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Combined variational iteration method with chebyshev wavelet for the solution of convection-diffusion-reaction problem

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ABSTRACT

The goal of the work is to solve the nonlinear convection-diffusion-reaction problem using the variational iteration method with the combination of the Chebyshev wavelet. This work developed a hybrid iterative technique named as Variational iteration method with the Chebyshev wavelet for the solutions of nonlinear convection-diffusion-reaction problems. The aim of applying the derived algorithm is to achieve fast convergence. During the solution of the given problem, the restricted variations will be mathematically justified. The effects of the scaling and other parameters like diffusion parameter, convection parameter, and reaction parameter on the solution are also focused on by their suitable selection. The approximate results include the error profiles and the simulations. The results of variational iteration with the Chebyshev wavelet are compared with variational iteration method, the Modified variational iteration method, and the Variational iteration method with Legendre wavelet. The error profiles allow us to compare the results with well-known existing schemes.

1. Introduction

The differential equations describe the physical phenomena in almost all fields of engineering and applied sciences. Various physical applications are expressed in the custom of mathematical equations called mathematical prototypes or models, which are used to simulate the physical behavior of the dynamical systems [1-3]. The solution of the partial differential equations (PDEs) based models using symbolic methods as well as the numerical approaches is a usual mathematical practice but due to nonlinear complexities and other ambiguous situations for the existence of the solutions, like effects of small scaling perturbations arising in various engineering models. The problems remain still challenging task for many researchers from all around the world. Therefore, they are keenly attracted to this area of research [4-6].

The convection-diffusion-reaction (CDR) partial differential equations (PDE's) provide a vital role of mathematical modeling for the wide-ranging real-world problems in natural sciences and engineering. These problems involve the air transport, absorption of pollutants in soil, neuron diffusion, processing of food, biological systems modeling, semiconductors modeling, chemical species reaction and oil reservoir flow transport, etc. The convection, diffusion and reaction processes have been used to describe a variety of physical issues, like the variation in concentration of one or more compounds in a medium. Convection depicts the movement of substances because of the transport medium whereas the reaction is a contact. The Diffusion also ensures the transportation of the material from a higher density to a lower density and maintains a uniform dispersion of the substance. The Convection-Diffusion-Reaction process describes the substance's concentration.

In various problems, the unknown solution of the governing PDE's indicates the physical quantities that do not take the negative results such as concentrations of chemical compounds, the pollutants, the population, etc. Generally, the applications of the convection-diffusion-reaction (CDR) type equations are divided into three modeling procedures. The first procedure is known as the convection process that occurs due to the materials movement. The diffusion process occurs due to the material movement from high concentrated to low concentrated regions. The third technique is the reaction process that occurs due to the decay, absorption, and reaction of the substances with other additives.

All the above three CRD modeling procedures define the quantity distribution and the changes in the given medium.

Generally, the determination of the analytical solution for the nonlinear diffusion Equations specifically with nonlinear internal source terms is not very straightforward. Many approaches have been proposed to solve the linear and nonlinear PDEs like Adomain Decomposition Methods (ADM) [7-9], Homotopy Perturbation techniques [10, 11], Legendre wavelet technique (LW) [5, 6, 12-16], Haar wavelet technique [17-19], Chebyshev wavelet (CW) technique [20, 21] and variational iteration method (VIM) [22-25]. The PDEs based problems are generally solved with known suitable boundary information, whereas the ordinary differential equations (ODEs) are generally solved as initial value problems (IVP) [1, 13]. The idea of variational strategies like He's variational iteration method (HVIM) is considered as very effective method for the solution of ODE's and PDE's [22, 26-28].

In the spirit of the suggested strategies and keeping in view the efficiency of the VIM along with the novel ideas of the solution strategies using wavelet-based methods (WBM), therefore, work is extended towards the coupling of the modern WBM with the latest techniques like VIM [23]. Combining the CW with the other various algorithms has several advantages moreover the main goal is good accuracy, which is possible in the spirit of [29-31]. This approach is based on the coupling the Chebyshev wavelets with VIM. The combining process yields a nonlinear system, which is then solved by the numerical method. The main contribution of our work is to derive the proposed VIM by combining the CW with VIM, which allows the classical solution as a convergent series for PDEs. Some problems have been tested with the proposed algorithm, which provides a strong computational framework for the solutions of the time dependent nonlinear PDEs.

2. Methodology

Consider the following model problem as a one-dimensional CDR equation as [8]

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(E(u(x,t)\frac{\partial u(x,t)}{\partial x}\right) + F(u(x,t))\frac{\partial u(x,t)}{\partial x} + Q(u) \quad (\because u \in R^{+} \times \Omega) \tag{1}$$

Here, $u(x,t)$ have been investigated, $F(u(x,t))$ is the convection velocity term in the horizontal direction, whereas $E(u(x,t))$ the diffusion and $Q(u)$ is the reaction term. The Ω is the one-dimensional spatial domain.

The work starts with the solution of some nonlinear PDEs by applying the VIM then extended work to the main part of the study, which is dedicated to coupling the VIM with the CW for the given specific class of the problems.

2.1 Variational Iteration Method

To discuss the elementary theories of VIM, we consider the following nonlinear problem.

$$L(u) + N(u) = g(x,t) \tag{2}$$

The L and $N$are denoted as the linear and nonlinear operators respectively. Here $g(x,t)$ is a known analytic function. We consider the following iterative scheme of He's VIM [23].

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\ \left[L(u_n(x,\tau)) + N(\tilde{u}_n(x,\tau)) - g(x,\tau)\right]d\tau \tag{3}$$

The function λ can be optimized by using the variational constraint $\delta\tilde{u}_n = 0$. The successive approximations of the solution $u(x,t)$ is modeled by $u_n(x,t)$, where (n) is the non-negative integer which can be determined with the help of the LMT in the availability of the initial solution u_0. The u_0 is a zeroth approximation generally considered as a given initial solution satisfying the given boundary conditions. The exact solution can be computed as

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \quad (4)$$

It has been observed from the results that VIM is an efficient algorithm for solving nonlinear problems.

