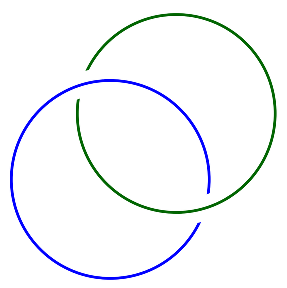
# The Delta-crossing Number of Links

Department of Mathematics

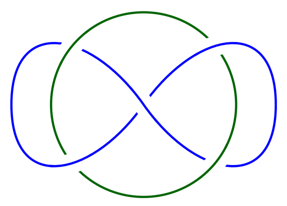
Zachary Duah
Andrews University, Advisor: Anthony Bosman

## Links and Tait Series

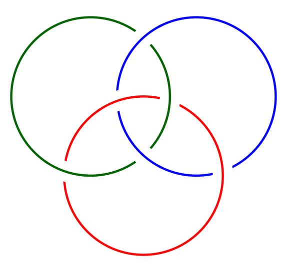
An $m$-component link is an embedding of $m$ circles into 3-dimensional space. Below are examples of links from the Thistlethwaite Link Table.



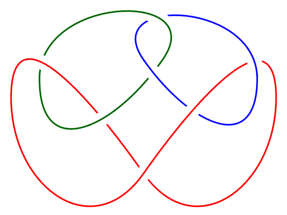
$L2a1$



$L5a1$

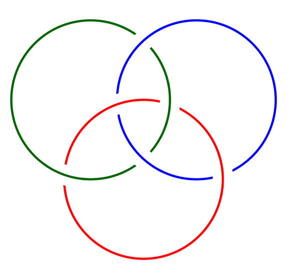


$L6a4$

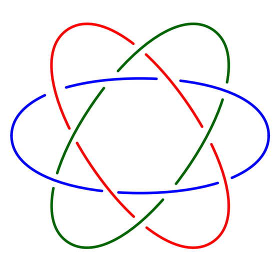


$L7a7$

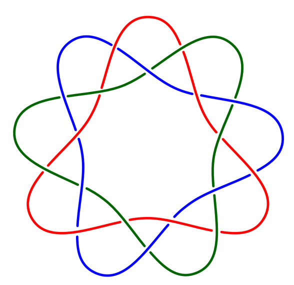
A Borromean ring is a link of three unknots such that one component is always over the second component and under the third. The Tait series is an infinite series of generalized Borromean rings.



$B_1$



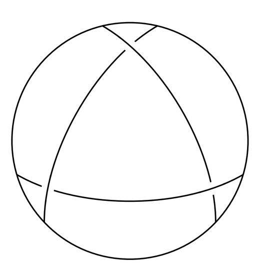
$B_2$

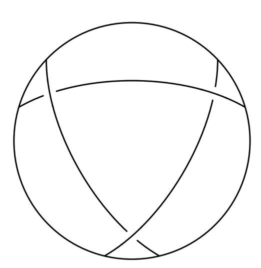


$B_3$

## Δ-crossing Tangle

The Δ-crossing tangle is a collection of three crossings as appear in a Δ move.





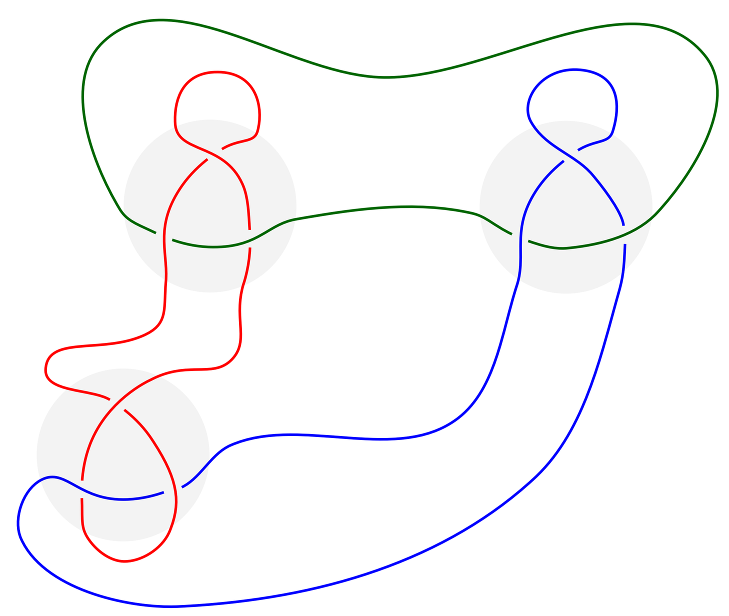
The Δ-crossing diagram of a link is a diagram of a link that is drawn such that all crossings occur in Δ-crossing tangles.

