pawns

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are necessary and sufficient to make a board whose rooks graph has diameter 5 or 6 or a board whose queens graph has diameter 4.

Several open questions arise from this work.

1. For a given d, what are $\lim_{n\to\infty} s_R(diam, n, d)$ and $\lim_{n\to\infty} s_Q(diam, n, d)$?

2. If we can place as many pawns as we like on the board, what are the maximum possible diameter of the rooks and queens graphs?

3. What can we say about rectangular, cylindrical, and other boards?

4. What can we say about separation for other distance-related parameters, such as the radius?

5. What can we say about other pieces, such as the knight?

6. In solutions to the classic n-queens problem of placing n mutually nonattacking queens on an $n \times n$ board [BS09], each queen is two moves away from every other queen. As indicated in Figures 1 and 2, we can place n queens and $2n-2$ pawns on an $n \times n$ board so each queen is 3 moves away from every other queen. How many pawns are needed on an $n \times n$ board to allow a placement of n queens at distance 3 from each other?

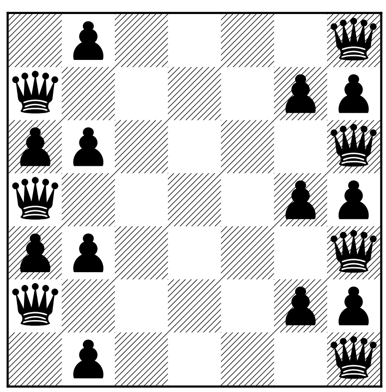

Figure 1: Seven queens and 12 pawns on an 7×7 board, with each queen needing three moves to reach any other queen

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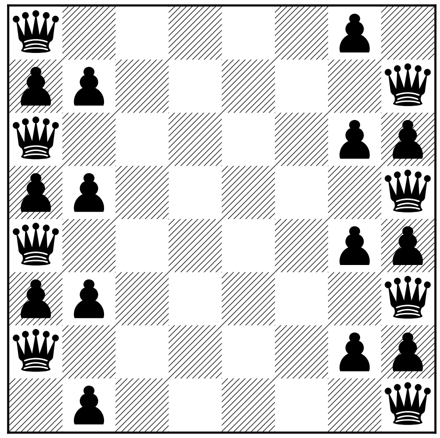

Figure 2: Eight queens and 14 pawns on an 8×8 board, with each queen needing three moves to reach any other queen

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Appendix

Below is a Python 3 program which computes the queens diameter separation number for a square board of input size n. The program uses algorithms for Boolean matrix multiplication and graph diameter determination presented in [Raz08].

```
from math import *
from itertools import *
def Identity(n): # produce identity matrix of order n
    return [[(i==j) for i in range(n)] for j in range(n)]

def Plus(a,b,n): # Boolean add n-by-n matrices a and b
    return [[(a[i][j] or b[i][j]) for i in range(n)] for j in range(n)]

def Power(b,irule,n): # kth Boolean power of adjacency matrix
    c = [[False for i in range(n)] for j in range(n)]
    for i in range(n):
        for j in range(i+1):
            for k in irule[i]:
                if c[i][j]:
                    break
                c[i][j]=b[k][j]
                c[j][i]=c[i][j] # We know A and its powers are symmetric
    return c

def Attack(pos1,pos2,pawns):
# determine whether (pos1,pos2) is an edge in the queens graph
    if pos1 in pawns or pos2 in pawns:
        return False
    if pos1 == pos2:
        return False
    if pos1[0] == pos2[0]:
        for i in pawns:
            if i[0]==pos1[0]:
                if min(pos1[1],pos2[1])<=i[1] and i[1]<=max(pos1[1],pos2[1]):
                    return False
        return True
    if pos1[1] == pos2[1]:
        for i in pawns:
            if i[1]==pos1[1]:
                if min(pos1[0],pos2[0])<=i[0] and i[0]<=max(pos1[0],pos2[0]):
```

```
                return False
        return True
    if pos1[1]+pos1[0] == pos2[1]+pos2[0]:
        for i in pawns:
            if i[1]+i[0]==pos1[1]+pos1[0]:
                if min(pos1[0],pos2[0])<=i[0] and i[0]<=max(pos1[0],pos2[0]):
                    return False
        return True
    if pos1[1]-pos1[0] == pos2[1]-pos2[0]:
        for i in pawns:
            if i[1]-i[0]==pos1[1]-pos1[0]:
                if min(pos1[0],pos2[0])<=i[0] and i[0]<=max(pos1[0],pos2[0]):
                    return False
        return True
    return False

def MakeBoard(n,pawns): # Form adjacency matrix for queens graph on
# n-by-n board with pawns at given positions
    c = [[False for i in range(n*n)] for j in range(n*n)]
    for i1 in range(n):
        for j1 in range(n):
            for i2 in range(n):
                for j2 in range(n):
                    if Attack([i1,j1],[i2,j2],pawns):
                        c[i1*n+j1][i2*n+j2]= True
    return c

def MakeLists(n,board): # For each row r of the adjacency matrix, make
# a list of column indices c for which the matrix has a True entry in
# row r and column c
    temp=[]
    for r in range(n*n):
        rowtemp=[]
        for c in range(n*n):
            if board[r][c]:
                rowtemp.append(c)
        temp.append(rowtemp)
    return temp

def Reach(board,n): # Count how many entries in board are True
    acc=0
```

```
    for i in range(n):
        for j in range(n):
            if board[i][j]==True:
                acc=acc+1
    return acc

def Diameter(board,irule,n,p):
    k = 0
    tot = Identity(n*n)
    po = Identity(n*n)
    oldreach = Reach(tot,n*n)
    while (oldreach<(n**2-p)**2+p):
        k=k+1
        po=Power(po,irule,n*n)
        tot=Plus(tot,po,n*n)
        newreach=Reach(tot,n*n)
        if oldreach==newreach:
            return n*n # Code for non-connected graph
        oldreach=newreach
    return k

n = int(input('Size of square:'))
pmax = int(input('Maximum number of pawns:'))
d = int(input('Desired diameter:'))
p = 0
doneflag = False
squares=[]
for i in range(n):
    for j in range(n):
        squares.append([i,j])
while (p<=pmax) and not(doneflag):
    for pawns in combinations(squares,p):
        board = MakeBoard(n,pawns)
        irule = MakeLists(n,board)
        d1=Diameter(board,irule,n,p)
        if d1==d:
            print("The diameter separation number is ",p,".",sep="")
            if len(pawns)>0:
                print("Placement of pawns: ",end="")
                if len(pawns)==1:
                    print("(",pawns[0],")\n",sep="")
```

```
                else:
                    print(pawns,"\n",sep="")
            doneflag = True
            break
    if not(doneflag):
        p = p+1
if (p>pmax):
    print("Did not get the desired diameter.")
y = input("Hit enter to close.")
```