2.2 Chebyshev Wavelets

The CW $\psi_{nm}(x)$ on [0,1] is defined as

$$\psi_{n,m} = \begin{cases} 2^{k/2}\tilde{T}_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ 0, & \text{elsewhere}, \end{cases} \quad (5)$$

Where, $k \in Z^+$, $\hat{n} = 2n - 1$, $(n = 1,2,3,\dots)$, m is known as the order of Chebyshev polynomials (CPs) and t is known as the normalized time.

$$\tilde{T}_m(x) = \begin{cases} \frac{1}{\sqrt{\pi}} & m = 0, \\ \sqrt{\frac{2}{\pi}} T_m & m > 0 \end{cases} \quad (6)$$

Where, $m = 0,1,\dots,M-1$, $n = 1,2,\dots,2^{k-1}$. The scaling coefficients are to meet the ortho-normality criteria. Here, $T_m(t)$are CPs of the first kind of degree *m*. These wavelets are orthogonal with respect to the function $w(t) = \frac{1}{\sqrt{1-t^2}}$on the interval $[-1,1]$, that satisfies the following recursive process.

$$\left.\begin{array}{l} T_0(t) = 1, \quad T_1(t) = t, \\ T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1,2,\dots \end{array}\right\} \quad (7)$$

One can note that in dealing with CW the weight function $w_n(t)$is defined as

$$w_n(t) = w\left(2^k t - 2n + 1\right) \quad (8)$$

2.3 Function Approximations

The series expansion of the function $f(t) \in L^2{}_w[0,1]$ in the form of CW series is given as

$$f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\psi(t) \quad (9)$$

Where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices

$$\left.\begin{array}{l} C = \begin{bmatrix} c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, c_{21}, \dots, c_{2M-1}, \dots, \\ c_{2^{k-1}M0}, c_{2^{k-1}M1}, \dots, c_{2^{k-1}M-1} \end{bmatrix}^T \\ \psi(t) = \begin{bmatrix} \psi_{10}, \psi_{11}, \dots, \psi_{1M-1}, \psi_{20}, \psi_{21}, \dots, \psi_{2M-1}, \dots, \\ \psi_{2^{k-1}M0}, \psi_{2^{k-1}M1}, \dots, \psi_{2^{k-1}M-1} \end{bmatrix}^T \end{array}\right\} \quad (10)$$

with $c_{nm} = \langle f(t), \psi_{nm} \rangle_{w_n}$ (11)

Where the notation $\langle\ ,\ \rangle$ is the inner product in $L^2{}_{w_n}[0,1]$ space.

The function $f(x,t) \in L^2([0,1] \times [0,1])$ can be determined by using CW series as follows.

$$f(x,t) = \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{k-1}M} c_{ij}\psi_i(x)\psi_j(t) = \psi^T(x)C\psi(t) \quad (12)$$

with

$$c_{ij} = \int_0^1 \int_0^1 \frac{f(x,t)\psi_i(x)\psi_j(t)}{\sqrt{1-\left(2^k t - 2n + 1\right)^2}\sqrt{1-\left(2^k x - 2n + 1\right)^2}}\ dx\ dt \quad (13)$$

Where C is a $2^{k-1}M \times 2^{k-1}M$ matrix.

2.4 Chebyshev Wavelet Operational Matrices of Integration

The integration of the vector $\psi(t)$ defined in (5) can give as

$$\int_0^t \psi(s)\ \ ds = P\ \psi(t) \quad (14)$$

Where P is the operational matrix of order $2^{k-1}M \times 2^{k-1}M$ for integration, for operational matrices reader is suggested to review [30]. The matrices $C$and S are $M \times M$matrices given.

$$C = \frac{1}{2^k}\begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{4\sqrt{2}} & 0 & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3\sqrt{2}} & \frac{-1}{4} & 0 & \frac{1}{12} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \cdots & \frac{-1}{4(M-3)} & 0 & \frac{-1}{4(M-1)} \\ \frac{-1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{4(M-2)} & 0 \end{bmatrix} \quad (15)$$

and

$$S = \frac{\sqrt{2}}{2^k}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{3} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{15} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{M(M-2)} & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (16)$$

2.5. Chebyshev Wavelet Based on Operational Matrices of Derivative

In this section, the operational matrices of the derivative$D$are derived to simplify the expansion of the derivative terms in the given PDE's, in terms of the wavelet series. The CW $\psi_{n,m}(x)$is defined on the interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ as [30]

$$\psi_{n,m}(x) = 2^{k/2}\tilde{T}_m(2^k x - 2n + 1) \quad (17)$$

The derivative of $\psi_{n,m}(x)$ with respect to *x*

$$\psi'_{n,m}(x) = \begin{cases} 2^{\frac{3k}{2}+1} m\ \sum_{k=1}^{m-1} \tilde{T}_k(2^k x - 2n + 1) & m\ \ even \\ 2^{\frac{3k}{2}}\left[2m\ \sum_{k=1}^{m-1} \tilde{T}_k(2^k x - 2n + 1) + m\tilde{T}_0(2^k x - 2n + 1)\right] & m\ \ odd \end{cases} \quad (18)$$

where, $m = 0,1,2,\dots,M-1$. The function $\psi_i(x)$ is zero outside the interval $\left[\frac{i-1}{2^{k-1}}, \frac{i}{2^{k-1}}\right]$ so

$$\psi_i'(x) = \psi_i(x)M \tag{19}$$

$$M = 2^k \begin{pmatrix} 0 & \sqrt{2} & 0 & 3\sqrt{2} & 0 & 5\sqrt{2} & \cdots & (M-1)\sqrt{2} \\ 0 & 0 & 4 & 0 & 8 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & \cdots & 2(M-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2(M-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{M\times M} \quad for\ even\ M$$

$$M = 2^k \begin{pmatrix} 0 & \sqrt{2} & 0 & 3\sqrt{2} & 0 & 5\sqrt{2} & \cdots & 0 \\ 0 & 0 & 4 & 0 & 8 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & \cdots & 2(M-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2(M-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{M\times M} \quad for\ odd\ M, \tag{20}$$