The Δ-crossing number $c_\Delta(L)$ is the minimal number of Δ-crossing tangles in a Δ-crossing diagram.

The Δ-unlinking number $u^\Delta(L)$ of an algebraically split link $L$ is the minimal number of Δ-moves needed to deform $L$ into the trivial link.

## Determining Δ-crossing Number

For an example of determining the Δ-crossing number, notice that link $L7a7$ can be drawn in such a way that all the crossings occur in a Δ-tangles:



Thus we know that $c_\Delta(L7a7) \leq 3$. To determine the lower bound, we proved:

**Proposition 1.** *Given an link $L$ with crossing number $c(L)$, $c_\Delta(L) \geq \frac{1}{3} \cdot c(L)$. Moreover, if we have equality, then $L$ can be turned into a Δ-crossing diagram without adding any crossings.*

**Proposition 2.** *Given the standard projection of a link $L$ with crossing number, $c(L)$, we must add $j$ crossings to the diagram to obtain a Δ-crossing diagram such that, $c(L)+j \equiv 0 \pmod 3$, where $j \in \mathbb{N}$.*

**Proposition 3.** *Given an link $L$, $c_\Delta(L) \leq 2c_3(L)$, where $c_3(L)$ denotes the triple-crossing number of a link.*

Therefore by Proposition 3, $c_\Delta(L7a7) \leq 8$. And by Proposition 1,

$$c_\Delta(L7a7) \geq \frac{1}{3} \cdot 7 = 2.33.$$

Thus $c_\Delta(L7a7) = 3$.
Arguing similarly, we determine the Δ-crossing numbers for prime links with up to 8 crossings.