Thus

$$\psi'(x) = D\ \psi(x) \tag{21}$$

Therefore, the operational matrices of the derivative "D" can be determined as $D = M^T, \quad for\ even\ M$ and $D = M^T, for\ odd\ M.$

2.6 Block Pulse Functions (BPF's)

A complete set of orthogonal functions, defined on the $[0,\ b]$ can be developed by BPF's as follows.

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m}b \le t \le \frac{i}{m}b \\ 0 & else\ where \end{cases} \tag{22}$$

For $i = 1,2,\dots,m$. An arbitrary function $f(t) \in L^1[0,b]$ has been described in BPFs as

$$f(t) \simeq \xi^T B_m(t) \tag{23}$$

Where, $B_m(t) = [b_1 \quad b_2 \quad \cdots \quad b_m]^T$ and $\xi^T = [f_1 \quad f_2 \quad \cdots \quad f_m]$, here f_i are the coefficients of BPF defined as:

$$f_i = \frac{m}{b}\int_{\frac{i-1}{m}b}^{\frac{i}{m}b} f(t)b_i(t)dt \tag{24}$$

The following are basic BPF properties, defined as

Disjoint: The BPF is disjointed with each other in the t, i.e., defined as the $t \in 0,T)$

$$b_i(t)b_j(t) = \delta_{ij}b_i(t)\ i,j = 1,2,\dots,m \tag{25}$$

Completeness: If m grow very large i.e., towards infinity, then the BPF is said to be complete, that is $f \in L^2 0,T)$ satisfy the following Parse-Val's identity.

$$\int_0^T f^2(t)\ dt = \sum_{i=1}^{\infty} f_i^2 \|b_i(t)\|^2,\ \because\ f_i = \frac{1}{h}\int_0^T f(t)b_i(t)\ dt \tag{26}$$

Orthogonality: The orthogonality in the interval $t \in 0,T)$:

$$\int_0^T b_i(t)b_j(t)\ dt = h\ \delta_{ij} \quad for\ i,j = 1,2,\dots,m \tag{27}$$

Lemma-I: Let the functions $f(t),\ g(t) \in L^1$, be extended in BPF as $f(t) = FB(t)$ and $g(t) = GB(t)$ respectively. One can define the product.

$$f(t)g(t) = FB(t)B^T(t)G^T = HB(t) \tag{28}$$

Where $H = F \otimes G = \left(f_{ij} \times g_{ij}\right)_{m\times m}$.

Lemma-II: Let $f(x,t),\ g(x,t) \in L^1$, it can be extended in BPF as $f(x,t) = B^T(x)FB(t)$, and $g(x,t) = B^T(x)GB(t)$ respectively, one has been defined as follows

$$f(x,t)g(x,t) = B^T(x)HB(t) \tag{29}$$

here, $H = F \otimes G = \left(f_{ij} \times g_{ij}\right)_{m\times m}$.

Proofs of Lemmas are given in [15].

2.7 Nonlinear Term Approximation (NTA)

The wavelets in the class of m -set of BPF functions are given as

$$\psi(t) = \phi_{m\times m}B_m(t) \tag{30}$$

$$\phi_{m\times m} \triangleq \left[\psi(t_1) \quad \psi(t_2) \quad \psi(t_3) \quad \cdots \quad \psi\left(t_{2^{k-1}M}\right)\right] \tag{31}$$

Where, t_i are the collocations points

$$t_i = \frac{i-0.5}{2^{k-1}M} \qquad \left(\because\ i = 1,2,\dots,2^{k-1}M\right) \tag{32}$$

The operational matrix of the product of wavelets can be designed with the help of the appropriate properties of BPF. Suppose that $f(x,t), g(x,t) \in L^1$ using the properties of BPF and expressed as

$$\begin{cases} f(x,t) = \psi^T(x)F\psi(t) \\ = B^T(x)\phi_{m\times m}F\phi_{m\times m}B(t) \\ g(x,t) = \psi^T(x)G\psi(t) \\ = B^T(x)\phi_{m\times m}G\phi_{m\times m}B(t) \end{cases} \tag{33}$$

Let

$$F_b = \phi_{m\times m}F\phi_{m\times m}$$
$$G_b = \phi_{m\times m}G\phi_{m\times m}$$
$$H_b = F_b \otimes G_b$$

Applying the Lemma-I & II the following results are obtained.

$$f(x,t)g(x,t) = \begin{cases} B^T(x)H_bB(t) \\ B^T(x)\phi_{m\times m}^T inv(\phi_{m\times m}^T)H_b inv(\phi_{m\times m}^T)\phi_{m\times m}B(t) \\ \psi^T(x)H\psi(t) \end{cases} \tag{34}$$

Where $H = inv(\phi_{m\times m}^T)H_b inv(\phi_{m\times m})$.

The following sections are dedicated to the results are obtained from the implemented strategies such as VIM, and VIM with CW (VIMCW). The model problem considered for this work is the CDR problems in one spatial domain.

3. Results and Discussions

In this section, the derived algorithm has been applied to the model problem for the solution of nonlinear PDEs. The comparison of the obtained results with the available well-rated methods and the error profiles of the proposed algorithm are the heart of this work. The effects of the diffusion and other scaling parameters like convection and reaction parameters on the obtained solution curves for these methods is another interesting aspect of this study, which is very clear from the given simulation results.

3.1 Solution of Convection-Diffusion-Reaction CDR PDEs' by VIM

We consider the following problem is from the literature [8], which authors have solved using the Adomain decomposition method (ADM).

$$\begin{cases} u_t = au_{xx} + buu_x + \frac{b^2}{9a}u(u-k)(u+k) \\ u(x,0) = \frac{k\left(-1+C_1 e^{\frac{bkx}{3a}}\right)}{1+C_1 e^{\frac{bkx}{3a}}+C_2 e^{\frac{bkx}{6a}}} \end{cases} \quad (35)$$

Where, C_1 and C_2 are arbitrary constants.

Applying the derived algorithm VIM, the following iterative process is developed.

$$u_{n+1} = u_n - \int_0^t \left(\frac{\partial u}{\partial \tau} - au_{xx} - buu_{xx} - \frac{b^2}{9a}(u^3 - k^2 u)\right) d\tau \quad (36)$$

Finally, the following results for the first three iterations are obtained. The iterative procedure is terminated after three iterations by truncating the higher powers of $t \in [0,1)$.

$$u_0[x,t] = \frac{k\left(-1+C_1 e^{\frac{bkx}{3a}}\right)}{1+C_1 e^{\frac{bkx}{3a}}+C_2 e^{\frac{bkx}{6a}}};$$

$$u_1[x,t] = \frac{k\left(-1+C_1 e^{\frac{bkx}{3a}}\right)}{1+C_1 e^{\frac{bkx}{3a}}+C_2 e^{\frac{bkx}{6a}}} - \frac{b^2 e^{\frac{bkx}{6a}} k^3 \left(-1+C_1 e^{\frac{bkx}{3a}}\right) C_2}{12a\left(1+C_1 e^{\frac{bkx}{3a}}+C_2 e^{\frac{bkx}{6a}}\right)^2} t$$

Thus, the final the result on the third iteration is given as

$$u_{VIM}[x,t] = \begin{cases} \frac{k\left(-1+C_1 e^{\frac{bkx}{3a}}\right)}{1+C_1 e^{\frac{bkx}{3a}}+C_2 e^{\frac{bkx}{6a}}} - \frac{b^2 e^{\frac{bkx}{6a}} k^3 \left(-1+C_1 e^{\frac{bkx}{3a}}\right) C_2}{12a\left(1+C_1 e^{\frac{bkx}{3a}}+C_2 e^{\frac{bkx}{6a}}\right)^2} t - \frac{b^4 e^{\frac{bkx}{6a}} k^5 \left(-1+C_1 e^{\frac{bkx}{3a}}\right) C_2 \left(1+C_1 e^{\frac{bkx}{3a}} - C_2 e^{\frac{bkx}{6a}}\right)}{288a^2\left(1+C_1 e^{\frac{bkx}{3a}}+C_2 e^{\frac{bkx}{6a}}\right)^3} t^2 \\ -\frac{b^6 e^{\frac{bkx}{6a}} k^7 \left(-1+C_1 e^{\frac{bkx}{3a}}\right)\left(C_2{}^2 e^{\frac{bkx}{3a}} - 4C_2 e^{\frac{bkx}{6a}}\left(1+C_1 e^{\frac{bkx}{3a}}\right) + \left(-1+C_1 e^{\frac{bkx}{3a}}\right)^2\right) C_2}{10368a^3\left(1+C_1 e^{\frac{bkx}{3a}}+C_2 e^{\frac{bkx}{6a}}\right)^4} t^3 + O(t^4) \end{cases} \quad (37)$$

For the evaluation of the performance of the given method and the comparative study of the obtained results the exact solution of the problem is known and given as:

$$u[x,t] = \frac{k\left(-1+C_1 e^{\frac{bkx}{3a}}\right)}{1+C_1 e^{\frac{bkx}{3a}}+C_2 e^{\frac{b^2k^2t}{12a}+\frac{bkx}{6a}}} \quad (38)$$

The error profiles for the above results are given in the following Tab. 1 as error analysis for VIM, MVIM and ADM [8]. It is observed from Table.1 that the results for all the discussed methods are very accurate and closed to the exact solution. The algorithm VIM significantly better performs than MVIM for this given problem and its accuracy is much closer to the ADM algorithm; moreover, both the methods VIM and ADM perform better than MVIM.

Table 1

Absolute Error for VIM, MVIM and ADM

T	x	VIM	MVIM	ADM [8]
t=0.2	x=0.2	1.0×10^{-19}	1.7×10^{-12}	0
t=0.4	x=0.4	6.5×10^{-19}	1.4×10^{-12}	4.3×10^{-19}
t=0.6	x=0.6	4.3×10^{-19}	4.7×10^{-11}	4.3×10^{-19}
t=0.8	x=0.8	3.9×10^{-18}	1.1×10^{-10}	2.6×10^{-18}
t=1.0	x=1.0	6.9×10^{-18}	2.1×10^{-10}	7.8×10^{-18}

Simulation results of the obtained results from various methods are given as under