| Link | $c_\Delta(L)$ | $\frac{1}{3}c(L)$ | $2c_3(L)$ | Link | $c_\Delta(L)$ | $\frac{1}{3}c(L)$ | $2c_3(L)$ | Link | $c_\Delta(L)$ | $\frac{1}{3}c(L)$ | $2c_3(L)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $L2a1$ | **1** | 0.67 | 2 | $L7n2$ | **3 or 4** | 2.33 | 6 | $L8a16$ | **3** | 2.67 | 10 |
| $L4a1$ | **2** | 1.33 | 4 | $L8a1$ | **3** | 2.67 | 10 | $L8a17$ | **3 or 4** | 2.67 | 10 |
| $L5a1$ | **2** | 1.67 | 6 | $L8a2$ | **3 or 4** | 2.67 | 10 | $L8a18$ | **3** | 2.67 | 8 |
| $L6a1$ | **2** | 2 | 6 | $L8a3$ | **3 or 4** | 2.67 | 8 | $L8a19$ | **3** | 2.67 | 10 |
| $L6a2$ | **2 or 3** | 2 | 8 | $L8a4$ | **3** | 2.67 | 10 | $L8a20$ | **3 or 4** | 2.67 | 8 |
| $L6a3$ | **2 or 3** | 2 | 6 | $L8a5$ | **3 or 4** | 2.67 | 10 | $L8a21$ | **3 or 4** | 2.67 | 8 |
| $L6a4$ | **2** | 2 | 8 | $L8a6$ | **3** | 2.67 | 8 | $L8n1$ | **3 or 4** | 2.67 | 6 |
| $L6a5$ | **2** | 2 | 8 | $L8a7$ | **3** | 2.67 | 10 | $L8n2$ | **3 or 4** | 2.67 | 6 |
| $L7a1$ | **3** | 2.33 | 10 | $L8a8$ | **3** | 2.67 | 10 | $L8n3$ | **3 or 4** | 2.67 | 6 |
| $L7a2$ | **3** | 2.33 | 8 | $L8a9$ | **3** | 2.67 | 10 | $L8n4$ | **3 or 4** | 2.67 | 6 |
| $L7a3$ | **3 or 4** | 2.33 | 10 | $L8a10$ | **3** | 2.67 | 10 | $L8n5$ | **3 or 4** | 2.67 | 8 |
| $L7a4$ | **3** | 2.33 | 8 | $L8a11$ | **3 or 4** | 2.67 | 8 | $L8n6$ | **3 or 4** | 2.67 | 8 |
| $L7a5$ | **3** | 2.33 | 8 | $L8a12$ | **3 or 4** | 2.67 | 10 | $L8n7$ | **3 or 4** | 2.67 | 8 |
| $L7a6$ | **3** | 2.33 | 8 | $L8a13$ | **3 or 4** | 2.67 | 10 | $L8n8$ | **3 or 4** | 2.67 | 8 |
| $L7a7$ | **3** | 2.33 | 8 | $L8a14$ | **4** | 2.67 | 8 | | | | |
| $L7n1$ | **3** | 2.33 | 6 | $L8a15$ | **3** | 2.67 | 8 | | | | |

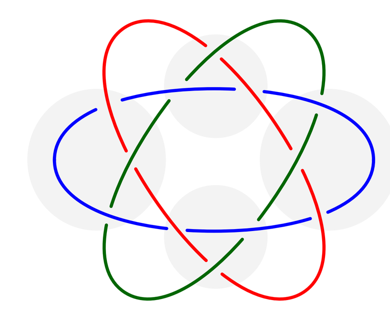
## Δ-crossing Number for Tait Series & Δ-unlinking Gap

We can determine the Δ-crossing number for every member of the Tait Series with the following theorem.

**Lemma 1.** *Given a generalized Borromean ring $B_n$ with crossing number $6n$,*

$$c_\Delta(B_n) = 2n$$

For instance, $B_2$ has Δ-crossing number of 4.



We can also obtain a bound for the Δ-unlinking number for the Tait Series.

**Lemma 2.** *Given a generalized Borromean ring $B_n$,*

$$u^\Delta(B_n) \leq n$$

By combining these two expressions, we obtain an arbitrarily large gap for the Δ-unlinking number.

**Theorem 1.** *Given a generalized Borromean ring $B_n$,*

$$c_\Delta(B_n) - u^\Delta(B_n) \geq n$$

For example, while $B_{10}$ can be transformed into the trivial link with no more than 10 moves, it cannot be drawn with less than 20 Δ-tangles, a gap of 10.

## References

Jablonowski, Michal (2022). *Upper bound on delta-crossing and tabulation of knots up to four delta-crossings.* DOI: 10.48550/ARXIV.2204.02161. URL: https://arxiv.org/abs/2204.02161.

Jablonowski, Michał and Lukasz Trojanowski (Feb. 2020). "Triple-crossing projections, moves on knots and links and their minimal diagrams". In: *Journal of Knot Theory and Its Ramifications* 29.04, p. 2050015. DOI: 10.1142/s0218216520500157. URL: https://doi.org/10.1142%2Fs0218216520500157.

Nakanishi, Yasutaka, Yoko Sakamoto, and Shin Satoh (2015). "Delta-crossing number for knots". In: *Topology and its Applications* 196, pp. 771–776. ISSN: 0166-8641. DOI: https://doi.org/10.1016/j.topol.2015.05.044.