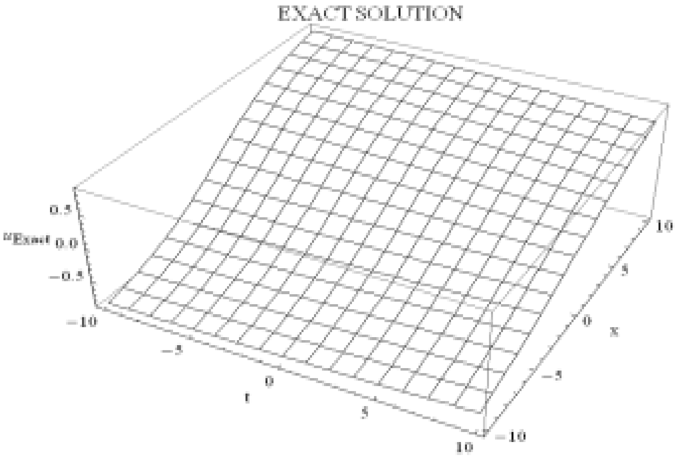

Fig. 1. Plot of Exact Solution of Eq. 47

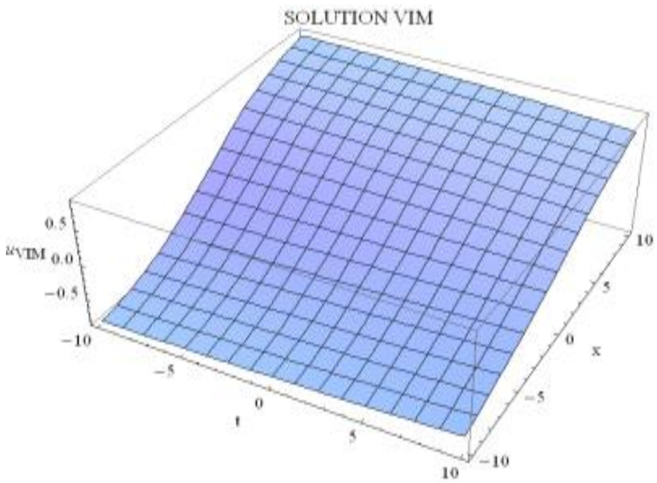

Fig. 2. Plot of the Eq. (47) by VIM

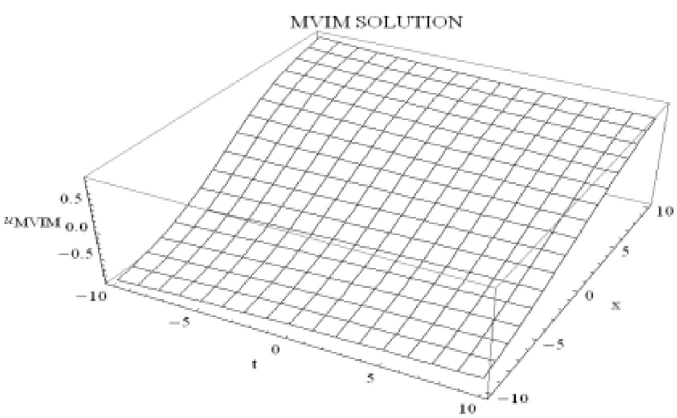

Fig. 3. Plot of Eq. (47) by MVIM

3.2 Results by the Combining the VIM with VIMCW

In the spirit of the above results of VIM and results are extended toward the main goal of this study by coupling the algorithm VIM with the latest wavelet functions like Chebyshev wavelets (VIMCW).

We consider the following problem available in the literature [7]

$$\begin{cases} u_t + \alpha u^{\delta} u_x - \varepsilon u_{xx} = \beta u(1-u^{\delta})(u^{\delta}-\gamma) \\ \qquad u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2} tanh(a_1 x)\right]^{\frac{1}{\delta}} \end{cases} \tag{39}$$

The exact solution to the problem (39) is directly borrowed from the reference [7] and is given by

$$u(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2} tanh\big(a_1(x - a_2 t)\big)\right]^{\frac{1}{\delta}} \tag{40}$$

Where $a_1 = \frac{-\alpha\delta \pm \sqrt{\alpha^2 + 4\beta(1+\delta)}}{4(1+\delta)}\gamma$

$a_2 = \frac{\alpha\gamma}{1+\delta} - \frac{(1+\delta-\gamma)\left(-\alpha\delta \mp \sqrt{\alpha^2+4\beta(1+\delta)}\right)}{2(1+\delta)}$

Case 01: When $\alpha = 0$, $\delta = 1$, $\beta = 1$ the Eq. (39) is reduced to the Fitzhugh-Nagumo Equation, a Reaction-Diffusion equation is generally applied in the circuit theory and estimated the transmembrane potential in the Axon. Here we are applying our strategy VIM with VIMCW and the obtained results are given in Table 2. From Table 2, it is observed that by the combination of VIM with CW, the high accuracy with the low computational cost is obtained. The results, in this case, are also much closed to the exact solution. From the error profiles as given in Table 2, one can observe that the method variational iteration method with Chebyshev wavelets (VIMCW) performs slightly better compared to the algorithm ADM [7]. Moreover, high accuracy is observed if the results are compared with the other methods.

Table 2

The Numerical Results for the ADM, VIMLW and VIMCW (Case-1)

X	T	Exact Solution	ADM [7]	VIMLW[15]	VIMCW	Error ADM [7]	Error VIMLW[15]	Error VIMCW
	0.05	0.00050003	0.000500218	0.000500128	0.000500000	1.88E-07	9.8E-08	3E-08
0.1	0.1	0.000500043	0.000500418	0.000500116	0.000499987	3.75E-07	7.3E-08	5.6E-08
	1	0.000500268	0.000504018	0.000499891	0.000499762	3.75E-06	3.77E-07	5.06E-07
	0.05	0.000500101	0.000500288	0.000500694	0.000500050	1.87E-07	5.93E-07	5.1E-08
0.5	0.1	0.000500113	0.000500488	0.000500682	0.000500037	3.75E-07	5.69E-07	7.6E-08
	1	0.000500338	0.000504089	0.000500457	0.000499812	3.751E-06	1.19E-07	5.26E-07
	0.05	0.000500172	0.000500359	0.000501260	0.0005001	1.87E-07	1.088E-06	7.2E-08
0.9	0.1	0.000500184	0.000500559	0.000501247	0.000500087	3.75E-07	1.063E-06	9.7E-08
	1	0.000500409	0.00050416	0.00050102	0.000499862	3.751E-06	6.11E-07	5.47E-07

Case 02: When $\delta = 1$, $\alpha \neq 0$ and $\beta \neq 0$ the Eq. (39) gives a prototype model for describing the interaction among the reaction mechanisms, diffusion and convection effect. Following the same lines as given above, the VIMCW yields the following numerical result (Table 3). From Table 3, it is observed that VIM with CW provides high accuracy and results are much closed to the exact solution. Moreover, these methods perform better than the ADM.

Table 3

The Numerical Results for the ADM, VIMCW (Case-2)

X		Exact Solution	ADM [7]	VIMLW [15]	VIMCW	Error ADM[7]	Error VIMLW[15]	Error VIMCW
	0.05	0.000500019	0.000500212	0.000500012	0.0005	1.93E-07	7E-09	1.9E-08
0.1	0.1	0.000500025	0.000500412	0.000500000	0.000499987	3.87E-07	2.5E-08	3.8E-08
	1	0.000500137	0.000504012	0.000499762	0.000499762	3.875E-06	3.75E-07	3.75E-07
	0.05	0.000500069	0.000500262	0.000500062	0.000500050	1.93E-07	7E-09	1.9E-08
0.5	0.1	0.000500075	0.000500462	0.000500049	0.000500037	3.87E-07	2.6E-08	3.8E-08
	1	0.000500187	0.000504063	0.000499812	0.000499812	3.876E-06	3.75E-07	3.75E-07
	0.05	0.000500119	0.000500312	0.000500112	0.0005001	1.93E-07	7E-09	1.9E-08
0.9	0.1	0.000500125	0.000500512	0.000500087	0.000500087	3.87E-07	3.8E-08	9.7E-08
	1	0.000500237	0.00050413	0.000499862	0.000499862	3.893E-06	3.75E-07	5.47E-07

3.3 CDR Equation with Homogeneous Boundaries And Effect Of The Scaling Parameters On The Solution

In this section, another aspect of the research is the sensitivity of the solution to the scaling coefficients discussed for the CDR equations. A more general CDR PDEs case has been considered with the homogeneous boundary conditions where the main goal is to study the effect of convection, diffusion and reaction parameters on the solution. The different combinations of the scaling parameters are considered a case study for the obtained solution from the derived algorithms VIMCW to select the problems given in this study.

$$u_t + \alpha u u_x - \varepsilon u_{xx} - \beta(1-u)(u-\gamma)u = 0 \quad \left(\because x \in (0,1)\right) \tag{41}$$

$$u(x,0) = sin(\pi x), \quad u(0,t) = u(1,t) = 0, \; 0 < t \leq T \tag{42}$$

The results for analyzing the effects of the convection, diffusion and reaction scaling parameters on the solution obtained from the proposed VIMCW with VIM algorithms are given in Tables 4-6.

Table 4

Effect of the Diffusion parameter on the solution, when α=β=γ=1and ε=0.1

X	T	VIM [31]	VIMLW[15]	VIMCW
	0.02	0.662906586	0.6632478110	0.6633897720
0.25	0.05	0.605471222	0.6062258200	0.6062907426
	0.1	0.533058172	0.5359022881	0.5357504370
	0.02	0.978505105	0.9783715082	0.9782226472
0.5	0.05	0.939899372	0.9401837872	0.9400889898
	0.1	0.859772696	0.8606183819	0.8606204326
	0.02	0.318734870	0.318901413	0.3188620393
0.9	0.05	0.334509962	0.336151406	0.3358843048
	0.1	0.364333749	0.369785657	0.368970441

Table 5

Analysis of the computed solution with different parameters, when $\alpha = \beta = \gamma = 1$ and for $\varepsilon = 0.01$

X	T	VIM [31]	VIMLW[15]	VIMCW
	0.02	0.6737738022	0.6741892803	0.6743629928
0.25	0.05	0.6263884372	0.6269184245	0.6270897822
	0.1	0.5545364193	0.5555862849	0.5557212602
	0.02	0.9960566781	0.9958626738	0.9956881105
0.5	0.05	0.9827805512	0.9831830066	0.9830295036
	0.1	0.9411525522	0.943148253	0.9430615633
	0.02	0.3248086356	0.324698387	0.3247817676
0.9	0.05	0.3519758496	0.353302645	0.3534087406
	0.1	0.4073554372	0.413604810	0.413736658

Table 6

Analysis of the computed solution with different parameters, when $\alpha = \beta = \gamma = 1$ and for $\varepsilon = 0.00001$

X	T	VIM [31]	VIMLW[15]	VIMCW
	0.02	0.6749934115	0.6754206971	0.6755977590
0.25	0.05	0.6287900677	0.6293176707	0.6294996607
	0.1	0.5572184911	0.5581594597	0.5583217752
	0.02	0.9980241078	0.9978210985	0.9976437121
0.5	0.05	0.9876581003	0.9880606002	0.9878995546
	0.1	0.9506424332	0.9527147813	0.9526135246
	0.02	0.3254891025	0.325357637	0.3254564968
0.9	0.05	0.3539562777	0.355318013	0.355466233
	0.1	0.4123139907	0.4189396763	0.419173732

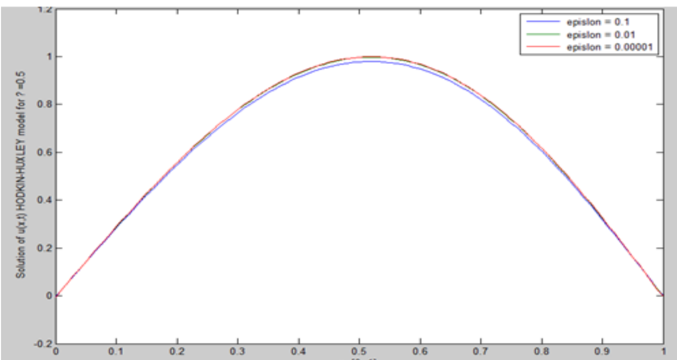

Fig. 4. Effect of Diffusion Co-efficient εon the solution $u(x,t)$

It is observed from Fig. 4 that at any known time$t$, the consistent and smooth curve for the given parabolic problem is observed parabolic symmetric solution with a maximum peak at $x = 0.5$. Also observed from Tabs 04-06 and Fig. 04, the obtained solution slightly increases as the scaling weight ε decreases from 0.1 to 0.01; however, no any significant effect is observed experimentally in the case, if the$\varepsilon < 0.01$. For example, in this case, the result is given in Fig. 04 at $\varepsilon = 0.00001$.

The approximate solution of (41) and VIMCW is estimated on various ε and γ values with α, and β as fixed weights. Following tables 07, 08, 09 and 10 show the comparison of the computed solution with VIMCW with VIM and the role of the reaction co-efficient γ with $\varepsilon = 0.1$ and $\varepsilon = 0.001$.

Table 7

Analysis of the computed solution with different parameters, when $\alpha = \beta = 1, \gamma = 0.01$and $\varepsilon = 0.1$

X	T	VIM [31]	VIMLW [15]	VIMCW
	0.02	0.6670784424	0.667455648666	0.667596736126
0.25	0.05	0.6160138406	0.617001684564	0.617065064632
	0.1	0.5536691817	0.557425730663	0.557272093308
	0.02	0.9788933699	0.978787984401	0.978641801385
0.5	0.05	0.9423019182	0.942533408286	0.942442774078
	0.1	k0.8692221544	0.869456480570	0.869460713328
	0.02	0.3228359713	0.322968604672	0.322930885759
0.9	0.05	0.3443273981	0.345288149504	0.345021960094
	0.1	0.3827278420	0.384629746723	0.383817641668

Table 8

Analysis of the computed solution with different parameters, when $\alpha = \beta = 1, \gamma = 0.9$and for$\varepsilon = 0.1$

X	T	VIM [31]	VIMLW [15]	VIMCW
	0.02	0.6670784424	0.667455648666	0.667596736126
0.25	0.05	0.6160138406	0.617001684564	0.617065064632
	0.1	0.5536691817	0.557425730663	0.557272093308
	0.02	0.9788933699	0.978787984401	0.978641801385
0.5	0.05	0.9423019182	0.942533408286	0.942442774078
	0.1	k0.8692221544	0.869456480570	0.869460713328
	0.02	0.3228359713	0.322968604672	0.322930885759
0.9	0.05	0.3443273981	0.345288149504	0.345021960094
	0.1	0.3827278420	0.384629746723	0.383817641668

Table 9

Analysis of the computed solution with different parameters, when $\alpha = \beta = 1, \gamma = 0.01$and for $\varepsilon = 0.001$

X	T	VIM [31]	VIMLW [15]	VIMCW
	0.02	0.6791313751	0.679568293661	0.679743741062
0.25	0.05	0.6397049627	0.640311460092	0.640489146652
	0.1	0.5804655667	0.58173261843	0.581886016511
	0.02	0.9978328741	0.997678598552	0.997504155418
0.5	0.05	0.9871981140	0.987665627665	0.987512492313
	0.1	0.9497956188	0.951703890379	0.951619335015
	0.02	0.3299199448	0.329948413546	0.33004377655
0.9	0.05	0.3660743434	0.367940162976	0.36808002838
	0.1	0.4402984102	0.448105396197	0.448320025468

Table 10

Analysis of the computed solution with different parameters, when $\alpha = \beta = 1, \gamma = 0.9$and for$\varepsilon = 0.001$

X	T	VIM [31]	VIMLW [15]	VIMCW
	0.02	0.6753039590	0.675731145512	0.675907736608
0.25	0.05	0.6296844687	0.630220016083	0.630400632227
	0.1	0.5593388359	0.560321154424	0.56048020832
	0.02	0.9978293608	0.997632470154	0.997455549656
0.5	0.05	0.9871761577	0.987587279157	0.987427653581
	0.1	0.9497078083	0.951764714775	0.951666479908
	0.02	0.3258745591	0.325762586512	0.32585887431
0.9	0.05	0.3549948213	0.356409044397	0.35655185817
	0.1	0.4146640268	0.42139503472	0.421617270271

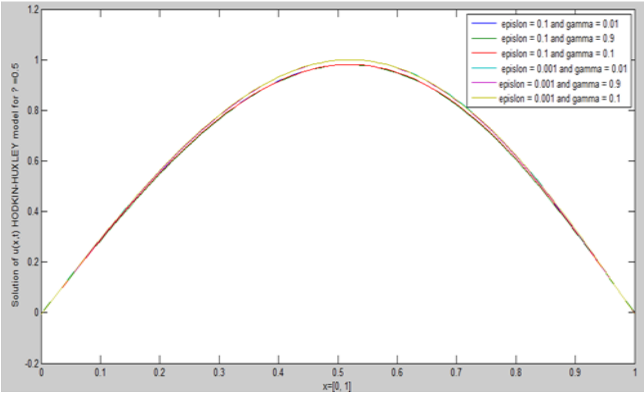

Fig. 5. Effect of Reaction γon the solution $u(x,t)$

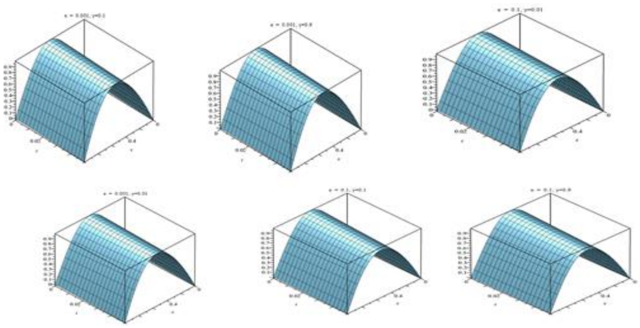

Fig. 6. Solution $u(x,t)$of CDR PDE for $\alpha=\beta=1;\ \gamma=0.01,0.1,0.9$ and $\varepsilon=0.001,0.1$

Fig.s 5-6, the solution is computed for $\boldsymbol{\alpha=\beta=1};\ \boldsymbol{\gamma=0.01,0.1,0.9}$ and $\boldsymbol{\varepsilon=0.001,0.1.}$ From the quantitative information given in tables (Tables-07-10) and the simulation curves (4.5), it is observed at the diffusion coefficient $\varepsilon=0.1$ and the variation in the value of reaction coefficient γ has almost no effect on the solution, but at the small value of the diffusion for example $\varepsilon=0.001$and with various combinations of the reaction coefficient, a minimal effect is observed on the solution.

In the following simulations (Fig. 7-8), the solution of the given model equation (4.8) is shown to select the various time steps.

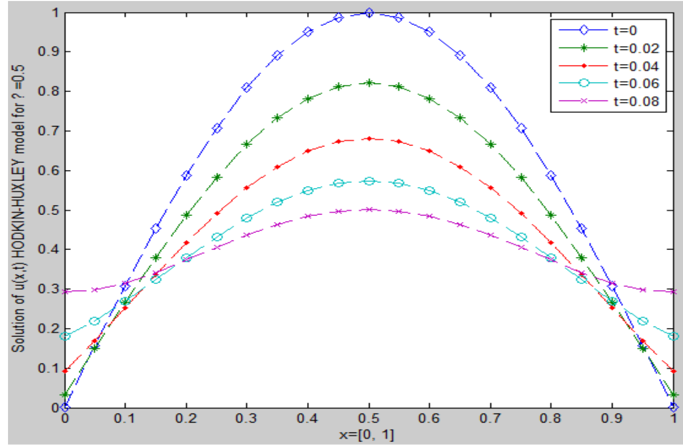

Fig. 7. Solution of Fitzhugh-Nagumo Equation, when $\alpha=0,\varepsilon=\beta=1;\ \gamma=(0,1)$by VIMLW

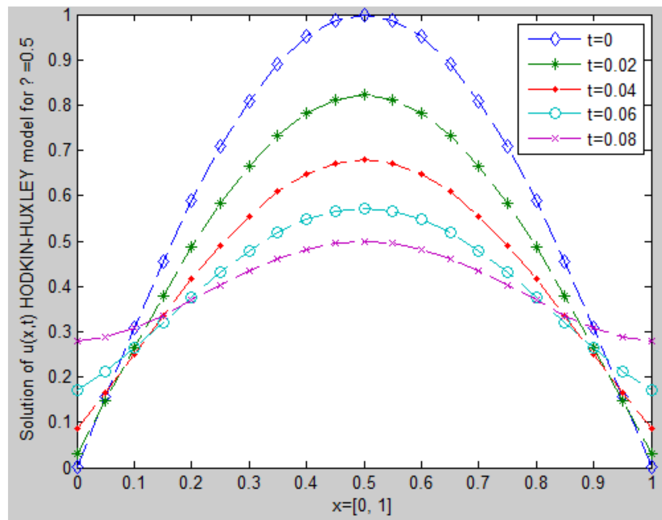

Fig. 8. Solution of Fitzhugh-Nagumo Equation, when $\alpha=0,\varepsilon=\beta=1;\ \gamma=0.5$by VIMLW

The simulations (07 and 08), show that the significant fall in delayed solutions. Moreover, it is also noticed that a parabolic profile is observed for a small value of time. For the large value of time, the solution deviates from the parabolic nature, which disagrees with the boundary conditions and consequently the oscillations appear in the solution.

4. Conclusion

The wavelet-based algorithm was designed and tested in this study to solve nonlinear PDEs, where the VIM is combined with CW to design the VIMCW algorithm. The models were considered CDR problems in one spatial domain for this study. The developed algorithm has been applied to the proposed model problem, where the error profiles and the effect on the solution of the scaling parameters are interesting aspects of the proposed study. The efficiency and accuracy of the method were evaluated, and it was observed that the VIM performs slightly better than the ADM. Also, it has been observed from the computational point of view that MVIM is the fast converging, but less accurate approach compared to ADM and VIM. In comparison to the VIM and ADM methods, it was observed that the proposed work VIMCW method can overcome the problem complexity of the integration and derivatives appearing in nonlinear terms and with no symbolic computations consequently. It was observed that VIMCW performs better than both the ADM and VIM, along with the less computational complexities and given error profiles.

It is further observed from prepared simulation results that the effect of the variations in small diffusion scaling parameters reveals a significant effect on the solution with the scaling parameters of reaction terms. It is observed that the larger time steps create the oscillations in the solution from the given simulations with the choice of different discrete time steps and the most suitable choice for the selection of the time step is 0.1 or less. The Operational matrices for the wavelets were designed and used to cope with nonlinear terms by using the applications of the BPFs. As a result, such modifications and developments speed the convergence and reduce the computational complexities as compared to the existing methods. The most interesting aspect is that the designed models have a good future for research. In view of this study, it is proposed that the work for the higher dimensions may be continued.

Conflict of Interests

The authors declared no potential conflicts of interests with respect to the research, authorship, and publication of this article.

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